Department of Electronic & Telecommunication Engineering
University of Moratuwa

BM4151 – Biosignal Processing

Assignment - III
Continuous and Discrete Wavelet Transforms

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This is submitted as a partial fulfilment for the module BM4151 – Biosignal Processing

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1 Continuous Wavelet Transform

1.1 Introduction

Fourier transform is a mathematical transformation process of time domain representation to frequency domain representation. Although the Fourier transform is capable of localizing in the frequency domain for stationary signals, it cannot do the localizing for non-stationary signals as loss of time information. Therefore, in an inverse Fourier transformation signal expected signal cannot be obtained for a non-stationary signal. As the solution, the wavelet transformation method can utilize for this. The continuous wavelet function can be defined as follows,

$$W(s,\tau) = \int x(t)\frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)dt$$

Here, s is the scaling factor, τ is the translation and ψ is the wavelet function. Harr, Mexican hat, Harr and Daubechie are a few of the waves. In the following filter implementation, the Mexican hat mother wavelet will be constructed and it will be applied to non-stationary signals.

1.1.1 Mexican hat function

$$Mexican\ Hat\ function = m(t) = -\frac{d^2g(t)}{dt^2}$$

Where,

$$g(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$$

$$\frac{d}{dt}g(t) = \frac{-t}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$$

$$\frac{d^2}{dt^2}g(t) = \frac{t^2}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2} + \frac{-1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$$

$$\therefore m(t) = \frac{(1-t^2)}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$$

Obtaining the normalizing factor of $m(t)$,

For the normalization factor calculation, we consider the Mexican hat function has unity energy.

$$E = \int_{-\infty}^{\infty} m(t)^2\,dt$$

$$E = \int_{-\infty}^{\infty} \frac{(1-t^2)^2e^{-t^2}}{2\pi}dt$$

$$E = \left[\frac{3\,\mathrm{erf}(t)}{16\sqrt{\pi}} - \frac{t(2t^2-1)e^{-t^2}}{8\pi}\right]_{-\infty}^{\infty} = \frac{3}{8\sqrt{\pi}}$$

$$\therefore E = \frac{3}{8\sqrt{\pi}}$$

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$$Normalization\ Factor = \frac{1}{\sqrt{E}} = \sqrt{\frac{8\sqrt{\pi}}{3}} = \sqrt{\frac{8\pi^{\frac{1}{2}}}{3}}$$

Therefore normalized Mexican hat function is,

$$m_{normalized}(t) = Normalized\ Factor\ .m(t)$$

$$m_{normalized}(t) = \frac{2\sqrt{2}\pi^{\frac{1}{4}}}{\sqrt{3}} \times \frac{(1 - t^2)}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

$$m_{normalized}(t) = \frac{2\,(1 - t^2)}{\sqrt{3}\pi^{\frac{1}{4}}} e^{-\frac{1}{2}t^2}$$

Normalized Mexican hat mother wavelet functions can be obtained as follows,

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

$$\psi_{s,\tau}(t) = \frac{2}{\sqrt{3s}\pi^{\frac{1}{4}}}\left[1 - \left(\frac{t-\tau}{s}\right)^2\right] e^{-\frac{1}{2}\left(\frac{t-\tau}{s}\right)^2}$$

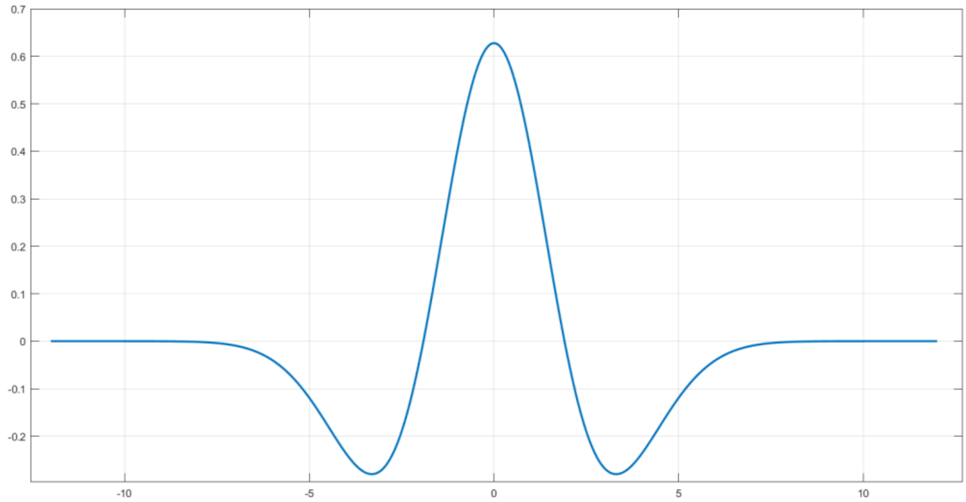

Figure 1: Normalized Mexican hat function

1.2 Wavelet properties

General properties of wavelet functions,

1. Zero mean function => $\int_{-\infty}^{\infty} \psi(t)dt = 0$

2. Unity Power => $\int_{-\infty}^{\infty} \psi^2(t)dt = 1$

3. Contact support (Limited in the time domain)

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The following figures show the daughter wavelet of the Mexican hat, wavelets are obtained using the given function called **wavelet_construction**. Here 0.01 – 2 scaling factors are used and those factors are increment with 0.1 within the range.

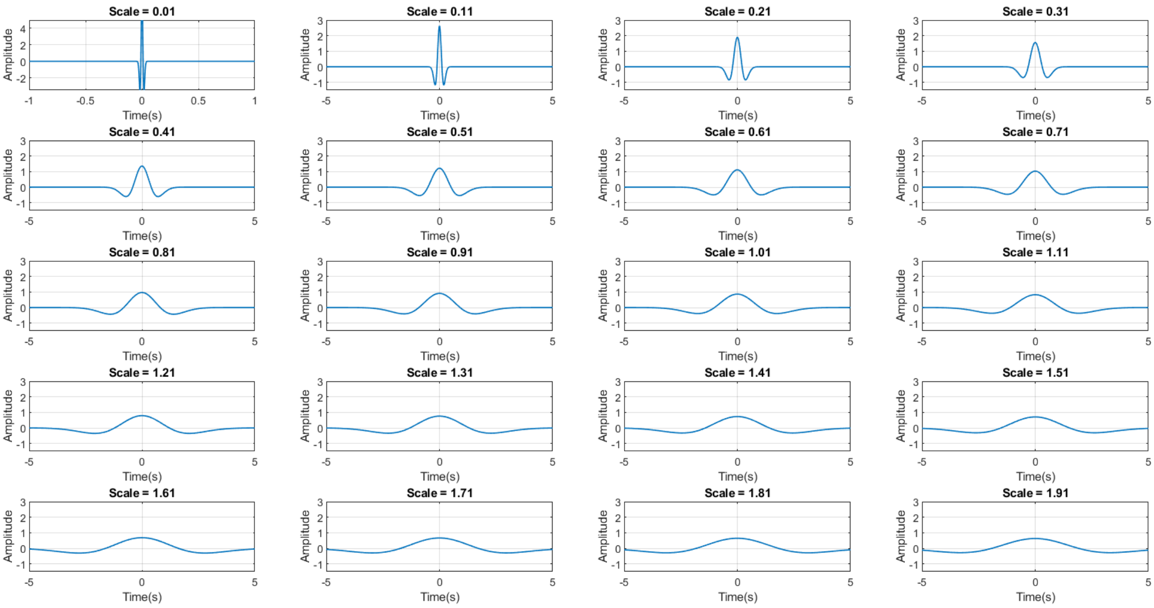

Figure 2: Time domain representation in Mexican hat wavelet

According to the above figures, when increasing the scaling factors waveform width is increasing and peak amplitude is reduced. Also, the details are limited in the time domain as most of the details are located in a small range. Therefore it satisfies 3rd property(**Compact support**) of the wavelet function.

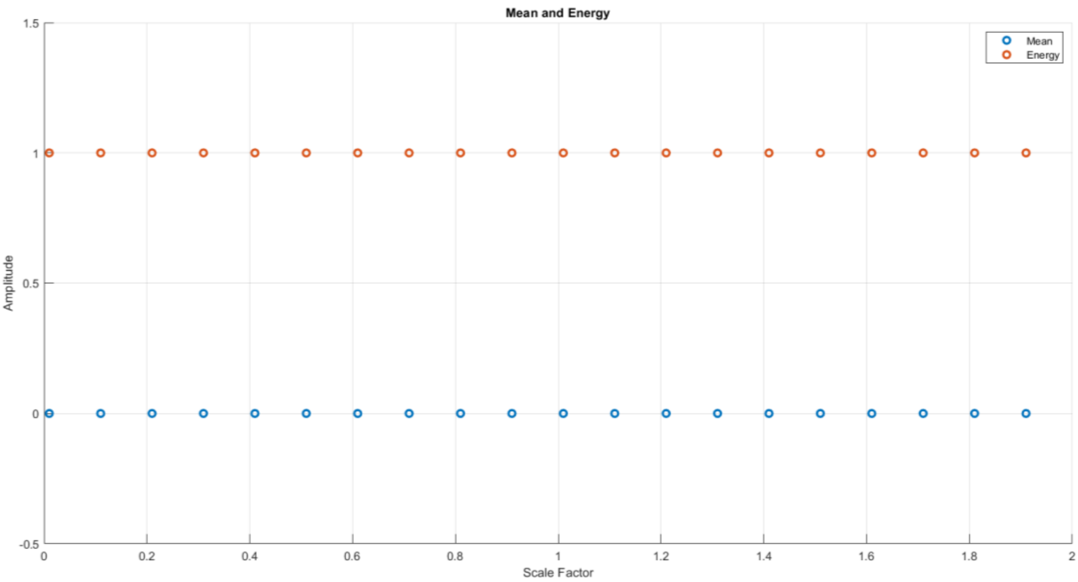

Figure 3: Mean and Energy at different scalling factors

According to the above figure, the graph shows that the mean value of the function equals zero and energy equals one. Therefore we can ensure that this wavelet function is satisfied the 2 and 3 general properties of the wavelet functions.

