# CS3063 Theory of Computing

## Semester 4 (20 Intake), Feb – Jun 2023

**Lecture 5**

**Regular Languages & Finite Automata – Session 4**

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# Other Announcements

- Assignment 1: due 24th April

- Mid-semester Test – tentative details:
  - Date, Time: 4th May, 8.15am-10.15am
  - Venue: Exam Hall 2

# Today’s Outline

## Lecture 5

- **FA with outputs (Moore, Mealy models)**
- **Pumping lemma for regular languages**
- **Applications of FA**
- **State Minimization**

[ **Conclusion of “FA + Regular Languages”** ]

# Overview of Topics Covered:

- Regular expressions/languages
- Finite automata (FA)
- Regular language $\leftrightarrow$ FA
- NFA
  - Given NFA → equivalent deterministic FA
- NFA-$\Lambda$
  - Given NFA-$\Lambda$ → equivalent NFA
- Equivalency among DFA, NFA, NFA-$\Lambda$

PART 1

# Today's Outline

## Lecture 5

- FA with outputs (Moore, Mealy models)
- Pumping lemma for regular languages
- Applications of FA
- State Minimization

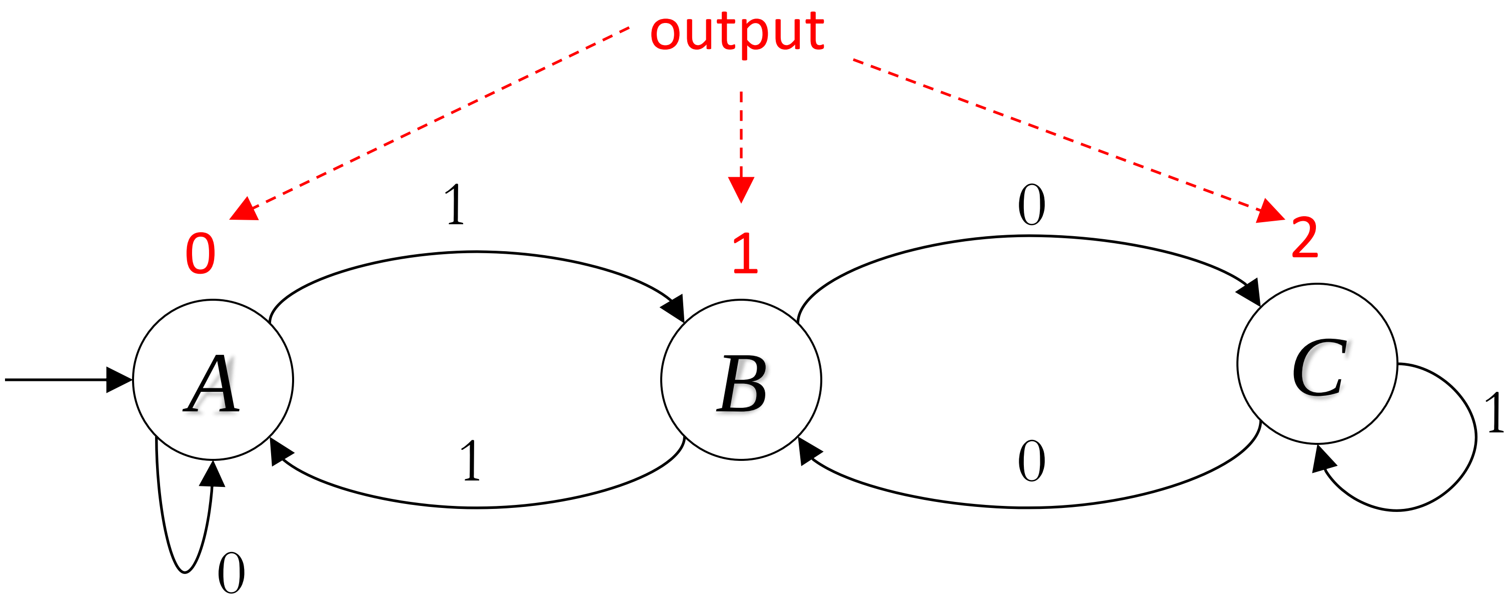
# 1. FA With Outputs

- So far we considered FA with a binary result: “accept” or “reject”
- Outputs from other alphabets are possible
- Two approaches
  - Moore model/machines
  - Mealy model/machines

# Moore Machines

- The output is associated with the state
- Formally, a Moore machine is a 6-tuple $(Q, \Sigma, \Delta, q_0, \delta, \lambda)$ where,
  - $Q, \Sigma, q_0, \delta$ are as in FA we studied
  - $\Delta$ is the output alphabet
  - $\lambda$ is a mapping from $Q$ to $\Delta$ (gives the output associated with each state)

# Example Moore Machine



DFA -> We will get the final output, at the end

# Example Moore Machine

- Transition Table
  - For the transition diagram in previous slide

doesn't depend on the input

| Present state | Next state | | Output |
|---|---|---|---|
| | Input=0 | Input=1 | |
| →A | A | B | 0 |
| B | C | A | 1 |
| C | B | C | 2 |

