Boundary Area

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We need to know

Handling Boundary edges

When partitioning a problem space (**such as a geometric space or a graph**), the elements near the **boundaries of partitions** may interact with elements in neighboring partitions. These interactions can create **edge effects**, where the behavior near the boundary is different from that in the interior.

Decomposing Global Problems

For some problems, the **global structure** is more important than the local details. By studying the boundaries, you can better understand how to decompose the global problem into manageable subproblems **without losing essential connections** or interactions between different parts of the problem.

We need to know

Dominating Set

A dominating set for a graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D.

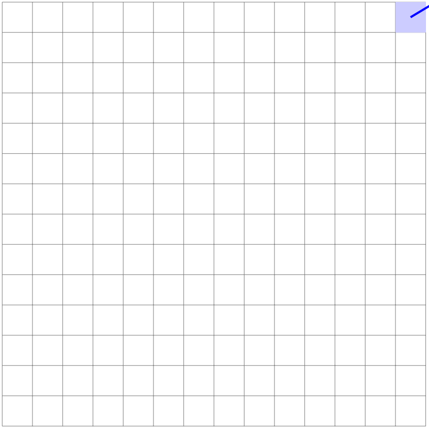
Unit Disk Graph

A unit disk graph is a graph where each vertex represents a disk of unit diameter in the Euclidean plane. There is an edge between two points u and v if and only if the two unit disks centered at u and v have a nonempty intersection.

CDS-UDG Problem

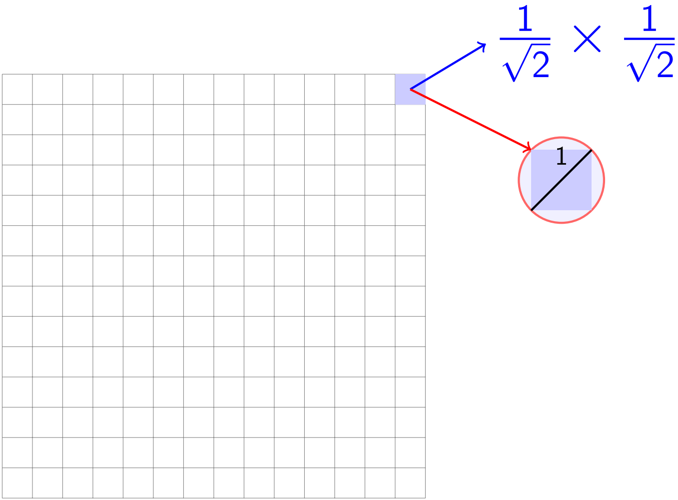
The Connected Dominating Set (CDS) problem in Unit Disk Graphs (UDGs) is to find a connected dominating set with minimum cardinality for the graph.

UDC_1 visualization for $m \times m$ grid



$$\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$

UDC_1 visualization for $m \times m$ grid



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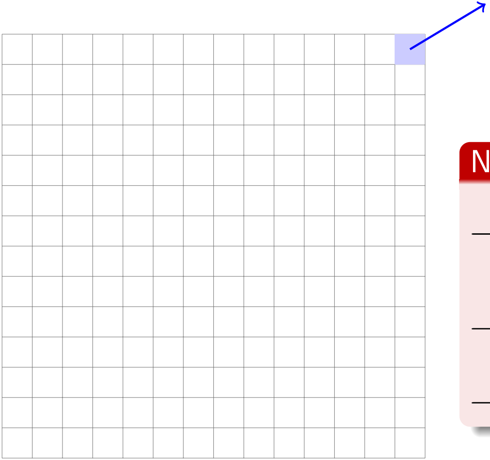


