

Numerical Orbital Mechanics Simulations of the Solar System

HÅKON TANSEM, NILS-OLE STUTZER, AND BERNHARD NORNES LOTSBERG

Abstract

We implement an N -body simulator to solve the motion of objects in the Solar System. We compare the performance of the Forward Euler and Velocity Verlet numerical schemes. In this case the Velocity Verlet outperforms the Forward Euler scheme as it has an error of ~ 3 orders of magnitude lower and due to its symplectic nature conserving constants of motion in a Hamiltonian system. Using our method the Earth's escape velocity from the Sun is found to be in the interval $v \in (8.86, 9.50)$ AU/yr, which corresponds well with the analytically derived value of $v \approx 8.88$ AU/yr. We find that a gravitational force $F \propto r^{-\beta}$ with $\beta = 2$ provides closed orbits, while systems with $\beta > 2$ seem to produce orbits that are not closed. Simulating all major bodies of the Solar System for a period of 250 yr with $dt = 10^{-5}$ yr yields stable results. We also implement a general relativistic correction term in Newton's gravitational force to calculate the precession of Mercury's orbit. Our results obtained with $dt = 10^{-7}$ yr of approximately 450400 arcseconds per century deviates a large amount from the observed precession motion of approximately 43 arcseconds when subtracting precession due to other effects like planetary interactions. Further investigations of systematic errors is needed for our method to resolve small scale motions like the perihelion precession of Mercury's orbit.

The Github repository containing source code for this paper can be found here:

<https://github.com/hakontan/FYS4150-Project-5>

1. INTRODUCTION

Celestial mechanics is a physical problem that has intrigued mankind since the day of time. It has fascinated astronomers since night time observations done by the ancient Greeks, through Galileo Galilei's studies of Jupiter's moons in the Renaissance, to today's detailed observations and simulations. At the present date one can do detailed numerical simulations of the motion of planets and other celestial objects, so as to predict their motion many centuries into the future.

In this paper we will study how the celestial bodies in our Solar System interact using two different numerical methods for solving the coupled differential equations of motion describing their movement. We will look at the classical Forward Euler and the more advanced Velocity Verlet methods and compare them. In addition we will study different versions of the gravitational force to see how they affect the planet's orbits, and also we will see what happens to the Earth's and Sun's motion if Jupiter's mass is changed. Last we will study Mercury's perihelion precession correcting for General Relativity.

We will present theory and its implementation in the Method section. The results of our study will be presented and discussed in the Results and Discussion section respectively.

2. METHOD

2.1. The Gravitational Force and the Equations of Motion

The motion of celestial bodies in the Solar System are governed by one single force, being the gravitational force. In Newtonian physics this is written as

$$\vec{F} = -G \frac{Mm}{r^3} \vec{r}, \quad (1)$$

where F is the force between two masses M and m separated by a distance \vec{r} ($r = |\vec{r}|$) and G is the gravitational constant. More generally when having more than just two celestial bodies (N bodies), the force on one of them m_i is simply the sum of the gravitational force from all the other bodies m_j . Thus we have the total gravitational force

$$\vec{F}_i = \sum_i^N \sum_{j \neq i}^N -G \frac{m_i m_j}{|\vec{r}_j - \vec{r}_i|^3} (\vec{r}_j - \vec{r}_i) = m_i \vec{a}_i = m_i \ddot{\vec{r}}_i, \quad (2)$$

where we used Newton's second law to find the acceleration. This way we have an N -body system of interacting particles. When simulating our Solar System, it is often common to let the Sun be static at the origin due to its mass being many orders of magnitude larger than that of the planets, limiting its motion to a small wiggle. We will however include the Sun's motion in our

simulations, as it reflects reality better than having it static.

When simulating the motions of the celestial bodies according to the force (2) it is convenient to chose a scaling more suited to the extension of the Solar System. To find such a scaling we consider the gravitational force between Earth and the Sun in a circular orbit. We then get

$$F = G \frac{M_{\oplus} M_{\odot}}{r^2} = \frac{v^2}{r} M_{\oplus}, \quad (3)$$

using that gravity equals the centripetal force in a circular orbit. Since we know that for a circular orbit $v = \frac{2\pi r}{T}$, for an orbital period $T = 1$ yr, we get that the gravitational constant becomes

$$G = 4\pi^2 \frac{\text{AU}^3}{\text{yr}^2 M_{\odot}}, \quad (4)$$

since the relative Sun-Earth distance in a circular orbit is 1 AU. Thus we use units solar masses M_{\odot} , astronomical units AU for distances and years yr for time.

In order to solve the equations of motion we can write the second order ordinary differential equation (ODE) as a system of two coupled first order equations. Furthermore as the equations are vector equations, we can write the coupled system of equations for body i component wise as

$$v_x^i = \frac{dx^i}{dt} = \dot{x}^i \text{ and } a_x^i = \frac{dv_x^i}{dt} = \dot{v}_x^i, \quad (5)$$

and similarly for the y and z components. Thus when simulating N bodies in three dimensions we need $6N$ coupled differential equations.

2.2. Discretisation and Numerical Solvers

When solving the system of $6N$ coupled ODEs numerically we need to discretise the equations. We do this by letting $x(t) \rightarrow x(t_i) = x_i$ where the time $t \rightarrow t_i = a + idt$, with $t \in [a, b]$ and $i = 0, 1, 2, \dots, n-1$. Then $a \rightarrow t_0$, $b \rightarrow t_n$ and the time step $dt = \frac{b-a}{n}$ for n time steps. Using this discretisation the position in the next time step is written as $x(t_i + dt) = x_{i+1}$.

From the Taylor expansion

$$x_{i+1} = x_i + dt\dot{x}_i + \frac{dt^2}{2}\ddot{x}_i + \mathcal{O}(h^3) \quad (6)$$

we can get Euler's Forward algorithm when only keeping first order terms. This then becomes

$$x_{i+1} = x_i + dtv_x^i + \mathcal{O}(h^2), \quad (7)$$

inserting that $v_x^i = \dot{x}_i$. Similarly, the second coupled ODE can be written

$$v_x^{i+1} = v_x^i + dt\dot{v}_x^i = v_x^i + dta_x^i + \mathcal{O}(dt^2). \quad (8)$$

This algorithm is very simple and requires only a few floating point operations (FLOPs) per time step, however, the trade-off is that it is quite inaccurate having an error term $\mathcal{O}(dt^2)$.

