

AST5220-Milestone IV: The CMB and Matter Power-Spectra

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Abstract

The codes for this paper can be found at:
<https://github.com/SagittariusA-Star/AST5220-Milestones>

1. INTRODUCTION

The aim of cosmology as a science has always been to unearth the secrets of how the Universe was formed and how it has evolved since. As modern high precision measurements by probes like Planck have shown is how successful the cosmological standard model with dark energy and cold dark matter (Λ CDM) is in describing the observed data of the CMB and its anisotropies (Planck Collaboration I 2018). One of the most common ways to estimate the parameter values of the Λ CDM model inferred from the data, is to compare the observed and the theoretical CMB and matter power spectra and find the best-fit by means of a statistical treatment.

In this paper we will focus on computing the CMB and matter power spectra, by building upon the foundation by Stutzer (2020a), Stutzer (2020b) and Stutzer (2020c) which computed the background evolution, recombination history and formation of structure from primordial perturbations of the Universe. The ultimate goal of computing the power spectra will be achieved using the same cosmological parameters as Callin (2006), also previously used in Stutzer (2020a), Stutzer (2020b) and Stutzer (2020c). Further more, we will for simplicity neglect the spatial curvature, neutrinos and polarization of photons.

2. METHOD

When solving for the CMB and matter power spectra in this paper we will mainly use the equations and relations that are provided by Winther (2020), Callin (2006) and Dodelson (2003). Unless otherwise stated the equations presented here are thus provided by these authors and refer the interested reader to them for detailed derivations, and only present the main outline towards computing the power spectra.

2.1. The Power Spectra

Before going on to actually computing the power spectra, a few words on what a power spectrum actually is. To find out this we consider the CMB as an example. The CMB as we see it today is build up of an average temperature of $T_{\text{CMB}} = 2.7255\text{K}$ upon which there

are small perturbations of the order $\delta T/T_{\text{CMB}} \sim 10^{-5}$ seen as anisotropies in the CMB. One can think of the CMB of a function on the celestial sphere that can be expanded in term of its anisotropies of different scales as basis functions. This is done using spherical harmonics as

$$T(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}), \quad (1)$$

where the temperature T in the direction \hat{n} on the sky is given by the sum over the spherical harmonics $Y_{\ell m}$ times the corresponding expansion coefficient $a_{\ell m}$. The indices ℓ and m quantify the scale and orientation of the perturbations. For instance $\ell = 0$, $\ell = 1$ and $\ell = 2$ represent the monopole (average temperature), dipole (hot and cold blobs separated by 180° on the sky) and the quadrupole (alternating hot and cold separated by 90° on the sky).

The angular power spectrum of the CMB anisotropies is simply related to this expansion and corresponds to the expected value of the expansion coefficient squared for each scale ℓ

$$C_\ell = \langle |a_{\ell m}|^2 \rangle = \langle a_{\ell m} a_{\ell m}^* \rangle. \quad (2)$$

In principle, though, there should be a dependence on the orientation m , however, since the CMB according to the underlying cosmological principle must be isotropic on large scales we can simply average out the directional dependence without large errors. The angular power spectrum C_ℓ hence quantifies how much contribution to the CMB there is on each angular scale, i.e. the amplitude, and can be understood as a correlation function between to points of a given angular separation.

In order to compute the power spectrum C_ℓ we need the coefficients $a_{\ell m}$ which are given by the temperature field at time $x = \ln a(t)$ $T(\hat{n}, x)$. The temperature fields evolution through time x and for different scales k were found in Stutzer (2020c) in the form of the photon perturbation multipole moments $\Theta_\ell(k, x)$. These multipole moments were however found in fourier \vec{k} -space, but we want to have them in real (\vec{n} -) space at the present time.

To get the perturbations $\Theta_\ell(\hat{n}, x)$ today, we simply inverse fourier transform and evaluate the result at the present time $x = 0$.

There is only one caveat to this idea. In order to compute all the coefficients $a_{\ell m}$ we would need the infinite series of multipole moments Θ_ℓ , however, [Stutzer \(2020c\)](#) only computed around 8 of them, but we would need around $\ell_{t\text{extmax}} \sim 1200$ to get a decent power spectrum ([Winther 2020](#)). Solving for all the multipole moments up to $\ell = 1200$ using the approach of [Stutzer \(2020c\)](#) would take a long time and be very inefficient.

