Inverse Square law from Kepler's First Law

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In this small note, we shall outline how Newton could have (or did) come to realise that gravity follows an inverse square law. The starting point is obviously Kepler's first law, which Kepler postulated from the observational data he and his mentor, Tycho Brahe, collected.

Kepler's first law. *The orbit of every planet is an ellipse with the Sun at one of the two foci.*

The fact that Kepler's first law encodes the force law (with the aid of Newton's laws of motion) is truly magnificent and Newton had already developed the necessary mathematical tools (read calculus) that can bring out the required result.

We start by writing Newton's second law under a central force $f(r)\hat{r}$ in polar form :

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

The derivations for the above are simple and hence we just sketch the proof procedure. One starts by setting up the coordinate system $\mathbb{R}^3$ such that the origin is at the cause of the force field. Then one notes that, under a central force field, the path is always a planar curve, and hence one can consider the coordinate system to be just the plane of motion and hence just $\mathbb{R}^2$. Then one places a particle at $\vec{r} = (r, \theta)$ and just calculates the acceleration $\vec{a}$ of such a particle in polar coordinates. Equating the components of $m\vec{a}$ with the force components $(f(r), 0)$, the equations are derived.

From the second equation, we note that :

$$0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)$$
$$\Rightarrow r^2\dot{\theta} = h$$

where h is a constant.

Our main goal is to find out the differential equation of the path of the particle $r(\theta)$ in terms of the central force-field $f(r)$. Once we have that, we can just solve one for the other. This can be found directly, but just to make our lives simpler, what we will actually find is $u(\theta)$ in terms of f(1/u) where $u = 1/r$.

From the above equation, after substituting $u = 1/r$, we get $\dot{\theta} = hu^2$. Also,

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = hu^2\frac{\mathrm{d}r}{\mathrm{d}\theta} = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}.$$

Similarly

$$\ddot{r} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}$$

Substituting all this into the first component of Newton's second law, we get:

$$m\left(-h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - h^2u^3\right) = f(1/u)$$

$$\Rightarrow f(1/u) = -mh^2u^2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right)$$

We are almost there. Now all we need to do this is plug into the equation of an ellipse into this and find f. The polar equation of an ellipse with the origin at one of the foci (taken such that the vector $(r, 0)$ points towards the center of the ellipse) is

$$r = \frac{a(1-e^2)}{1-e\cos\theta}$$

where e is the eccentricity and a is the semi-major axis (the derivation of this is left as an exercise to the reader xD). Then,

$$u = 1/r = \frac{1-e\cos\theta}{a(1-e^2)}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \frac{e\sin\theta}{a(1-e^2)}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \frac{e\cos\theta}{a(1-e^2)}$$

Putting everything in. we get:

$$
\begin{aligned}
f(1/u) &= -mh^2 \frac{(1 - e\cos\theta)^2}{a^2(1-e^2)^2} \left\{ \frac{e\cos\theta}{a(1-e^2)} + \frac{1 - e\cos\theta}{a(1-e^2)} \right\} \\
&= -mh^2 \frac{(1 - e\cos\theta)^2}{a^3(1-e^2)^3} \\
&= -\frac{mh^2}{a(1-e^2)} \frac{(1 - e\cos\theta)^2}{a^2(1-e^2)^2} \\
&= -\frac{mh^2}{a(1-e^2)} u^2
\end{aligned}
$$

Now, reverting the substitution by putting $u = 1/r$, we finally have

$$
f(r) = \frac{mh^2}{a(1-e^2)r^2} \propto 1/r^2
$$

This completes the derivation.