Another numerical method more commonly used is the Velocity Verlet algorithm. It has the advantage of being more accurate than the Forward Euler, having a mathematical error term of $\mathcal{O}(dt^3)$, in addition to requiring about the same amount of FLOPs. Also it is tailored towards conserving the total mechanical energy and angular momentum of a Hamiltonian system such as an N -body system, because it is a symplectic integration scheme (Holmes (2007)). We can write the two coupled ODEs as

$$x_{i+1} = x_i + hv_x^i + \frac{h^2}{2}a_x^i + \mathcal{O}(h^3) \quad (9)$$

$$v_x^{i+1} = v_x^i + \frac{h}{2}[a_x^{i+1} + a_x^i] + \mathcal{O}(h^3) \quad (10)$$

Jensen (2015). As opposed to the Forward Euler we see that the two equations in this scheme are not independent of each other. To solve for the velocity at the next time step one needs the acceleration for the next time step as well. This acceleration is found through the next position x_{i+1} . These two equations thus always have to be solved together. Comparing the amount of FLOPs per time step, we find that the Forward Euler algorithm has about 4 FLOPs per step, while the Velocity Verlet scheme has 7 if $dt/2$ and $dt^2/2$ are pre-calculated. This is remarkable, as one can construct a scheme with a superior error and conservation of energy and angular momentum with only a few additional FLOPs. The drawback is of course that one has to compute an acceleration two times per step using the Velocity Verlet scheme, which was not taken into account when counting the FLOPs as the FLOPs in the acceleration are dependent on how many bodies are simulated. As both schemes have similar amounts of FLOPs we expect them to perform similarly in a timing of the algorithms.

2.3. Testing the Algorithms

To test that the ODE solvers work properly we plot the trajectory of the Sun-Earth system with a circular orbit. This is done by letting the Earth and Sun start at a separation of 1 AU and we give them the initial velocity corresponding to a circular orbit. This is found using that the gravitational and centripetal forces are equal for a circular orbit so that

$$F = m \frac{v^2}{r} = G \frac{M_{\oplus} M_{\odot}}{r^2} \quad (11)$$

$$\implies v = \sqrt{\frac{GM_{\odot}}{r}}. \quad (12)$$

We know from classical mechanics that there are certain quantities that are constant over time, the so-called constants of motion. In our case where we consider a system of interacting bodies in a conservative force potential, the kinetic energy K , potential energy V and total mechanical energy E as well as the angular momentum per mass l are such constants of motion. If we consider a Sun-Earth system in the plane of the motion we get that Earth has a Lagrangian

$$L = K + V = \frac{1}{2}M_{\oplus}(\dot{r}^2 + r^2\dot{\phi}^2) + G\frac{M_{\oplus}M_{\odot}}{r}, \quad (13)$$

for an angular velocity $\dot{\phi}$. We see that since the Lagrangian L is independent of the azimuth angle ϕ we get from Lagrange's equation that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0, \quad (14)$$

gives us that

$$\Rightarrow l = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = \text{constant}. \quad (15)$$

This is a consequence of Noether's theorem, where all symmetries of a system result in a constant of motion (Leinaas (2018)), such as the invariance under an azimuth rotation in our case. Furthermore, since there are no rigid-body constraints in our system that are time-dependent the total energy of the system is given by the Hamiltonian $H = E = K + V$. If we have an explicitly time-independent Lagrangian it follows that the implicit time dependence of the Hamiltonian is given as

$$\frac{dH}{dt} = \frac{dE}{dt} = \frac{\partial L}{\partial t} = 0, \quad (16)$$

which implies that the total energy of our system must be conserved (Leinaas (2018)). In the special case of a circular orbit the kinetic and potential energies are also conserved, because the constant distance r to the center of mass (CM) gives a constant potential energy and a constant orbital speed

$$v = \sqrt{\frac{GM_{\odot}}{r}}. \quad (17)$$

To check whether our numerical schemes conserve the constants of motion we simulate a Sun-Earth system over several years and plot the energies and angular momentum against time. When doing this we must correct for the motion of the CM as it is the angular momentum around the CM that is constant. The center of mass is

given by the position \vec{R}_{CM} and the velocity \vec{V}_{CM} as

$$\vec{R}_{\text{CM}} = \frac{1}{M_{\text{tot}}} \sum_i^N m_i \vec{r}_i \quad (18)$$

$$\vec{V}_{\text{CM}} = \frac{1}{M_{\text{tot}}} \sum_i^N m_i \vec{v}_i \quad (19)$$

for the total mass M_{tot} .

Also to research the stability of the two algorithms over time we simulate several Sun-Earth systems using different time steps dt and check how close to their starting point they get after one orbit.

2.4. Escape Velocity and Modified Gravitational Force

Next we will consider a Sun-Earth system where the Earth starts 1 AU from the Sun. We now want to find which initial velocity the Earth must have for it to escape the Sun's gravitational field. To do this we run several simulations, each with different initial velocities. In order to determine whether the Earth has left the gravitational field of the Sun, escaping to infinity, we simulate the system over a large amount of time. This is, of course, not the best method since the Earth may simply orbit the Sun on a very eccentric orbit with a period longer than the simulated time. However, we will still get a rough estimate of the escape velocity when simulating over a large amount of time.

The analytical value of the escape velocity can easily be found. Consider a planet of mass m initially at escape velocity v_{esc} at radius r from the Sun. If it is to escape to infinity, where it is at rest, it will have energy $E_{\infty} = 0$ at $r \rightarrow \infty$. Energy conservation then gives us

$$E_0 = \frac{1}{2}mv_{\text{esc}}^2 - G\frac{M_{\odot}m}{r} = 0 = E_{\infty}, \quad (20)$$

which gives

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}. \quad (21)$$

The planet m must thus initially have a speed of v_{esc} to escape the Solar System.

Further, we want to find how the orbit of the Earth would behave if one changes the gravitational force to

$$F = -G\frac{M_{\odot}M_{\oplus}}{r^{\beta}}, \quad (22)$$

for some $\beta \in [2, 3]$. To do this we simulate the orbit of several different values of β and compare them. Our findings can then be compared to theoretical expectations. According to Bertrand's theorem (Goldstein et al. 2001, ch. 3.6) only inverse-square force (and harmonic

potentials), such as Newton’s law of Universal gravity, exhibit closed orbits. Generally to have a bound orbit one needs to have $\beta > 3$ so that the effective potential $U_{\text{eff}} = \frac{l^2}{2mr^2} - G\frac{M_{\odot}m}{(1-\beta)r^{\beta-1}}$ has a local minimum around which the planet can oscillate (Ray & Shamanna (2004)).

2.5. The Three-Body Problem

Now that we have looked at several simpler two-body systems, it is time to consider a three-body system of the Sun, Earth and Jupiter. We will simulate the behaviour of this system for the regular masses of the involved celestial bodies, and what happens when increasing the mass of Jupiter by a factor 10 and 1000 respectively. The motion of the system as a whole was corrected by transforming into the CM frame.

We expect that the system will be quite stable when Jupiter has its regular mass, however when increasing its mass by a factor 10 we expect Jupiter’s pull to affect both the Earth’s and Sun’s motion. The Sun, even though it is very massive compared to Jupiter (about 1000 times more massive), will feel the pull of Jupiter and thus orbit the common CM. Increasing the mass of Jupiter by a factor 10 will thus enlarge the orbital motion of the Sun, and the now increased pull from Jupiter may change the motion of the Earth significantly.

When increasing the mass of Jupiter by a factor 1000, we essentially simulate a double-star system, as Jupiter is now about as massive as the Sun. In this case it would be especially unrealistic to keep the Sun static, as the now added new star to the system will have a non-negligible affect on the Sun. The Sun and Jupiter should now orbit each other. The Earth may be thrown out of the system by a sudden boost in angular momentum caused by one of the two more massive objects.

2.6. The Full Solar System

We have now considered a N-body simulation with two and three bodies. Next we add all planets from Mercury to Neptune, including the Sun and the dwarf planet Pluto, to our System. To see how low-mass objects behave in the Solar System simulation, we include Elon Musk’s Tesla Roadster launched into orbit by SpaceX. All initial values for the celestial bodies were kindly provided by the Horizon-Web interface by NASA (2019). To get closed orbits for all bodies included, even Pluto, we simulate about 250 yr of time.

2.7. Mercury’s Perihelion Precession

When looking closely at Mercury’s orbit one can see that it is not simply a static closed ellipse, but that the

semi-major axis of the ellipse is rotating slowly. This is the so-called perihelion precession of Mercury and is approximately 43 arcseconds per century (Jensen, M. H. (2019)). Regular Newtonian gravity cannot account for this, however including general relativistic effects may describe the precession better. In order to simulate this we implement the relativistic correction to Newtonian gravity written as

$$F = \frac{GM_{\odot}M_{\oplus}}{r^2} \left(1 + \frac{3l^2}{rc^2} \right), \quad (23)$$

where $l = |\vec{r} \times \vec{v}|$ is the magnitude of Mercury’s angular momentum per mass and c is the speed of light (Jensen, M. H. (2019)).

In order to compute the perihelion angle θ_p we use $\tan \theta_p = \frac{y_p}{x_p}$, where (x_p, y_p) is the plane position of the perihelion of Mercury. This is the point in Mercury’s orbit closest to the CM of the system. Since the perihelion precession of Mercury is so small, we need to simulate the orbits with a sufficiently small time step dt and we need to simulate long enough. Also to avoid large amounts of saved data, we only save 0.5 yr (Earth years) of data at the beginning and end of the simulation. The difference in the angle θ_p then gives the perihelion precession. The numerically found result can then be compared to the theoretical value.

3. RESULTS

The following results were produced running on an Intel Core i7-6700HQ CPU with a clock speed of 2.60 GHz and 8GB RAM.

The two integration algorithms were compared testing the stability of the solutions in circular orbit. Using (17), with $r = 1$ AU and $M = M_{\odot}$, one finds the velocity for circular orbit to be 6.28 AU/yr. Figure 1 shows the solution of the Earth-Sun system using both the Velocity Verlet and the Forward Euler algorithms with a time step of 1500 steps per year. Here the earth was initialised with a distance of 1 AU from the Sun with the calculated circular velocity. The simulations were run over two years. The same simulations were done varying the amount of steps per year and simulating over two years. This is shown in Figure 2. Here we plot the discrepancy in the distance between the initial starting point and the point after the Earth has finished one full orbit after being initialised with circular velocity as a function of time step. The potential and kinetic energy as well as the angular momentum for the Earth was also calculated for both methods in the Earth-Sun system. The potential energies, kinetic energies and total energies are shown in Figure 3 and the angular momenta

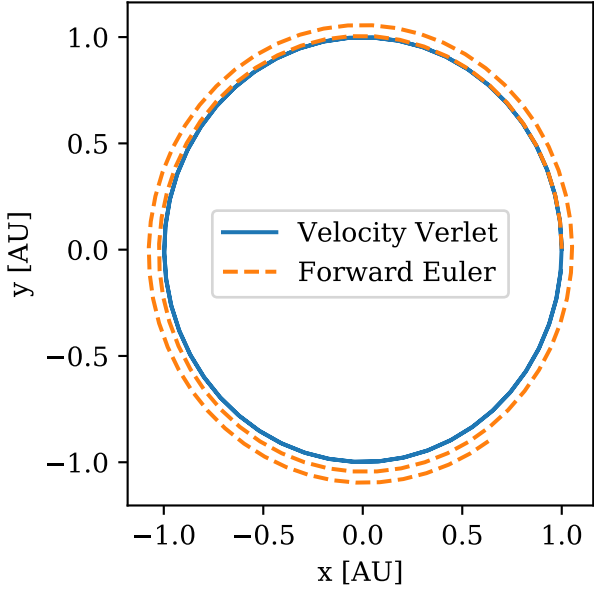


Figure 1. Figure showing the trajectory for the Sun-Earth system initialised with the Earth at a distance of 1 AU from the Sun with an initial circular velocity of 6.28 AU/yr using both the Forward Euler and Velocity Verlet integration methods. The simulation was run for two years with a time step of 1500 steps per year.

are shown in Figure 4. The simulation was done over 200 years using 10^5 steps per year. Here the Earth was also initialised with a distance of 1 AU from the Sun with the calculated circular velocity. When calculating the energies and angular momenta, both integration methods were timed. The Velocity Verlet scheme had a calculation time of 10.17 seconds, while the Forward Euler method had a calculation time of 10.35 seconds.

