1 Motivation

Highway System

Following figure shows the highway system in a city. A road inspector, a person who is responsible for inspecting, must travel all of these roads and file reports on road conditions, visibility of lines on the roads, status of traffic signs, and so on. Since the road inspector lives in city A, the most economical way to inspect all of the roads would be to start in city A, travel each of the roads exactly once, and return to city A. Is this possible?

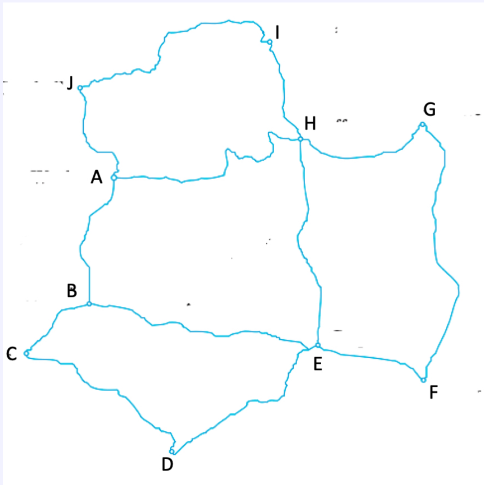

Figure 1.1

- The problem can be modeled as a graph as follows.

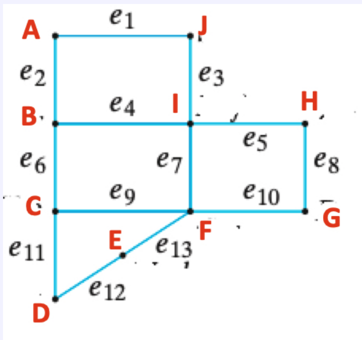

Figure 1.2

- The dots in above figure are called vertices, and the lines that connect the vertices are called edges.
- If we start at a vertex A, travel along an edge to vertex B, travel along another edge to vertex C, and so on, and eventually arrive at vertex J, we call the complete tour a path from A to J.
- Suppose that there is a path start in city A, travel each of the roads exactly once, and return to city A.

- Consider city B. Each time we arrive at city B on some edge, we must leave it on a different edge.
- Furthermore, every edge that touches city B must be used.
- Thus an even number of edges must touch city B .
- Since three edges touch city B ,a contradiction occurs.
- Therefore, there is no path from city A to city A in highway system that traverses every road exactly once.

2 Graph and Di Graph

Definition 2.0.1 (Undirected Graph) *A graph (or undirected graph) G consists of a set V of ver tices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an unordered pair of vertices.*

Definition 2.0.2 (Directed Graph) *A directed graph (or digraph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an ordered pair of vertices.*

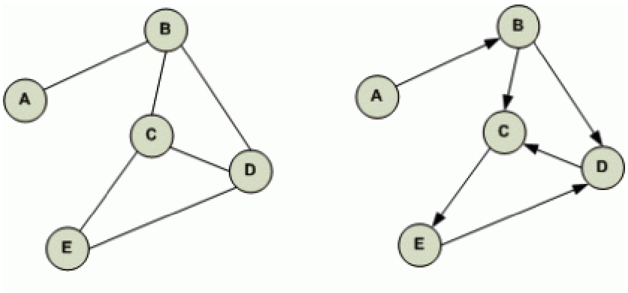

Figure 2.3: Undirected Graph and Di Graph

- If edge e is associated with the vertices v and w in an undirected graph, we write $e = (v, w)$ or $e = (w, v)$. (order doesn't matter)
- If e is associated with the ordered pair (v, w) of vertices in directed graph, we write $e = (v, w)$, which denotes an edge from v to w.
- In a directed graph, the directed edges are indicated by arrows.
- An edge e in a graph (undirected or directed) that is associated with the pair of vertices v and w is said to be incident on v and w, and v and w are said to be incident on e and to be adjacent vertices.
- If G is a graph (undirected or directed) with vertices V and edges E, we write $G = (V, E)$.

Example 2.1 *In the graph shown in figure* 1.2 *consists of the set*

$$V = \{A, B, C, D, E, F, G, H, I, J\}$$

of vertices and the set

$$E = \{e_1, e_2, \ldots, e_{13}\}$$

of edges.

- *Edge e_1 is associated with the unordered pair $\{A, J\}$ of vertices.*
- *Edge e_1 is denoted (A, J) or (J, A). Edge e_4 is incident on B and I, and the vertices B and I are adjacent.*

- Distinct edges to be associated with the same pair of vertices are called parallel edges.
- An edge incident on a single vertex is called a loop.
- A vertex that is not incident on any edge is called an isolated vertex.
- A graph with neither loops nor parallel edges is called a simple graph.

Example 2.2 *Find the parallel edges, loops and isolated vertices, if there any, of following graph.*

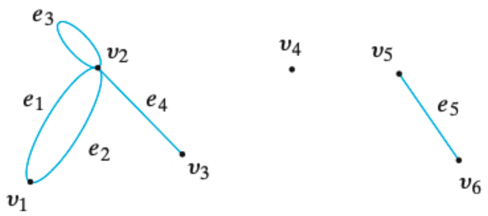

Figure 2.4

Example 2.3 *Frequently in manufacturing, it is necessary to bore many holes in sheets of metal as shown in figure below. Components can then be bolted to these sheets of metal. The holes can be drilled using a drill press under the control of a computer. To save time and money, the drill press should be moved as quickly as possible. This situation is modeled as a graph shown below. In the graph, the the numbers on the edges are indicated the time to move the drill press.*

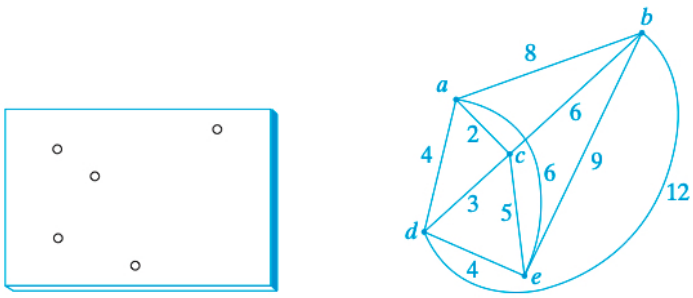

Figure 2.5

- *A graph with numbers on the edges (such as above graph) is called a weighted graph.*
- *If edge e is labeled k, we say that the weight of edge e is k. As an example, the weight of edge (c, e) is 5.*

- *In a weighted graph, the length of a path is the sum of the weights of the edges in the path. As an example, the length of the path that starts at a, visits c, and terminates at b is 8.*

Find a path of minimum length that visits every vertex exactly one time (the optimal path for the drill press to follow).

3 Paths and Cycles

Definition 3.0.1 *Let v_0 and v_n be vertices in a graph. A path from v_0 to v_n of length n is an alternating sequence of $n+1$ vertices and n edges beginning with vertex v_0 and ending with vertex v_n,*

$$(v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n),$$

in which edge e_i is incident on vertices v_{i-1} and v_i for $i = 1, \ldots, n$.

Example 3.1 *Find the path of length 4 from vertex 1 to vertex 2.*

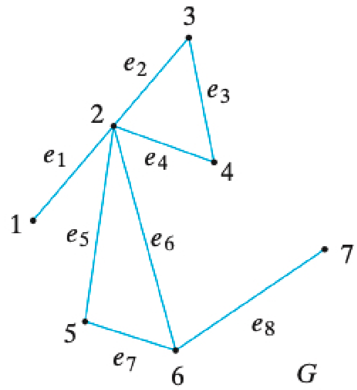

Figure 3.6

Definition 3.0.2 *A graph G is connected if given any vertices v and w in G, there is a path from v to w.*

Definition 3.0.3 *Let $G = (V, E)$ be a graph. We call (V', E') a subgraph of G if*

(i) $V' \subset V$ and $E' \subset E$.

(ii) For every edge $e' \in E'$, if e' is incident on v' and w', then $v', w' \in V'$.

Definition 3.0.4 *Let G be a graph and let v be a vertex in G. The subgraph G' of G consisting of all edges and vertices in G that are contained in some path beginning at v is called the component of G containing v.*

Example 3.2 *Consider the graph shown in figure* 3.7

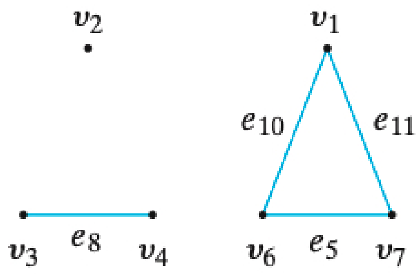

Figure 3.7

- *The component of the graph containing v_2 is the subgraph*

$$G_1 = (V_1, E_1)\ ,\ V_1 = \{v_2\}\ ,\ E_1 = \emptyset$$

- *The component of the graph containing v_1 is the subgraph*

$$G_2 = (V_2, E_2)\ ,\ V_2 = \{v_1, v_6, v_7\}\ ,\ E_1 = \{e_5, e_{10}, e_{11}\}$$

Definition 3.0.5 *Let v and w be vertices in a graph G.*
A simple path from v to w is a path from v to w with no repeated vertices.
A cycle (or circuit) is a path of nonzero length from v to v with no repeated edges.
A simple cycle is a cycle from v to v in which, except for the beginning and ending vertices that are both equal to v, there are no repeated vertices.

Example 3.3 *Check whether the following given paths of the graph in figure* 3.6 *are simple path, cycle or simple cycle.*

(a) $(6, 5, 2, 4, 3, 2, 1)$

(b) $(6, 5, 2, 4)$

(c) $(2, 6, 5, 2, 4, 3, 2)$

(d) $(5, 6, 2, 5)$

(e) (7)

- The **degree of a vertex** v, $\delta(v)$, is the number of edges incident on v.
- In a directed graph the in-degree of a vertex is the number of edges coming to the vertex and the out-degree of a vertex is the number of edges leaving the vertex.

Theorem 3.0.1 *If G is a graph with m edges and vertices $\{v_1, v_2, \ldots, v_n\}$, then*

$$\sum_{i=1}^{n} \delta(v_i) = 2m.$$

In particular, the sum of the degrees of all the vertices in a graph is even.

Theorem 3.0.2 *In any graph, the number of vertices of odd degree is even.*

Theorem 3.0.3 *A graph has a path with no repeated edges from* v *to* w $(v \neq w)$ *containing all the edges and vertices if and only if it is connected and* v *and* w *are the only vertices having odd degree.*

Theorem 3.0.4 *If a graph* G *contains a cycle from* v *to* v, G *contains a simple cycle from* v *to* v.

[0]**REFERENCES**

(i) *Discrete Mathematics*, Richard Johnsonbaugh.

(ii) *Discrete Mathematics and its Applications*, Kenneth H. Rosen