Draft version December 12, 2019
Typeset using LaTeX twocolumn style in AASTeX62

Orbital Mechanics Simulations by Means of Numerical Methods

Håkon Tansem, Nils-Ole Stutzer, and Bernhard Nornes Lotsberg

Abstract

1. INTRODUCTION

Celestial mechanics is a physical problem that has intreeged mankind since the day of time. It has fascinated astronomers since night time observations done by the antient greeks, through Galileo Galilei's studies of Jupiters moons in the Renaissance, to todays detailed observations and simulations. At the present date one can do detailed numerical simulations of the motion of planets and other celestial objects, so as to predict their motion many centuries into the future.

In this paper we will study how the celestial bodies in our Solar System interact using two different numerical methods for solving the coupled differential equations of motion discribing their movement. We will look at the classical Forward Euler and the more advanced Velocity Verlet methods, and compare them. In addition we will study different versions of the gravitational force to see how it affects the planets orbit, and also we will see what happens to the Earth's and Sun's motion if Jupiters mass is changes. Last we will study Mercury's perihelion precession.

We will present theory and its implementation in the Method section. The results of our study will be presented and discussed in the Results and Discussion section respectively.

2. METHOD

2.1. *The Gravitational force and the Equations of Motion*

The motion of celectial bodies in the solar system are governed by one single force, being the gravitational force. In Newtonian physics this is written as

$$\vec{F} = -G\frac{Mm}{r^3}\vec{r}, \tag{1}$$

where F is the force between two masses M and m separated by a distance $\vec{r}$ ($r = |\vec{r}|$) and G is the gravitational constant. More generally when having more than just two celectial bodies (N bodies), the force on one of them m_i is simply the sum of the gravitational force from all the other bodies m_j. Thus we have the total gravitational force

$$\vec{F}_i = \sum_i^N \sum_{j\neq i}^N -G\frac{m_i m_j}{|\vec{r}_j - \vec{r}_i|^3}(\vec{r}_j - \vec{r}_i) = m_i\vec{a}_i = m_i\ddot{\vec{r}}_i, \tag{2}$$

where we used Newton's second law to find the acceleration. In this N-body problem celectial bodies are interacing with each other, affecting each others motions. When simulating our Solar System, it is often common to let the Sun be static at the origin due to its mass being many orders of magnitude larger than that of the planet, limiting its motion to a small wiggle. We will however include the Suns motion in our simulations, as it reflects reality better than having it static.

When simulating the motions of the celectial bodies according to the force (2) it is convenient to chose a scaling more suited to the scales of the Solar System. To find such a scaling we consider the gravitational force between Earth and the Sun in a circular orbit. We then get

$$F = G\frac{M_\oplus M_\odot}{r^2} = \frac{v^2}{r}M_\oplus, \tag{3}$$

using that gravity equals the centripetal force in a circular orbit. Since we know that for a circualr orbit $v = \frac{2\pi r}{T}$, for an orbital periode $T = 1\text{yr}$, we get that the gravitational constant becomes

$$G = 4\pi^2\frac{\text{AU}^3}{\text{yr}^2\text{M}_\odot}, \tag{4}$$

since the relative Sun-Earth distance in a circular orbit is 1AU. Thus we use units solar masses $M_\odot$, astronomical units AU for distances and years yr for time, as they are far easier to handle.

In order to solve the equations of motion we can write the second order ordinary differentail equation (ODE) as a system of two coupled first order equations. Further more as the equations are vector equations, we can write the coupled system of equations for body i component wise as

$$v_x^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} = \dot{x}^i \text{ and } a_x^i = \frac{\mathrm{d}v_x^i}{\mathrm{d}t} = \dot{v}_x^i, \tag{5}$$

and similarly for the y and z components. Thus when simulation N bodies in three dimensions we would need $6N$ coupled differential equations.

2.2. *discretization and Numerical Solvers*

When solving the system of $6N$ coupled ODEs numically we need to discretize the equations. We do this by letting $x(t) \to x(t_i) = x_i$ where the time $t \to t_i = a+ih$, with $t \in [a,b]$ and $i = 0,1,2,\dots n-1$. Then $a \to t_0$, $b \to t_n$ and the time step $h = \frac{b-a}{n}$ for n time steps. Using this discretization the position in the next time step is written as $x(t_i + h) = x_{i+1}$.