When calculating the escape velocity for the Earth with respect to the Sun, we found using 21 that the escape velocity for the earth at a distance of 1 AU to be 8.88 AU/yr. The escape velocity was also approximated numerically. This is shown in Figure 5. This result was simulated using the the Velocity Verlet method with an initial distance of 1 AU from the Sun along the x -axis with a varying initial velocity in the y -direction. The simulations were run for 10^4 years with a time step of 10^4 steps per year. The simulation time of 10^4 years was chosen to make sure the Earth really escapes and does not get pulled back into orbit after a few years. The Velocity Verlet method was also tested with a varying β for the exponent in the radial term of the gravitational law as described in section 2.4. This result is shown in Figure 6. Here we initialised the Earth with a distance of 1 AU from the Sun with a velocity slightly higher

than circular velocity, namely 6.7 AU/yr. The system was simulated for one year using 10^4 time steps per year.

The method was also tested using the Velocity Verlet scheme for a three body system including the Sun, Earth and Jupiter. These simulations were initialised using the initial data for the three objects given by the Horizon-Web interface provided by NASA (2019). The simulations were run for 15 years with a time step of 10^5 steps per year. We also studied the effect of changing the mass of Jupiter by a factor of 10 and 1000. These results are shown in Figure 7, 8 and 9 respectively. The method was also expanded to include all the major planets in the Solar system including Pluto as well as the Tesla Roadster launched by SpaceX. This system was simulated for 250 years with a time step of 10^5 steps per year. Figure 10 shows the full system while Figure 11 shows the inner Solar system as the orbits of the inner planets are barely visible when including the orbits of the outer bodies. For the inner solar system the simulation was run for three years with a time step of 10^5 steps per year.

The perihelion precession of Mercury was also calculated using the general relativistic correction term to the Newtonian force given by 23 as discussed in section 2.7. The precession per century was found to be approximately 450399 arcseconds with the relativistic correction term and 450420 without the correction term. This result was calculated over 100 years with a step size of 10^7 steps per year.

4. DISCUSSION

As seen in the orbits in Figure 1, the Velocity Verlet scheme keeps the Earth in a circular orbit, while the Euler method seems to increase the total energy of the system as its orbital radius increases with time. This is expected behaviour, as the Velocity Verlet scheme is known to be better at conserving energy than the Forward Euler scheme. This observation is also consistent with Figure 3. This figure shows that Forward Euler increases the total energy over time, decreasing the kinetic energy and increasing the potential energy as the Earth moves away from the Sun. We also see in Figure 4 that Forward Euler does not conserve angular momentum over time compared to Velocity Verlet. For Velocity Verlet the constants of motion are conserved even over a timespan of 200 years. An interesting detail in the discrepancies shown in Figure 2 is that the error for Velocity Verlet increases with smaller time steps within the range shown. This makes sense as errors in numerical schemes can be decomposed into

$$\text{err}_{\text{tot}} = \text{err}_{\text{Taylor}} + \text{err}_{\text{Numerical}}, \quad (24)$$

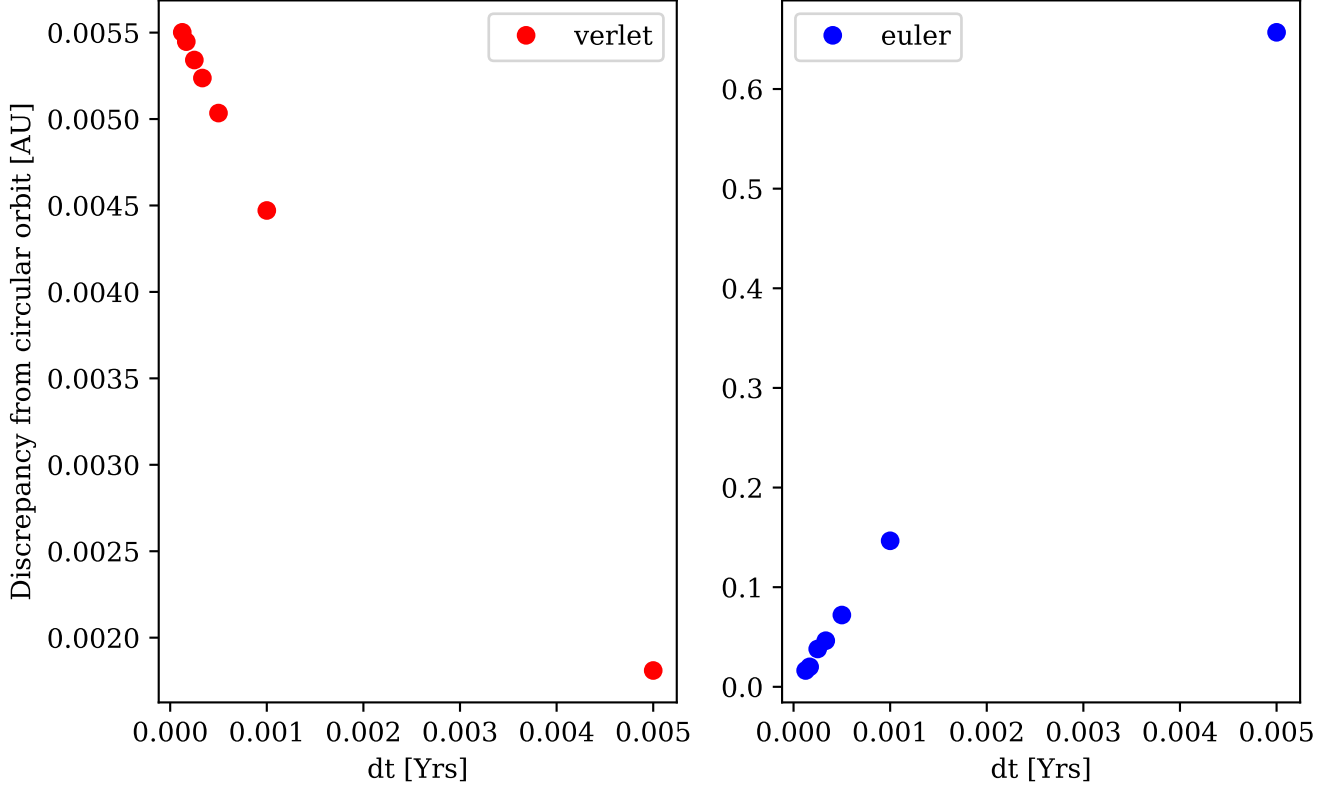


Figure 2. Figure showing the discrepancy between initial starting position in a circular orbit for the Earth in the Sun-Earth system after one orbit as a function of time step for the Forward Euler method and the Velocity Verlet method.