Fortunately Zaldarriaga and Seljak solved this problem for us by inventing the *line-of-sight* integration. To obtain the coupled equations for the Θ_ℓ 's in the first place one used the photon perturbation $\Theta(k, \mu, x)$, being a function of angle $\mu = \cos \theta$. Thus one can, instead expanding the perturbation equation into multipoles and then solving the long coupled system of differential equations as in [Stutzer \(2020c\)](#) (but with $\ell_{\text{max}} = 1200$), integrate up the equation for the underlying Θ subsequently find the multipoles. This is shown in detail by [Callin \(2006\)](#) and [Dodelson \(2003\)](#) and yields the following integral for the multipoles

$$\Theta_\ell(k, x = 0) = \int_{-\infty}^0 \tilde{S}(k, x) j_\ell[k(\eta_0 - \eta)] dx, \quad (3)$$

where j_ℓ are the spherical Bessel functions being highly oscillatory functions function as an orthogonal basis here that basically project the project the 3D spatial temperature field onto the 2D celestial sphere (as we observe it) and η (subscript 0) is the conformal time (today).

The quantity $\tilde{S}(k, x)$ is called the source function and is given by

$$\begin{aligned} \tilde{S}(k, x) = & \tilde{g} \left[\Theta_0 + \Psi + \frac{1}{4}\Pi \right] + e^{-\tau} [\Psi' - \Phi'] \\ & - \frac{1}{ck} \frac{d}{dx} (\mathcal{H} \tilde{g} v_b) + \frac{3}{4c^2 k^2} \frac{d}{dx} \left[\mathcal{H} \frac{d}{dx} (\mathcal{H} \tilde{g} \Pi) \right], \quad (4) \end{aligned}$$

where the metric potential perturbations Ψ and Φ , the baryon velocity v_b and the photon mono- and quadrupole Θ_0 and Θ_2 ($\Pi = \Theta_2$ if polarization is neglected) are all computed by [Stutzer \(2020c\)](#). The optical depth τ and visibility function \tilde{g} , and the reduced Hubble parameter \mathcal{H} are found by [Stutzer \(2020b\)](#) and [Stutzer \(2020a\)](#) respectively. The source function quantifies how light is affected (both extinction and emission) when traveling through a medium, which in our case is the universe.

This approach of finding the multipoles Θ_ℓ is much faster as we only need Θ_0 and Θ_2 from the coupled set of ODEs and it has the advantage of being easier to understand. Intuitively one can interpret the

line-of-sight integration as integrating up a local radiation monopole through the universes history, which is represented by the term $\tilde{g}\Theta_0$. The terms $\tilde{g}\Psi$ and $\tilde{g}\frac{1}{4}\Pi$ are corrections from the photons loosing energy through the potential wells and the correction of the local quadrupole (polarization if included) respectively. Secondly one can spot how the radiation field is affected by the change of the gravitational potentials, seen in the second part of \tilde{S} , and represents the gain in energy through the so-called Integrated Sachs-Wolf (ISW) effect. Basically the ISW is that photons that fall into a potential well will gain energy, but lose less than they gained when exiting, since the gravitational wells have decayed due to spatial expansion while the photon traveled through it. The third part of \tilde{S} is simply a doppler effect term.

Now that we have all the Θ_ℓ 's we can finally find the power spectrum C_ℓ . As computed by [Stutzer \(2020c\)](#) the metric perturbations are initially of order $\Psi \sim 1$ which was the underlying initial condition for the whole set of coupled equations for the perturbations. To get the actual initially condition set up by inflation the multipoles (squared) are simply multiplied by the primordial power spectrum, given by the Harrison-Zel'dovich spectrum $P_{\text{primordial}}(k)$ given by

$$\frac{k^3}{2\pi^2} P_{\text{primordial}}(k) = A_s \left(\frac{k}{k_{\text{pivot}}} \right)^{n_s - 1}. \quad (5)$$

