Graceful Labeling

Exploration and Extension for Trees & Graphs

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Definition

Let $G = (V, E)$ be a graph with m edges. A **graceful labeling** is a injective function $f : V \to \{0, 1, 2, \ldots, m\}$, such that:

1. f is injective, i.e., for every vertex $v \in V$, $f(v) \in \{0, 1, 2, \ldots, m\}$, and no two vertices share the same label.
2. For each edge $e = (u, v) \in E$, the edge label $\ell(e)$ is defined as the absolute difference between the labels of its endpoints:

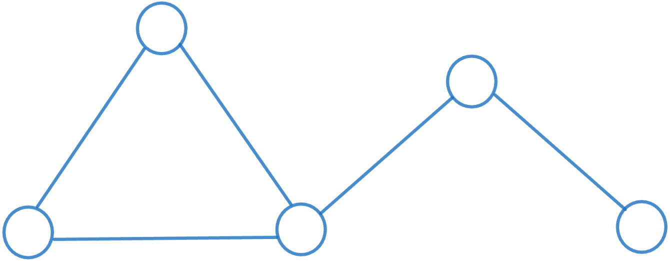
$$\ell(e) = |f(u) - f(v)|$$

3. The edge labels $\ell(e)$ for all $e \in E$ are distinct and satisfy:

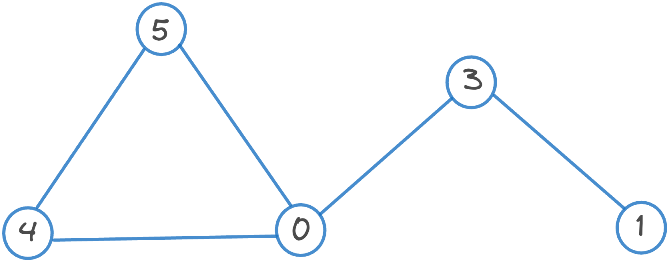
$$\ell(e) \in \{1, 2, \ldots, m\}$$

A graph that admits such a labeling is called a **graceful graph**.

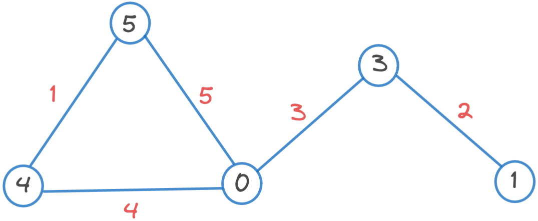
Graceful Labeling Example



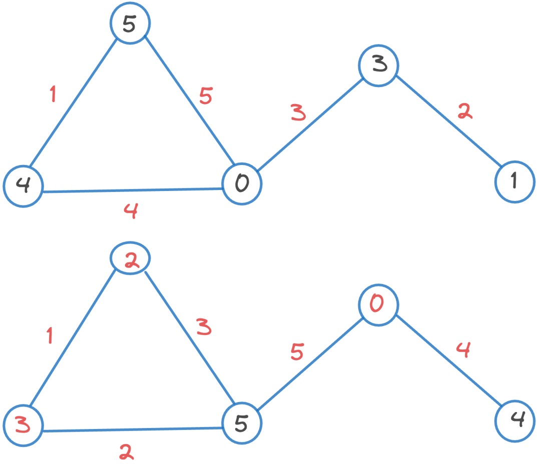
Graceful Labeling Example



Graceful Labeling Example



Graceful Labeling Example



Special Graph Classes

Path Graphs

- The path graph P_n is graceful for all $n \geq 1$.

Complete Graphs

- The complete graph K_n is graceful if, and only if, $n \leq 4$.

Eulerian Graphs

- Let G be an Eulerian graph. If $m \equiv 1, 2 \pmod 4$, then G is not graceful.

Cycle Graphs

- The cycle graph C_n is graceful if, and only if, $n \equiv 0, 3 \pmod 4$.

Special Graph Classes (Contd.)

Wheel Graphs

- The wheel graph W_p is graceful for all $p \geq 3$.

Caterpillar Graphs

- All caterpillar trees are graceful.

Complete Bipartite Graphs

- All complete bipartite graphs $K_{p,q}$ are graceful.

Reference:
Graceful Labeling of Graphs (student thesis), Rodrigo Ming Zhou, Federal University of Rio de Janeiro, 2016.
Available at: Gracelful Labeling of Graphs

Gracefulness of Trees

Graceful Tree Conjecture : Every tree is Graceful.

- The conjecture was introduced by Rosa et al. as an attempt to solve Ringel's conjecture, which states that K_{2m+1} can be decomposed into $2m+1$ subgraphs, each of which is isomorphic to a given tree with m edges.

Other known results

- All trees with diameter 4 and 5 are graceful.
- All trees up to 35 vertices are graceful.
- All symmetric trees are graceful.

Reference:

Alternative construction of graceful symmetric trees, P Sandy et al 2018, J. Phys.: Conf. Ser. 1008
Available at: Alternative construction of graceful symmetric trees

Definition

Given a graph $G = (V, E)$ with m edges, a k**-graceful labeling** is a function $f : V \to \{0, 1, 2, \ldots, k\}$ for some $k \geq m$ that satisfies all the original conditions of graceful labeling. The objective is to find the minimum value of k for which G satisfies the following :

1. f is injective.
2. For each edge $e = (u, v) \in E$, the edge label $\ell(e)$ is defined as the absolute difference between the labels of its endpoints:

$$\ell(e) = |f(u) - f(v)|$$

3. The edge labels $\ell(e)$ for all $e \in E$ are distinct and satisfy:

$$\ell(e) \in \{1, 2, \ldots, m\}$$

ILP Formulation

Decision Variables

- $$x_{ij} \in \{0,1\}, \quad \text{where} \quad x_{ij} = \begin{cases} 1 & \text{if node } i \text{ has label } j, \\ 0 & \text{otherwise.} \end{cases}$$
- $$y_{ijk} \in \{0,1\}, \quad \text{where} \quad y_{ijk} = \begin{cases} 1 & \text{if the edge } (i,j) \text{ has induced label } k, \\ 0 & \text{otherwise.} \end{cases}$$

ILP Formulation (Contd.)

Constraints

- Node Label Uniqueness

$$\sum_{j=0}^{m} x_{ij} = 1 \quad \forall i \in V$$

$$\sum_{i \in V} x_{ij} \leq 1 \quad \forall j \in \{0, 1, \ldots, m\}$$

- Edge Label Uniqueness

$$\sum_{k=1}^{m} y_{ijk} = 1 \quad \forall (i,j) \in E$$

$$\sum_{(i,j) \in E} y_{ijk} = 1 \quad \forall k \in \{1, 2, \ldots, m\}$$

ILP Formulation (Contd.)

- Node-Edge Relationship

$$k \cdot y_{ijk} = \sum_{\substack{p \neq q \\ p,q \in \{0,1,\ldots,m\}}} x_{ip} \cdot x_{jq} \cdot |p-q| \quad \forall (i,j) \in E, \quad \forall k \in \{1,2,\ldots,m\}$$

Targeted Challenges Addressed by ILP

- To solve K - graceful problem for any general graph G, using binary search construction on variable K.
- Another objective is to explore the potential for extending the result of gracefulness of trees to 36 vertices.
- The above ILP formulation can be modified to solve for trees and other relaxed versions of Graceful labeling

Conclusion and Future work

Conclusion

- The ILP formulation is one approach to contribute to graceful labeling problem. It can be effectively used to solve various relaxed versions of graceful labeling problems for general graphs and trees.
- Graceful labeling on trees can have applications in network addressing and graph databases.

Future Work

- To count number of graceful labelings for perfect binary trees.

Thank you for listening !

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