Faraday Waves at the interface of two liquids: Equations and Boundary Conditions

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1 Governing Equations

The Faraday waves at the interface of two fluids can be described using the vorticity-stream function formulation of the Navier-Stokes equations in 2D, along with additional equations for pressure and interface tracking:

1.1 Vorticity Function Equation

$$\nabla \times \vec{u} = \omega \tag{1}$$

where ω is the vorticity and $\vec{u}$ is the velocity of the fluid.

1.2 Stream Function Equation

$$\nabla^2 \psi = -\omega \tag{2}$$

where ψ is the stream function and ω is the vorticity.

The stream function automatically guarantees **INCOMPRESSIBILITY** i.e.

$$\nabla \dot{\vec{u}} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \tag{3}$$

1.3 Vorticity Transport Equation

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = \frac{1}{\rho} \nabla \cdot (\mu \nabla \omega) + \frac{1}{\rho^2} J(\rho, p) + \frac{1}{\rho} \nabla \times \mathbf{f} + \frac{\sigma}{\rho^2} \nabla \times (\kappa \nabla \sigma) \tag{4}$$

where:

- ρ is the fluid density
- μ is the dynamic viscosity
- p is the pressure
- $\mathbf{f}$ represents body forces
- σ is the surface tension coefficient
- κ is the curvature of the interface

1.4 Jacobian Terms

The vorticity transport equation contains two important Jacobian terms:

1.4.1 Advection Term $J(\psi, \omega)$

This term represents the advection of vorticity by the flow field and is defined as:

$$J(\psi, \omega) = \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} \tag{5}$$

1.4.2 Baroclinic Torque Term $J(\rho, p)$

This term represents the baroclinic torque, which arises from misaligned pressure and density gradients:

$$J(\rho, p) = \frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x} \tag{6}$$

This term is important in flows with variable density, as it can generate vorticity due to density stratification or compressibility effects.

1.5 Pressure Equation

The pressure can be obtained by solving a Poisson equation derived from the divergence of the momentum equation:

$$\nabla^2 p = -\rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \mu \nabla^2 (\nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{f} \tag{7}$$

For incompressible flow ($\nabla \cdot \mathbf{u} = 0$), this simplifies to:

$$\nabla^2 p = -\rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \cdot \mathbf{f} \tag{8}$$

In terms of the stream function:

$$\nabla^2 p = -\rho \left[\frac{\partial^2 (\psi_y^2)}{\partial x^2} + 2\frac{\partial^2 (\psi_x \psi_y)}{\partial x \partial y} - \frac{\partial^2 (\psi_x^2)}{\partial y^2} \right] + \nabla \cdot \mathbf{f} \tag{9}$$

where $\psi_x = \frac{\partial \psi}{\partial x}$ and $\psi_y = \frac{\partial \psi}{\partial y}$.

1.6 Interface Tracking Equation

We use the level set method for interface tracking:

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0 \tag{10}$$

where ϕ is the level set function, with $\phi > 0$ in one fluid and $\phi < 0$ in the other, and the interface is defined by $\phi = 0$.

1.7 Density and Viscosity

The density and viscosity are defined using the Heaviside function $H(\phi)$:

$$\rho(\phi) = \rho_1 H(\phi) + \rho_2(1 - H(\phi)) \tag{11}$$
$$\mu(\phi) = \mu_1 H(\phi) + \mu_2(1 - H(\phi)) \tag{12}$$

where ρ_1, μ_1 and ρ_2, μ_2 are the densities and viscosities of the two fluids respectively.

1.8 Surface Tension Term

The surface tension term can be rewritten using the level set function:

$$\frac{\sigma}{\rho^2}\nabla \times (\kappa \nabla \sigma) = \sigma \kappa \delta(\phi) \frac{\nabla \phi}{|\nabla \phi|} \tag{13}$$

where $\delta(\phi)$ is the Dirac delta function and $\kappa = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|}$ is the curvature.

1.9 Body Forces

For Faraday waves, the body force term includes gravity and the oscillatory forcing:

$$\mathbf{f} = \mathbf{g} + A(2\pi f)^2 \cos(2\pi f t)\hat{\mathbf{y}} \tag{14}$$

where A is the amplitude of oscillation, f is the frequency, and $\mathbf{g}$ is the gravitational acceleration.

2 Boundary Conditions

For a rectangular domain $[0, L_x] \times [0, L_y]$, we apply the following boundary conditions:

2.1 Stream Function

No-slip condition on all boundaries:

$$\psi = 0 \quad \text{on all boundaries} \tag{15}$$

2.2 Vorticity

Zero vorticity flux at the boundaries:

$$\frac{\partial \omega}{\partial n} = 0 \quad \text{on all boundaries} \tag{16}$$

where n is the direction normal to the boundary.

2.3 Pressure

For the pressure Poisson equation, we typically use Neumann boundary conditions:

$$\frac{\partial p}{\partial n} = \rho(\mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \mathbf{n} \quad \text{on all boundaries} \tag{17}$$

where $\mathbf{n}$ is the unit normal vector to the boundary.

2.4 Level Set Function

For the level set function, we typically use Neumann boundary conditions:

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on all boundaries} \tag{18}$$

This ensures that the interface remains perpendicular to the walls when it reaches them.

3 Numerical Considerations

1. The Heaviside and delta functions in the density, viscosity, and surface tension terms are typically smoothed over a few grid cells for numerical stability.

2. The level set function ϕ may need to be reinitialized periodically to maintain it as a signed distance function.

3. The pressure Poisson equation needs to be solved at each time step to enforce incompressibility.

4. Special care must be taken when discretizing the Jacobian terms to avoid numerical instabilities.

5. The surface tension term can impose severe time step restrictions, especially for high surface tension coefficients or fine grids.

4 Solution Procedure

A typical solution procedure for each time step might look like:

1. Solve the stream function equation to update ψ.

2. Use ψ to compute the velocity field $\mathbf{u}$.

3. Solve the level set equation to update ϕ.

4. Update ρ and μ based on the new ϕ.

5. Solve the pressure Poisson equation to update p.
6. Solve the vorticity transport equation to update ω.
7. If necessary, reinitialize the level set function ϕ.