where $\text{err}_{\text{Taylor}}$ is the error from the Taylor series used to derive the numerical scheme and $\text{err}_{\text{Numerical}}$ is the error caused by numerical round off. A possible explanation for this is that $\text{err}_{\text{Taylor}} \ll \text{err}_{\text{Numerical}}$, and the numerical round off error increases with decreasing time step. The total error is clearly better for the Velocity Verlet scheme in this case, the error for Forward Euler being several orders of magnitude higher. In addition the Velocity Verlet and Forward Euler schemes seem to have very similar run times, as predicted by the amount of FLOPs discussed in section 2.2, meaning there is no argument for not using Verlet instead of Forward Euler in this case. For much longer simulations with more bodies there could be a larger discrepancy in timing favouring Forward Euler, however Forward Euler’s lack of precision does not justify this speed-up.

When researching gravitational forces with different radial power laws, the results shown Figure 6 seem to be consistent with Bertrand’s theorem as discussed in section 2.4, showing that only the Sun-Earth-system with gravity $F_g \propto r^{-2}$ has a closed orbit. As shown by Ray & Shamanna (2004) the shown systems will be bound, but only the Newtonian gravitational force with $\beta = 2$ will have a closed orbit. For $\beta > 2$ we expect a perihelion

precession. The Earth’s orbit for $\beta > 2$ will experience a weaker gravitational force and will drift further away before returning towards the sun. It will not return to same initial starting point, but will instead have a precession motion around the Sun if it doesn’t escape.

Figure 5 shows that the escape velocity is somewhere in the range $v \in (8.86, 9.5)$ AU/yr, which corresponds to the analytical escape velocity $v \approx 8.88$ AU/yr. Whether this theoretical velocity is the escape velocity in this simulated system is unclear though, due to numerical errors.

When simulating the Sun-Earth-Jupiter system with regular masses and initial values from NASA (2019) as shown in Figure 7 we see that system behaves as observed in nature where all three bodies exhibit closed, stable orbits around the CM. Increasing the mass of Jupiter in the Sun-Earth-Jupiter system makes the system behave in an increasingly chaotic manner, especially at $1000 \cdot M_{\text{J}}$. At ten times the normal mass, Jupiter visibly affects the orbit of the Earth. This is easily spotted when comparing Figure 7 and 8. Every time the Earth aligns with Jupiter it will receive a larger gravitational tug than if Jupiter has its regular mass. This system may settle in a new equilibrium in the future, but our

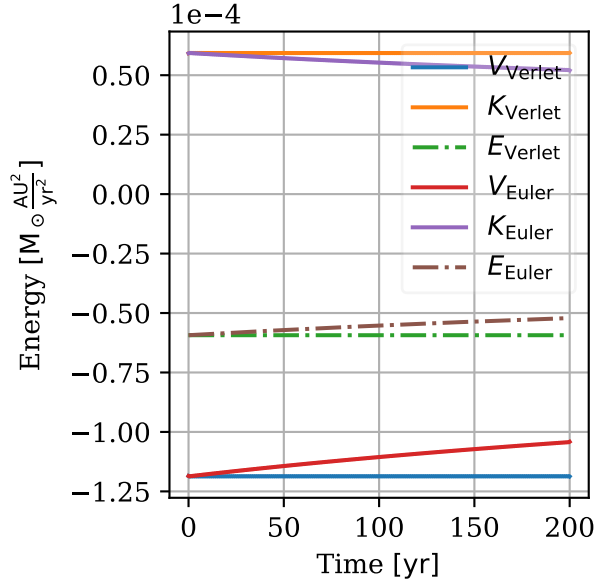


Figure 3. Figure showing the potential and kinetic energy as well as the total energy for the Earth in circular orbit using the Forward Euler and the Velocity Verlet methods. The simulation was run over 200 years using a time step of 10^5 steps per year.

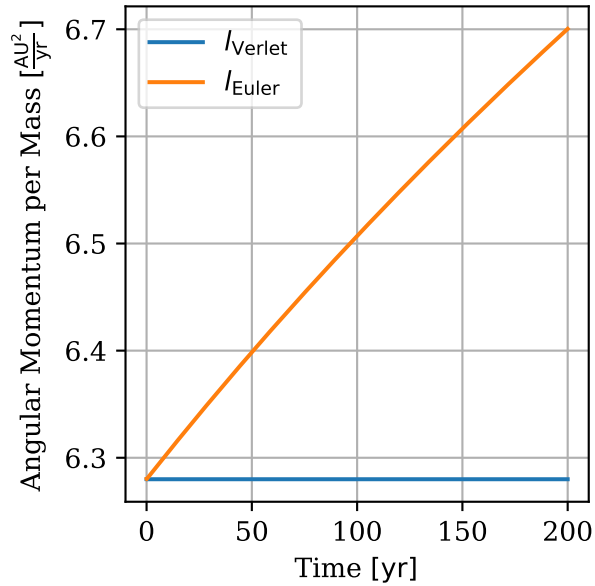


Figure 4. Figure showing the angular momentum of the Earth in circular orbit using the Forward Euler and the Velocity Verlet methods. The simulation was run over 200 years using a time step of 10^5 steps per year.

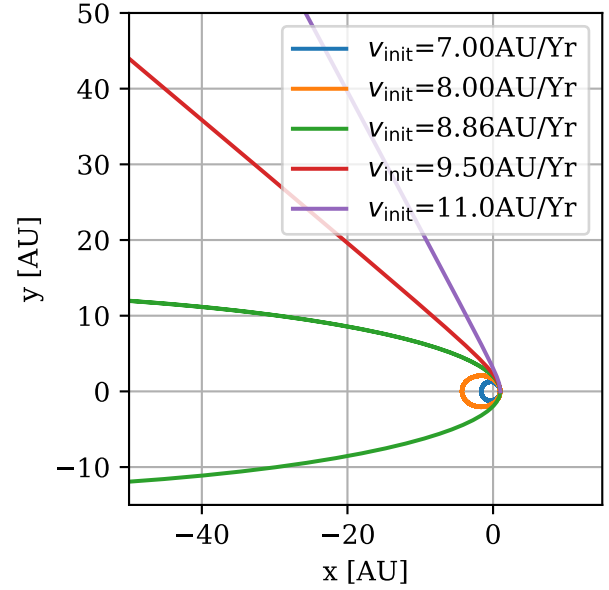


Figure 5. Figure showing the trajectory of the Earth in the Earth-Sun system with a varying initial velocity to see whether it reaches escape velocity. The earth is initialized with a distance of 1AU of the Sun. The simulation was run using the Velocity Verlet scheme for 10^4 years with a time step of 10^4 steps per year.