From the taylor expansion

$$x_{i+1} = x_i + h\dot{x}_i + \frac{h^2}{2}\ddot{x}_i + \mathcal{O}(h^3) \tag{6}$$

we can get Eulers Forward algorithm when only keeping first order terms. This then becomes

$$x_{i+1} = x_i + hv_x^i + \mathcal{O}(h^2), \tag{7}$$

inserting that $v_x^i = \dot{x}_i$. Similarly, the second coupled ODE can be written

$$v_x^{i+1} = v_x^i + h\dot{v}_x^i = v_x^i + ha_x^i + \mathcal{O}(h^2). \tag{8}$$

This algorithm is very simple and requires only a few floating point operations (FLOPs) per time step, however, the traid-off is that it is quite inacurate having an error term $(O)(h^2)$.

Another numerical method more comonly used is the Velocity Verlet algoritm. It has the advantage of being more accurate than the Forward Euler, having a mathematical error term of $\mathcal{O}(h^3)$, in addition to requireing about the same amont of FLOPs. Also it is taylored towards conserving the total mechanical energy and angular momentum of a Hamiltonian system such as a N-body system, because it is a symplectic integration scheme (KILDER). We can write the two coupled ODEs as

$$x_{i+1} = x_i + hv_x^i + \frac{h^2}{2}a_x^i + \mathcal{O}(h^3) \tag{9}$$

$$v_x^{i+1} = v_x^i + \frac{h}{2}[a_x^{i+1} + a_x^i] + \mathcal{O}(h^3). \tag{10}$$

As oppose to the Forward Euler we see that the two equations in this scheme are not independent of eachother. To solve for the velcity at the next time step one needs the acceleration for the next time step as well. This acceleration is found through the next position x_{i+1}. These two equations thus always have to be solved together. Comparing the amount of FLOPs per time step, we find that the Forward Euler algorithm has about 4 flops per step, while the Velocity Verlet scheme has 7 if $h/2$ and $h^2/2$ are precalculated. This is remarcable, as one can constuct a scheme with a supperior error conserving energy and angular momentum with only a few FLOPs extra. The drawback is of course that one has to compute an acceleration two times per step using the Velocity Verlet scheme, which was not taken into account when counting the FLOPs as the FLOPs in the acceleration are dependent on how many bodies are simulated. As both schemes have similar amounts of FLOPs we expect them to perform similarly in a timing of the algorithms.

2.3. *Testing the algorithms*

We know from classical mechanics that there are certain quantities that are constant over time, the so-called constants of motion. In our case where we consider a system of interacting in a conservative force potensial, the kinetic K, potensial V and total mechanical energy E as well as the angular momentum l are such constants of motion. If we consider a Sun-Earth system in the plane of the motion we get that Earth has a Lagranian

$$L = K + V = \frac{1}{2}M_\oplus(\dot{r}^2 + r^2\dot{\phi}^2) + G\frac{M_\oplus M_\odot}{r}, \tag{11}$$

for an angular velocity $dot\phi$. We see that since the Lagranian L is independent of the azimuth angle ϕ we get from Lagranges equation that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\phi}} = 0, \tag{12}$$

gives us that

$$\implies l = \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi} = \text{constant}. \tag{13}$$

This is a constant of Noether's theorem, where all symetries of a system result in a constant of motion (KILDE), such as the invatiance under an azimuth rotation in our case. Furthermore, since there are no rigid-body constaints in our system that are time-dependent the total energy of the system is given by the Hamiltonian $H = E = K + V$. If we have a explicitly time-independent Lagrangian it follows that the implicit time dependence of the Hamiltonian is given as

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial L}{\partial t} = 0, \tag{14}$$

which implies that the total energy of our system must be conserved (KILDER). In the special case of a circular orbit the kinetic and potensial energies are also conserved, because the constant distance r to the center of mass (CM) gives a constant potential energy and a constant orbital speed $v = \sqrt{\frac{GM_\odot}{r}}$, i.e. a constant kinetic energy.

