MA2054 Graph Theory **Lecturer: Dr. Aruna Bandara**

Semester 4 (Intake 2022)

Lecture Note 2

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2 Representation of Graphs and Graph Isomorphisms

In graph theory, there are different ways to represent graphs, each with its own benefits and applications. In week 1, we utilized two forms of representation: set-theoretic representation and diagrammatic representation. The diagrammatic representation is convenient only when the number of vertices and the number of edges are reasonably small.

Now, we are ready to introduce two more representations called adjacency matrix and incidence matrix representation.

The choice of representation depends on the specific requirements and characteristics of the graph as well as the operations to be performed. For instance, adjacency matrices are useful for dense graphs.

Note. *A **dense graph** is a simple graph in which the number of edges is close to the maximal number of edges. The opposite, a simple graph with only a few edges, is called a **sparse graph**.*

2.1 Adjacency Matrix Representation

Definition 2.1.1. *Let G be a graph (simple or multi) with n vertices (say $V(G) = \{v_1, \cdots, v_n\}$). Then the adjacency matrix $A(G)$ of G is an $n \times n$ matrix and $A(G) = (a_{ij})$ where a_{ij} is the number of edges between v_i and v_j for each $i, j \in \{1, \cdots, n\}$.*

Remark.

- *For each G, $A(G)$ is a symmetric matrix.*
- *If G is a simple graph then the entries are either 0 or 1 and the entries on the main diagonal are always zero.*

Example 1. *Consider the diagram of G given below. Write its set-theoretic representation and $A(G)$.*

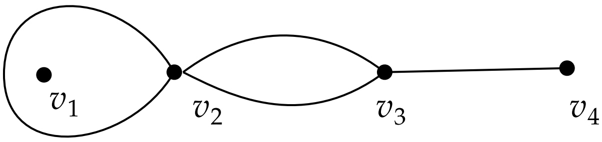

Figure 1: graph G

Example 2. *Write a set-theoretic representation and give a graphical representation if*

$$A(G) = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 1 & 0 \end{pmatrix}.$$

Exercise 1. *Write the adjacency matrix for each graph given below.*

(i). $G = (V, E)$ *where* $V = \{v_1, v_2, v_3\}$ *and* $E = \varnothing$.

(ii). $H = (V, E)$ *where* $V = \{1, 2, 3, 4, 5, 6\}$ *and* $E = \{\{a, b\} | a, b \in V \text{ and } \gcd(a, b) = 1\}$.

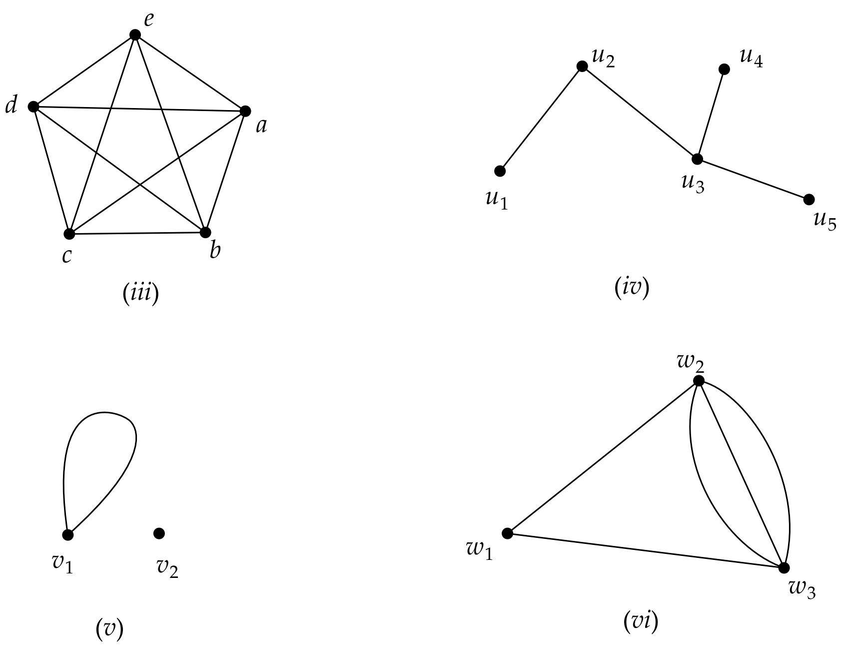

Figure 2:

Exercise 2. *Represent the graph G graphically if $A(G)$ is*

(i). $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. *(ii).* $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{pmatrix}$. *(iii).* $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

2.2 Incidence Matrix Representation

The adjacency matrix does not differentiate between edges; it only indicates the number of edges connecting each pair of vertices in the graph.