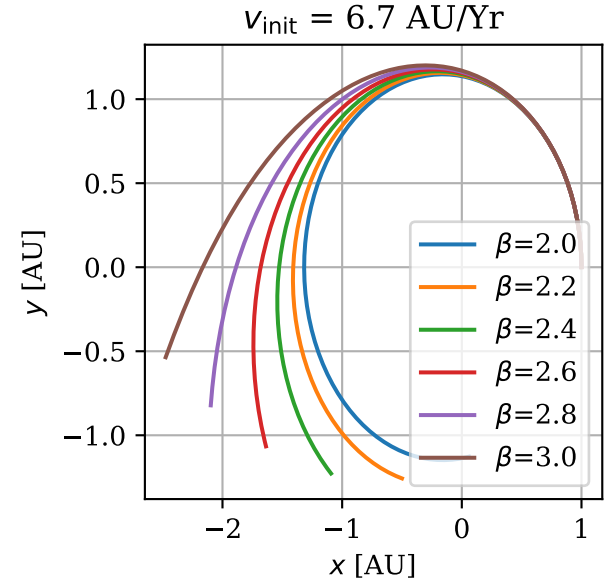


Figure 6. Figure showing the trajectory of the Earth in the Earth-Sun system for different β as described in section 2.4. Here the Earth is initialized at a distance of 1 AU with an initial velocity of 6.7 AU/yr. This is slightly higher than circular orbit. The simulation was run using the Velocity Verlet scheme for one year using a time step of 10^5 time steps per year.

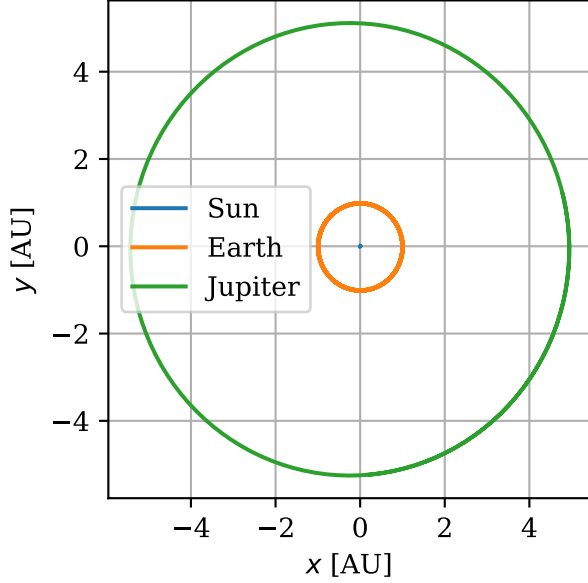


Figure 7. Figure showing the three-body simulation of the Sun, Earth and Jupiter system with Jupiter having its normal mass. The simulation was initialised using the initial data for the three objects given by the Horizon-Web interface provided by [NASA \(2019\)](#). The simulation was run for 15 years with a time step of 10^5 steps per year.

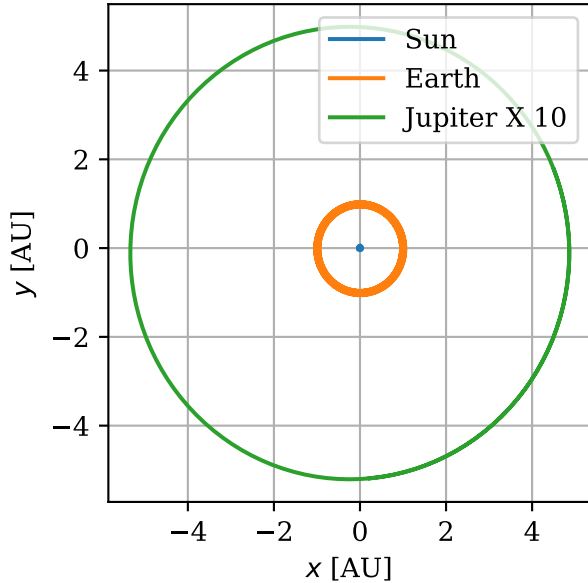


Figure 8. Figure showing the three-body simulation of the Sun, Earth and Jupiter system with Jupiter having ten times its normal mass. The simulation was initialised using the initial data for the three objects given by the Horizon-Web interface provided by [NASA \(2019\)](#). The simulation was run for 15 years with a time step of 10^5 steps per year.

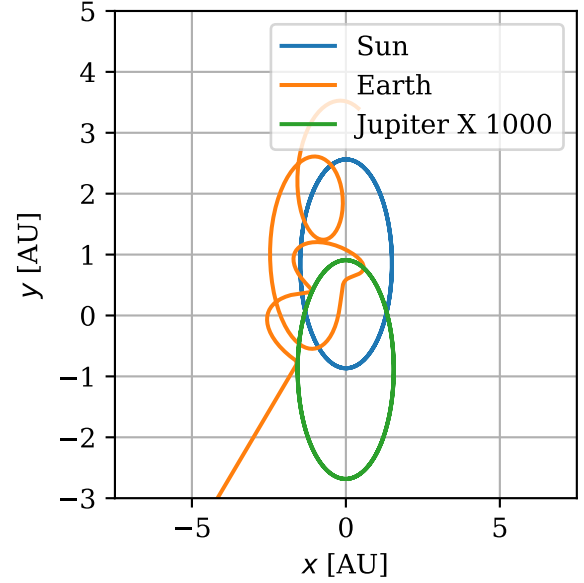


Figure 9. Figure showing the three-body simulation of the Sun, Earth and Jupiter system with Jupiter having 1000 times its normal mass. The simulation was initialised using the initial data for the three objects given by the Horizon-Web interface provided by [NASA \(2019\)](#). The simulation was run for 15 years with a time step of 10^5 steps per year.

