Theorem 3.0.1 (Handshaking Theorem)**.** *If G is a graph with m edges and vertices $\{v_1, v_2, \ldots, v_n\}$, then*

$$\sum_{i=1}^{n} \delta(v_i) = 2m.$$

In particular, the sum of the degrees of all the vertices in a graph is even.

Proof. Since the degree of a vertex is the number of edges incident with that vertex, the sum of degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices,each edge gets counted twice,once at each end. Thus the sum of the degrees is equal twice the number of edges. □

Note: This theorem applies even if multiple edges and loops are present.

Theorem 3.0.2. *In any graph, the number of vertices of odd degree is even.*

Proof. Let V_1 be the set of vertices of even degree and V_2 be the set of vertices of odd degree in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} \delta(v) = \sum_{v \in V_1} \delta(v) + \sum_{v \in V_2} \delta(v)$$

Since $\delta(v)$ is even for $v \in V_1$, $\sum_{v \in V_1} \delta(v)$ is even.

Also, the sum two summation in the previous equation is even by the Handshaking theorem.

Hence, $\sum_{v \in V_2} \delta(v)$ must be even.

Since $\delta(v)$ is odd for $v \in V_2$, there should be even number of vertices in the set V_2. Thus, there are an even number of vertices of odd degree.

□

Theorem 3.0.3. *A graph has a path with no repeated edges from v to w $(v \neq w)$ containing all the edges and vertices if and only if it is connected and v and w are the only vertices having odd degree.*

Proof. Left for the student □

Theorem 3.0.4. *If a graph G contains a cycle from v to v, G contains a simple cycle from v to v.*

Proof. Left for the student □

4 Euler Path, Cycle and Graph

- ***Euler path*** in a graph G is a path in a finite graph that visits every edge exactly once.
- An Euler path that starts and ends on the same vertex of G is called an ***Euler cycle***.
- The ***Euler graph*** is a graph with an Euler circuit.

Example 4.1 (Königsberg Bridge Problem)**.** *The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands - Kneiphof and Lomse - which were connected to each other, or to the two mainland portions of the city, by seven bridges.*

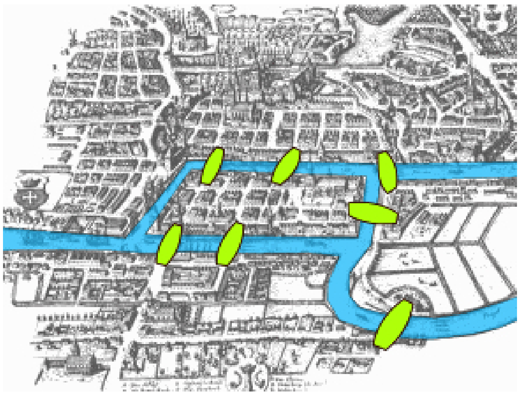

Figure 4.1: Map of Königsberg in Euler's time

The problem was to devise a path through the city that would cross each of those bridges once and only once. Euler proved that the problem has no solution.

- *The bridge configuration can be modeled as a graph, as shown in Figure below.*
- *The vertices represent the locations and the edges represent the bridges.*
- *The Königsbergg bridge problem is now reduced to finding a cycle in the graph of figure 4.2 that includes all of the edges and all of the vertices.*

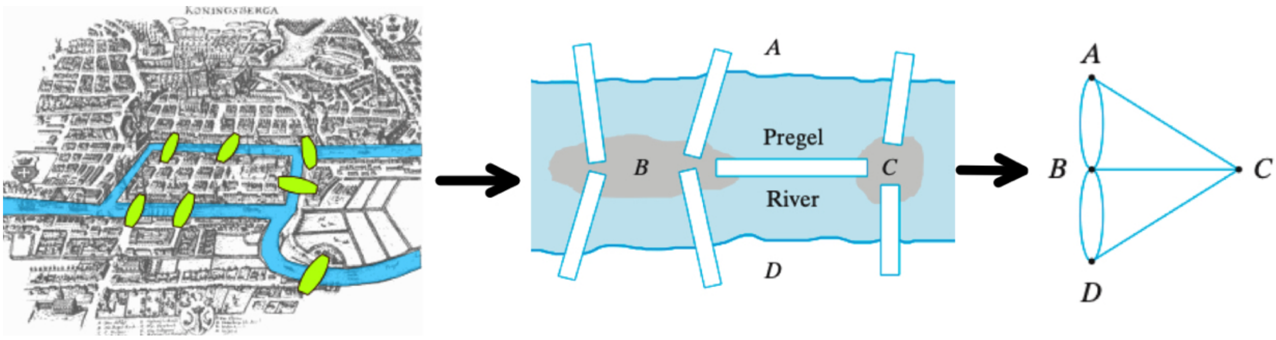

Figure 4.2

- *We see that there is no Euler cycle in the graph of figure 4.2 because the number of edges incident on vertex A is odd.*

Theorem 4.0.1. *If a graph G has an Euler cycle, then G is connected and every vertex has even degree.*

Theorem 4.0.2. *If G is a connected graph and every vertex has even degree, then G has an Euler cycle.*

- Fleury's Algorithm is used to display the Euler path or Euler circuit from a given graph.

