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GRACEFUL LABELING OF GRAPHS

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GRACEFUL LABELING OF GRAPHS

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Orientadores: Celina Miraglia Herrera de Figueiredo
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Em 1966, A. Rosa propôs uma nova coloração de grafos chamada *coloração-β* em que os vértices são coloridos com números distintos entre 0 a m, onde m é o número de arestas, tal que cada aresta é rotulada com o módulo da diferença das cores dos seus vértices extremos e cada um é único no grafo. Alguns anos depois, S. W. Golomb renomeou essa coloração de *coloração graciosa*, como é conhecida hoje em dia.

Esta definição permitiu que Rosa mostrasse que se toda árvore admitisse uma coloração graciosa, então uma conjectura de G. Ringel seria verdadeira. A partir disso, foi conjecturado que toda árvore fosse graciosa, a Conjectura das Árvores Graciosas.

Este trabalho apresenta alguns dos principais resultados em coloração graciosa de grafos e também apresenta esforços computacionais na direção da Conjectura das Árvores Graciosas. Inspirado por isso, nós também tomamos a abordagem computacional para estender a graciosidade de grafos cones generalizados.

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GRACEFUL LABELING OF GRAPHS

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In 1966, A. Rosa introduced a new graph labeling called *β-labeling* in which the vertices are labeled with distinct numbers chosen from 0 to m, where m is the number of edges, such that each edge is labeled with the absolute difference of the labels of its end vertices and it is unique in the graph. A few years later, S. W. Golomb renamed β-labeling as *graceful labeling* as it is known today.

This definition allowed Rosa to show that if every tree admits a graceful labeling, then a conjecture from G. Ringel would hold, from which it was conjectured that every tree is graceful, the Graceful Tree Conjecture.

This work presents some of the main results on graceful labeling of graphs and also presents computational efforts in the direction of the Graceful Tree Conjecture. Inspired by that, we also took the computational approach to extend the gracefulness of generalized cone graphs.

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Chapter 1

Introduction

Suppose we want to decompose a complete graph G into trees, all of them isomorphic between themselves. In other words, we want to partition the edges of G such that the subgraph induced by each set of edges of the partition is isomorphic to a given tree T. Ringel [20] conjectured that, for any tree T with n vertices, the complete graph K_{2n-1} can be decomposed into $2n-1$ trees isomorphic to T.

Rosa [21] introduced graceful labeling in 1966, and, back then, he called it β-labeling. The term "graceful" was introduced by Golomb [12] in 1972. Rosa showed that if every tree is graceful, then Ringel's conjecture holds. Since then, researchers have been trying to prove Ringel's conjecture through the Graceful Tree Conjecture, which claims that every tree is graceful.

However, graceful graphs gained their own merit of study over the years. David S. Johnson, in his NP-completeness column of 1983 [16], includes the decision problem of graceful labeling as the "Open Problem of the Month". Moreover, there is the International Workshop on Graph Labeling in which graceful labeling is one of the main themes, and a complete survey on the subject from Gallian [11] that is constantly updated.

In Section 1.1, we give the definitions used throughout the text. In Chapter 2, we present the formal definition of graceful labeling of a graph and present the gracefulness of some graph classes as well as some general results about graceful labeling of graphs. In Chapter 3, we focus on results towards the Graceful Tree Conjecture, presenting different approaches to tackle the conjecture. Finally, in Chapter 4 we change our focus to generalized cone graphs, a graph class defined by the join of two graphs. We review known theoretical results and propose new computational results which establish the gracefulness of families of generalized cone graphs and suggest a conjecture regarding the non-graceful ones.

1.1 Definitions

In this section, we give most of the definitions and notation of graph theory used in this text. For any missing definition, see Bondy and Murty [7].

A *graph* G is an ordered pair (V, E) where V is a set of elements called vertices and E is a set of unordered pairs of distinct vertices from V called edges. We say an edge e connects two vertices u and v, denoting as $e = uv$, and we say u and v are adjacent if they are connected by an edge. The set of adjacent vertices of a vertex u is denoted as $N(u)$, and it is also called the set of neighbors of u. The degree of a vertex u is $d(u) = |N(u)|$, the number of neighbors of u.

For a given graph G, when the vertex set and the edge set are not given explicitly, we refer to them as $V(G)$ and $E(G)$, and we use the letters n and m as the number of vertices and edges, respectively.

A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph $G = (V, E)$ and a subset $W \subseteq V$, the subgraph of G *induced* by W, denoted as $G[W]$, is the graph $H = (W, F)$ such that, for all $u, v \in W$, if $uv \in E$, then $uv \in F$. We say H is an *induced subgraph* of G. Equivalently, we can define subgraphs and induced subgraphs in terms of deletion of vertices and edges: H is an induced subgraph of G if it is obtained by deletion of vertices, and H is a subgraph of G if it is obtained by deletion of vertices and edges.

A *walk* in a graph is a finite sequence of vertices $W = (v_0, v_1, \ldots, v_k)$ such that $v_i v_{i+1}$ is an edge of the graph. If the walk W does not go through an edge twice, we say W is a *trail*, and if it does not go through a vertex twice, we say W is a *path*. A path starting in u and ending in v is called a *uv-path*.

The *length* of a path is the number of its edges and the *distance* between two vertices u and v is the length of the shortest path between them and denoted as $dist(u, v)$. If there is no path between u and v, then $dist(u, v) = \infty$.

A walk is said to be *closed* if the first and the last vertices are the same. A *cycle* is a closed trail in which all vertices, but the last, are distinct.

An *Eulerian trail* (or *Eulerian path*) of a graph is a trail that traverses each edge of the graph exactly once. Similarly, an *Eulerian tour* (or *Eulerian cycle*) is a cycle that traverses each edge exactly once. A graph is *Eulerian* if it admits an Eulerian cycle.

A graph G is said to be *connected* if every pair of vertices is connected by a path. If there is exactly one path connecting each pair of vertices, we say G is a *tree*. Equivalently, a tree is a connected graph with $n - 1$ edges (see [7]).

A *path graph* P_n is a connected graph on n vertices such that each vertex has degree at most 2. A *cycle graph* C_n is a connected graph on n vertices such that every vertex has degree 2.

A *complete graph* K_n is a graph with n vertices such that every vertex is adjacent to all the others. On the other hand, an *independent set* is a set of vertices of a graph in which no two vertices are adjacent. We denote I_n for an independent set with n vertices.

A *bipartite graph* $G = (V, E)$ is a graph such that there exists a partition $\mathcal{P} = (A, B)$ of V such that every edge of G connects a vertex in A to one in B. Equivalently, G is said to be bipartite if A and B are independent sets. The bipartite graph is also denoted as $G = (A, B, E)$.

The *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets is the graph $G = (V, E)$ such that $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$, that is, G is obtained by connecting every vertex of G_1 to every vertex of G_2.

Finally, for a given graph $G = (V, E)$, a *vertex labeling* (or *vertex coloring*) of G is a function $f\colon V \to \mathbb{N}$, and an *edge labeling* (or *edge coloring*) of G is a function $g\colon E \to \mathbb{N}$. Intuitively, we are assigning labels (colors) to vertices and/or edges of the graph. Throughout this text, we have the codomains as a finite subset of $\mathbb{N}$, and we denote $[a, b] = \{a, a + 1, \ldots, b\}$.

Many problems of graph theory consist in finding a vertex or an edge labeling for a graph satisfying certain properties. For example, a *proper vertex coloring* is a vertex coloring such that adjacent vertices have different colors, and a very well known problem is to find for a given graph G the minimum k such that there exists a proper vertex coloring f of G with $|\mathrm{Im}(f)| = k$. In our case, we are interested in graceful labeling.

Chapter 2

Graceful Labeling

A *graceful labeling* of a graph G is a vertex labeling $f\colon V \to [0, m]$ such that f is injective and the edge labeling $f_\gamma\colon E \to [1, m]$ defined by $f_\gamma(uv) = |f(u) - f(v)|$ is also injective. If a graph G admits a graceful labeling, we say G is a *graceful graph*.

Although it has been studied for 50 years, not many general results are known about graceful labeling. Most of the results are about asserting the gracefulness of a graph class since it suffices to show a graceful labeling for each graph in the class. On the other hand, results on non-gracefulness of a graph rely basically on a necessary condition only valid for Eulerian graphs (see Theorem 2.4) or on trying to label the graph gracefully until reaching a contradiction, which is not very effective in most of the cases.

Figure 2.1: Graceful labeling of P_3 and $K_{1,3}$.

To gain some intuition on how to label a graph gracefully, let us show how to label a path graph. So, take a path graph P_n and let $V(P_n) = \{u_0, u_1, \ldots, u_{n-1}\}$ be the set of vertices such that $u_{k-1}u_k \in E(P_n)$ for $0 < k < n$. Since P_n has $m = n - 1$ edges, we must label the vertices with numbers from 0 to $n - 1$ so that every number in $[1, n-1]$ appears as an edge label. We start with edge label $n - 1$ since there is only one way to get an absolute difference equal to $n - 1$, which is having a vertex with label 0 adjacent to a vertex with label $n - 1$. Thus, let us try labeling u_0 with 0 and u_1 with $n - 1$. Next, let us try to get an edge label with value $n - 2$. There are only two possible ways to get $n - 2$ as an absolute difference: $n - 2 = |(n - 2) - 0| = |(n - 1) - 1|$. Since u_0 has no more unlabeled adjacent vertices, we can only get the edge label $n-2$ by labeling u_2 with 1. Going on with this strategy, our resulting labeling will be as follows:

$$f(u_k) = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ n - \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

Now, to show that f is indeed a graceful labeling of P_n, it suffices to show that the edge label 1 appears, which is expected to appear on the last edge $u_{n-2}u_{n-1}$. If n is even, then $f(u_{n-1}) = \frac{n}{2}$ and $f(u_{n-2}) = \frac{n-2}{2}$. Hence, $f_\gamma(u_{n-1}u_{n-2}) = \frac{n}{2} - \frac{n-2}{2} = 1$. If n is odd, an analogous argument establishes the edge label 1. Therefore, the following proposition holds.

Proposition 2.1. *The path graph P_n is graceful for all $n \geq 1$.*

For a second example, we try to find a graceful labeling for the complete graph K_n. Since K_1 and K_2 are also path graphs, they are graceful. For K_3 and K_4, Figure 2.2 presents a graceful labeling for each one.

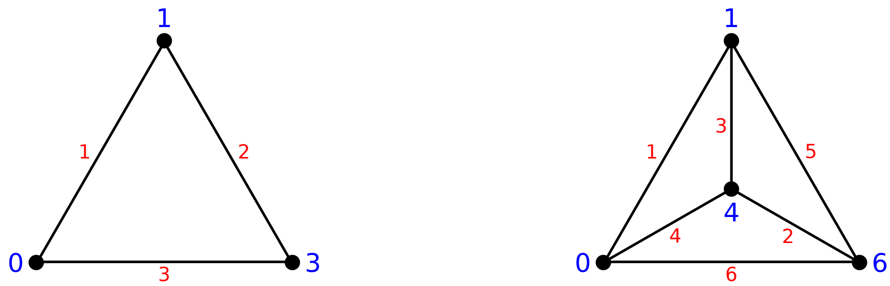

Figure 2.2: Graceful labeling of K_3 and K_4.

Before analysing the general case, let us first introduce a property of graceful labelings. Given a graph with a graceful labeling, if we swap every vertex label k with $m-k$, the resulting labeling is also graceful since the edge labels will not have changed: the end vertices of an edge with labels a and b become $m-a$ and $m-b$, and $|a-b| = |(m-a)-(m-b)|$. This is called the *complementarity property*.