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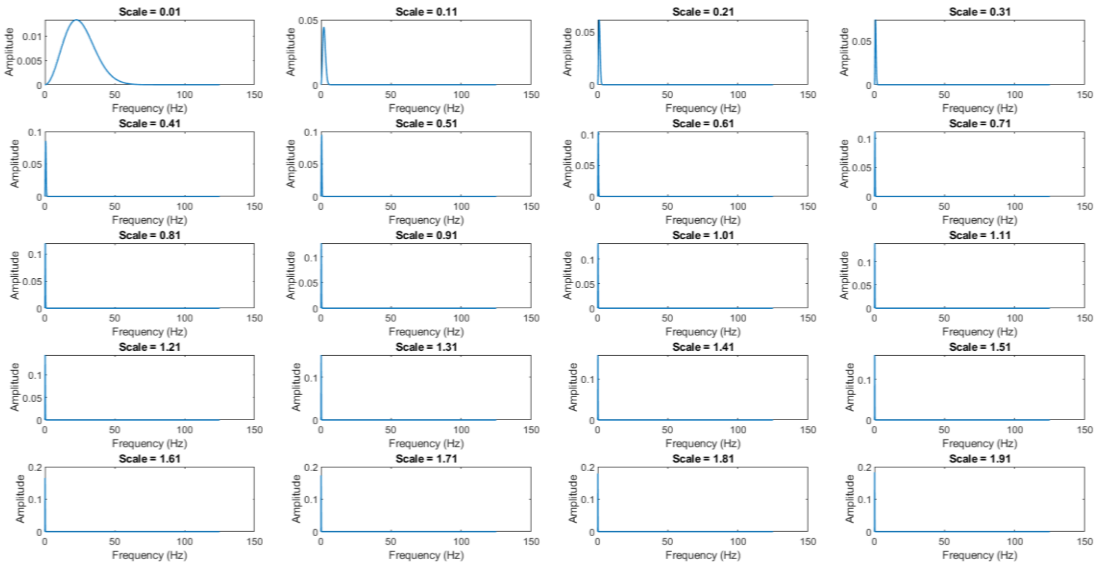

Figure 4: Spectrum of wavelets at different scalling factors

According to the above figures, we can identify bandpass filter characteristics that are included in wavelets. In addition to that when increasing the scaling factor the centre of frequency is reduced. Therefore, they have an inversely proportional relationship and with small scaling factors high-frequency components can be captured(Because when decreasing the scaling factor bandwidth is increasing and the centre of frequency shifts to the right side).

1.3 Continuous Wavelet Decomposition

According to the given instructions in the assignment following waveform should be created,

$$x(n) = \begin{cases} \sin(0.5\pi n) & 1 \leq n < \dfrac{3N}{2} \\ \sin(1.5\pi n) & \dfrac{3N}{2} \leq n < 3N \end{cases}$$

After following the instruction following figures are observed.

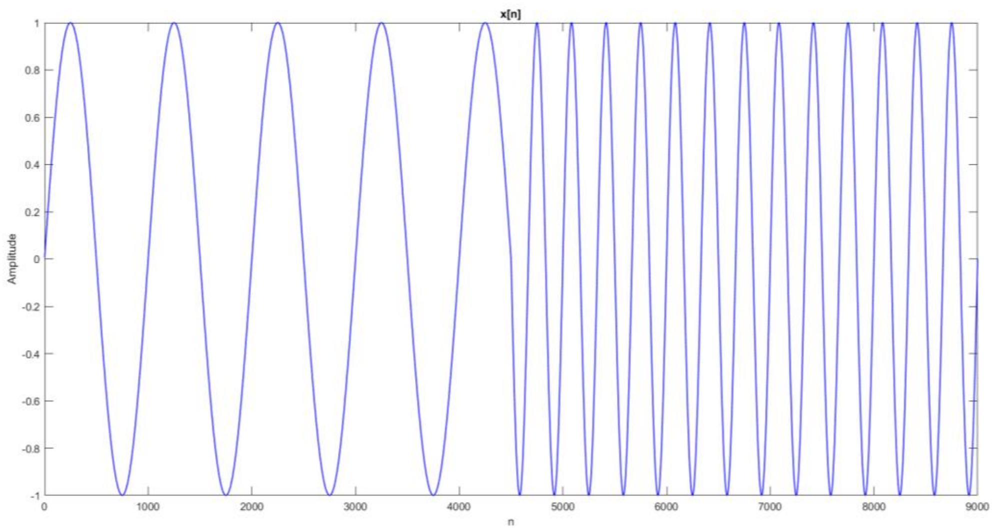

Figure 5:Input stationary signal x[n]

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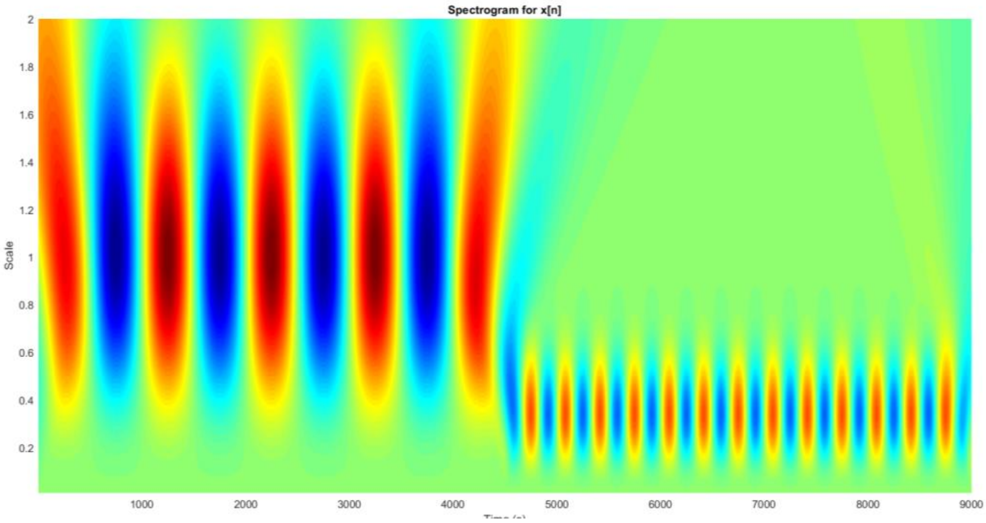

Figure 6: Spectrogram of x[n]

By observing the above spectrogram first half of the spectrogram's lower frequency parts are located near the higher scale values. The next half contains higher frequency parts and they are located in lower scale values. Wavelets which are near the scaling factor 1 have $0.5\pi\, rads^{-1}$(0.25 Hz) centre of frequency. Wavelets which are near the scaling factor 0.32 have $1.5\pi\, rads^{-1}$ (0.75 Hz)centre of frequency. Through the above figures, we can observe that a higher coefficient at higher scaling factors is related to low frequencies with low resolution and a lower coefficient at lower scaling factors is related to a higher frequency with higher resolution. Therefore continuous wavelet decomposition represents the time domain and frequency domain variation correctly.

2 Discrete Wavelet Transform

2.1 Introduction

The drawbacks of CWT include highly redundant computations which lead to the requirement of additional computational power and time consumption. Avoiding this, in discrete wavelet transform (DWT), the scaling and translation are performed discretely.

For DWT, the equation for CWT is modified as follows.

$$\psi_{m,n}(t) = \frac{1}{\sqrt{S_0^m}}\psi\left(\frac{t - n\tau_0 s_0^m}{S_0^m}\right)$$

Here S_0 represent the scaling step size and τ_0 represent the translation step size. Usually $S_0 = 2$ and $\tau_0 = 1$ are used for efficient analysis. m and n are corresponding multiplier integers.

2.2 Applying DWT with the Wavelet Toolbox in MATLAB

Waveforms are generated using the following instructions

$$x_1[n] = \begin{cases} 2\sin(20\pi n) + \sin(80\pi n) & 0 \le n < 512 \\ 0.5\sin(40\pi n) + \sin(60\pi n) & 512 \le n < 1024 \end{cases}$$

$$x_2[n] = \begin{cases} 1 & 0 \leq n < 64 \\ 2 & 192 \leq n < 256 \\ -1 & 256 \leq n < 512 \\ 3 & 512 \leq n < 704 \\ 1 & 704 \leq n < 960 \\ 0 & otherwise \end{cases}$$

After obtaining the above signals both are corrupted with AWGN of 10dB and the figure shows the corrupted signals called y_1 and y_2.

