MA2054 Graph Theory **Lecturer: Dr. Aruna Bandara**

Semester 4 (Intake 2022)

Lecture Note 4

4 Types of Graphs

During our discussion so far, we have covered different types of graphs such as simple graphs, multi-graphs (pseudo graphs), bipartite graphs, and directed graphs. In this section, we will introduce additional types and terminologies that are necessary to know before exploring real-world applications.

4.1 Digraphs

Recall that we discussed digraphs and their incidence matrices in week two. This section continues the discussion of digraphs.

Definition 4.1.1. *Let G be a simple or multi-digraph and $v \in V(G)$. Then the number of edges* ***incident into*** *v is called the* ***in-degree*** *and the number of edges* ***incident out of*** *v is called the* ***out-degree*** *of v. In-degree and out-degree of v are respectively denoted by* $\text{indeg}(v)$ *and* $\text{outdeg}(v)$*.*

Then, the total degree (or simply degree) of v (denoted $\deg(v)$) is

$$\deg(v) = \text{indeg}(v) + \text{outdeg}(v).$$

Remark. *If G is an undirected graph and $v \in V(G)$, then the degree of v ($\deg(v)$) is the number of incidences on v. The total degree of a vertex in a digraph is the same as the degree of it when disregarding the orientation of the graph.*

Example 1. *Consider two digraphs depicted in figure 1. Find the in-degree, out-degree, and total degree for each vertex.*

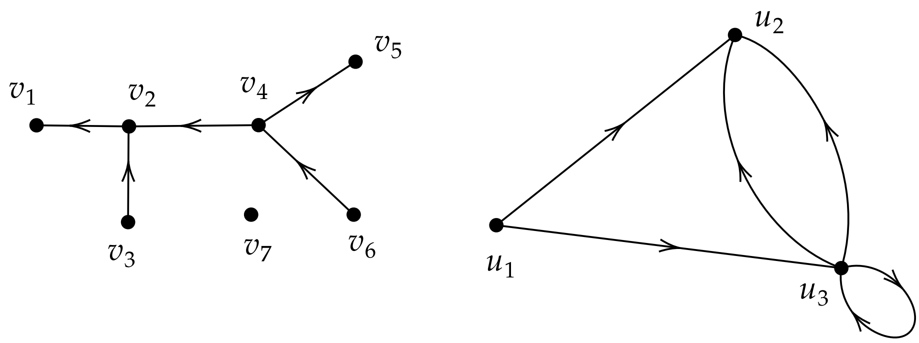

Figure 1:

The handshaking lemma can be modified for a digraph as follows.

Theorem 4.1.1 (The Handshaking Lemma). *Let G be a simple or multi-digraph with n vertices (say $V(G) = \{v_1, v_2, \cdots v_n\}$) and m edges. Then,*

$$\sum_{i=1}^{n} \text{indeg}(v_i) = \sum_{i=1}^{n} \text{outdeg}(v_i) = m.$$

(That is in-degree sum and out-degree sum of G are equal to the number of edges of G.)

Exercise 1. *Prove the Handshaking Lemma for a digraph.*

Exercise 2. *Verify that the graphs illustrated in figure 1 satisfy the Handshaking Lemma.*

In graph theory, problems involving finding paths or cycles in a graph that satisfy certain conditions or optimize specific criteria are known as traveling problems. Euler and Hamiltonian graphs are important in their study.

4.2 Euler Graphs

As discussed in the first lesson, Leonhard Euler first addressed this concept while solving the famous Seven Bridges problem in 1736. In what types of connected graphs is it possible to find a walk that traverses every edge exactly once?
Before answering the question, let's first define an Eulerian graph.