To chenck whether our numerical schemes conserve the constants of motion we simulate a Sun-Earth system over several years and plot the energies and angular momentum against time. When doing this we must correct for the motion of the CM as it is the angular momentum around the CM that is constant. The center of mass is given by the position $\vec{R}_{\mathrm{CM}}$ and the velocity $\vec{V}_{\mathrm{CM}}$ as

$$\vec{R}_{\mathrm{CM}} = \frac{1}{M_{\mathrm{tot}}} \sum_{i}^{N} m_i \vec{r}_i \tag{15}$$

$$\vec{V}_{\mathrm{CM}} = \frac{1}{M_{\mathrm{tot}}} \sum_{i}^{N} m_i \vec{v}_i \tag{16}$$

for the total mass M_{tot}.

Also to research the stability of the two algorithms over time we simulate several Sun-Earth systems using different time steps h and chenck how close to thir staring point thay get after one orbit.

2.4. *Escape Velocity and Modifies Gravitational Force*

Next we will consider a Sun-Earth system where the Earth starts 1 AU from the Sun. We now want to find which initial velocity the Earth must have for it to escape the Sun's gravitational field. To do this we run several simulations, each with different initial velocities. In order to determain whether the Earth has left the gravitational field of the Sun, escaping to infinity, we simply simulate the system over a large amount of time. This is, of course, not the best method since the Earth may simply orbit the Sun on a very eccentric orbit with a periode langer than the simulated time. However, we will still get a rough estimate of the escape velocity when simulating over a large amount of time.

The numerical value of the escape velocity can easily be found. Consider a planet of mass m initially at escape velocity v_{esc} at radius r from the Sun. If it is to escape to infinity, where it is at rest, it will have energy $E_\infty = 0$ at $r \to \infty$. Energy conservation then gives us

$$E_0 = \frac{1}{2} m v_{\mathrm{esc}}^2 - G \frac{M_\odot m}{r} = 0 = E_\infty, \tag{17}$$

which gives $v_{\mathrm{esc}} = \sqrt{\frac{2GM}{r}}$. The planet m must thus initially have a speed of v_{esc} to escape the solar system.

Further, we want to find how the orbit of the Earth would behave like if changing the gravitational force to

$$F = -G \frac{M_\odot M_\oplus}{r^\beta}, \tag{18}$$

for some $\beta \in [2, 3]$. To do this we simulate the orbit of several different values of β and compare them. Our findings can than be theoretical expectations. According to Bertrand's theorem (KILDER) the attractional central force must have a power law index $n = -\beta > -3$ for an orbit to be closed. This is so that the effective potential $U_{\mathrm{eff}} = \frac{l^2}{2mr^2} - G \frac{M_\odot m}{(1-\beta) r^{\beta - 1}}$ has a local minimum around which the planet can oscillate. This is only the case when $\beta > 3$.

2.5. *The Three-Body Problem*

Now that we have looked at several simpler two-body systems, it is time to consider a three-body system of the Sun, Earth and Jupiter. We will simulate the behaviour of this system for the regular masses of the involved celectial bodies, and what happens when increasing the mass of Jupiter by a factor 10 and 1000 respectively. The motion of the system as a whole was corrected by transforming into the CM frame.

We expect that the system will be quite stable when Jupiter has its regular mass, however when increaing its mass by a factor 10 we expect Jupiter's pull to affect both the Earth and Sun's motion. The Sun, eventhough it is very massive compared to Jupiter (about 1000 times more massive), will feel the pull of Jupiter and thus orbit the CM (which is withing the Sun). Increasing the mass of Jupiter by a factor 10 will thus enlargen the orbital motion of the Sun, and the now increast pull from Jupiter may change the motion of the Earth significalty.

When increasing the mass of Jupiter by a factor 1000, we essentially simulate a double-star system, as Jupiter is now about as massive as the Sun. In this case it would be espessially unrealistic to keep the Sun static, as the now added new star to the system will have a non-negligible affect on the Sun. The Sun and Jupiter should now orbit each other. The Earth may now be thrown out of the system by a sudden boost in angular momentum of one of the two more massive objects.

2.6. *The Full Solar System*

We have now consideres a N-body simulation with two and three bodies. Next we add all planets from Mercury to Neptune, including the Sun and the dwarf planet Pluto, to our System. To see how low-mass objects behave in the Solar System simulation, we include Elon Musk's Tesla Roadster launched into orbit by Space-X. To get closed orbits for all bodies included, even Pluto, we simulate about 250 yr of time.

