MA2054 Graph Theory **Lecturer: Dr. Aruna Bandara**

Semester 4 (Intake 2022)

Lecture Note 1

1 Introduction

Graph theory provides a powerful mathematical framework for modeling, analyzing, and solving problems related to connectivity, relationships, and structures in diverse applications. Its versatility and applicability make it a crucial tool in various scientific, technological, and real-world domains.

If we look at the field of computer science, we can see that it plays a central role in various algorithms, data structures, and applications.

1.1 The History of Graph Theory

The history of graph theory can be traced back to the 18th century, but the formal development of the field began in the 19th century.

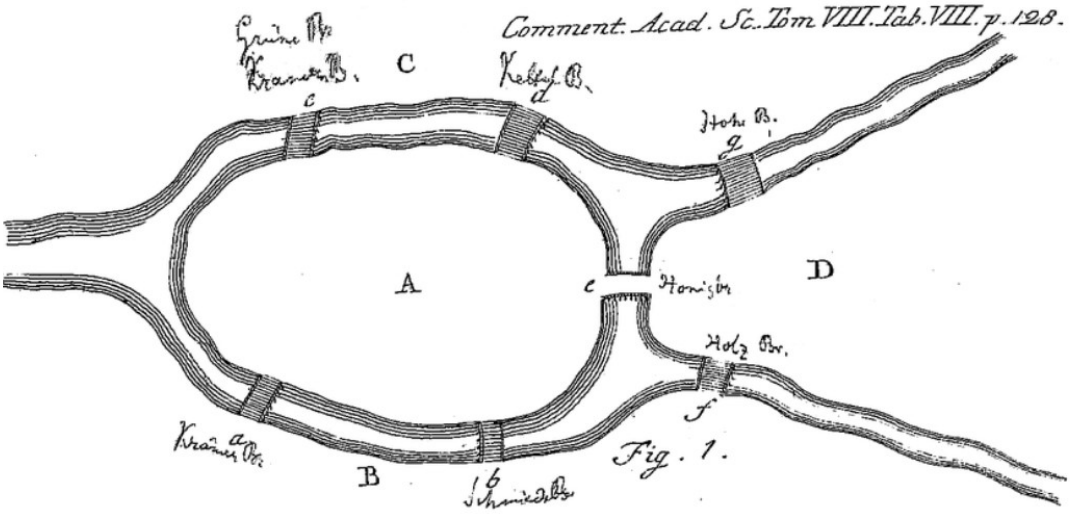

Figure 1: seven bridges of Königsberg

The Seven Bridges of Königsberg problem is a famous historical puzzle that played a crucial role in the development of graph theory. In the 18th century, the city of Königsberg in Prussia (now Kaliningrad, Russia) was situated on both sides of the Pregel River, and the river included two large islands connected to each other and the mainland by seven bridges as shown in figure 1. **The problem was to determine whether it was possible to take a walk through the city in such a way that each bridge would be crossed exactly once.**

The problem was first addressed by the Swiss mathematician **Leonhard Euler** in 1736. Euler approached the problem abstractly by representing the land masses (the four regions of Königsberg) as points and the bridges as lines connecting these points. This abstraction led to the creation of what we now call a "graph", and Euler's solution laid the foundation for graph theory.

Figure 2: Leonhard Euler (1707-1783)

Euler realized that the key to solving the problem was not in the specific layout of the land masses and bridges but in the connectivity and structure of the underlying graph. He proved that **it was impossible to find a continuous path that crossed each bridge exactly once if more than two land masses had an odd number of bridges connecting to them.** In the case of Königsberg, each land mass had an odd number of bridges (three of them had three bridges each, and one had five), making the problem unsolvable.

Euler's solution demonstrated the power of abstract thinking and the use of graphs to solve real-world problems. This marked the beginning of graph theory as a mathematical discipline. The concept of Eulerian paths and circuits, which Euler introduced while solving the Seven Bridges of Königsberg problem, has become fundamental to the field of graph theory.

1.2 Basic Definitions and Results

When proving the result Euler first represented the essential features of the bridge problem by a graph, as illustrated in the figure 3.

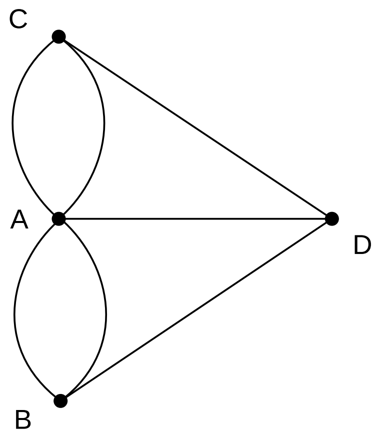

Figure 3: abstracted version

A picture of this abstracted version of the bridge problem kind is called a **pseudograph**. Now, let us consider the more formal definition.