Now, for K_n with $n > 4$, as before, we must have a vertex with label 0 adjacent to a vertex labeled m to get the edge label m. But, in this case, every vertex is adjacent to every other vertex. Thus, we can label any vertex with 0 and any other one with m without loss of generality. To get the edge label $m-1$, we have two options: $m-1 = |(m-1)-0| = |m-1|$. However, the complementarity property allows us to choose either one without loss of generality. Choosing to label a vertex with 1, we get edge labels 1 and $m-1$. Now we need to get the edge label $m-2 = |(m-2)-0| = |(m-1)-1| = |m-2|$. We can not label a vertex with $m-1$ or 2 because it would create a duplicate edge label. Hence, our only option is to label a vertex with $m-2$, obtaining edge labels 2, $m-3$ and $m-2$.

Since $m-3$ has already appeared on an edge, the next edge label we must obtain is $m-4 = |(m-4)-0| = |(m-3)-1| = |(m-2)-2| = |(m-1)-3| = |m-4|$. Again, we only have one option without creating duplicate edge labels, which is to label a vertex with 4, obtaining edge labels 3, 4, $m-6$ and $m-4$. At this point, we have labeled five vertices. However, for K_5, we would have $m-6=4$ as a duplicate edge label. For $n \geq 6$, the next edge label to get is $m-5$. But, all the five possible ways to get $m-5$ lead to a duplicate edge label. Therefore, there is no way to get label $m-5$ on an edge and the following proposition holds.

Proposition 2.2. *The complete graph K_n is graceful if, and only if, $n \leq 4$.*

Given the initial intuition on how to gracefully label a graph, Section 2.1 presents some general results on graceful graphs and Section 2.2 shows the gracefulness of some graph classes.

2.1 General results

We start by showing a couple of results concerning necessary conditions to the existence of a graceful labeling of a graph. The first one is a straightforward condition given by Golomb [12].

Proposition 2.3. *If $G = (V, E)$ is graceful, then there exists a partition $\mathcal{P} = (A, B)$ of V such that the number of edges with one end in A and the other in B is $\lceil \frac{m}{2} \rceil$.*

Proof. Let $G = (V, E)$ be a graph with a graceful labeling f and consider the partition $\mathcal{P} = (A, B)$ of V such that $A = \{u \in V : f(u) \equiv 0 \pmod 2\}$. Since there are $\lceil \frac{m}{2} \rceil$ odd values between 1 and m, and an odd difference is only possible by subtracting an even value from an odd one, the number of edges connecting two vertices with different parities must be exactly $\lceil \frac{m}{2} \rceil$. □

Although Proposition 2.3 gives a necessary condition to the existence of a graceful labeling for a graph, it has no practical use since it would be necessary to check all the 2^{n-1} possible partitions of V to decide if a graph can admit a graceful labeling.

A more useful necessary condition was given by Rosa [21], but it only applies to Eulerian graphs. It is known as the *parity condition*.

Theorem 2.4. *Let G be an Eulerian graph. If $m \equiv 1, 2 \pmod 4$, then G is not graceful.*

Proof. Suppose $G = (V, E)$ is a graceful Eulerian graph. Let $f : V \to [0, m]$ be a graceful labeling of G and $C = (u_0, u_1, \ldots, u_{m-1}, u_m = u_0)$ be an Eulerian cycle of

G. Taking the sum of the edge labels of C modulo 2, we have:

$$\begin{aligned}\sum_{i=1}^{m} f_\gamma(u_{i-1}u_i) &= \sum_{i=1}^{m} |f(u_{i-1}) - f(u_i)| \\ &\equiv \sum_{i=1}^{m} f(u_{i-1}) - f(u_i) \equiv 0 \pmod 2\end{aligned} \tag{2.1}$$

And, since C is an Eulerian cycle, i.e., the cycle C goes through each edge exactly once, and f is a graceful labeling of G, we have:

$$\sum_{e\in E} f_\gamma(e) = \sum_{k=1}^{m} k = \frac{m(m+1)}{2} \overset{(2.1)}{\equiv} 0 \pmod 2 \tag{2.2}$$

Thus, we must have $m \equiv 0, 3 \pmod 4$ in order to satisfy equation (2.2). $\square$

The parity condition, unlike Proposition 2.3, provides a simple way to test if an Eulerian graph can be graceful or not. And an interesting question arises: is there a graph class for which the parity condition is also a sufficient condition? As we will see, the parity condition does characterize at least one graph class.

In graph theory, it is natural to think of substructures that make a graph not satisfy a certain property, in this case being graceful. Such substructures can be subgraphs, induced subgraphs, or others, and they are called *forbidden substructures* for the graph class. Thus, one might think of finding forbidden substructures for the class of graceful graphs. However, Arumugam and Bagga [3] proved that every graph is an induced subgraph of a graceful graph.

Theorem 2.5. *Every graph is an induced subgraph of a graceful graph.*

Proof. Given a graph $G = (V, E)$, let us construct a graph H from G such that H is graceful and G is an induced subgraph of H. Consider a vertex labeling $f\colon V \to [0, k]$ injective for some $k \geq m$ such that the edge labeling $f_\gamma\colon E \to \mathbb{N}$ is also injective, and there exist $u, v \in V$ with $f(u) = 0$ and $f(v) = k$. Let $\{x_1, x_2, \ldots, x_r\}$ be the set of missing edge labels. Without loss of generality, $x_1, x_2, \ldots, x_s$ are not vertex labels and $x_{s+1}, \ldots, x_r$ are vertex labels. For each x_i, $1 \leq i \leq s$, add a vertex w_i with label x_i and add an edge connecting w_i to u so that $f_\gamma(uw_i) = x_i$. For each x_i, $s + 1 \leq i \leq r$, add a vertex w_i with label $k + x_i$ and connect w_i to u and v so that $f_\gamma(uw_i) = k + x_i$ and $f_\gamma(vw_i) = x_i$. Note that the last step might have introduced new missing edge labels by creating vertex labels with values greater than k. However, these new missing edge labels are not vertex labels. So, for each new missing edge label y, add a new vertex z_y with label y and connect z_y to u so that $f_\gamma(uz_y) = y$. The resulting graph H is graceful and it contains G as an induced subgraph. $\square$

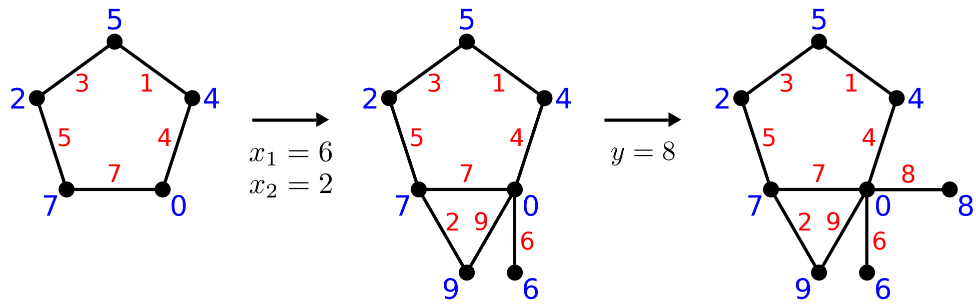

Figure 2.3: Constructing a graceful graph from C_5.

Theorem 2.5 says that a graph G being non-graceful does not matter for graphs for which G is an induced subgraph. It also says that we can always construct a graceful graph from any graph.

So far, we have characterized the gracefulness of two families of graphs: the path graphs and the complete graphs. The first one is a family of graceful graphs and the second one, for $n \geq 5$, is a family of non-graceful graphs. We have also shown that we can construct a graceful graph from any graph, graceful or not.

Next, we present an unpublished result of Erdős [12]. The following proof was given by Graham and Sloane[13].

Theorem 2.6. *Almost all graphs are not graceful.*

Proof. We show that for a fixed number m, almost all graphs with n vertices and m edges are not graceful as $n \to \infty$.

First, note that there are $\binom{n(n-1)/2}{m}$ labeled graphs with n vertices and m edges. So, the number of unlabeled graphs is at least $\frac{1}{n!}\binom{n(n-1)/2}{m}$.

Let f be a vertex labeling on n vertices with distinct numbers from $[0, m]$. There are $\frac{(m+1)!}{(m-n+1)!} \leq (m+1)^n$ such labelings. Let us count how many graphs there are for which f is a graceful labeling. Let p_i be the number of pairs of vertices $\{u, v\}$ with $|f(u) - f(v)| = i$. Clearly, $\sum_{i=1}^{m} p_i = \binom{n}{2}$. If we construct a graph by taking one edge from each class counted by p_i, the resulting graph is graceful. Thus, there are $\prod_{i=1}^{m} p_i$ labeled graphs for which f is a graceful labeling. Since this product is maximized when all p_i's are equal, $\prod_{i=1}^{m} p_i \leq (\frac{n(n-1)}{2m})^m$. Therefore, there are at most $(m+1)^n(\frac{n(n-1)}{2m})^m$ graceful labeled graphs, and this is also an upper bound for the number of graceful unlabeled graphs. Finally, we show that the ratio

$$\rho = \frac{(m+1)^n(\frac{n(n-1)}{2m})^m}{\frac{1}{n!}\binom{n(n-1)/2}{m}}$$

goes to 0 as $n \to \infty$. Writing $m = (\frac{1}{2} - \mu)\binom{n}{2}$ with $\mu \in (-\frac{1}{2}, \frac{1}{2})$, we have

$$\rho = \frac{(m+1)^n n!}{(\frac{1}{2}-\mu)^m \binom{\binom{n}{2}}{m}} < \frac{(m+1)^n n! \sqrt{8\binom{n}{2}(\frac{1}{2}-\mu)(\frac{1}{2}+\mu)}}{(\frac{1}{2}-\mu)^m 2^{\binom{n}{2} H_2(1/2-\mu)}}$$

where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ (cf. [18, p. 309]). Simplifying the denominator,

$$\rho < \frac{(m+1)^n n! \sqrt{8\binom{n}{2}(\frac{1}{2}-\mu)(\frac{1}{2}+\mu)}}{2^{-\binom{n}{2}(\frac{1}{2}+\mu)\log_2(\frac{1}{2}+\mu)}}$$

Taking the logarithm, it is easy to show that the right hand side of the inequality goes to $-\infty$ as $n \to \infty$. Therefore, $\rho \to 0$ as $n \to \infty$. $\square$

We finish this section by giving another construction of graceful graphs given by Acharya [1].

The *full augmentation* of a graceful graph $G = (V, E)$ is the addition of an isolated vertex to G for each vertex label not used. Formally, $G_f = G \cup I_{m-n+1}$ is the full augmentation of G. Clearly, G_f is also graceful and, in particular, graceful trees are already full augmented.

Theorem 2.7. *If G is a graceful graph and G_f is its full augmentation, then $G_f + I_q$ is graceful for all $q \geq 1$.*

Proof. Let $f\colon V(G_f) \to [0, m]$ be a graceful labeling of G_f and $V(I_q) = \{v_0, v_1, \ldots, v_{q-1}\}$. Then, we can extend the labeling f to $V(I_q)$ as follows: label v_i with $f(v_i) = m + (i+1)(m+1)$.

We have $|E(G_f + I_q)| = m + q(m+1)$ and, since we are extending f, we already have $\mathrm{Im}(f|_{V(G_f)}) = [0, m]$ and $\mathrm{Im}(f_\gamma|_{E(G_f)}) = [1, m]$. By definition of f, every vertex label is unique, and the set of labels on the edges incident with v_i is exactly $[m+i(m+1), m+(i+1)(m+1)]$. Then, every label in $[1, m+q(m+1)]$ appears once on some edge. Therefore, the extension of f is a graceful labeling of $G_f + I_q$. $\square$

2.2 Gracefulness of graph classes

In this section, we present the gracefulness of some graph classes. Most of the results asserting the gracefulness of a graph class are given by explicit graceful labelings. For the non-gracefulness of a graph class, there are only a few tools for that. Basically, we only have Proposition 2.3 and theorem 2.4. We can also prove by trying to label the graph and finding a contradiction. For instance, Rosa [21] showed Proposition 2.2 this way. Although the last method is not effective if done by hand, if it is done computationally, it may result in something useful, as we will see later in following chapters.

It was already shown that the path graph P_n is graceful and the complete graph K_n is graceful if, and only if, $n \leq 4$. Next, we present the gracefulness cycle graphs, which was characterized by Rosa [21].

Proposition 2.8. *The cycle graph C_n is graceful if, and only if, $n \equiv 0, 3 \pmod 4$.*

Proof. Cycle graphs are Eulerian graphs. Therefore, by the parity condition, if $n \equiv 1, 2 \pmod 4$, then C_n is not graceful. Otherwise, let us call $V(C_n) = \{u_0, u_1, \ldots, u_{n-1}\}$ such that $u_k u_{k+1} \in E(C_n)$ for $0 \leq k \leq n-1$ and $u_n = u_0$.

If $n \equiv 0 \pmod 4$, then label the vertices according to the following formula:

$$f(u_i) = \begin{cases} \frac{i}{2} & \text{if } i = 0, 2, 4, \ldots, n-2 \\ n - \frac{i-1}{2} & \text{if } i = 1, 3, 5, \ldots, \frac{n}{2} - 1 \\ n - \frac{i-1}{2} - 1 & \text{if } i = \frac{n}{2} + 1, \frac{n}{2} + 3, \ldots, n-1 \end{cases}$$

If $n \equiv 3 \pmod 4$, then label $V(C_n)$ as follows:

$$f(u_i) = \begin{cases} \frac{i}{2} & \text{if } i = 0, 2, 4, \ldots, n-1 \\ n - \frac{i-1}{2} & \text{if } i = 1, 3, 5, \ldots, \frac{n+1}{2} - 1 \\ n - \frac{i-1}{2} - 1 & \text{if } i = \frac{n+1}{2} + 1, \frac{n+1}{2} + 3, \ldots, n-2 \end{cases}$$

□

Note that the parity condition characterizes the gracefulness of cycle graphs.

The *wheel graph* W_p is the join of a cycle graph C_p with a singleton graph, i.e., $W_p = C_p + K_1$. Frucht [10] showed that all wheels are graceful.

Proposition 2.9. *The wheel graph W_p is graceful for all $p \geq 3$.*

Proof. Let $V(W_p) = \{u_0, u_1, \ldots, u_{p-1}, v\}$ be the set of vertices where v is the vertex joined with the cycle and consider the following two cases.

1. If $p \equiv 0 \pmod 2$, then the following formula gives a graceful labeling:

$$f(v) = 0$$
$$f(u_i) = \begin{cases} 2p & \text{if } i = 0 \\ 2 & \text{if } i = p-1 \\ i & \text{if } i = 1, 3, 5, \ldots, p-3 \\ 2p - i - 1 & \text{if } i = 2, 4, 6, \ldots, p-2 \end{cases}$$

2. If $p \equiv 1 \pmod 2$, then the following formula gives a graceful labeling:

$$f(v) = 0$$
$$f(u_i) = \begin{cases} 2p & \text{if } i = 0 \\ 2 & \text{if } i = 1 \\ p + i & \text{if } i = 2, 4, 6, \ldots, p-1 \\ p + 1 - i & \text{if } i = 3, 5, 7, \ldots, p-2 \end{cases}$$

□

A *caterpillar* is a tree in which the removal of all leaves results in a path graph. It was proven by Rosa [21] that they are all graceful.

Proposition 2.10. *All caterpillar trees are graceful.*

Proof. Draw the caterpillar tree as a planar bipartite representation and label it as shown in Figure 2.4. It is easy to check that such drawing scheme is always possible. □

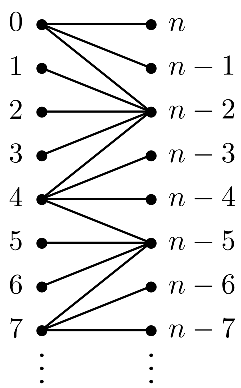

Figure 2.4: Graceful labeling of caterpillar tree.

Note that a path graph P_n is also a caterpillar tree and the labeling scheme given by Proposition 2.10, when applied to a path graph, yields the same labeling constructed before.

The *complete bipartite graph* $K_{p,q}$ is a bipartite graph $G = (A, B, E)$ such that $|A| = p$, $|B| = q$, and if $u \in A$ and $v \in B$, then $uv \in E$. In particular, the *star graph* is the complete bipartite graph $K_{1,q}$.

It was shown that for all positives values of p and q, the complete bipartite graphs are graceful [12, 21].

Proposition 2.11. *The complete bipartite graph $K_{p,q}$ is graceful for all $p, q \geq 1$.*

Proof. Let $G = (A, B, E)$ be a bipartite graph with $a = |A|$ and $b = |B|$. Assign the vertices from A with numbers 0, 1, $\ldots$, $a - 1$, and assign the vertices from B with numbers a, $2a$, $\ldots$, ba. $\square$

We can generalize the concept of bipartite graph to *multipartite graph* and, in a similar fashion, we have the *complete multipartite graph*. It was proven the following proposition regarding the gracefulness of complete multipartite graphs [5].

Proposition 2.12. *The complete multipartite graphs $K_{p,q}$, $K_{1,p,q}$, $K_{2,p,q}$, and $K_{1,1,p,q}$ are graceful.*

Proof. The graceful labelings are given in Figure 2.5. $\square$

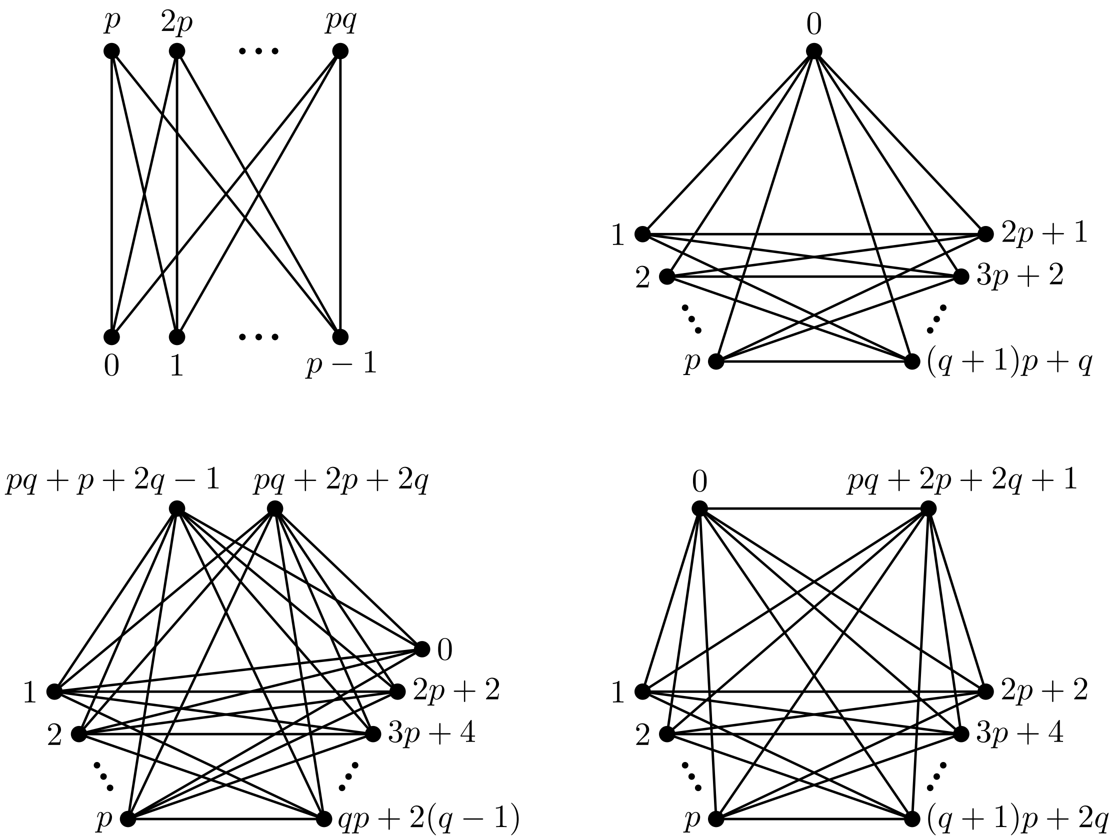

Figure 2.5: Graceful labelings of $K_{p,q}$, $K_{1,p,q}$, $K_{2,p,q}$, and $K_{1,1,p,q}$.

Furthermore, Beutner [5] conjectured that these graphs are the only complete multipartite graphs which are graceful, and showed computationally that it is valid for all complete multipartite graphs up to 23 vertices.

Chapter 3

Trees

The Graceful Tree Conjecture remains unsolved to these days and there have been a few different approaches researchers have been trying to prove the conjecture. In this section, we present results on the gracefulness of trees and the different ways in which the conjecture has been tackled.

Conjecture 3.1 (Graceful Tree Conjecture)**.** *Every tree is graceful.*

As shown in Chapter 2, paths and caterpillars are graceful. A first approach would be to extend the definition of caterpillars to new families of trees, i.e., look at the class of trees in which the removal of all leaves results in a caterpillar tree—the *lobsters*—, and so on. However, even the lobster trees have not been characterized yet. Bermond [4] conjectured in 1979 that all lobsters are graceful. This chapter presents others approaches which have shown to be more interesting.

3.1 Trees with limited diameter

The *diameter* of a tree T is the maximum distance between two vertices, i.e., $\mathrm{diam}(T) = \max\{dist(u,v) : u,v \in V(T)\}$. Trees with small diameter have been proved to be graceful. We already showed that trees with diameter 1 (only K_2), diameter 2 (star graphs), and diameter 3 (a subclass of caterpillar trees) are graceful since they are all also caterpillar trees. For greater diameters, Zhao [28] proved in 1989 that all trees with diameter 4 are graceful, Hrnčiar and Haviar [15] proved in 2001 that all trees with diameter 5 are graceful, and Superdock [23, 24] proved more recently that some subclasses of trees with diameter 6 are graceful.

We show in this section that all trees with diameter 4 are graceful. The proof presented here was given by Hrnčiar and Haviar [15] since it is simpler than the original proof of Zhao [28].

