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AST5220-Milestone I: The Background Cosmology

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Abstract

We have simulated the large scale evolution of the universe. The expansion rate of the universe was computed, by the Friedmann equation, the particle horison scale (the conforal time) was computed. Using these and/or combined with the matter-energy density parameters made it possible to estimate in which each matter-energy component domminated the universe's evolution at any given time. The matter-radiation equality was found to be at log-scale factor $x \sim -7$ and the matter-dark energy equality at $x \sim 0$. The behavior of the computed quantities was consistant with known approximations and known results.

1. INTRODUCTION

In order to discribe the universes evolution the first, perhaps most fundamental part, is to discribe its large scale dynamics. Thus in this project we will define some consepts and quantities used to discribe the large scale dynamics of the universe as a whole, since the Big Bang untill today, the so-called Background Cosmology. The main equation used for this is the Friedmann equation. Furthermore, we will study how the different components of the matter-energy content of the universe evolves and how the paticle horizon evolves as the universe expands.

2. METHOD

2.1. *Consepts and Quantities*

Before starting on how to solve for the evolution of the universe as a whole, we start introducing some consepts and quantities. Because we know from previously conducted cosmological experiments (KILDER) that the universe is nearly flat, we will here only consider the case of a flat universe filled with a homogenous and isotropically distributed matter-energy content. The latter of which is called the cosmological principal. Doing theis the invariant line-element is given by the Friedmann–Lemaître–Robertson–Walker metric (FLRW metric)

$$ds^2 = -c^2dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \tag{1}$$
$$= a^2(t)(-d\eta^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{2}$$

where $a(t)$ and eta denote the scale factor and the conformal time respectively. The scale factor quantifies the expansion of the universe, being a translation factor between proper (physical) and comoving distances. For convinience we introduce the log-scale facor $x \equiv \log a$ (base e), because we will consider a wide range of universe scales. The universe scale today at $t = t_0$ is normalized to $a(t = t_0) = a_0 = 1$, or in log-scale $x_0 = 0$. The conformal time is the total time a photon is able to travle since the Big Bang at $t = 0$ until a time t, and is thus also a measure of cosmic time. It is thus equivalent to the particle horizon scale of the universe at any given time, and we will here define it by a ordinary differential equation (ODE)

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{c}{a}, \tag{3}$$

which we can rewrite into

$$\frac{\mathrm{d}\eta}{\mathrm{d}a} = \frac{c}{a^2H} = \frac{c}{a\mathcal{H}}. \tag{4}$$

Here $H(a) \equiv \frac{\dot{a}}{a}$ is the Hubble parameter measuring the expansion rate of the universe. We define the scaled Hubble parameter $\mathcal{H}(a) \equiv aH(a)$.

The Hubble parameter is given by the Friedmann equation

$$H = H_0\sqrt{(\Omega_{b,0} + \Omega_{CDM,0})a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0}}, \tag{5}$$

where $H_0 = 100h\ \mathrm{kms^{-1}Mpc^{-1}}$ is the Hubble parameter today (Hubble constant, and the dimensionless Hubble parameter is usually set to $h = 0.7$) and the $\Omega_{x,0}$'s are the matter-energy density parameters today defined as $\Omega_{x,0} \equiv \frac{\rho_{x,0}}{\rho_{c,0}}$ for a energy component x. The critical density $\rho_c \equiv \frac{3H^2}{8\pi G}$, is the density needed in order to have a flat universe, and is today equal to $\rho_{c,0} \equiv \frac{3H_0^2}{8\pi G}$.