2.7. *Mercurie's Perihelion Precession*

When looking closely at Mercury's orbit one can see that it is not simply a static closed ellipse, but that the semi-major axis of the ellipse is rotating slowly. This is the so-called Perihelion Precession of Mercury, and is of the order of 43 arcseconds per century. Regular Newtonian gravity cannot accound for this, however, including general relativistic effects may describe the precession

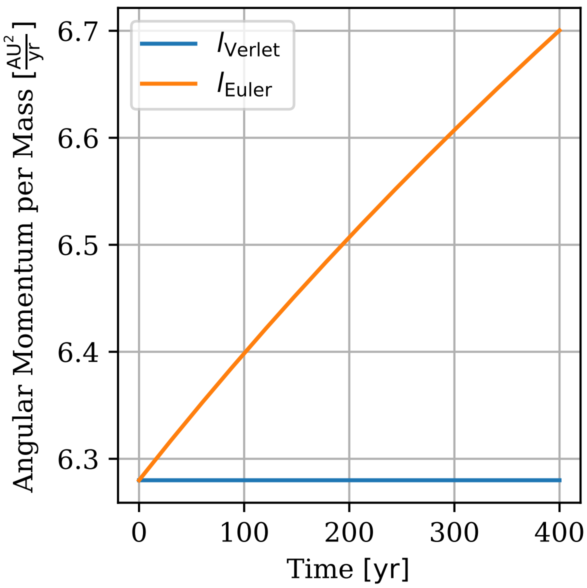

Figure 1. Angular momentum for Euler and Verlet.

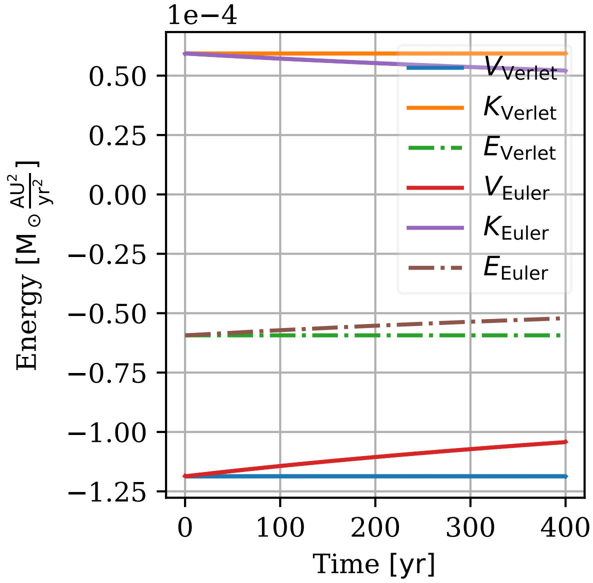

Figure 2. Energies for Euler and Verlet

better. In order to simulate this we implement the relativistic correction to Newtonian gravity written as

$$F = \frac{GM_\odot M_\oplus}{r^2}\left(1 + \frac{3l^2}{rc^2}\right), \tag{19}$$

where $l = |\vec{r} \times \vec{v}|$ is the magnitude of Mercury's angular momentum per mass and c is the speed of light (KILDER).

In order to compute the perihelion angle θ_p we use $\tan\theta_p = \frac{y_p}{x_p}$, where (x_p, y_p) is the plane position of the perihelion of Mercury. This is the point in Mercury's orbit closest to the CM of the system. Since the perihelion precession of Mercury is so small, we need to simulate the orbits with a sufficiently small time step h and we need to simulate long enough, for intance over 100 yr. Also to avoid large amounts of saved data, we only save 0.5 yr (Earth years) of data at the beginning and end of the simulation. The difference in the angle θ_p then gives the perihelion precession. The numerically found result can than be compared to the theoretical value.

3. RESULTS

The two integration algorithms were compared testing the stability of the solutions in circular orbit. Using (likning for sirkulær bane), with $r = 1AU$ and $M = M_\odot$, one finds the velocity for circular orbit to be 6.28AU/Yr.