Lemma 3.1. *Let T be a tree with a graceful labeling f and let $u \in V(T)$ the vertex with $f(u) = 0$. If T' is the tree obtained from T by adding a new vertex v only adjacent to u, then T' is graceful.*

Proof. If m is the number of edges of T, then the vertex labeling f' such that $f'|_{V(T)} = f$ and $f'(v) = m + 1$ is a graceful labeling of T'. □

Corollary 3.1.1. *If $w \in V(T)$ has label m, then adding a new vertex only adjacent to w also results in a graceful tree.*

Proof. Just consider the complementary graceful labeling of f. □

Corollary 3.1.2. *If $u \in V(T)$ has label 0 (or m) and H is a caterpillar tree, then adding an edge between u and a vertex of H with maximum eccentricity also results in a graceful tree.*

Proof. Apply iteratively Lemma 3.1 giving preference to adding leaves first whenever it is possible. Also note that the corollary is valid for any graceful graph G as long as $u \in V(G)$ has label 0 (or m). □

Lemma 3.1 allows us to obtain new graceful graphs from smaller ones by adding a vertex. Then, it is reasonable to ask if this could be used to prove the Graceful Tree Conjecture, i.e., somehow show that for any tree, there is a finite sequence of graceful trees starting from a single vertex such that each tree is the previous one in the sequence plus a vertex, and the last tree of the sequence is the target tree itself.

One sufficient condition to the existence of such sequence is if every tree admits a graceful labeling in which the label 0 can be assigned to any vertex. In the general context, such graphs are called *0-rotatable* graceful graphs. However, it is not true that every tree is 0-rotatable graceful [26].

Let T be a tree and $uv \in E(T)$. We denote by $T_{u,v}$ the subtree of T containing v after the removal of the edge uv. Precisely, if $S = \{w \in V(T) : v \text{ is on the } uw\text{-path}\}$, then $T_{u,v} = T[S]$.

Lemma 3.2. *Let T be a tree with a graceful labeling f and let $u \in V(T)$ be a vertex adjacent to u_1 and u_2. Consider $T' = T - (V(T_{u,u_1}) \cup V(T_{u,u_2}))$ and let $v \in V(T')$, $v \neq u$.*

(a) *If $u_1 \neq u_2$ and $f(u_1) + f(u_2) = f(u) + f(v)$, then the tree obtained by a disjoint union of T', T_{u,u_1} and T_{u,u_2}, and connecting v to u_1 and u_2 is graceful with the same graceful labeling f.*

(b) *If $u_1 = u_2$ and $2f(u_1) = f(u) + f(v)$, then the tree obtained by a disjoint union of T' and T_{u,u_1}, and connecting v to u_1 is graceful with the same graceful labeling f.*

Proof. It suffices to show that the edge labels of uu_1 and uu_2 are the same as of vu_1 and vu_2.

(a) $|f(u_1) - f(u)| = |f(u) + f(v) - f(u_2) - f(u)| = |f(v) - f(u_2)|$
$|f(u_2) - f(u)| = |f(u) + f(v) - f(u_1) - f(u)| = |f(v) - f(u_1)|$

(b) $|f(u_1) - f(u)| = |\frac{f(u)+f(v)}{2} - f(u)| = |\frac{f(v)-f(u)}{2}|$
$|f(u_1) - f(v)| = |\frac{f(u)+f(v)}{2} - f(v)| = |\frac{f(u)-f(v)}{2}|$ □

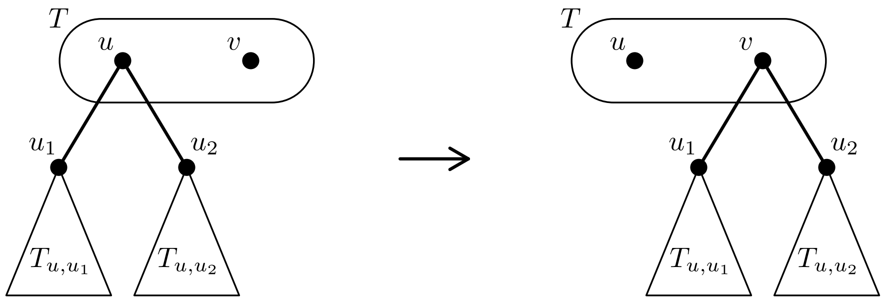

Figure 3.1: Transfer of subtrees from u to v.

This operation is called a transfer and we mostly do transfers of leaves from one vertex to another. For the remaining of this section, for a graceful tree, we no longer distinguish the vertex label from the vertex itself since in a tree every number from $[0, n-1]$ must appear as a vertex label.

As an example, take the star graph $K_{1,m}$. We can transfer some leaves, which is connected to vertex 0, to the vertex m (see Figure 3.2). For an example, we can transfer k and $m-k$ from 0 to m since $k + (m-k) = 0 + m$. As said before, the subtree being transferred is usually a leaf and we denote a sequence of transfers of leaves adjacent to u to v as $u \to v$. Although the notation is not precise, the context will make clear how many and which leaves are being transferred.

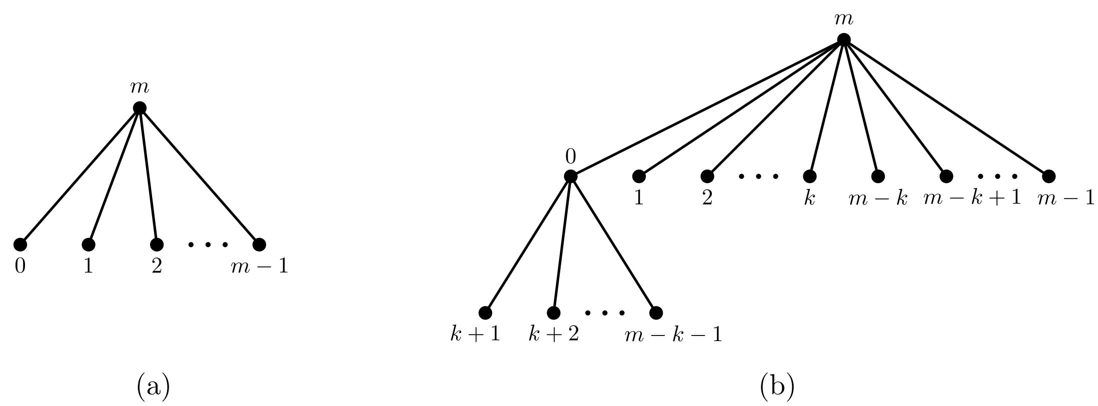

Figure 3.2: Transfer of leaves from m to 0 ($m \to 0$ transfer).

Proposition 3.3. *All trees with diameter 4 are graceful.*

Proof. Consider the following types of transfers.

A $u \to v$ transfer is of type 1 if the leaves being transferred are $k, k+1, \dots, k+s$. This type of transfer can be realized if $u + v = k + (k + s)$. We use this type of transfer when we want to leave an odd number of vertices connected to u.

A $u \to v$ transfer is of type 2 if the leaves being transferred are $k, k+1, \dots, k+s$ and $l, l+1, \dots, l+s$ with $k + s < l$. This type of transfer can be realized if $u + v = k + (l + s)$. We use this type of transfer when we want to leave an even number of vertices connected to u.

By Lemma 3.1, it is sufficient to show that every tree T of diameter 4 with central vertex (which is unique in T) of odd degree has a graceful labeling with the central vertex having the maximum label. This is true because, in a tree of diameter 4, any subtree rooted at one of the children of central vertex is a caterpillar tree.

Let w be the central vertex of T, x be the number of vertices adjacent to w with even degree, and y be the number of vertices adjacent to w with odd degree greater than 1. Let $d(w) = 2k + 1$ and consider the tree of Figure 3.2b. We can obtain T from that tree by the following sequence of transfers: $0 \to m - 1 \to 1 \to m - 2 \to 2 \to m - 3 \to \cdots$, where the first x transfers (or $x - 1$ if $y = 0$) are of type 1 and the next $y - 1$ transfers (if $y > 1$) are of type 2.

In order to verify that this sequence works, let us analyse the first transfer. Suppose $\{u_1, \dots, u_x\}$ is the set of vertices adjacent to w with even degree. Starting with the tree on Figure 3.2b, the central vertex w is the one with label m. The first transfer is $0 \to m-1$. Then, u_1 is the vertex 0 and we want to leave $d(u_1)-1$ vertices attached to it. Initially, we have the vertices $k+1, k+2, \dots, m-k-2, m-k-1$ adjacent to 0. Since $0 + (m - 1) = (k + 1) + (m - k - 2)$, it is possible to leave $d(u_1) - 1$ vertices by doing a type 1 transfer of a continuous sequence of vertices to $m - 1$. Going on with an analogous analysis, it can be seen that this sequence works. $\square$

Proposition 3.4. *All trees with diameter 5 are graceful.*

The proof of Proposition 3.4 also uses the transfers operations used in the proof of Proposition 3.3. However, since it is divided in several cases and it does not add much to the discussion, we omit it.

3.2 All trees up to 35 vertices are graceful

Given that the Graceful Tree Conjecture has remained open for a long time, it is valid to question if it can be false. For that, it would suffice to come up with a tree that does not admit a graceful labeling. In order to show that a tree does not admit such labeling, one must verify an exponential number of possible ways to label it. Thus, a computational approach is more suited for the task.

Fang [9] took this approach and proved in 2010 that all trees up to 35 vertices are graceful. Fang's result replaces previous ones in this direction: Aldred and McKay [2] established in 1998 that all trees up to 27 vertices are graceful, and Horton [14] verified in 2003 that all trees with at most 29 vertices are graceful.

Proposition 3.5. *All trees up to 35 vertices are graceful.*

For the verification of Proposition 3.5, Fang used the algorithm described by Wright et al. [27] to enumerate all trees which has amortized constant time complexity to generate each of them.

For each tree, the algorithm to find a graceful labeling is divided in two parts. First, it tries to find a graceful labeling using a backtracking search with a fixed maximum number of iterations. If it does not find one, then it tries to find a graceful labeling through a combinatorial optimization approach, and it uses a hill-climbing tabu search combined with ideas from simulated annealing.

The backtracking search tries to construct a graceful labeling f for the tree with $f(r) = 0$ where r is the root, which is the center vertex in a central tree or one of the centers in a bicentral tree. Then, at each iteration, it tries to create a new edge label k by labeling a not yet labeled vertex u adjacent to an already labeled vertex v such that $|f(u) - f(v)| = k$. In order to avoid branching the decision tree, the search goes from edge label $n - 1$ to 1. As noted before, the higher the value, the less the number of possible ways to get that value as an absolute difference.

As usual of backtracking search algorithms, the decision tree can grow exponentially as n increases. Then, Fang added a threshold to the number of backtracks, preventing searching for very long time. This threshold was chosen empirically and set to $(n - 19) * 11000 - 1000$. Algorithm 3.1 is a pseudocode for the backtracking search algorithm.

If the backtracking search does not return a graceful labeling, a combinatorial optimization approach is taken. Solving a decision problem by this approach requires formulating an evaluation function such that the answer is "yes" if, and only if, the function reaches a certain extreme value. For deciding if a tree admits a graceful labeling, the following function is taken:

$$h(f) = \sum_{k \in [1,n-1] \setminus \mathrm{Im}(f_\gamma)} k$$

where f is an injective vertex labeling of the tree.

Given a vertex labeling f, the evaluation function h is summing the edge labels

Algorithm 3.1: Backtracking search

```
Function Search(k):
    if k = 0 then
        return true
    if iterations exceed threshold then
        return false
    for every vertex v without label with its parent v′ labeled do
        if label f(v′) + k is valid and not yet used then
            label v with f(v′) + k
            if Search(k − 1) then
                return true
            unlabel v
        if label f(v′) − k is valid and not yet used then
            label v with f(v′) + k
            if Search(k − 1) then
                return true
            unlabel v
    return false
```

that did not appeared on any of the edges of the tree. Hence, f is a graceful labeling of the tree if, and only if, $h(f) = 0$. Thus, since h is always non-negative, we are interested in minimizing h.

