Price Of Anarchy

EE6418 : Dynamic Games : Theory and Applications

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   - Definition
   - Variations of PoA
2. Motivation
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   - Braess Paradox
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   - Coping with Selfishness
   - Marginal Cost Pricing
   - Capacity Augmentation

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Definition

Price of Anarchy(PoA) measures the degradation of a system's efficiency due to selfish, non-cooperative behaviour of the players. It is a measure of the inefficiency of equilibria in a game.

Mathematically, PoA is defined as the ratio between the worst possible equilibrium and the social optimum of the game.

Definition

The **price of stability** is another measure of inefficiency of equilibria designed to differentiate between games in which all equilibria are inefficient and those in which some equilibrium is inefficient.

Mathematically, the price of stability is defined as the ratio between the best possible equilibrium and the social optimum of the game.

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In a game with unique equilibrium, its price of anarchy and price of stability are equal.

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Variations of PoA

- Pure Price of Anarchy(deterministic Nash Equilibrium)
- Mixed Price of Anarchy(randomized Nash Equilibrium)
- Bayes-Nash Price of Anarchy(incomplete information games)

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Motivation : Prisoners' Dilemma

B A	B stays silent	B betrays
A stays silent	-1 -1	0 -3
A betrays	-3 0	-2 -2

Here, the overall optimal strategy for both the players is to **remain silent**

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But the Nash Equilibrium occurs when both betray.

Here, the players are worse off being non-cooperative.

$$\text{PoA}=\frac{\textit{Payoff of Nash Equilibrium}}{\textit{Payoff of social optimum}}=\frac{-2-2}{-1-1}=2$$

The problem here isn't lack of information or lack of computational power. It is **lack of coordination**.

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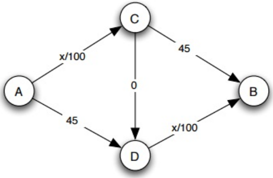
Consider the following routing game with 4000 players.



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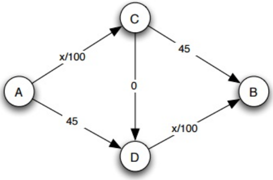
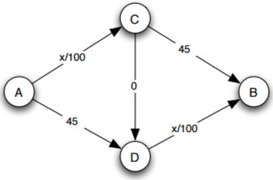


Here, the Nash equilibrium is for all players to take the path A$\rightarrow$C$\rightarrow$D$\rightarrow$B.

All the players get a payoff of -80.

Motivation : Braess Paradox

Consider the following routing game with 4000 players.



But, consider the case where 2000 players take the path A→C→B and the rest take the path A→D→B.

Now, they all get a payoff of -65.

Motivation

$$\text{PoA}{=}{-80}/-65 = 16/13$$

Again, the Nash Equilibrium is worse than the social optimum.

So, there are situations where if individuals make decisions on their own **without communicating**, they are worse as a group and as individuals.

Motivation

There are many cases where this situation is observed.

We analyse such cases, mathematically and intuitively as **Price of Anarchy**.

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Routing Networks

Routing Networks are one of the most studied games under PoA.

A large number of papers and theorems exist for analyzing the PoA of Routing Games, their bounds and corresponding proofs.

Routing Networks

In this section :

- We analyze the bounds for Pigou's network

Routing Networks

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- We analyze the bounds for Pigou's network
- We extend the proof to all classical multicommodity flow networks

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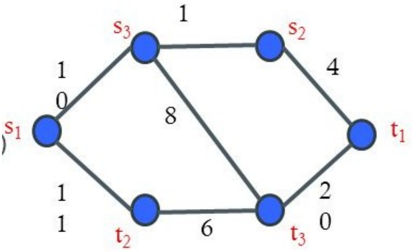
Notation : Multicommodity Flow Networks



Directed Graph $G = (V, E)$

Source-Sink pairs (s_i, t_i) (called commodities)