4. DISCUSSION

5. CONCLUSION

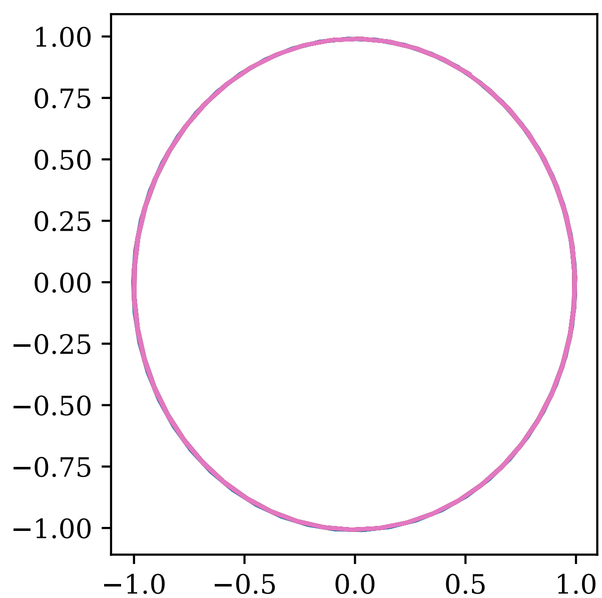

Figure 3. Trajectory Sun and Earth

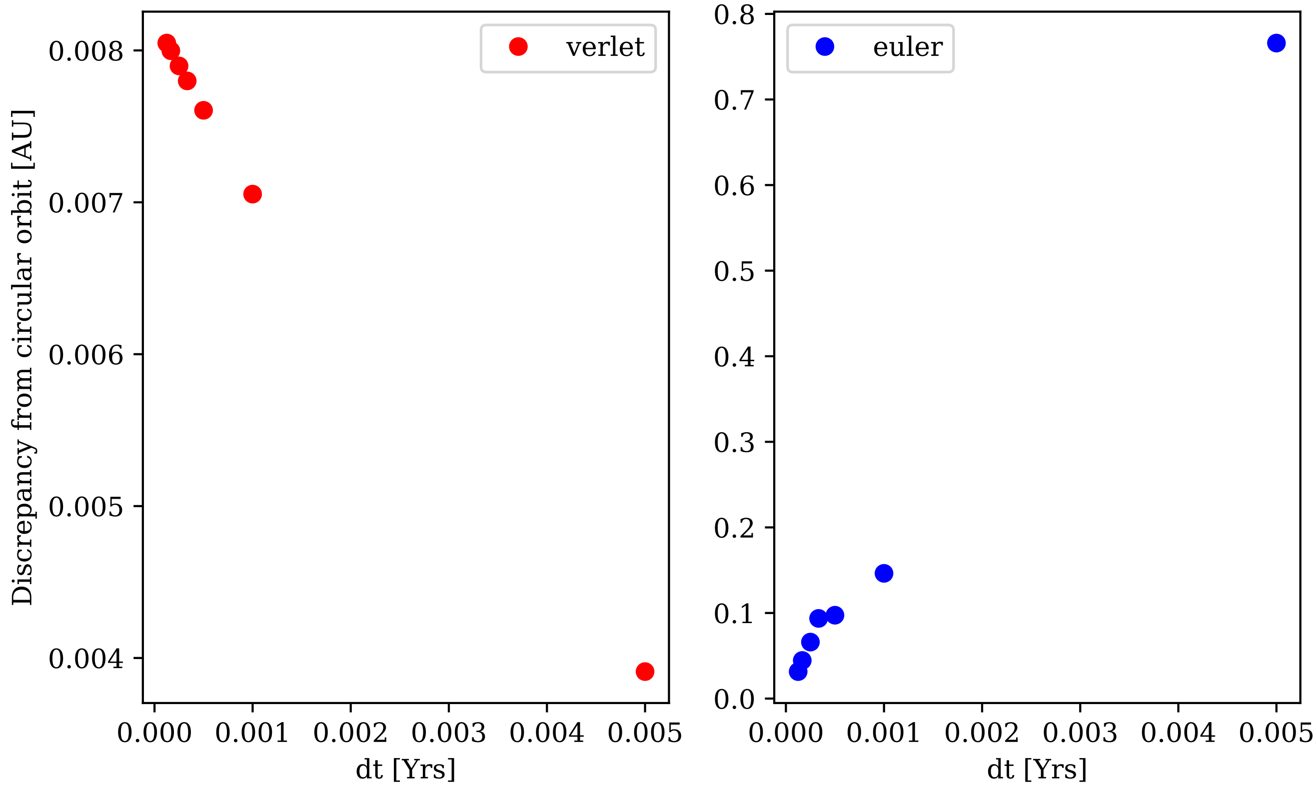

Figure 4. Errors Euler and Verlet

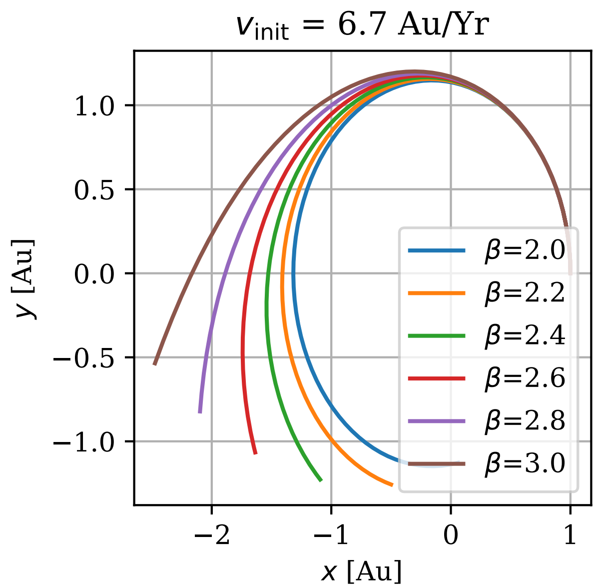