Since the graph is a tree, an injective vertex labeling is also a permutation of $[0, n-1]$ on its vertices. Then, the domain of exploration of h is all permutations of $[0, n-1]$. The local search uses the hill-climbing method: at each iteration, it selects a number of random pairs of vertices, swaps their labels and picks the best one that improves the current solution.

As it is known, the hill-climbing method purely can get stuck in a local minimum. To avoid this problem, two strategies are adopted. The first one is the use of tabu search which forbids certain moves if they were made very recently, unless it results in a graceful labeling. The second strategy is based on an idea from the simulated annealing technique in which it is allowed to worsen the solution with a certain probability. Algorithm 3.2 is a pseudocode of these ideas.

This hybrid algorithm combining backtracking search and combinatorial optimization approach allowed the verification of the gracefulness of all trees up to 35 vertices. It is worth mentioning that the task was accomplished with the help of a community of volunteers in which the task was divided and distributed between them. Details of the performance of the algorithm can be found in the Fang's paper [9].

Algorithm 3.2: Local search using metaheuristics

let f be the vertex labeling corresponding to the identity permutation
$v \leftarrow h(f)$
while $v \neq 0$ **do**
  randomly choose $2n$ pairs of vertices
  foreach pair of vertices (x, y) chosen **do**
    swap the values of $f(x)$ and $f(y)$
    evaluate h for the modified labeling
    swap back $f(x)$ with $f(y)$
  choose the pair (x, y) that minimizes h
  let f' be the labeling obtained by swapping $f(x)$ with $f(y)$
  $v' \leftarrow h(f')$
  if $f(x)$, $f(y)$ was not swapped in the last $\lfloor n/3 \rfloor$ iterations **then**
    if $v > v'$ **then**
      swap $f(x)$ with $f(y)$, and update v
    else
      with probability p, swap $f(x)$ with $f(y)$, and update v
  else if $v' = 0$ **then**
    swap $f(x)$ with $f(y)$, and update v
return f

3.3 Relaxed versions

Relaxed versions of graceful labeling have been studied for as long as graceful labeling itself. Rosa himself introduced together with graceful labeling some variants of it, both stronger and weaker versions of graceful labeling.

Usually, one only consider relaxed versions when the graph is not graceful. However, here we consider a couple of relaxed versions of graceful labeling for trees, and, with the purpose of getting closer to the Graceful Tree Conjecture, the goal has been in trying to improve bounds for these labelings.

Probably, the following relaxed graceful labelings are the most intuitive ones.

1. *Edge-relaxed*: f_γ can be non-injective.

2. *Vertex-relaxed*: f can be non-injective (f_γ must still be injective).

3. *Range-relaxed*: $f : V(G) \to [0, k]$ for some $k \geq m$.

Bounds have been established for all these three versions. Rosa and Širáň [22] showed that every tree has a edge-relaxed graceful labeling with at least $5m/7$ different edge labels. Van Bussel [25] showed two results, one concerning vertex-relaxed graceful labeling of trees and the other concerning the range-relaxed graceful labeling of trees, which we present next.

Theorem 3.6. *Every tree T has a range-relaxed graceful labeling with vertex labels in the range* $[0, 2m - \mathrm{diam}(T)]$.

Proof. Let $T = (V, E)$ be a tree and $v_0 \in V$, and consider the tree T rooted at v_0. Also consider that the longest path from v_0 is at the leftmost in a planar representation of T. Let the length of this path be ℓ, the vertices of the path be $v_0, v_1, \ldots, v_\ell$, and h_i be the number of vertices at level i. The following construction provides a vertex labeling for T in the range $[0, 2m - \ell]$.

1. Label v_0 temporarily with α and v_1 with $\alpha + 1$. After labeling all vertices, we shift all labels by a constant so that the smallest value is 0.

2. For $i > 1$, label v_i as follows:

$$f(v_i) = \begin{cases} f(v_{i-2}) - h_{i-2} - h_{i-1} + 1 = \alpha - \sum_{j=0}^{i-1} h_j + \frac{i}{2} & \text{if } i \text{ is even} \\ f(v_{i-2}) + h_{i-2} + h_{i-1} - 1 = \alpha + \sum_{j=0}^{i-1} h_j - \frac{i-1}{2} & \text{if } i \text{ is odd} \end{cases}$$

3. At each level i, consider the order in which the vertices are represented in the place. Label the k-th vertex $u_{i,k}$ at level i, $k \in [0, h_i - 1]$, as follows:

$$f(u_{i,k}) = \begin{cases} f(v_i) - k & \text{if } i \text{ is even} \\ f(v_i) + k & \text{if } i \text{ is odd} \end{cases}$$

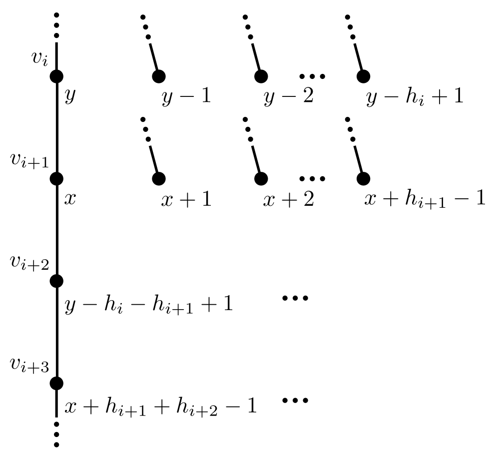

Figure 3.3: Labeling of Theorem 3.6 at level i even.

It is clear that all vertex labels are distinct: as we go from top to bottom, left to right, on even levels they are decreasing, and on odd levels they are increasing. Moreover, all edge labels are distinct. Indeed, the edge labels are increasing as we go from top to bottom, left to right.

Consider two edges u_iu_{i+1} and w_iw_{i+1} where u_i and w_i are at the same level i. If i is even, then $f(u_i) > f(w_i)$ and $f(u_{i+1}) < f(w_{i+1})$. Then,

$$f_\gamma(u_iu_{i+1}) = f(u_{i+1}) - f(u_i) < f(w_{i+1}) - f(w_i) = f_\gamma(w_iw_{i+1})$$

Analogously, the same holds if i is odd.

Now, let u_i and u_{i+1} be the rightmost vertices at levels i and $i+1$, respectively, and consider the edge u_iu_{i+1}, which is not necessarily an edge of the tree. By what we just showed, it has the largest edge label from level i to $i+1$. So, it suffices to show that $f_\gamma(u_iu_{i+1}) < f_\gamma(v_{i+1}v_{i+2})$, since $v_{i+1}v_{i+2}$ has the smallest edge label from level $i+1$ to $i+2$. Assuming i even, we have

$$\begin{aligned}
f_\gamma(u_iu_{i+1}) &= f(u_{i+1}) - f(u_i) \\
&= f(v_{i+1}) + h_{i+1} - 1 - (f(v_i) - h_i + 1) \\
&< f(v_{i+1}) - (f(v_i) - h_i - h_{i+1} + 1) \\
&= f_\gamma(v_{i+1}v_{i+2})
\end{aligned}$$

Again, the same holds if i is odd by an analogous proof.

Finally, we check that the labels are inside the range. Let f_{max} and f_{min} be the maximum and the minimum vertex labels, respectively. If ℓ is even, then the largest vertex label is at level $\ell - 1$ and the smallest one is at level ℓ.

$$\begin{aligned}
f_{max} &= f(v_{\ell-1}) + h_{\ell-1} - 1 \\
&= \alpha + \sum_{j=0}^{\ell-2} h_j - \frac{\ell-2}{2} + h_{\ell-1} - 1 \\
&= \alpha + m + 1 - h_\ell - \frac{\ell}{2} \\
f_{min} &= f(v_\ell) - h_\ell + 1 \\
&= \alpha - \sum_{j=0}^{\ell-1} h_j + \frac{\ell}{2} - h_\ell + 1 \\
&= \alpha - m + \frac{\ell}{2} \\
f_{max} - f_{min} &= \alpha + m + 1 - h_\ell - \frac{\ell}{2} - \left(\alpha - m + \frac{\ell}{2}\right) \\
&= 2m - l - h_\ell + 1 \\
&\leq 2m - l
\end{aligned}$$

Thus, if we choose our root as one of the end vertices of a longest path in the tree, we obtain a range-relaxed graceful labeling in the range $[0, 2m - \text{diam}(T)]$. $\square$

Theorem 3.7. *Every tree T has a vertex-relaxed graceful labeling with more than $\frac{n}{2}$ distinct vertex labels.*

Instead of proving Theorem 3.7 directly, Van Bussel proved a stronger result. Before that, we must define the following labeling. We say a vertex labeling f is *locally bipartite* if there is a bipartition of $V(G) = A \cup B$ such that

1. $\forall u \in A\ \forall v \in N(u) : f(u) < f(v)$
2. $\forall v \in B\ \forall u \in N(v) : f(u) < f(v)$

Note that if a graph G admits such labeling, then G must be bipartite.

Theorem 3.8. *Let $T = (V, E)$ be a tree with a bipartition of $V = A \cup B$, and let $v \in A$ be an arbitrary vertex. Then, there exists a vertex-relaxed graceful labeling f of T satisfying the following properties:*

1. f is locally bipartite;

2. $f(v) = 0$;

3. $f(x) \neq f(y)$ for all $x, y \in B$.

Proof. We prove by induction on n. For $n = 1$ and $n = 2$, it is clear that such labeling exists. Suppose $n > 2$, and let v be an arbitrary vertex of T. We divide in two cases.

Case 1. Assume $d(v) \geq 2$. Since v has at least two adjacent vertices, we can split v into two vertices v_1 and v_2 and obtain two trees T_1 and T_2 strictly smaller than T such that T is the union of T_1 and T_2 by identifying v_1 with v_2. By induction

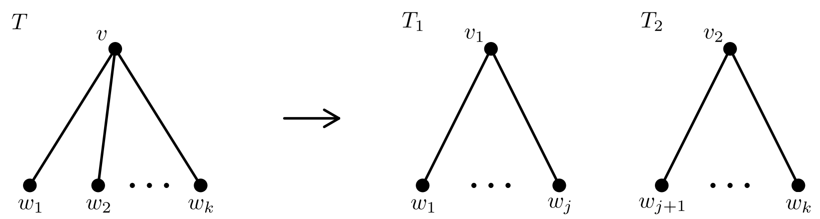

Figure 3.4: Splitting the tree at the vertex v.

hypothesis, T_1 has a bipartition $V(T_1) = A_1 \cup B_1$ with $v_1 \in A_1$ and a vertex-relaxed graceful labeling f_1 satisfying those properties with respect to v_1. Similarly, we have A_2, B_2, and f_2 for T_2. Thus, if $m_1 = |E(T_1)|$, the labeling f of T defined by

$$f(u) = \begin{cases} f_1(u) & \text{if } u \in V(T_1) \\ f_2(u) & \text{if } u \in A_2 \\ f_2(u) + m_1 & \text{if } u \in B_2 \end{cases}$$

is the required labeling.

1. $f(v) = f(v_1) = f(v_2) = 0$
2. Since we are adding a constant to all vertex labels in B_2, f_2 remains locally bipartite. Hence, f is also locally bipartite.
3. The edge labels in T_1 remains the same in T and those in T_2 are shifted by m_1, generating edge labels $\{m_1 + 1, \ldots, m_1 + m_2\}$. Hence, f is a vertex-relaxed graceful labeling of T.
4. All vertex labels in B_1 and B_2 are distinct. Thus,

$$\min\{f_2(B_2)\} + m_1 \geq m_1 + 1 > m_1 = \max\{f_1(B_1)\}$$

and we have that all vertex labels in $B_1 \cup B_2$ are distinct.

