In [1]:

```python
import talk.config as con

% matplotlib inline

con.config_mosek()
con.config_configManager()
con.config_matplotlib()
```

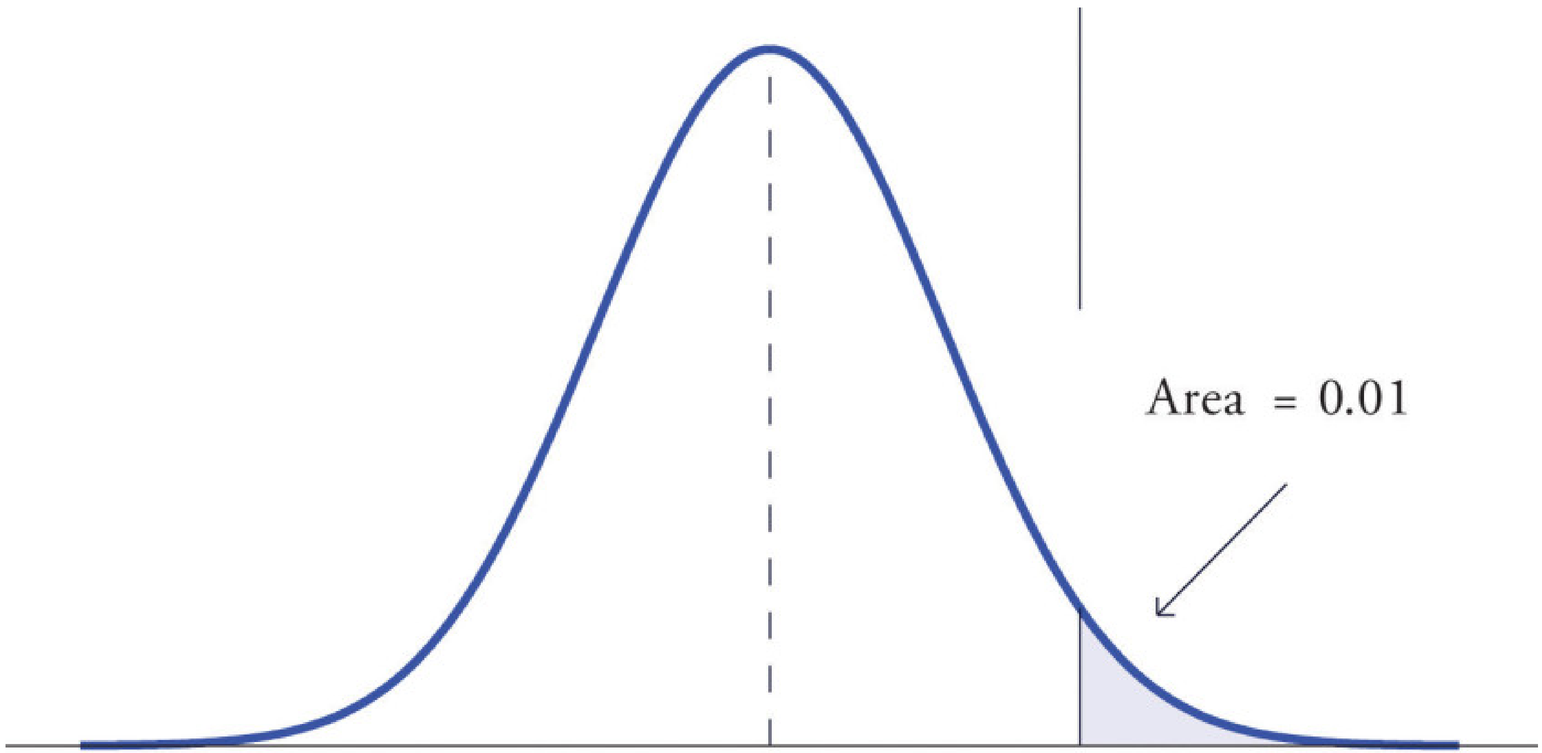
```
Set MOSEKLM_LICENSE_FILE environment variable
Update ConfigManager
```

The Conditional Value at Risk

https://en.wikipedia.org/wiki/Expected_shortfall
(https://en.wikipedia.org/wiki/Expected_shortfall)

Thomas Schmelzer

The $\alpha = 0.99$ tail of a loss distribution



- In this talk we assume losses are postive. Larger losses, more pain... We want negative losses!
- The value at risk $\mathtt{VaR}_\alpha$ at level α is (the smallest) loss such that $\alpha\%$ of losses are smaller than $\mathtt{VaR}_\alpha$.
- This does not say anything about the magnitude of the losses larger than the $\mathtt{VaR}_\alpha$. We can only make statements about their number: $n(1-\alpha)$
- To describe the tail of the return distribution better one could use the Conditional Value at Risk $\mathtt{CVaR}_\alpha$, defined as the mean of the losses larger (or equal) than the $\mathtt{VaR}_\alpha$.

