Notes for Graceful Labeling

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※ Introduction

This is my notes for the various ideas generated during the discussions with Sahaj and Srinibas.

※ Problem Description

We are given a graph G on n vertices and m edges. We consider labeling the vertices using labels $\{0, 1, \ldots, m\}$. The vertex labels induce an edge labeling as follows; The an edge (u, v) is labeled $\mid u - v \mid$. A labeling is called **graceful** if the induced edge labels form the set $\{1, 2, \ldots, m\}$. You can find more information here.[2][1]

We are interested in counting the number of graceful labellings for for a cycle C_n.

※ Approximate Counting

Let $\sigma = \sigma_0\sigma_1 \ldots \sigma_m$ be a permutation chosen uniformly at random. Each permutation will correspond to a labeling on C_m as follows. The label σ_0 is an unused label. Vertex i gets the label σ_i. Lets call this labeling as l_σ. We are interested in knowing if l_σ is graceful. Let p be the probability that l_σ is graceful. Then, the number of graceful labelings are $p \cdot (m+1)!$ You can find more about approximate counting and uniform generation here[3].

With the aim of computing/approximating p, we define the following random variables. Let X_i denote the induced label of the edge $(i, i+1)$ where $1 \leq i \leq m-1$. Let X_m denote the label of the edge $(m, 1)$. In other words, $X_i = \mid \sigma_{i+1} - \sigma_i \mid$ Next, we define random variables $Y_{i,j}$ where $i \neq j$ and

$1 \le i, j \le m$. $Y_{i,j} = \mathbb{I}_{\{X_i == X_j\}}$. Finally, we define the random variable $Y = \sum_{i,j} Y_{i,j}$. Note that if the random labeling is graceful then Y is 0.

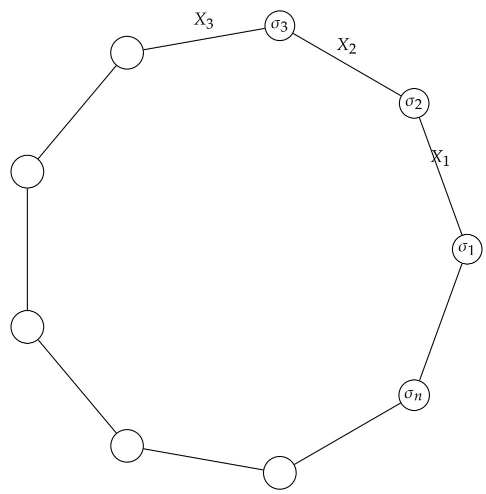

Figure 1: Random Labeling of C_n

3.1 Computing the expectation of Y

Let $Y = \sum_{i \le j} Y_{i,j}$. We compute $\mathbb{E}[Y]$ as follows:

$$
\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}\Big[\sum_{i<j} Y_{i,j}\Big] \\
&= \sum_{i<j} \mathbb{E}[Y_{i,j}] && \text{(Linearity of expectation)} \\
&= \sum_{i} \mathbb{E}[Y_{i,i+1}] + \sum_{i+1<j} \mathbb{E}[Y_i] && \text{(Splitting the summation)}
\end{aligned}
$$

$Y_{i,i+1}$ depends on X_i and X_{i+1} which in turn are determined by σ_i, σ_{i+1} and σ_{i+2}. The first term of the summation has m terms. Once σ_i and σ_{i+1} are fixed, there are a fixed number of choices for σ_{i+2} which makes $Y_{i,i+1}$ equal to 1. Thus the probability of this is some $\frac{c}{n}$. The second term can also be treated similarly. The overall sum can thus be approximated to cn.

We can now consider the following question.

What is the probability that a random variable whose expectation is at aleast cn takes the value 0? If this probability can be shown to be very small, then the number of graceful labelings will be small as well.

Observation: Y depends on $\binom{n}{2}$ random variables. If any of these random variables take a non zero value, then Y is non zero. Can we show that Y is zero with only a very small probability? This would have been easy if the random variables were independent. What can we salvage with the kind of dependency we have?

References

[1] Kintali. Graceful tree conjecture, June 23 2009.

[2] Alexander Rosa. On certain valuations of the vertices of a graph. *Journal of Graph Theory - JGT*, 01 1967.

[3] Alistair Sinclair and Mark Jerrum. Approximate counting, uniform generation and rapidly mixing markov chains. *Inf. Comput.*, 82(1):93–133, 1989.