FE527 W5

Solving PDE's using CUDA

- Intro to PDEs in FE
- Explicit Finite Difference
- Implementation in CUDA

- Backward Equation

$$dS_t = \mu(S_t,t)dt + \sigma(S_t,t)dW_t$$

- If we define

$$C(x,t) = E\left[e^{-\int_t^T b(S_u,u)du}\,\phi(S_T)\,\middle|\,S_t = x\right]$$

then C solves

$$C_t + \mu(x,t)C_x + \tfrac{1}{2}\sigma^2(x,t)C_{xx} - b(x,t)\cdot C = 0$$

Δ A special case:

$$\left.\begin{aligned} &\mu(S_t,t) = r\cdot S_t \\ &\sigma(S_t,t) = \sigma\cdot S_t \\ &b(S_t,t) = r \\ &E \text{ w.r.t } \mathbb{Q} \\ &\phi(S_T) = (S_T - K)^+ \end{aligned}\right\} \text{Black-Scholes PDE}$$

- We use such PDE's (Backward Kol...) to price financial derivatives.

- Forward Equation

$$dS_t = \mu(S_t,t)dt + \sigma(S_t,t)dW_t$$

If $P(S_t,t)$ solves

$$-P_t - \frac{\partial}{\partial S_t}\left(\mu(S_t,t)\cdot P\right) + \frac{1}{2}\frac{\partial^2}{\partial S_t^2}\left(\sigma^2(S_t,t)\cdot P\right) = 0$$

with initial condition

$$P(S_0, 0) = \begin{cases} 1, & S_0 = x \\ 0, & \text{otherwise} \end{cases}$$

then $P(S_t,t)$ is the p.d.f of S_t

Explicit Finite Difference

- Forward Equation $t=0 \rightarrow t=T$

let's take $\mu(S_t,t) = 0$, $\sigma(S_t,t) = 1$

the $dS_t = dW_t \leftarrow$ 1D BM

- $-P_t + \frac{1}{2}\frac{\partial^2 P}{\partial S^2} = 0$

$$0 = -\frac{P_{i+1,j} - P_{i,j}}{\Delta t} + \frac{1}{2}\frac{P_{i,j+1} - 2P_{i,j} + P_{i,j-1}}{(\Delta S)^2}$$

$$\Leftrightarrow P_{i+1,j} = \frac{\Delta t}{2\Delta S^2}P_{i,j+1} + \frac{\Delta t}{2\Delta S^2}P_{i,j-1} + \left(1 - \frac{\Delta t}{\Delta S}\right)P_{i,j}$$

- Backward Equation $t=T \rightarrow t=0$

- BS example

$$C_{ij} = P_d C_{i+1,j+1} + P_m C_{i+1,j} + P_u C_{i+1,j-1}$$

W6. 2D-PDE

- We use 2D BM as an example. In general case, the gridding is similar, but the scheme is more complicated. (refer to Anderson & Piterbarg 2004)

2D BM:

- $dS_t = dW_t^1$
  $dV_t = dW_t^2$

$$\Leftrightarrow \begin{pmatrix} dS_t \\ dV_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dW^1 \\ dW^2 \end{pmatrix}$$

- forward PDE:

$$P_t = \frac{1}{2}\frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\frac{\partial^2 P}{\partial V^2} = 0$$

- $P_{ij}(t+\Delta t) - P_{ij}(t)$

$$= \frac{1}{2}\,\frac{P_{i+1,j}(t) - 2P_{ij}(t) + P_{i-1,j}(t)}{\Delta S^2} + \frac{1}{2}\,\frac{P_{i,j+1}(t) - 2P_{ij}(t) + P_{i,j-1}(t)}{\Delta V^2}$$

$\Leftrightarrow$

$$P_{ij}(t+\Delta t) = \underbrace{\frac{\Delta t}{2\Delta S^2}}_{\alpha}\left[P_{i+1,j}(t) - 2P_{ij}(t) + P_{i-1,j}(t)\right] + \underbrace{\frac{\Delta t}{2\Delta V^2}}_{\beta}\left[P_{i,j+1}(t) - 2P_{ij}(t) + P_{i,j-1}(t)\right] + P_{ij}(t)$$

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