In [2]:

```python
import numpy as np


def tail(ts, alpha=0.99):
    return np.sort(ts)[int(len(ts) * alpha):]


def value_at_risk(ts, alpha=0.99):
    return tail(ts, alpha)[0]


def cvalue_at_risk(ts, alpha=0.99):
    return np.mean(tail(ts, alpha))


R = np.array([-1.0, 2.0, 3.0, 2.0, 4.0, 2.0, 0.0, 1.0, -2.0, -2.0])
print("Length: {0}".format(len(R)))
print("Tail:   {0}".format(tail(R, alpha=0.80)))
print("VaR:    {0}".format(value_at_risk(R, alpha=0.80)))
print("CVaR:   {0}".format(cvalue_at_risk(R, alpha=0.80)))
```

```
Length: 10
Tail:   [ 3.  4.]
VaR:    3.0
CVaR:   3.5
```

Sub-additivity:

If $Z_1, Z_2 \in \mathcal{L}$, then $\varrho(Z_1 + Z_2) \leq \varrho(Z_1) + \varrho(Z_2)$

The risk of two portfolios together cannot get any worse than adding the two risks separately: this is the diversification principle.

The $\mathtt{VaR}_\alpha$ **is not** sub-additive. It is not convex either.

The $\mathtt{CVaR}$ **is** sub-additive and convex.

Other convex risk measures: Entropic Value-at-Risk, ...

In [3]:

```python
import numpy.random as nr
import pandas as pd

# number of draws...
n = 10000000
alpha = 0.99

# Let's define two portfolios A and B
frame = pd.DataFrame(data=nr.uniform(0, 10, size=(n, 2)) * nr.binomial(1, p=0.0075,
size=(n, 2)), columns=["A", "B"])
frame["A+B"] = frame.sum(axis=1)

print("VALUE AT RISK")
print(frame.apply(value_at_risk, alpha=alpha))

print("CONDITIONAL VALUE AT RISK")
print(frame.apply(cvalue_at_risk, alpha=alpha))
```

```
VALUE AT RISK
A      0.000000
B      0.000000
A+B    3.358835
dtype: float64
CONDITIONAL VALUE AT RISK
A      3.769197
B      3.775107
A+B    6.706661
dtype: float64
```

We introduce a free variable γ and define the function f as:

$$f(\gamma) \quad = \gamma + \frac{1}{n\,(1-\alpha)} \sum (r_i - \gamma)^+$$

This is a continuous and convex function (in γ). The first derivative is:

$$f^{'}(\gamma) = 1 - \frac{\#\,\{r_i \geq \gamma\}}{n\,(1-\alpha)}$$

If γ such that $\#\{r_i \geq \gamma\} = n\,(1-\alpha)$:

- γ is a minimizer of f.
- $f(\gamma)$.
  $= \mathtt{CVaR}_\alpha(\mathbf{r})$

In particular:

- $f(\mathtt{VaR}_\alpha(\mathbf{r}))$.
  $= \mathtt{CVaR}_\alpha(\mathbf{r})$

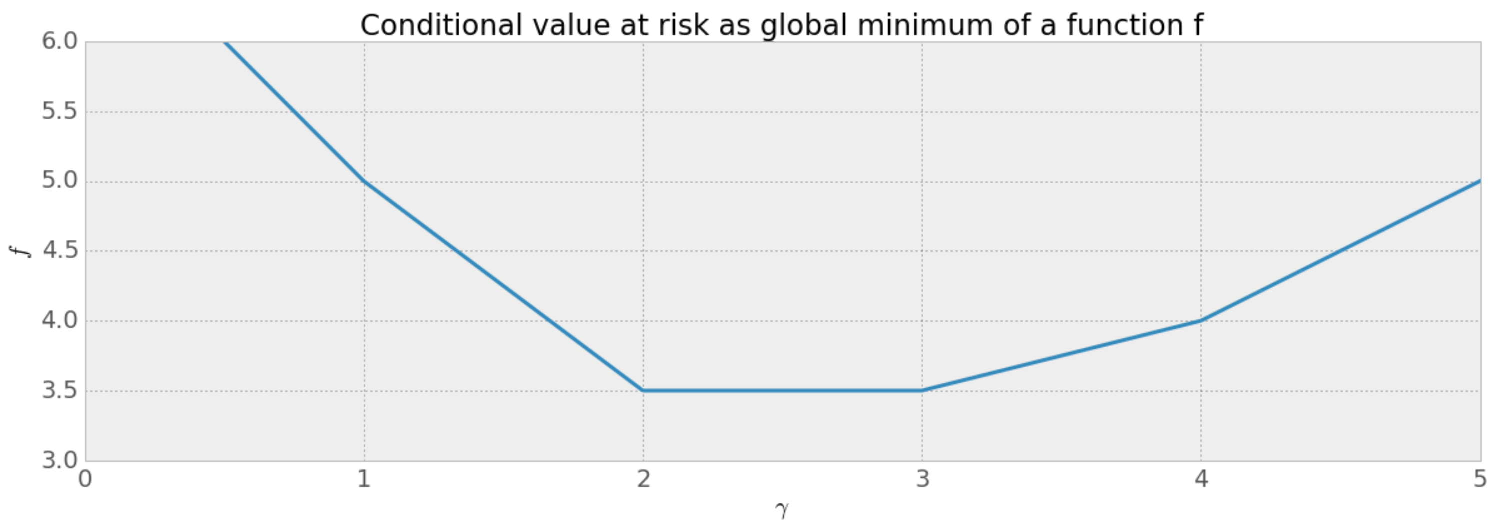
In [7]:

```
import numpy as np
import matplotlib.pyplot as plt

def f(gamma, returns, alpha=0.99):
    excess = returns - gamma
    return gamma + 1.0 / (len(returns) * (1 - alpha)) * excess[excess > 0].sum()

r = np.array([-1.0, 2.0, 3.0, 2.0, 4.0, 2.0, 0.0, 1.0, -2.0, -2.0])
x = np.linspace(start=-1.0, stop=5.0, num=1000)
v = np.array([f(gamma=g, returns=r, alpha=0.80) for g in x])

plt.plot(x, v), plt.grid(True), plt.xlabel('$\gamma$'), plt.ylabel('$f$')
plt.title('Conditional value at risk as global minimum of a function f')
plt.axis([0, 5, 3, 6])
plt.show()
```