# FA → Moore Machine?

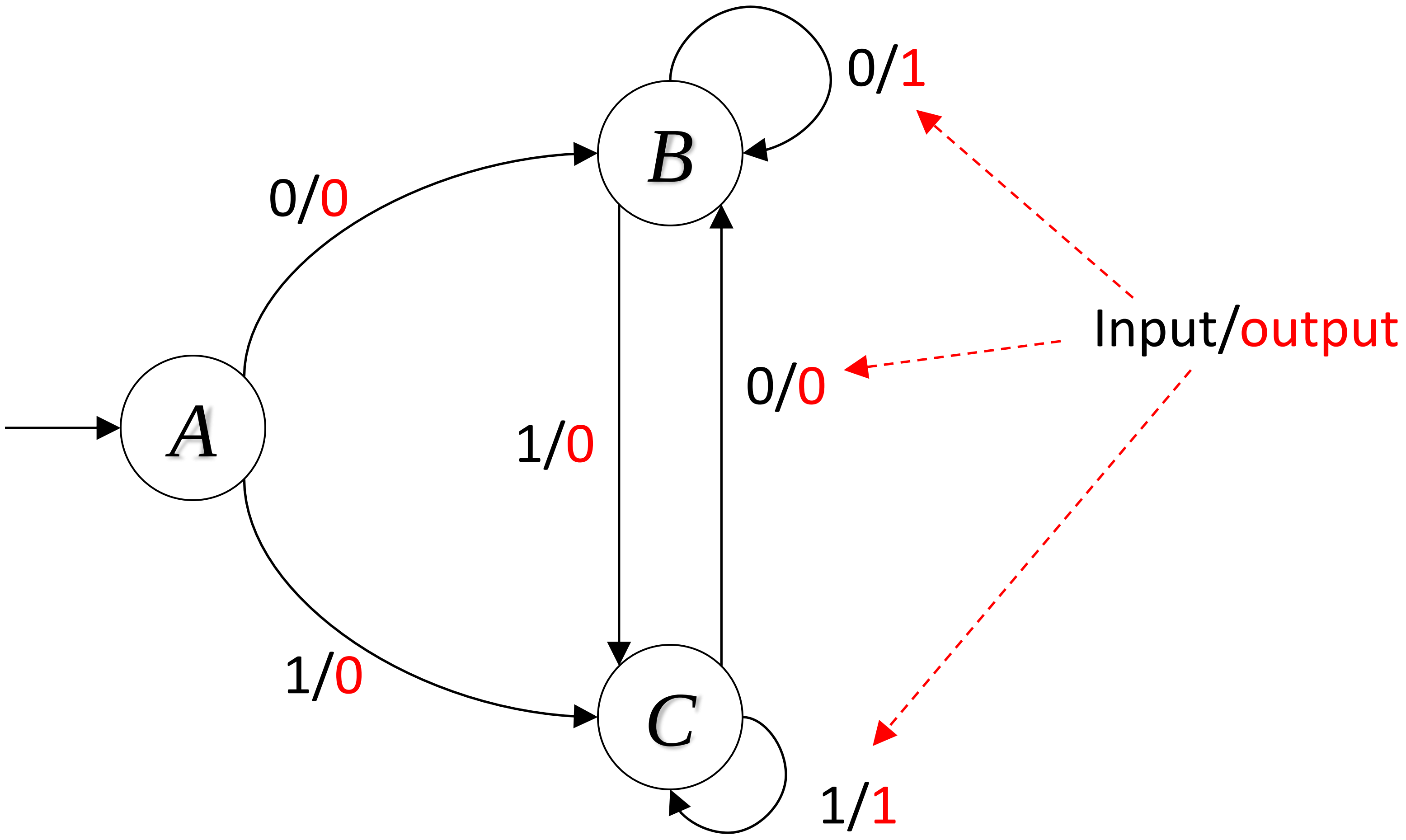
- Given an FA, we can get an "equivalent Moore machine" as follows
  - $\Delta = \{0, 1\}$
  - $\lambda(q)=1$ if q is an accepting state
  - $\lambda(q)=0$ if q is not an accepting state

# Mealy Machines

- The output is associated with the transition

- A Mealy machine is a 6-tuple $(Q, \Sigma, \Delta, q_0, \delta, \lambda)$ where,
  - All elements are as in the Moore machine, …
  - Except $\lambda$ maps $Q \times \Sigma$ to $\Delta$
  - That is, $\lambda(q, a)$ gives the output associated with the transition from state $q$ on input $a$

# Example Mealy Machine



# Example Mealy Machine

- Transition Table
  - For the transition diagram in previous slide

Depend on the input as well

| Present State | Input=0 | | Input=1 | |
|---|---|---|---|---|
| | Next State | Output | Next State | Output |
| →A | B | 0 | C | 0 |
| B | B | 1 | C | 0 |
| C | B | 0 | C | 1 |

# Moore vs. Mealy Models

- If the input string is of length ***n***, the length of the output string is:
  - For a Moore machine → ***n***+1
    - $\lambda(q_0)$ is the same for all cases
  - For a Mealy machine → ***n***

- What is the output for input $\Lambda$ ?
  - Moore machine gives output $\lambda(q_0)$
  - Mealy machine gives output $\Lambda$

# Moore-Mealy Equivalence

- Ignoring the output of a Moore machine for input $\Lambda$, *for a given Moore machine there is an equivalent Mealy machine* (and vice versa)
  – i.e., for a given input string, the output strings would be the same for the two machines
- Homework
  – Find how to convert between the two types

PART 2

# Today's Outline

## Lecture 5

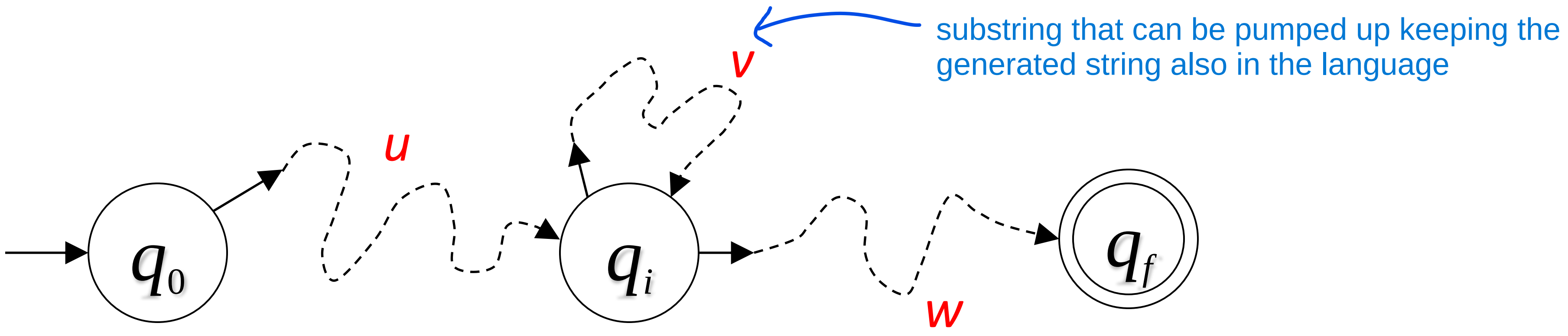
- FA with outputs (Moore, Mealy models)
- **Pumping lemma for regular languages**
- Applications of FA
- State Minimization

# 2. Pumping Lemma

- Allows us to prove non-regularity (i.e., that a language is not regular)
- A theorem that says all regular languages have a special property
  - Suppose $M=(Q, \Sigma, q_0, A, \delta)$ is an FA that recognizes a language $L$
  - Strings with sufficient length (pumping length) in the language can be "pumped" up
  - These strings correspond to "loops" in the path of transitions from start state to accepting state