Figure 5. Trajectory Earth Sun different β.

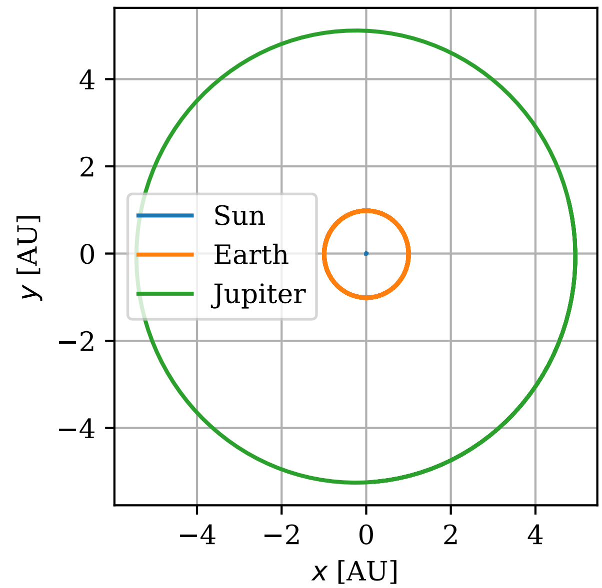

Figure 7. Jupiter normal weight

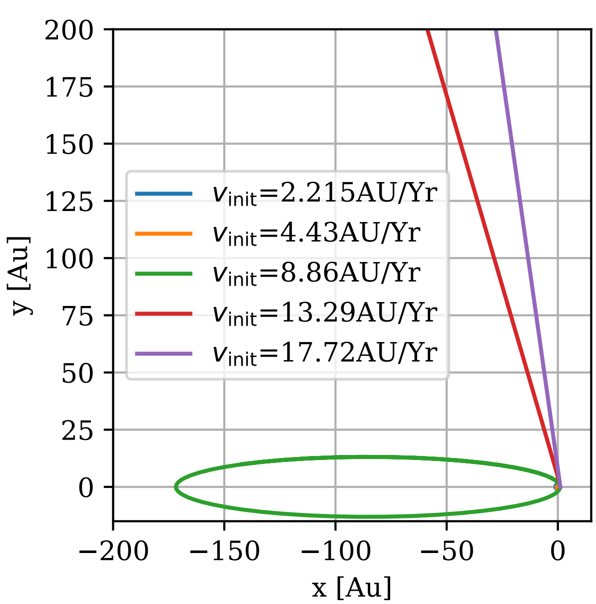

Figure 6. Different velocities for Earth Sun system.

