CS3063 Theory of Computing

Semester 4 (20 Intake), Feb – Jun 2023

Lecture 3

Regular Languages & Finite Automata – Session 2

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Announcement on Quizzes

- Students must be present in the lab
- Quiz attempts must only be from the lab computers
- If there are N quizzes in the semester, the best N-2 quizzes will be counted for each student
  – 2 spare quizzes for each student

- Unexpected/unfortunate issues (e.g., computer getting stuck)
  – 2 spare quizzes are meant to address such issues

Today's Outline

Lecture 3

- FA $\leftrightarrow$ Regular expressions: How?
- Distinguishing strings
- Set operations on regular languages
- Non-deterministic Finite Automata (NFA)
- Equivalence between NFA and FA (DFA)

PART 1

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Review: Regular Languages and FA

- **Kleene's Theorem**
  - *A language $L \subseteq \Sigma^*$ is regular if and only if there is an FA with alphabet Σ that accepts L*
- This means:
  - If M is an FA, there is a regular expression corresponding to the language $L(M)$
  - Given a regular expression, there is an FA that accepts the corresponding language

Regular Languages and FA

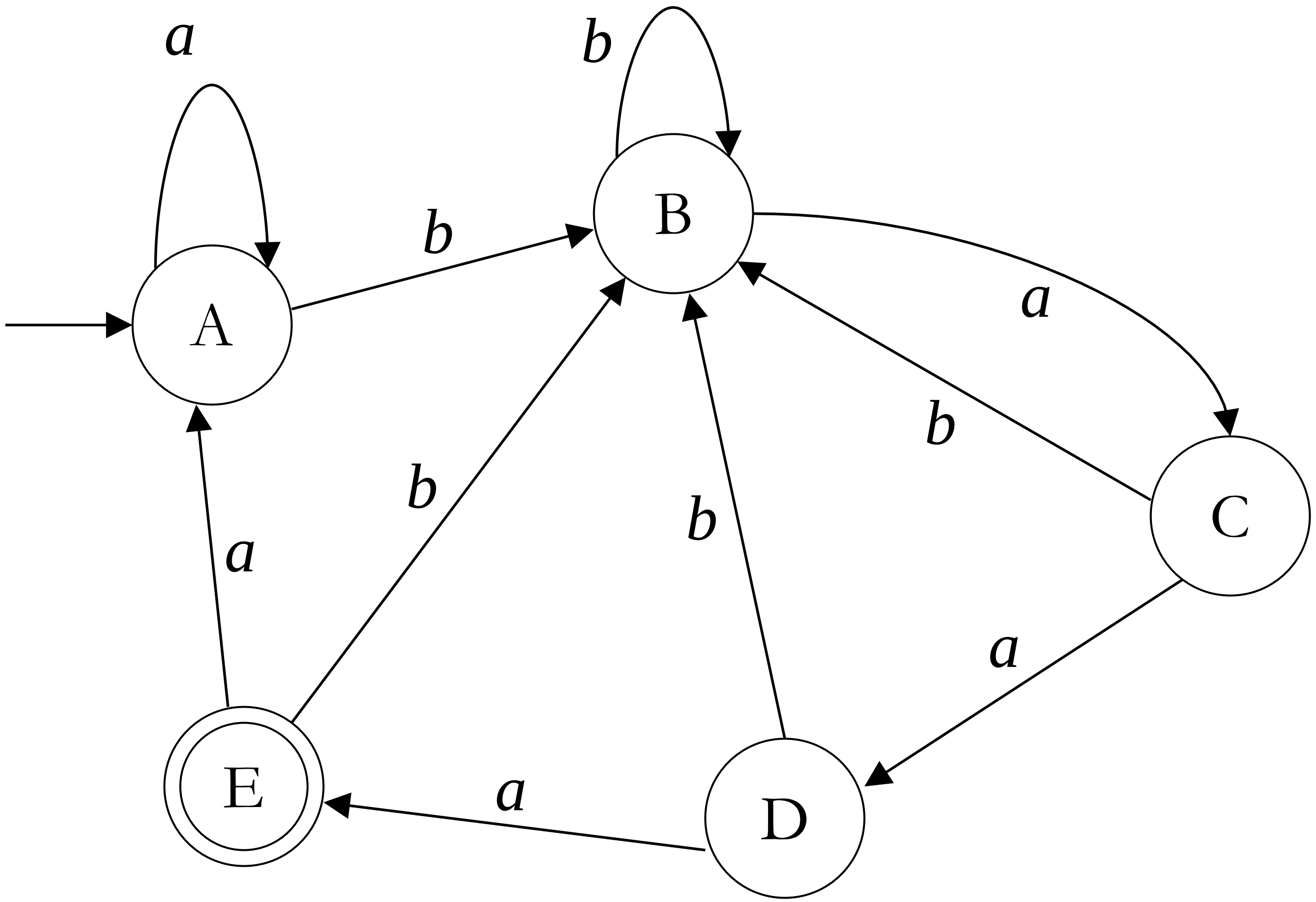
- How to get FA, given regular expression (and vice-versa)?
  - Will discuss later
  - When discussing proof of Kleene’s Theorem