The primordial amplitude parameter $A_s \sim 2 \cdot 10^{-9}$ at a pivot scale $k_{\text{pivot}} = 0.05/\text{Mpc}$ and the spectral index of scalar perturbations $n_s \approx 0.96$. The primordial amplitude thus quantifies the overall height of the spectrum, while the spectral index and the pivot scale quantify the tilt and whether the primordial spectrum is scale invariant (which is not since $n_s \neq 1$). This simple rescaling we can simply do since all equations are linear. The final power spectrum is then simply found by integrating up all the squared multipoles weighted by $P_{\text{primordial}}(k)$ over all of \vec{k} -space

$$C_\ell = C_\ell = \frac{2}{\pi} \int k^2 P_{\text{primordial}}(k) \Theta_\ell^2(k) dk \quad (6)$$

$$= 4\pi \int_0^\infty A_s \left(\frac{k}{k_{\text{pivot}}} \right)^{n_s - 1} \Theta_\ell^2(k) \frac{dk}{k}, \quad (7)$$

where we could simply transform the 3D integral to a 1D integral because of isotropy. It is common to present the CMB angular power spectrum $\frac{\ell(\ell+1)}{2\pi} C_\ell$ in units μK^2 , which is obtained by multiplying it with $(10^6 T_{\text{CMB}})^2$. The prefactor $\frac{\ell(\ell+1)}{2\pi}$ is used for cosmetic purpose, to eliminate some of the tilt of the spectrum.

The next thing to compute is the matter (both dark and baryonic) power spectrum. This turns out, is the

easiest part of the work in this paper, as it is simply found by things we already know from Stutzer (2020c).

$$P_M(k, x) = |\Delta_M(k, x)|^2 P_{\text{primordial}}(k), \quad (8)$$

where the comoving density contrast $\Delta_M(k, x)$ (which is gauge invariant) is given by

$$\Delta_M(k, x) \equiv \frac{c^2 k^2 \Phi(k, x)}{\frac{3}{2} \Omega_{M0} a^{-1} H_0^2} \quad (9)$$

(Winther 2020). This we can simply compute at the present time $x = 0$ by using the splined quantities from Stutzer (2020c) that go into 9.

The power spectra for the CMB C_ℓ and for matter $P_M(k)$ are then plotted versus multipole index ℓ and (comoving) wavenumber k .

2.2. The Implementation

3. RESULTS

4. DISCUSSION

5. CONCLUSION

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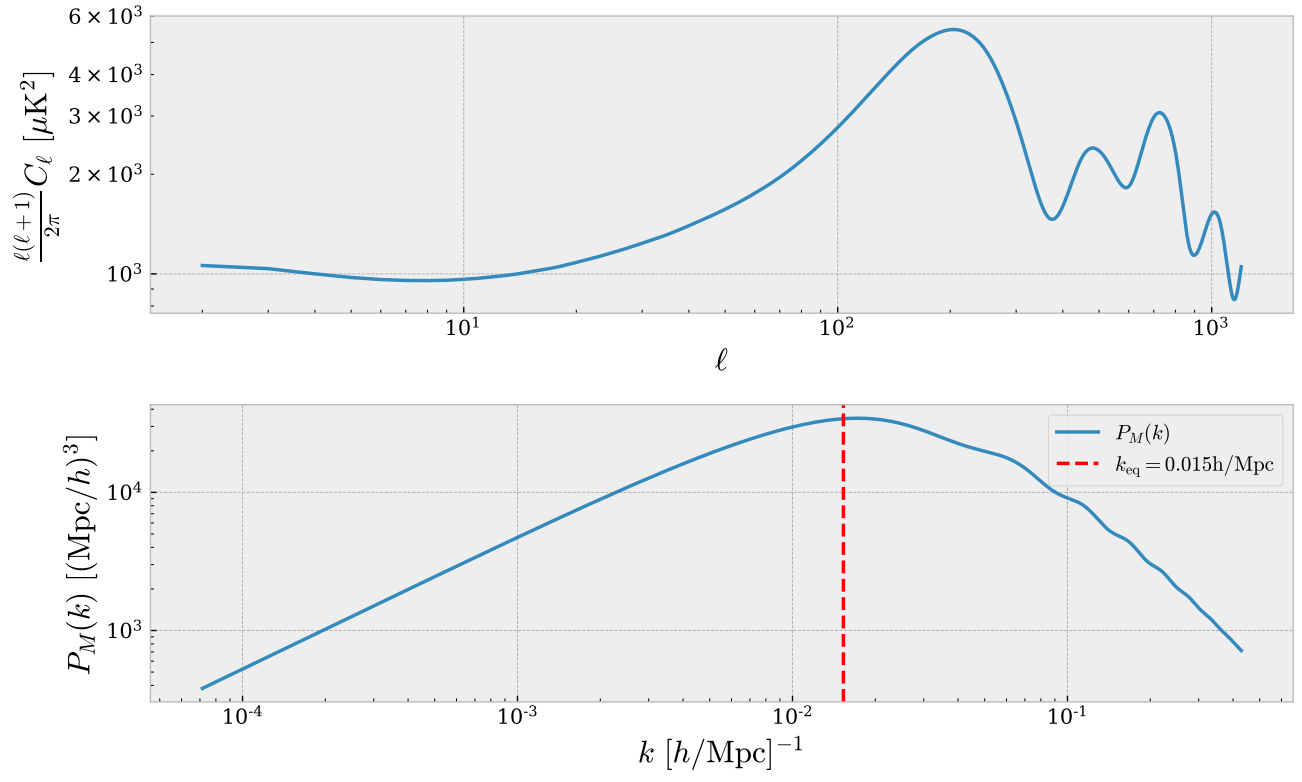