The incidence matrix representation is particularly useful when dealing with sparse graphs with relatively few edges and when analyzing edge connectivity in a graph, such as in-network flow algorithms or certain optimization problems.

Definition 2.2.1. *Let G be an undirected simple graph or a multi-graph with n vertices, $V(G) = \{v_1, \cdots, v_n\}$, and m edges, $E(G) = \{e_1, \cdots, e_m\}$. Then the incidence matrix $I(G)$ of G is an $n \times m$ matrix and $I(G) = (a_{ij})$ where*

$$a_{ij} = \begin{cases} 1 & \textit{if } v_i \textit{ is adjacent to distinct vertex by } e_j \\ 2 & \textit{if } e_j \textit{ is a loop at } v_i \\ 0 & \textit{otherwise.} \end{cases}$$

Remark. *The rows and columns of the incidence matrix correspond to vertices and edges, respectively.*

Example 3. *Consider the graph $G = (V, E)$ where $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}\}$. Represent G graphically and hence write the incidence matrix $I(G)$.*

Exercise 3. *Consider the following diagram of a multi-graph G. Write its incidence matrix $I(G)$.*

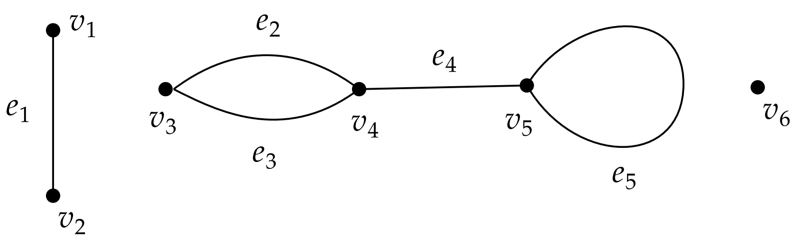

Figure 3: multi-graph G

Exercise 4. *Represent G diagrammatically if* $I(G) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.

Incidence matrices are commonly used for directed graphs (digraphs), where edges have a direction from one vertex to another. The formal definition of digraph is as follows.

Definition 2.2.2. *A* ***digraph (directed graph)*** *is an ordered pair $G = (V, E)$ where V is the set of all vertices of G and E is a set of* ***ordered pairs*** *of vertices, called arcs, directed edges, arrows, or directed lines.*

Remark. *Both simple digraphs and multi-digraphs can be defined.*

Example 4. *Figure 4 depicts a simple digraph and a multi-digraph.*

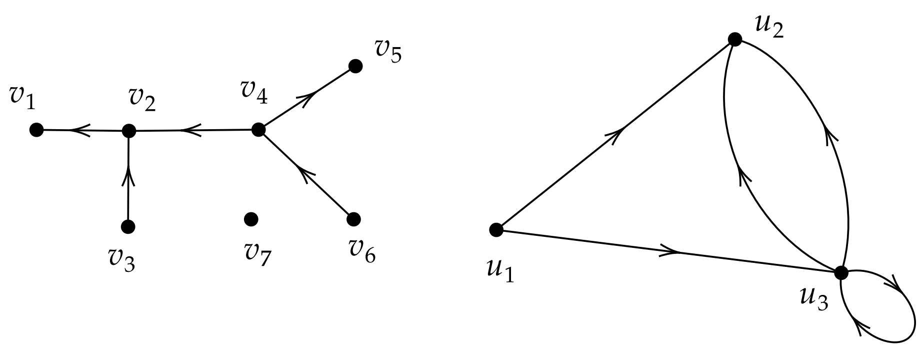

Figure 4:

Now let's define the incidence matrix for a digraph.

Definition 2.2.3. *Let G be a simple digraph or a multi-digraph with n vertices, $V(G) = \{v_1, \cdots, v_n\}$, and m edges, $E(G) = \{e_1, \cdots, e_m\}$. Then the incidence matrix $I(G)$ of G is an $n \times m$ matrix and $I(G) = (a_{ij})$ where*

$$a_{ij} = \begin{cases} 1 & \textit{if } v_i \textit{ is the tail of } e_j \\ -1 & \textit{if } v_i \textit{ is the head of } e_j \\ 2 & \textit{if } e_j \textit{ is a loop at } v_i \\ 0 & \textit{otherwise.} \end{cases}$$

Exercise 5. *Write the incidence matrices for the graphs shown in figure 4.*

The adjacency matrix $A(G)$ can also be defined for directed graphs (digraphs). Its formal definition is as follows.