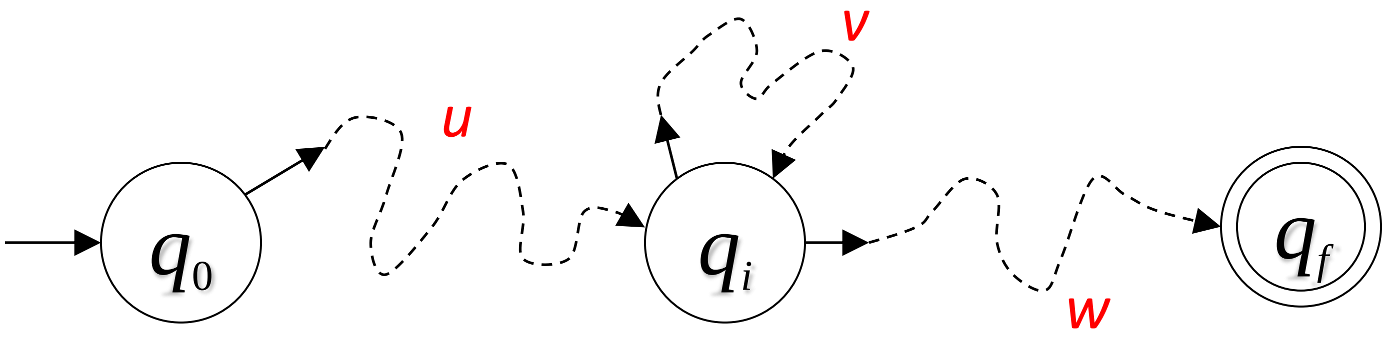
# Pumping Lemma ...contd

- For a string $x \in L$, if $\mathrm{M}$ enters a state twice then we have a path with a loop
  - $x$ is of the form $uvw$ where $v$ corresponds to the loop



# Pumping Lemma ...contd

- If $|Q|=n$, for a string $x$ in $L$ with length at least $n$
  - We can write, $x=a_1a_2...a_ny$
  - The sequence of $n$+1 states $q_0=\delta^*(q_0,\Lambda)$, $q_1=\delta^*(q_0, a_1)$, $q_2=\delta^*(q_0, a_1a_2)$,…, $q_n=\delta^*(q_0, a_1a_2...a_n)$ must contain some state at least twice (where loop exists)
  - $\delta^*(q_i, v) = q_i$ means $\delta^*(q_i, v^m) = q_i$ for every $m \geq 0$
  - So, $\delta^*(q_0, uv^mw) = q_f$ for every $m \geq 0$



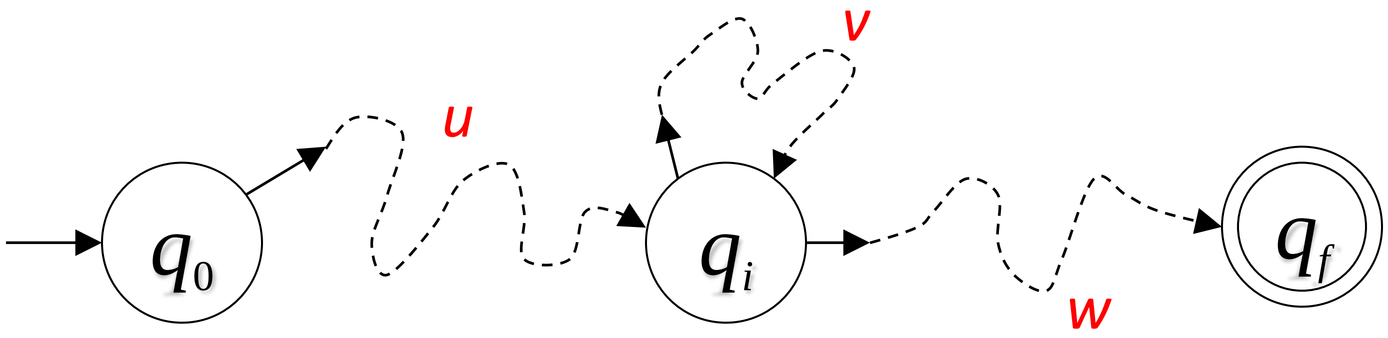
# Pumping Lemma ...contd

- Pumping Lemma: Version 1
  - Suppose $L$ is a regular language recognized by an FA with $n$ states. For any string $x$ in $L$ with $|x| \geq n$, $x$ may be written as $x=uvw$ for some strings $u$, $v$ and $w$ satisfying

$|uv| \leq n$

$|v| > 0$ V must not be empty

for any $m \geq 0$, $uv^mw$ is in $L$



# Pumping Lemma ...contd

- Pumping Lemma: Version 2 (more common)
  - Suppose $L$ is a regular language. Then there is an integer $n$ so that for any $x$ in $L$ with $|x| \geq n$, there are strings $u$, $v$ and $w$ so that

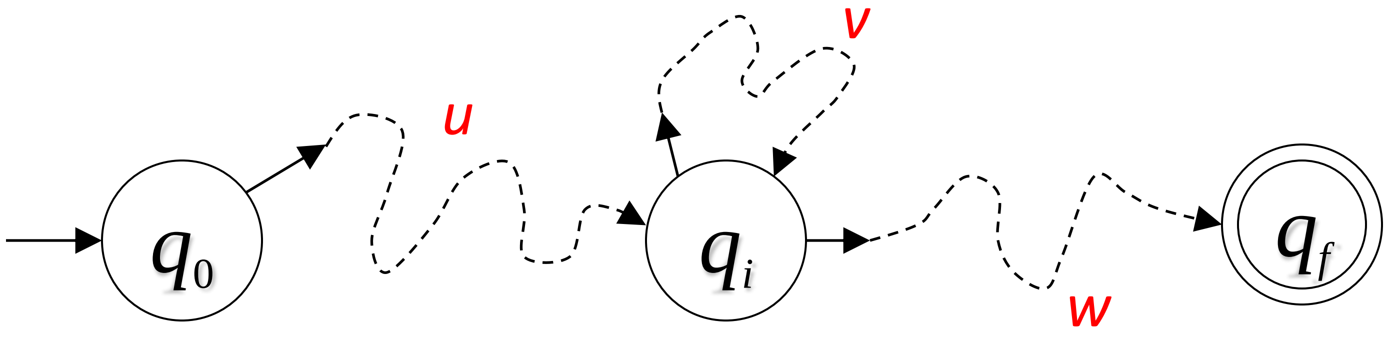
$x=uvw$

$|uv| \leq n$

$|v| > 0$

for any $m \geq 0$, $uv^m w$ is in $L$

Either ***u*** or ***w*** may be Λ, but ***v*** can't be Λ



# Pumping Lemma ...contd

- Idea: for an arbitrary string of sufficient length in *L*, a portion of it can be pumped up
  - Lemma gives a *necessary condition* to be regular
- To prove that a language is not regular using this lemma, we must show that the language does not have the property described in it
  - Can assume property holds and show contradiction
  - E.g., assume there is an ***n*** (although we do not know it), then find a string ***x***, with $|x| \geq n$, that will lead to a contradiction