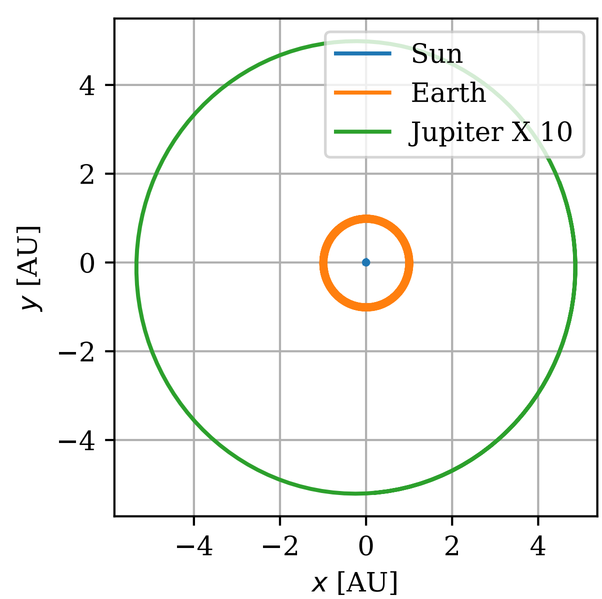

Figure 8. Jupiter ten times size

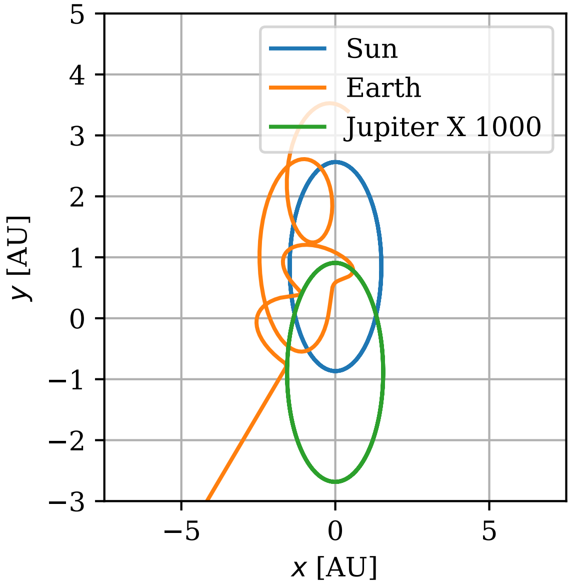

Figure 9. Jupiter 1000 times size

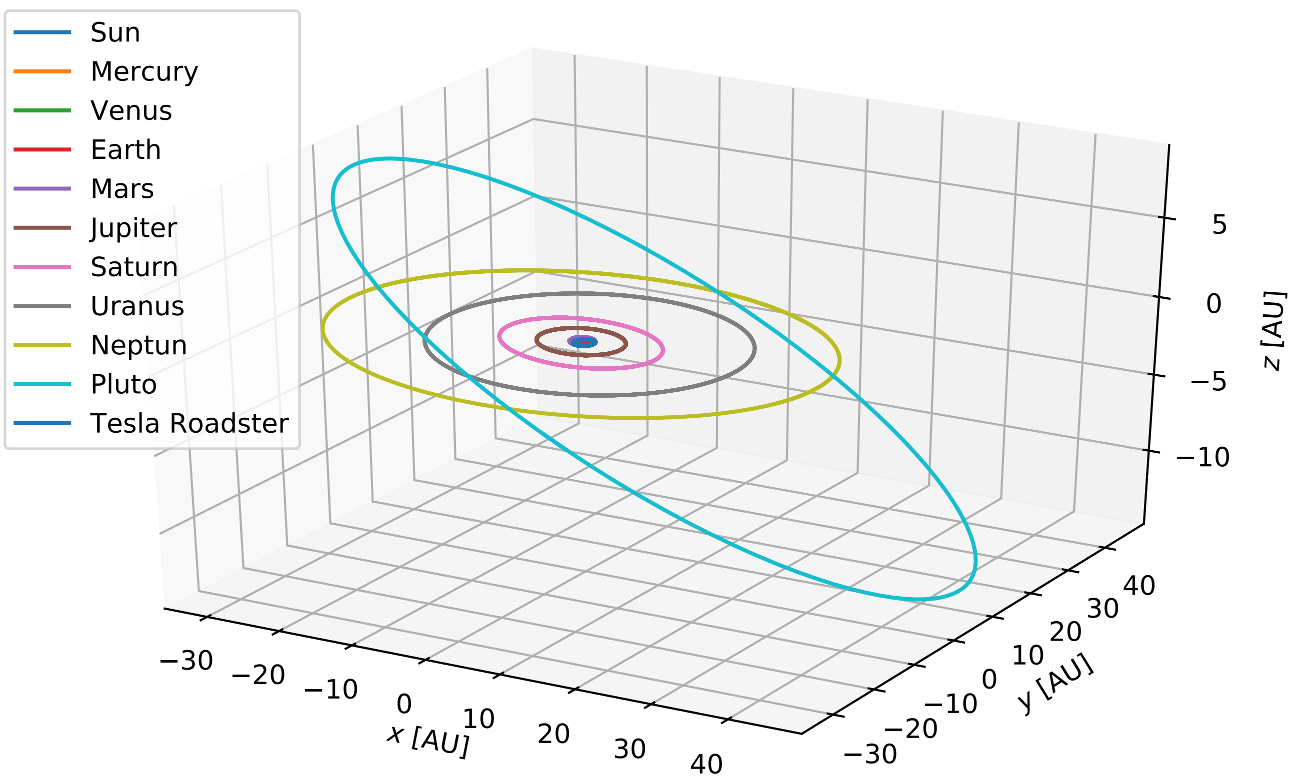

Figure 10. Full solar system

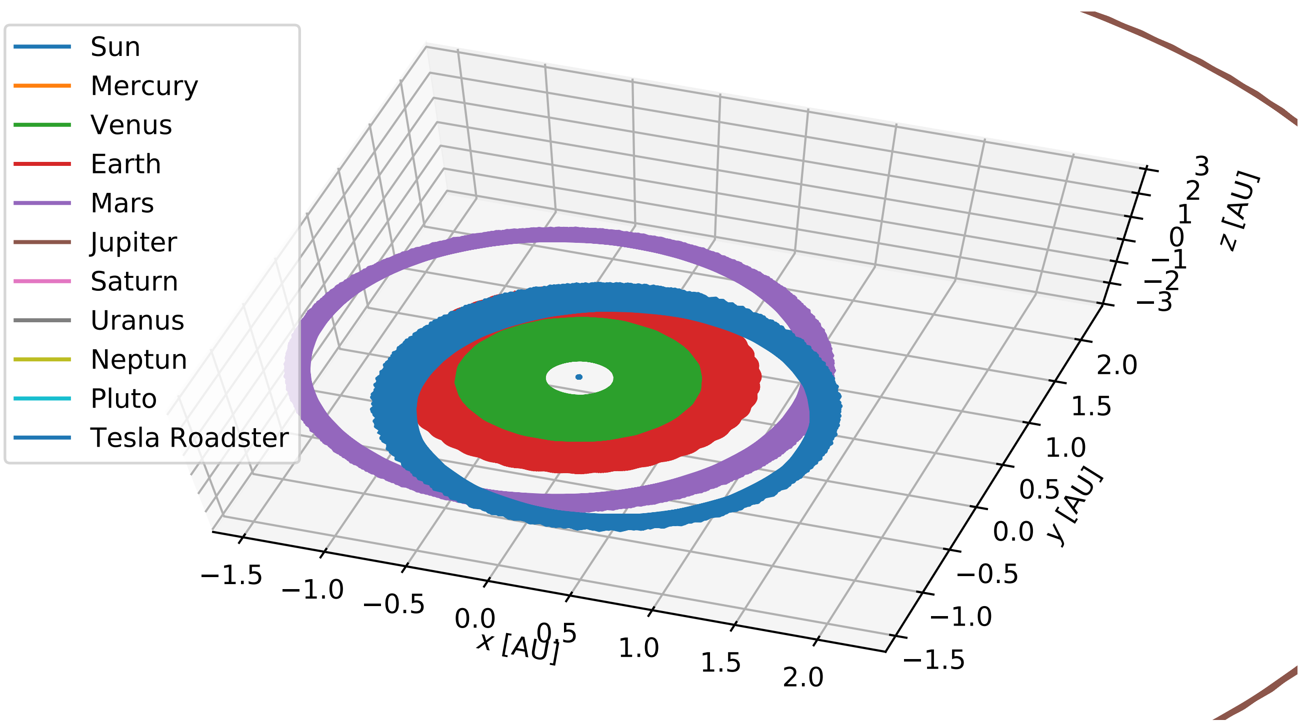

Figure 11. Zoom-in of full solar system