Notation : Multicommodity Flow Networks

Let ρ_i be the set of paths connecting s_i and t_i

$$\rho = \bigcup \rho_i$$

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Flow on edge e,$f_e = \sum_{P \in \rho : e \in P} f_P$

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Cost function for edge e : $c_e(.)$

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Cost function for edge e : $c_e(.)$

(G, r, c) represents the multicommodity network

$$c_p(f) = \sum_{e \in P} c_e(f_e)$$

Wardrop Equilibrium :

Let f be a feasible flow for (G, r, c). The flow f is a *Wardrop equilibrium* if for every commodity (s_i, t_i) and every path $P, \tilde{P} \in \rho_i$,

$$c_P(f) \leq c_{\tilde{P}}(f)$$

Let (G, r, c) be an instance.

(a)The instance admits atleast one wardrop Equilibrium

(b)If f and $\tilde{f}$ are Wardrop equilibria for (G, r, c), then $c_e(f_e) = c_e(\tilde{f}_e)$

Basics : Multicommodity Flow Networks

A flow f is a Wardrop Equilibria only if :

$$\sum_{e \in E} c_e(f_e) f_e \leq \sum_{e \in E} c_e(f_e) f_e^*$$

for every flow f_e^* feasible for (G, r, c)

PoA Equation

Let f be a Wardrop Equilibrium and f^* be optimal flow for (G, r, c). Then the price of anarchy $\psi(G, r, c)$ is :

$$\psi(G, r, c) = \frac{C(f)}{C(f^*)}$$

where $C(f) = \sum_{e \in E} c_e(f_e) f_e$

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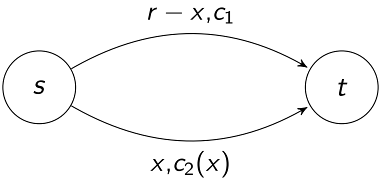
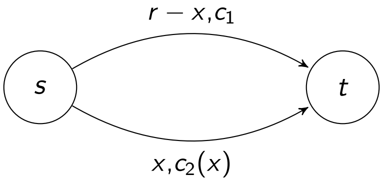
Pigou Bound



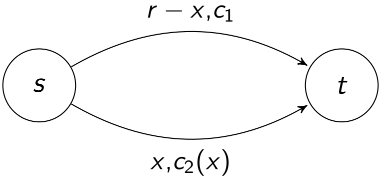
Traffic Rate $= r$.
c_1, c_2 represent the cost functions as shown, with $c_1 = c_2(r)$ always.

Pigou Bound



$$\psi = max_x \frac{r.c_2(r)}{x.c_2(x)+(r-x).c2_{(}r)}$$

Pigou Bound



$$\textbf{Pigou Bound} : \alpha(C) = sup_{c \in C} sup_{x,r \geq 0} \left(\frac{r.c_2(r)}{x.c_2(x) + (r-x).c2_{(}r)} \right)$$

where C is the set of cost functions.

Pigou Bound

If C is the set of linear cost functions $ax + b : a, b \geq 0$, then the Pigou bound $\alpha(C)$ is $\frac{4}{3}$.

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If C is the set of polynomials with non-negative coefficients and degree p, then

$$\alpha(C) = [1 - p.(1 + p)^{\frac{-(p+1)}{p}}]^{-1}$$

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As p grows large, above expression tends to ∞ as $\frac{p}{lnp}$

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Extending to Multicommodity Networks

From the above definition of the Pigou Bound $\alpha(c)$:

$$x.c(x) \geq \frac{r.c(r)}{\alpha(C)} + (x - r).c(r)$$

Theorem : (**Tightness of the Pigou Bound**) *Let C be a set of cost functions and $\alpha(C)$ be the Pigou Bound for C. If (G, r, c) is a non-atomic instance with cost functions in C, the price of anarchy (G, r, c) is atmost $\alpha(C)$.*