# Example

- Show that $L = \{0^i1^i \mid i \geq 0\}$ is not regular
  – Assume properties in pumping lemma hold for $L$
  – Choose ***x*** with |***x***|$\geq n$; a reasonable choice is ***x***= $0^n1^n$
  – Lemma says ***x*** can be split into 3 as ***x***=***uvw*** for some ***u***, ***v***, ***w*** and for any ***m*** $\geq$ 0, ***uv<sup></sup>***$^{m}$***w*** is in $L$

even the u and w are empty, v^m != 0^n1^n

    - We can show this is not possible, as follows
    - Note: either ***u*** or ***w*** may be $\Lambda$, but ***v*** can't be $\Lambda$

- <u>**Case 1**</u>: <u>The string ***v*** with only 0s</u>

V = 0

  – For any ***u***, ***w***, the string ***uvvw*** has more 0s than 1s; → ***uvvw*** is not in $L$
  – Similarly, for any ***m*** $\geq$ 0, **uv**$^{\mathbf{m}}$**w** is not in $L$
  – This case is a contradiction

# Example ...contd

- Show that $L = \{0^i 1^i \mid i \geq 0\}$ is not regular
- **Case 2**: The string ***v*** with only 1s

  V =1

  - For whatever ***u***, ***w***, for any ***m*** $\geq$ 0, the string **uv$^{m}$w** has more 1s than 0s; so **uv$^{m}$w** is not in $L$
  - This case is a contradiction
- **Case 3**: String ***v*** consists of both 0s and 1s

  V = 01 => V^n = (01)^n != 0^n1^n

  - In this case, the string ***uvvw*** may have the same number of 0s and 1s, but they will be out of order (some 1s before 0s)
  - A contradiction

# Example ...contd

- Show that $L = \{0^i1^i \mid i \geq 0\}$ is not regular
  - [Cases 2 and 3 can be eliminated by considering the condition $|\boldsymbol{uv}| \leq$ n ]

  - Contradictions for all cases of ***v***
  - $L$ cannot be regular

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- Programming languages are not regular
  - E.g., main( ) $\{^n\}^n$

PART 3

# Today's Outline

## Lecture 5

- FA with outputs (Moore, Mealy models)
- Pumping lemma for regular languages
- Applications of FA
- State Minimization

# 3. Applications of FA

- Modeling of *reactive systems*
  - Reactive system
    - A system that changes its actions, outputs and status in response to stimuli from within or outside
    - Maintains an ongoing interaction with the environment rather than produce some final value upon termination
  - Examples
    - Vending machines, ATMs, communication protocols
    - Systems for air-traffic control
    - Control systems for trains, planes, nuclear plants

# Applications of FA

- Some software design problems simplified by using regular expressions or converting regular expressions to FA

- Though programming languages are not regular, *tokens* (identifiers, literals, operators, reserved words, punctuation) can be described by regular expressions

# Applications of FA

- Lexical analysis/analyzers
  - First phase in compiling a program
  - Identifying and classifying the tokens
  - Lexical-analyzer generator
    - Input: sequence of regular expressions (for tokens)
    - Output: a lexical analyzer (an FA) to recognize any token
    - E.g., *lex* and *flex*

# Applications of FA ...contd

- Text editors
  - Operations based on regular expressions
  - For searching, substitution
  - E.g., **vi** editor
    - `%s/\s\s\s*/\s/` → substitute two or more spaces by a single space
- “grep”: utility to search for reg. expressions
- Other similar tools, situations…

PART 4

# Today's Outline

## Lecture 5

- FA with outputs (Moore, Mealy models)
- Pumping lemma for regular languages
- Applications of FA
- **State Minimization**

# 4. (State) Minimization of DFA

- Minimization of DFA means *minimizing the number of states* of an DFA
- Detailed discussion on this requires understanding of equivalence relations and equivalence classes of states
- *Myhill-Nerode Theorem*
  - ***Reading assignment***
  - Provides a necessary and sufficient condition for a language to be regular https://youtu.be/Dx2RJ2DXRYs

# Minimization ...contd

- Myhill-Nerode theorem implies that there is a unique minimum-state DFA for every regular language

- Idea is to identify pairs of *equivalent states*
  - Two states $q_i$ and $q_j$ are equivalent if some language L takes the DFA from either state to an accepting state (same or different)

# Minimization ...contd

- In practice, rather than looking for pairs of equivalent states we find pairs ($p$, $q$) of *distinguishable states*, which is easier
  - i.e., $\delta^*(p, x)$ is an accepting state and $\delta^*(q, x)$ is not, or vice versa, for some string $x$
- **If two states are not *equivalent*, they are *distinguishable***
  - All pairs of states are presumed equivalent until they are proved distinguishable

# State Minimization ...contd

- Initially we have *two equivalence classes* or *two distinguishable sets of states*
  - The *set of accepting states*, and
  - The *set of non-accepting states*
- But we initially don't know the equivalence relation between 2 states in one class
  - So, next we consider pairs of states presumed equivalent (not yet distinguishable)
  - For this, consider transitions from states

# State Minimization ...contd

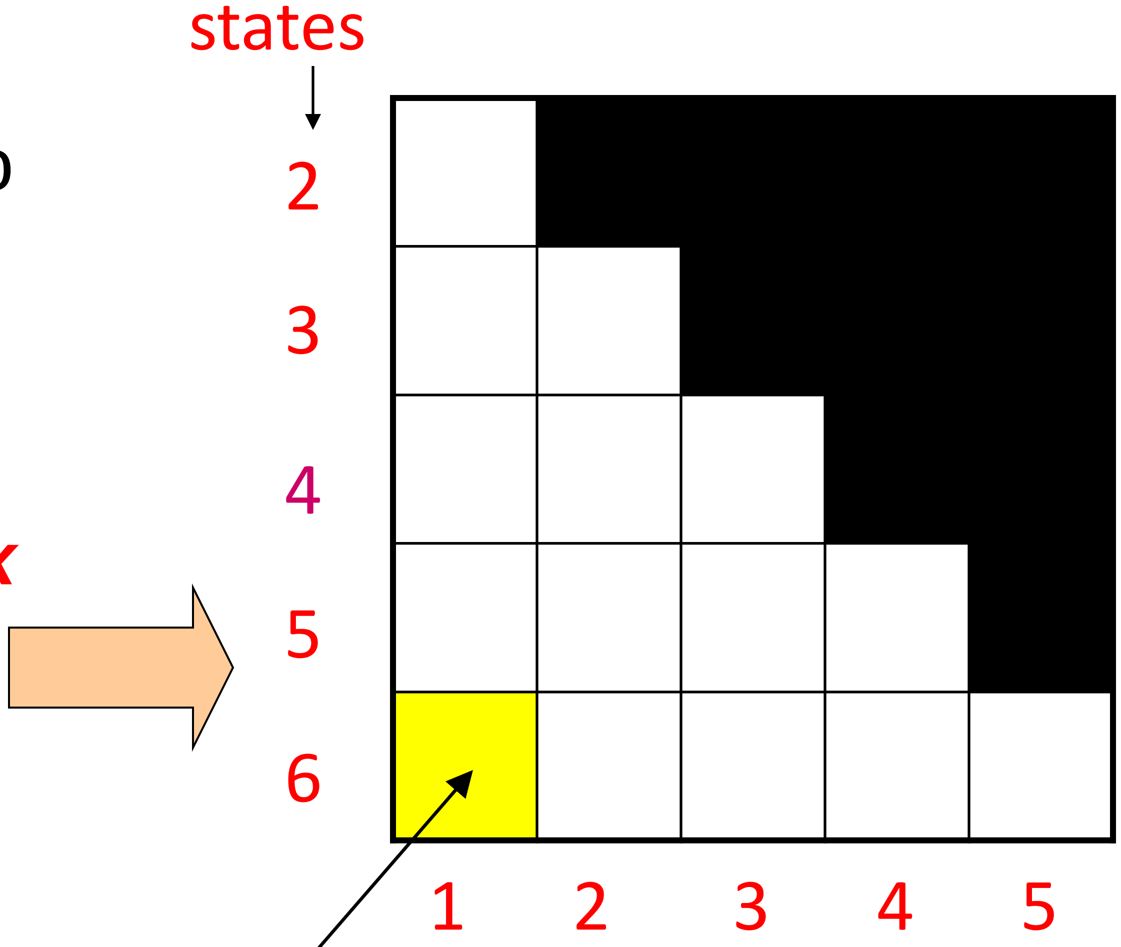
- We look at single symbols from $\Sigma$ to check the transitions from pairs of states
  - If all symbols in $\Sigma$ take a DFA from states *p* and *q* to accepting states, then *p* and *q* are equivalent
  - Even if one symbol in $\Sigma$ takes a DFA from states *p* and *q* to a pair of states already known to be distinguishable, then *p* and *q* are also distinguishable

# Minimization Algorithm

- To identify distinguishable pairs of states
  - List all (unordered) pairs of states
  - Make a sequence of passes through these
  - **1st pass**: mark each pair of which exactly one is an accepting state
  - **Next passes**: mark any pair $(p, q)$ if there is an $a$ in $\Sigma$ for which $\delta(p, a)=r$, $\delta(q, a)=s$ and also $(r, s)$ is already marked
  - After a pass with no new pair marked, ***stop***
  - Marked states→distinguishable, else→equivalent

# Minimization Algorithm

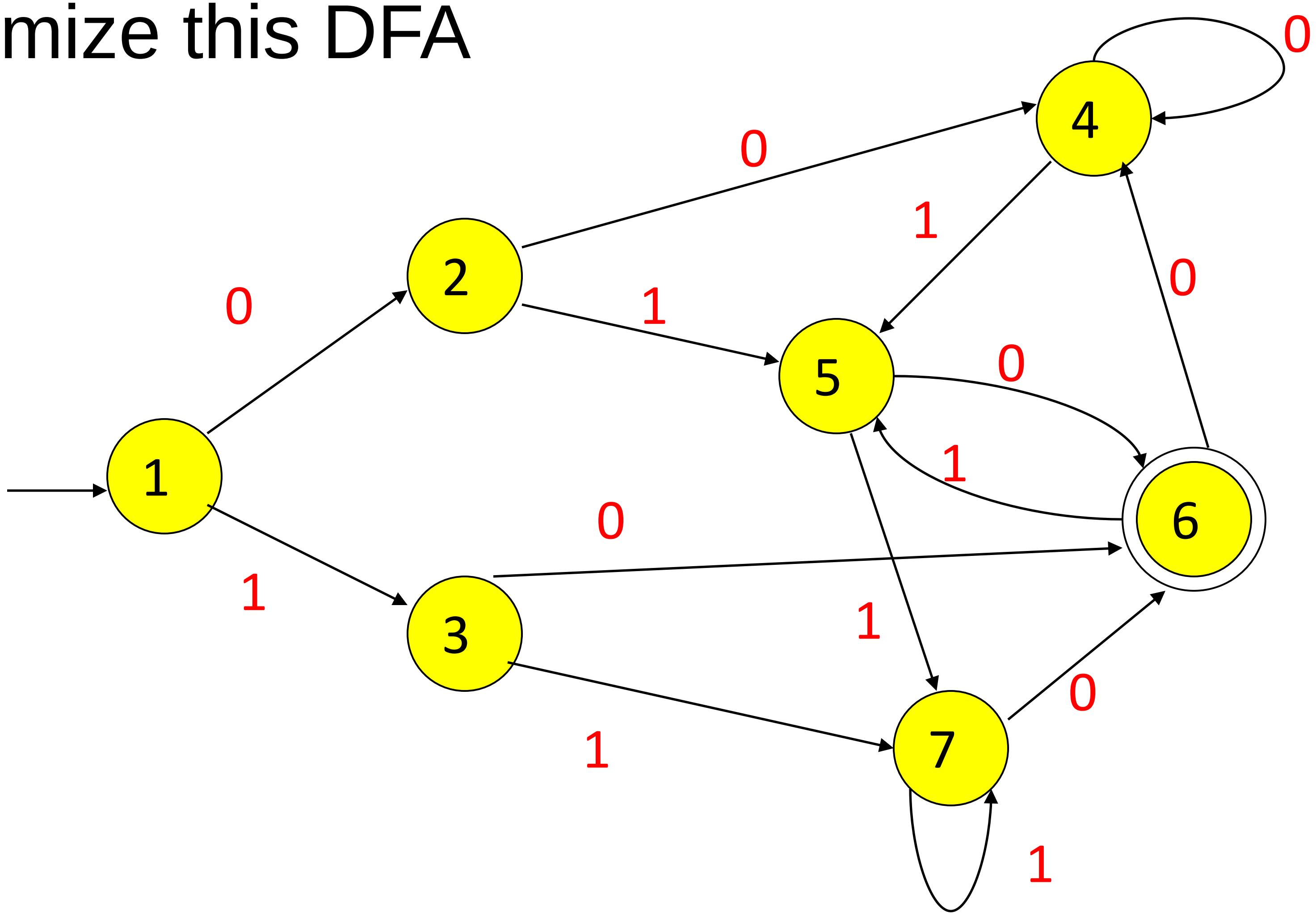
- Can use a lower (or upper) triangular matrix to mark the pairs in passes
- This is the ***distinguishability matrix***
  - Example is for a DFA with 6 states




