ROUGH NOTES ON SMOOTH MANIFOLDS

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1. Introduction

Historically, the space which we seem to be embedded in has been modeled in many ways. The Euclidean space can be regarded as the first formal and well structured mathematical model. It was described through a set of axioms. In modern terms, we can define it as the underlying space of $\mathbb{R}^3$ with no fixed origin but still has the concept of straight lines, angles and distances. It is also infinite in all directions. It does seem very close to the reality. The continuous space around us can be modeled as a euclidean space. And several eminent thinkers such as Galileo, Newton, Maxwell and so on, have explained our everyday realities such as cycles of heavenly bodies, curved path of projectiles, composition of light using this structure. But now we know that the universe may be too exotic to be modeled by such a simple object. We may be wrong in assuming the universe to be infinite! Maybe it is so large compared to us, but still finite. And we may be wrong in assuming it is flat as the euclidean space.

In this situation, one possible resolution can be made to the structure of the space without compromising much. The most important aspect of the euclidean space which helped us in the above mentioned explanations was its geometric structure. And more subtly the calculus which emerges from it. A manifold is a space, which resembles the euclidean space around every point of it. But it is a very flexible object and can with hold many exotic geometric structures as we will see in the course of this discussion.

More technically it is a quotient space obtained by gluing an arbitrary collection of open sets of the euclidean space, $\mathbb{R}^n$ through homeomorphisms. If all such gluing maps are assumed to be indefintely differentiable with indefinitely differentiable inverses (called diffeomorphisms) then we call the resulting quotient a *smooth manifold* of dimension n. We shall eventually see that these spaces inherit a richer calculus than its local model euclidean space.

The gluing procedure can be indicated as follows. Consider two open sets U_α, U_β of $\mathbb{R}^n$. Let

$$\phi : U_\alpha \supset V_\alpha \to V_\beta \subset U_\beta$$

be a diffeomorphism. Then the quotient space,

$$M = U_\alpha \sqcup U_\beta / \{p \sim \phi(p) : p \in V_\alpha\},$$

is said to be obtained by gluing U_α and U_β through ϕ. Thus smooth manifold is a space obtained by gluing an arbitrary collection of open sets of the euclidean space.

2. The tangent bundle

From multivariable calculus, we know that a map such as ϕ described above, has a derivative map

$$D\phi_p : \mathbb{R}^n \to \mathbb{R}^n$$

for every $p \in V_\alpha$ which is linear. Since ϕ has a differentiable inverse, the above derivative map is a linear isomorphism. Notice that any linear map is infinitely differentiable and hence a linear isomorphism is a diffeomorphism. Hence one can naturally consider a diffeomorphism, $T\phi : V_\alpha \times \mathbb{R}^n \to V_\beta \times \mathbb{R}^n$

$$T\phi(p, v) = (\phi(p), D\phi_p(v))$$

and glue $U_\alpha \times \mathbb{R}^n$ and $U_\beta \times \mathbb{R}^n$. Thus from the gluing data of any manifold M of dimension n one can obtain a new manifold TM of dimension $2n$ by the above prescription. TM can be viewed as a space obtained by attaching a copy $\mathbb{R}^n$ at every point $p \in M$.

Consider the natural projection maps $\pi_\alpha : U_\alpha \times \mathbb{R}^n \to U_\alpha$. The set of all points of $U_\alpha \times \mathbb{R}^n$ which maps to $p \in U_\alpha$ is a copy of $\mathbb{R}^n$. This fibre of p is mapped by $D\phi_p$ isomorphically onto the corresponding fibre of $\phi(p)$ under the map $\pi_\beta : U_\beta \times \mathbb{R}^n \to U_\beta$. And thus one can see that we can define a map,

$$\begin{array}{c} TM \\ \big\downarrow{\scriptstyle \pi} \\ M \end{array}$$

and the fibre of a point $p \in M$ under this map will be denoted by T_pM, called the tangent space of p. It is easily seen to be a real vector space of dimension n. In each coordinate chart U_α around p this space can be naturally identified with the fibre $\mathbb{R}^n$ of the projection π_α. And thus the operations of vector addition and scalar multiplication in T_pM can be done by descending to a coordinate chart around p and performing the corresponding operation in the fibre $\mathbb{R}^n$ and then transcending back to T_pM. And these operations are easily seen to be independent on the choice of the chart.

Let M and N are two manifolds of dimensions m and n respectively and let $f : M \to N$ be a function. Now for any point $p \in M$ by descending to a chart U_α around p and a chart U'_γ around $f(p)$ we can obtain a function from $\tilde{f}_{\{\alpha,\gamma\}} : U_\alpha \to U'_\gamma$. If $\tilde{f}_{\{\alpha,\gamma\}}$ is an infinitely differentiable function then we say *f is smooth at p*. And f is said to be a *smooth map* if it is smooth at every $p \in M$. And ofcourse we can define the derivative linear map of f at p,

$$T_pf : T_pM \to T_{f(p)}M,$$

by descending to a chart as the classical derivative $D_p\tilde{f}_{\{\alpha,\gamma\}} : \mathbb{R}^m \to \mathbb{R}^n$ and then transcending back. But notice that T_pf doesn't depend on the particular choices of charts and there is no natural notion of a *Jacobian matrix* since there is no chosen bases. But the the determinant of the linear map can be defined since it is independent of a basis, and hence the *Jacobian*, $|T_pf|$ is well defined.

A smooth map $X : M \to TM$ such that $\pi \circ X(p) = p$ for each $p \in M$, is called a vector field on M. At every $p \in M$, $X(p)$ is a vector in T_pM. If X, Y are both vector fields on M and λ is a real number then,

$$[X + \lambda Y](p) := X(p) + \lambda Y(p)$$

defines a new vector field. Thus the space $\mathfrak{X}(M)$ of all vector fields on M is a real vector space. For all non-empty manifolds of dimension atleast 1, this is an infinite dimensional vector space. Let $C^\infty(M)$ be the ring of all smooth real valued functions on M, with addition and multiplication operations defined pointwise. Now for an $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, then we can define a new vector field, fX by,

$$fX(p) := f(p)X(p)$$

and it is easy to see that this turns $\mathfrak{X}(M)$ into a module over $C^\infty(M)$. Notice that in a coordinate chart, any tangent vector at a point p can be interpreted as a directional derivative operator. Thus a $v \in T_pM$ acts on functions f defined in a neighbourhood of p and produces a number, which we call the directional derivative of f at p in the direction of v.
Then given a vector field X, we can produce a tangent vector at every point p. And thus from a smooth function f we can produce a number at every point by calculating its directional derivative in the direction of $X(p)$ and hence we can produce a new function, which we will denote by Xf. And hence every vector field X can be seen a map,

$$X : C^\infty(M) \to C^\infty(M)$$

From standard rules of calculus in $\mathbb{R}^n$ it is obvious that the above action satisfies the following properties, for every $\in \mathbb{R}$, $f, g \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$.

- $X(f + \lambda g) = Xf + \lambda X(g)$ (linearity)
- $X(f.g) = (Xf).g + f.(Xg)$ (Leibniz rule)

And it can be shown that any map $K : C^\infty(M) \to C^\infty(M)$ which satisfies the above two properties is always a vector field. That is one could define a vector field as an endomorphism $X : C^\infty(M) \to C^\infty(M)$ which is linear and satisfies the Leibniz rule. Thus these properties characterizes vector fields in a purely algebraic way. Thus a vector field has a dual nature of being a *geometric object* (directional derivative at every point) and an *algebraic object* (an endomorphism of $C^\infty(M)$). This duality makes it a powerful object which can play a huge spectrum of roles in different situations and posses several generalizations.

3. The endless river!

We all have a visual conception of what motion is. Everything around us is moving in several ways. But what exactly is motion? How do we define motion? There are three fundamental ideas of *space, time and matter*. Space is a concept which arises when one tries to represent the ''visual'' universe.

Time arises from the ‘‘order’’ of occurrence of events. And matter is the content of the fabric created by these two. Motion is some kind of an interaction of the three.
Here we would like to concern ourselves with a simplistic model of motion starting with a model of the motion of a point particle. A manifold, M will play the role of space. Each point of M is a region of space for the particle to occupy. For some $\epsilon > 0$ a continuous map

$$\gamma : (-\epsilon, \epsilon) \to M$$

is called a path in M. We can interpret this as the ‘‘data of the motion of a point particle’’, where the numbers $t \in (-\epsilon, \epsilon)$ are thought of as time measured by a clock. The point $\gamma(0)$ is a place of interest for us, and the other negative and positive values of t corresponds to the past and future of the particle at $\gamma(0)$. By continuity of the map γ we prohibit mystical events of a particle vanishing from some point and appearing somewhere the next moment and other similar pathological possibilities.
Now as we know $(-\epsilon, \epsilon)$ is a manifold. Hence we can consider smooth paths ,imbeddings of $(-\epsilon, \epsilon)$, in M. On $(-\epsilon, \epsilon)$ we have a smooth vector field $\frac{\partial}{\partial t}$ which can be pushed forward to a vector field on the path. This vector field, which is defined only on the image of γ will be referred to as a velocity field of γ.
But one should be careful here. This velocity vector field is a little primitive than the velocity we see in Physics. For example, this velocity doesn’t have a well defined magnitude since the vectors in T_pM doesn’t have a well defined length. So in different coordinate charts we may get different magnitudes, as the derivative maps of the transition functions need not be distance preserving. But this is good enough for a lot of purposes as we will see in future. Which in a sense shows that many of the properties of motion which depended on velocity, didn’t really have much to do with its magnitude. The properties of many such quantities like acceleration, momentum and more general fields are arising mostly from their quality of being a vector.
Now consider the group $(\mathbb{R}, +)$ of real numbers under addition. A smooth action of this group on a manifold M is called a 1-parameter group on M. It is just a smooth map,

$$\phi : M \times \mathbb{R} \to M$$

such that the map $\phi_t(x) := \phi(x, t)$ is a diffeomorphism for each $t \in \mathbb{R}$ and

$$\forall t, s \in \mathbb{R}, \phi_t \circ \phi_s = \phi_{t+s}.$$

Given a $p \in M$, one can consider $t \mapsto \phi_t(p)$ as a smooth path in M which is at p when $t = 0$. It will be called the ‘‘flow line’’ of p. At an arbitrary point of time, say $\mathfrak{t}$, it is at a point $\phi_{\mathfrak{t}}(p)$. We can think of this as the point where a particle at p would have reached by ‘‘*flowing*’’ for time $\mathfrak{t}$. If it flows additionally for $\mathfrak{s}$ time then it will reach the point $\phi_{\mathfrak{s}}(\phi_{\mathfrak{t}}(p))$ which is the same as the point, $\phi_{\mathfrak{s}+\mathfrak{t}}$, reached by flowing for $\mathfrak{s}+\mathfrak{t}$ time from p by the above condition. And ofcourse p is just an arbitrary point in M and this is the state of every point in M. This matches perfectly our visual conception of a flow. Every point in M is flowing for limitless time. The manifold has become an endless river! For this reason, we will call a 1-parameter group as a ''flow'' on M.

Illustrative examples for $M = \mathbb{R}^2$:

- $\phi_t(x, y) = (x + t, y)$.
  This is a river uniformly flowing in the x direction with unit velocity. The flow lines are horizontal lines.
- $\phi_t(x, y) := (Re(e^{i2\pi t}(x + iy), Im(e^{i2\pi t}(x + iy))$.
  Here the origin is static and every other point in the plane is orbiting the origin in counter-clockwise direction with each cycle taking unit time. The flow lines are circles centered at origin.
- $\phi_t(x, y) = (e^t x, e^{-t} y)$.
  Well, figure it out!