  - Until then, try to do this without an algorithm

Obtaining RE, given the FA?

- Intuitive approach (brute force)
- Can study the set of states and inputs on the transition diagram
- Start with simple strings
- Consider all paths/cases
- But some FAs can be difficult, experience will help

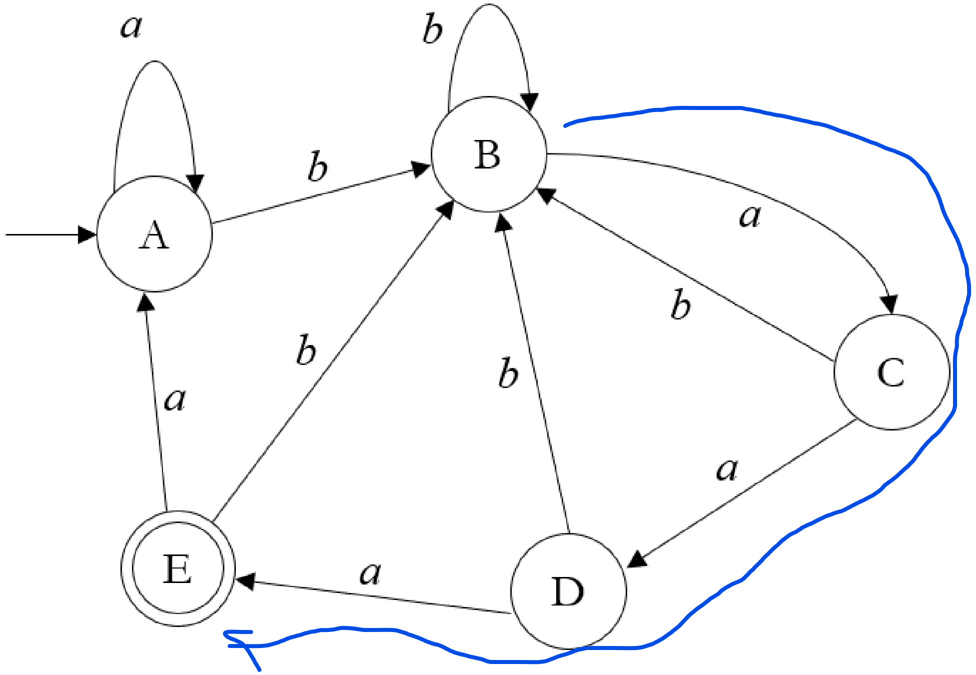
Example



What are the strings accepted by this FA?

Solution / Observations

- For any string ending in $b \Rightarrow$ go to state B
- The only way to get to state:
  - E is from state D with input a
  - D is from state C with input a
  - C is from state B with input a
  - B is with input b from any state
- The language accepted
  - Set of all strings ending in $baaa \Rightarrow (a\,|b)^{*}baaa$



This path should be here anyway

Obtaining FA, given the RE

- Observe the given set for key patterns
- May be able to identify the states quickly
  - Depends on the given regular set
- Example
  - Construct the FA that accepts the language *L* corresponding to (11 | 110)*0