Algorithm 4.0.1 (Fleury's Algorithm)**.** *Let $G = (V, E)$ be connected graph in which degree of each vertex is even*

Step 1: Initially, $i = 0$, start from vertex v_0 and define $T_0 = (v_0)$

Step 2: Let $T_i = (v_0, e_1, v_1, e_2, \ldots, e_i, v_i)$ be the path between v_0 and v_i at stage i. Select an edge e_{i+1} joining v_i and v_{i+1} from the set $E_i = E - \{e_1, e_2, \ldots, e_i\}$. If edge e_{i+1} is an edge in the subgraph obtain from G after deleting the edges belonging to E_i from E, select it for inclusion in the updated path $T_{i+1} = (v_0, e_1, v_1, e_2, \ldots, e_i, v_i, e_{i+1}, v_{i+1})$ only if there is no other choice. If there is no such edge, stop

Step 3: Replace i by $i + 1$ and go to Step 2.

Example 4.2. *Use Fleury's Algorithm to find the Euler cycle of following graph.*

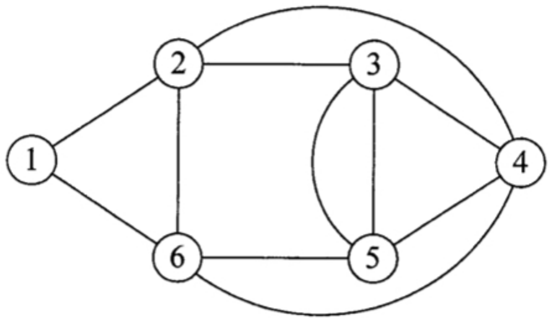

Figure 4.3

- Directed Euler cycle is a directed cycle that contains all the directed edges of di-graph G.
- The di graph G is Euler graph if it contains a directed Euler cycle

Example 4.3. *Find the Euler cycle and Euler path of following di graphs. Are they Euler graphs?*

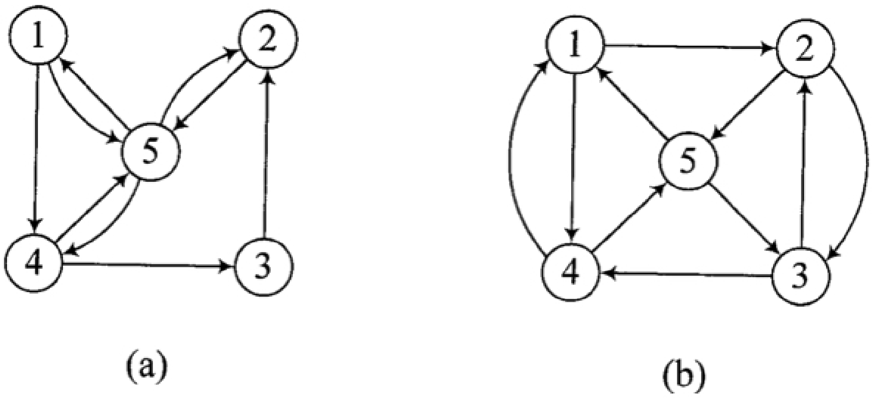

Figure 4.4

5 Complete Graph

- The ***complete graph*** on n vertices, denoted K_n, is the simple graph with n vertices in which there is an edge between every pair of distinct vertices.

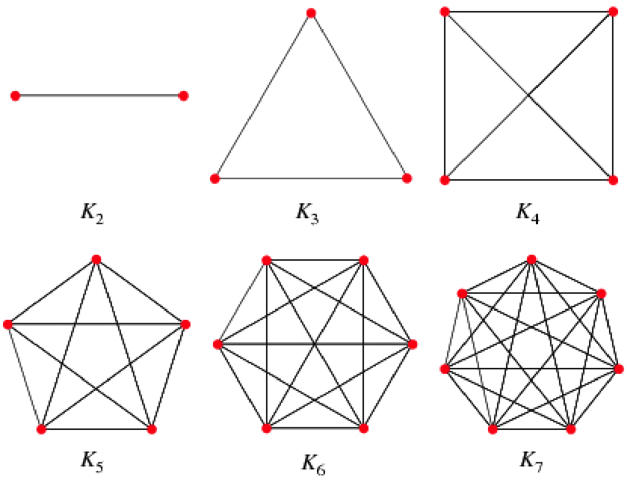

Figure 5.5

- A graph $G = (V, E)$ is ***bipartite*** if there exist subsets V_1 and V_2 (either possibly empty) of V such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$, and each edge in E is incident on one vertex in V_1 and one vertex in V_2.

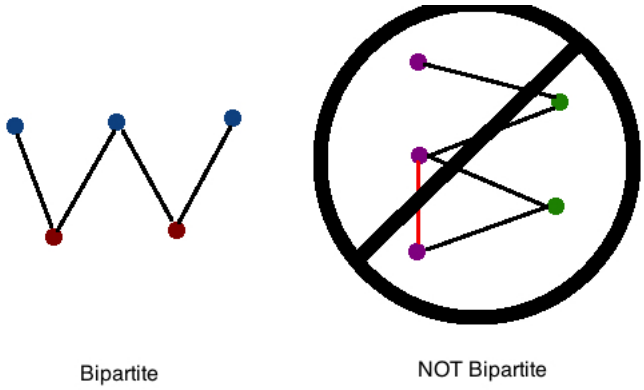

Figure 5.6

- The ***complete bipartite graph on*** m ***and*** n ***vertices***, denoted $K_{m,n}$, is the simple graph whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which the edge set consists of all edges of the form (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$.

Example 5.1. *Find the complete bipartite graph on two and four vertices,* $K_{2,4}$.

[0]**REFERENCES**

(i) *Discrete Mathematics*, Richard Johnsonbaugh.

(ii) *Discrete Mathematics and its Applications*, Kenneth H. Rosen