In order to know wheter the conformal time is computet right, one can compare it to the analytical approximations for each epoch of domminance. These are

$$\eta_r(a) = \frac{c}{aH} = \frac{c}{\mathcal{H}(a)} \tag{6}$$

$$\eta_m(a) = \eta(a_*) + 2c\left(\frac{1}{\mathcal{H}(a)} - \frac{1}{\mathcal{H}(a_*)}\right) \tag{7}$$

$$\eta_\Lambda(a) = \eta(\tilde{a}) + c\left(\frac{1}{\mathcal{H}(\tilde{a})} - \frac{1}{\mathcal{H}(a)}\right), \tag{8}$$

for the conformal time in the epoch of domminance for radiation, matter (baryons + CDM) and dark matter respectively. Here a_* and $\tilde{a}$ denote the scale factor when $\Omega_m \approx 1$ and $\Omega_\Lambda \approx 1$ respectively.

In order to know how much each component of the matter-energy content of the universe contributes to the total energy content, we can compute the matter-energy density of each component. This is done when solving the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0, \tag{9}$$

having the solution

$$\rho_x = \rho_{x,0} a^{-3(1+\omega)}, \tag{10}$$

where ρ_x, $\rho_{x,0}$ and $\omega = P/\rho$ are the density at a given time $a(t)$, the density today and the equation of state (EOS) parameter for a matter energy component x, respectively. For pressureless fluids like baryons and cold dark matter (CDM) $\omega = 0$, for relativistic particles like radiation $\omega = 1/3$ and for dark energy (the cosmological constant) $\omega = -1$.

It is, however, more convinient to instead compute the energy density parameters given as $\Omega_x(a) = \rho_x/\rho_c$ at any time. Writing out these we get for a component x that

$$\Omega_x = \frac{\rho_x}{\rho_c} = \frac{\rho_{x,0} a^{-3(1+\omega)} 8\pi G}{3H^2} \tag{11}$$

$$= \frac{\rho_{x,0}}{\rho_{c,0}/H_0^2} \frac{a^{-3(1+\omega)}}{H^2} \tag{12}$$

$$= \frac{\Omega_{x,0}}{(H/H_0)^2} a^{-3(1+\omega)}. \tag{13}$$

Inserting the respective EOS parameters we get that the energy density parameter for baryonic matter, CDM, radiation and dark energy (Λ) at any given universe scale a is given as

$$\Omega_b(a) = \frac{\Omega_{b,0}}{a^3 (H/H_0)^2} \tag{14}$$

$$\Omega_{CDM}(a) = \frac{\Omega_{CDM,0}}{a^3 (H/H_0)^2} \tag{15}$$

$$\Omega_r(a) = \frac{\Omega_{r,0}}{a^4 (H/H_0)^2} \tag{16}$$

$$\Omega_\Lambda(a) = \frac{\Omega_{\Lambda,0}}{(H/H_0)^2}, \tag{17}$$

where we have neglected the curvature parameter Ω_k, sometimes included in the energy-density parameters, as we only consider a flat universe here. Also we have not included the neutrinos on the calculations.

Table 1. Table showing the energy density parameter values at the current time Callin (2006).

x	$\Omega_{x,0}$
CDM	0.224
b	0.046
Λ	0.72995
r	$5.042 \cdot 10^{-5}$

We know that at any given time these density parameters must sum to 1, as they respectively represent the fraction of matter-energy contribution to the total content of the universe. This can for instance be seen from the Friedmann equation (5), when inserting the scale facor today $a_0 = 1$, we must recover $H = H_0$ or else it would not make sence. Thus the density parameters today sum to 1, and for any other time one can simply sum the above density parameters and check wheter they sum to unity. The values of the density parameters today are well known from cosmological surveyes like Planck, and we will here use the values provided by Callin (2006). These can be found in Table 1

Note that the radiation density parameter is given by

$$\Omega_r = 2\frac{\pi^2}{30} \frac{(k_B T_{CMB})^4}{\hbar^3 + c^5} \frac{8\pi G}{3H_0^2}, \tag{18}$$

where the temperature of the Cosmic Micowave Background (CMB) $T_{CMB} = 2.7255$K, and the Boltzmann constant, reduced Planck constant and the speed of light take their regular SI values in our calculations.