Mark this for the pair (1, 6)

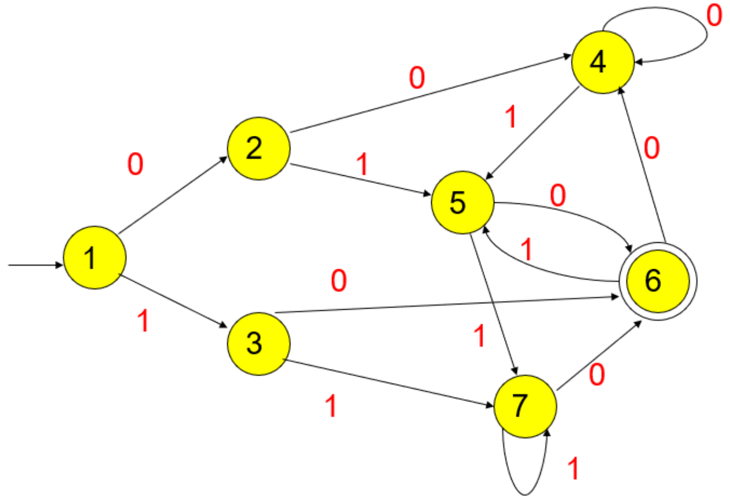
# Example 5.6 in Book (p. 179)

- Minimize this DFA



# Solution – Step 1

- First pass
  - Pairs marked as “1” are those with exactly one element being an (the only) accepting state



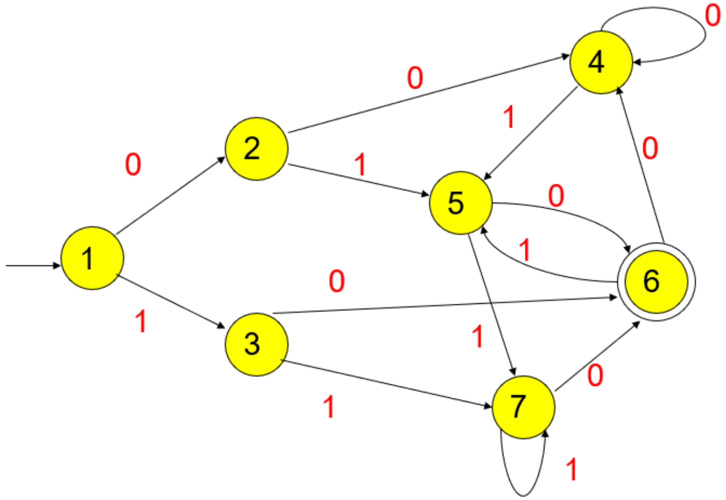

This row is not needed

~~Distinguishability Matrix~~

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | 1 | 1 | 1 | 1 | 1 | |
| 7 | | | | | | 1 |
| | 1 | 2 | 3 | 4 | 5 | 6 |

# Solution – Step 2

- $2^{nd}$ pass
  - Pairs marked as “2”
  - (2,5) is marked because $\delta(2,0)=4$, $\delta(5,0)=6$ and (4, 6) is already marked
  - Similarly for other cases




**Distinguishability Matrix**

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 2 | | | | | | |
| 3 | 2 | 2 | | | | |
| 4 | | | 2 | | | |
| 5 | 2 | 2 | | 2 | | |
| 6 | 1 | 1 | 1 | 1 | 1 | |
| 7 | 2 | 2 | | 2 | | 1 |