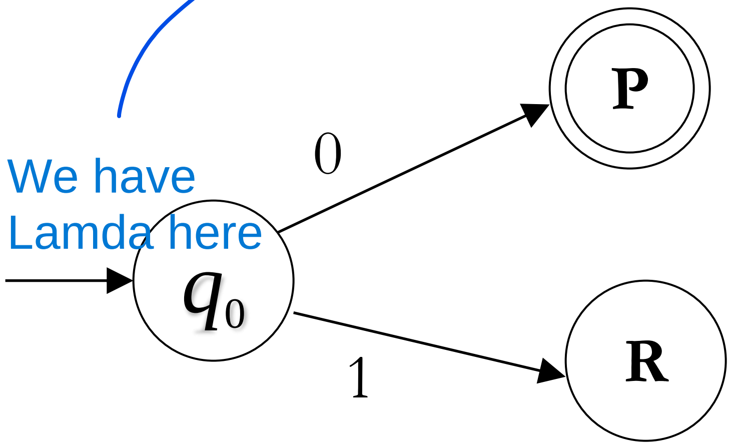
- [Note: easier method using NFAs discussed later]

Solution: the FA for $L \equiv (11|110)^*0$

- Λ is not in $L \Rightarrow q_0$ is not an accepting state
- 0 is in $L \Rightarrow$ from q_0 input 0 takes us to an accepting state
- 1 is not in L; 1 and Λ needs to be distinguished
  - 110 is in L, but 1110 is not in L

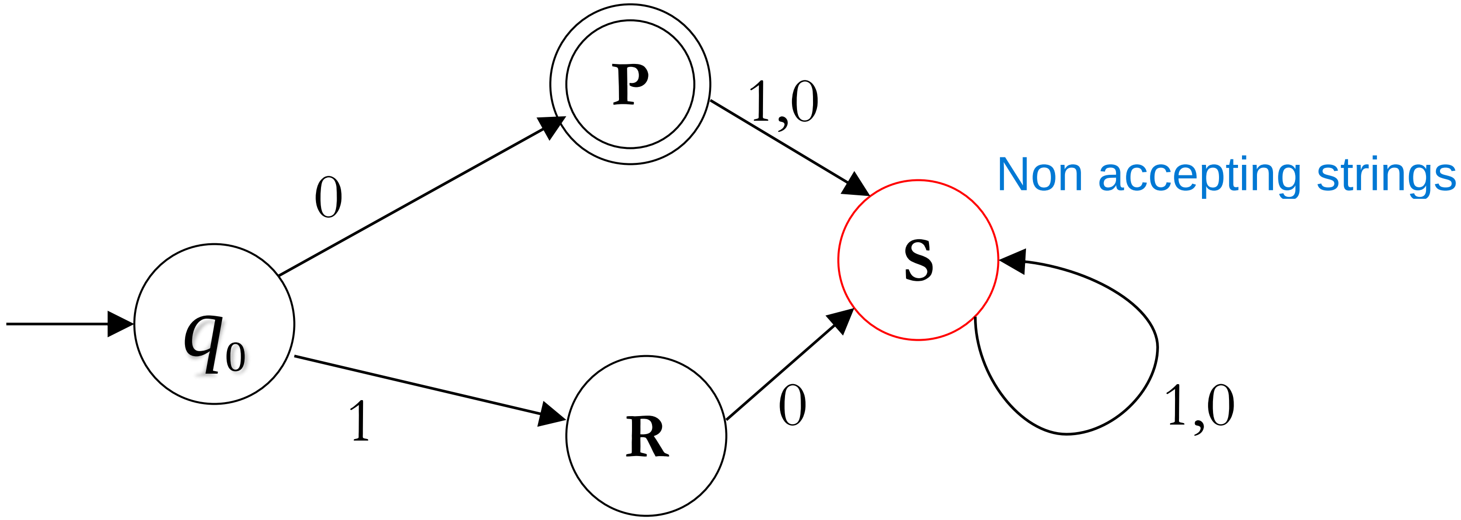
So with the next input we have to go to another state