Case 2. Assume $d(v) = 1$. Let w be the adjacent vertex of v. Since $n \geq 3$, we have $d(w) \geq 2$. Let $r_1, r_2, \ldots, r_k$, where $k = d(w) - 1$, be the vertices adjacent to w except for v. Let the trees of $T - w$ rooted at r_i be T_i with m_i edges, bipartition (A_i, B_i), and $r_i \in A_i$.

Since T_i is smaller than T by at least 2 vertices, by induction hypothesis, T_i has a vertex-relaxed graceful labeling f_i satisfying those properties with respect to r_i. The labeling f of T defined below is as required.

$$\begin{aligned} f(v) &= 0 \\ f(w) &= m \\ f(u) &= \begin{cases} f_i(u) + i & \text{if } u \in A_i \\ f_i(u) + \sum_{j=1}^{i-1} m_j + i & \text{if } u \in B_i \end{cases} \end{aligned}$$

For the verification, let us denote $M_i = \sum_{j=1}^{i-1} m_j$.

1. For each tree T_i, f adds the constant i to all vertices and also adds M_i to the vertices in B_i, which means that f is locally bipartite in T_i. And, since w gets the largest vertex label possible, we have that f is locally bipartite in T.
2. For each tree T_i, f shifts all edge labels by M_i. Together with edges incident with w, which has labels $\{m, m-1, \ldots, m-k\}$, we have that each edge label in $[1, m]$ appears in some edge. Hence, f is a vertex-relaxed graceful labeling.
3. It is clear that in each B_i, all vertices have different labels. And, since $\max\{f(B_i)\} = M_i + m_i + i < M_{i+1} + i + 1 = \min\{f(B_i)\}$, we have that all labels in $\bigcup_{i=1}^{k} B_i$ are distinct, and the maximum of these labels is $M_k + m_k + k = m - 1$. Hence, all the vertex labels in $B = \{w\} \cup \bigcup_{i=1}^{k} B_i$ are distinct.

Therefore, every tree T admits a vertex-relaxed graceful labeling satisfying those properties. If we take A as the smallest set of the bipartition of T, we have $|B| \geq \frac{n}{2}$. And, since the vertex label 0 can not appear in B, we have at least $\frac{n}{2} + 1$ distinct vertex labels, as required in Theorem 3.7. $\square$

Although it is clear that every graph admits an edge-relaxed and a range-relaxed graceful labelings, not all graphs have a vertex-relaxed graceful labeling [25]. Furthermore, it is still unknown a connected non-graceful graph that has a vertex-relaxed graceful labeling.

Chapter 4

Generalized Cone Graphs

In Chapter 2, we presented the gracefulness of some graph classes and how to construct bigger graceful graphs from smaller ones. In this chapter, we generalize the wheel graphs, also known as cone graphs, and study its gracefulness. This graph class was first studied by Bhat-Nayak and Selvam [6] in 2003 and not much progress has been made since then.

A *generalized cone graph* is the join of a cycle graph C_p and an independent set I_q, where $p \geq 3$ and $q \geq 0$. For instance, for $q = 0$ and $q = 1$, we simply have the cycle graphs and the wheel graphs, respectively.

Throughout this chapter, we denote the vertices of the generalized cone graphs as $V(C_p + I_q) = \{u_0, u_1, \ldots, u_{p-1}, v_0, v_1, \ldots, v_{q-1}\}$ where $u_k \in V(C_p)$, $u_k u_{k+1} \in E(C_p)$ for $0 \leq k < p$ and $u_p = u_0$, and $v_k \in V(I_q)$. Also, from now on, we simply call generalized cone graphs as cone graphs.

The first result we show is concerning the non-graceful cone graphs. As we said in Chapter 2, the only useful theoretical tool for proving the non-existence of graceful labeling for a given graph is the parity condition, which only applies to Eulerian graphs. Thus, applying the parity condition to Eulerian cone graphs, the following holds.

Proposition 4.1. *The cone graph $C_p + I_q$ is not graceful for $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 2$.*

Proof. For $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 2$, the cone $C_p + I_q$ is Eulerian since the degree of every vertex is even (cf. [7]), and it has $m = p(q + 1)$ edges. Writing $p = 4s + 2$ and $q = 2t$, we have $m = (4s + 2)(2t + 1) \equiv 2 \pmod 4$. Hence, by the parity condition, $C_p + I_q$ is not graceful. $\square$

4.1 Graceful cones

For $q = 0$ and $q = 1$, we have the cycle graphs and the wheel graphs, respectively, and their gracefulness is already characterized in Chapter 2. For $q = 2$, we have the *double cones*, and it is still an open problem to characterize them. By Proposition 4.1, the double cone $C_p + I_2$ is not graceful for $p \equiv 2 \pmod 4$, and so far they are the only non-graceful double cones [6, 11, 19].

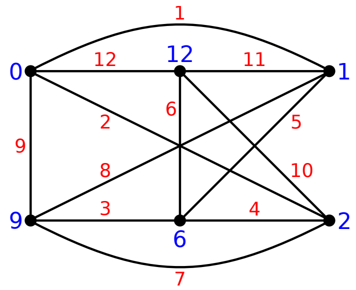

Figure 4.1: Graceful labeling of $C_4 + I_2$.

For the general case, Bhat-Nayak and Selvam [6] proved the following theorem.

Proposition 4.2. *The cone graph $C_p + I_q$ is graceful for $p \equiv 0, 3 \pmod{12}$ and $q \geq 1$.*

For the proof of Proposition 4.2, Bhat-Nayak and Selvam introduced a new graph labeling and showed a more general result similar to Theorem 2.7.

A vertex labeling f of a graph G with n vertices is said to be a *special labeling* if it satisfies the following conditions:

1. For every $i \in [1, n]$, there exists a vertex $u_i \in V(G)$ such that $f(u_i)$ is either $2i - 1$ or $2i$.

2. $\mathrm{Im}(f_\gamma) = [1, 2n] \setminus \mathrm{Im}(f)$.

3. If $f(x)$ and $f_\gamma(xy)$ are odd, then $f(x) < f(y)$.

Note that conditions 1 and 2 imply that the number of vertices must be the same as the number of edges, i.e., $n = m$.

Theorem 4.3. *If a graph G has a special labeling, then the graph $G + I_q$ is graceful for all $q \geq 1$.*

Proof. Let G be a graph on p vertices and f be a special labeling of G. Define the vertex labeling g for $G + I_q$ as follows, where $V(G) = \{u_1, \ldots, u_p\}$ and $V(I_q) = \{v_1, \ldots, v_q\}$:

$$g(v_j) = j - 1$$

$$g(u_i) = \begin{cases} i(q+1) & \text{if } f(u_i) = 2i \\ i(q+1) - 1 & \text{if } f(u_i) = 2i - 1 \end{cases}$$

We claim g is a graceful labeling of $G + I_q$. As noted before, since G has a special labeling, G has p edges. Thus, the number of edges of $G + I_q$ is $p + pq$. Clearly, $g\colon V(G + I_q) \to [0, p(q+1)]$ and it is injective. So, we have to prove that g_γ is onto $[1, p(q+1)]$. For that, we show that for each $i \in [1, p]$ and $j \in [1, q+1]$, there is an edge e with $g_\gamma(e) = (i-1)(q+1) + j$.

Consider a pair (i, j). Since f is a special labeling of G, by condition 1, there is a vertex $u_i \in V(G)$ with $f(u_i) = 2i - 1$ or $f(u_i) = 2i$.

Case 1. $f(u_i) = 2i - 1$ and $1 \le j \le q$.

We have $g(u_i) = i(q+1) - 1$ and $g(v_{q-j+1}) = q - j$. Since $q - j < i(q+1) - 1$, the edge label on $u_i v_{q-j+1}$ is $i(q+1) - 1 - (q - j) = (i-1)(q+1) + j$.

Case 2. $f(u_i) = 2i - 1$ and $j = q + 1$.

By condition 2, there is an edge $e = xy \in E(G)$ with $f_\gamma(xy) = 2i$. Hence, $f(x)$ and $f(y)$ have the same parity. Suppose $f(x) = 2a + r$ and $f(y) = 2b + r$, where $r \in \{0, 1\}$ is the parity. Then, $f_\gamma(xy) = 2i = |(2a + r) - (2b + r)| = 2|a - b|$, and $i = |a - b|$. Therefore, $g_\gamma(xy) = |(a(q+1) - r) - (b(q+1) - r)| = (q+1)|a - b| = i(q+1) = (i-1)(q+1) + (q+1)$

Case 3. $f(u_i) = 2i$ and $2 \le j \le q + 1$.

We have $g(u_i) = i(q+1)$ and $g(v_{q-j+2}) = q - j + 1$. Since $q - j + 1 < i(q+1)$, the edge label on $u_i v_{q-j+2}$ is $i(q+1) - (q - j + 1) = (i-1)(q+1) + j$.

Case 4. $f(u_i) = 2i$ and $j = 1$.

By condition 2, there is an edge $e = xy \in E(G)$ with $f_\gamma(xy) = 2i - 1$. Now, $f(x)$ and $f(y)$ have different parities. Without loss of generality, suppose $f(x)$ odd and let $f(x) = 2a - 1$ and $f(y) = 2b$. By condition 3, we have $f(x) < f(y)$ which implies $g(x) < g(y)$. Thus, $f_\gamma(xy) = 2i - 1 = 2b - (2a - 1)$ implies $i - 1 = b - a$. Finally, $g_\gamma(xy) = b(q+1) - (a(q+1) - 1) = (b - a)(q+1) - 1 = (i-1)(q+1) - 1$.

Thus, we have proved that $\mathrm{Im}(g_\gamma) = [1, p(q+1)]$ and therefore g is a graceful labeling of $G + I_q$. $\square$

We do not present here the complete proof of Proposition 4.2. Here, we only show a partial result which says that $C_{24k} + I_q$ is graceful. For that, Bhat-Nayak and Selvam proved the following lemmas.

Lemma 4.4. *For $k \geq 2$, P_{4k-3} has a vertex labeling f such that* $\mathrm{Im}(f) = [k+2, 2k] \cup [2k+3, 3k+1] \cup [5k+1, 7k-1]$, $\mathrm{Im}(f_\gamma) = [2k+1, 6k-4]$, *and the end vertices receive the labels* $5k+1$ *and* $7k-1$.

Proof. Let $P_{4k-3} = u_1 u_2 \cdots u_{4k-3}$ and define the vertex labeling f as follows:

$$\begin{aligned} f(u_{2i-1}) &= 5k+i && \text{for } 1 \leq i \leq 2k-1 \\ f(u_{2i}) &= k+2 && \text{for } i = 1 \\ &= 3k+3-i && \text{for } 2 \leq i \leq k \\ &= 3k+1-i && \text{for } k+1 \leq i \leq 2k-2 \end{aligned}$$

Now, it is easy to verify directly that $\mathrm{Im}(f_\gamma) = [2k+1, 6k-4]$. $\square$

Remark 4.1. For $k = 1$, consider the single vertex of P_1 labeled with 6.

Lemma 4.5. *For $k \geq 1$, P_{8k-1} has a vertex labeling f such that* $\mathrm{Im}(f) = [1, k] \cup [k+2, 8k]$, $\mathrm{Im}(f_\gamma) = [1, 8k-2]$, *and the end vertices receive the labels* $2k+1$ *and* $8k$.

Proof. Let $P_{8k-1} = u_1 u_2 \cdots u_{8k-1}$ and define the vertex labeling f as follows:

$$\begin{aligned} f(u_1) &= 2k+1 \\ f(u_{2i+1}) &= 4k+1+i && \text{for } 1 \leq i \leq k \\ f(u_{2i}) &= 4k+2-i && \text{for } 1 \leq i \leq k \\ f(u_{8k+1-2i}) &= 8k+1-i && \text{for } 1 \leq i \leq k+2 \\ f(u_{8k-2}) &= 2k+2 \\ f(u_{8k-2-2i}) &= i && \text{for } 1 \leq i \leq k \end{aligned}$$

Thus, we labeled the vertices $u_1, \ldots, u_{2k+1}, u_{6k-3}, \ldots, u_{8k-1}$ with labels in $[1, k] \cup [2k+1, 2k+2] \cup [3k+2, 5k+1] \cup [7k-1, 8k]$, and obtained edge labels in $[1, 2k] \cup [6k-3, 8k-2]$. For the remaining subpath $u_{2k+1} u_{2k+2} \cdots u_{6k-3}$, label it as given by Lemma 4.4 to obtain the desired labeling. $\square$

Lemma 4.6. *For $k \geq 1$, P_{8k-1} has a vertex labeling g such that* $\mathrm{Im}(g) = \{16k+2, 16k+4, \ldots, 18k\} \cup \{18k+4, 18k+6, \ldots, 32k\}$, $\mathrm{Im}(f_\gamma) = \{2, 4, \ldots, 16k-4\}$, *and the end vertices receive the labels* $20k+2$ *and* $32k$.

Proof. Let f be the vertex labeling obtained from Lemma 4.5. Then, defining g as $g(u) = 2f(u) + 16k$ gives the required labeling. $\square$

Lemma 4.7. *For $k \geq 1$, P_{16k+3} has a vertex labeling f such that* $\mathrm{Im}(f) = \{1, 3, \ldots, 16k-1, 18m+2, 20k+2, 32k, 32k+2, \ldots, 48k\}$, $\mathrm{Im}(f_\gamma) = \{16k-2, 16k, 16k+1, 16k+3, \ldots, 48k-1\}$, *and the end vertices receive the labels* $20k+2$ *and* $32k$.

Proof. Let $P_{16k+3} = u_1u_2\cdots u_{16k+3}$ and define the vertex labeling f as follows:

$$
\begin{aligned}
f(u_{2i-1}) &= 20k+2 && \text{for } i = 1\\
&= 48k+4-2i && \text{for } 2 \le i \le 8k+2\\
f(u_{2i}) &= 2i-1 && \text{for } 1 \le i \le 7k\\
&= 18k+2 && \text{for } i = 7k+1\\
&= 2i-3 && \text{for } 7k+2 \le i \le 8k+1
\end{aligned}
$$

Now, it is easy to verify that $\mathrm{Im}(f_\gamma)$ is as required. □

Proposition 4.8. *The cone graph $C_{24k} + I_q$ is graceful for all $k \ge 1$.*

Proof. Consider P_{8k-1} and P_{16k+3} labeled as given by Lemmas 4.6 and 4.7 respectively. By joining the paths by identifying the end vertices with the same label, we get a C_{24k} with a vertex labeling f such that $\mathrm{Im}(f) = \{1, 3, \ldots, 16k-1, 16k+2, 16k+4, \ldots, 48k\}$ and $\mathrm{Im}(f_\gamma) = \{2, 4, \ldots, 16k, 16k+1, 16k+3, \ldots, 48k-1\}$. Furthermore, the largest odd vertex label is less than the smallest even vertex label. Therefore, f satisfies all three conditions of being a special labeling for C_{24k}.

Therefore, by Theorem 4.3, $C_{24k} + I_q$ is graceful. □

For the proof of Proposition 4.2, Bhat-Nayak and Selvam proved not only Proposition 4.8, but also that $C_p + I_q$ is graceful for $p \equiv 3, 12, 15 \pmod{12}$, each of them following the same strategy as shown before: prove the existence of a specific vertex labeling of some specific paths and then join their end vertices to form a cycle graph.

Besides Proposition 4.2, Bhat-Nayak and Selvam also proved the following proposition.

Proposition 4.9. *The cone graph $C_p + I_q$ is graceful for $p = 7, 11, 19$ and $q \ge 1$.*

Proof. The following vertex labelings are special labelings for their respective cycle.

C_7: 1, 14, 5, 7, 10, 4, 12.

C_{11}: 1, 22, 5, 18, 7, 15, 9, 12, 14, 4, 20.

C_{19}: 1, 36, 3, 34, 5, 32, 7, 30, 12, 26, 16, 22, 20, 24, 13, 28, 9, 17, 38. □

Brundage [8] also worked on this problem and showed the following result.

Proposition 4.10. *The cone graphs $C_5 + I_q$ and $C_8 + I_q$ are graceful for all $q \ge 1$.*

Proof. Brundage gives a graceful labeling $f\colon V \to [0, m]$ for each case.

For $C_5 + I_q$, label the vertices of C_5 with $0, m, m-3, 3, m-1$ consecutively along the cycle, where $m = 5(q+1)$ is the total number of edges. Now, label the vertices of I_q as follows:

$$
f(v_k) = \begin{cases} 2 & \text{if } k = 0\\ 5k+3 & \text{if } k = 1, 2, \ldots, q-1 \end{cases}
$$

Thus, for $0 < k < q$, as $3 < 5k+3 < m-3$, the incident edges of v_k have labels $5k+3, m-(5k+3), (m-3)-(5k+3), 5k, (m-1)-(5k+3)$, which are all distinct since they have different residues modulo 5:

$$\begin{aligned} 5k+3 &\equiv 3 \pmod 5 \\ m-(5k+3) &\equiv 2 \pmod 5 \\ (m-3)-(5k+3) &\equiv 4 \pmod 5 \\ 5k &\equiv 0 \pmod 5 \\ (m-1)-(5k+3) &\equiv 1 \pmod 5 \end{aligned}$$

It is now easy to see that the labels in the edges incident with v_k, $0 < k < q$, cover the whole interval $[4, m-7]$. Along with the labels of the edges in C_5 $(m, 3, m-6, m-4, m-1)$ and those incident with v_0 $(2, m-2, m-5, 1, m-3)$, all the labels in $[1, m]$ appear exactly once. Thus, f is a graceful labeling of $C_5 + I_q$.

For $C_8 + I_q$, label the vertices of C_8 with $0, m, 2, 3, m-2, 1, m-3, m-1$ along the cycle, where $m = 8(q+1)$, and label each v_k in I_q with $4k+6$. The proof that this is indeed a graceful labeling is analogous to the previous case. □

Remark 4.2. Note that the graceful labeling for some families of cone graphs is often not unique. For instance, a graceful labeling for $C_8 + I_q$ distinct from the one given by Brundage goes as follows. Label C_8 with $0, m, \frac{m}{2}, \frac{3m}{4}+1, \frac{m}{2}+1, \frac{3m}{4}, \frac{m}{4}-1, m-1$, and label I_q with $2k+2$ for $0 \leq k < q$, where $m = 8(q+1)$.

Brundage [8] organized the gracefulness of cone graphs in a table (see Table 4.1) and made a conjecture characterizing this class.

Conjecture 4.1 (Brundage, 1994)**.** *The generalized cone graph $C_p + I_q$ is graceful if, and only if, the parity condition holds.*

$q \backslash p$	$3,4$	5	6	$7,8$	9	10	$11,12$	13	14	comments
0	Y	N	N	Y	N	N	Y	N	N	Y iff $p \equiv 0, 3 \pmod 4$
1	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y $\forall p$
2	Y	Y	N	Y	Y	N	Y	?	N	?, N $\forall p = 6+4k$
3	Y	Y	Y	Y	Y	?	Y	?	?	?
4	Y	Y	N	Y	Y	N	Y	?	N	?, N $\forall p = 6+4k$
5	Y	Y	?	Y	?	?	Y	?	?	?
6	Y	Y	N	Y	?	N	Y	?	N	?, N $\forall p = 6+4k$
7	Y	Y	?	Y	?	?	Y	?	?	?
8	Y	Y	N	Y	?	N	Y	?	N	?, N $\forall p = 6+4k$
9	Y	Y	?	Y	?	?	Y	?	?	?
comments	Y	Y $\forall q \geq 1$	?, N $\forall q$ even	Y	?	?, N $\forall q$ even	Y	?	?, N $\forall q$ even	?, N $\forall p = 6+4k, q$ even, Y $\forall p \equiv 0, 3 \pmod{12}$

Table 4.1: Gracefulness of $C_p + I_q$ (updated as of 2014).

4.2 Computational results

Questioning the validity of Conjecture 4.1, we started looking for counterexamples, i.e., find a cone graph for which the parity condition does not hold and it is not graceful. For this task, a backtracking search algorithm similar to the Fang's algorithm presented in Chapter 3 was implemented.

The strategy is the same as in Fang's algorithm: it tries to create a new edge label at each iteration by labeling a not yet labeled vertex. For reducing the search tree, some optimizations were made due to the inherent symmetries of cone graphs. The following observations eliminate most of search through equivalent labelings given by the symmetries of the graph.

Force $f(u_0) = 0$ without loss of generality. Since the edge labeling function f_γ is a bijection, i.e., in a graceful labeling, every possible edge label from 1 to m must appear as a label of some edge, and an edge label is obtained as the absolute value of the difference of the labels of its incident vertices, it follows that the vertices labeled 0 and m must be adjacent in the graph. Otherwise, no edge would be assigned label m. Furthermore, since all edges are incident with at least one vertex of the cycle, one of the vertices of the cycle must be labeled 0 or m. By symmetry, let u_0 be that vertex. Now, the complementarity property allows us to assume without loss of generality that $f(u_0) = 0$.

Just two candidate recipients for vertex label m. Assuming $f(u_0) = 0$, the vertex label m must be assigned to a vertex that is adjacent to u_0, i.e., to either u_1, u_{p-1} or v_k for some $k \in [0, q-1]$. Again owing to the symmetries in both the cycle and the independent set, we can narrow down our options, without loss of generality, to only two among those vertices, say u_1 and v_0.

Constrained recipients for edge label $m-1$. If, in the previous step, we chose vertex u_1 to receive label m, then, because we had already assigned label 0 to vertex u_0, the edge label $m-1$ can only appear on an edge that is incident with either u_1 (a neighbor of u_1 would receive label 1) or u_0 (a neighbor of u_0 would receive label $m-1$). Owing to the symmetries (rotation, reflection) of the cycle and the complementarity property, these two cases are actually equivalent. We can therefore consider, without loss of generality, that the edge labeled $m-1$ will be incident with u_1. We must now pick a neighbor of vertex u_1 to assign label 1. Since vertex u_0 is already labeled with 0, the possible neighbors are u_2 or v_k. However, by the symmetry of the independent set, we can consider v_0 as the sole candidate to receive label 1, and our search is limited to just two cases. If, on the other hand, we chose vertex v_0 to receive label m, then we must either assign label $m-1$ to a neighbor of u_0 (namely u_1 or v_1 without loss of generality), or assign label 1 to a neighbor of v_0 (namely u_k, where we can impose $1 \leq k \leq \lfloor \frac{p}{2} \rfloor$ owing to the reflection symmetry of the cycle).

Establish an order of labeling in I_q. Since all vertices in the independent set are indistinguishable between themselves (both from the standpoint of some vertex in the independent set, since there are no edges between any of them, and from the standpoint of some vertex in the cycle, since each vertex in the cycle is adjacent to all vertices in the independent set), we may assume an order in which the vertices of I_q are labeled. This prevents looking for labelings that are identical up to a permutation of the vertices in I_q.

Putting together these ideas, Algorithm 4.1 shows a pseudocode for the backtracking search algorithm to find a graceful labeling for a cone graph.

Algorithm 4.1: Backtracking search for generalized cone graphs

---

Function *Search(upper)*:
    if $upper = 0$ **then**
        return true
    $lbl \leftarrow$ largest edge label $\leq upper$ not present yet
    foreach pair (k, kc) with $|k - kc| = lbl$ **do**
        if both k and kc are not vertex labels yet **then**
            foreach edge uv with both ends unlabeled **do**
                label u with k and v with kc
                if `Check()` and `Search(`$lbl - 1$`)` **then return true**
                label u with kc and v with k
                if `Check()` and `Search(`$lbl - 1$`)` **then return true**
                unlabel u and v
        else
            let k be the unused vertex label and u be the vertex with label kc
            if $u \in V(C_p)$ **then**
                foreach $v \in N(u) \cap V(C_p)$, v unlabeled **do**
                    label v with k
                    if `Check()` and `Search(`$lbl - 1$`)` **then return true**
                    unlabel v
                if there are unlabeled vertex in I_q **then**
                    $v \leftarrow$ next unlabeled vertex from I_q
                    label v with k
                    if `Check()` and `Search(`$lbl - 1$`)` **then return true**
                    unlabel v
            else
                foreach $v \in V(C_p)$, v unlabeled **do**
                    label v with k
                    if `Check()` and `Search(`$lbl - 1$`)` **then return true**
                    unlabel v

The function `Check` in the pseudocode checks if the current labeling is valid, i.e., it checks if there are no repeated edge or vertex labels. Unlike Fang's backtracking search algorithm, this check is necessary here because labeling a vertex can create more than just one edge label. So, a verification is necessary every time we label a new vertex before continuing the search.

Running the search for a graceful labeling for $C_6 + I_5$, the smallest cone graph which was still unknown to be graceful or not, the algorithm returned no possible graceful labeling, refuting, therefore, Brundage's conjecture. Moreover, the algorithm did not find a graceful labeling for $C_6 + I_q$ with $5 \leq q \leq 35$. Notice that we are only interested in odd values of q since, for even values, the parity condition already settles that $C_6 + I_q$ is not graceful.

Searching for more non-graceful cone graphs, it makes sense to look for cone graphs $C_p + I_q$ with $p \equiv 2 \pmod 4$ as they are the only ones that, together with an even q, are not graceful by the parity condition. Then, the next subclass to search for non-graceful cones is $C_{10} + I_q$. We found that $C_{10} + I_3$ and $C_{10} + I_5$ are graceful. However, the algorithm returned no graceful labeling for $C_{10} + I_q$ with $7 \leq q \leq 25$. A similar result was gotten with $p = 14$: the cones $C_{14} + I_3$ and $C_{14} + I_5$ are graceful, but the cones $C_{14} + I_7$ and $C_{14} + I_9$ are not. The following propositions summarize these results.

Proposition 4.11. *The cone graphs $C_{10} + I_q$ and $C_{14} + I_q$ are graceful for $q = 3, 5$.*

Proof. We have the following labelings where the first p labels are from the cycle and the last q are from the independent set.

$C_{10} + I_3$: 0, 40, 25, 3, 33, 13, 6, 29, 10, 21; 37, 38, 39.

$C_{10} + I_5$: 0, 27, 1, 57, 14, 13, 2, 16, 3, 15; 23, 32, 51, 55, 60.

$C_{14} + I_3$: 0, 56, 6, 1, 28, 5, 2, 30, 34, 3, 33, 11, 22, 55; 40, 47, 54.

$C_{14} + I_5$: 0, 84, 33, 17, 82, 34, 47, 54, 64, 68, 69, 32, 49, 83; 2, 5, 8, 11, 14. □

Proposition 4.12. *The cone graphs $C_6 + I_q$, $5 \leq q \leq 35$, $C_{10} + I_q$, $7 \leq q \leq 25$, $C_{14} + I_7$, and $C_{14} + I_9$ are not graceful.*

Proof. Proven computationally. □

Proposition 4.12 not only disproves Conjecture 4.1, but also gives a stronger feeling about how the non-gracefulness of generalized cone graphs behaves, from which we conjecture the following.

Conjecture 4.2. *For every $p \equiv 2 \pmod 4$, there exists a $q_p > 1$ such that the cone graph $C_p + I_q$ is not graceful for all $q \geq q_p$.*

One might think of trying to prove it computationally, implementing an algorithm to do something similar to the proof of Proposition 2.2, exhausting all possibilities for all values of q greater than a threshold. However, as it was noted, the running times of the algorithm to establish the non-gracefulness were growing exponentially, which indicates that it is not possible to prove it in this way.

Besides the non-graceful cone graphs, we also searched for new families of cones which are graceful. We have seen two approaches to tackle this class: fixing the size of the independent size or fixing the size of the cycle. By taking the last one, we started to find graceful labelings for $C_9 + I_q$, the smallest family of this kind which was still open, and tried to find a pattern in the labelings while increasing the size of the independent size. As seen in Proposition 4.10, a simple rule could be possible, and indeed we found a scheme of labeling, not only for $C_9 + I_q$, but also for $C_{13} + I_q$.

Proposition 4.13. *The cone graphs $C_9 + I_q$ and $C_{13} + I_q$ are graceful for all $q \geq 1$.*

Proof. For $C_9 + I_q$, label the vertices of C_9 with $0, m, 5, m-7, 3, m-8, m-3, 4, m-2$ along the cycle, where $m = 9(q+1)$ is the number of edges, and label I_q as follows:

$$f(v_k) = \begin{cases} 1 & \text{if } k = 0 \\ 9k + 4 & \text{if } k = 1, 2, \ldots, q-1 \end{cases}$$

For $C_{13} + I_q$, label the vertices of C_{13} as $0, m, m-8, 6, m-9, 10, m-6, 7, m-4, m-7, 5, m-1, m-3$ along the cycle, where $m = 13(q+1)$ is the number of edges, and label I_q as follows:

$$f(v_k) = \begin{cases} 1 & \text{if } k = 0 \\ 13k + 4 & \text{if } k = 1, 2, \ldots, q-1 \end{cases}$$

As for the verification, since it is analogous to the proof of Proposition 4.10, it is omitted. □

On the other hand, finding a pattern after having fixed the size of the independent set (and allowing the size of the cycle to grow freely) seems to be much harder. For instance, it seems that, for $p > 5$ and $p \equiv 1 \pmod 4$, the cone graph $C_p + I_q$ has a graceful labeling f such that $f(v_0) = 1$ and $f(v_k) = pk + 4$ for $1 \leq k < q$, as it can be seen in Proposition 4.13; we have also verified it for several cones with $p = 17$ and $p = 21$. However, no pattern has been found for the cycles. Another example is the family of cone graphs $C_p + I_q$ with $p \equiv 0 \pmod 4$: each of them seems to have a graceful labeling with $f(v_k) = \frac{p}{4}(k+1)$ for $0 \leq k < q$. That is known to be true for $p = 4$ [6], and now $p = 8$ (see Remark 4.2); we have also verified that several cones with $p = 12, 16, 20$ admit such labeling.

Table 4.2 summarizes the current state of the gracefulness of generalized cone graphs for small values and gives a comment for the state of each row and column.

q \ p	$3, 4$	5	6	$7, 8$	9	10	$11, 12$	13	14	comments
0	Y	N	N	Y	N	N	Y	N	N	Y iff $p \equiv 0, 3 \pmod 4$
1	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y $\forall p$
2	Y	Y	N	Y	Y	N	Y	Y	N	?, N $\forall p = 6 + 4k$
3	Y	Y	Y	Y	Y	Y	Y	Y	Y	?
4	Y	Y	N	Y	Y	N	Y	Y	N	?, N $\forall p = 6 + 4k$
5	Y	Y	N	Y	Y	Y	Y	Y	Y	?
6	Y	Y	N	Y	Y	N	Y	Y	N	?, N $\forall p = 6 + 4k$
7	Y	Y	N	Y	Y	N	Y	Y	N	?
8	Y	Y	N	Y	Y	N	Y	Y	N	?, N $\forall p = 6 + 4k$
9	Y	Y	N	Y	Y	N	Y	Y	N	?
10	Y	Y	N	Y	Y	N	Y	Y	N	?, N $\forall p = 6 + 4k$
11	Y	Y	N	Y	Y	N	Y	Y	?	?
comments	Y	Y $\forall q \geq 1$	?, N $\forall q$ even	Y	Y $\forall q \geq 1$	?, N $\forall q$ even	Y	Y $\forall q \geq 1$	?, N $\forall q$ even	?, N $\forall p = 6 + 4k, q$ even Y $\forall p \equiv 0, 3 \pmod{12}$

Table 4.2: Gracefulness of $C_p + I_q$ (shaded entries are new results).

Chapter 5

Conclusion

The graceful labeling of graphs has been a topic of research for 50 years and it still has many properties to be found. Although its primary interest was the graceful labeling of trees in order to solve Ringel's conjecture, graceful labeling of graphs gained over the years its own beauty and interest.

This work gives a brief overview of the subject, presenting not only theoretical results from the literature, but also some computational results. Furthermore, we give some contributions to this problem.

In Chapter 2, the problem is presented, as well as the gracefulness of some rather simple graph classes like cycles and wheels. We also show necessary conditions to the existence of a graceful labeling for a graph, and two methods of constructing graceful graphs. In particular, one of them shows that any graph is an induced subgraph of some graceful graph.

In Chapter 3, we focus on graceful labeling of trees, more specifically, on different ways to approach the Graceful Tree Conjecture. The first one tackle the trees by limiting the diameter by introducing the transfer operation to modify a tree keeping it graceful. The second one reinforces the conjecture by showing computationally that all trees up to 35 vertices are graceful. Finally, we present some relaxed version of graceful labeling in which the better the bound, the closer to the conjecture we are.

In Chapter 4, we move our focus to generalized cone graphs. Their gracefulness was first tackled by Bhat-Nayak and Selvam, although some particular cases were already known. Later, Brundage also worked on this graph class and made a conjecture characterizing the gracefulness of cone graphs.

We tackled the gracefulness of cone graphs computationally and were able to disprove Brundage's conjecture. We also establish the gracefulness of new families of cone graphs and make a new conjecture regarding the non-graceful cone graphs.

For future work on the subject, we could consider looking for a way to prove Conjecture 4.2, or even characterize the gracefulness of generalized cone graphs. As we showed in Chapter 4, it seems that $C_p + I_q$ is graceful for $p \equiv 0, 1, 3 \pmod 4$ and $q \geq 1$. For $p \equiv 2 \pmod 4$, our conjecture says there is a $q_p > 1$ such that the cone graph is not graceful for all $q \geq q_p$. If, moreover, we could find out the parameter q_p for each $p \equiv 2 \pmod 4$, we would have a characterization of the gracefulness of generalized cone graphs.

Another class of interest is the class of trees, being the main open class on this topic. It is already settled that many classes of trees are graceful, but also there are many classes, even simple ones like lobsters, that are still open. Finally, another approach to the problem is to relax the conditions of graceful labelings and find nearly graceful labelings. This approach by approximating the labeling is also a topic of research for both trees and graphs in general.

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