$$
\begin{aligned}
\texttt{CVaR}(\mathbf{r}) = \min_{\gamma\in\mathbb{R},\mathbf{t}\in\mathbb{R}^n}\ & \gamma + \frac{1}{n\,(1-\alpha)} \sum t_i \\
\text{s.t. } & t_i \geq r_i - \gamma \\
& \mathbf{t} \geq 0
\end{aligned}
$$

In [5]:
```
from mosek.fusion import *
import numpy as np

R = [-1.0, 2.0, 3.0, 2.0, 4.0, 2.0, 0.0, 1.0, -2.0, -2.0]

n = len(R)
# We are interested in CVaR for alpha=0.80, e.g. what's the mean of the 20% of the b
iggest losses
alpha = 0.80

with Model('cvar') as model:
    # introduce the variable for the var
    gamma = model.variable("gamma", 1, Domain.unbounded())

    # Auxiliary variable
    t = model.variable("t", n, Domain.greaterThan(0.0))

    # t > r-gamma <===> t + gamma > r
    model.constraint(Expr.add(t, Variable.repeat(gamma, n)), Domain.greaterThan(R))

    # gamma + 1/[n*(1-alpha)] * \sum{t}
    cvar = Expr.add(gamma, Expr.mul(1.0 / (n * (1 - alpha)), Expr.sum(t)))

    # minimization of the conditional value at risk
    model.objective(ObjectiveSense.Minimize, cvar)
    model.solve()

    print("A minimizer of f (<= VaR):  {0}".format(gamma.level()[0]))
    print("Minimum of f (== CVaR):     {0}".format(gamma.level()[0] + 1.0 / (n * (1
- alpha)) * np.array(t.level()).sum()))
```

```
A minimizer of f (<= VaR):  3.00000000011
Minimum of f (== CVaR):     3.50000000007
```

Summary

- The Conditional Value at Risk $\mathtt{CVaR}$ is a coherent risk measure.
- We could compute the $\mathtt{CVaR}$ for a vector of length n by solving a convex program in $n+1$ dimensions.
- We do not need to sort the elements nor do we need to know the Value at Risk $\mathtt{VaR}$.

In practice the vector $\mathbf{r}$ is not given. Rather we have m assets and try to find a linear combination of their corresponding return vectors such that the resulting portfolio has minimal Conditional Value at Risk.

In [6]:
```
from mosek.fusion import *
import numpy as np
import matplotlib.pyplot as plt

R = np.random.randn(2500,200)

n,m = R.shape
n = int(n)
m = int(m)
# We are interested in CVaR for alpha=0.95, e.g. what's the mean of the 5% of the bi
ggest losses
alpha = 0.95

with Model('cvar') as model:
    # introduce the variable for the var
    gamma = model.variable("gamma", 1, Domain.unbounded())
    weight = model.variable("weight",m, Domain.inRange(0.0, 1.0))
    # Auxiliary variable
    t = model.variable("t", n, Domain.greaterThan(0.0))

    # e'*w = 1
    model.constraint(Expr.sum(weight), Domain.equalsTo(1.0))

    # r = R*w
    r = Expr.mul(DenseMatrix(R), weight)

    # t > r-gamma <===> t + gamma - r > 0
    model.constraint(Expr.sub(Expr.add(t, Variable.repeat(gamma, n)),r), Domain.grea
terThan(0.0))

    # gamma + 1/[n*(1-alpha)] * \sum{t}
    cvar = Expr.add(gamma, Expr.mul(1.0 / (n * (1 - alpha)), Expr.sum(t)))

    # minimization of the conditional value at risk
    model.objective(ObjectiveSense.Minimize, cvar)
```

```
    model.solve()

    var = gamma.level()[0]
    cvar = gamma.level()[0] + 1.0 / (n * (1 - alpha)) * np.array(t.level()).sum()
    print("Minimizer of f (<= VaR):  {0}".format(var))
    print("Minimum of f (= CVaR):  {0}".format(cvar))
```

Back to Overview (http://localhost:8888)