Complexity of Turbulent Flows

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Chapter 1

> Aristotle said a bunch of stuff that was wrong. Galileo and Newton fixed things up. Then Einstein broke everything again. Now we′ve basically got it all worked out except for small stuff, big stuff, hot stuff, cold stuff, fast stuff, heavy stuff, dark stuff, *turbulence* and the concept of time.
>
> Semi-popular 21^{st} century meme

1.1 Introduction

Dynamics of fluids, at scales much greater than mean free path, is believed to be described, in general by the celebrated Navier-Stokes Equations.

$$\frac{\partial v}{\partial t} + v\nabla v = -\frac{\nabla P}{\rho} + f + \nu\nabla^2 v \tag{1.1}$$

$$\nabla . v = 0 \tag{1.2}$$

The importance of this problem stems essentially from the vast, wide range of phenomena, for which understanding and prediction remains a challenging puzzle till date. Nearly, two centuries have passed after the *governing equation* was written down the first time by Navier in 1822. Still, we don′t understand the motion of the water that gushes out from tap, we have no clue how exactly clouds form and travel, what exactly happens to the surrounding air when a jet passes through and in turn how it shakes it once in a while, we don't know how the smoke from an incense stick (even in a closed room with no fans) transits from the well behaved streams to the confused convolution, and, of course, we cant predict monsoons even qualitatively, to adequate accuracy.

The studies on Turbulence and Navier-Stokes equations, or the present flow thereof, broadly have three genres. The first approach is that of *Analysis*, that of attempts of constructing explicit solutions or coming up with conditions in which a finite-time-singularity could occur or would occur. The famous Clay Millennium problem [1] belongs to this set. The problem statement is to show the existence and uniqueness of smooth velocity ($v^i(x,t)$) and pressure ($P(x,t)$) fields for the whole space (or for periodic cubes) and all time, that satisfies unforced version of 1.1,1.2 starting from a divergence free initial smooth velocity field ($v^0(x,t)$). Otherwise to show that there exists at least one smooth divergence free velocity field for which, when coupled with a smooth forcing ($f^i(x,t)$), there exists no smooth, divergence free velocity field on whole space (or on periodic cubes) that satisfies 1.1,1.2.

The most general case for 3D is still unsolved, but there are a lot of analytic progress for simpler versions. In the 2 dimensions, the solutions are known since 1969 (Ladyzhenskaya, [2]). These solutions exhibit finite time singularity and vortices. But due to topological restrictions this flows