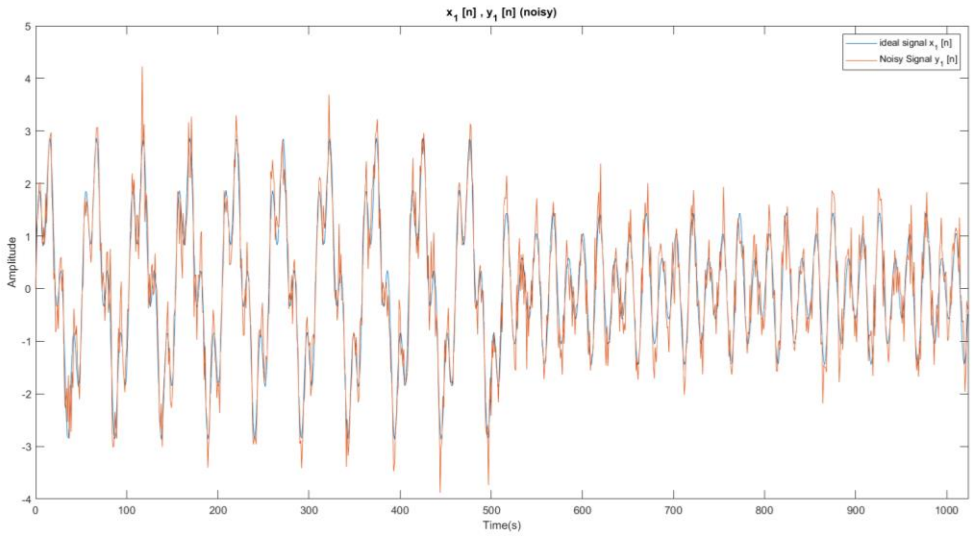

Figure 8: Noise y1 signal and Ideal x1 signal

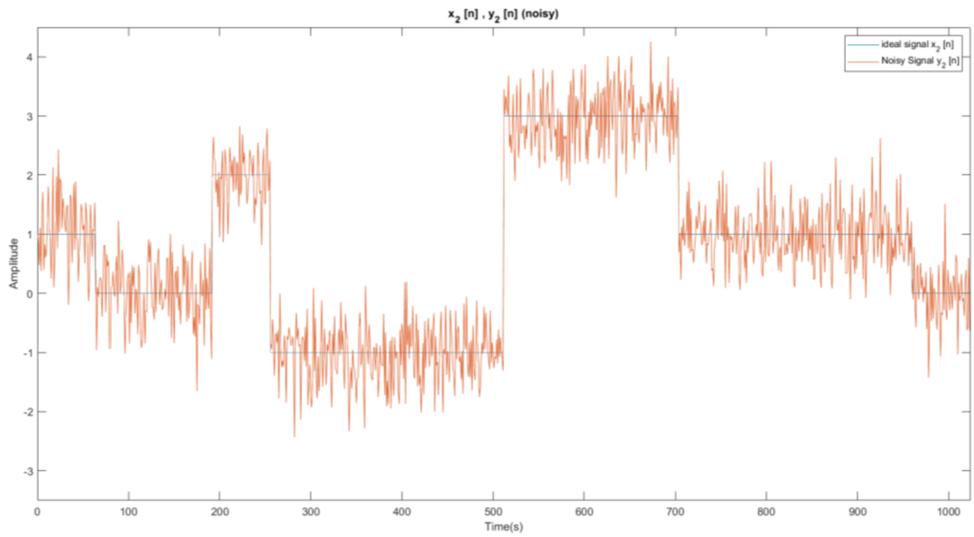

Figure 7: Noise y2 signal and ideal x2 signal

The following figure shows the morphology of the wavelet and scaling functions of Haar and Daubechies tap 9.

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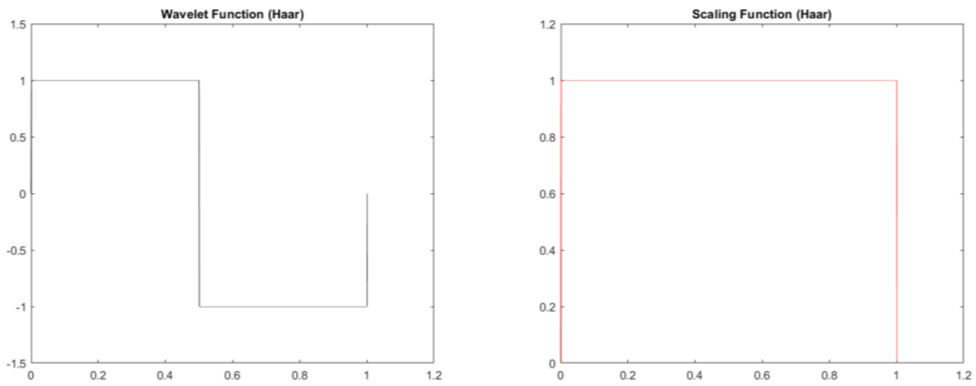

Figure 10: Wavelet function and Scalling function (Harr Wavelet)

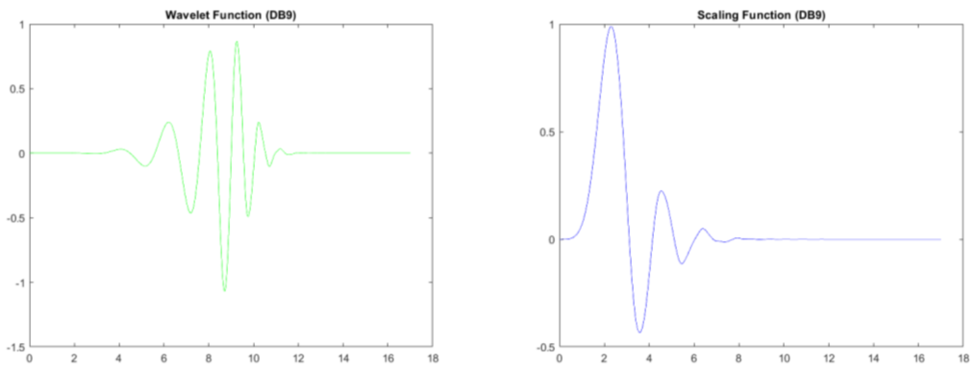

Figure 9: Wavelet function and Scalling function (DB9 Wavelet)

The following Figure 11 and Figure 14 show the wavelet properties of harr and db9 wavelets (the waveletAnalyzer tool is used for the observation)

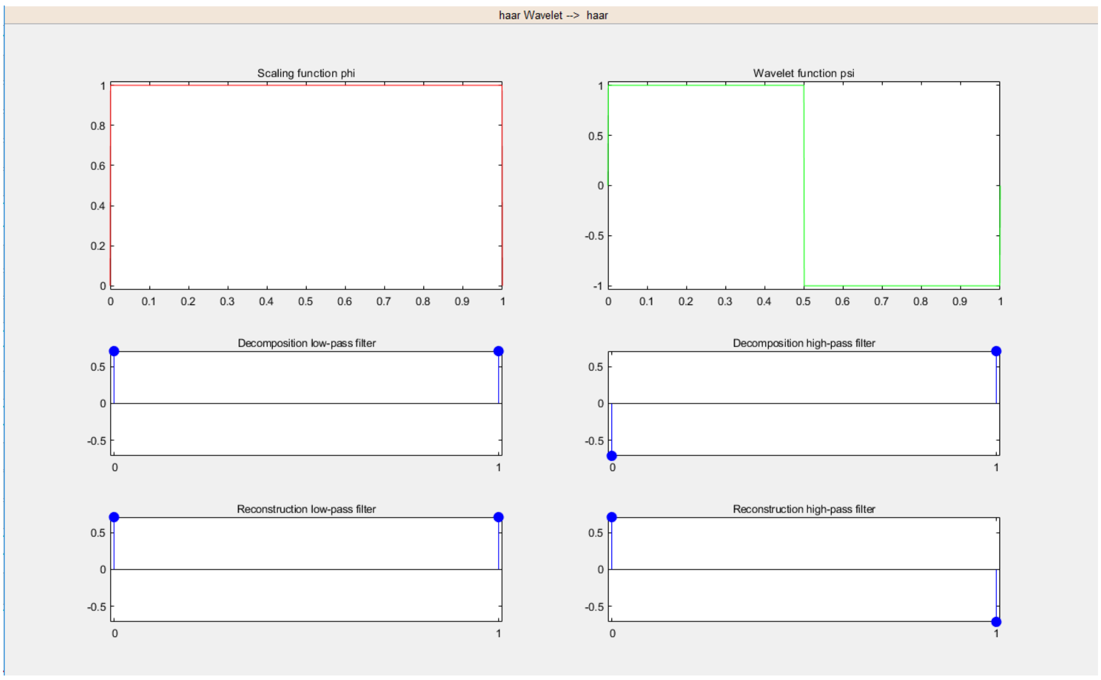

Figure 11: Wavelet Analyzer results(Harr Wavelet)

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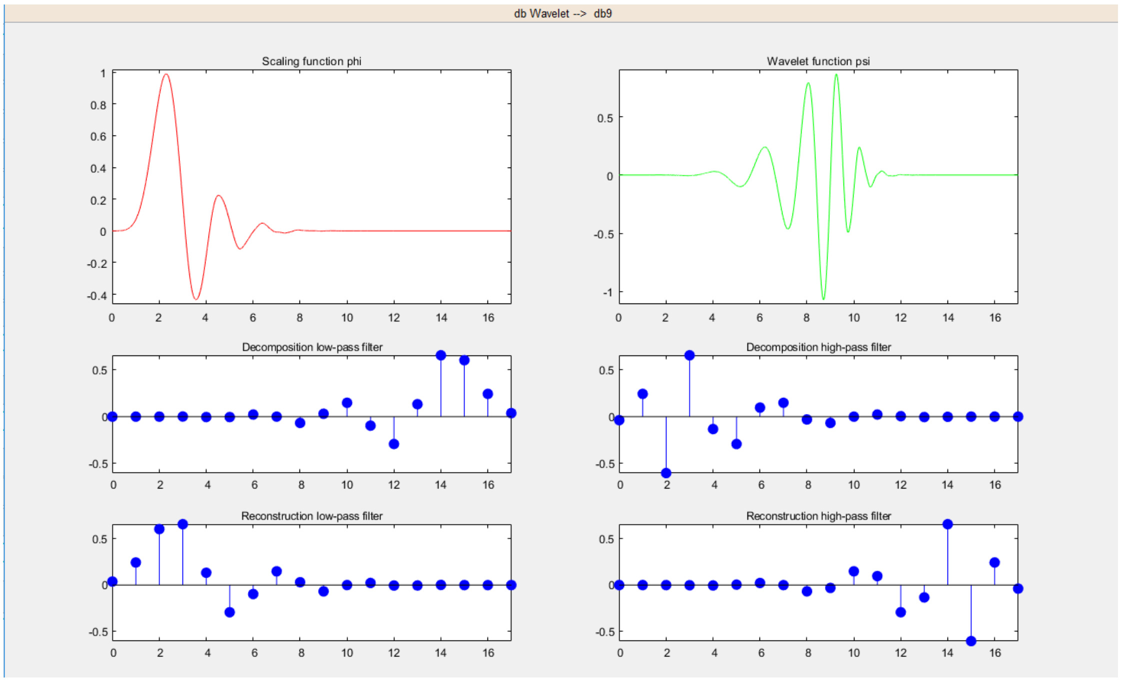

Figure 14: Wavelet Analyzer result (DB9 Wavelet)

Then 10-level wavelet decomposition of the signal using wavelets ‘db9’ and ‘haar’ is calculated using the wavedec() command.