2.2. *Implementation*

We want to know how the universe as a whole evolves from the Big Bang until today. To do that we want to compute the evolution of the regular and the scaled Hubble parameters as a function of the log-scale factor x (as we consider a wide range of scales a). This is simply done by generating an array of x values and compute the Hubble parameters from the Friedmann equation and the scaled Hubble parameter by simply multiplying the regular Hubble parameter by the scale factor $a = e^x$. One can simply use the Friedmann equation on the form (5), only having to change the scale factors a to log-scale factors $a = e^x$. Next, one can compute the density parameters from their definition given in the previous subsection, by simply also exchanging the scale factor with an exponential of the log-scale factor.

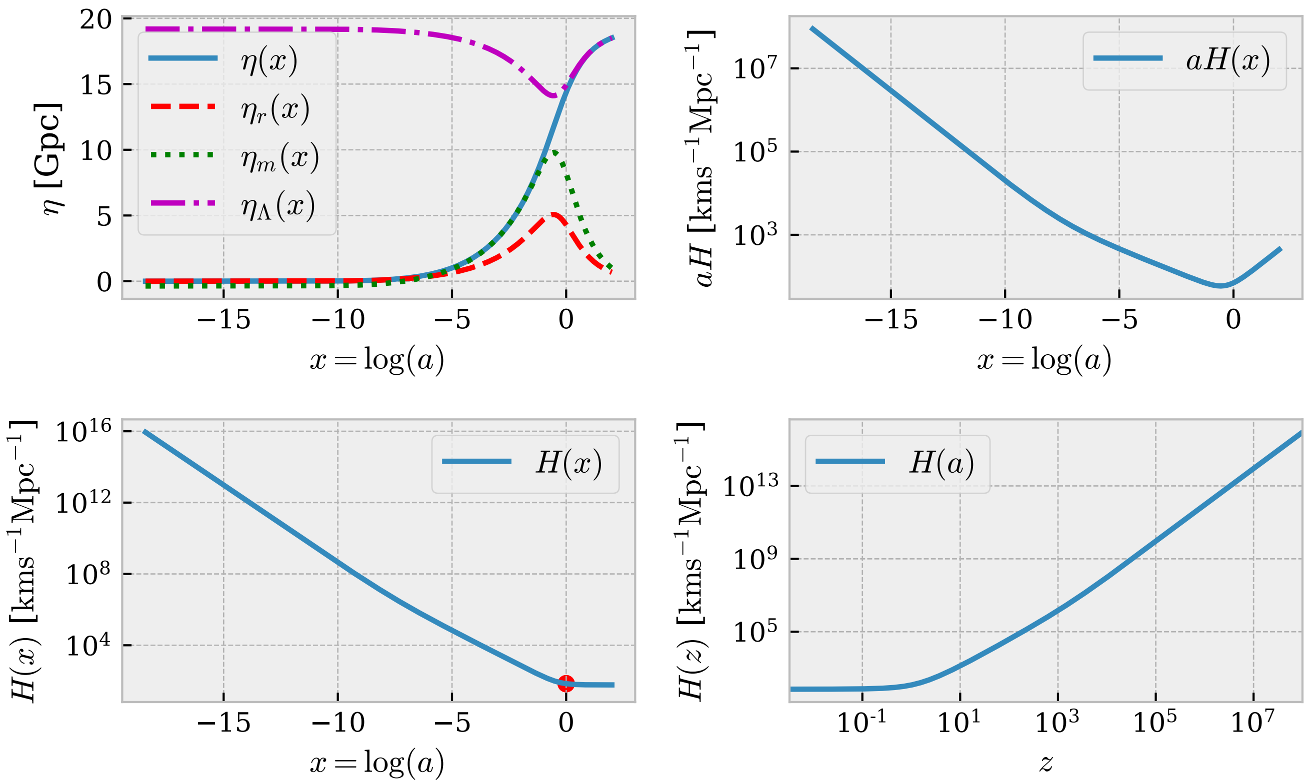

Figure 1. Upper left: The figure shows the conformal time η(horizon scale) in Gpc as a function of the log-scale factor x. Overplotted are the analytical approximations for each era of dommination. **Upper right:** The figure shows the scaled Hubble parameter (expansion rate $\dot{a}$) as a function of the log-scale factor x. **Lower pannels:** Here the Hubble parameter H is shown as a function of the log-scale facor x (**left pannel**) and as a function of the scale factor a and the redshift z (**right pannel**), in addithon to a red dot illustrating the Hubble parameters value today. Note that the Hubble parameter as a function of the redshift is only plotted from early times until today, while the remaining plots go a bit further.

Further, we want to compute the conformal time (particle horizon scale). This is done by simply solving the ODE given in equation (4) using the `ODESolver` (C++) module cindly provided by Hans A. Winther. We use initial conditions $\eta(x) = 0$, as the horizon was very small at early times. We