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Alternative construction of graceful symmetric trees

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Abstract. Graceful labeling is one of the interesting topics in graph theory. Let $G = (V, E)$ be a tree. The injective mapping $f : V \to \{0, 1, ..., |E|\}$ is called graceful if the weight of edge $w(xy) = |f(x) - f(y)|$ are all different for every edge xy. The famous conjecture in this area is all trees are graceful. In this paper we give alternative construction of graceful labeling on symmetric tree using adjacency matrix.

1. Introduction

Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Rosa [5] called a function f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, ..., q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are all different. Golomb [3] subsequently called such labeling graceful and this term is used in this paper. Rosa [5] introduced β-valuations as well as a number of other labeling as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of proving the conjecture of Ringel that $K_(2n + 1)$ can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. Since the graceful labeling introduced, there are many results have been published. More results on graph labeling, including graceful labeling, can be found in Gallian's survey [2, 4].

Cavalier [1] has established several requirements that give an easy way to check the gracefulness of a labeling given the graph's adjacency matrix. The graph that we discussed in this paper is a symmetric tree, which is a root tree that every edge in the same level has the same degree.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then the matrix $A_G = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

is called an adjacency matrix of G.

Let G be a graph with $m = |E(G)|$ and a mapping $f : V(G) \to \{0, 1, , m\}$. Then the $(m + 1) \times (m + 1)$ matrix $A_G = [a_i j]$ defined by:

$$a_{ij} = \begin{cases} 1 & \text{if } xy \in E(G), \text{and } f(x) = i \text{ and } f(y) = j \\ 0 & \text{otherwise,} \end{cases}$$

where where $i, j = 0, 1, , m$, is called the generalized adjacency matrix of G induced by the mapping f. The generalized adjacency matrix allows (all zero) rows/columns corresponding to missing labels; while in the adjacency matrix such rows correspond to vertices of degree 0. Since there is no orientation on the edges, the corresponding adjacency matrices will be symmetric and thus there is no distinction drawn between edges v_iv_j and v_jv_i. The property, that all the graphs considered are simple, guarantees that the main diagonal entries of the adjacency matrices will be zeros as a one on the main diagonal corresponds to a loop in the graph. Let A be an $n \times n$ matrix. Then the k^th diagonal line of A is the collection of entries $D_k = a_{ij} : |j - i| = k$, counting multiplicity. Let $A = A_G$ be a (generalized) adjacency matrix, A is symmetric and $D_k = D_(- k)$. The entry 1 corresponding to an edge between vertices v_i and v_j lies in the $|j - i|^{th}$ diagonal line and the edge label for that edge is $|j - i|$, assuming $f(v_i) = i$ for each $i = 0, 1, , n - 1$ and f is a mapping on G.

Theorem 1.1 *[1] Let G be a labeled graph and let A_G be the generalized adjacency matrix for G. Then A_G has exactly one entry 1 in each diagonal line, except the main diagonal of zeros, if and only if the valuation f on G that induces A_G is graceful.*

Proof. We begin by noting that we need only consider the upper triangular part of A_G since it is symmetric. That is, for edges v_iv_j, we may assume $j > i$. Suppose A_G has exactly one entry of 1 in each diagonal line, other than the main diagonal of zeros. Suppose to the contrary that the labeling of G that induces A_G is not a graceful labeling. Then there are distinct edges v_gv_h and v_qv_l with edge labels $|h - g| = |l - q| = k > 0$. This implies

$$\sum_{i,j,|i-j|=k} a_{ij} \geq 2$$

contradicting the assumption that A_G has exactly one entry in each diagonal (not including main diagonal). Now suppose G is gracefully labeled by f and consider A_G. Then for all $k = 1, 2, , |E(G)|$, there is exactly one non-zero entry $a_ij = 1$, such that $j > i$ and $|j - i| = k$, contributing to $|D_k|$ since each edge has a unique label. That is, A_G has exactly one entry of 1 in each non-main diagonal. □

2. Alternative Construction of Graceful Symmetric Trees

Let G be a rooted tree with k level. We define this rooted tree as a symmetrical tree which every level contains vertices of the same degree. It has been shown in [2, 4] that all symmetrical trees are graceful. We are going to exhibit an alternative proof by constructing the graceful symmetrical tree from a graceful star.