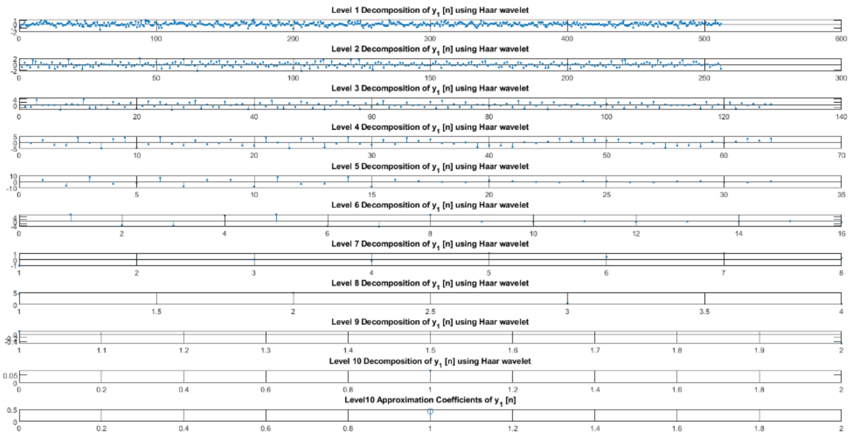

Figure 12: 10-level wavelet decomposition of y1 signal (Harr Wavelet)

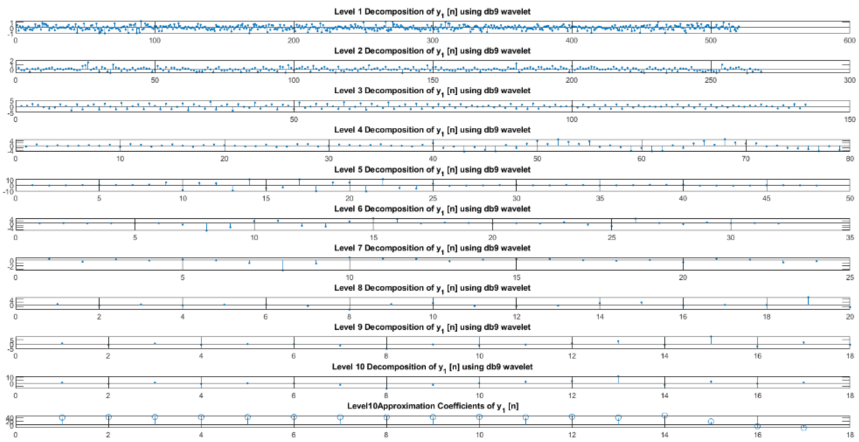

Figure 13: 10-level wavelet decomposition of y1 signal (Db9 Wavelet)

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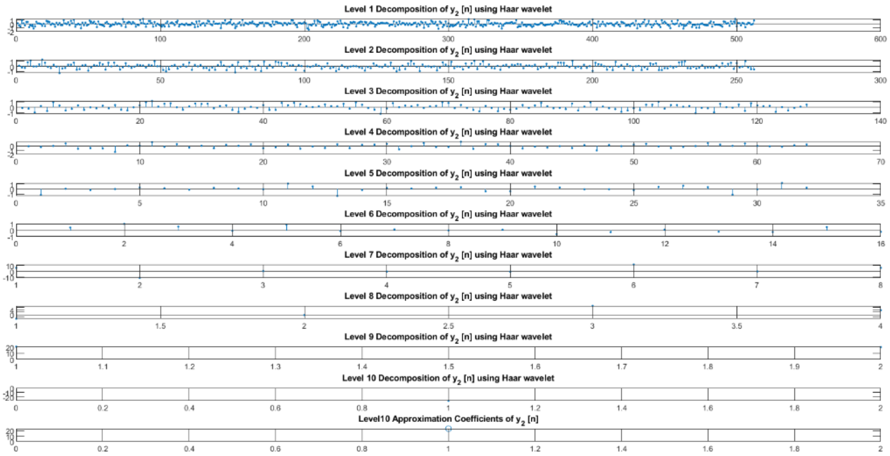

Figure 16: 10-level wavelet decomposition of y2 signal (Harr Wavelet)

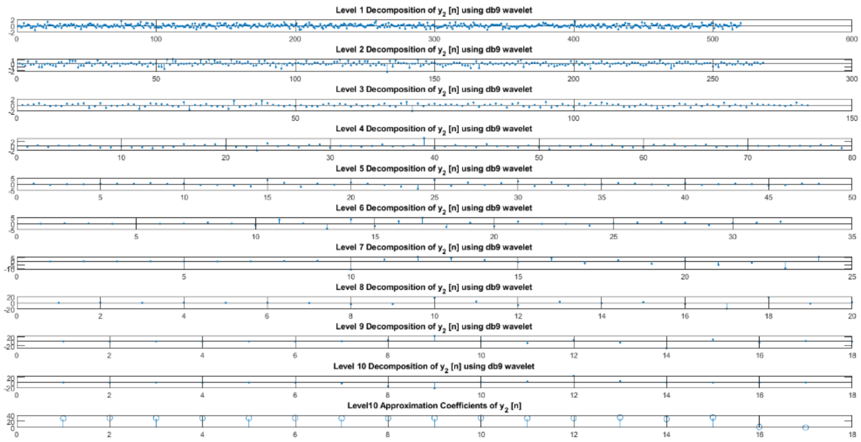

Figure 15: 10-level wavelet decomposition of y2 signal (Db9 Wavelet)

Inverse DWT is used to reconstruct A10, D10, D9, …., D2, and D1 using $y = \Sigma\ D\ i + A$. Here the reconstructed signal is much similar to the original signal. Then a recursive calculation method was used to obtain the signal.

A10 , D10 =>A9 | A9 , D9 => A8 | ….| A1 , D1 => original signal

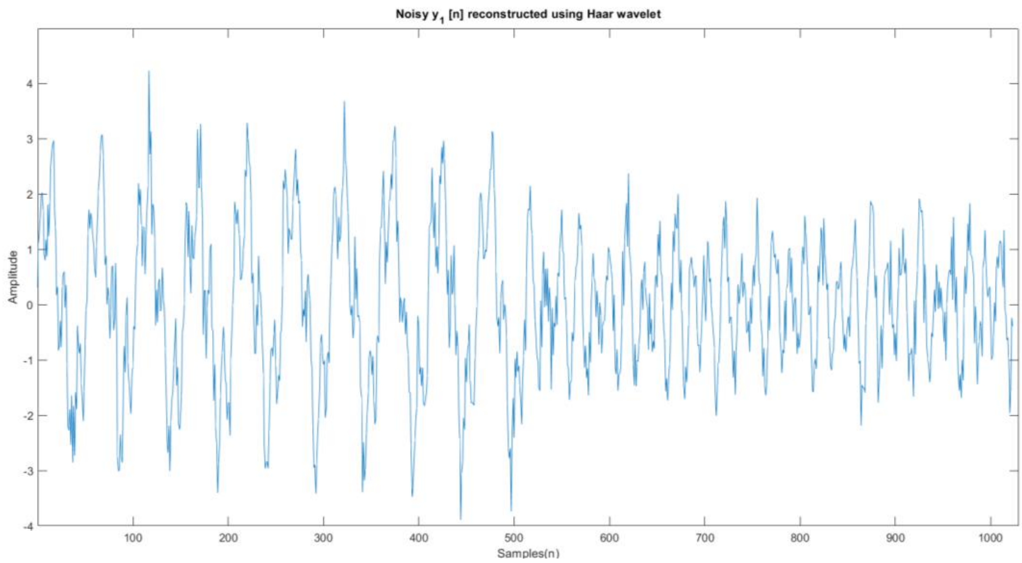

Figure 17: y1[n] signal reconstruction using Harr Wavelet

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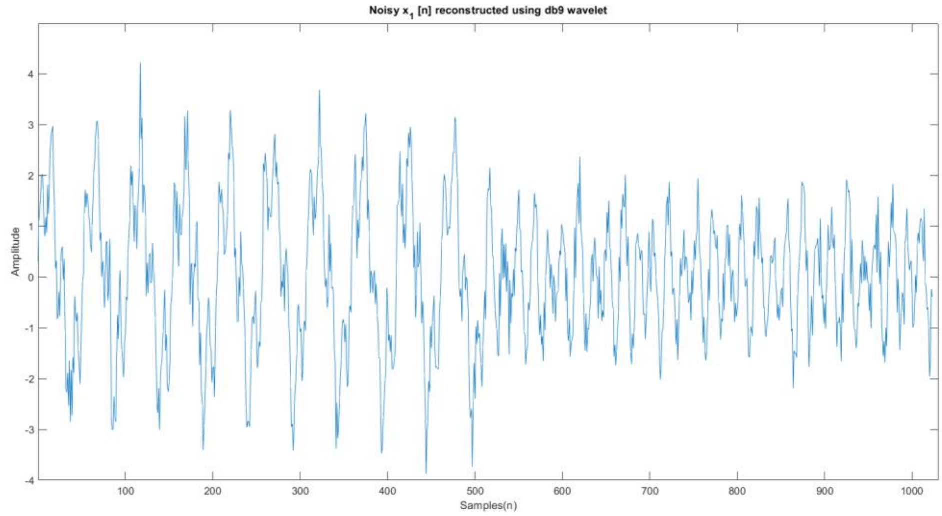