$$\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$

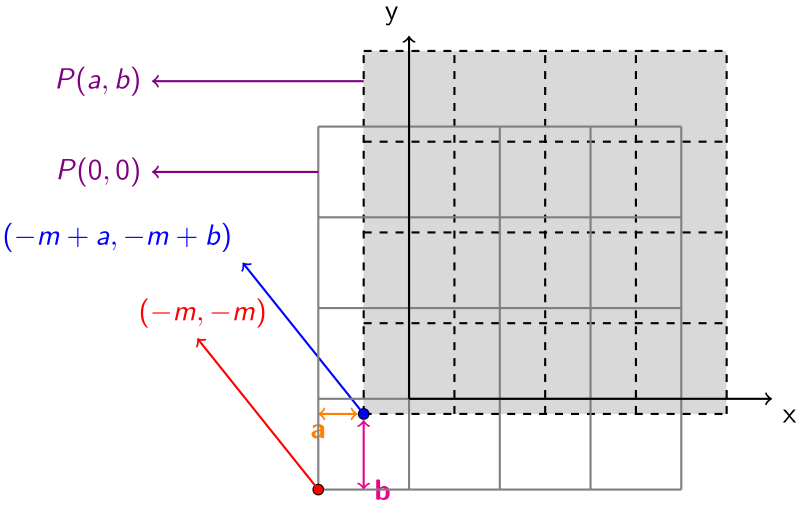
Number of disks

$$\rightarrow \left(\frac{m}{\frac{1}{\sqrt{2}}}\right)^2$$

$$\rightarrow \lceil \sqrt{2}m \rceil^2$$

$$\rightarrow \lceil 2m^2 \rceil$$

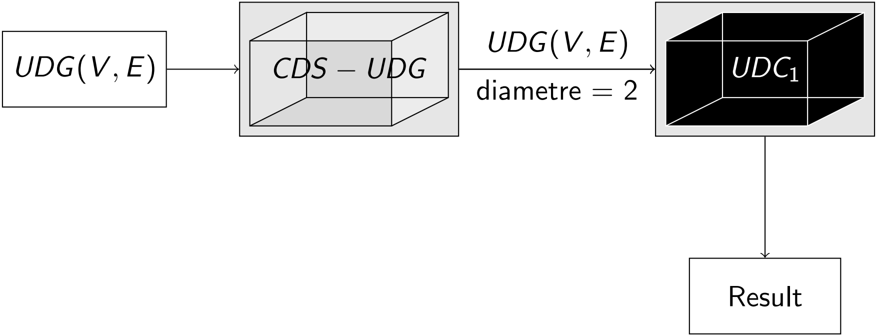
UDC$_1$ visualization for $m \times m$ grid

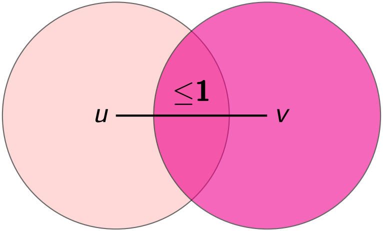


Theorem 4.4

For any $\varepsilon > 0$, there is a $(1 + \varepsilon)$-approximation for the minimum dominating set problem in unit disk graphs that runs in time $n^{O(1/\varepsilon^2)}$.

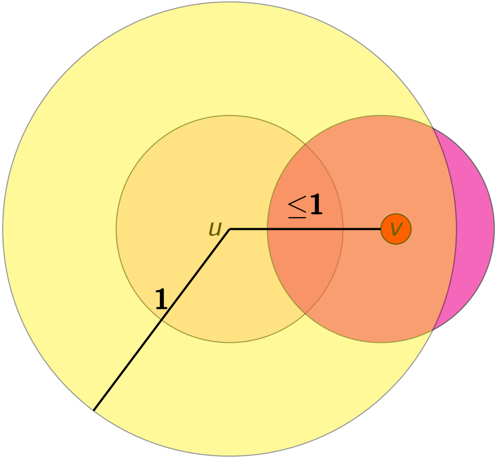
Theorem 4.4



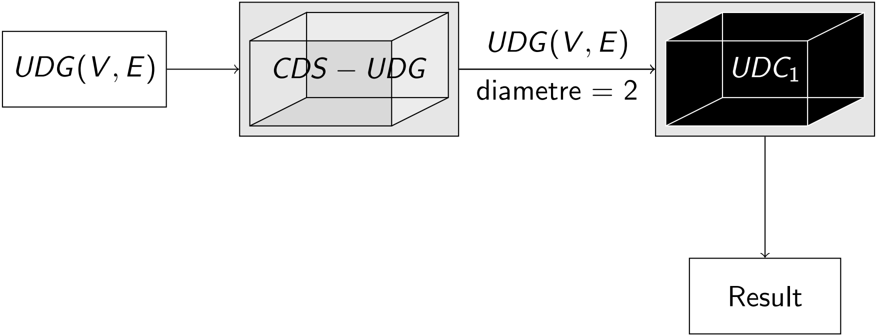


$\leq \mathbf{1}$
u
v

Theorem 4.4 contd...

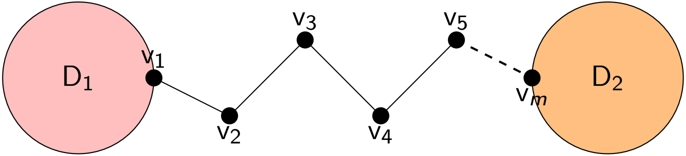


Theorem 4.4 contd...



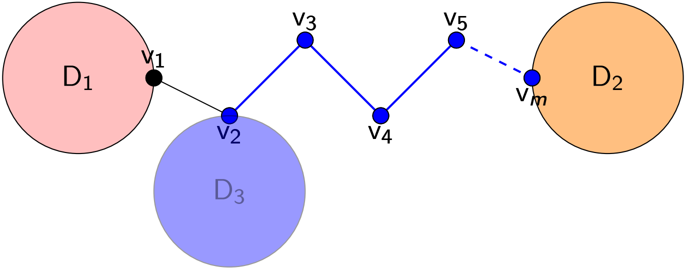
A Lemma We need to know

If a Dominating set D of some graph G is not connected, then the number of its connected components can be reduced by adding at most two vertices to D



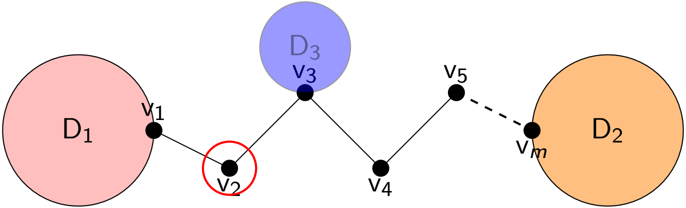
D_1 and D_2 are the two connected components of the entire dominating set of vertices D of some graph G that are connected with the shortest possible path among connecting components $\{v_1, v_2, v_3, \ldots, v_m\}$ where $v_1 \epsilon D_1$ and $v_m \epsilon D_2$.

