5 Graph Coloring

Graph coloring is a fundamental concept in graph theory. Graph coloring involves assigning colors to the vertices/edges of a graph, subject to certain rules or constraints.

Graph coloring has various applications, including scheduling problems, map coloring, and resource allocation in computer networks. The study of graph coloring helps researchers and practitioners understand and solve problems related to the efficient use of resources and the organization of interconnected systems.

Note. *Throughout this session the term "graph" refers to undirected simple graphs.*

5.1 Vertex Coloring Problem

Definition 5.1.1.

- ***Vertex coloring*** *refers to the problem of coloring vertices (or assigning labels to vertices) of a graph in such a way that* ***no two adjacent vertices have the same color****.*
- *If the coloring is done using m colors, it is called an m****-coloring****.*

Example 1. *Figure 1 describes a 5-coloring.*

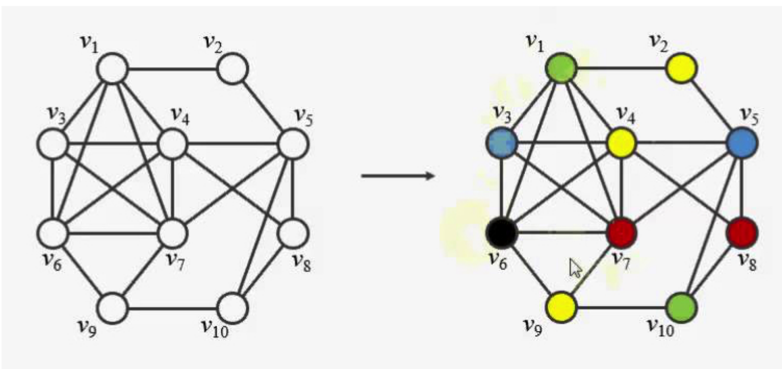


Figure 1: a 5-coloring

Definition 5.1.2. *The **minimum number of colors** needed to color the vertices of a graph by following the rule described in definition 5.1.1 is called its **chromatic number**. For a given graph G, its chromatic number is denoted by $\chi(G)$.* Kai

Example 2. *For instance, the following graph can be colored using a minimum of 3 colors. Therefore $\chi(G) = 3$.*

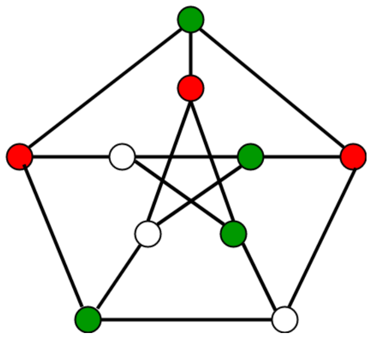


Figure 2: graph G

For certain types of graphs, such as complete graph (K_n) or complete bipartite graph $(K_{m,n})$, there are very few choices possible.

Exercise 1. *Find $\chi(K_n)$ and $\chi(K_{m,n})$.*

X(Kn) = n
X(Km,n) = 2

Now let us consider n-cycle C_n where $n > 2$.

Exercise 2. *Prove that* $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

Theorem 5.1.1. *Suppose G and G' are two undirected simple graphs. If $G \cong G'$ then $\chi(G) = \chi(G')$.*

G = G' => X(G) = X(G')

Proof. Obvious. □

Exercise 3. *Is the converse of theorem 5.1.1 true? Justify your answer.* No

Remark. *The chromatic number is a graph invariant.*

Definition 5.1.3. *A graph G is called **k-colorable** if there exists a vertex coloring on G with k colors.*

Theorem 5.1.2. *If G is k-colorable and $k < n \leq |V(G)|$ then G is n-colorable.*

Proof. Obvious. $\square$

Exercise 4. *Suppose G and H are two undirected simple connected graphs with at least two vertices.*

(i). The graph I is constructed by connecting one vertex of G with one vertex of H. Find $\chi(I)$. Justify your answer. X(I) = max { X(G), X(H) }

(ii). The graph J is constructed by connecting each vertex of G to each vertex of H. Find $\chi(J)$. Justify your answer. X(I) = X(G) + X(H)

Greedy Coloring Algorithm

Algorithms are crucial in vertex coloring as they provide systematic and efficient approaches to solving complex problems. One of the simplest and most widely used algorithms for vertex coloring is the Greedy Coloring Algorithm.

Here is a step-by-step explanation of the Greedy Coloring Algorithm:

1. **Initialization:**
   - Start with an empty set of colored vertices and an empty set of available colors.
   - Initialize a list of vertices to be colored.
2. **Ordering of Vertices:**
   - Choose an ordering for the vertices. This ordering can significantly impact the resulting coloring but is often based on vertex degree (number of edges incident to a vertex). When the order is not there
3. **Color Assignment:**
   - Iterate through each vertex in the chosen order.
   - For each vertex, assign the smallest available color that does not conflict with the colors of its already colored neighbors.