Definition 1.2.1. *A pseudograph consists of a set of vertices (or points or nodes) and some pairs of these (not necessarily all pairs) are connected by edges (or line segments).*

Suppose G is a pseudograph. The set of vertices of G is usually denoted by V; the set of edges is denoted by E. Then G can be indicated as $G = (V, E)$.

Remark.

- *In pseudographs, **loops** (edges that connect a vertex to itself) and **multiple edges** (more than one edge connecting two vertices) may exist.*
- *Some authors refer to pseudographs as **multi-graphs**.*
- *It does not matter whether the edges are straight lines or curvy lines.*

Example. *A figure 4 illustrates a psedograph.*

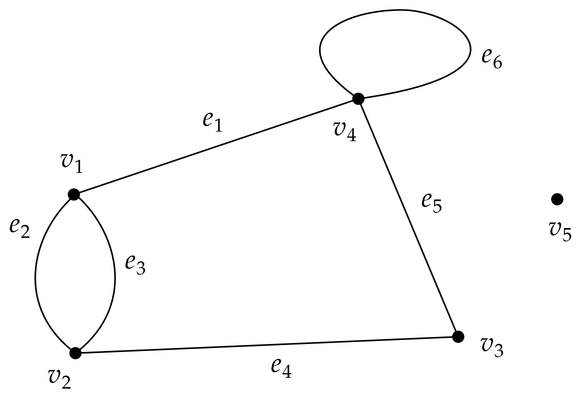

Figure 4: a pseudograph

Definition 1.2.2. *A pseudograph with no loops and multiple edges is called a **simple graph**.*

Remark. *Some authors use the term "graph" to refer to simple graphs, while others use it to refer to pseudographs.*

1.2.1 Adjacency, Incidence, and Degree

Definition 1.2.3.

- *Two vertices are said to be **adjacent** if they are connected to each other by an edge.*
- *If a vertex (v) is an end vertex of some edge (e) then they (v and e) are said to be **incident** on each other.*
- *The number of incidences on a vertex is called the **degree** of that vertex. The degree of the vertex v is denoted by $\deg(v)$.*
- *A vertex with degree zero is called an **isolated vertex**.*

Theorem 1.2.1 (The Handshaking Lemma)**.** *Let G be a simple graph (or a pseudograph) with n vertices (say $V(G) = \{v_1, v_2, \cdots v_n\}$) and m edges. Then,*

$$\boxed{\sum_{i=1}^{n} \deg(v_i) = 2m.}$$

Proof. An edge is incident with two distinct vertices if it is not a loop. Otherwise, it is incident to the same vertex twice. In both cases, an edge gets counted twice. Therefore, if there are m edges in G, then the degree sum is $2m$. □

Exercise 1. *Show that the graph given below agrees with the handshaking lemma.*

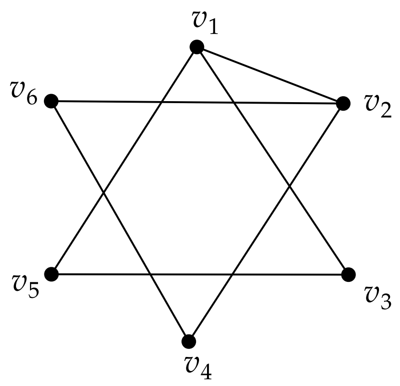

Figure 5: a simple graph

Theorem 1.2.2. *In every simple graph (pseudograph), the number of vertices with odd degrees is even.*

Exercise 2. *Prove theorem 1.2.2.*

1.2.2 Regular Graph, Null Graph, and Complete Graph

Definition 1.2.4.

- *If all the vertices of a graph have the same degree, then it is called a* ***regular graph****. More precisely, if the degree of each vertex is r then the corresponding is called an* ***r-regular graph****.*