Definition 2.2.4. *Let G be a digraph (simple or multi) with n vertices (say $V(G) = \{v_1, \cdots, v_n\}$). Then the adjacency matrix $A(G)$ of G is an $n \times n$ matrix and $A(G) = (a_{ij})$ where for each $i, j \in \{1, \cdots, n\}$,*

$$a_{ij} = \begin{cases} 1 & \textit{if there is a directed edge from } v_i \textit{ to } v_j \\ 0 & \textit{otherwise.} \end{cases}$$

Remark. *Unlike undirected graphs, where the adjacency matrix is symmetric, in a digraph, $a_{ij} \neq a_{ji}$ in general.*

Exercise 6. *Write the adjacency matrices for the graphs shown in figure 4.*

In the second half of the lesson, we are going to discuss about isomorphism of graphs, which was used already in week 1 when describing definition 1.2.7.

2.3 Graph Isomorphism

Graph isomorphism is a concept in graph theory that describes a **structural similarity** between two graphs. Two graphs are considered isomorphic if there exists a bijective function (a one-to-one correspondence) between their vertices that preserves the edge relationships. Formally, it can be defined as follows.

Definition 2.3.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. G_1 and G_2 are said to be isomorphic if*

1. *there exists a bijection $f : V_1 \to V_2$ and*

2. *for each $u, v \in V_1$,*

$$(u, v) \in E_1 \text{ if and only if } (f(u), f(v)) \in E_2.$$

If G_1 and G_2 are isomorphic then it is denoted by $G_1 \cong G_2$.

Remark. *When a map f satisfies two conditions given in the above definition, it becomes an isomorphism.*

Example 5. *Here are a few examples of isomorphic graphs.*

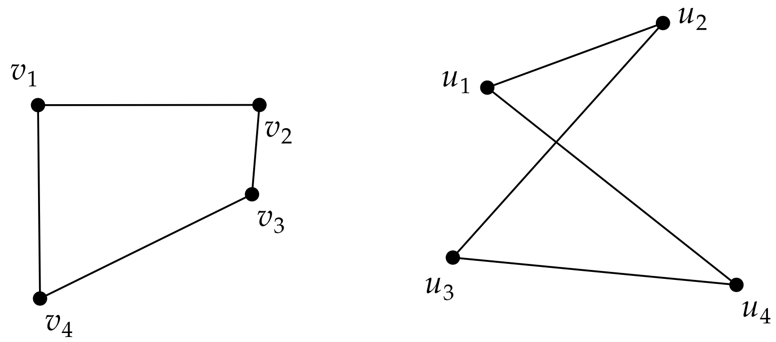

Figure 5:

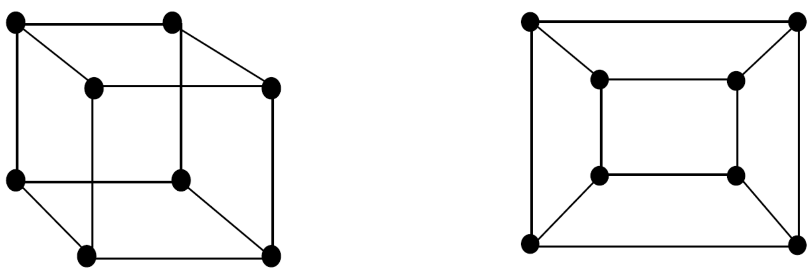

Figure 6: cube graphs

Remark. *In the above definition, G_1 and G_2 can be either* ***undirected graphs*** *or* ***di-graphs****. That is why the* ***ordered pairs*** *are used in the second condition.*

Remark. *It is easy to see that if two graphs G_1 and G_2 are isomorphic then they have the* ***same number of vertices, the same number of edges, and the same degree sequence****.*

Exercise 7. *Suppose G_1 and G_2 are two simple graphs that have the same number of vertices, the same number of edges, and the same degree sequence. Are they isomorphic? Justify your answer.*

Alternatively, isomorphic graphs can be defined in the following way.

Definition 2.3.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. G_1 and G_2 are said to be isomorphic if there exists a continuous bijection $f : (V_1, E_1) \to (V_2, E_2)$.*

It is important to note that graph isomorphism is an **equivalence relation**, which means it satisfies three properties:

- **Reflexivity:** Every graph is isomorphic to itself.
- **Symmetry:** If G_1 is isomorphic to G_2, then G_2 is isomorphic to G_1.
- **Transitivity:** If G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3.