Need at least these states



Solution: the FA for $L \equiv (11|110)^*0$

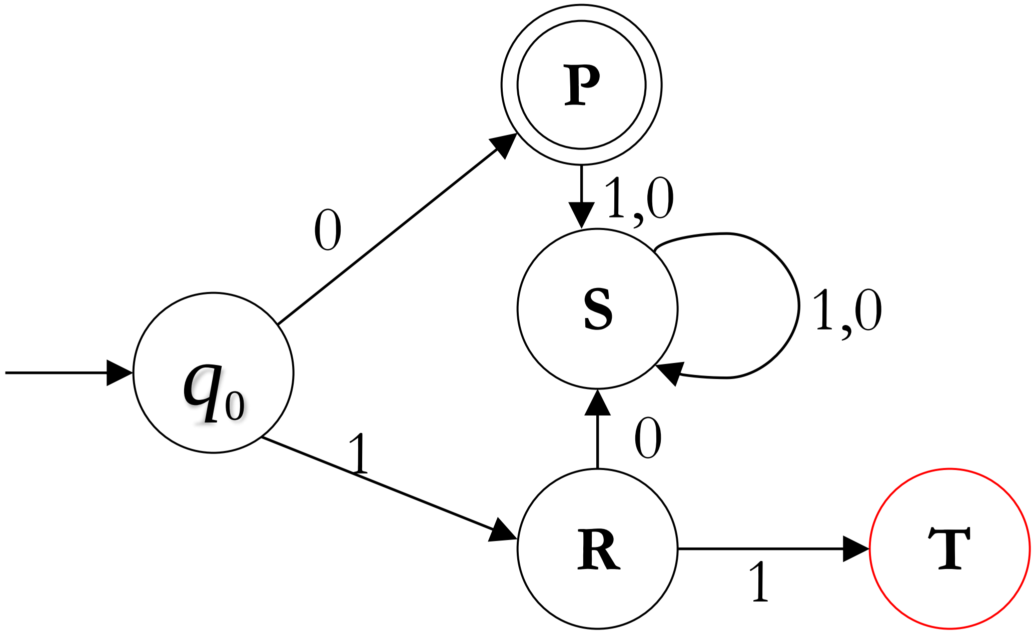
- *L* contains 0 but no string of the form 0x
- *L* contains no string of the form 10x
- Can add another state S to represent above 2 unaccepted forms; when you go there, you never leave



Solution: the FA for $L \equiv (11|110)^*0$

- What happens at R for input 1?
  - Shouldn't stay at R (1 and 11 to be distinguished)
  - Shouldn't return to q_0 (Λ and 11 to be distinguished)
  - Need a new state, T

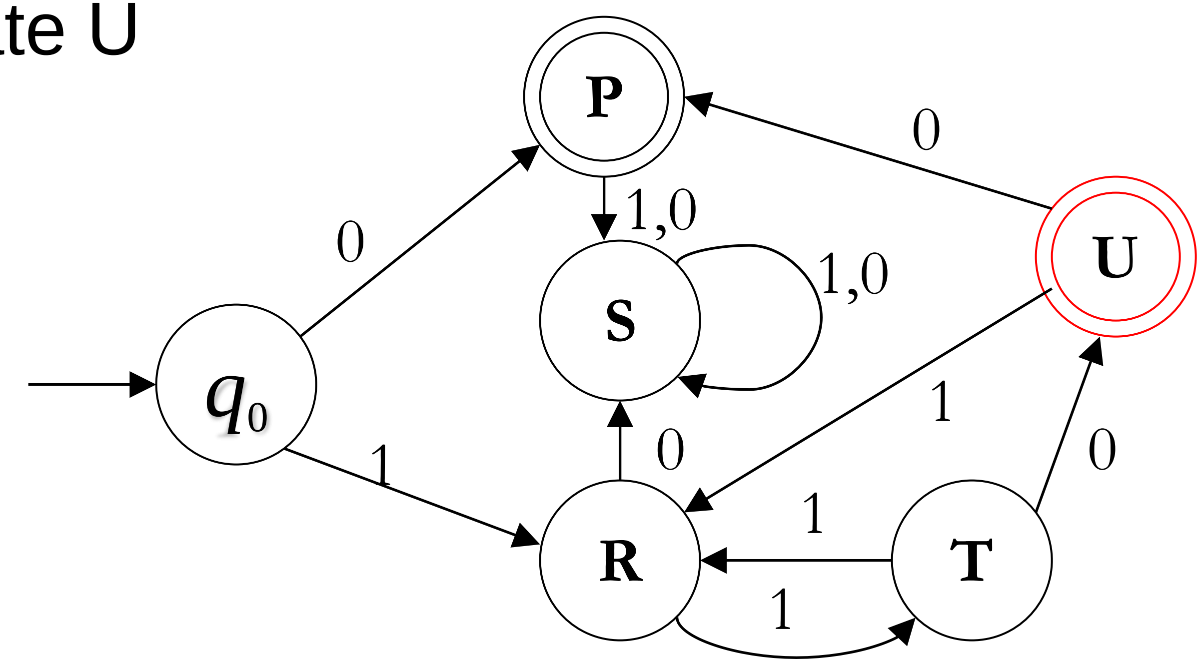
We can't separate 1 and 11, so need to go to another state



Solution: the FA for $L \equiv (11|110)^*0$

- What happens at T?
  - As we did so far, consider all inputs
  - Need an accepting state U

- Final solution



Obtaining FA, given the RE

- May not be able to identify the states quickly

- As we saw in the example, can keep on adding states
  - Eventually ends because language is regular
  - If language is not regular, with the same approach, we will continue forever

PART 2

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- FA ↔ Regular expressions: How?
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Distinguishing Strings

- Consider an FA recognizing a language L
  - There are *groups of strings* where strings within the same group need not be distinguished from each other by FA
  - Remembering which group a string belongs to is enough when it is reading a string
  - The *number of distinct states the FA needs* to recognize L is related to the *number of distinct strings to be distinguished from each other*

a,b,c,... => for symbols
x,y,z,... => for strings

Distinguishable Strings

- **Definition** (Definition 3.5, p. 105 in text)
  - Let L be a language in Σ^* and x and y be any strings in Σ^*. The set L $/x$ is defined as

    $$L\,/x = \{z \text{ in } \Sigma^* \mid xz \text{ is in } L\}$$

  - Two strings x and y are distinguishable with respect to L if L $/x \neq$ L $/y$. Any z that is in one of the two sets but not the other is said to distinguish x and y w.r.t. L
  - If L $/x$=L $/y$, x and y are indistinguishable with respect to L

Two Important Properties

- Property 1 (Theorem 3.2 in text, p. 106)
  - Suppose $L \subseteq \Sigma^*$ and for some +ve integer ***n***, there are ***n*** strings in Σ*, any two of which are distinguishable w.r.t. L. Then every FA recognizing L must have at least ***n*** states
- Property 2 (Theorem 3.3 in text, p. 108)
  - The language ***pal*** of palindromes over the alphabet {0,1} cannot be accepted by any FA and therefore not regular

Need memory to identify, that means finite automaton can't do it, So palindromes are not regular

Shows a lower bound on the memory requirements of an FA to recognize a language

Set Operations and Languages

- Suppose $M_1=(Q_1, \Sigma, q_1, A_1, \delta_1)$ and $M_2=(Q_2, \Sigma, q_2, A_2, \delta_2)$ accept languages L_1 and L_2
- Let $M=(Q, \Sigma, q_0, A, \delta)$ where

$$Q = Q_1 \times Q_2$$

$$q_0 = (q_1 , q_2)$$

$$\delta((p, q), a)= (\delta_1(p, a), \delta_2(q, a))$$

- Then the following hold

Set Operations & Languages ...contd

1. If A={(p, q) | p is in A_1 or q is in A_2}, then M accepts the language $L_1 \cup L_2$

2. If A={(p, q) | p is in A_1 and q is in A_2}, then M accepts the language $L_1 \cap L_2$

3. If A={(p, q) | p is in A_1 and q is not in A_2}, then M accepts the language $L_1 - L_2$

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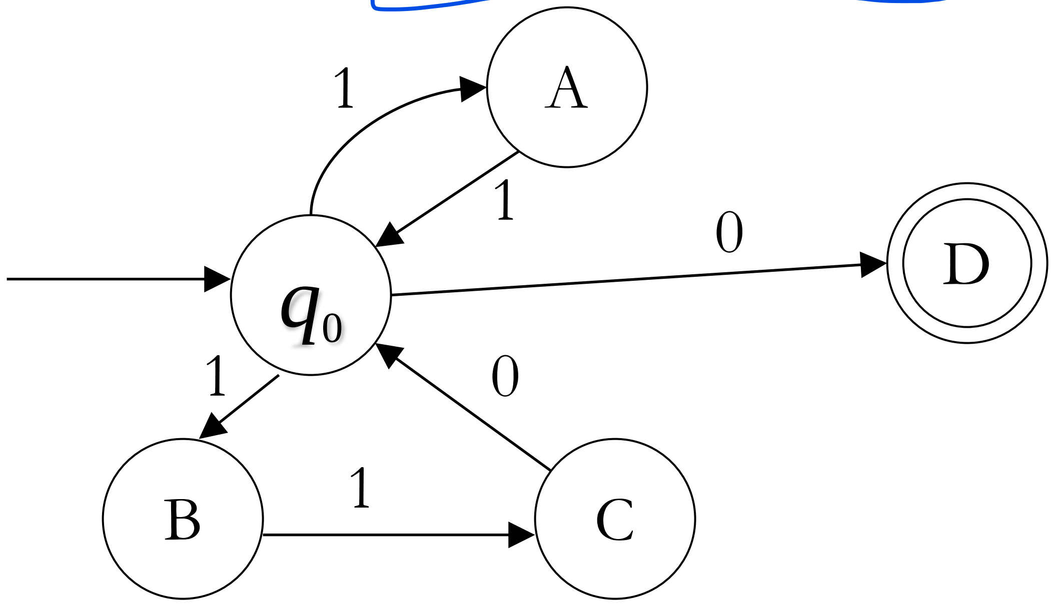
Nondeterministic Finite Automata (NFA)