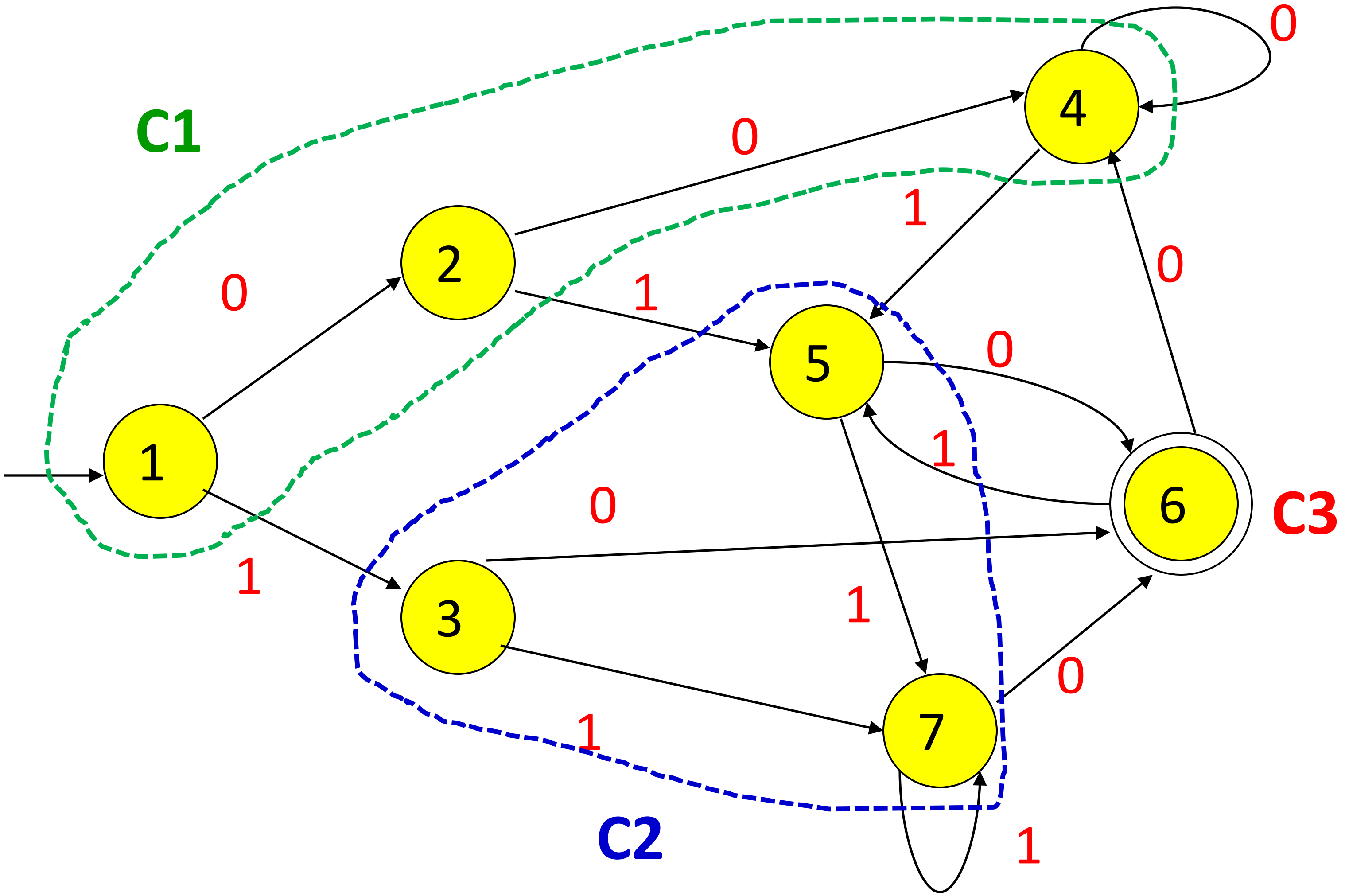
# Solution – Step 3

- 3rd pass
  - No new pairs marked
- Stop !!
- Equivalence classes
  - {1, 2, 4}
  - {3, 5, 7}
  - {6}

**Distinguishability Matrix**

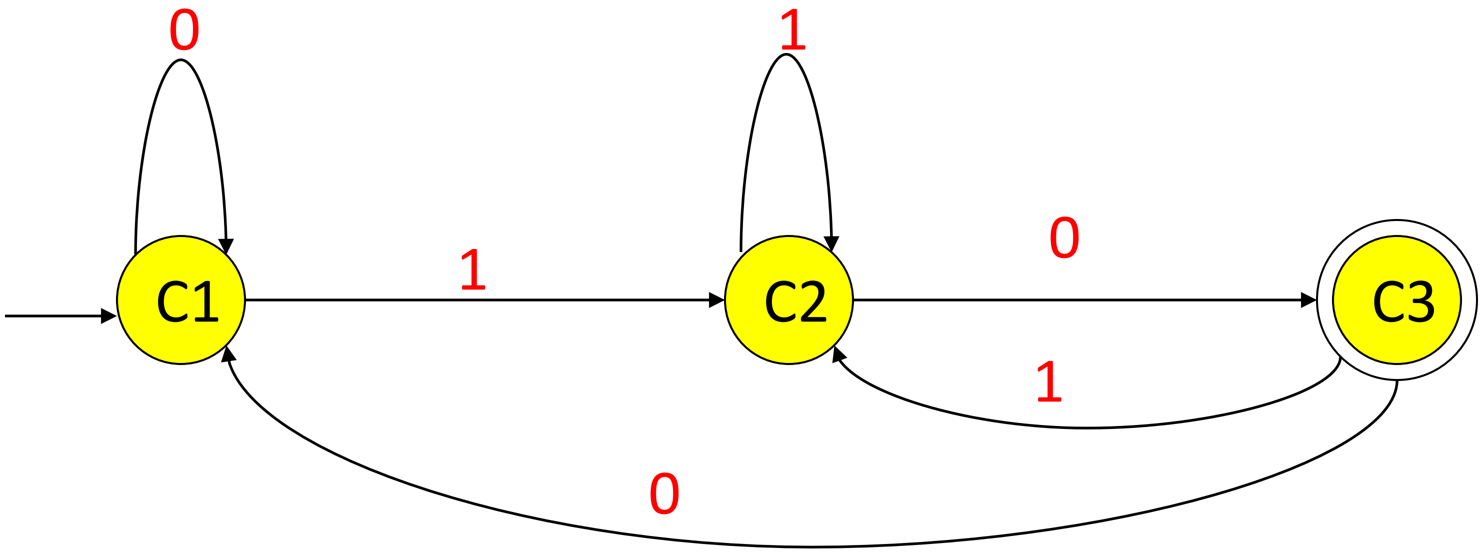
| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | | | | | | |
| 3 | 2 | 2 | | | | |
| 4 | | | 2 | | | |
| 5 | 2 | 2 | | 2 | | |
| 6 | 1 | 1 | 1 | 1 | 1 | |
| 7 | 2 | 2 | | 2 | | 1 |
| | 1 | 2 | 3 | 4 | 5 | 6 |

# Solution – Step 4



# Solution – Final Answer

- Minimum state DFA

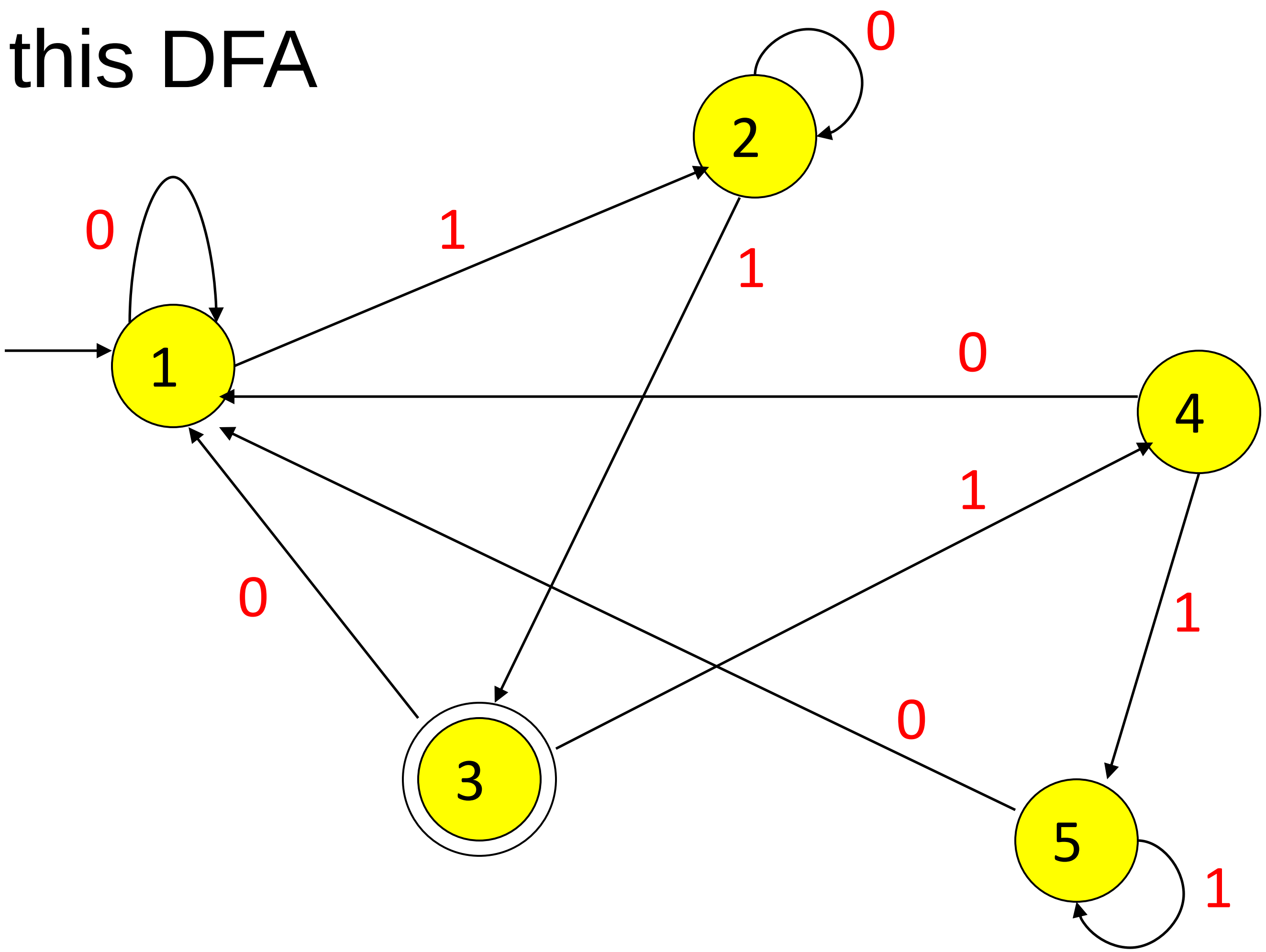


# More on Minimization

- Within a pass, the following is possible
  - A pair ($p$,$q$) is unmarked while every pair ($r$, $s$) such that $\delta(p, a)=r$, $\delta(q, a)=s$ for every $a$ in $\Sigma$ is also unmarked
  - Add ($p$,$q$) to a linked list for each ($r$, $s$); if later ($r$, $s$) is marked, then mark ($p$,$q$) also
- At the end
  - an unmarked-pair means the 2 states are equivalent and can be merged
  - # of equivalent classes = # of minimum states

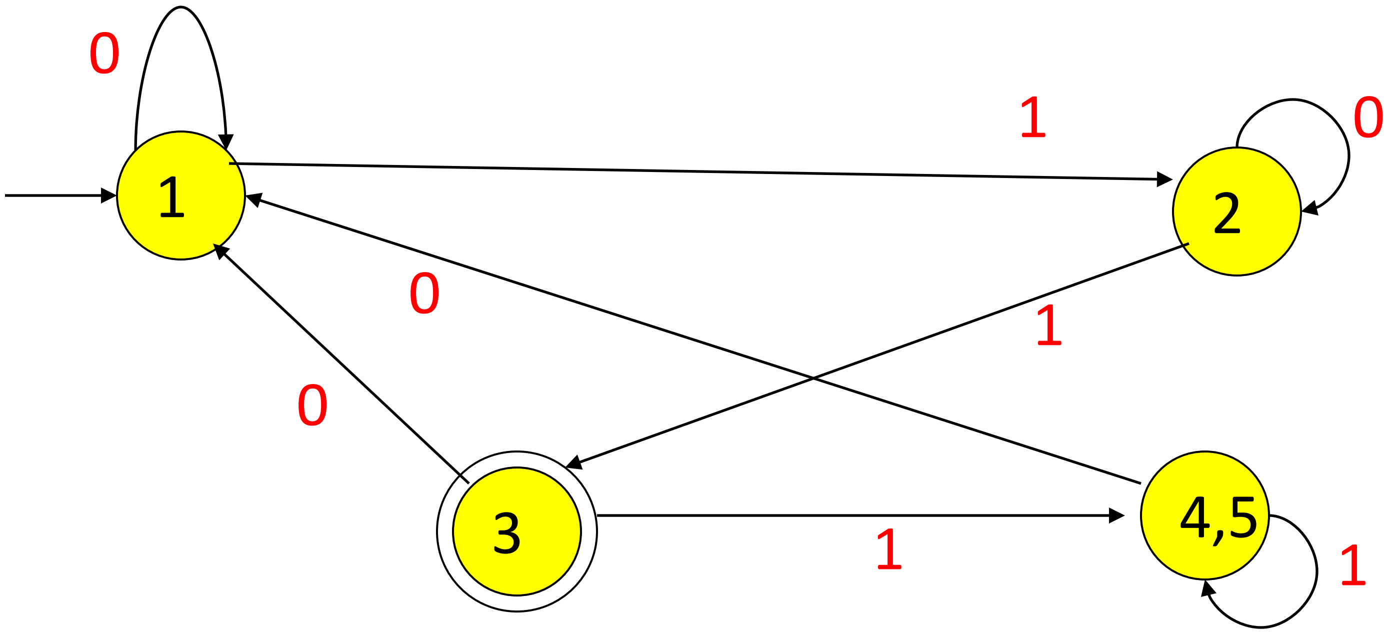
# Exercise

- Minimize this DFA



# Solution

- States 4 and 5 are equivalent (in the same equivalence class, indistinguishable)
  - Can merge 4 and 5



# Conclusion

- Today we discussed
  - FA with output
  - Pumping lemma
  - Applications of FA
  - State Minimization
- We conclude “FA+Regular Languages”

- Next topic: Context-free Languages