- If all the colors used are taken by neighbors, create a new color and assign it to the current vertex.

4. **Termination:**

- Continue this process until all vertices are colored.

Exercise 5. *Color the graph G shown below using the Greedy algorithm. Then, find the chromatic number $\chi(G)$.*

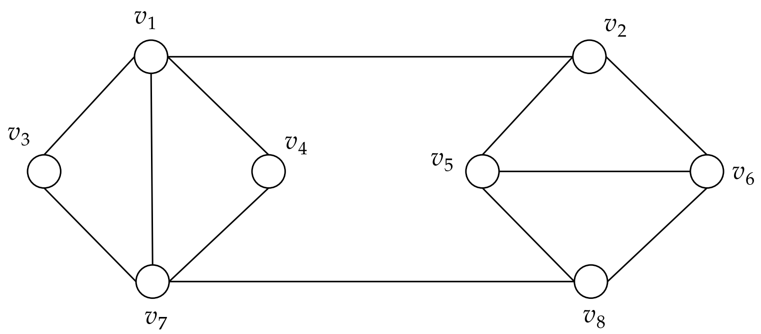


Figure 3: greph G

Remark. *The Greedy algorithm does not guarantee an optimal solution. It makes locally optimal choices at each step without considering the global consequences. As a result, the final solution may not be the best possible.*

Depend on the order

5.2 Edge Coloring Problem

Edge coloring is analogous to vertex coloring. Each edge of a graph has a color assigned to it in such a way that **no two adjacent edges are the same color**. If edge coloring is done using m colors, it is called an **m-edge coloring**.

Example 3. *Figure 4 describes a 4-edge coloring.*

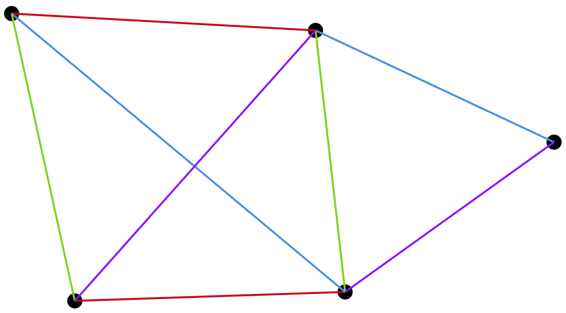


Figure 4: a 4-edge coloring

Definition 5.2.1. *The minimum number of colors needed to properly edge color of a graph is called the* ***edge chromatic number****. For a given graph G its edge chromatic number is denoted by $\chi'(G)$.*

Example 4. *Find $\chi'(C_n)$ where $n > 2$.* odd => 2 , even => 3

Theorem 5.2.1. *Suppose $n > 2$. Then* $\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$

Exercise 6. *Prove theorem 5.2.1.*

Exercise 7. *Find the edge chromatic number of the cube graph.* 4

Exercise 8. *Find $\chi'(K_{n,m})$.* => max(m , n)

5.3 Applications of Graph Coloring

Graph coloring has a wide range of applications in various fields due to its ability to model and solve problems related to assignment, scheduling, and resource allocation. Here are some notable applications of graph coloring:

- Map coloring.
- Timetable scheduling.
- Register allocation in the compiler.
- Mobile radio frequency assignment.

- Sudoku.

5.3.1 Map Coloring

Here the goal is to assign colors to regions on a map in such a way that no two adjacent regions share the same color. This problem can be represented as a graph, where the vertices correspond to regions, and edges connect regions that share a common boundary. The resulting graph is a planar graph.