Figure 1.

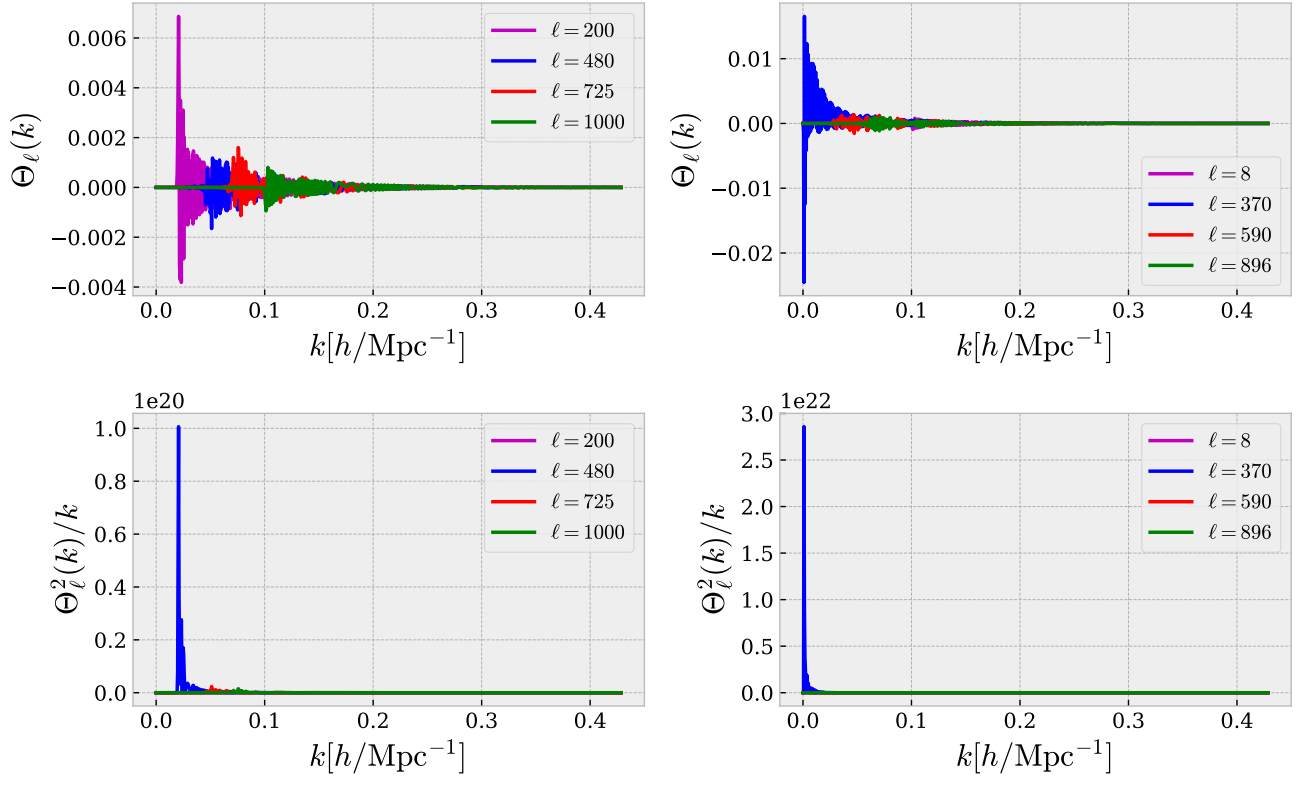


Figure 2.