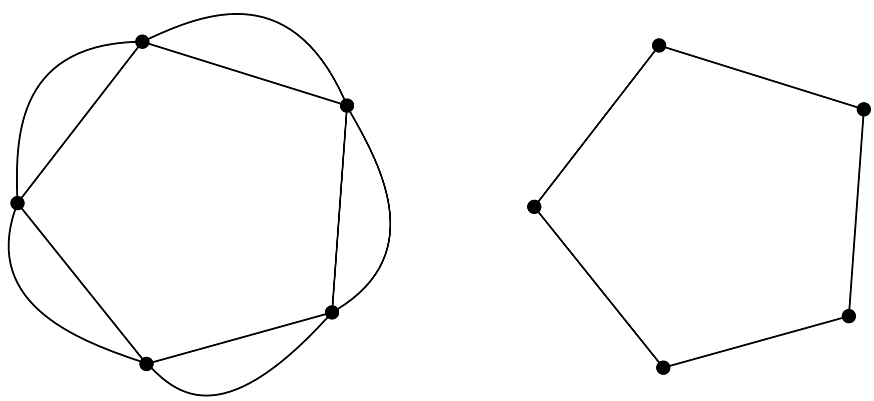

Figure 6: a 4-regular multi-graph and a 2-regular simple graph

- *A graph with no edges is called a* ***null graph****. (Vertices of a null graph are isolated.)*

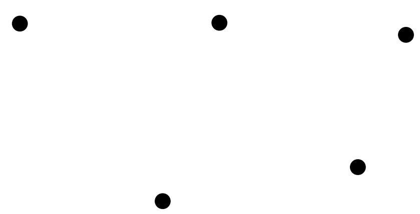

Figure 7: a null graph

- *A* ***complete graph*** *is a simple graph in which each pair of vertices is connected by an edge. (A complete graph contains the maximum possible number of edges.) The complete graph with n vertices is denoted by K_n.*

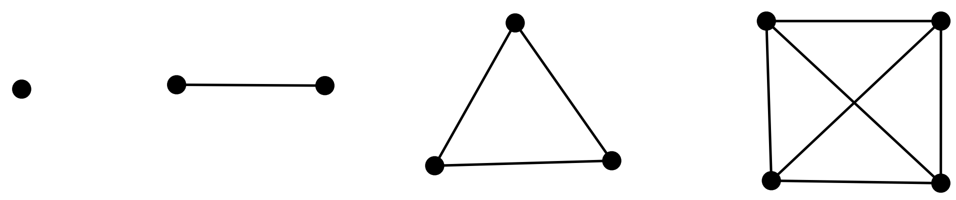

Figure 8: the first four complete graphs

Theorem 1.2.3. *For each $n \in \mathbb{N}$, the complete graph K_n contains $\boxed{\dfrac{n(n-1)}{2}}$ edges.*

Exercise 3. *Prove theorem 1.2.3.*

1.2.3 Subgraphs and Supergraphs

Definition 1.2.5.
- *A graph $G' = (V', E')$ is said to be a* ***subgraph*** *of a graph $G = (V, E)$ if whose vertex set and edge set are subsets of those of G. That is $V' \subseteq V$ and $E' \subseteq E$.*

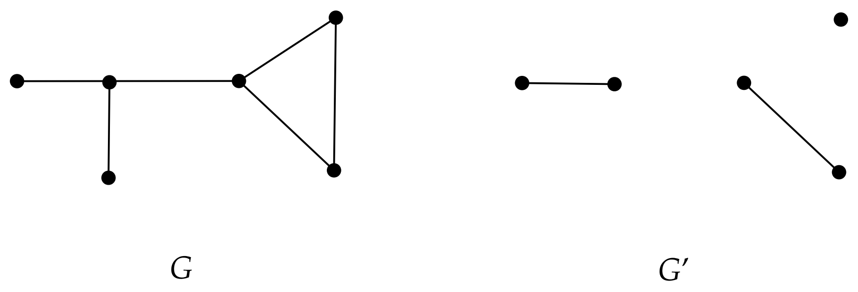

Figure 9: a subgraph and its supergraph

- *If G' is a subgraph of G, then G is said to be a* ***supergraph*** *of G'.*

1.2.4 Complement or Inverse

Definition 1.2.6. *The* ***complement*** *or* ***inverse*** *of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G. The complement (or inverse) of G is denoted by G^C (Here, graphs are assumed to be simple.)*

Exercise 4. *Find the complement of the graph G shown in the figure.*

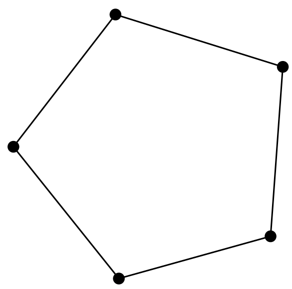

Figure 10: the graph G

Theorem 1.2.4. *For any simple graph G, $(G^C)^C = G$.*

Exercise 5. *Prove the above result.*

Definition 1.2.7. *A simple graph G is said to be* ***self-complementary*** *if $G \cong G^C$.*