Extending to Multicommodity Networks

Let f and f^* be the Wardrop Equilibrium and the optimal flow for above (G, r, c).

$$C(f^*) = \sum_{e \in E} c_e(f_e^*) f_e^*$$

$$\geq \frac{1}{\alpha(C)} \sum_{e \in E} c_e(f_e) f_e + \sum_{e \in E} (f_e^* - f_e) c_e(f_e)$$

$$\geq C(f)/\alpha(C)$$

Extending to Multicommodity Networks

Hence for all classical multicommodity networks with C of a particular type, the upper bound on PoA is the same as for a simple Pigou's network with the same C.

For example, the upper bound for a (G, r, c) with C being linear is $4/3$.

Table for common classes of Cost Functions

Table 1
The price of anarchy for common classes of edge latency functions

Description	Typical Representative	Price of Anarchy
Linear	$ax + b$	$\frac{4}{3} \approx 1.333$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3} - 2} \approx 1.626$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4} - 3} \approx 1.896$
Polynomials of degree $\leqslant p$	$\sum_{i=0}^{p} a_i x^i$	$\Theta\left(\frac{p}{\log p}\right)$
M/M/1 delay functions	$(u - x)^{-1}$	$\frac{1}{2}\left(1 + \sqrt{\frac{u_{\min}}{u_{\min} - R_{\max}}}\right)$

The price of anarchy is independent of the network topology, Roughgarden, Tim, *Journal of Computer and System Sciences*

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Bounds for Braess Paradox

Braess's Paradox shows that adding (intuitively helpful) edges to a network can **increase** the cost of its equilibrium flow.

The PoA for the simple example demonstrated before, increased on adding an edge(can increase upto $\frac{4}{3}$ for affine cost functions).

We can extend the ideas we have learnt for networks to other domains as well.

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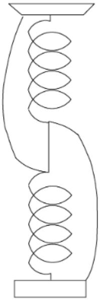
Mechanical Analogue

Braess's paradox is not particular to traffic routing networks. Here we briefly describe an analogous mechanical network of strings and springs, constructed by Cohen and Horowitz [1991].

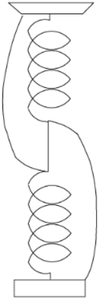


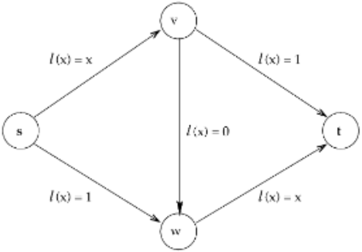
One end of a spring is attached to a fixed support and the other end to a string. Another identical spring is hung from the free end of the string and carries a heavy weight. Finally, the strings are connected (with some slack) from the support to the upper end of the second spring and from the lower end of the first spring to the weight.

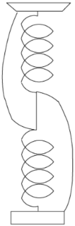
Mechanical Analogue

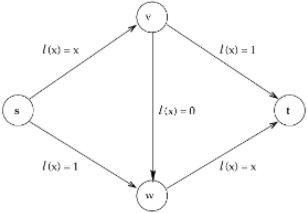
Viewing the support as source, suspended weight as sink, with each string and spring as an arc, the equilibrium position of the mechanical device can be modelled as a Nash equilibrium in the traffic network shown below, with the support-weight distance corresponding to the common latency of the source-sink flow path.

Strings are modelled as links with constant latency functions while the springs correspond to linear latency functions *ax* (assuming that the springs are ideal).



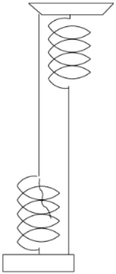


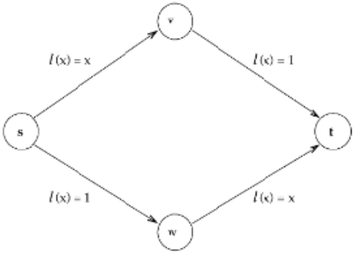




The equilibrium occurs when all the traffic is routed through the linear latency path, then the zero latency path followed by the linear latency path. Equivalently, the springs are extended and the middle string is taut at mechanical equilibrium.

