MA2054 Graph Theory **Lecturer: Dr. Aruna Bandara**

Semester 4 (Intake 2022)

Lecture Note 3

3 Connectedness

Before discussing the connectedness of a graph, it is important to have a clear understanding of the different types of graph traversals and their terminology.

Note. *Unless otherwise specified, it is assumed that all graphs described in this lesson are undirected.*

3.1 Walk, Trail, and Path

Definition 3.1.1. *Let G be a simple or multi-graph. A **walk** in G is a finite alternating sequence of vertices and edges, which begins and ends with a vertex, so that each edge is incident with the vertices preceding and following it, and **any vertex or edge can appear more than once.***

Example 1. *Consider the graph given below in figure 1. Write five walks in G.*

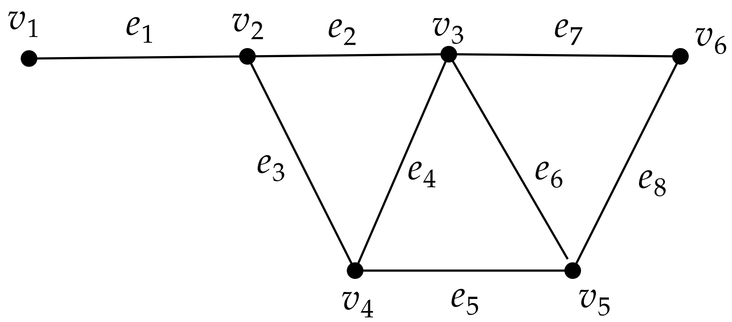

Figure 1: the graph G

Definition 3.1.2. *Let G be a simple or multi-graph. A **trail** in G is defined as a finite alternating sequence of vertices and edges, which begins and ends with a vertex, so that* *each edge is incident with the vertices preceding and following it, and* ***any edge cannot appear more than once, but vertices can be repeated*** *in a trail.*

Exercise 1. *Consider the graph depicted in figure 1. Which of the following are trails?*

(i). $v_1, e_1, v_2, e_3, v_4, e_5, v_5, e_8, v_6$

(ii). $v_1, e_1, v_2, e_2, v_3, e_4, v_4, e_5, v_5, e_6, v_3, e_7, v_6$

(iii). $v_1, e_1, v_2, e_2, v_3, e_4, v_4, e_3, v_2, e_2, v_3, e_7, v_6$

Remark. *A walk is a trail where repetition of edges is allowed.*

Definition 3.1.3. *A trail is called a* ***circuit*** *if its initial and terminal vertices are the same.*

Remark. *A circuit is a closed trail.*

Exercise 2. *Write three circuits in the graph shown in figure 1.*

Definition 3.1.4. *A* ***path*** *is a trail in which* ***no vertex appears more than once****.*

Exercise 3. *Write three paths in the graph shown in figure 1.*

Definition 3.1.5.

- *A path is said to be a* ***cycle*** *if it is closed.*
- *A graph containing at least one cycle is called a* ***cyclic graph****, and otherwise the graph is called an* ***acyclic graph****.*
- *Furthermore, a simple graph consisting of only a cycle with n vertices is called an n-****cycle*** *and it is denoted by C_n.*

Exercise 4. *Write three cycles in the graph shown in figure 1.*

Example 2. *Figure 2 describes the first three n-cycles.*

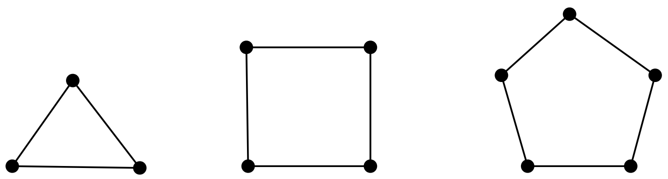

Figure 2: n-cycles

As you read through this text, you may wonder why a 2-cycle (C_2) cannot exist. As multiple edges are not allowed between the same pair of vertices in a simple graph, the only way to connect the two vertices is by joining them with a single edge, which creates an open path. That means C_2 cannot exist if the graph is simple. However, if the graph is a multi-graph, it is possible to construct C_2.

Exercise 5. *Consider the diagram of multi-graph G depicted in figure 3 and write*

(i). all trails from w_1 to w_2.

(ii). all paths from w_1 to w_2.

(iii). all cycles in G.

(iv). the number of 2, 3, and 4-cycles in G.

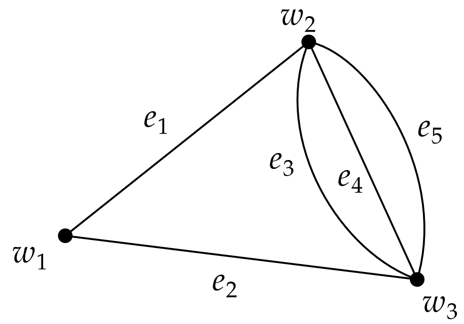

Figure 3: graph G

Remark. *When dealing with directed graphs (digraphs), the definitions of trails, walks, paths, circuits, and cycles are adjusted to account for the directed nature of the edges.*

Exercise 6. *Consider the digraph G illustrated in figure 4 and write*

(i). all trails from u_1 to u_3.

(ii). all paths from u_1 to u_3.

(iii). all cycles in G.

(iv). the number of 1, 2, and 3-cycles in G.

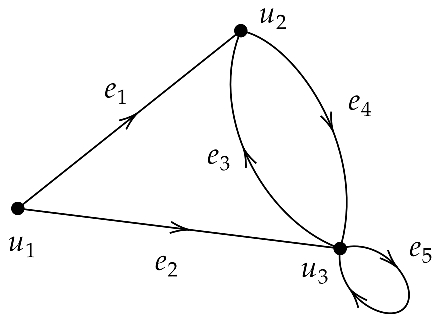

Figure 4: digraph G

3.2 Connectedness

We have now prepared to introduce the concept of connectedness.

Definition 3.2.1. *Let G be a graph (simple or multi). G is said to be a* ***connected graph*** *if each pair of vertices can be connected by a path in G. Otherwise, it is called a* ***disconnected graph****.*

Example 3. *Look at the graphs shown below. G is a connected graph, while H is disconnected.*

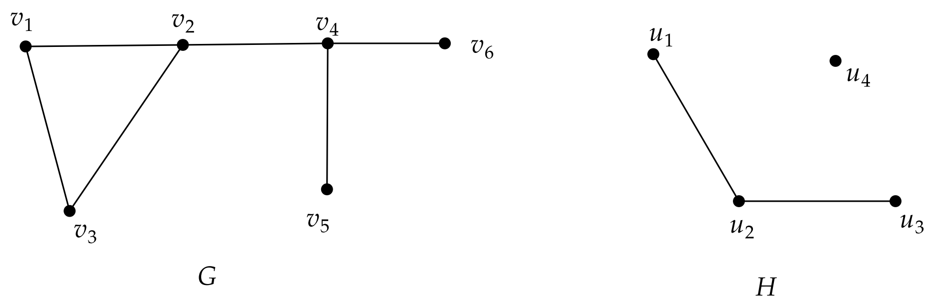

Figure 5:

Remark. *If G is a directed graph, we treat it as an undirected graph and apply the definition 3.2.1 to determine its connectivity.*

Theorem 3.2.1. *A simple or multi-graph G is disconnected if and only if its vertex set $V(G)$ can be partitioned into two non-empty, disjoint subsets V_1 and V_2 so that there is no edge whose one end is in V_1 and the other end is in V_2.*

Exercise 7. *Use theorem 3.2.1 to show that the graph H in figure 5 is disconnected.*

Exercise 8. *Prove theorem 3.2.1.*

Theorem 3.2.2. *Let G be a connected simple or multi-graph with n vertices. Then, any two vertices of G are joined by a path of length at most $n-1$.*

Proof. Obvious. □

Example 4. *If a connected graph G has 7 vertices, then each pair of vertices of G can be connected by a path of at most 6 edges.*

The next result describes the relationship between walks in a graph and its adjacency matrix.

Theorem 3.2.3. *Let G be an simple or multi-graph with n vertices (say $V(G) = \{v_1, \cdots v_n\}$), A be its adjacency matrix, and m be a positive integer. If $A^m = (c_{ij})$ then there are c_{ij} distinct walks from v_i to v_j of length m.*

Proof. Omit. □

Exercise 9. *Suppose G is a graph with 3 vertices, $V(G) = \{v_1, v_2, v_3\}$, and*

$$A(G) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(i). Determine whether G is a simple graph or not.

(ii). Determine the number of walks of length three between each pair of vertices.

(iii). Graphically represent G and precisely write each walk of length three from v_1 to v_2.

The adjacency matrix can be utilized to ascertain whether the given graph is connected or not.

Theorem 3.2.4. *Let G be a simple or multi-graph with n vertices ($n > 1$) and A be its adjacency matrix. Then, G is connected if and only if each entry of $A + A^2 + \cdots + A^{n-1}$ is non-zero.*

Proof. Omit. □

Exercise 10. *Use the above result to show that the graph G in exercise 9 is connected.*

Exercise 11. *Suppose G is a graph and* $A(G) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

(i). Determine whether G is a connected graph or not.

(ii). Determine the number of walks of length three between each pair of vertices.

It is easy to see that a disconnected graph consists of two or more connected sub-graphs and those connected sub-graphs are called components. Now, let's provide a formal definition for components.

Definition 3.2.2. *Let G be a simple or multi-graph, and let H be a sub-graph of G. Then, H is called a* ***connected component*** *of G if every pair of vertices in H is connected by a path, and these vertices are not connected to the other vertices of G.*

Example 5. *The graph depicted in figure 6 consists of three connected components.*

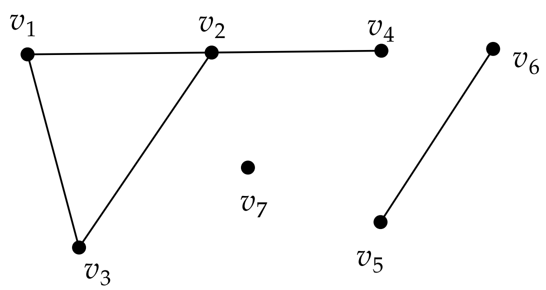

Figure 6:

Remark. *If G is connected, then G is considered a connected component of itself, and it has only one connected component.*

3.3 Distance and Diameter of a Graph

Definition 3.3.1. *Let G be a simple or multi-graph and $u, v \in V(G)$. Then the **distance** between u and v is the **number of edges in the shortest path** between u and v, and it is denoted by* $\mathrm{dist}(u, v)$*. If u and v belong to two connected components then the distance is considered as infinity. That is* $\mathrm{dist}(u, v) = \infty$.

Example 6. *Consider the graph shown in figure 6. Find the distance between each pair of vertices.*

Observe that, for a given graph G, the distance preserves the following rules.

- **Non-negativity:** For each $u, v \in V(G)$, $\mathrm{dist}(u, v) \geq 0$, and $\mathrm{dist}(u, v) = 0$ if and only if $u = v$.
- **Symmetry:** For each $u, v \in V(G)$, $\mathrm{dist}(u, v) = \mathrm{dist}(v, u)$.
- **Triangle Inequality:** For each $u, v \in V(G)$, $\mathrm{dist}(u, w) \leq \mathrm{dist}(u, v) + \mathrm{dist}(v, w)$.

Definition 3.3.2. *The **diameter** of G is the **maximum distance between any two vertices** of G and it is denoted by* $\mathrm{diam}(G)$*.*
That is

$$\mathrm{diam}(G) = \max\{\mathrm{dist}(u, v) : u, v \in V(G)\}.$$

Example 7. *Find the diameter of the graph shown in figure 7.*

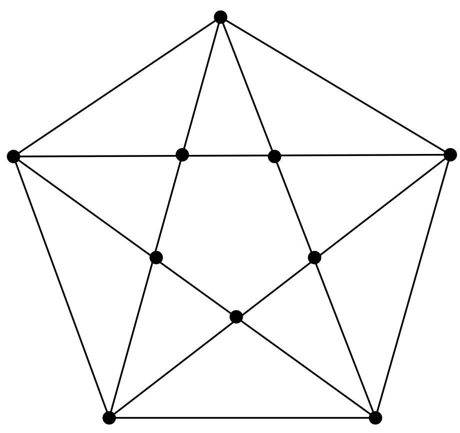

Figure 7:

Homework

1. Find a circuit of length 7 that is not a cycle.

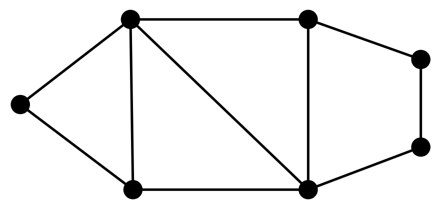

Figure 8:

2. Suppose G is a simple graph with $V(G) = \{v_1, \cdots v_n\}$ and $\delta(G)$ is the minimum degree in G. Prove that if $\delta(G) \geq \dfrac{n-1}{2}$ then G is connected.

3. Give an example of a disconnected multi-graph with $\delta(G) \geq \dfrac{n-1}{2}$.

4. Wickremesinghe states that if a simple graph G is not connected, then its complement, G^C, is connected. Is this statement true? Justify your answer.

5. Suppose G is a simple graph. Prove that if G has exactly two vertices of an odd degree then there must be a path joining these two vertices.

6. Provide an example to show that $\text{dist}(u, w) \neq \text{dist}(u, v) + \text{dist}(v, w)$ in general.

7. Let G be a simple graph with $V(G) = \{1, 2, \cdots, 15\}$ so that for each $i, j \in V(G)$, i and j are connected via an edge if and only if the greatest common divisor of i and j is greater than 1. Count the number of components of G and determine the diameter of G.