Figure 20: y1 signal reconstruction Db9 wavelet

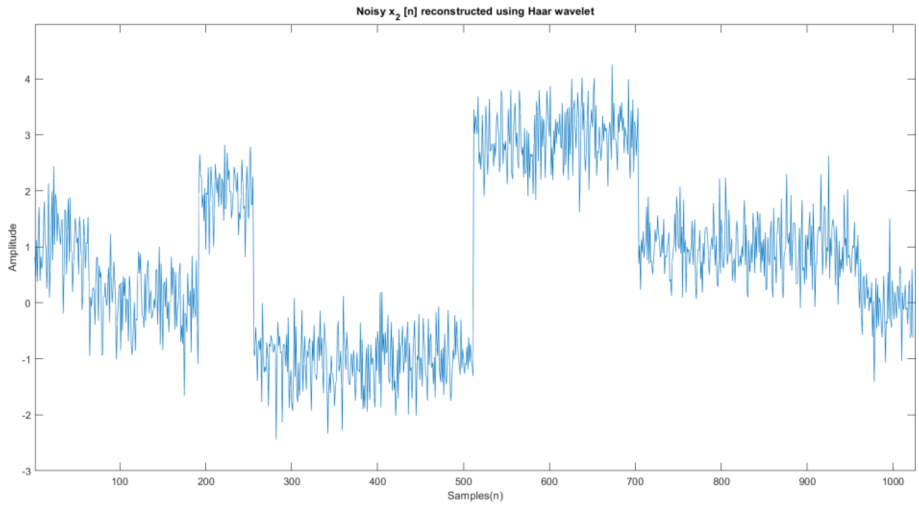

Figure 19: y2 signal reconstruction Harr wavelet

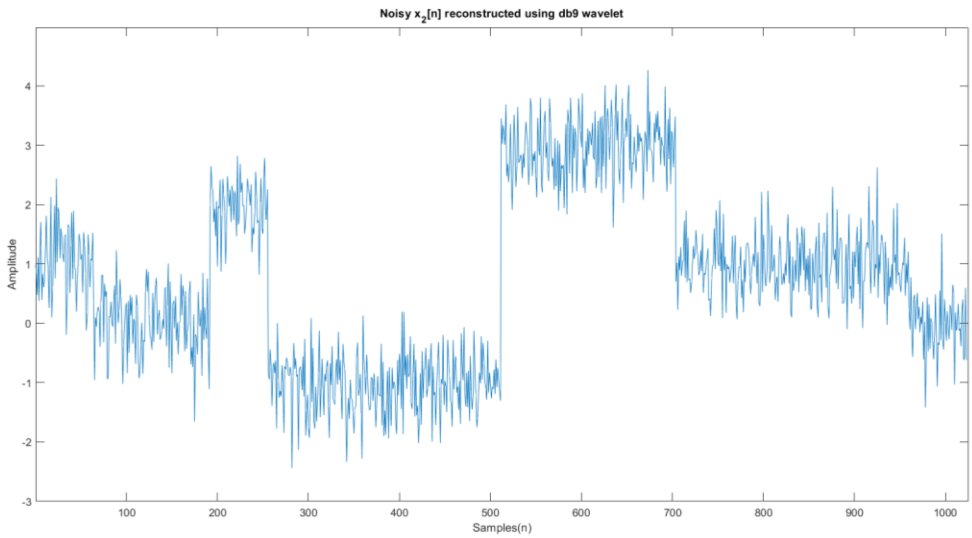

Figure 18: y2 signal reconstruction Db9 wavelet

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Table 1: Energy calculation of obtained signals

	y1[n] signal	y2[n] signal
The energy of the original signal	1815.8017	2783.4412
The energy of the reconstructed signal (Harr wavelet)	1815.8017	2783.4412
The energy of the reconstructed signal (Db9 wavelet)	1815.8017	2783.4412

According to the table, the reconstructed signal using both the harr wavelet and db9 wavelet has the same energy as the original signal. I proved that $y = \Sigma\, D\, i + A$ is reconstructed the original signal with minimum loss (approximately zero).

2.3 Signal Denoising with DWT

We can obtain a set of detailed coefficients at different translations for each scale for the given signal from Wavelet decomposition. Certain features are neglected by setting some coefficients to zero and then performing wavelet reconstruction. Those removing coefficients are selected by using a threshold value λ. (Coefficient values lower than the threshold value are considered as a noise component).

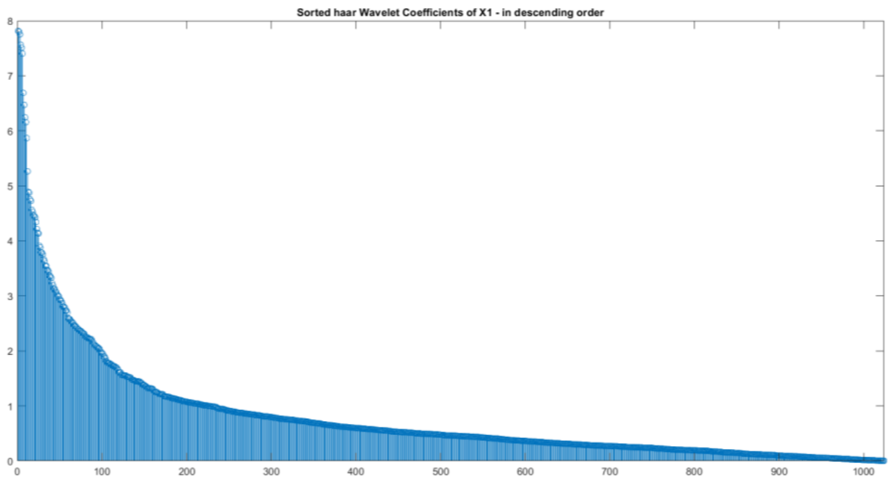

Figure 21: Harr Wavelet coefficients of x1 signal

The following figure shows the reconstructed noisy $x1$ signal with a harr and a db9 wavelet. Here the selected threshold value is 1.

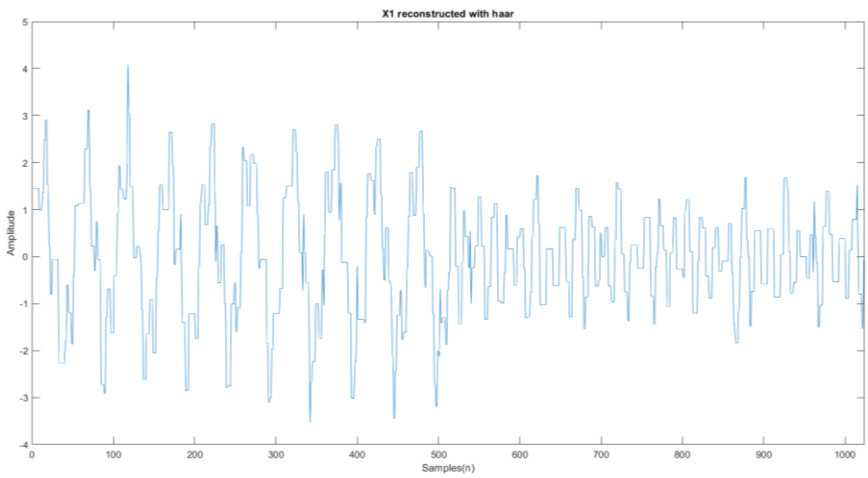

Figure 22: x1 reconstruction harr(RMSE =0.42585,Threshold=1)

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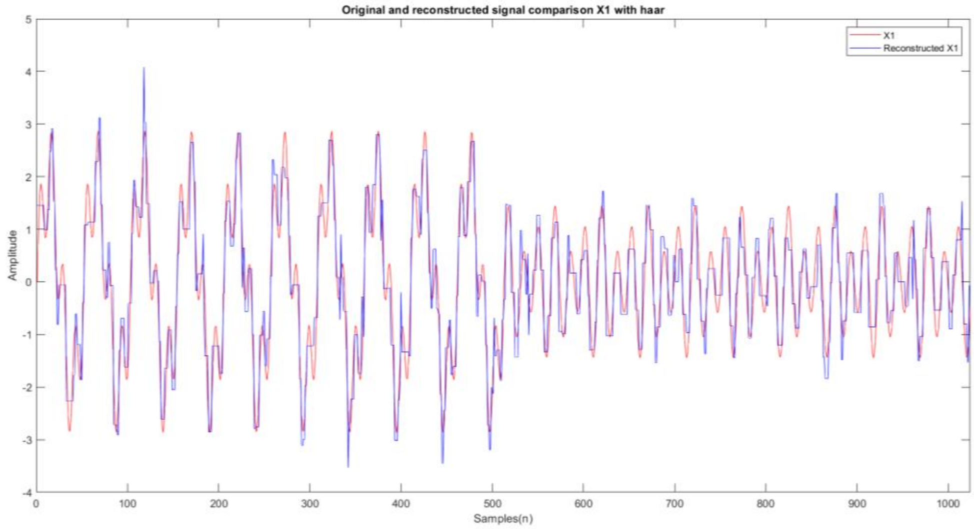

Figure 23: Original and reconstructed signal comparison (Harr)

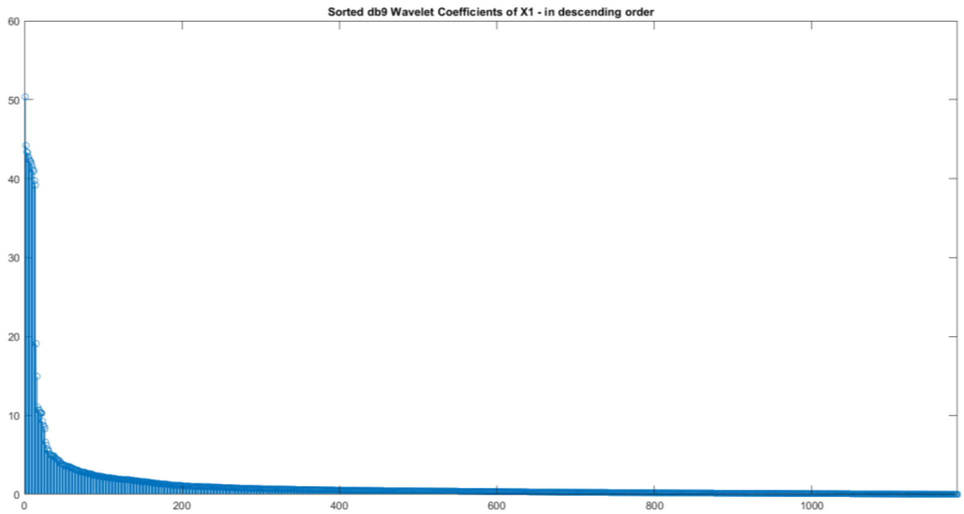

Figure 24: DB9 Wavelet coefficients of x1 signal

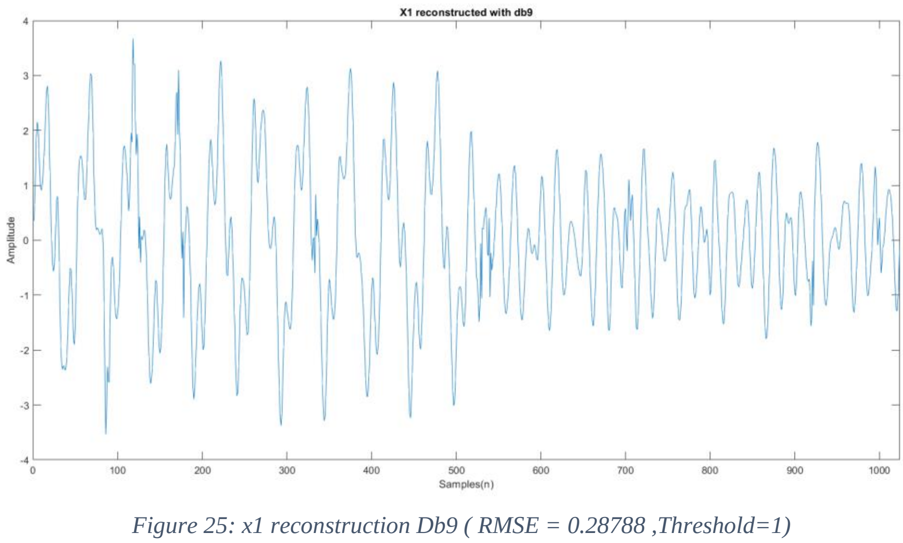

Figure 25: x1 reconstruction Db9 (RMSE = 0.28788 ,Threshold=1)

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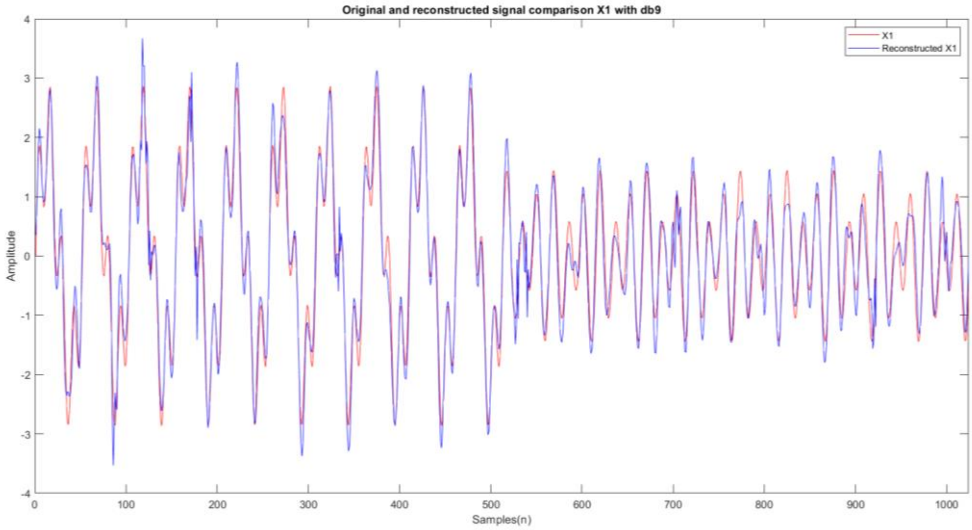

Figure 26: Original and reconstructed signal comparison (DB9)

Table 2: RMSE comparison of two methods

Wavelet	Root Mean Squared Error (RMSE)
Harr	0.42585
Db9	0.28788

According to the above figures and tables, the Db9 wavelet performed well in the $y1$ signal compared to the Harr wavelet. The reason for this is having rapid transitions in the Harr wavelet and smooth transitions in the Db9 wavelet. Therefore for reconstructing sinusoidal varying signals most suitable selection among the two is the db9 wavelet. The above figures show the Db9 reconstructed signal is more similar to the original signal.

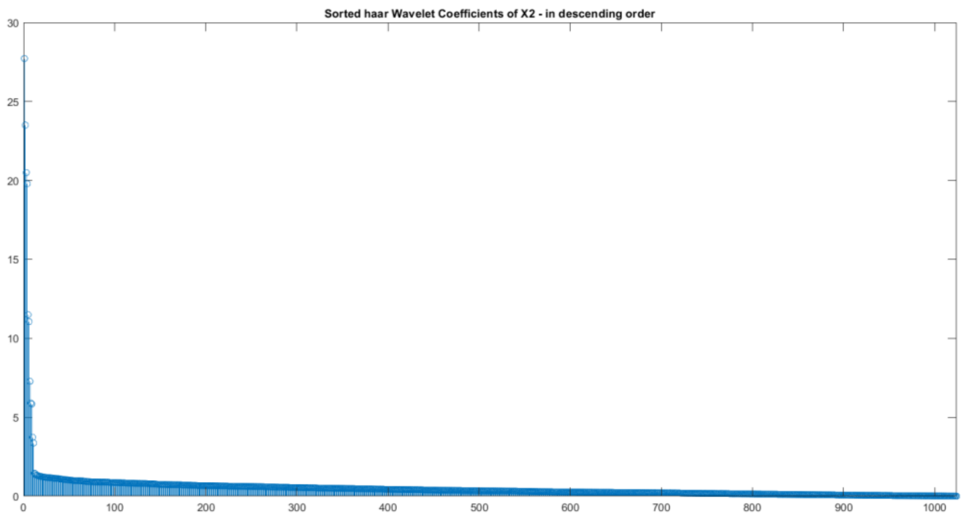

Figure 27: Harr Wavelet coefficients of x2 signal

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The following figure shows the reconstructed noisy $x2$ signal with a harr and a db9 wavelet. Here the selected threshold value is 2.

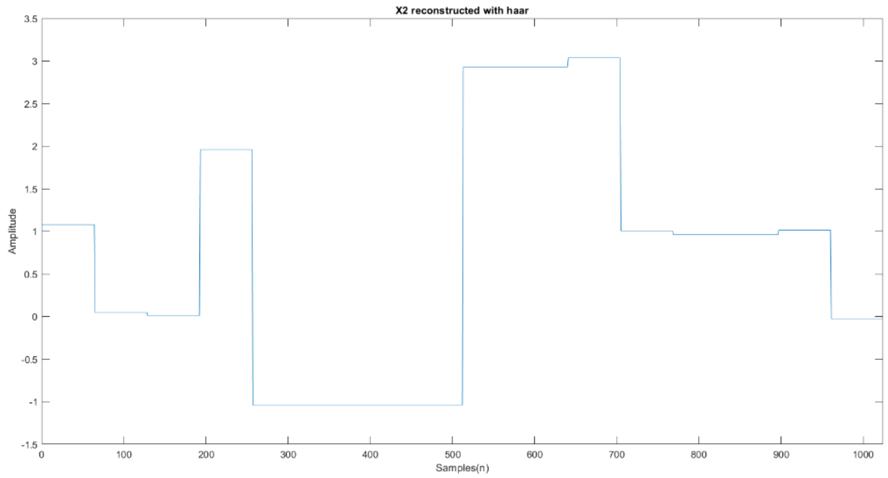

Figure 29: x1 reconstruction harr(RMSE =0.04426, Threshold=2)

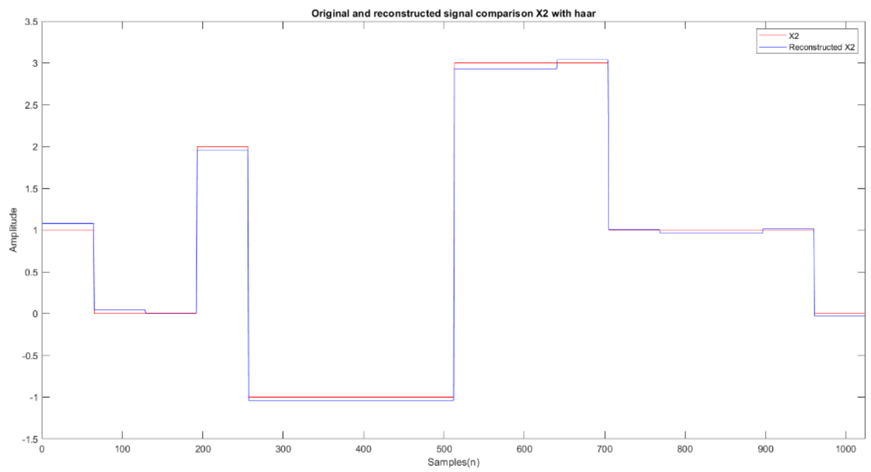

Figure 28: Original and reconstructed signal comparison (Harr)

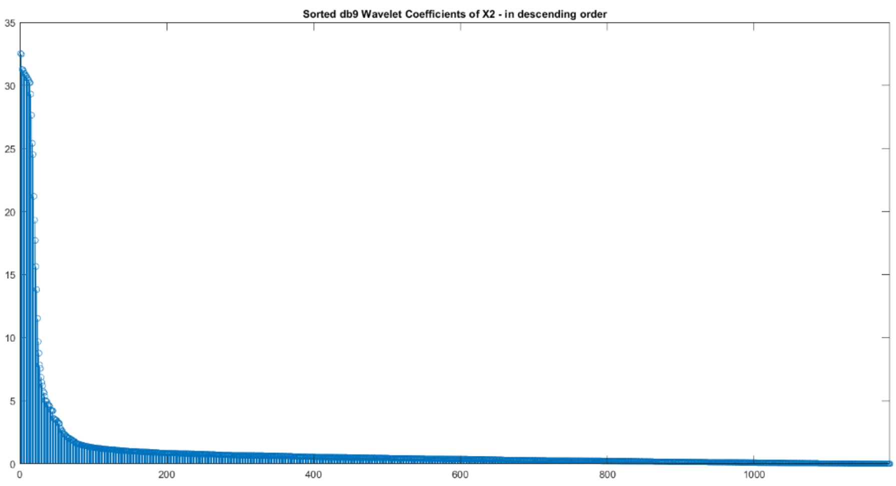

Figure 30: DB9 Wavelet coefficients of x2 signal

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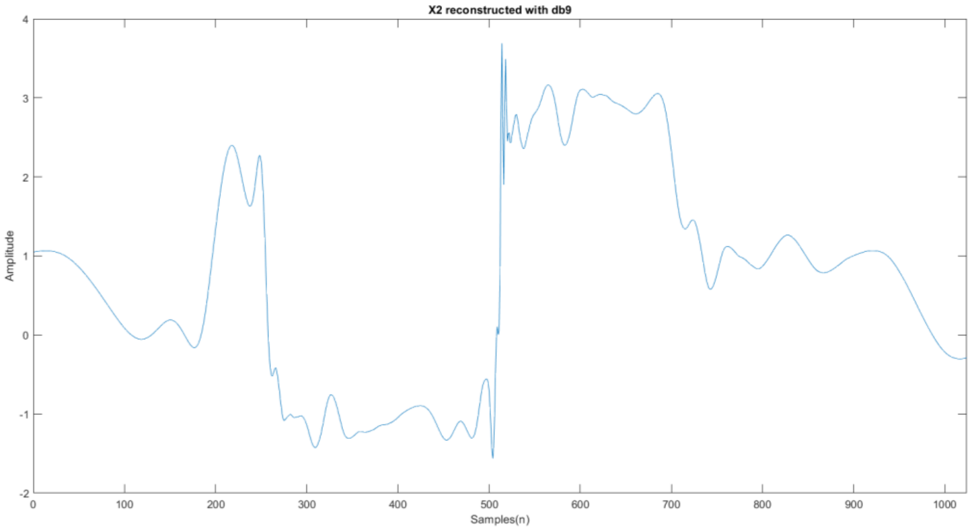

Figure 31: x2 reconstruction Db9 (RMSE = 0.28607, Threshold=2)

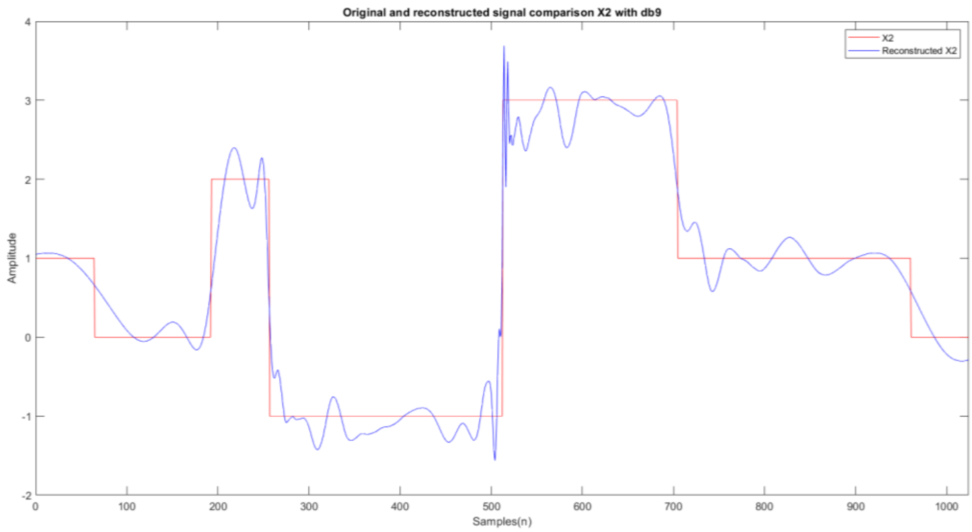

Figure 32: Original and reconstructed signal comparison (DB9)

Table 3: RMSE comparison of two methods

Wavelet	Root Mean Squared Error (RMSE)
Harr	0.04426
Db9	0.28607

According to the above figures and tables, the Harr wavelet performed well in the $y2$ signal compared to the Db9 wavelet. The reason for this is having rapid transitions in the Harr wavelet and smooth transitions in the Db9 wavelet. Therefore for reconstructing rapid transition signals most suitable selection among the two is the Harr wavelet. The above figures show the Harr reconstructed signal is more similar to the original signal.

Signal	Most suitable wavelet
y1	Db9 Wavelet
y2	Harr Wavelet

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2.4 Signal Compression with DWT

The negligibly small wavelet coefficients can be considered noise, but in the signal compression process, coefficients are kept until those components preserve the given percentage of the energy. Other components are removed by setting them to zero. For the signal compression process aVR lead of an ECG sampled at 257 Hz in ‘ECGsig.mat’ is given and they are applied the two wavelet transforms. The maximum number of dyadic wavelet decompositions that can be applied is equal to log2 (2570) ≈ 12. The following figures show the time domain representations and the wavelength decomposition coefficients for both haar and db9 wavelet cases.

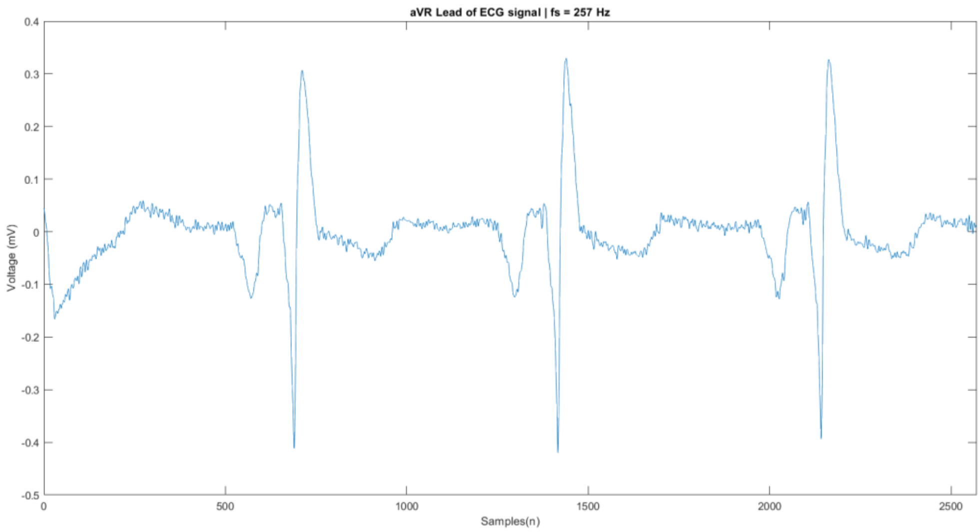

Figure 33: aVR signal in Time domain

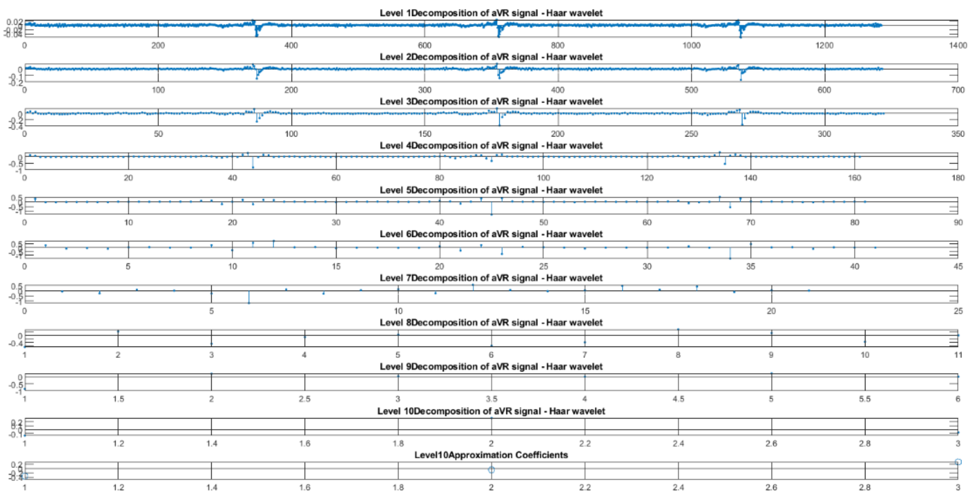

Figure 34: aVR decompossion Harr Wavelet

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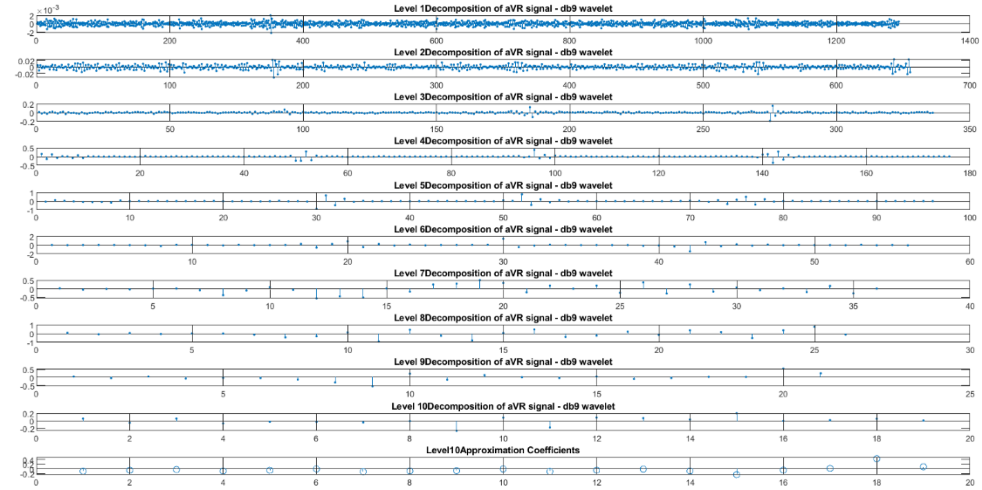

Figure 36: aVR decompossion DB9 Wavelet

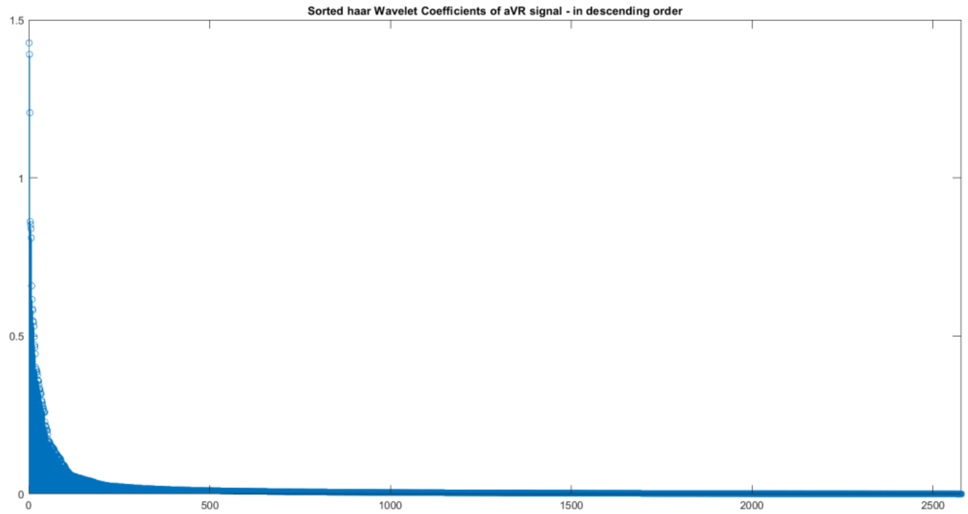

Figure 37: Sorted coefficient (Harr)

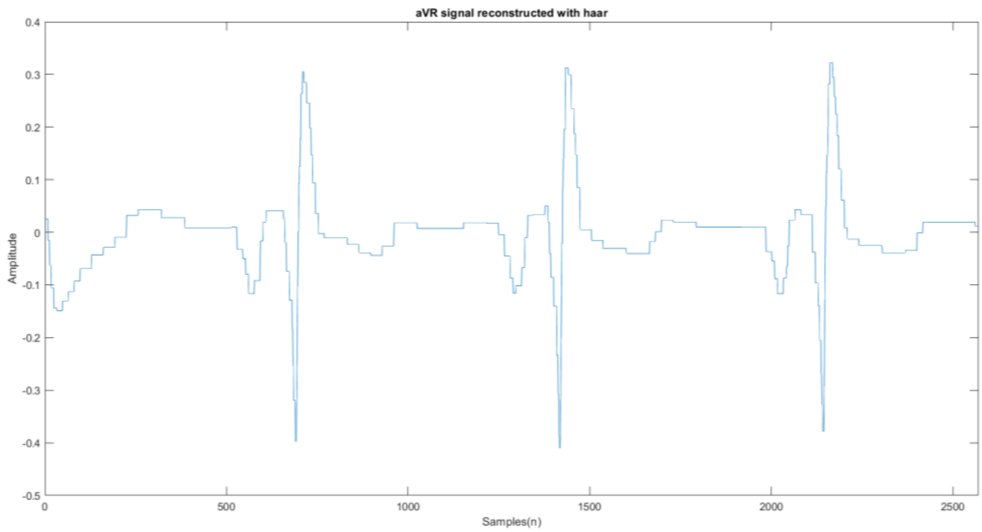

Figure 35: Constructed aVR using compressed data (Harr)

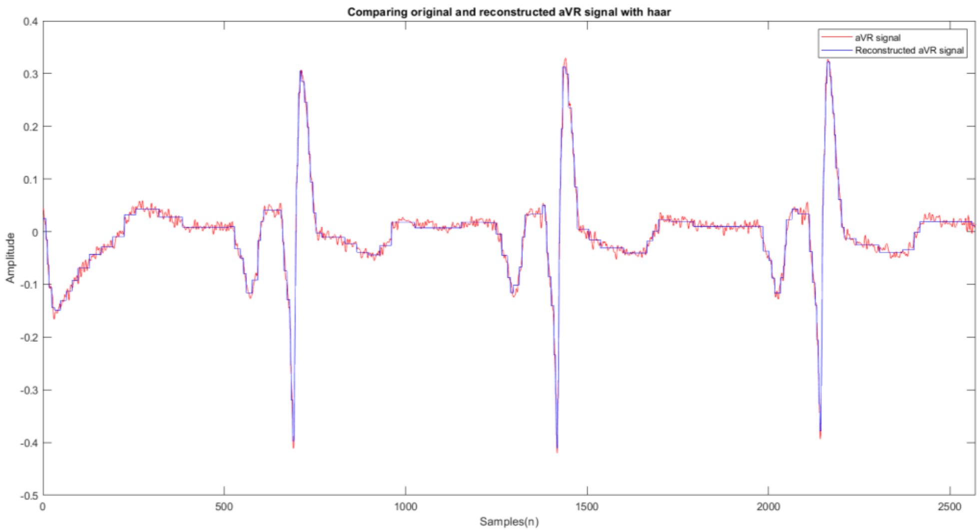

Figure 38: Signal comparison (Harr wavelet)

Compression Ratio = 18.6232: 1
Number of coefficients required for 99% of the energy of the signal = 138
RMSE (Harr Wavelet): 9.5818×10^{-3}

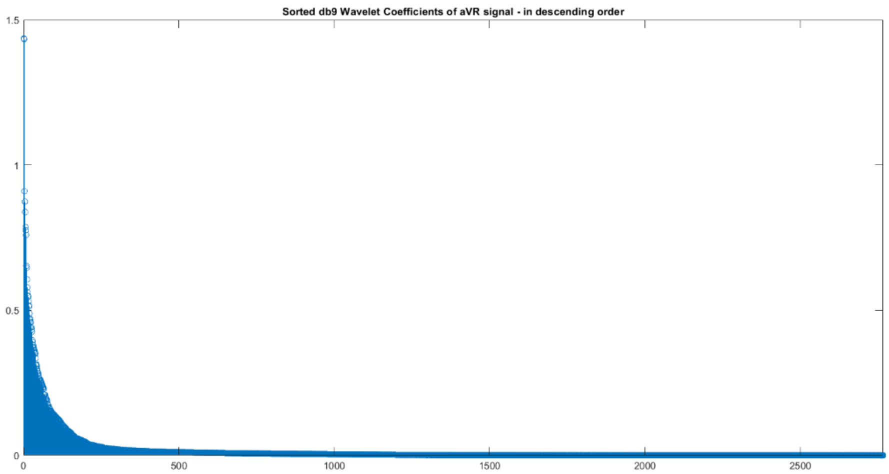

Figure 39: Sorted coefficient (DB9)

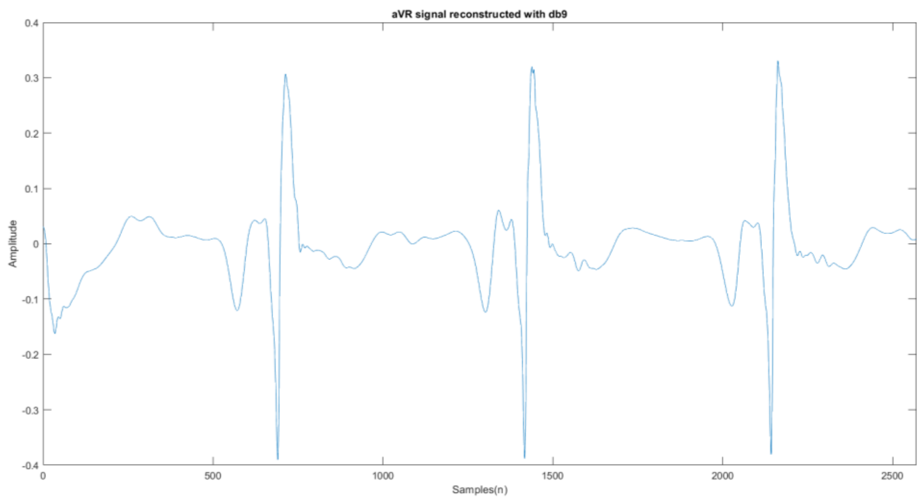

Figure 40: Constructed aVR using compressed data (DB9)

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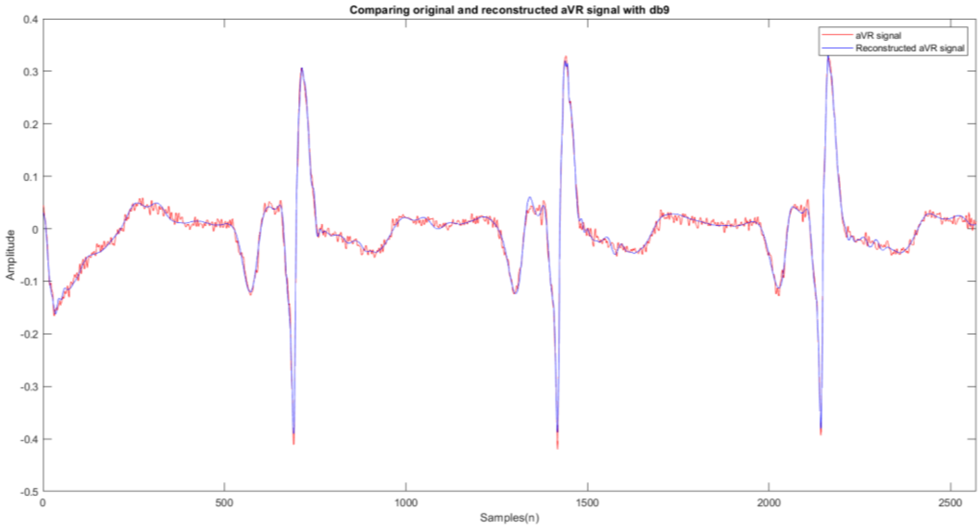

Figure 41: Signal comparison (DB9 wavelet)

Compression Ratio: 15.9627: 1
Number of coefficients required for 99% of the energy of the signal = 161
RMSE (DB9 Wavelet): 8.1062×10^{-3}

Wavelet	Compression Ratio	Number of coefficients	RMSE
Harr Wavelet	18.6232: 1	138	9.5818×10^{-3}
DB9 Wavelet	15.9627: 1	161	8.1062×10^{-3}

Table 4: Summary of signal compression

According to the above observations, The RMSE value of the db9 wavelet is comparatively smaller than the RMSE value of the Harr wavelet. Also, the reconstructed signal using the db9 wavelet is more similar to the original signal. As mentioned previously very small coefficients have low energy and those can be considered noise components. Therefore, they are removed in this process. In addition to that harr wavelet claims a higher compression ratio compared to the db9 wavelet.

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