- An NFA differs from a deterministic FA (or DFA) on δ
  - Allows zero, one or more transitions from a state on the same input symbol
  - So, the value of δ is a *set of states*

This is the only change from the DFAs



Easy to build FAs with this method, other than the DFAs

This NFA accepts (11 | 110)*0 as the DFA on slide 14

Definition of NFA

- An NFA is a 5-tuple $(Q, \Sigma, q_0, A, \delta)$, where:
  - Q, Σ, q_0 and A have the same meaning as for a DFA, but…
  - δ, the transition function, maps $\boxed{Q \times \Sigma \text{ to } 2^Q}$
  - (2^Q is the power set of Q, the set of all subsets of Q)

Including empty set

Extended Transition Function δ*

- We can extend δ, as with DFA, to describe the status of an NFA on an input string x
- Definition: The function $\delta^* : Q \times \Sigma^* \to 2^Q$ is such that:
  - For any $q \in Q$, $\delta^*(q, \Lambda) = \{q\}$ Now we get set of states from the transition function
  - For any $q \in Q$, $y \in \Sigma^*$ and $a \in \Sigma$,

$$\delta^*(q, ya) = \bigcup_{r \in \delta^*(q,y)} \delta(r, a)$$

r = set of states

Properties of NFAs

- Any language accepted by an NFA is also accepted by a DFA (iff)
- Constructing an NFA for a regular expression is often simpler
- NFA are useful for proving theorems
- There is a procedure to convert an NFA into an equivalent DFA
- DFA (Deterministic FA) is a special case of NFA

general case of FA

Acceptance by an NFA

- A string is accepted by an NFA if there is a sequence of transitions for it leading from the initial state to an accepting state

- That is, an NFA, M, accepts a string x if the set of states M can end up after processing x contains at least one accepting state

Example

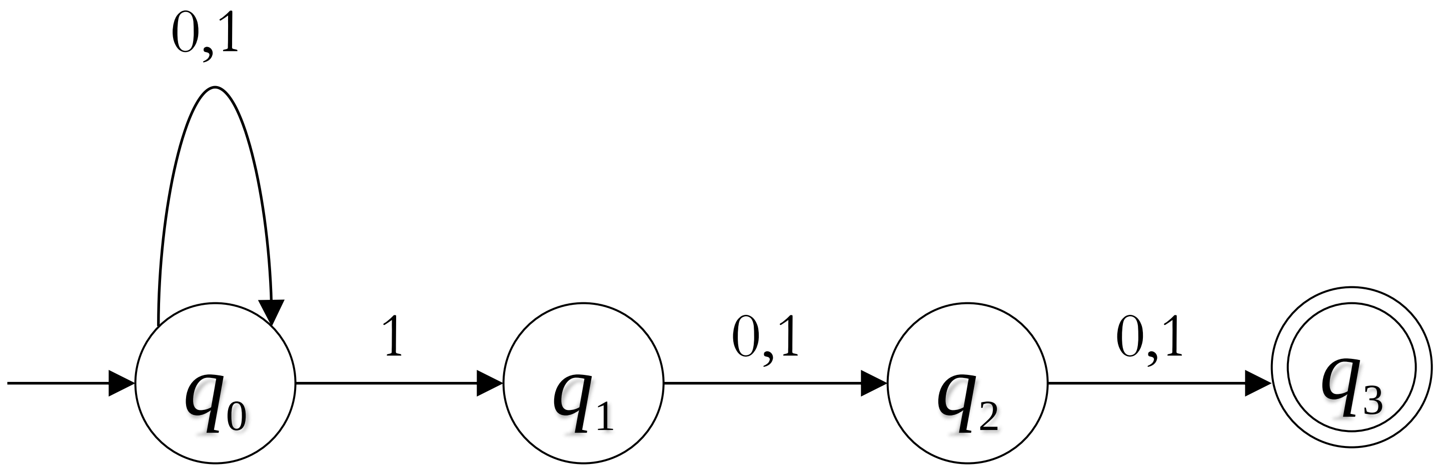
- Suppose the NFA, M=(Q, Σ, q_0, A, δ) where Q={q_0, q_1, q_2, q_3}, Σ={0,1}, A={q_3} and δ specified as follows is given.

for 0,1 =>
q0 is achievable

q	$\delta(q, 0)$	$\delta(q, 1)$
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\{q_2\}$
q_2	$\{q_3\}$	$\{q_3\}$
q_3	$\varnothing$	$\varnothing$

1. Draw the transition diagram
2. Determine the language accepted by M

Solution



Language accepted? $(0 \mid 1)^*1(0 \mid 1)^2$

PART 4

Today's Outline

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- FA ↔ Regular expressions: How?
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- Set operations on regular languages
- Non-deterministic Finite Automata (NFA)
- **Equivalence between NFA and FA (DFA)**

Equivalence Between NFA & DFA

- **Theorem**: For an NFA, $M=(Q, \Sigma, q_0, A, \delta)$ accepting a language $L \subseteq \Sigma^*$, there is a DFA, $M_1=(Q_1, \Sigma, q_1, A_1, \delta_1)$ that accepts L
- M_1 can be defined such that:

$Q_1 = 2^Q$, $q_1=\{q_0\}$,

for $q \in Q_1$ and $a \in \Sigma$, $\delta_1(q,a) = \bigcup_{r \in q} \delta(r,a)$

$$A_1 = \{q \in Q_1 \mid q \cap A \neq \emptyset\}$$

DFA Equivalent to an NFA: How?

- From the theorem on equivalency
  - Proof gives a method to obtain equivalent DFA
  - Proof by induction (on length of input string)

- Method based on *subset construction*
  - *A set of states in the NFA* is considered as *a state in the DFA*
  - DFA keeps track of all states that the NFA could be in after reading the same input as the DFA has read

DFA can have state corresponding to the empty set as well

Example

- Consider the NFA (Example on slide 28): $M=(Q, \Sigma, q_0, A, \delta)$ where $Q=\{q_0, q_1, q_2, q_3\}$, $\Sigma=\{0,1\}$, $A=\{q_3\}$ and δ specified as follows;

2^Q = 2^4 = 16

q	$\delta(q, 0)$	$\delta(q, 1)$
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\{q_2\}$
q_2	$\{q_3\}$	$\{q_3\}$
q_3	$\varnothing$	$\varnothing$

Find an equivalent DFA

Solution: Approach

- Subset construction could produce a DFA with 16 (2^4) states
  - Because the NFA has 4 states
- But we may get fewer states if we consider only states reachable from initial state
  - Start from q_0
  - Each time a new state (subset) S appears, then compute new state from S for each input

Solution ...contd

q	$\delta_1(q, 0)$	$\delta_1(q, 1)$
$\{q_0\}$	$\{q_0\}$	$\{\underbrace{q_0 , q_1}\}$
$\{q_0 , q_1\}$	$\{q_0 , q_2\}$	$\{q_0 , q_1 , q_2\}$

Will be a single state

Solution ...contd

Make sure to include all states, including the generated states here

Start Sate

Keep adding new states

Accepting State

q	$\delta_1(q, 0)$	$\delta_1(q, 1)$
$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
$\{q_0, q_1\}$	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_3\}$	$\{q_0, q_1, q_3\}$
$\{q_0, q_1, q_2\}$	$\{q_0, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$
$\{q_0, q_3\}$	$\{q_0\}$	$\{q_0, q_1\}$
$\{q_0, q_1, q_3\}$	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$
$\{q_0, q_2, q_3\}$	$\{q_0, q_3\}$	$\{q_0, q_1, q_3\}$
$\{q_0, q_1, q_2, q_3\}$	$\{q_0, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$

Conclusion

- We discussed today
  - FA $\leftrightarrow$ Regular expressions
  - Distinguishing strings
  - Set operations on regular languages
  - Non-deterministic FA (NFA)
  - NFA$\leftrightarrow$DFA Equivalency
  - Finding an equivalent DFA for a given NFA