result is not sufficient enough to determine whether this will happen. It is also worth noting that our initial data is gathered from a stable system where Jupiter has its regular mass. Therefore it is expected that the system initially behaves unstably as it not initially in an equilibrium state when increasing the mass of Jupiter. Note also that the Sun, orbiting a common CM, now orbits at a larger radius from the new CM in its orbital motion. In Figure 9 Jupiter has a mass very similar to M_{\odot} , meaning the system devolves into a chaotic three body system. One can see that the Sun and Jupiter settle into a stable system where both orbit a common CM. The Earth however is put into a very unstable orbit experiencing gravitational force from two bodies approximately 10^6 times as massive. The initial starting point for the Earth would be an equilibrium if Jupiter had its regular mass. Instead with Jupiter having 1000 times its original mass, we have effectively added another Sun into the system. This will massively perturb the motion of Earth and it will sooner or later experience a close encounter with one of the other bodies, being ejected from the system. This may also have resulted in a collision. Our method has not implemented any collisional effects and since we treat our bodies as point masses a real collision would result in a small radial distance resulting in a large force amplifying the slingshot effect.

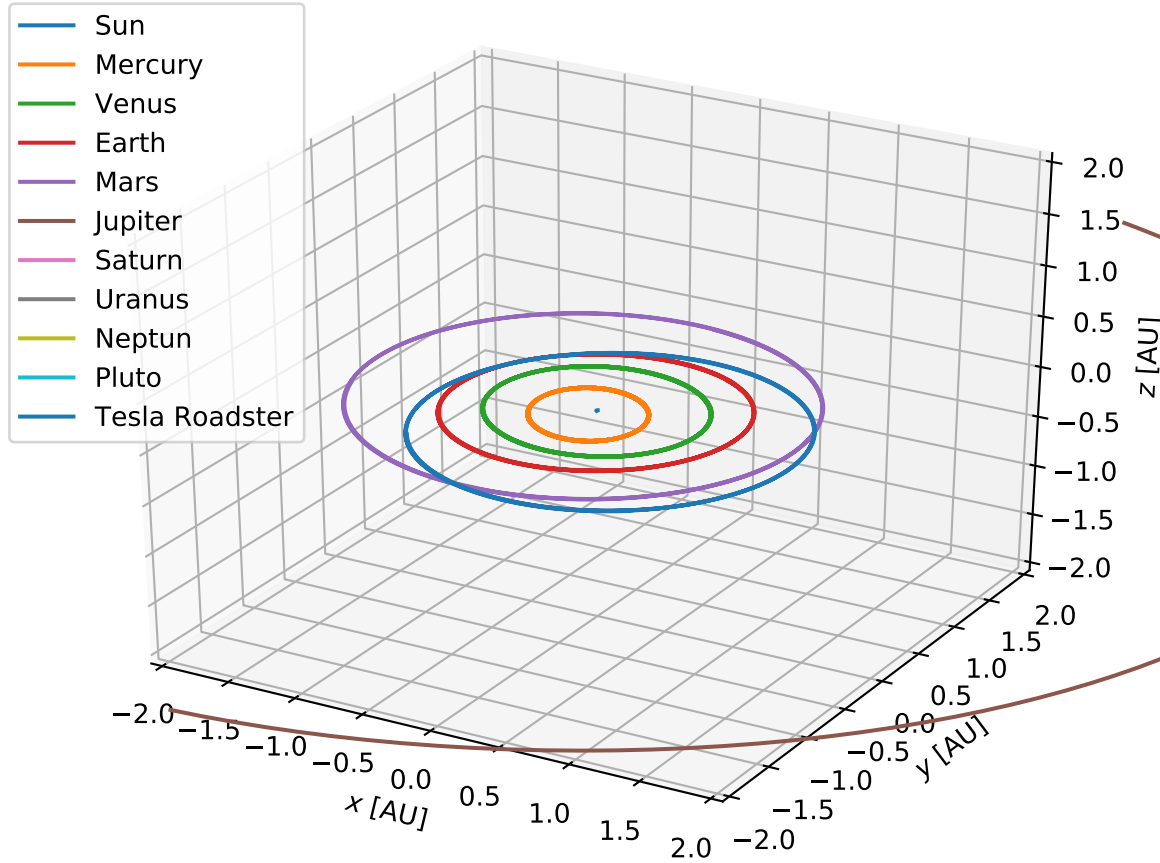


Figure 10. Figure showing a simulation of the inner Solar system using the Velocity Verlet scheme. The simulation was run for three years with a time step 10^5 steps per year. The initial data was given by the Horizon-Web interface provided by [NASA \(2019\)](#)

The simulation of all planets in the Solar System in addition to Pluto and the Tesla Roadster launched by SpaceX, as seen in Figure 10 and 11, stays visibly stable for several years. Pluto seems to have a closed orbit, and keeps its inclination. This indicates that the Velocity Verlet scheme is well enough suited for visualisations of this scale. If the simulations shown in Figure 10 and 11 were run using Forward Euler, we have seen from Figure 1 that the orbits would spiral outwards due to Forward Euler adding to the total energy. In order to have Pluto complete one full orbit we had to simulate for 250 years. During this time the inner planets of the Solar System would have completed many orbits and would have been perturbed by the spiralling motion. This could result in possible close encounters with either each other or the outermost massive planets destabilising the system. However the fact that the inner Solar System stays stable for 250 years is a strong indication that the Velocity Verlet scheme remains stable. The added low-mass Tesla Roadster behaves very similar to the other planets in its orbital motion. However, in a possible future fly-by, it

could gain enough angular momentum from one of the planets to achieve escape velocity.

As mentioned when discussing the results in Figure 6, a inverse-power $\beta > 2$ will lead to a perihelion precession. This is effectively what happens when adding the relativistic correction term as shown in (23), as the correction term has $\beta = 4$. This will slightly perturb the otherwise closed elliptical orbit. We expected the precession of the Sun-Mercury system to be $\sim 43''$, but our results vary dramatically from this. This could imply that our simulations aren't accurate enough to resolve details down to this small scale. However, as we have simulated Mercury's orbit with a time step of 10^{-7} years, we should observe an accurate estimate of the perihelion precession. This not being the case, and the additional fact that the regular Newtonian gravity also exhibits a precession of similar size, indicate that there are systematic errors in our method or in the post-processing of the data. This is something that is important to look more into in the future. Other potential sources of the observed discrepancy could be a perihelion precession

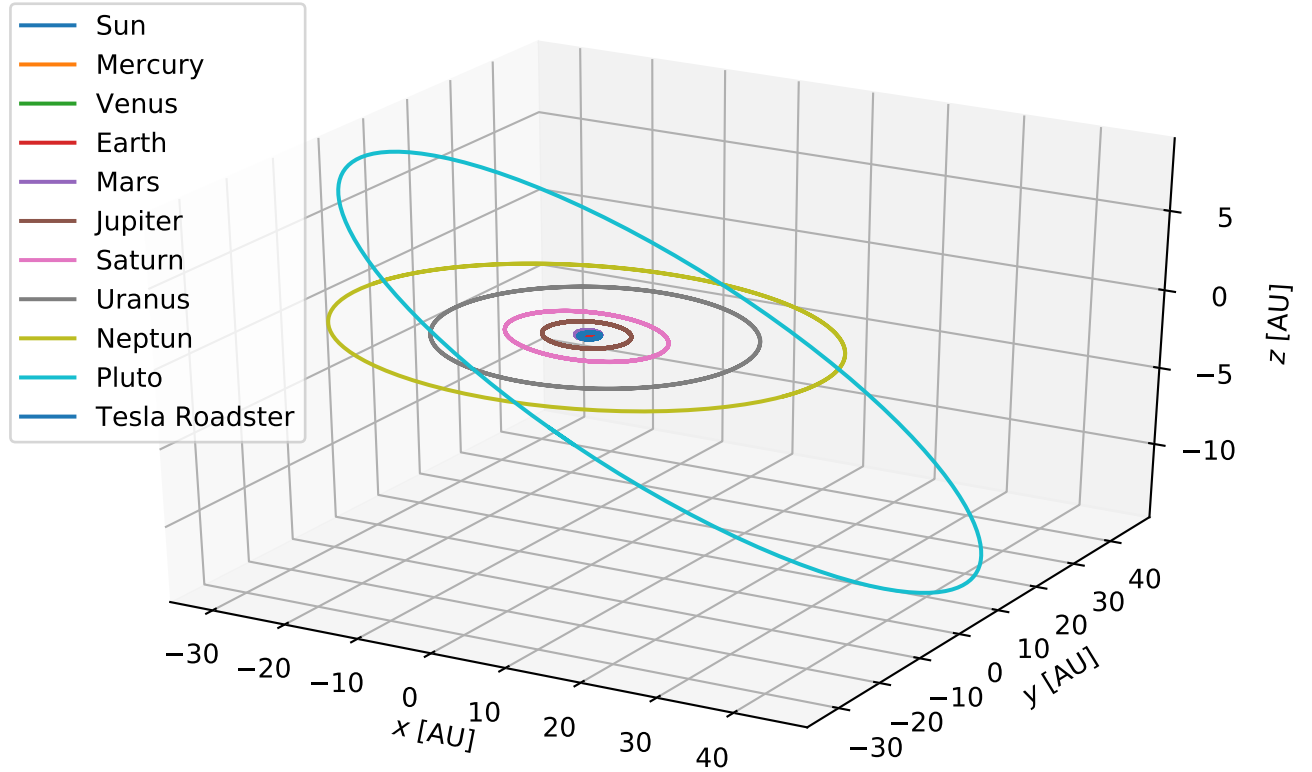


Figure 11. Figure showing a simulation of the full Solar system using the Velocity Verlet scheme. The simulation was run for 250 years with a time step 10^5 steps per year. The initial data was given by the Horizon-Web interface provided by (NASA 2019)

inherent to the Velocity Verlet scheme. In order to resolve the motion in a high enough detail the time step had to be low enough in order to measure any significant small scale motions like the precession of Mercury. We can also see from Figure 2 that in our implementation of the Velocity Verlet scheme, this will also increase the numerical round-off error in the orbital motion resulting in motions that may drown the precession motion we were supposed to measure.

5. CONCLUSION

We have created an N -body simulator implementing Newton’s law of gravity and tested it on a Solar System scale. Our results imply that our method performs well on these larger scales, but struggles to resolve small scale motions needed to estimate the perihelion precession of Mercury.

In the future collision modelling should be added to better simulate chaotic systems such as a three body problem with large masses. Simulations over longer time periods should be run to find whether $3 > \beta > 2$ would indeed produce bound orbits as predicted by (Ray & Shamanna 2004), looking into $\beta > 3$ could also be an

interesting topic of research. Time should be invested in exploring possible reasons for the systematic errors causing us to not properly estimate the perihelion precession of Mercury. Also, the timing comparing the Euler and Velocity Verlet schemes should be done many times and then averaged to get a better estimation of their performance relative to each other.

REFERENCES

- Goldstein, H., Charles, P., & Safko, J. 2001, Classical Mechanics, Addison Wesley,
https://detritus.fundacioace.com/pub/books/Classical_Mechanics_Goldstein_3ed.pdf, Visited: 12.12.2019
- Holmes, M. H. 2007, Introduction to Numerical Methods in Differential Equations, Springer-Verlag New York.
<https://books.google.no/books?id=k53dvgEACAAJ>
- Jensen, M. H. 2015, Computational Physics, Lecture notes Fall 2015, , , <https://github.com/CompPhysics/ComputationalPhysics/blob/master/doc/Lectures/lectures2015.pdf>, Visited: 7.11.2019
- Jensen, M. H. 2019, Project 5, deadline December 15, Computational Physics I FYS3150/FYS4150, Norway: University of Oslo, <http://compphysics.github.io/ComputationalPhysics/doc/Projects/2019/Project5/SolarSystem/html/SolarSystem.html>, Visited: 12.12.2019
- Leinaas, J. 2018, Classical Mechanics and Electrodynamics, World Scientific Publishing Company Pte. Limited.
<https://books.google.no/books?id=k53dvgEACAAJ>
- NASA. 2019, HORIZONS Web-Interface, , , <https://ssd.jpl.nasa.gov/horizons.cgi#top>, Visited: 12.12.2019
- Ray, S., & Shamanna, J. 2004, Orbits in a central force field: Bounded orbits, ,