Definition 4.2.1. *Let G be a connected simple or multi-graph.*

- *If a closed walk in G, running through each and every edge of G exactly once, then the walk is called an* ***Euler circuit*** *and the graph G is called an* ***Euler graph*** *(or an* ***Eulerian graph****).*

- *An* ***Eulerian path*** *is a walk in the graph that covers every edge exactly once but does not necessarily return to the starting vertex.*

Remark. *Euler circuit is a closed Eulerian path.*

Example 2. 3*-cycle (C_3) is the smallest Eulerian simple graph.*

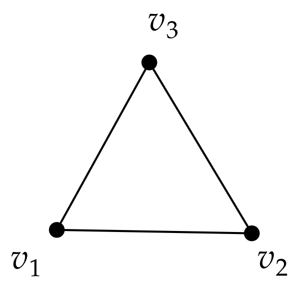

Figure 2: C_3

The closed path v_1, v_2, v_3, v_1 is an Euler circuit.

Theorem 4.2.1. *Let G be a* ***connected simple or multi-graph****. Then, G is an Euler graph if and only if every vertex of G has an even degree.*

Proof. Suppose G is an Euler graph. Let $v \in V(G)$ be arbitrary. Since G is Eulerian, every time an edge arrives at v, another edge departs from v. Therefore $\deg(v)$ must be even and hence every vertex of G has an even degree.

Now conversely suppose every vertex of G has an even degree. Construct a closed walk starting at an arbitrary vertex v, going through the edges of G so that no edge is traced more than once, passing through every vertex of G and ending with the same vertex v (clearly such a walk is exist since every vertex of G has an even degree.) Suppose W is the largest such a walk. Assume there exists an edge e in G so that it is not included in W. Then both vertices that are adjacent with e must be odd degrees. But this is a contradiction. Therefore W consists of all the edges of G and hence G is an Euler graph. $\square$

Exercise 3. *Determine whether the following graph is Eulerian or not. If it is Eulerian then find an Euler circuit.*

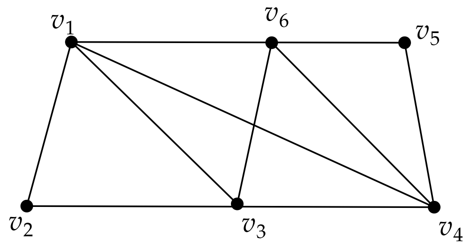

Figure 3:

4.3 Hamiltonian Graphs

The origin of this term was introduced in 1857 by William Hamilton based on the construction of cycles containing all vertices in the graph of the dodecahedron.

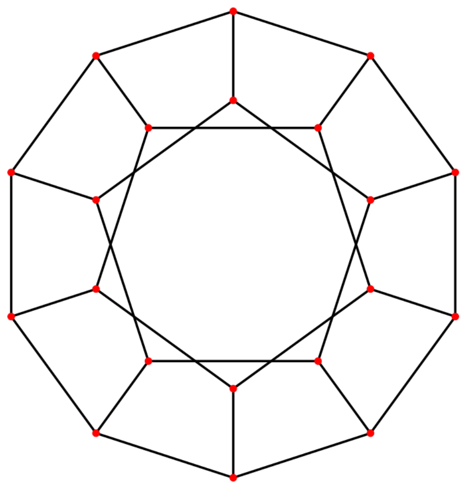

Figure 4: a dodecahedron

Definition 4.3.1. *Let G be a simple or multi-graph. A cycle in G is said to be a* ***Hamiltonian cycle*** *if it includes every vertex of G. The graph G is called a* ***Hamiltonian graph*** *if it has a Hamiltonian cycle.*

Example 3. *Consider the graphs shown in Figure 5. Graph G is a Hamiltonian graph, while graph H is not.*

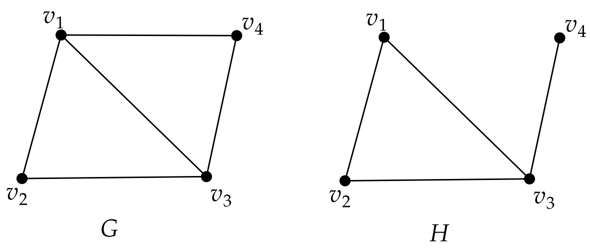

Figure 5:

Exercise 4. *By constructing a Hamiltonian circle, show that the dodecahedron (see figure 4) is a Hamiltonian graph.*

Determining the necessary and sufficient conditions for a graph to be Hamiltonian is a complex problem, and a complete characterization is not yet known. However, there are some well-known results and theorems that provide insight into this question.