cannot use $a = 0$ here, though, as this results in a singularity. We thus use $a = 10^{-8}$, corresponding to $x \approx -18.42$, to represent the scale at early times. We let the simulation run until $x = 2$ so as to see what happens beyond the current age. We solve the ODE using 1000 points and save them to a file together will the corresponding other quantities (the Ω's, H etc.) After solving for $\eta(x)$ we have a discrete set of conformal times and corresponding log-scale factors. To get a more continous representation, we then perfrom a cubic spline interpolation, so as to enable computation of the conformal time between the previously found discrete values. This is done using the `Spline` modulde cindly provided by Hans A. Winther.

To illustrate the evolution of the large scale universe we now can plot the density parameters as a function of the log-scale factor x, as well as the horizon scale, the regular and scaled Hubble parameters as functions of x. Also we plot the Hubble parameter as a function of redshift z, being another measure of time. It is related to the scale factor by $a^{-1} = 1 + z$, and measures how much a wavelength of light is streached as light travels through an expanding universe.

3. RESULTS/DISCUSSION

The conformal time (horizon scale) as well as the Hubble and scaled Hubble parameters as functions of x (z and a) can be seen in Figure 1. As one can see the horizon scale stayes very small for a long while, from early times until $x \approx -7$, then starting to grow exponentially and finally starting to flatten out towards the end of the simulated period at around $x \sim 0$. Also we have overplotted the analytical approximations for conformal time in each of the epochs of dominnance to check whether our solution to the conformal time makes sence. See Appendix for derivation of the approximations. We see that the approximations overlap the con-