Homework

1. Write $A(G)$ and $I(G)$ if the graph G is as shown in figure 7.

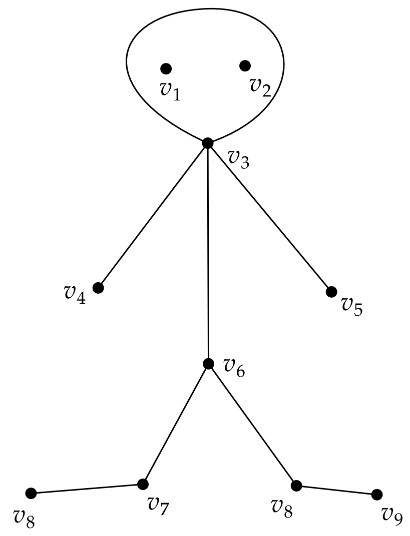

Figure 7:

2. Draw seven non-isomorphic subgraphs of K_3.

3. Anura is talking about two simple graphs G and H. He says that G and H are isomorphic if and only if G^C and H^C are isomorphic. Is his statement true? Justify your answer.

4. Let G_1 and G_2 be two undirected simple graphs with three vertices. Prove that G_1 and G_2 are isomorphic if and only if G_1 and G_2 have the same number of edges.

5. Are the graphs shown in figure 8 isomorphic? Justify your answer.

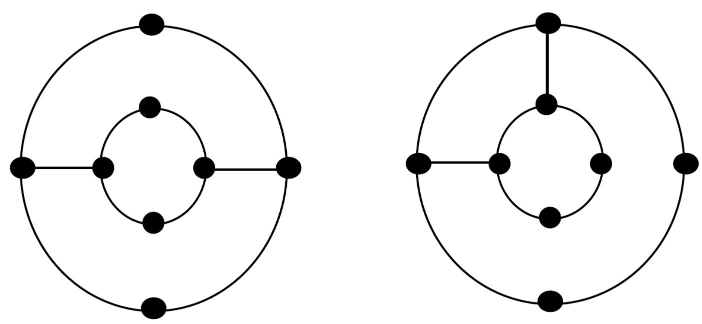

Figure 8:

6. The graph depicted in figure 9 is called the **Petersen graph**, which is a well-known graph in graph theory.

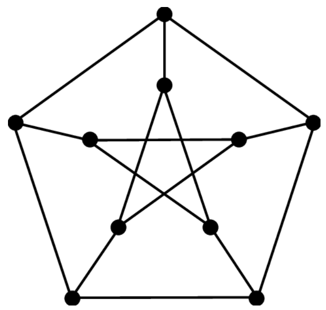

Figure 9:

Which of the following graph(s) is/are isomorphic to the Petersen graph?

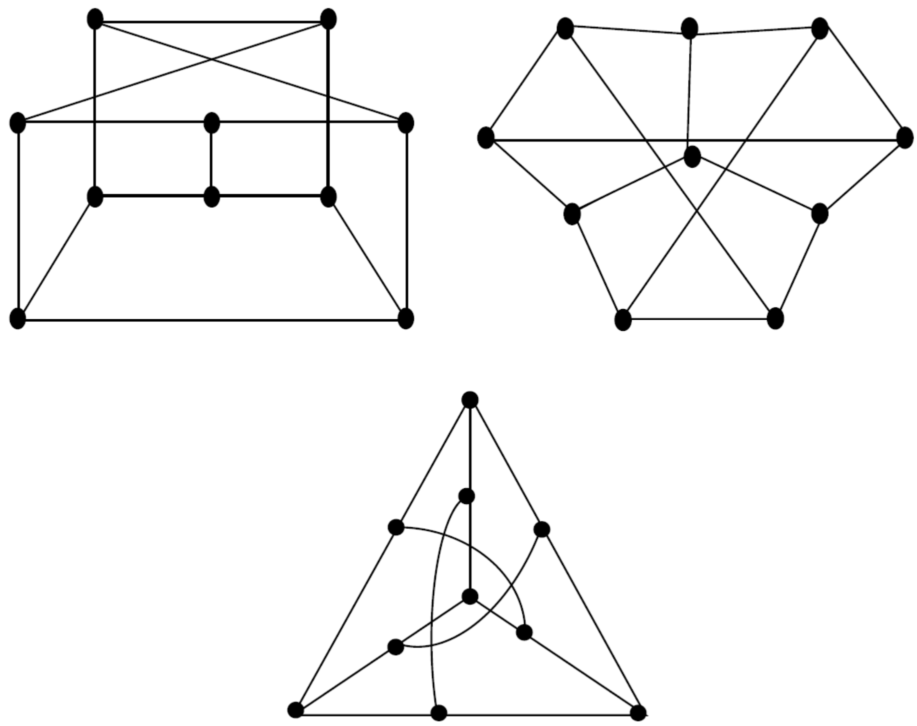

Figure 10:

7. Find a necessary and sufficient condition for $K_{a,b} \cong K_{c,d}$.

8. G_1, G_2, and G_3 are three undirected graphs. Suppose G_1 is not isomorphic to G_2, and G_2 is not isomorphic to G_3. Is necessarily G_1 non-isomorphic to G_3? Justify your answer.