Exercise: Show that if flow lines of two distinct points p, q intersect, then they represent the same path in M. That is every point in M lies on a unique flow line.

Thus at every point in M we get a velocity vector corresponding to the flow and hence a smooth vector field on M. That is every flow on M generates a vector field on M. Are all vector fields on M generated by a flow?

The answer turns out to be a 'no'. But infact by the existence and uniqueness theorem for solutions to first-order equations, it follows that all vector fields give rise to local flows. Which lasts for short intervals of time, but follows all the other requirements needed for a flow. These flows can be calculated using cordinate charts, and seen to be independent of the choice of chart due to the uniqueness of solutions. But the only trouble is that the solutions in general may not be defined for all $t \in \mathbb{R}$. And hence in general may not give rise to a flow on M. But on a compact manifold, all vector fields indeed generate flows.

But the key observation is that vector fields can generate motion in space. For example, a radially inward directed field on $\mathbb{R}^3 \setminus \{0\}$ with vectors of length proportional to the inverse square of the distance from origin, can create flows in the three dimensional space. Here every point will flow towards the origin, but as it goes closer to origin the velocity of flow blows up. The origin is a singularity. This flow models the motion of a charged point particle in the field of another oppositely charged particle fixed at the origin. And here the two charged particles are not in physical contact. Thus this is indeed a way to model action at a distance. But this is not the only way as we will see that forces like gravity needs more complicated fields. Also we will use the idea of local flows of a vector field to model observables genetating translations in the classical phase space.

4. The infinitesimal world

In the earlier sections we studied the tangent bundle of a manifold, M. The instantaneous velocity of a moving particle at a $p \in M$, is a vector in T_pM. A smooth map between two manifolds induces linear maps between the corresponding tangent spaces. Vector fields taking values in the tangent spaces can set points in M to motion by generating local flows. In many ways, the tangent space at a point can be interpreted as an infinitesimal approximation of a manifold near that point. The goal of this section will be study some possible structures on this infinitesimal manifold. And later we will study how these structures reflect on the manifold. Since the infinitesimal manifold is a vector space, what we will be doing is linear algebra. In what follows, unless otherwise mentioned all vector spaces are real.

Suppose V is a vector space and W is an arbitrary subspace. Which means W is closed under vector addition and scalar multiplication. For any vector $x \in V$ we can define,

$$x + W := \{x + a | a \in W\}$$

a subset of V. On careful analysis, one can realise this is nothing but the hyperplane in V obtained by translating W by the vector x. We will write $0 + W$ as simply W. Also if $x + W = y + W$, then for every $a \in W$ there exists some $b \in W$ such that

$$x + a = y + b \implies x - y \in W.$$

Thus for any $a \in W$, we have $a+W = W$. Geometrically it is obvious that if we translate the hyperplane W by a vector in itself, the translate is superposed on the same hyperplane W. Now consider the set,

$$V/W := \{x + W | x \in V\}$$

the collection of all hyperplanes in V which are translates of W. One can define an addition on this set by,

$$(x + W) + (y + W) := (x + y) + W.$$

It is obvious that it is commutative and associative. And the element W acts as the identity of the operation. and ever element $x+W$ has an additive inverse, $-x + W$. Thus this is a well defined vector addition on V/W. Similarly one can define a scalar multiplication by defining,

$$\lambda * (x + W) = (\lambda x) + W.$$

It is an easy exercise that the above defined vector addition and scalar multiplication turns V/W into a vector space. We call it the quotient of V by the subspace W. And the natural map,

$$q : V \longrightarrow V/W$$
$$x \mapsto (x + W)$$

is clearly linear and surjective. It is called the quotient map.

Exercise: Show that,

$$dim(V/W) = dim(V) - dim(W).$$

In the case of $V = \mathbb{R}^3$ and W being the XY-plane, the space V/W can be seen as the space of all horizontal planes in $\mathbb{R}^3$. Which is clearly seen to be parametrized by the Z-axis which is one dimensional. Similarly if W was the X-axis, then V/W is the space of all lines parallel to the X-axis. This is clearly parametrized by the YZ-plane, which is 2-dimensional. If $W = (0)$ then V/W is the space of all points in V. Which is naturally isomorphic to V itself. The quotient map itself is a natural isomorphism. And when $W = V$ then V/W has exactly one element which is W. Thus V/W is the zero vector space.

Let U, V and W be vector spaces. Then the cartesian product, $U \times V$ is a vector space with coordinate wise defined operations. A function

$$f : U \times V \longrightarrow W$$

is said to be bilinear if and only if for every $v, v' \in V$, $u, u' \in U$ and $\lambda \in \mathbb{R}$ we have,

$$\begin{aligned} f(\lambda u, v) &= f(u, \lambda v) = \lambda f(u, v) \\ f(u + \lambda u', v) &= f(u, v) + \lambda f(u', v), \\ f(u, v + \lambda v') &= f(u, v) + \lambda f(u, v'). \end{aligned} \tag{1}$$

Notice that such a function is not a linear map. For any set X we will denote the free vector space over X by $F(X)$. X is a basis for this vector space. Consider the subspace H of $F(U \times V)$ generated by the words of the form,

$$\begin{gathered} (u + \lambda u', v) - (u, v) - \lambda(u', v) \\ (u, v + \lambda v') - (u, v) - \lambda(u, v') \\ (\lambda u, v) - \lambda(u, v) \\ (u, \lambda v) - \lambda(u, v) \end{gathered}$$

for every $v, v' \in V$, $u, u' \in U$ and $\lambda \in \mathbb{R}$. Then we can define,

$$U \otimes V := F(U \times V)/H,$$

is called the ‘‘tensor product of U and V’’. Let $q : F(U \times V) \to U \otimes V$ be the quotient map. Then we denote,

$$u \otimes v := q(u, v).$$

Elements of the form $u \otimes v$ are called elementary tensors. The ones of the form $u \otimes v + u' \otimes v'$ are called mixed tensors. Notice that since all the elements of H are mapped by the quotient map to $0 \in U \otimes V$, we have the following relations.

$$\begin{aligned} \lambda u \otimes v = (\lambda u) \otimes v = u \otimes \lambda v, \\ (u + \lambda u') \otimes v = u \otimes v + \lambda u' \otimes v, \\ (u, v + \lambda v') = u \otimes v + \lambda u \otimes v. \end{aligned} \tag{2}$$

Notice the apparent similarity of the expressions in (1) and (2). Now suppose $f : U \times V \to W$ is a bilinear map. Since the set $U \times V$ is a basis for $F(U \times V)$, the above bilinear map extends to a linear map,

$$f' : F(U \times V) \to W.$$

The bilinearity of f implies that f vanishes on each element of the subspace H, since it vanishes on each of the generators. Thus f descends to a linear map,

$$\tilde{f} : U \otimes V \longrightarrow W.$$

That is a bilinear map from f on $U \times V$ induces a linear map $\tilde{f}$ on $U \otimes V$.This is one of the most desirable properties of the tensor product. The following are some good exercises on tensor products.

Exercise 1: Show that $dim(U \otimes V) = dim(U) \times dim(V)$.
Exercise 2: Construct the following isomorphisms,

$$\begin{aligned} U \otimes (V \otimes W) \cong (U \otimes V) \otimes W, \\ U \otimes V \cong V \otimes U, \\ U \otimes (0) \cong (0), \\ U \otimes \mathbb{R} \cong U. \end{aligned}$$

One can ofcourse extend these notions and define n-linear maps inducing linear maps on an n-fold tensor product.

References

[1] Hermann Weyl; Space, Time, Matter.
[2] S.Ramanan; Global Calculus; Graduate series in Mathematics Volume 65, AMS.