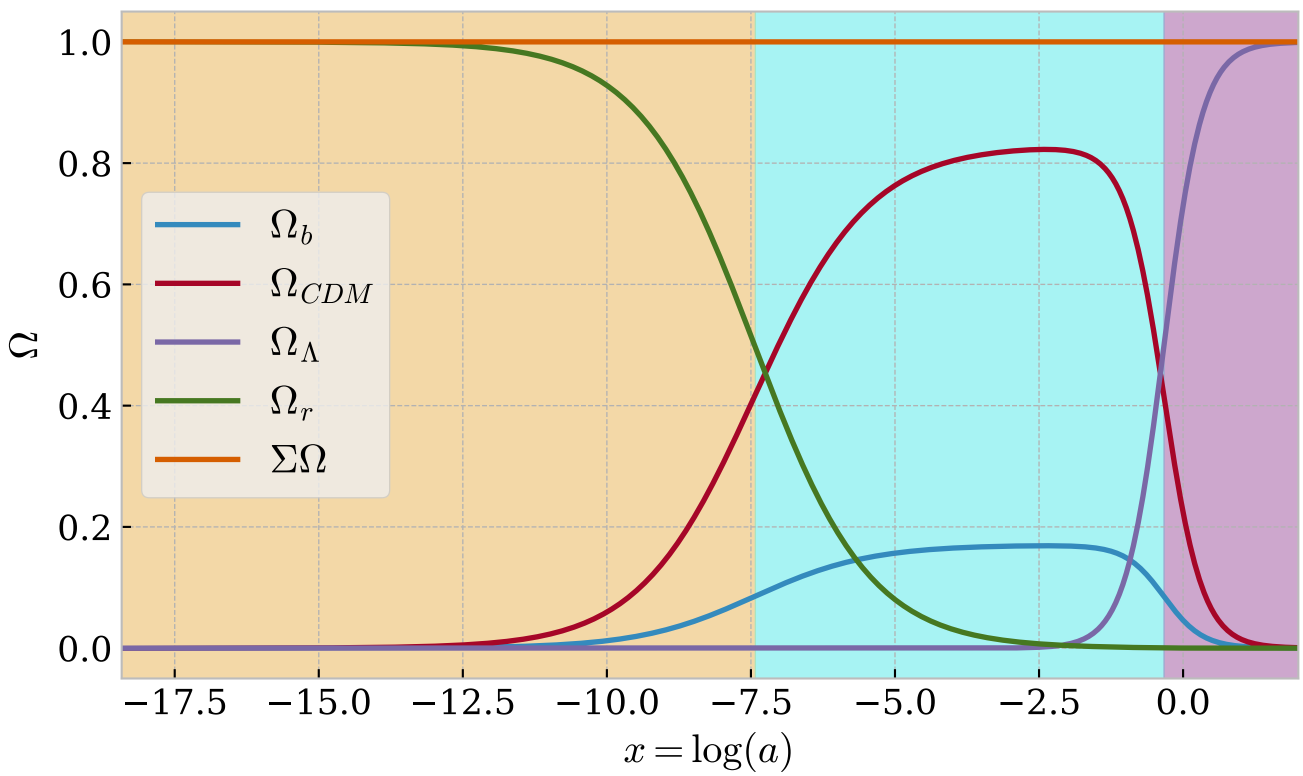

Figure 2. The figure shows the matter-energy density parameters of each component of the total matter-energy content of the universe. Also shown is the sum of all density parameters. To illustrate which component dominates the energy content of the universe at each time, we have colored the radiation domminated era yellow, the matter domminated era blue and the era domminated by dark energy by purple.

formal time resonalbly well withing their respective intervals of valitity. (DISKUTER)

The expansion rate quantified by the scaled Hubble parameter $\mathcal{H}(x) = aH(x)$ is also seen in Figure 1. We can clearly see from its shape in which era of the universe we are in. At early times, when the universe was radiation domminated the scaled Hubble parameter $\mathcal{H} \propto a^{-1}$. When matter (baryons and CDM) eventually started dominating, the expansion rate scaled differently; $\mathcal{H} \propto a^{-0.5}$, having a somewhat shallower slope compared to the expansion rate at radiation domminance. This transition seems to happen at $x \sim -7$, coinsiding roughly with the sudden growth of η, as seen for the change in slope of $\mathcal{H}$. Finally we see the transition from a decelerating universe, to an accelerating one. The expanding universe halts, as seen by the extremum of $\mathcal{H}$, after which the curve turns upwards. This corresponds to the era of dark matter, where the universe gradually turns to an exponential expansion rate. This hypothisis is further supported by the fact that the regular Hubble parameter seen in the bottom left pannel of Figure 1 becomes almost constant (in the log-log) after crossing into the era of dark energy. This is easily seen from the Friedmann equation, assuming that the other densty parameters are negligable. Another notworthy thing is that the Hubble parameter seems to hit its known current value (see red dot in Figure 1) pretty well, putting further evidence on that the solving of the equations are done correctly. The plot in the lower right pannel of Figure 1 tells the same story as the lower left one, however, it is nice to see the redshift (scale factor) dependence of the Hubble parameter directly.

The evolution of the density parameters is shown in Figure 2 and effectively illustrates at which point each of the components domminate. We see as expected that at any given time the density parameters sum to unity. At early times the universe was domminated by radiation as seen by the fact that $\Omega_r \approx 1$. This epoch is marked by a yellow background. Then when the matter starts to domminate, i.e. $\Omega_B + \Omega_{CDM} \approx 1$, we see a gradually decelerating radiation contribution. This is marked by a blue background. The epoch accelerating expansion of the universe when dark energy domminates is marked by a purple background. Note that the transitions between one to another epoch of domminance is not sharp, but rather a smooth transition, something which is not illustrated here. Also noteworthy is that the approximate time (scale) of transition between each era seems to coinside well with the time of transition earlier discussed. All in all we seem to recover results cosistant with known science and approximations.

4. CONCLUSION

We have simulated the large scale motion of the universe as a whole, and seen how the expansion rate and particle horizon scale of the universe is affected by the different matter-energy contributions contained within the universe. Also the evolution of the matter-energy contribution of each component was simulated. The results of the simulations where found to be consistant with known approximations and known results, therefore justifying the conclusion that the results are significant.

REFERENCES

Callin, P. 2006, How to calculate the CMB spectrum, , , arXiv:astro-ph/0606683

5. APPENDIX

When within the epoch of radiation domminance we can from the Friedmann equation see that the Hubble parameter

$$H^2 = H_0^2 \frac{\Omega_{r,0}}{a^4}. \tag{19}$$

The conformal time then becomes simplified making it

$$\eta_r(a) = \int_0^a \frac{cda}{a^2 H(a)} \approx \int_0^a \frac{cda}{a^2 H_0 \sqrt{\Omega_r}} a^2 = \frac{ca}{H_0 \sqrt{\Omega_r}} = \frac{c}{aH} = \frac{c}{\mathcal{H}(a)}. \tag{20}$$

This expression is valid for $\Omega_r(a) \approx 1$. At scales a_* where $\Omega_m(a_*) \approx 1$, where $\Omega_m = \Omega_B + \Omega_{CDM}$ we have a similar approximation

$$\eta_m(a) = \eta(a_*) + \int_{a_*}^a \frac{cda}{a^2 H} \approx \eta(a_*) + \int_{a_*}^a \frac{cda}{a^2 H_0 \sqrt{\Omega_m}} a^{3/2} = \eta(a_*) + \int_{a_*}^a \frac{da}{\sqrt{a}} \tag{21}$$

$$= \eta(a_*) + 2c \left(\frac{1}{aH} - \frac{1}{a_* H(a_*)} \right) = \eta(a_*) + 2c \left(\frac{1}{\mathcal{H}(a)} - \frac{1}{\mathcal{H}(a_*)} \right). \tag{22}$$

Here $\eta(a_*)$ is the conformal time when matter dommination starts. Note that since we looked at the matter domminated era, we could use the Friedmann equation on the form

$$H^2 = H_0^2 \frac{\Omega_m}{a^3}. \tag{23}$$

Finally we can also consider the case where the matter-energy is domminated by dark energy at $\tilde{a}$ so that $\Omega_\Lambda(\tilde{a}) \approx 1$. Then we can write

$$H^2 = H_0 \Omega_\Lambda, \tag{24}$$

enabeling us to write

$$\eta_\Lambda(a) = \eta(\tilde{a}) + \int_{\tilde{a}}^a \frac{cda}{aH} = \eta(\tilde{a}) + \frac{c}{H_0 \sqrt{\Omega_\Lambda}} \int_{\tilde{a}}^a \frac{da}{a^2} \tag{25}$$

$$= \eta(\tilde{a}) + c \left(\frac{1}{\mathcal{H}(\tilde{a})} - \frac{1}{\mathcal{H}(a)} \right). \tag{26}$$

Here we let $\eta(\tilde{a})$ denote the conformal time when the dark energy domminated era started. These three expressions can now be used to check whether the full computed conformal time is resonalble.