A Lemma We need to know



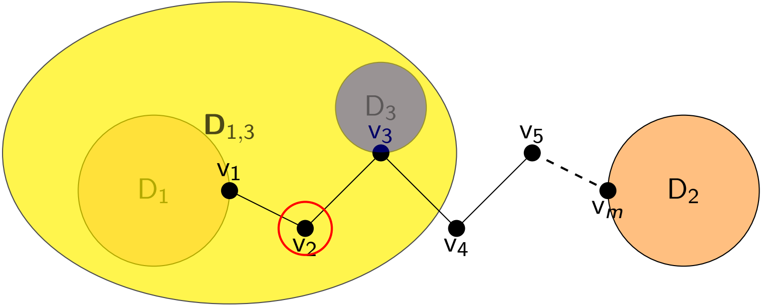
Now v_2 cannot be a part of the Dominating set D of the graph G as this will give a shorter path $\{v_2, v_3, v_4, \ldots, v_m\}$ contradicting to our assumption.

A Lemma We need to know



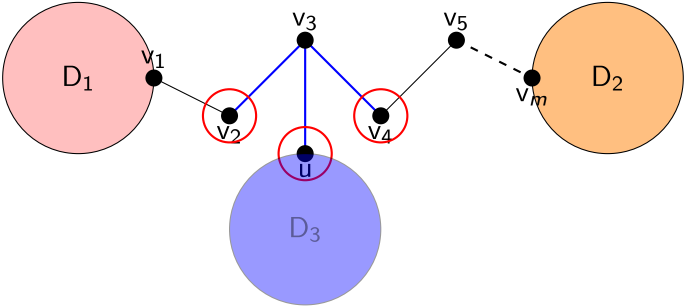
If $v_3 \epsilon D$, v_2 can be included in D in order to reduce the number of connected components.

A Lemma We need to know



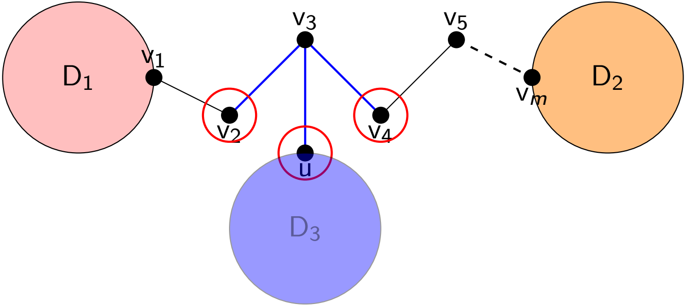
D_1 and D_3 are merged upon joining v_2 hence number of connected components reduced.

A Lemma We need to know



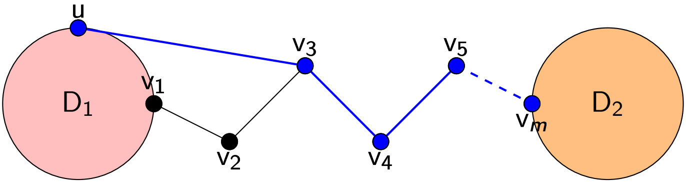
If $v_3 \notin D$, then v_3 must be dominated by one of its neighbours $\{v_2, v_4, u\}$

A Lemma We need to know



→ We already know $v_2 \notin D$
→ If $v_4 \epsilon D$ or some $u \epsilon D_3$ ($D_3 \neq D_1$), then v_2 and v_3 are added to reduce a component

A Lemma We need to know



Again $u \notin D_1$ as this will give yet another shorter path $\{u, v_3, v_4, \ldots, v_m\}$ contradicting to our assumption.

Theorem 4.5

There is a polynomial-time 4-approximation for CDS-UDG.

Theorem 4.5 contd...

1. Suppose the approximated Dominating set of a Unit Disk Graph $G(V, E)$, $D \subseteq V$ is a (4/3)-approximation for the minimum connected Dominating set D^*
   $|D| \leq 4|D^*|/3$
2. Our claim is that D is not connected
3. We have already proved that, If a Dominating set D of some graph G is not connected, then the number of its connected components can be reduced by adding at most two vertices to D
4. Hence, as per our claim, we need to add at most $2(c-1)$ vertices to D to get a connected dominating set, where c is the number of connected components in D.

Theorem 4.5 contd...

Calculation for 4-approximation

Clearly, $c - 1 < c \leq |D|$

$\rightarrow 2(c-1) < 2|D|$

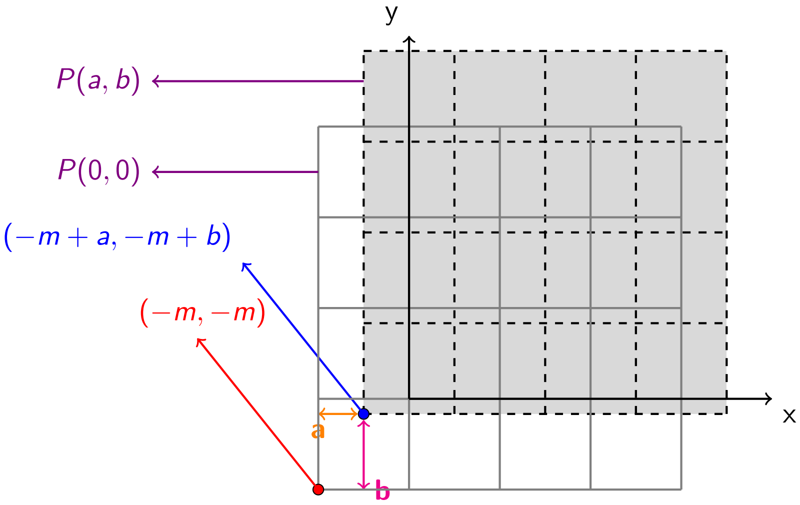
$\rightarrow |D| + 2(c-1) < 3|D|$

$\rightarrow |C| < 3|D|$

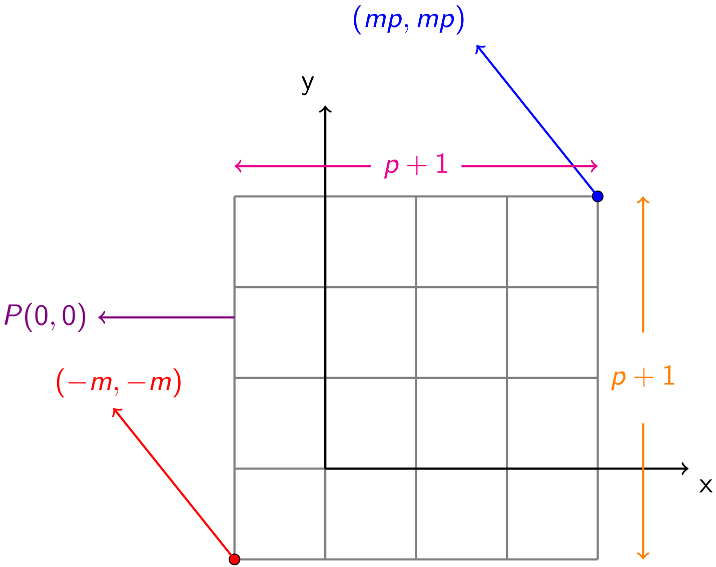
$\rightarrow |C| < 3(4|D^*|/3)$

$\rightarrow |C| < 4|D^*|$

We need to know more



Why Partition shifting?



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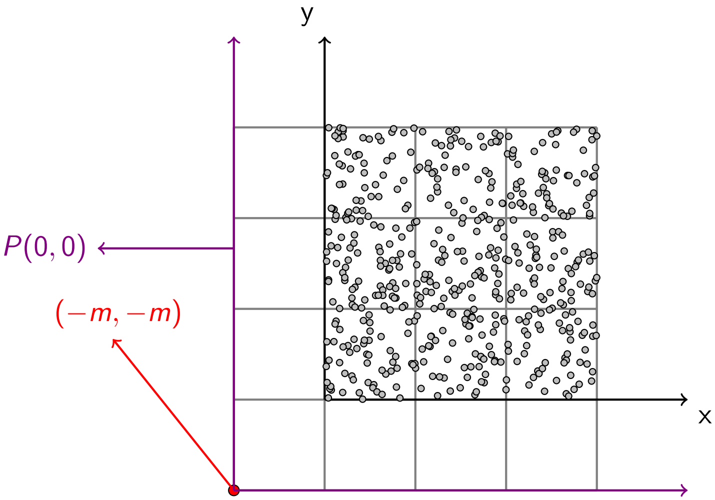

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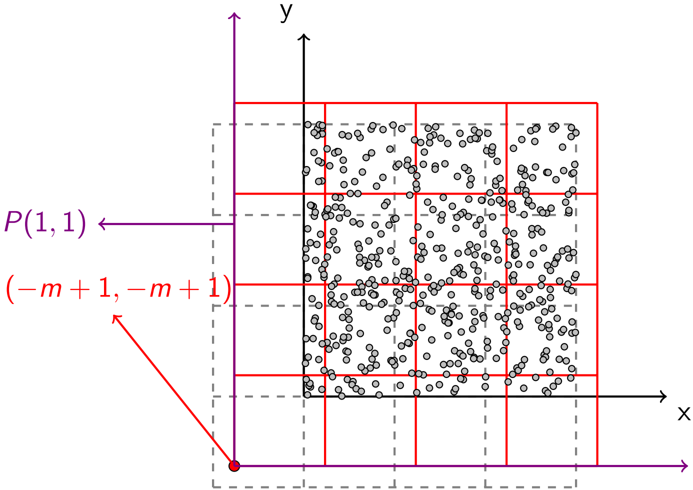

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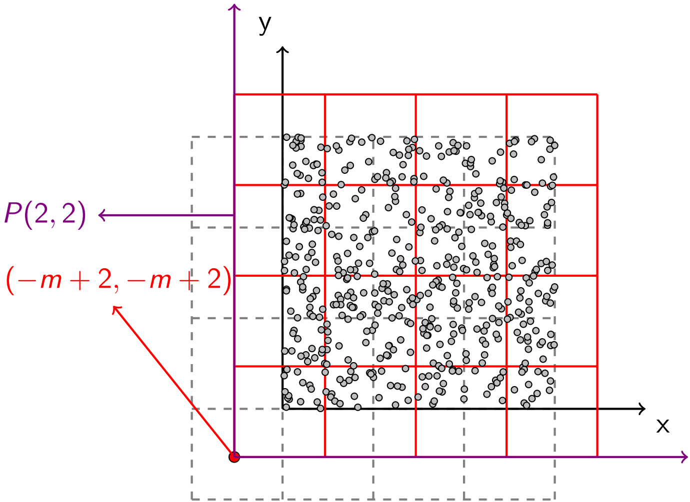