Theorem 2.1 *All symmetric tree are graceful.*

Proof. We begin by considering a subgraph of the rooted tree, in which it has m vertices at level k connected to a vertex at level $k - 1$, as a star. It is clear that this star has $m + 1$ vertices.

This star is constructed by the generalized adjacency matrix $(m + 1) \times (m + 1)$ as follows.

$$A_{k-1} = \begin{bmatrix} 0 & ... & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & ... & 0 & 1 \\ 1 & ... & 1 & 0 \end{bmatrix}$$

By Theorem 1.1, this star is graceful. Now consider a subgraph of the rooted tree in which it has every center vertices of p previous stars connected to a vertex at level $k - 2$.

This graph is constructed by the generalized adjacency matrix

$$A_{k-2} = \begin{pmatrix} 0 & \dots & 0 & A'_{k-1} & v \\ \vdots & \ddots & A'_{k-1} & 0 & v \\ 0 & \dots & 0 & \dots & \dots \\ A'_{k-1} & 0 & \dots & v & \\ v^t & v^t & \dots & v^t & 0 \end{pmatrix}.$$

The matrix has order $((m+1)p+1) \times ((m+1)p+1)$, while the p sub-matrices of the form

$$A'_{k-1} = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}$$

where O is zero matrix, and

$$v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with order $(m+1) \times 1$, and v^t is the transpose of v. Note that the sub-matrix A'_{k-1} is the generalized adjacency matrix A_{k-1} with entries displaced by rotating the matrix 180 degree clockwise. It means that if $A'_{k-1} = (a'_{ij})$ and $A_{k-1} = (a_{ij})$ then $a'_{ij} = a_{(n-1+i)(n+1-j)}$.

See the symmetric tree graph of level k in Figure 1.

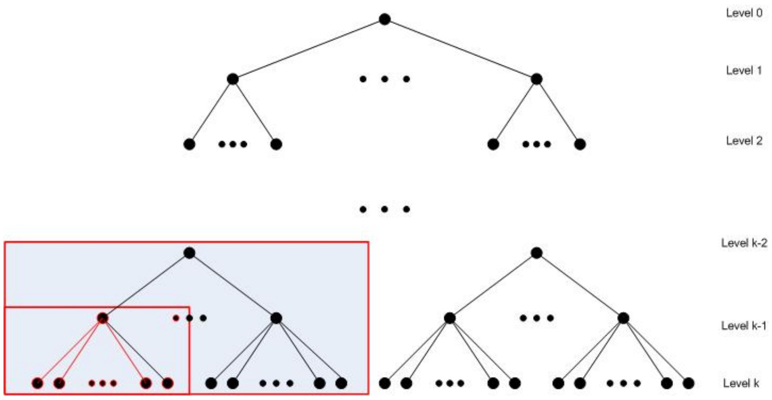

Figure 1. Symmetric Tree's construction.

By Theorem 1.1, this graph is graceful. Now consider a subgraph of the rooted tree in which it has every root vertices of q previous graphs connected to a vertex at level $k-3$.

This graph is constructed by the following generalized adjacency matrix

$$A_{k-3} = \begin{pmatrix} 0 & \dots & 0 & A'_{k-2} & v \\ \vdots & \ddots & A'_{k-2} & 0 & v \\ 0 & \ddots & \vdots & \vdots & v \\ A'_{k-2} & 0 & \dots & 0 & v \\ v^t & v^t & \dots & v^t & v \end{pmatrix}.$$

with order $(((m+1)p+1) \times (((m+1)p+1_q+1)))$ and the q sub-matrices has the form

$$A_{k-2} = \begin{pmatrix} 0 & v^{t'} & \dots & v^{t'} & v^{t'} \\ v' & 0 & \dots & 0 & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v' & 0 & A_{k-1} & \vdots & 0 \\ v' & A_{k-1} & 0 & \dots & 0 \end{pmatrix}.$$

where O is zero matrix, and

$$v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and v^t is the transpose of v. Note that the sub-matrix A'_{k-2} is the generalized adjacency matrix A_{k-2} with entries displaced by rotating the matrix 180 degree clockwise.

By Theorem 1.1, this graph is graceful. This process can be repeated until we finally obtain a rooted tree with k level, which is constructed by the generalized matrix adjacency A_0.

$$A_0 = \begin{pmatrix} 0 & ... & 0 & A'_1 & v \\ \vdots & \ddots & A'_1 & 0 & v \\ 0 & ... & 0 & ... & ... \\ A'_1 & 0 & ... & v & \\ v^t & v^t & ... & v^t & 0 \end{pmatrix}.$$

By the Theorem 1.1, this graph is graceful. Thus the symmetric tree is graceful. □

To make the construction clearer, we can see the following example.

Let we start from S_3, a star with three leaves with the label as in Figure 2.

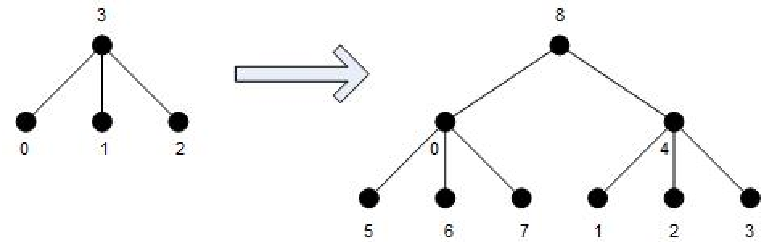

Figure 2. Tree's construction from star.

The adjacency matrix of the star is as follows.

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

In the construction in Figure 2, we can have the adjacency matrix of the tree on the right side is as follows.

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we continue the process to have A_0 then the resulting labeling of symmetric tree of three level can be seen in Figure 3.

$$A_0 = \begin{pmatrix} 0 & 0 & A_1' & v \\ 0 & A_1' & 0 & v \\ A_1' & 0 & 0 & v \\ v^t & v^t & v^t & 0 \end{pmatrix}.$$

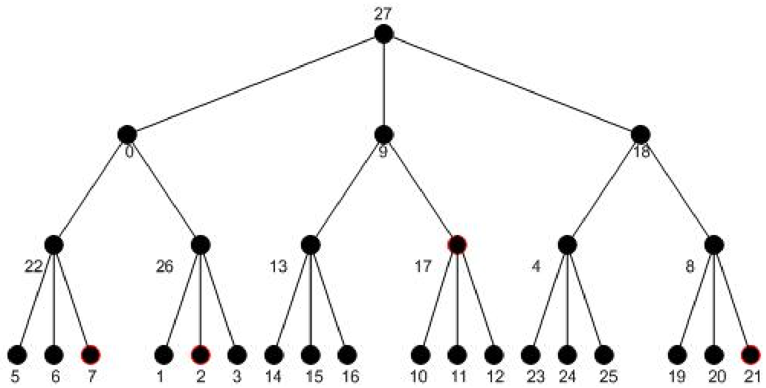

Figure 3. Example of graceful labeling.

3. Conclusion and Open Problem

The graceful construction using adjacency matrix can be use to prove that symmetric tree is graceful. The original conjecture on " all trees are graceful" is still open. The construction using adjacency matrix is promising to solve conjecture for the semi regular tree.

Acknowledgments

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References

[1] Cavalier C M 2006 *Graceful Labelings* Thesis Lousiana State University
[2] Gallian J A 2016 Dynamic Survey of Graph Labeling it The Electronic Journal of Combinatorics 1-79
[3] Golomb S W 1972 *How to Number a Graph, Graph Theory and Computing* R C Read ed Academic Press New York 23-37
[4] Robeva E 2011 *An Extensive Survey of Graceful Trees* Thesis Stanford University
[5] Rosa A 1966 *On Certain Valuations of the Vertices of a Graph* Theory of Graphs (Internat. Symposium, Rome, July 1966) 349-355