Activity 1.2.1. *Draw three self-complementary graphs.*

1.2.5 Bipartite Graphs

Definition 1.2.8. *A **bipartite graph** is a simple graph where the vertices can be divided into two disjoint sets V_1 and V_2 such that all edges connect a vertex in one set to a vertex in another set. There are no edges between vertices in the set V_1 or in V_2.*

Note. *Two sets V_1 and V_2 in definition 1.2.8 are called **bipartition sets**.*

Definition 1.2.9. *A **complete bipartite graph** is a bipartite graph in which every vertex in V_1 is joined to every vertex in V_2. The complete bipartite graph on bipartition sets of m and n vertices is denoted by $\boxed{K_{m,n}}$.*

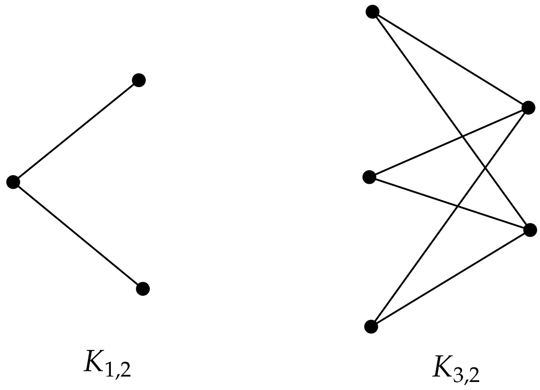

Figure 11: a few complete bipartite graphs

Exercise 6. *Find the number of edges in $K_{m,n}$. It is provided that $m, n \in \mathbb{N}$.*

Homework

1. Draw all simple graphs with 2, 3, 4, and 5 vertices.

2. Draw a simple graph with vertices representing the numbers $1, 2, \cdots 10$ in which two vertices are connected by an edge if and only if their greatest common divisor is greater than one.

3. Depict the following data by a graph, assuming all acquaintances are mutual:

   There is a group of five people in a mafia gang: Wimal, Chamal, Namal, Nimal, and Neomal. When they first met, Wimal knew everybody else, Chamal and Namal knew each other, and also Namal knew Neomal.

4. Is there a simple graph on 7 vertices with degrees $2, 3, 4, 5, 6, 7, 8$? Justify your answer.

5. How many edges does K_{10} have?

6. Suppose G is a simple graph with n vertices. Is $G \cup G^C = K_n$? Justify your answer.

7. Prove that, at a party with two or more than two people, it is always possible to find two people having the same number of friends.

8. Prove that, at a party with six people, it is always possible to find three mutual friends or three mutual strangers.

9. Can there be a self-complementary regular graph with

   (i). 5 vertices?

   (ii). 6 vertices?

   (iii). 7 vertices?

   (iv). 8 vertices?

   (v). 9 vertices?

10. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum possible degrees of the vertices of a graph $G = (V, E)$, and $|V|$ and $|E|$ denote the number of vertices and edges of G respectively. Prove that

$$\delta(G) \leq 2\frac{|E|}{|V|} \leq \Delta(G).$$

11. Which of the following graphs are bipartite?

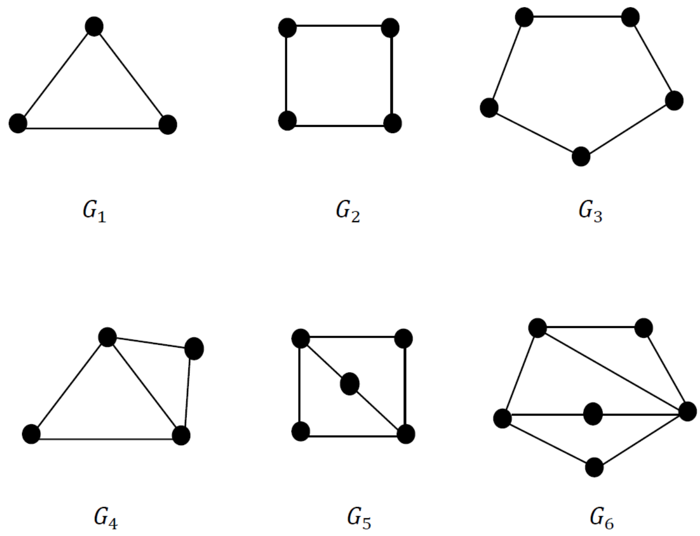

Figure 12:

12. Sajith says that a subgraph of a bipartite graph is always bipartite. Is his statement true? Justify your answer.

13. Assuming n is an even integer, prove that the number of edges in a bipartite graph with n vertices is at most $\frac{n^2}{4}$.

14. What is the largest possible number of vertices in a graph with 35 edges, all vertices having degree at least three?