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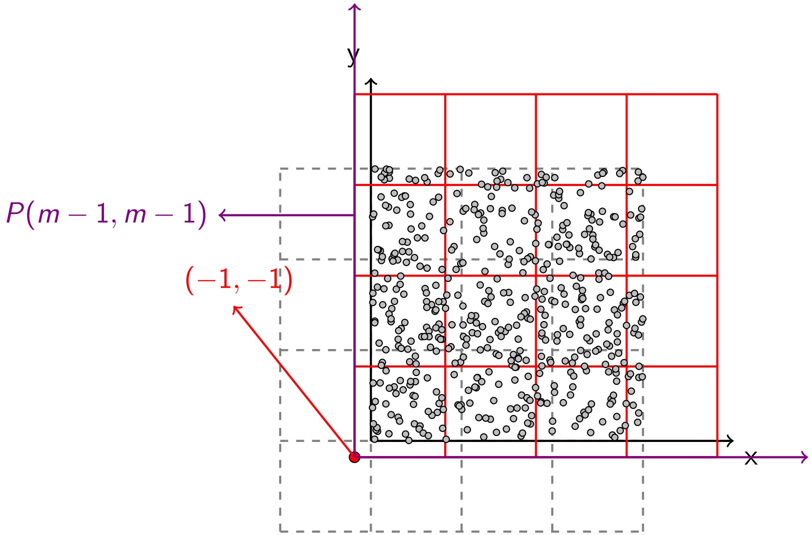
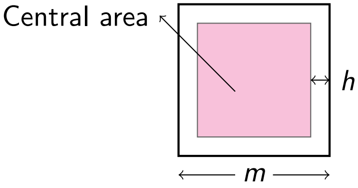
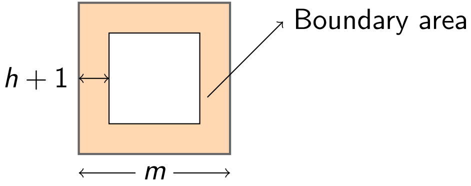
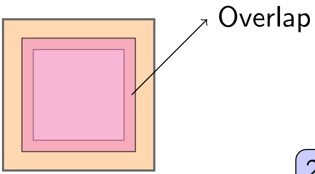


Central Area and Boundary Area

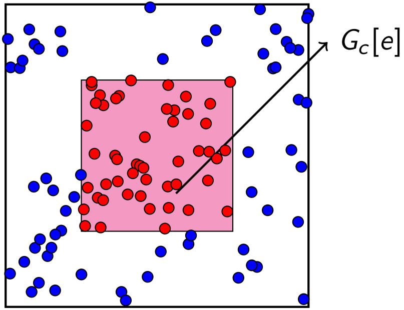






$$\boxed{2h + 2 < m}$$

Central Area and Boundary Area



→ One of the $(p+1)*(p+1)$ cells is considered e
→ For each connected component H of $G_c[e]$, the subgraph of G induced by $C[e]$ has a connected component dominating H

Theorem 4.6

For each cell e in a partition $P(a, a)$, the set $C[e]$ can be computed in time $n_e^{O(m^2)}$, where n_e is the number of vertices in e

Theorem 4.6

1. We have already known that that each cell can be covered with $2m^2$ unit disks without considering boundaries, hence there are at most $2m^2$ dominating vertices in $C[e]$
2. We have already proved that, If a Dominating set D of some graph G is not connected, then the number of its connected components can be reduced by adding at most two vertices to D, hence for making the set connected we add at most $2 * 2m^2 = 4m^2$ more vertices to $C[e]$
3. Hence, as per our claim, there at most $6m^2$ vertices in $C[e]$ to get a connected dominating set, which are selected from n_e vertices in cell e

Theorem 4.6 contd...

Calculation for $C[e]$

At most $6m^2$ vertices are exhaustively selected from n_e vertices.

$$\rightarrow \binom{n_e}{1} + \binom{n_e}{2} + \binom{n_e}{3} + \ldots + \binom{n_e}{6m^2}$$

$$\rightarrow \sum_{k=1}^{6m^2} \binom{n_e}{k}$$

$$\rightarrow n_e^{O(m^2)}$$

Whether $C[e]$ dominates each connected dominating set H in subgraph $G_c[e]$ can be determine in linear time. So $C[e]$ is calculated in time $n_e^{O(m^2)}$

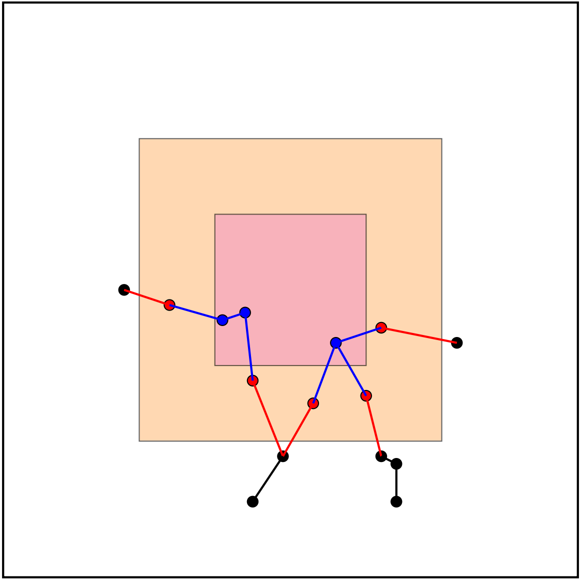
PTAS for CDS-UDG

Algorithm 4.B PTAS for CDS-UDG

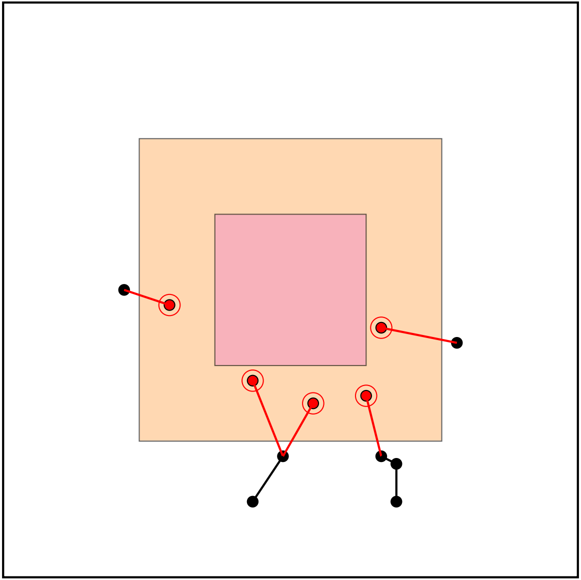
Input: A unit disk graph $G = (V, E)$, with all vertices lying in square Q.
Output: An approximate connected dominating set A_{a^*}.

```
Let h ← 3 and m ← ⌈160/ε⌉.
Let D ⊆ V be a 4-approximation to the minimum connected dominating set
  for G (obtained by the algorithm of Corollary 4.5).
for a ← 0 to m − 1 do
    Let D_a ← {v ∈ D | v lies in the boundary area of P(a, a)}.
    for each cell e of P(a, a) do
        Compute set C[e] (by exhaustive search of Lemma 4.6).
    end for
    Let A_a ← D_a ∪ ( ⋃_{e∈P(a,a)} C[e] ).
end for
Let a* ← arg min_{0≤a<m} |A_a|.
Return A_{a*}.
```

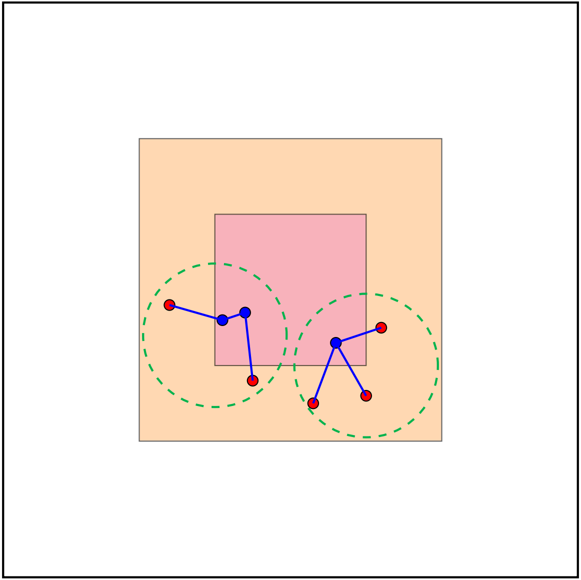
And we need to know again



How do we achieve D_a ?



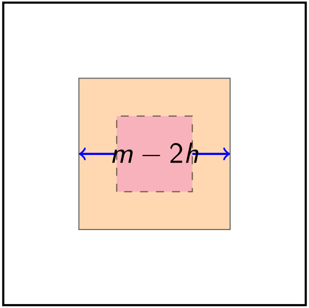
How do we achieve $G_c[e]$?



Lemma 4.7

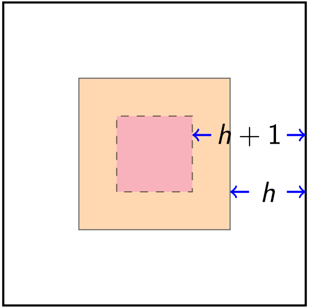
For each $a \epsilon \{0, 1, ..., m - 1\}$, set A_a computed by Algorithm $4.B$ in step (8) is a connected dominating set for input unit disk graph G.

Lemma 4.7



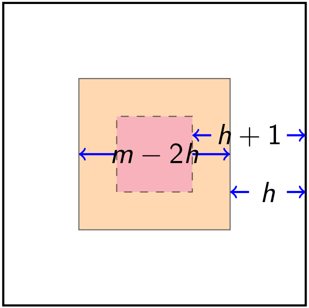
The area within the white boundary, that is the Central Area of dimension$(m - 2h) * (m - 2h)$ is dominated by $C[e]$ as per the condition of $C[e]$, where e denotes the cell

Lemma 4.7



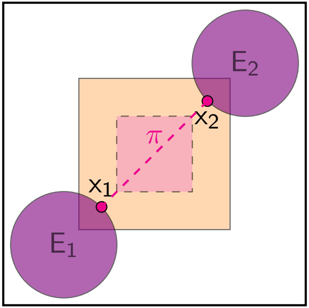
The white boundary, that denotes the area within h unit of the boundary of cell e is dominated by the boundary area as the boundary area is 1 unit wider from each side

Lemma 4.7



Thus the set A_a is a Dominating set

Lemma 4.7



Both x_1 and x_2 are inside the central area, hence are dominated by some connected component C' of the subgraph of G induced by $C[e]$ as per the condition of $C[e]$, where e denotes the cell

Algorithm 4.B is a PTAS

Runtime of $C[e]$ computation for each cell e in partition $P(a, a)$

$$\sum_{e \in P(a,a)} n_e^{O(m^2)} \leq \left(\sum_{e \in P(a,a)} n_e \right)^{O(m^2)} = n^{O(m^2)} \tag{1}$$

Runtime of algorithm 4.B

for $a \leftarrow 0$ to $m-1$ **do** $\rightarrow O(m)$

$D_a \leftarrow \{v \in D \mid v \text{ lies in the boundary area of } P(a,a)\} \rightarrow O(n)$

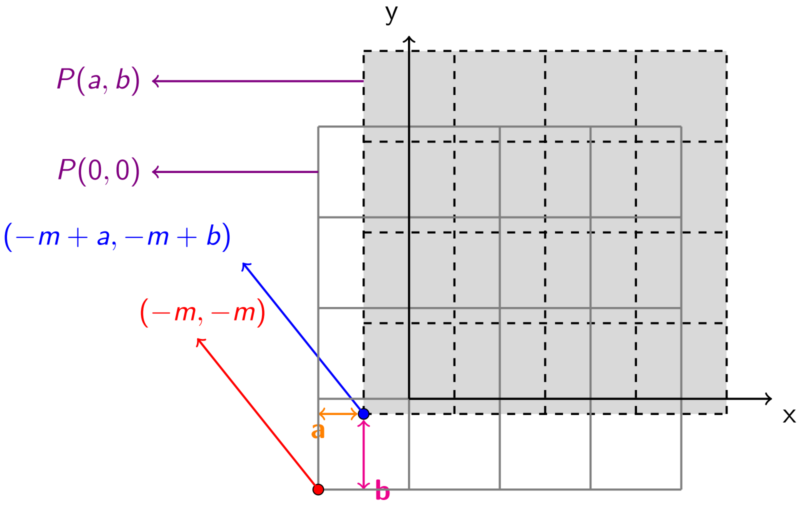
Compute $\bigcup_{e \in P(a,a)} C[e] \rightarrow n^{O(m^2)}$

$Runtime = O(mn) + m \cdot n^{O(m^2)}$

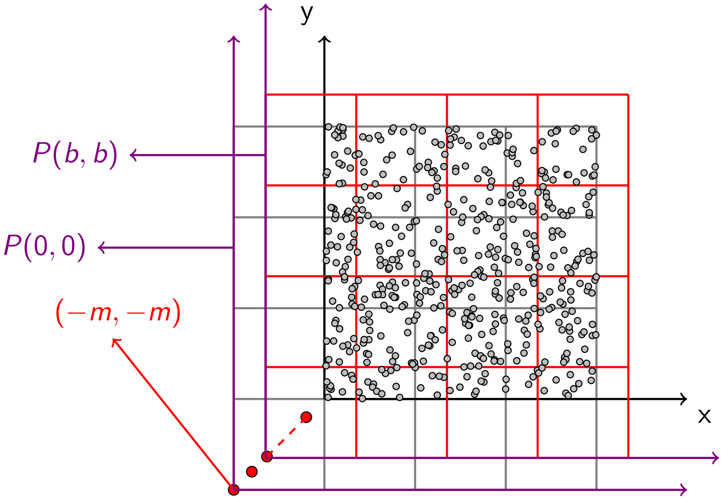
Theorem 4.8

Output A_a^* of algorithm $4.B$ is a $(1+\varepsilon)-$ *approximation* for CDS-UDG with computation time $n^{O(1/\varepsilon^2)}$.

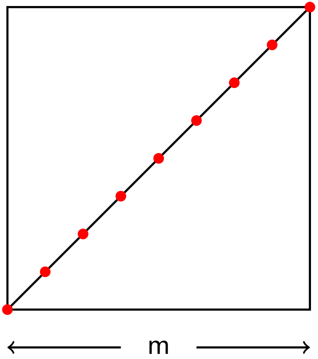
Theorem 4.8



Theorem 4.8 contd...



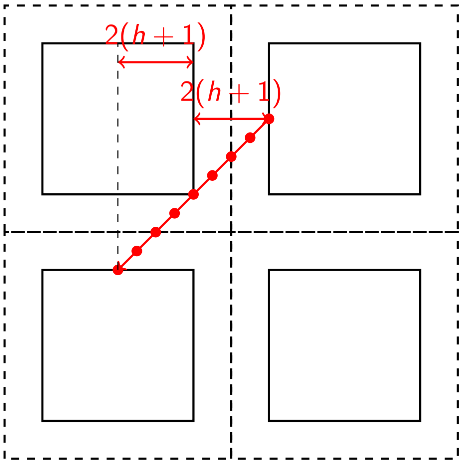
Theorem 4.8 contd...



$\rightarrow$ The slope of the line created by the change of the position of points is 1

$\rightarrow$ There are m possible positions for a point in a cell for partition transition, each of them are consecutively spaced by $\sqrt{2}$ unit

Theorem 4.8 contd...



Theorem 4.8 contd...

Claim

$$6 \cdot |D_b^*| + |D_b| \leq \varepsilon \cdot |D^*| \tag{2}$$

For any vertex v in D^*, it belongs to at most $4(h+1)$ of the sets $D_0^*, D_1^*, ..., D_{m-1}^*$. Therefore, by the pigeonhole principle, we have

$$\sum_{a=0}^{m-1} |D_a^*| \leq 4(h+1)|D^*| \tag{3}$$

For any vertex v in D, it belongs to at most $4(h+1)$ of the sets $D_0, D_1, ..., D_{m-1}$. Therefore, by the pigeonhole principle, we have

$$\sum_{a=0}^{m-1} |D_a| \leq 4(h+1)|D| \leq 16(h+1)|D^*| \tag{4}$$

Theorem 4.8 contd...

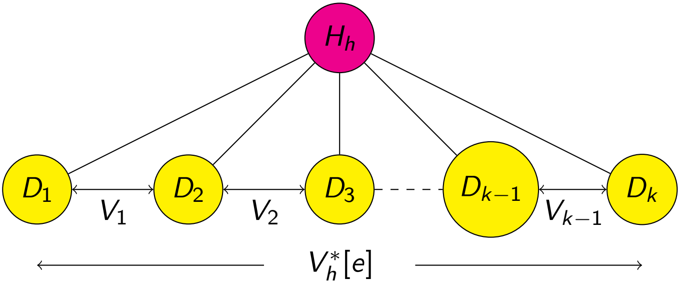
Adding $6 * (3)$ and (4)

$$6 \cdot \sum_{a=0}^{m-1} |D_a^*| + \sum_{a=0}^{m-1} |D_a| \leq 40(h+1)|D^*| \tag{5}$$

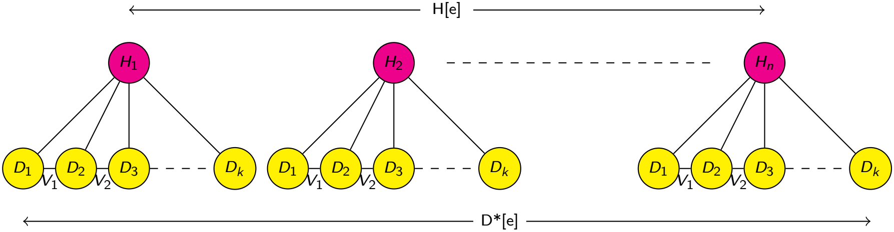
Therefore, again applying the pigeonhole principle, there must exist an integer $b \ \epsilon \ \{0, 1, ..., m-1\}$ such that,

$$6 \cdot |D_b^*| + |D_b| \leq \frac{40(h+1)}{m}|D^*| \leq \varepsilon \cdot |D^*| \tag{6}$$

Theorem 4.8 contd...



Theorem 4.8 contd...



Theorem 4.8 contd...

For any cell e that has n connected components in central Area,

$$|V^*[e]| = \sum_{h=0}^{n} |V_h^*[e]| \leq 2(k-1)n$$

$$|D'[e]| = |D^*[e]| + |V^*[e]| \leq |D^*[e]| + 2(k-1)n$$

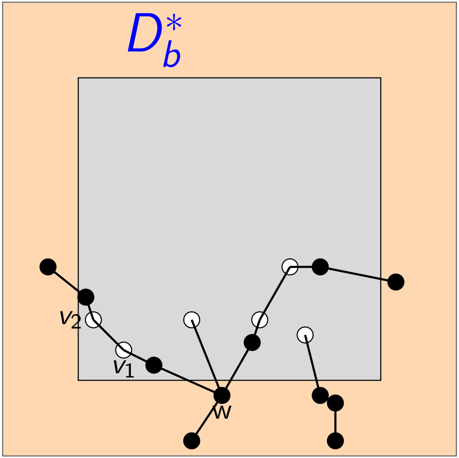
We define D' for some $b \epsilon \{0, 1, 2, ..., m-1\}$,

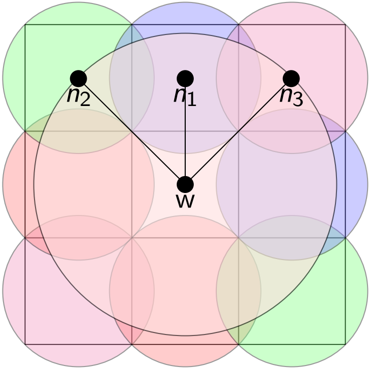
$$D' = D_b \cup \left(\bigcup_{e \in P(b,b)} D'[e] \right)$$

Clearly for each connected component H of central area $G_c[e]$, D' has a connected component dominating H. However, not necessarily with minimum number of vertices.

$$|A_b| \leq |D'| \tag{7}$$

Theorem 4.8 contd...





Theorem 4.8 contd...

Since vertex w in D_b^* can be charged only twice (by vertices $\{v_1, v_2\}$) for each connected component induced by independent neighbour in central Area, and in case of an unit disk graph vertex w can have at most 3 independent neighbours $\{n_1, n_2, n_3\}$, it implies that w can be charged 2*3 = 6 times. This phenomenon is only possible for vertices in D_b^*, hence,

$$|D'[e]| \leq |D^*[e]| + 6 \cdot |D_b^*[e]|$$

$$|D'| \leq |D_b| + \sum_{e \epsilon P(b,b)} |D^*[e]| \leq \sum_{e \epsilon P(b,b)} (|D^*[e]| + 6 \cdot |D_b^*[e]|)$$

$$= |D_b| + |D^*| + 6 \cdot |D_b^*|$$

Theorem 4.8 contd...

Using our claim for equation (2), we conclude

$|D_b| + |D^*| + 6 \cdot |D_b^*| \leq |D^*| + \varepsilon \cdot |D^*|$

$|D'| \leq (1 + \varepsilon) \cdot |D^*|$

$$|A_b| \leq |D'| \leq (1 + \varepsilon) \cdot |D^*| \tag{8}$$