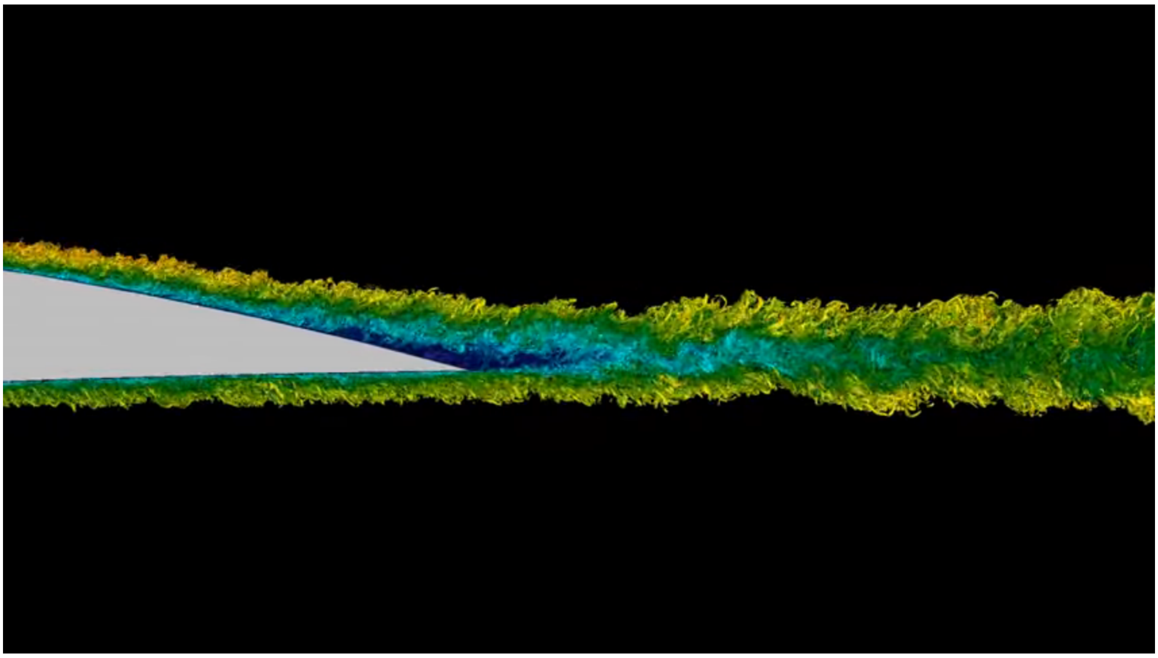

Figure 1.1: A three dimensional direct numerical simulation using high-order methods has been performed to study the flow around the asymmetric NACA-4412 wing at a moderate chord Reynolds number (Re=400,000), with an angle of attack of 5 degrees. This flow regime corresponds approximately to the flow around a small glider. Linn FLOW Centre, Submission to APS DFD Gallery of Fluid Motion 2015

are far simpler than the three dimensional case. The *weak*[1] solutions to the equations were given by Leray in 1934 [3]. But there smoothness remains an unsolved problem. However, the existence of smooth solutions for small initial velocities is known (The low Reynolds number case). It is also known that smooth solutions exists at least for a finite time, for any initial velocity distribution. The case of turbulence occurs essentially outside these two regimes, so turbulent solutions to the Navier-Stokes equations are unknown.

The practical needs of designing aeroplanes and predicting climate accurately in absence of analytic solutions essentially forms the second genre of study. The approach here is to iterate the Navier-Stokes equation in powerful clusters and super-computers for very specific case of boundary conditions (like around wings of an aeroplane or in an unit cube in case of climate) and using this simulations for practical aforementioned needs as well as to shed some light to the behaviour of the solutions for the particular cases. These direct numerical simulations **(DNS)** have not only served as successful models of the air-tunnel experiments like those performed in ONERA, but with sufficient confidence they have replaced them in most of the cases. This essentially has lead to study of fluid flow for very ideal and very complicated boundaries and for an wide range of Reynolds numbers, and forms another pillar in the whole brunch of study. The reason that in most of the cases planes do not crash land due to turbulence, the trajectories of jet do not get hugely perturbed because of the shock wave the generates is credited to these super computer simulations.

For the purpose of the project, which the thesis summarizes, we have taken essentially the third approach, the approach of Dynamical Systems theory. We see the equations 1.1,1.2 as a system of (unaccountably) infinite dimensional non-linear equations, and adapt the approach of qualitative study and geometric thinking, first introduced by Poincare. The whole report discusses the crucial features that we had discussed for past three and half months. The study was broadly based on the first 5 chapters of the book titled "Turbulence" by Uriel Frisch (published by Cambridge University Press), a chronology that is largely reflected in the logical sequence of the report. Often, to get a hang of the way too complicated actuality, we have digressed to the study of simpler and finite dimensional

[1] The idea of constructing an weak solution is to essentially integrate the equations over a smooth test field in the same domain. An formal integration by parts would then lead the *derivatives* to shift to the test field from the actual sought for solution. A solution to this integral system is called weak solution to the actual differential equation. Its existence and regularity essentially implies the existence of the solution to the original equations as long as they are smooth.

cases. These were largely referred to the sections 26, 30-34 of Volume 5 of Course on Theoretical Physics by Landau and Lifshitz [4], chapters 9-12 of the Nonlinear Dynamics book by Strogatz[5] [2] and in some cases (like Burger Equation) some research papers (referenced as appropriate). All of these have shaped the logical flow in their own ways.

1.2 Properties of the Navier-Stokes equations

Here at the very onset we review the significance of the various terms of the PDE under our scopes. The first part of the equation is the Newton′s Law for an individual fluid element subjected to the local pressure gradient, external forcing and viscous drag from an Lagrangian (*go-with-the-flow*) frame of reference. In the framework of theoretical fluid dynamics there exist another type of frame of reference, the Eulerian, which is fixed on a point, and studies the dynamics of a quantity there as a function of time as the fluid flows. The Eulerian frame of reference is relevant in most of the experimental cases. The conversion between the velocity v measured in the Lab Frame (Eulerian, *overlined*) and the theoretical prediction from Navier-Stokes (Lagrangian) is given by,

$$v(t, x) = \overline{v}(t, x - Ut) + U \tag{1.3}$$

where U is the meanflow.

The prime suspect in the complexity of its solution is the non-linear term in velocity (second-term of the LHS.) Principle of superposition fails here and having uncountably many degrees of freedom, this non-linearity can build up into serious complicated structures in principle. From an analytic viewpoint, the existence of the dissipation term makes carrying out the explicit solution slightly trickier as we can no more use the total energy as an integral of motion to our favour. Moreover, dissipative non-linear dynamical systems are known to exhibit chaos at certain ranges of their parameter values, even with three degrees of freedom (minimum topological requirement for showing such behavior)

Another property which adds to the complexity is the non-local nature of the equations, which essentially makes the solution critically dependent on the exact details of the boundary. This implies, the time rate of change of the velocity at a given point depends not only on the values of the required quantities (pressure, velocity, forcing etc) at that point in the previous moment, but also requires data from neighbors. Consider a spanning grid in the space. Each component of velocity at a given lattice point of is an independent degree of freedom for the systems. The time evolution of each of the components then becomes coupled with all the other lattice points, in general in a non-linear way. That how many neighbors actually significantly contributes would depend on the strength of the coupling and off hand we can not restrict it to nearest neighbor-cases, in general. Contrarywise, if it was not non-local, it will be break up into trivial finite-dimensional evolution equations, separate for each space point.

The non locality makes the kinks even more dangerous. It states that the behavior of the air flow around an smooth aeroplane wing, and a very similar one with a slight kink on it can be very different. And for chaotic systems this effects can be magnified so much that it can bring the whole jet down, in principle. This is precisely one of the reasons, why an exact solution is so much preferred and sought for even though the Numerical Simulations seems to work so well. The numerical simulations can not be trusted completely as they are always associated with some error. For chaotic systems the non-linearity of the problem would magnify them very fast, and the non-locality would spread them everywhere.

An standard technique in physics for solving PDEs are to initially break the whole space in cubes of length L, solve it inside a cube and apply periodic boundary condition. The solution of whole space then, can be in principle obtained by taking the limit L goes to ∞. As far as, local equations are concerned we can equivalently solve it for a sphere and take the R goes to ∞ limit to obtain the solution to whole space, and this would be identical to the cube case. This statement is not so obvious for the non-local cases, as for them even the effect of the boundary at ∞ might make an observable difference depending on the strength of the coupling. For a non-local equation the solution inside a sphere and inside an apple (roughly a sphere with two conical protrusion which makes it non-integrable) both with infinite volume wont look similar in general, even though other than very few points their boundary were the same.

One important theorem that makes the situation slightly better is the principle of dynamical similarity. It states that for a given shape of boundary, the only parameter that governs the geometry of the Navier Stokes equation is the Reynolds number. Solutions inside a big sphere and a small

[2]Chapters 2-8 were referred to shortly for basic understanding of finite-dimensional non-linear systems and is not reviewed in the report

sphere having same Reynolds Number $R = \frac{vL}{\nu}$ would essentially be the same[3] This principle of dynamical similitude states, we don't actually need to make bigger / smaller air tunnels. Just by varying the velocities we can get away with the study, as the Reynolds number uniquely specifies the flow.

For non-linear equations the topology of the solutions might drastically change as the value of concerned parameters are varied. Stable points might break into two, disappear, replaced by limit cycle, limit cycles can become stable points or can break into two limit cycles, all sorts of this things are possible and are officially termed as bifurcation.

Turbulence can be seen as chaos in an infinite dimensional system of non-linear coupled ODEs. One possible model for transition to this chaos is the bifurcation route (discussed later in detail.) And the similarity theorem tells us the only parameter that is involved is the Reynolds Number.

Basic Symmetries of the Navier Stokes Equation:

The quantities involved in the Navier-Stokes equation has the following continuous symmetries: Spatial translations, rotations, time translation, Galilean transformations, the following parity,

$$P : t, r, v > t, -r, -v.$$

and the following scaling law.[4]

$$G : t, r, v > \lambda^{1-h} t, \lambda r, \lambda^h v; h = -1$$

This symmetries of the equation may or may not be reflected in the solutions in general, as the boundaries and the mechanism producing turbulence often breaks one or multiple spatial symmetries. Studying and keeping track of this symmetries is important for the following reason. Oftentimes for streamlined flows, depending on if it is already broken by the boundary conditions and the mechanism producing turbulence or not, the symmetries are usually reflected in the solutions. (like in the case of the flow around a cylinder discussed in the onset of chapter 1 of [6]) This symmetries are then broken one by one as the Reynolds Number is increased. One very important observed property of the solutions is that this symmetries starts getting restored in a statistical sense, as the flow transits to more and more developed Turbulence. The Kolmogorov 1941 theory, essentially states that at fully developed turbulence all of these symmetries are statistically restored. This theory forms the content of the next section.

Given a specific boundary condition and smooth initial data, the streamlined (low Reynolds number) solution to the Navier Stokes equation in 3D is well understood. There is, however, no successful attempt of a complete, and rigorous theory which deduces the Kolmogorov results from Navier-Stokes. Transition to turbulence is thus a very weakly understood phenomena, till date.

[3] discussed in detail in Appendix A

[4] If the viscous term is dropped this would be true for any positive real h, thus the non-viscous version has infinitely many scaling groups.

Chapter 2

The Kolmogorov 1941 Theory

> Experiments are the only truth,
> rest is poetry
>
> Enrico Fermi

At very high Reynolds number the wind tunnel experiments, performed at ONERA and many other labs, for various different boundaries seems to exhibit quite similar behavior, on an average. Other than the foundations of Navier-Stokes this subject thus have two experimental laws of fully developed turbulence, which are universal, and is reproducible. This two laws essentially forms one basis of the celebrated Kolmogorov 1941 theory.

The nth structure function of the velocity field is defined as,

$$S_n(l) =< (\delta v_{||}(l)^n) > \tag{2.1}$$

$$\delta v_{||}(r, l) = [v(r + l) - v(r)].\frac{l}{|l|} \tag{2.2}$$

The $\delta v_{||}(r, l)$ essentially the difference functions of velocities at two points separated by a distance l projected along the line of separation. In general it depends of the location of the points as well as the separation. But whenever, the structure functions $S_n(l)$ becomes relevant in the discussion, the turbulence is already fully developed, and acquired a state of homogeneity, and thus they are only a function of the separation.

The two-thirds law: In the case of fully developed turbulence, away from the boundaries, and when length scale of the study is far smaller than the integral length scale of the flow causing the turbulence, the second order structure function behaves approximately as the two thirds power of separation.

$$S_2(l) =< (\delta v_{||}(l)^2) > \propto l^{2/3} \tag{2.3}$$

Law of finite energy dissipation: If in an experiment of turbulent flow, all the conditions barring the viscosity is held the same, in the limit of very low viscosity the mean rate of energy dissipation per unit mass reaches a finite positive limit.

Richardson had an model of the scale by scale energy balance of turbulent flows using the eddies that occurs. According to him the energy after entering the flow essentially is distributed to eddies which have the length scale similar to the integral lengths scale causing the turbulent flows. This eddies then break down to smaller eddies, and then even smaller eddies and this series of decreasing length scale goes on until a length where the viscosity term become way more significant than the other terms and transfer this kinetic energy to the internal energy of the flow (zittering of the constituents molecules.) This length scale is essentially identified as the length where the probabilistic theory of fully developed turbulence starts making sense. According to Kolmogorov, since this behavior is quite universal and independent if the details of the boundaries causing the turbulence, the statistical properties of the flow will essentially be universal. Since, the geometric properties wont play any role, the only two physical quantities on which the flow can depend on, according to Komogorov are the kinematc viscosity ν and the mean energy dissipation rate ϵ. The length scale that can be cooked up from this two quantities is,

$$\eta = (\frac{\nu^3}{\epsilon})^{(\frac{1}{4})} \tag{2.4}$$

This is known as the Kolmogorov length, and usually whenever the concept of a far smaller length scale than the integral scale is invoked in the following statements this is the length that is referred to.

The first hypothesis of Kolmogorov, H1 At the limit of infinite Reynolds number, away from the boundaries and the mechanism producing turbulence, essentially all the symmetries of the Navier-Stokes equation is restored in a statistical sense.

This hypothesis of Kolmogorov essentially stems from the experimental facts. The restoration of the time translation, occurs if the boundary conditions are time independent and essentially means all the moments and structure functions are independent of time. It appears though, even if the forcing is time dependent but localized or works at a length scale larger than the Kolmogorov length, then too there is no problem of restoring the time translation symmetry for the moments. The case of the other symmetries depends essentially on the mechanisms and boundary conditions for the low Reynolds case, all of these are however restored in the most of the cases, as the flow becomes homogeneous and isotropic. The only law that requires slightly more delicacy in interpretation is the scaling. The flows remain self similar as long as the flow is in the Kolmogorov regime.

H2: When H1 applies the turbulent flow is self similar at small scales, i.e, posseses an unique scaling exponent h.

$$\delta v(r, \lambda l) = \lambda^h \delta v(r, l) \tag{2.5}$$

The third hypothesis essentially assumes the energy dissipation law to hold,

H3: For a turbulent flow satisfying H1, in the limit of very low viscosity, the mean rate of energy dissipation per unit mass reaches a finite positive limit.

The main result of Kolmogorov theory is to fix the exponent h to be $\frac{1}{3}$.[8]

The second universality assumption: At the limit of infinite Reynolds number, with H1 assumed, all the statistical properties at small scales are uniquely and universally determined by the separation length and the energy dissipation rate ϵ.

A straightforward dimensional analysis would show that the second order structure function goes as,

$$S_2(l) =< (\delta v_{||}(l)^2) >= Cl^{2/3}\epsilon^{2/3} \tag{2.6}$$

now according to H2 the second order structure function should be proportional to l^{2h} and that fixes the h to be 1/3 The second result that follows is,

The Kolmogorov four-fifth law: Under the conditions of H1 and assuming H3, the third order structure function for homogeneous and isotropic turbulence is given as the

$$S_3(l) =< (\delta v_{||}(l)^3) >= -\frac{4}{5} l\epsilon \tag{2.7}$$

In general, the general result of Kolmogorov 1941 theory goes as, when H1, H2, H3 is satisfied,

$$S_p(l) =< (\delta v_{||}(l)^p) >= C_p l^{p/3}\epsilon^{p/3} \tag{2.8}$$

for p=3 the constant is $-\frac{4}{5}$ is an universal experimental fact. The assertion is that for all other moments, the C_ps have the universality for all sorts of Turbulent flows.

Landau had discussed an objection in his book on the second universality assumption. [4, 6] Quoting him, It might be thought that the possibility exists in principle of obtaining a universal formula, applicable to any turbulent flow, which should give $S_2(l)$ for all distance s.t. that are small compared with $S_2(l)$ In fact, however, there can be no such formula,as we see from the following argument. The instantaneous value of $\delta v_{||}(l)^p$ might in principle be expressed as a universal function of the dissipation s at the instant considered. When we average these expressions, however, an important part will be played by the manner of variation of v over times of the order of the periods of the large eddies (with size l_0), and this variation is different for different flows. The result of the averaging therefore cannot be universal.

What rescues Kolmogorov theory is the second assumption is actually not necessary for the results, a fact that even Kolmogorov seems to be aware of and explicitly mentioning in [8] That assumption is only used to simplify the proof. The actual results can be recovered without that as is elaborated in [6] and Kolmogorov seemed to be well aware of that derivation even in 1941.

However, the universality of the dimensionless constants C_ps, has been questioned by various further studies. Various numerical simulations has reported values which of C_2 and C_3 for specific cases, which are way off from the theoretical predictions. These fact suggests that the K41 actually might be an approximate theory. One of the suggested way is to use wavelet analysis, in place of Fourier transforms, which completely neglects the spatial information. However this was not discussed further during the course of the project.

Specifications of the moments at high Reynolds number restricts the possible allowed velocity fields enormously, but actually does not uniquely determine it for a given boundary conditions, as the result is only true about moments. Interestingly, in the whole discussion Kolmogorov never invokes the Navier-Stokes equations, and its an open possibility other than a probable deduction or a finite time blow up, if it is shown that any solutions to Navier-Stokes equation in general can not satisfy all of the Kolmogorov results. In that case Such a result would indicate that modifications are required to the K41 theory. Any such studies however were not cited and not discussed during the study.

Chapter 3

Route to Turbulence

> Suppose you are studying human heart. Do you necessarily need to cut open a human?
>
> Anonymous professor of Biology, IISER Pune

Since Navier-Stokes works so well in the low-R flow, there are two disjoint possibilities for a theory of transition to turbulence. As we increase the Reynolds number the solution might develop a finite time singularity, and thus the framework might breakdown. In that case, in an attempt of having a theory at all levels, we would need to generalize the solution-space or the framework itself. Kolmogorov theory puts possible restrictions to the allowed possibilities. Any *theory-of-all-levels* should either reproduce Kolmogorovs results, at least approximately, or explicitly violate its assumptions. On a very different track attempt has been made at least at qualitative levels to deduce a theory of transition by examining the possible nature of solutions, when bifurcation occurs to the Navier-Stokes equation, exploiting the machinaries of Dynamical Systems Theory. One of such attempts is the Landau-Hopf[9, 10] version of transition to turbulenece, which is based on linear stability analysis of the Navier Stokes Equation, analogous to the period doubling route to Chaos. In the first section of the present chapter we would discuss the nitty-gritties of the the Landau-Hopf theory which was studied mainly from reference([4]). Then we go on to discuss a few instances of finite time singularities in differential equations, with special reference to the context of Burger Equation.

3.1 Landau theory of n-tori: Linear stability analysis of the Navier-Stokes equation.

Consider a space of solutions to the the Navier Stokes equations, and name it *state space* in a way that each point of the abstract space denotes a spatial solution. The solutions at neighboring time is denoted by neighboring points in this space. In general the space is countably infinite dimensional, irrespective of boundary conditions, as given a solution at a particular instance, in the next moment it can change in countably infinite ways, corresponding the change of the velocity vector at each point of the space, which itself has three degrees of freedom.

However for some special cases, the solution might be allowed to traverse only a subspace of finite dimension. Steady solutions are represented by points in the state space, periodic solutions by circles and so on. In this description, however the *dimensionality* of the available subspace and the embedding is not really well-defined. Consider a periodic flow which has two distinct characteristic angular frequencies. In general, the trajectory would be restricted to the surface of a 2-tori, the two degrees of freedom corresponding to the phases corresponding to two frequencies. Now, if the frequencies are commensurate, the trajectory would be a closed loop, which can as well be described as a closed loop embedded in 2D, with a new frequency, equal to the LCM of the actual ones. Objects in this state space are thus unique only up-to homeomorphism. For incommensurate case, the trajectory never really closes and thus cant be reduced further from a torus.

For a given shape of boundary, consider a steady solution $v_0(r)$ to the NS equation. This of course satisfies,

$$v_0(r)\nabla v_0(r) = -\frac{\nabla P_0}{\rho} + \nu\nabla^2 v_0(r) \tag{3.1}$$

A steady solution is essentially one which does not depend on time. Such a solution might however need not be steady. Consider a slight perturbation $v_1(r,t)$ to the velocity and correspondingly a perturbation $P_1(r,t)$ in the pressure field. This Perturbation can either be stable, unstable or can lead to a bifurcation. $v_1(r,t)$ satisfies,

$$\frac{\partial v_1(r,t)}{\partial t} + v_0(r)\nabla v_1(r,t) = -\frac{\nabla P_1}{\rho} + f + \nu\nabla^2 v_1(r,t) \tag{3.2}$$

which is homogeneous equation of $v_1(r,t)$ with time independent coefficients. The general solution to such equation can be written in the form

$$v_1(r,t) = A(r)e^{i(\omega_1+i\lambda)t}$$

The permitted values of ω and λ are determined by the boundary condition.

The real part of the frequency gives a periodicity to the solution, while depending on the sign, the complex part can damp the solution back to $v_0(r)$ or make the flow unstable. Such instabilities would tend to increase the velocity indefinitely. This result is not valid however outside the regime of linear stability. Usually for the naturally occurring flows, such instabilities would finally be damped by higher order stabilities. Interesting case happens, if for a Reynolds number, λ is identically zero. This essentially makes the flow periodic in nature, and in the state space, the point is replaced by a circle. This is exactly an Hopf bifurcation, suppose this phenomena occurs at $R_{cr,1}$. We can further increase the flow and similar results will be expected. At some $R_{cr,2} > R_{cr,1}$ the flow would undergo another Hopf Bifurcation, with another characteristic frequency ω_2 making its appearance in the flow. If ω_1, ω_2 are incommensurate the flow becomes aperiodic and is replaced by tori in the state space. The trajectory in this case would cover the whole surface of the tori without intersecting itself as time goes to ∞. This process can continue and and respectively $R_{cr,3}, R_{cr,4}, R_{cr,5}$ and so on the manifold in the state space will be replaced by first a 3-tori, then a 4-tori, 5-tori so on, with new and new frequencies coming into picture. Now the claim of Landau is that the sequence of this Reynolds Numbers at which this bifurcation occurs, essentially forms a Cauchy Sequence. So it would converge to some finite R. Above this R the solution changes no more, due to the similarity theorem. That such a claim is qualitatively true can be seen from the existence of Kolmogorov theory. After the turbulence is developed fully, it by definition looks the same. The manifold to which this sequence of n-tori converges to is completely aperiodic in nature as infinitely many frequencies has gone into its structure, and thus is capable of imposing the chaotic nature to the flow. ‘

Essentially this final manifold forms a strange attractor in the state space. This idea is more illustrative from the finite dimensional cases. As the opening quote suggests it is often illusive to study simpler (but not too simple) cases in the attempt of gaining knowledge about a way too complicated system like human heart. The rest main body of the thesis essentially delves in the same spirit. For dissipative systems in the state space, in finite dimensional system of ODEs,

$$\frac{\partial \overline{x}}{\partial t} = \overline{f\overline{x}}$$

the gradient of f is always negative and thus with time, the volume in phase space would continue to shrink. Thus for dissipative systems, essentially there are two fates, either it can come to a stable point or limit cycle, or for dimensions greater equal to three they can converge into a surface which has zero volume in the state space but infinite surface area. The second requirement comes from the fact that the trajectories that comes to this closed set of points remains bound for eternity without intersecting itself or each other. This surface embeded in the original space is of course of lower dimension and usually the dimension can be fractional[5]. The strange attractors for dissipative systems usually exhibits this fractal nature, which results into its chaotic motion.

There is an algorithm due to Grassberger and Procaccia [11]which tells just from an enough long time series of an single variable how one can deduce the equations of motions. The idea for finite dimension goes like the following. One plots the time series in an N dimensional surface such that the N is the minimum dimensions at which the time series do not intersect itself. This dimension is essentially the embedding dimensions. The equation of the hypersurface that the time series crawls can be in principle then inverted to get back the equations of motion. This method is extremely successful for lower dimensional cases, especially and an analogue for the infinite dimension would be a bliss, as more than often the only real data you have for fluid flows, is time series of one component of velocity at a point.

3.2 Finite time singularities of Differential Equations

Occasionally, in modelling physical phenomena with differential equations, we encounter the fact that the solution develops into a (or more) singularity, starting from a smooth initial data, within a finite time frame. In that case, the equation can not be iterated anymore for all times, but depending on what is being modelled, the equation usually still is effective other than the times the singularity is there. If we start with an initial normal distribution and evolve the diffusion equation back in time, it will go to a Dirac delta after finite time t_0. But that is fine, because the initial normal data can be thought of as a resultant of time evolution of the delta function for time t_0 and it is the forward evolution equation that carries physical significance. Consider, on the other hand, the type differential equation,

$$\frac{\partial^2 x}{\partial t^2} = -x^2 \tag{3.3}$$

The solution x(t) blows up once and for all at a finite time. That this situation can be made sense of depends on what it is modelling. If it is modelling a particle motion under quadratic force then we are fine as the particle has already gone to -infinity and the notion of its co-ordinate after that instant anyhow does not makes any sense. But this same equation also models the congruence of geodesics according to the famous Raychaudhuri equations [12] in general relativity. There it forms caustic once and for all and the theoretical framework breaks down.

The fact the second possibility for the Clay Problem is entertained inquires a study of possibility of the such things occurring for Navier-Stokes equations. And it turns out this behaviour is actually shown at least for a simplified version of the same.

The Burgers equations are essentially the Navier Stokes equation for one dimension with no forcing and the pressure term dropped.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = +\nu\frac{\partial^2 u}{\partial x^2} \tag{3.4}$$

For the non-viscous case starting from a Gaussian Wavelet it forms a singularity after some blow-up time and essentially forms shock-wave that travels. Interesting things like two shock waves existing simultaneously and colliding is also exhibited. For the viscous case the shock-waves decays down. However, due to dimensional restrictions, Burger Equation can neither show chaos or exhibit vorticity.

All of these are possible complicated structures that can occur in the turbulent solutions to Navier-Stokes equation, in principle. Studies have shown some smallness criteria of initial data for not blowing up and it still remains in general not understood.

Chapter 4

Summary

During the project we have mainly treated the Navier-Stokes equation as the foundation. The crux of the discussion is that the symmetries of the solutions for low Reynolds number flows, which are broken as the Reynolds number is increased, are statistically restored. The velocity fluctuation at any point is aperiodic and chaotic, but the histogram for equal intervals are reproducible, for long enough intervals. The K41 theory attempts to leap from two experimental statistical law, and frame a universal theory of moments, independent of route to turbulence. However there are various objections and studies which questions the validity of the universality, and thus it is not not considered as an exact result in the field. There are a few models, like the Landau-Hopf bifurcation theory which uses the bifurcation route to chaos of dynamical systems theory, and gives qualitative approximate models as candidates of route to chaos. However at a quantitative level there is no such general result, and a theory of route to turbulence starting from Navier-Stokes equations is yet to be formulated. Meanwhile, the practical needs of the understanding in absence of a general theory is supplemented by direct numerical simulations, which also provides us with bettered understanding of the transition and the possibilities of occurrence of finite time singularity.

Appendix A

Similarity Theorem

The similarity theorem in the context of fluid dynamics, which crowns the Reynolds number as the *only* parameter to have determined the topology of the fluid streamlines for a given shape of boundary (and not the size of it), is consequence to a more general result, which provides us with a way to *nondimensionalise* a physical equation. The principle is that two systems having the same values of all the dimensionless parameters would behave exactly in the same way. The theorem rescues the theorists from anxiety by ruling out other plausible *hidden* parameters causing the *trouble* in the Navier-Stokes equations, as well as allows the experimentalists, engineers, sheepbuileders, to test scaled down models first and strongly hope that the original model would work out. Especially for us (and for Landau and Hopf), the theorem had been at the foundation of discussion, whenever we have invoked bifurcation of the Navier-Stokes equation only based on differed values of the Reynolds Number. We, thus find it important to present the intricacies of the result on which so many bold claims discussed in the thesis is based upon and include it as the topic of Appendix A.

Buckinghum Π theorem: If there is a physically meaningful equation involving n quantities and k basic physical dimensions (such as mass, length, time, luminescence, charge etc), then the equation can be re-written in terms of p dimensionless quantities, where $p = n - k$.

proof: Consider a vector space of physical quantities defined over the rational numbers, with the fundamental units $q_1, q_2....q_k$ as the basis vectors, multiplication as the vector addition operation and raising a rational number as power as the scalar addition operation. Any derived quantity can be written as a vector in the space, with the powers of the fundamental quantities as *co-ordinates*

Given a system involving, n physical quantities in total, construct an (n,k) matrix whose (i,j) entry is the exponent of the j-th fundamental quantity in the expression of the i-th physical quantity concerned. The required proof then follows from the rank-nullity theorem. For n column vectors the rank, that is the number of independent vectors, equals the number of fundamental units by construction. The number of available dimensionless quantities are then given by the nullity of the matrix which equals, p=n-k.

Lemma: Rank-Nullity theorem: For a given system of n vectors, k being the rank (maximum number of linearly independent vectors) and p being the nullity(number of vectors that spans the null space of the system.) then

rank+nullity=no of vectors in the system (the dimension)

proof: refer to Hoffman, Kunze, page 71.

The Π theorem states only the numbers of *dimensionless* quantities that can be cooked up, given a choice of a set of *fundamental* units. Since the later is not unique, so is not the previous. But once the dimensionless version of an equation has been obtained, the principle of similarity follows. Specifying the values of the dimensionless parameters specifies the system completely. If same values of the dimensionless parameters are same for two different systems, they would essentially have the exact similar behavior.

Consider the Navier-Stokes Equation in the following form:

$$\frac{\partial u}{\partial t} + u\nabla u = -\frac{\nabla P}{\rho} + f + \nu\nabla^2 u \tag{A.1}$$

Consider the length and the velocity as the fundamental units, defining L as the characteristic Length of the system (arising from boundary conditions) and v as the characteristic velocity (coming in the system from initial data), we have this rescaled dimensionless quantities,

$$u = v\overline{u}, x = L\overline{x}, \frac{P}{\rho} = v^2\frac{\overline{P}}{\overline{\rho}}, f(x) = \frac{v^2}{L}\overline{f(x)}$$

substituting them in A.1, and dividing all through by $\frac{v^2}{L}$ one obtains the dimensionless equation

$$\frac{\partial\overline{u}}{\partial\overline{t}} + \overline{u}\overline{\nabla}\overline{u} = -\frac{\overline{\nabla P}}{\overline{\rho}} + \overline{f} + \frac{\nu}{vL}\overline{\nabla}^2\overline{u} \tag{A.2}$$

The reciprocal of the factor $\frac{\nu}{vL}$ is the Reynolds number, and specifying the value of it completely specifies the dynamical system. Since it depends on only the ratio of Kinematic viscosity (which is property of the fluid) to L,v and and does not contain any info of the shape of the boundary, the flow in two different shaped boxes (say a sphere and a pipe) with same Reynolds number can have very different flow. But once the shape is specified, Reynolds number becomes the only parameter to determine the topology of the flow.

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