Mechanical Analogue





Contrary to intuition, severing the taut string causes the weight to rise. This corresponds to deleting the zero latency arc, thereby obtaining the second network with its improved Nash equilibrium.

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Coping with Selfishness

The price of anarchy of selfish routing can be large in networks with highly nonlinear cost functions.

Therefore we devise techniques to design or modify a selfish routing network to reduce the price of anarchy and minimize the inefficiency of its equilibria.

Coping with Selfishness

We briefly discuss some techniques for mitigating the inefficiency of selfish routing in nonatomic instances.

- Marginal Cost Pricing
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Marginal Cost Pricing

This approach to reduce PoA involves introducing **marginal cost taxes** on the edges of the network to influence traffic.

The idea is to charge each network user on each edge for the additional cost the user's presence causes for the other users of that edge.

Marginal Cost Pricing

Consider a nonatomic selfish network where each edge e is assigned a non-negative tax value τ_e.

Let the original network be represented by (G, r, c) then the new network with edge tax τ will be represented by $(G, r, c + \tau)$ or (G, r, c^{τ}), where the cost function c_e^{τ} is the shifted version of the original cost function c_e such as

$$c_e^{\tau}(x) = c_e(x) + \tau_e \quad \forall \quad x \geq 0$$

Marginal Cost Pricing

An equilibrium flow for the modified network $(G, r, c + \tau)$ will have traffic travelling through routes that minimize the sum of the edge costs and the edge taxes.

Marginal Cost Pricing

For simplicity lets assume the cost functions are differentiable.

For a feasible flow f in a network (G, r, c),

the marginal increase in cost caused by one user of the edge e is $c_e^{'}(f_e)$ and f_e is the amount of traffic on the edge that suffers this increase. (where $c_e^{'}$ denotes the derivative of c_e)

Marginal Cost Pricing

By the principle of marginal cost pricing, the tax τ_e assigned to the edge e should be

$$\tau_e = f_e . c_e^{'}(f_e)$$

(where $c_e^{'}$ denotes the derivative of c_e)

Marginal Cost Pricing

$$\tau_e = f_e . c_e^{'}(f_e)$$

These taxes correct for the failure of selfish users to account for the additional cost that their presence causes the other users using that edge, thereby reducing the inefficiency of the equilibrium.

Marginal Cost Pricing

Theorem:
Let (G, r, c) be a nonatomic instance with continuously differentiable cost functions. Let f^* be an optimal flow for (G, r, c) and let $\tau_e = f_e^*.c_e^{'}(f_e^*)$ denote the marginal cost tax for edge e with respect to flow f^*. Then f^* is an equilibrium flow for $(G, r, c + \tau)$

Source: Algorithmic Game Theory
By Noam Nisan, Tim Roughgarden, Eva Tardos, Vijay V. Vazirani

Marginal Cost Pricing

Marginal cost taxes thus induce an optimal flow as an equilibrium flow, reducing the price of anarchy to 1.

The above theorem implicitly assumes that all network users trade off cost and taxes in an identical way, when in reality each network user chooses a path that minimizes a weighted sum of the edge costs and the edge taxes - also known as heterogenous traffic.

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Capacity Augmentation

This concept involves increasing the capacity of the network to reduce the inefficiency of equilibria.

The cost of an equilibrium flow is at most that of an optimal flow that is forced to route twice as much traffic between each source-sink pair.

Capacity Augmentation

Theorem:
If f is an equilibrium flow for (G, r, c) and f^* is a feasible flow for $(G, 2r, c)$, then $C(f) \leq C(f^*)$

Source: Algorithmic Game Theory
By Noam Nisan, Tim Roughgarden, Eva Tardos, Vijay V. Vazirani

Capacity Augmentation

The above theorem states that the cost of optimal flow when the traffic on each edge is doubled, is atleast the cost of equilibrium flow before increasing the capacity.

Thus the price of anarchy is reduced by potentially increasing the cost of optimal flow by increasing the capacity.

Capacity Augmentation

Corollary:
Let (G, r, c) be a nonatomic instance and define the modified cost function $c'(x) = [c(x/2)]/2$ for each edge e. Let f be an equilibrium flow for (G, r, c') and f^* be a feasible flow for (G, r, c). Then $C'(f) \leq C(f^*)$

This is equivalent to the theorem stated previously.

Source: Algorithmic Game Theory
By Noam Nisan, Tim Roughgarden, Eva Tardos, Vijay V. Vazirani

Capacity Augmentation

Another interpretation is that the benefit of centralized control is equalled or exceeded by the benefit of a sufficient improvement in link technology.

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Cournot Equilibrium

Let us now briefly revisit the case of Cournot Equilibrium in a Duopoly market in the context of analysing price of anarchy.

Using quantity as strategies,

$$P = A - B(Q_1 + Q_2)$$

Cournot Equilibrium

$$P = A - B(Q_1 + Q_2)$$

The profits for each firm is given by

$$\pi_1(Q_1, Q_2) = PQ_1 - CQ_1$$

$$\pi_2(Q_1, Q_2) = PQ_2 - CQ_2$$

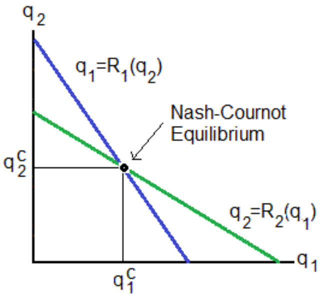
Assume the firms are symmetrical/identical
i.e. $A_1 = A_2, B_1 = B_2, C_1 = C_2$

Maximizing individual profits, we get Cournot Equilibrium.

$$\frac{\partial \pi_1}{\partial Q_1} = 0 \text{ and } \frac{\partial \pi_2}{\partial Q_2} = 0$$

$$(Q_1^*, Q_2^*) = \frac{(A-C)}{3B}, \frac{(A-C)}{3B}$$

$$(\pi_1^*, \pi_2^*) = \frac{(A-C)^2}{9B}, \frac{(A-C)^2}{9B}$$



Cournot Equilibrium

Maximizing combined profits, we get social optimum.

$$\frac{\partial(\pi_1+\pi_2)}{\partial Q_1} = 0 \ , \ \frac{\partial(\pi_1+\pi_2)}{\partial Q_2} = 0$$

$$(Q_1^*, Q_2^*) = \frac{(A-C)}{4B}, \frac{(A-C)}{4B}$$

$$(\pi_1^*, \pi_2^*) = \frac{(A-C)^2}{8B}, \frac{(A-C)^2}{8B}$$

Cournot Equilibrium

Firm1/Firm2	NC	C
NC	$\frac{(A-C)^2}{9B}, \frac{(A-C)^2}{9B}$	$\frac{9(A-C)^2}{64B}, \frac{3(A-C)^2}{32B}$
C	$\frac{3(A-C)^2}{32B}, \frac{9(A-C)^2}{64B}$	$\frac{(A-C)^2}{8B}, \frac{(A-C)^2}{8B}$

Clearly, the profits are higher for both firms if they cooperate. But the equilibrium occurs when both firms don't cooperate.

Cournot Equilibrium

Due to the self-interested nature of the firms, one of the firms stops cooperating and deviates from the social optimum (C, C) in hopes of gaining higher profits and moves to either (NC, C) or (C, NC), then the other firm also stops cooperating and moves to the Counot equilibrium (NC, NC).

$$\text{PoA} = \frac{\textit{Payoff of social optimum}}{\textit{Payoff of Cournot Equilibrium}} = \frac{\frac{1}{8}+\frac{1}{8}}{\frac{1}{9}+\frac{1}{9}} = \frac{9}{8}$$

This case clearly illustrates the price of anarchy concept.

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Job Scheduling

A job scheduling game is a game that models a scenario in which multiple selfish users wish to utilize multiple processing machines.

There are N players and each of them has a job to run. They can choose one of M machines to run their job. The incentive of each player is to have his job run as fast as possible.

Job Scheduling

The strategy set of each player is the set of machines. Given a strategy for each player, the total load on each machine is the sum of processing times of the jobs that chose that machine.

Usually each player seeks to minimize the total load on its chosen machine. The standard objective function is minimizing the total load on the most-loaded machine which would lead to social optimum.

Job Scheduling

The Price of Anarchy compares the situation where the selection of machines is guided/directed centrally so as to obtain the social optimum, to the situation where each player chooses the machine that will make his job run fastest.

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Public Goods Game

In the public goods game, each of the N players is in possession of a certain number of tokens, say e_i. Each player chooses how many of his tokens to contribute to a public pot, say x_i.

Finally, the tokens in this pot are multiplied by a factor r $(1 < r < N)$ and this public good payoff is evenly divided among the players. Also each player gets to keep the tokens that he does not contribute.

The payoff of player i $u_i = e_i - x_i + \frac{r}{N} . \sum_{j=1}^{N} x_j$

Public Goods Game

The total payoff of the group is maximized when every player contributes all of his tokens to the public pool.

However, the Nash equilibrium in this game is simply zero contributions by all the players since any rational agent does best contributing zero tokens, regardless of what the other players do.

The price of anarchy compares the above two situations.

Public Goods Game

The payoff of player i $u_i = e_i - x_i + \frac{r}{N} . \sum_{j=1}^{N} x_j$

Total payoffs $\sum_{i=1}^{N} u_i = \sum_{i=1}^{N} [e_i - x_i + \frac{r}{N} . \sum_{j=1}^{N} x_j]$

$$= E - X + N.[\frac{r}{N}.X]$$

$$= E + (r-1)X$$

where $E = \sum_{i=1}^{N} e_i$ and $X = \sum_{i=1}^{N} x_i$

Public Goods Game

Totall payoffs:

Social Optimum $(X = E) = r.E$
Nash Equilibrium $(X = 0) = E$

$$\text{PoA} = \frac{Payoff\ \ of\ \ social\ optimum}{Payoff\ \ of\ \ Nash\ equilibrium} = \frac{r.E}{E} = r$$

Conclusion

Price of Anarchy is a useful concept which highlights the need for cooperation and communication between players.

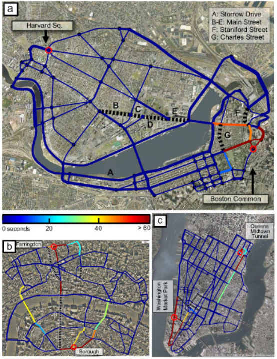
We've analyzed the intuition behind Price of Anarchy using examples from various domains. We've also seen that ideas formed by analysing PoA in one domain can be easily extended to other domains.

We have also analysed certain proofs and theorems concerning PoA.

Looking Further

Price of Anarchy is a vast topic with many more theoretical and practical applications.

Looking Further



Price of Anarchy used in the analysing and assessing the transportation networks and traffic patterns in Boston.

Source: The Price of Anarchy in Transportation Networks Paper By Hyejin Youn, Michael T. Gastner and Hawoong Jeong

Looking Further

Price of Anarchy is a vast topic with many more theoretical and practical applications.

- PoA analysis for mixed and bayesian Nash equilibria
- Proofs about the lower bound, and matching upper/lower bounds of PoA
- Other methods to analyse PoA(such as graphically)
- Other examples such as Tragedy of the Commons, Auctions etc.

We hope this presentation was informative and entertaining.

THANK YOU