Theorem 4.3.1 (Dirac's Theorem)**.** *Suppose G is a simple graph with n vertices where $n \geq 3$. If for each $v \in V(G)$, $\deg(v) \geq \frac{n}{2}$ then G is Hamiltonian.*

Proof. Omit. □

Theorem 4.3.2 (Ore's Theorem)**.** *Suppose G is a simple graph with n vertices where $n \geq 3$. If for each pair of non-adjacent $u, v \in V(G)$, $\deg(u) + \deg(v) \geq n$ then G is Hamiltonian.*

Proof. Omit. □

Exercise 5. *Use Dirac's theorem or Ore's theorem to prove that the graph G in Figure 5 is Hamiltonian. Can you apply these theorems to the graph H in the same figure to determine whether it is Hamiltonian?*

Remark. *Dirac's theorem and Ore's theorem provide* ***sufficient conditions*** *for a graph to be Hamiltonian, they are not necessary conditions. There are Hamiltonian graphs that do not satisfy these conditions.*

4.4 Weighted Graphs

In graph theory, a weighted graph is a graph in which each edge has an associated numerical value called a weight. These weights often represent distances, costs, or some other measure associated with the edges. Weighted graphs are used to model and analyze various real-world scenarios where the relationships between vertices have quantitative values.

The formal definition of a weighted graph is as follows.

Definition 4.4.1. *Suppose $G = (V, E)$ is a graph, which can be simple, multi, oriented, or unoriented. Then G is said to be a* ***weighted graph*** *if each edge $e \in E$ has an associated numerical value w_e, called a* ***weight****.*

That is a weighted graph may be denoted as $G = (V, E, w)$, where V is the set of vertices, E is the set of edges, and w is a ***function*** *that assigns weights to edges.*

Example 4. *Represents each graph depicted in figure 6, set theoretically.*

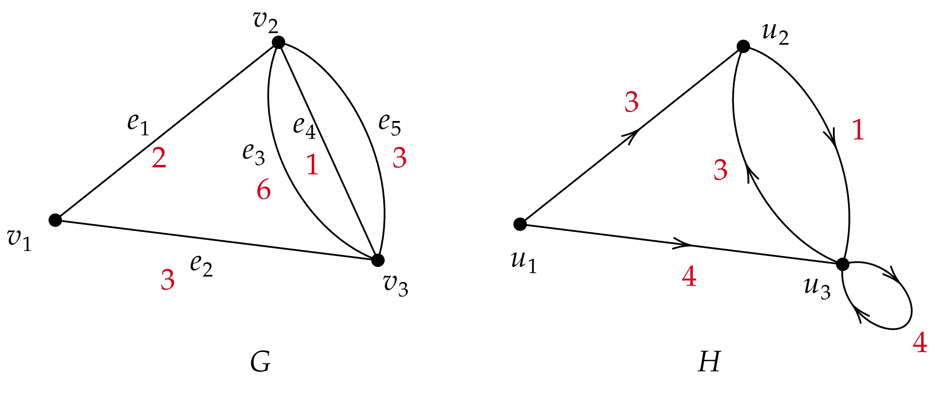

Figure 6:

Exercise 6. *Provide a real-world example for a weighted graph.*

4.5 Planar Graphs

Definition 4.5.1. *A* ***planar graph*** *is a graph that can be drawn on a plane without any of its edges crossing.*

Example 5. *Consider the graphs shown in figure 7. G_1 is a planar graph because it can be drawn as G_2 without crossing its edges. G_2 is called a **planar representation** (or **planar embedding**) of G_1.*

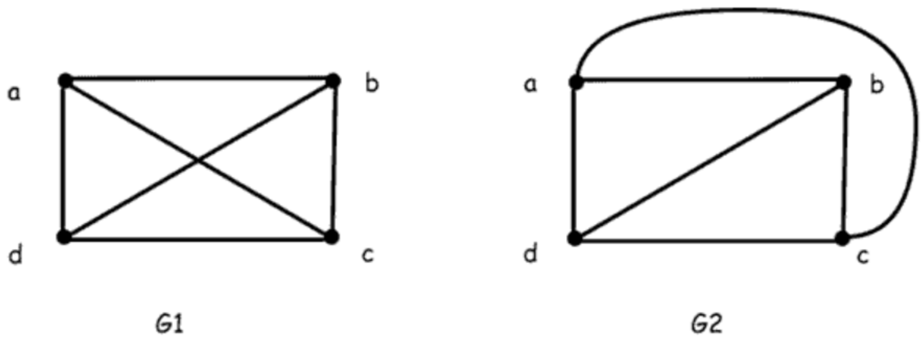

Figure 7:

Remark. *Any planar graph has at least one planar representation.*

The planar representation of a graph is important in real-world applications for several reasons. In electronic circuit design, planar graphs are often used to model connections between components. In urban planning, planar graphs are useful for modeling transportation networks. Minimizing edge crossings can lead to more efficient road layouts and public transportation systems.

Here are a few interesting results related to planar graphs.

Theorem 4.5.1 (Euler's Formula). *Suppose $G = (V, E)$ is a connected planar simple or multi-graph with $|V| = v$ and $|E| = e$ and G is drawn in a planar representation. If there are f regions (faces) in the representation (including the outer region) then*

$$v - e + f = 2.$$

Example 6. *Show that the planar graph G_2 in Figure 7 satisfies Euler's formula.*

Euler's Formula has some important corollaries.

Corollary 4.5.1. *Every planar representation of a planar graph G has the same number of regions.*

Proof. Obvious. □

Corollary 4.5.2. *If G is a connected planar simple graph with $v \geq 3$ vertices and e edges then*

$$e \leq 3v - 6.$$

Exercise 7. *Prove corollary 4.5.2.*

Exercise 8. *Show that two graphs K_5 and $K_{3,3}$ are not planar. (See figure 8.)*

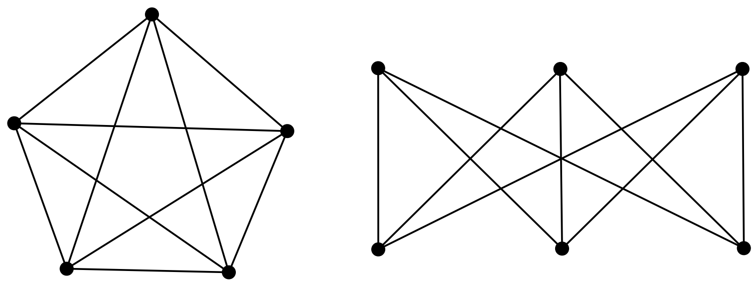

Figure 8: K_5 and $K_{3,3}$

Remark. *K_5 and $K_{3,3}$ are the* ***smallest*** *non-planar graphs.*

Kuratowski's Theorem is a fundamental result in graph theory, which is named after the Polish mathematician Kazimierz Kuratowski and was formulated in the 1930s.

Theorem 4.5.2 (Kuratowski's Theorem)**.** *A graph is non-planar if and only if it contains a subgraph homeomorphic to either K_5 and $K_{3,3}$.*

Proof. Omit. □

The following are the consequences of Kuratowski's theorem.

1. If a graph has neither K_5 nor $K_{3,3}$ as a subgraph then it is planar.

2. If a graph is non-planar then it must contain either K_5 or $K_{3,3}$ as a subgraph.

Homework

1. Determine whether the following graphs are Eulerian or not.

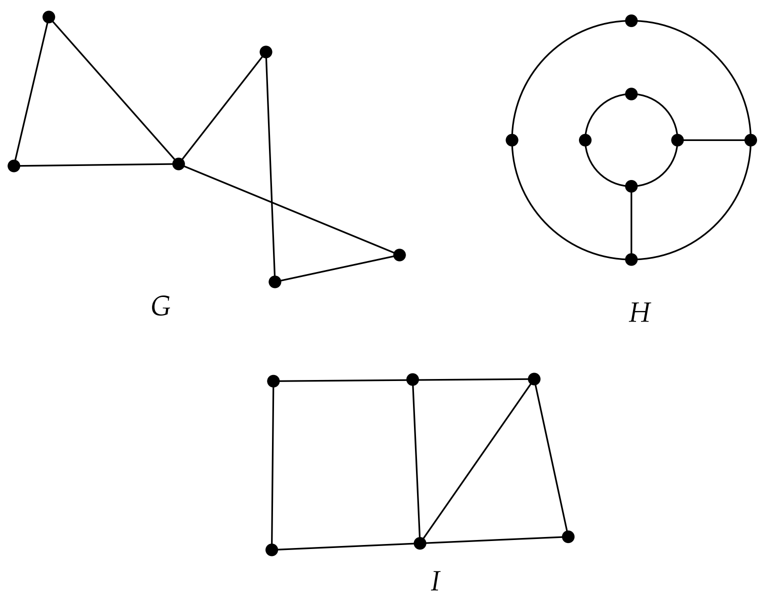

Figure 9:

2. Which graphs shown in Figure 9 are Hamiltonian? Prove your answers.

3. Provide an example of a Hamiltonian graph that does not satisfy the conditions of Dirac's theorem and Ore's theorem.

4. Describe the relationship between Dirac's theorem and Ore's theorem.

5. Use the induction to prove Euler's formula.

6. Modify Euler's formula for a simple or multi graph with k connected components.

7. Prove that a simple connected planar graph G with $|V(G)| \geq 3$ has a vertex of degree five or less.

8. Prove that **any subgraph of a planar graph is planar** and **any super graph of non-planar graph is non-planar**.

9. Prove that the Petersen graph is non-planar.

10. Determine whether each graph is planar or non-planar. Prove your answer.

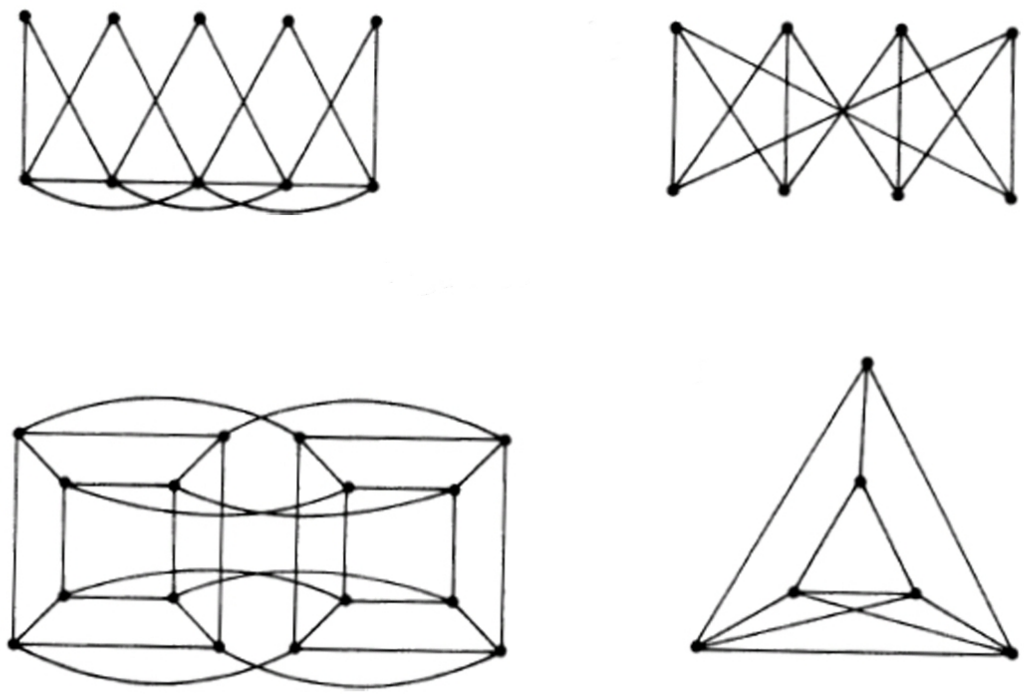

Figure 10: