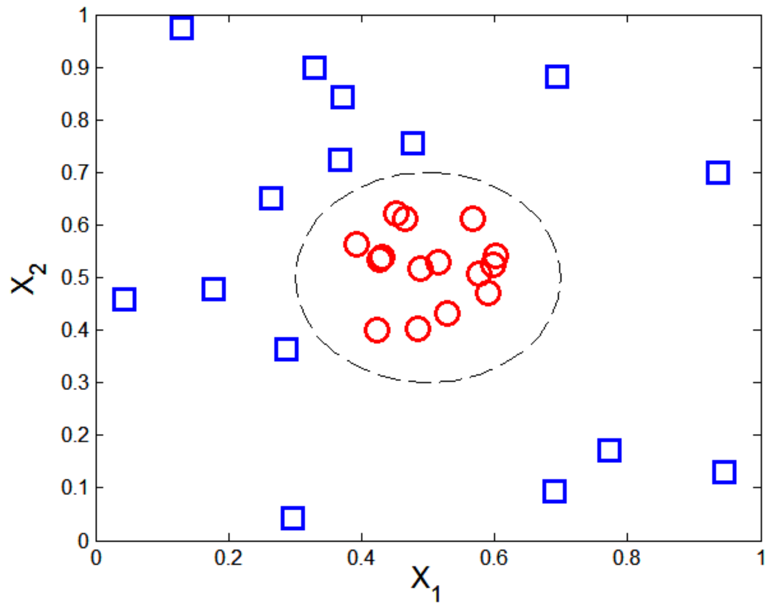
Nonlinear Support Vector Machines

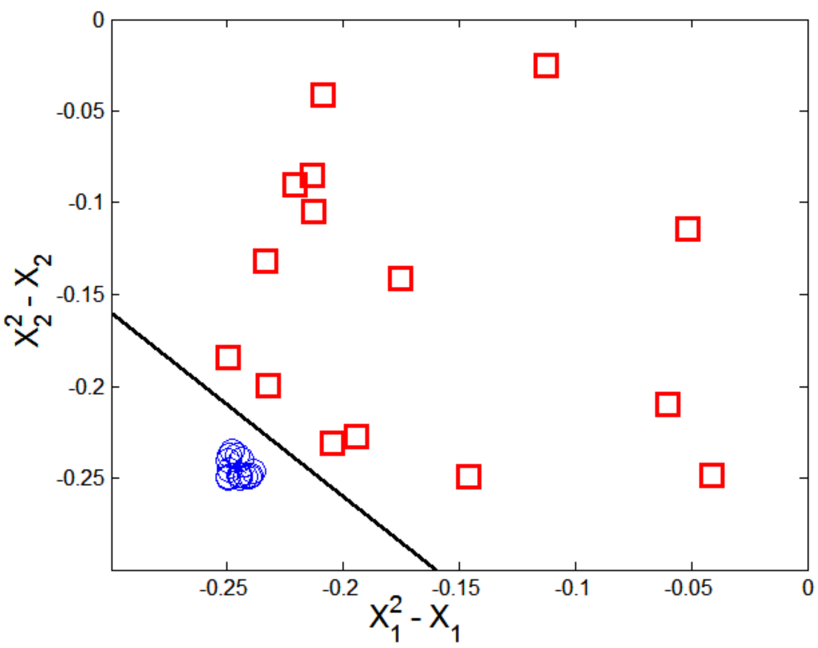
- What if decision boundary is not linear?



$$y(x_1, x_2) = \begin{cases} 1 & \text{if } \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2} > 0.2 \\ -1 & \text{otherwise} \end{cases}$$

Nonlinear Support Vector Machines

- Transform data into higher dimensional space



$$x_1^2 - x_1 + x_2^2 - x_2 = -0.46.$$

$$\Phi : (x_1, x_2) \longrightarrow (x_1^2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2, 1).$$

$$w_4 x_1^2 + w_3 x_2^2 + w_2 \sqrt{2} x_1 + w_1 \sqrt{2} x_2 + w_0 = 0.$$

Decision boundary:

$$\mathbf{w} \,.\, \Phi(\mathbf{x}) + b = 0$$

Learning Nonlinear SVM

- Optimization problem:

$$\min_{w} \frac{\|\mathbf{w}\|^2}{2}$$
$$\textit{subject to} \quad y_i(\boldsymbol{w} \cdot \Phi(\boldsymbol{x}_i) + b) \geq 1, \ \forall \{(\boldsymbol{x}_i, y_i)\}$$

- Which leads to the same set of equations (but involving Φ(x) instead of x)

$$L_D = \sum_{i=1}^{n} \lambda_i - \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j y_i y_j \Phi(\mathbf{x}_i) \cdot \Phi(\mathbf{x}_j)$$

$$\mathbf{w} = \sum_{i} \lambda_i y_i \Phi(\mathbf{x}_i)$$

$$\lambda_i \{ y_i (\sum_{j} \lambda_j y_j \Phi(\mathbf{x}_j) \cdot \Phi(\mathbf{x}_i) + b) - 1 \} = 0,$$

$$f(\mathbf{z}) = sign(\mathbf{w} \cdot \Phi(\mathbf{z}) + b) = sign(\sum_{i=1}^{n} \lambda_i y_i \Phi(\mathbf{x}_i) \cdot \Phi(\mathbf{z}) + b).$$

Learning Nonlinear SVM

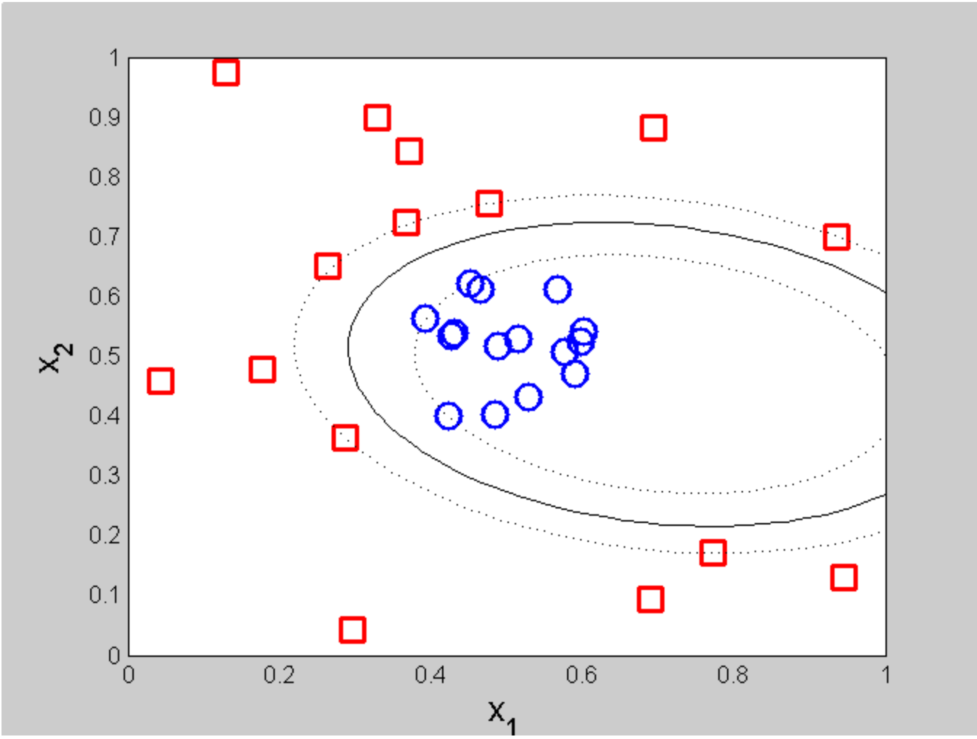
- Issues:
  - What type of mapping function Φ should be used?
  - How to do the computation in high dimensional space?
    - Most computations involve dot product $\Phi(x_i) \bullet \Phi(x_j)$
    - Curse of dimensionality?

Learning Nonlinear SVM

- Kernel Trick:
  - $\Phi(x_i) \bullet \Phi(x_j) = K(x_i, x_j)$
  - $K(x_i, x_j)$ is a kernel function (expressed in terms of the coordinates in the original space)
    - Examples:

$$K(\mathbf{x}, \mathbf{y}) = (\mathbf{x} \cdot \mathbf{y} + 1)^p$$
$$K(\mathbf{x}, \mathbf{y}) = e^{-\|\mathbf{x}-\mathbf{y}\|^2/(2\sigma^2)}$$
$$K(\mathbf{x}, \mathbf{y}) = \tanh(k\mathbf{x} \cdot \mathbf{y} - \delta)$$

Example of Nonlinear SVM



SVM with polynomial degree 2 kernel

Learning Nonlinear SVM

- Advantages of using kernel:
  - Don’t have to know the mapping function Φ
  - Computing dot product $\Phi(x_i) \bullet \Phi(x_j)$ in the original space avoids need for very high dimensional transforms “kernel trick”

- Not all functions can be kernels
  - Must make sure there is a corresponding Φ in some high-dimensional space
  - Mercer’s theorem (see textbook)

Characteristics of SVM

- The learning problem is formulated as a convex optimization problem
  - “Efficient” algorithms are available to find the global minima
  - Many of the other methods use greedy approaches and find locally optimal solutions
  - High computational complexity for building the model

- Robust to noise
- Overfitting is handled by maximizing the margin of the decision boundary,
- SVM can handle irrelevant and redundant features better than many other techniques
- The user needs to determine the kernel function and C parameter