Example 5. *A map of the world can be colored only using four colors.*

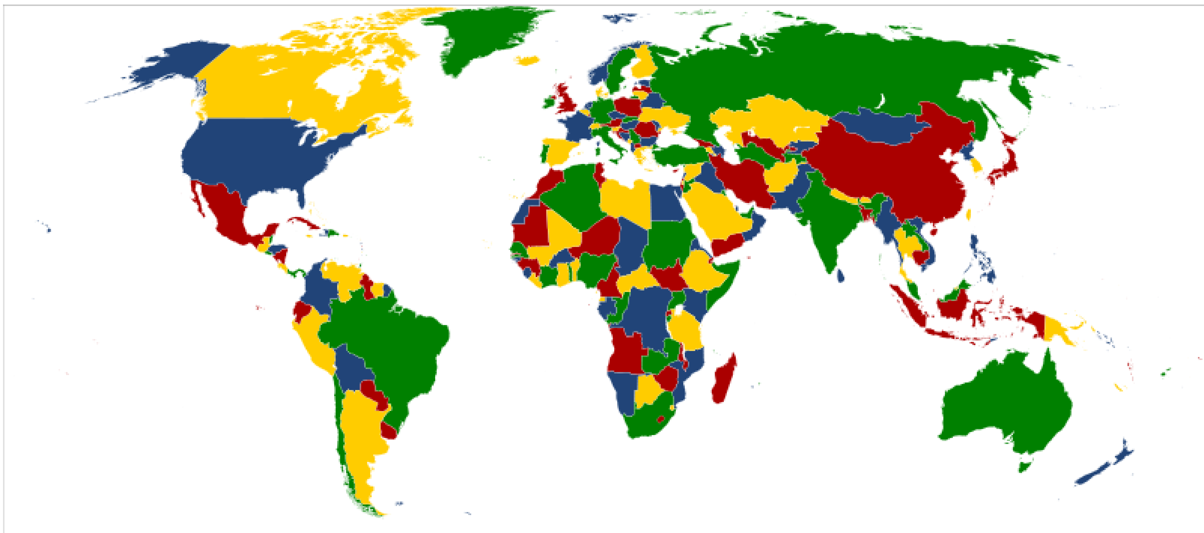


Figure 5:

The following theorem is called the **Four-Color Theorem**, which was proposed by Francis Guthrie in 1852 and proven in 1976.

Theorem 5.3.1 (Four-Color Theorem)**.** *Any planar map can be colored using* ***at most four colors*** *in such a way that no two adjacent regions share the same color.*

The corollary 5.3.1 immediately follows from theorem 5.3.1.

Corollary 5.3.1. *Suppose G is a planar graph drawn in a planar representation. Then,* $\chi(G) \leq 4$.

Example 6. *Consider the graph shown in figure 6.*

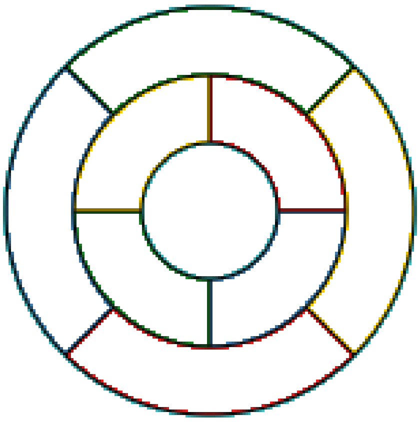


Figure 6:

(i). Represent it as a graph.

(ii). Find the smallest number of colors necessary to perform the coloring. (It is not necessary to color the outer region.)

5.3.2 Timetable Scheduling

Example 7. *Suppose that the final exams at a university have to be scheduled so that no student has two exams at the same time.*

- *Let's model it! Here, exams are represented by vertices. Two vertices are connected by an edge if the corresponding exams have at least one student in common.*
- *Each time slot for a final exam is represented by a different color.*
- *Scheduling of the exams corresponds to a coloring of the associated graph.*
- *For instance, suppose there are seven finals to be scheduled. Let the courses be numbered* 1 *through* 7*. Suppose that the following pairs of courses have common students:* 1 *and* 2, 1 *and* 3, 1 *and* 4, 1 *and* 7, 2 *and* 3, 2 *and* 4, 2 *and* 5, 2 *and* 7, 3 *and* 4, 3 *and* 6, 3 *and* 7, 4 *and* 5, 4 *and* 6, 5 *and* 6, 5 *and* 7, *and* 6 *and* 7*. This can be modeled by a graph as shown below.*

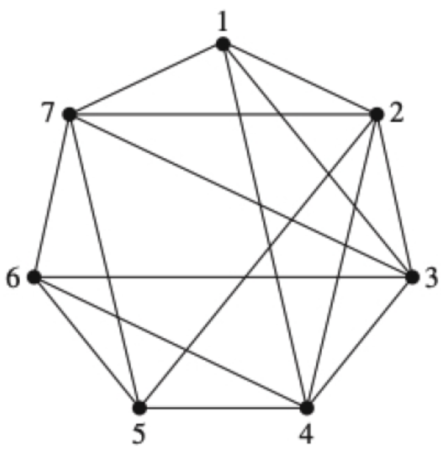


Figure 7:

- *A schedule corresponds to a coloring of this graph.*
- *The chromatic number of this graph is* 4*.*

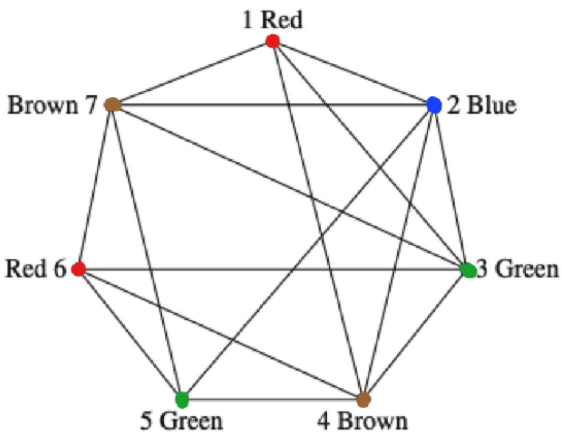


Figure 8:

- *Thus, four time slots are required in order to no student has two exams at the same time.*

Homework

1. Find the chromatic number of each graph.

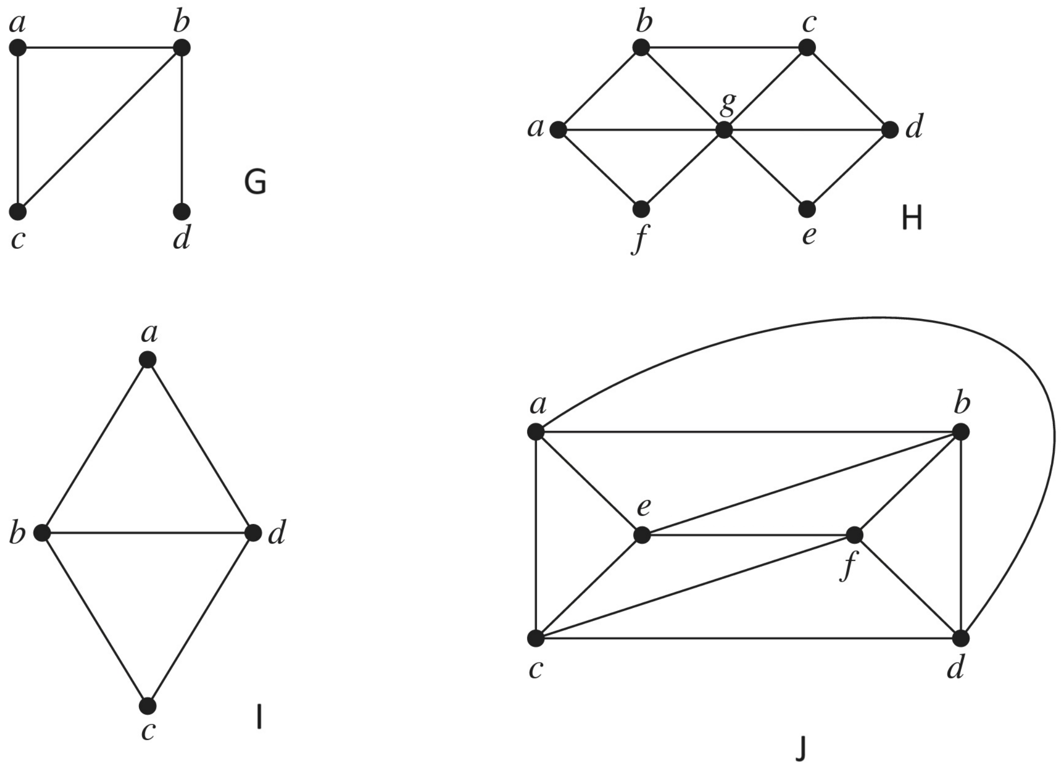
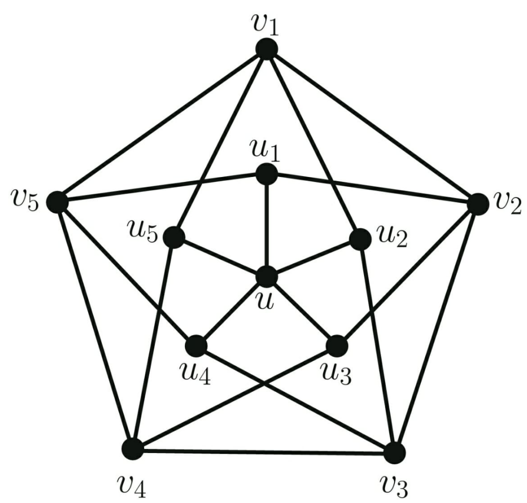


Figure 9:

2. The graph shown in figure 10 is called the **Grötzsch graph**.



G_4 : Grötzsch graph

Figure 10:

Find $\chi(G_4)$ and $\chi'(G_4)$.

3. Consider the Petersen graph shown in figure 2. Show that $\chi'(G) = 4$.

4. Draw up an examination schedule involving the minimum number of days for the following problem:

Set of students	Examination subjects
S_1	Algebra, real analysis, and topology
S_2	Algebra, operations research, and complex analysis
S_3	Real analysis, functional analysis, and complex analysis
S_4	Algebra, graph theory, and combinatorics
S_5	Combinatorics, topology, and functional analysis
S_6	Operations research, graph theory, and coding theory
S_7	Operations research, graph theory, and number theory
S_8	Algebra, number theory, and coding theory
S_9	Algebra, operations research, and real analysis

Figure 11: