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THE ASYMPTOTIC EXPANSION OF THE INCOMPLETE GAMMA FUNCTIONS*

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Abstract. Earlier investigations on uniform asymptotic expansions of the incomplete gamma functions are reconsidered. The new results include estimations for the remainder and the extension of the results to complex variables.

1. Introduction. We consider the incomplete gamma functions ratios P and Q defined by

$$P(a,x)=\frac{1}{\Gamma(a)}\int_0^x t^{a-1}\,e^{-t}\,dt,\qquad Q(a,x)=\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}\,e^{-t}\,dt. \tag{1.1}$$

We suppose first that x and a are real with

$$x\geqq 0,\qquad a>0. \tag{1.2}$$

In Temme [4] we derived asymptotic expansions of P and Q for $a\to\infty$, uniformly valid for $x\geqq 0$. In this paper we reconsider these expansions. Our new results concern the representations of the remainder in the asymptotic expansion, representations for the coefficients of the expansion for numerical applications, numerical upper bounds for the remainder in the case of real variables, and extension of the asymptotic expansions to the case of complex variables. These problems are mentioned by Olver in [2].

To describe the expansions given in [4] we introduce the following parameters

$$\lambda=x/a,\qquad \mu=\lambda-1,\qquad \eta=\{2[\mu-\ln(1+\mu)]\}^{1/2}, \tag{1.3}$$

with the convention that the square root has the sign of μ ($\mu>-1$). As a function of μ, η is monotone and infinitely differentiable on $(-1,\infty)$. Analytic properties of $\eta(\mu)$ for complex μ are considered in § 5.

The asymptotic expansions of P and Q derived in [4] follow from the representations

$$\begin{aligned}P(a,x)&=\tfrac12\operatorname{erfc}[-\eta(a/2)^{1/2}]-R_a(\eta),\\ Q(a,x)&=\tfrac12\operatorname{erfc}[\eta(a/2)^{1/2}]+R_a(\eta)\end{aligned} \tag{1.4}$$

with

$$R_a(\eta)\sim(2\pi a)^{-1/2}\,e^{-(1/2)a\eta^2}\sum_{k=0}^{\infty}c_k(\eta)a^{-k} \tag{1.5}$$

for $a\to\infty$, uniformly valid with respect to $\eta\in\mathbb{R}$; erfc is the complementary error function defined by

$$\operatorname{erfc}(x)=2\pi^{-1/2}\int_x^\infty e^{-t^2}\,dt. \tag{1.6}$$

The expansion (1.5) was derived by using saddle point methods. In § 2 we use a different method which yields recurrence relations for the coefficients c_k and a representation for the remainder of (1.5). In § 3 we discuss representations for c_k that can be used for numerical applications. In § 4 numerical error bounds are constructed

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for the remainder of the series in (1.5) when the first n terms in the series are used. Bounds are given up to $n=10$. As a side result this section gives bounds for the remainder of the asymptotic expansion of the reciprocal gamma function $1/\Gamma(x)$ for real x. In § 5 the results are extended to complex values of a and x.

2. Recurrence relations for the coefficients and representations of the remainder. First we remark that the asymptotic expansion for $a\to\infty$, of $dR_a(\eta)/d\eta$ may be obtained by formal differentiation of (1.5). This is not proved here, but it follows from the representation of $R_a(\eta)$ in our previous paper (formula (2.10) of Temme [4]). The result is

$$\frac{dR_a(\eta)}{d\eta}\sim a(2\pi a)^{-1/2}e^{-(1/2)a\eta^2}\sum_{k=0}^{\infty}c_k^{(1)}(\eta)a^{-k} \tag{2.1}$$

with

$$\begin{aligned} c_0^{(1)}(\eta)&=-\eta c_0(\eta),\\ c_k^{(1)}(\eta)&=-\eta c_k(\eta)+\frac{dc_{k-1}(\eta)}{d\eta}; \qquad k\geqq 1. \end{aligned} \tag{2.2}$$

Secondly, we need the coefficients of the asymptotic expansion of the complete gamma function. Let us define

$$\Gamma^*(a)=(a/(2\pi))^{1/2}e^a a^{-a}\Gamma(a), \qquad a>0. \tag{2.3}$$

Then Γ^* and $1/\Gamma^*$ have the well-known asymptotic expansions for $a\to\infty$

$$\begin{aligned} \Gamma^*(a)&\sim\sum_{k=0}^{\infty}(-1)^k\gamma_k a^{-k},\\ 1/\Gamma^*(a)&\sim\sum_{k=0}^{\infty}\gamma_k a^{-k}. \end{aligned} \tag{2.4}$$

The first few coefficients are

$$\gamma_0=1, \qquad \gamma_1=-\frac{1}{12}, \qquad \gamma_2=\frac{1}{288}, \qquad \gamma_3=\frac{139}{51840}.$$

Further coefficients follow from Spira [3] and Wrench [5]. Wrench gives $(-1)^k\gamma_k$ up to $k=20$ in rational form, Spira the remaining up to $k=30$. Decimal representations are also given in both references.

With these preparations we have

THEOREM 1. *Let* $\{\gamma_k\}$ *be defined by* (2.4). *Then the coefficients* c_k *of* (1.5) *satisfy the recurrence relation*

$$\begin{aligned} c_0(\eta)&=\frac{1}{\mu}-\frac{1}{\eta},\\ \eta c_k(\eta)&=\frac{dc_{k-1}(\eta)}{d\eta}+\frac{\eta}{\mu}\gamma_k, \qquad k\geqq 1. \end{aligned} \tag{2.5}$$

Proof. By differentiating one of the formulas in (1.4) with respect to η and by using (1.1) it follows that

$$\frac{d}{d\eta}R_a(\eta)=(a/(2\pi))^{1/2}\left(1-\frac{1}{\mu+1}\frac{1}{\Gamma^*(a)}\frac{d\mu}{d\eta}\right)e^{-(1/2)a\eta^2}. \tag{2.6}$$

From (1.3) we have

$$\frac{d\mu}{d\eta}=\frac{(\mu+1)\eta}{\mu}, \tag{2.7}$$

and substituting (2.1) and the second relation of (2.4) we obtain (2.5) by collecting equal powers of a^{-1} and using (2.2). □

As follows from [4], the coefficients c_k are holomorphic in a neighborhood of $\eta=0$. In fact the singularities of $1/\mu$ and $1/\eta$ in c_0 cancel each other. So the limiting value of c_0 for $\eta\to 0$ is well defined.

Owing to the presence of the derivative of c_{k-1} in (2.5) this formula cannot be handled easily from a numerical point of view. Further, the above mentioned cancellation of singular parts in c_0 occurs in all c_k when working with (2.5). Therefore other representations are given for these coefficients. In the next section we discuss some aspects of the Taylor expansions for small $|\eta|$-values, while for larger $|\eta|$-values a recurrence relation is constructed from which the coefficients can be computed directly. But first we give representations of the remainder in the asymptotic expansion (1.5).

From (1.4) it follows that $R_a(\infty)=R_a(-\infty)=0$. Hence, integration of (2.6) gives

$$\begin{aligned} R_a(\zeta)&=(a/(2\pi))^{1/2}\int_{-\infty}^{\zeta}\left[1-\frac{\eta}{\mu}\frac{1}{\Gamma^*(a)}\right]e^{-(1/2)a\eta^2}\,d\eta \\ &=-(a/(2\pi))^{1/2}\int_{\zeta}^{\infty}\left[1-\frac{\eta}{\mu}\frac{1}{\Gamma^*(a)}\right]e^{-(1/2)a\eta^2}\,d\eta, \end{aligned} \tag{2.8}$$

where μ as a function of η is defined implicitly in (1.3). From these representations and the recurrence relations for c_k a simple expression for the remainder follows. For this purpose we introduce the notation

$$R_a(\eta)=(2\pi a)^{-1/2}\,e^{-(1/2)a\eta^2}\left[\sum_{k=0}^{N-1}c_k(\eta)a^{-k}+a^{-N}G_N(\eta;a)\right], \tag{2.9}$$

$a>0, \eta\in\mathbb{R}, N=0,1,2,\cdots$. Furthermore, we need a notation for the remainder in the asymptotic expansion of $1/\Gamma^*(a)$, which is written as

$$1/\Gamma^*(a)=\sum_{k=0}^{N-1}\gamma_k a^{-k}+a^{-N}H_N(a),\qquad a>0,\qquad N=0,1,2,\cdots. \tag{2.10}$$

THEOREM 2. *Let G_N and H_N be defined by* (2.9) *and* (2.10). *Then*

$$\begin{aligned} e^{-(1/2)a\zeta^2}G_N(\zeta;a)&=a\int_{\zeta}^{\infty}\eta c_N(\eta)\,e^{-(1/2)a\eta^2}\,d\eta \\ &\quad+H_{N+1}(a)\int_{\zeta}^{\infty}\frac{\eta}{\mu}e^{-(1/2)a\eta^2}\,d\eta. \end{aligned} \tag{2.11}$$

Proof. The proof follows immediately from substitution of (2.9) and (2.10) in (2.6) (and by use of (2.5) and (2.7)). □

The second integral in (2.11) can be expressed in terms of $Q(a,x)$. The first one can be bounded if we have estimations for $|c_k(\eta)|$. From representations of c_k given in the following section it follows that $[2+\mu(\eta)]^{\kappa}|c_k(\eta)|$ is a bounded function of $\eta\in\mathbb{R}$, with

$\kappa=\frac{1}{2}$ if $k=0$, $\kappa=1$ if $k\geqq 1$. For estimating the second integral of (2.11) we define

$$C_k=\sup_{\eta\in\mathbb{R}}\,[2+\mu(\eta)]^{\kappa}|c_k(\eta)|,\qquad k=0,1,2,\cdots,$$
(2.12)
$$\kappa=\begin{cases}\frac{1}{2} & \text{for } k=0,\\ 1 & \text{for } k\geqq 1.\end{cases}$$

For numerical applications the following is important.

COROLLARY. *Let C_k be defined by* (2.12). *Then for* $N=0,1,2,\cdots$,

(2.13)
$$\left|Q(a,x)-\tfrac{1}{2}\operatorname{erfc}[\eta(a/2)^{1/2}]-e^{-(1/2)a\eta^2}(2\pi a)^{-1/2}\sum_{k=0}^{N-1}c_k(\eta)a^{-k}\right| \leqq Q_N(\eta;a)(2\pi a)^{-1/2}a^{-N},$$

with

(2.14)
$$Q_N(\eta;a)=(\mu+2)^{-\kappa}C_N\begin{cases}e^{-(1/2)a\eta^2}\\ (2-e^{-(1/2)a\eta^2})\end{cases}+|H_{N+1}(a)|e^{a}a^{-a}\Gamma(a)Q(a,x),$$

where the upper term is for $\eta\geqq 0$, the lower one for $\eta\leqq 0$.

In § 4 we give more (numerical) information on C_N and H_N. With numerical values of C_N and H_N we have strict and realistic error bounds for the remainder of the uniform asymptotic expansion of $Q(a,x)$. Similar results hold for the function $P(a,x)$. For $N=0,1,2,\cdots$ we have

(2.15)
$$\left|P(a,x)-\tfrac{1}{2}\operatorname{erfc}[-\eta(a/2)^{1/2}]+e^{-(1/2)a\eta^2}(2\pi a)^{-1/2}\sum_{k=0}^{N-1}c_k(\eta)a^{-k}\right| \leqq P_N(\eta;a)(2\pi a)^{-1/2}a^{-N},$$

with

(2.16)
$$P_N(\eta;a)=(\mu+2)^{-\kappa}C_N\begin{cases}(2-e^{-(1/2)a\eta^2})\\ e^{-(1/2)a\eta^2}\end{cases}+|H_{N+1}(a)|e^{a}a^{-a}\Gamma(a)P(a,x),$$

where the upper term is for $\eta\geqq 0$, the lower one for $\eta\leqq 0$.

Remark 1. The functions multiplying the constants C_N in (2.14) and (2.16) have quite different behavior for $\eta<0$ and $\eta>0$. This, however, is in agreement with the behavior of the functions P and Q in the same formula. In fact, the bounds P_N and Q_N give a measure for the relative accuracy for the error in the uniform expansions.

Remark 2. The asymptotic expansion (1.5) and the representation for the remainder is easily obtained by partial integration of one of the integrals in (2.8) and by using the recursions (2.5) and $H_k(a)=\gamma_k+(1/a)H_{k+1}(a)$. To demonstrate this we write in the second of (2.8) $1/\Gamma^*(a)=1+a^{-1}H_1(a)$ and we use the first line of (2.5). Then we obtain

(2.17)
$$R_a(\zeta)=(a/(2\pi))^{1/2}\int_{\zeta}^{\infty}\eta c_0(\eta)\,e^{-(1/2)a\eta^2}\,d\eta +(2\pi a)^{-1/2}H_1(a)\int_{\zeta}^{\infty}\frac{\eta}{\mu}\,e^{-(1/2)a\eta^2}\,d\eta.$$

Writing the first integral as $-a^{-1}\int c_0(\eta)\,d\,e^{-(1/2)a\eta^2}$, we obtain for this integral (after partial integration and using the second line of (2.5))

$$a^{-1}c_0(\zeta)\,e^{-(1/2)a\zeta^2}+a^{-1}\int\eta c_1(\eta)\,e^{-(1/2)a\eta^2}\,d\eta-\gamma_1a^{-1}\int\frac{\eta}{\mu}\,e^{-(1/2)a\eta^2}\,d\eta.$$

Combining the second integral of this expression with the second integral of (2.17) and using $H_1(a)-\gamma_1=a^{-1}H_2(a)$ we arrive at (2.9) and (2.11) with $N=1$. So the process can be continued.

3. Representations of c_k. Using (2.5) with $k=1$ we obtain

(3.1) $$\eta c_1(\eta)=-\frac{1}{\mu^2}\frac{d\mu}{d\eta}+\frac{1}{\eta^2}+\frac{\eta}{\mu}\gamma_1,$$

and using (2.7) we have

(3.2) $$c_1(\eta)=\frac{1}{\eta^3}-\frac{1+\mu+\mu^2/12}{\mu^3}.$$

Computing higher order coefficients we notice the following structure

(3.3) $$c_k(\eta)=(-1)^k\left\{\frac{Q_k(\mu)}{\mu^{2k+1}}-\frac{A_k}{\eta^{2k+1}}\right\},$$

where Q_k is a polynomial in μ of degree $2k$ and $A_k=2^k=2^k\Gamma(k+\frac{1}{2})/\Gamma(\frac{1}{2})$.

The first few polynomials are

(3.4) $$\begin{aligned}Q_0(\mu)&=1,\\ Q_1(\mu)&=1+\mu+\tfrac{1}{12}\mu^2,\\ Q_2(\mu)&=3+5\mu+\tfrac{25}{12}\mu^2+\tfrac{1}{12}\mu^3+\tfrac{1}{288}\mu^4.\end{aligned}$$

In order to preserve accuracy near $\mu=-1$ we write

(3.5) $$Q_k(\mu)=(1+\mu)P_k(\mu)+(-1)^k\gamma_k\mu^{2k}.$$

P_k is a polynomial of degree $2k-2$ $(k\geqq 1)$. Writing

(3.6) $$P_k(\mu)=p_0^{(k)}+p_1^{(k)}\mu+\cdots+p_{2k-2}^{(k)}\mu^{2k-2}$$

we have the relation (which is easily obtained by substituting (3.6), (3.5) and (3.3) in (2.5))

(3.7) $$\begin{aligned}&p_0^{(k)}=(2k-1)p_0^{(k-1)}\\ &p_j^{(k)}=(2k-1-j)[p_j^{(k-1)}+p_{j-1}^{(k-1)}], \qquad j=1,2,\cdots,2k-4,\\ &p_{2k-3}^{(k)}=2p_{2k-4}^{(k-1)}, \qquad p_{2k-2}^{(k)}=(-1)^{k-1}\gamma_{k-1},\end{aligned}$$

with as starting polynomial $P_1(\mu)=1$, or $p_0^{(1)}=1$.

In Table 1 we give the coefficients $p_j^{(k)}$ of (3.7) for $k=1,2,\cdots,5$, $j=0,1,\cdots,2k-2$.

TABLE 1

k	$p_j^{(k)}$
1	1
2	3, 2, 1/12
3	15, 20, 25/4, 1/6, 1/288
4	105, 210, 525/4, 77/3, 49/96, 1/144, −139/51840
5	945, 2520, 9555/4, 1883/2, 12565/96, 149/72, 221/17280, −139/25920, −571/2488320

As remarked in the previous section, when computing c_k via (3.3) near $\eta=0$, one must pay attention to the cancellation of the singular parts. It may be convenient (and, when using a fixed number of word lengths on a computer, even necessary) to use representations which preserve the accuracy near $\eta=0$.

If $|\eta|$ is small it is preferred to use expansions either in terms of η or in terms of μ. We advise expansions in η, since it gives better convergence properties. When expanding c_k in powers of μ we need (among others) the expansion of η in powers of μ. Owing to the singularity of the logarithm in (1.3), the radius of convergence of this series is 1. Other singularities for η are zeros of $\mu-\ln(1+\mu)$, but they are outside the domain $|\mu|\leqq 1$. This follows from straightforward analysis. The reader may also consult an interesting note of Diekmann [1]. The expansion of μ in powers of η has radius of convergence $2\sqrt{\pi}\simeq 3.54$. This follows from the analysis of § 5. From the recurrence relation (2.5) it is easily seen that the radius of convergence of the power series for c_k either in μ or in η is the same for all k.

We conclude this section with some information on the construction of the coefficients for the expansion of c_k in powers of η. It is convenient to start with the computation of the α_k in

$$\mu(\eta)=\alpha_1\eta+\alpha_2\eta^2+\cdots, \tag{3.8}$$

where μ is defined implicitly in (1.3). Substitution of (3.8) in (2.7) yields the recurrence relation

$$(k+1)\ \alpha_k=\alpha_{k-1}-\sum_{j=2}^{k-1} j\alpha_j\alpha_{k-j+1}, \qquad k\geqq 2.$$

The first few are $\alpha_1=1$, $\alpha_2=\frac{1}{3}$, $\alpha_3=\frac{1}{36}$, $\alpha_4=-\frac{1}{270}$, $\alpha_5=\frac{1}{4320}$. With α_k we also have available the γ_k of (2.4), which are also needed in (2.5). The relation between α_k and γ_k is $\gamma_k=(-1)^k 1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)\alpha_{2k+1}$, $(k=0,1,2,\cdots)$.

The coefficients $c_n^{(k)}$ in the expansion $c_k(\eta)=\sum_{n=0}^{\infty} c_n^{(k)}\eta^n$ follow now from (2.5). For $k=0$ we have

$$c_0^{(0)}=-\tfrac{1}{3}, \qquad c_k^{(0)}=(k+2)\alpha_{k+2}, \qquad k\geqq 1$$

and for general $k\geqq 1$ the recursion is

$$c_n^{(k)}=\gamma_k c_n^{(0)}+(n+2)c_{n+2}^{(k-1)}, \qquad n\geqq 0,$$

or, in terms of $c_n^{(0)}$,

$$c_n^{(k)}=\gamma_k c_n^{(0)}+\gamma_{k-1}(n+2)c_{n+2}^{(0)}+\cdots+\gamma_0(n+2)\cdots(n+2k)c_{n+2k}^{(0)}. \tag{3.9}$$

As follows from the rate of convergence of the series for c_k (with radius $2\sqrt{\pi}$) successive terms in (3.9) are decreasing in absolute value. Hence no instability problems arise when using (3.9) for the computation of $c_n^{(k)}$.

4. Bounds for the remainder in the asymptotic expansion. In Table 2 we give the numbers C_k defined in (2.12). These bounds were obtained numerically by using representations of c_k given in the foregoing section. From (3.3) and (1.3) it follows that

$$\lim_{\eta\to-\infty} c_0(\eta)=-1, \qquad \lim_{\eta\to+\infty}=[2+\mu(\eta)]^{1/2}c_0(\eta)=-2^{-1/2},$$

$$\lim_{\eta\to\pm\infty}[2+\mu(\eta)]c_k(\eta)=\pm\gamma_k, \qquad k\geqq 1.$$

TABLE 2

k	c_{2k}	c_{2k+1}
0	1	0.083
1	0.010	0.0027
2	0.0024	0.00092
3	0.0016	0.00083
4	0.0021	0.0014
5	0.0045	

Next we give details for computing the bounds H_k (defined in (2.10)) for $k = 0, 1, \cdots, 10$. It is convenient to start with details for obtaining the asymptotic expansion of $1/\Gamma(a)$. Again, a is a positive number. Our starting point is Hankel's integral

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^t t^{-a}\, dt. \tag{4.1}$$

This can be written as

$$\frac{1}{\Gamma(a)} = a^{1-a} e^a \pi^{-1} \int_{-\infty}^{\infty} e^{-(1/2)au^2} g(u)\, du$$

with

$$f(u) = \frac{ut}{1-t}, \qquad -\tfrac{1}{2}u^2 = t - 1 - \ln t, \qquad g(u) = \frac{f(u)+f(-u)}{2i}, \tag{4.2}$$

where $u \in \mathbb{R}$ and t follows the steepest descent line for (4.1) in the t-plane. More information on the relations in (4.2) is found in our previous paper [4].

The asymptotic expansion of $1/\Gamma(a)$ is obtained by expanding $g(u)$ in powers of u and integrating term by term. Let us define (for $N = 0, 1, 2, \cdots$) the functions g_N by writing

$$g(u) = \sum_{k=0}^{N-1} a_k u^{2k} + a_N u^{2N} g_N(u), \qquad a_k = \frac{1}{(2k)!} g^{(2k)}(0);$$

all a_k are different from zero. Then the function H_N of (2.10) is given by

$$H_N(a) = (a/(2\pi))^{1/2} a_N a^N \int_{-\infty}^{\infty} e^{-(1/2)au^2} u^{2N} g_N(u)\, du.$$

It appears that g, and hence g_N, is bounded on $\mathbb{R}$. Let us define the bounds

$$G_N = \sup_{u \in \mathbb{R}} |g_N(u)|,$$

then a bound for H_N is given by

$$|H_N(a)| \leqq |\gamma_N| G_N, \qquad a > 0, \tag{4.3}$$

where γ_k are the coefficients in (2.10). Table 3 gives the value of G_N for $N = 0, 1, \cdots, 11$.

TABLE 3

N	G_{2N}	G_{2N+1}
0	1	1
1	1.95	1
2	3.33	1
3	5.05	1
4	6.95	1
5	8.90	1

For $N=0, 1, 3, 5, 7, 9, 11$ the maximal function values of $|g_N(u)|$ occur at $u=0$; for $N=2, 4, 6, 8, 10$ the maxima occur in the neighborhood of $u=\pm 2\sqrt{\pi}$. These latter points are the points on the real axis marking the domain of convergence of the Taylor series of g.

With the data of Table 2 and Table 3 and relation (4.3) the bounds Q_N and P_N defined in (2.14) and (2.16) are easily computed.

5. Extension to complex variables. In this section we will show that the asymptotic expansion for P and Q given by (1.4) and (1.5) are valid for $a \to \infty$ uniformly in $|\arg a| \leqq \pi - \varepsilon_1$, $|\arg x/a| \leqq 2\pi - \varepsilon_2$ where ε_1 and ε_2 are positive numbers, $0<\varepsilon_1<\pi$, $0<\varepsilon_2<2\pi$.

The condition on the argument of a follows from the validity of the expansions in (2.4), which are known to be uniformly valid when $|\arg a| \leqq \pi - \varepsilon_1$. As noticed in Remark 2 of § 2 the asymptotic expansion of $R_a(\eta)$ can be obtained by partial integration of one of (2.8). If we consider the second integral, one of the assumptions by partial integration will be that $\exp(-\frac{1}{2}a\eta^2)$ vanishes at infinity in a certain direction of the η-plane. If $|\arg a| < \pi$ and if it is allowed to use η-values at infinity with $\arg(a\eta^2) < \pi/2$ then the convergence of the integral is established for $|\arg a| \leqq \pi - \varepsilon_1$. From these inequalities it follows that it is sufficient to show that for large $|\eta|$ we can take $\arg \eta$ in $(-\frac{3}{4}\pi, \frac{3}{4}\pi)$. A second aspect of using the second integral of (2.8) is the possibility of joining the point ζ with ∞ such that the function $\mu(\eta)$ of the integral is holomorphic along this path and such that the point ζ can be associated unequivocally with a point in the μ-plane. In order to settle this we discuss the relation between η and the parameter μ (or λ) for complex values.

It is convenient to consider

$$\eta = [2(\lambda - 1 - \ln \lambda)]^{1/2}. \tag{5.1}$$

For $\lambda > 0$ the function η is to be interpreted as drawn in Fig. 1. This implies a choice of the square root.

We obtain a clear insight in the mapping $\lambda \to \eta(\lambda)$ and its inverse if we draw images of the half-lines l_ϕ defined by

$$l_\phi = \{\lambda | \lambda = \rho e^{i\phi}, \rho > 0\}$$

where ϕ is real, $|\phi| \leqq 2\pi$. Writing $\eta = \alpha + i\beta$ we find that the image of l_ϕ in the η-plane is governed by the equations

$$\tfrac{1}{2}(\alpha^2 - \beta^2) = \rho \cos \phi - 1 - \ln \rho,$$

$$\alpha\beta = \rho \sin \phi - \phi.$$

Taking into account the convention about the choice of the square root in (5.1) we

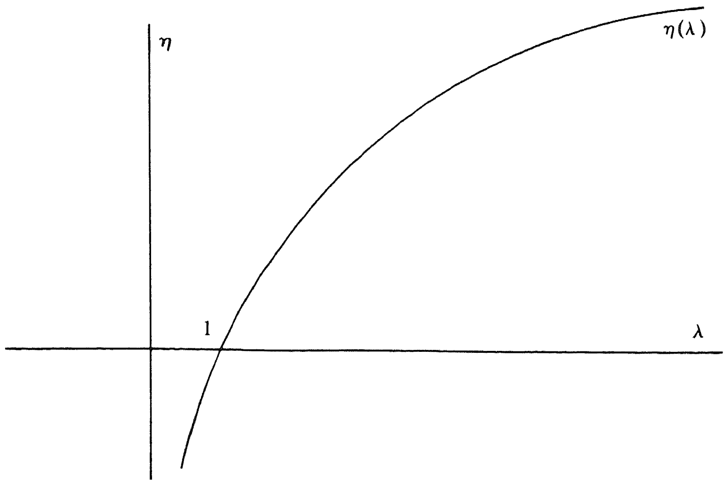

FIG. 1

obtain Fig. 2, which contains images of l_ϕ for $0 \leqq \phi \leqq 2\pi$. The complete picture for $-2\pi \leqq \phi \leqq 2\pi$ is symmetric with respect to the α-axis.

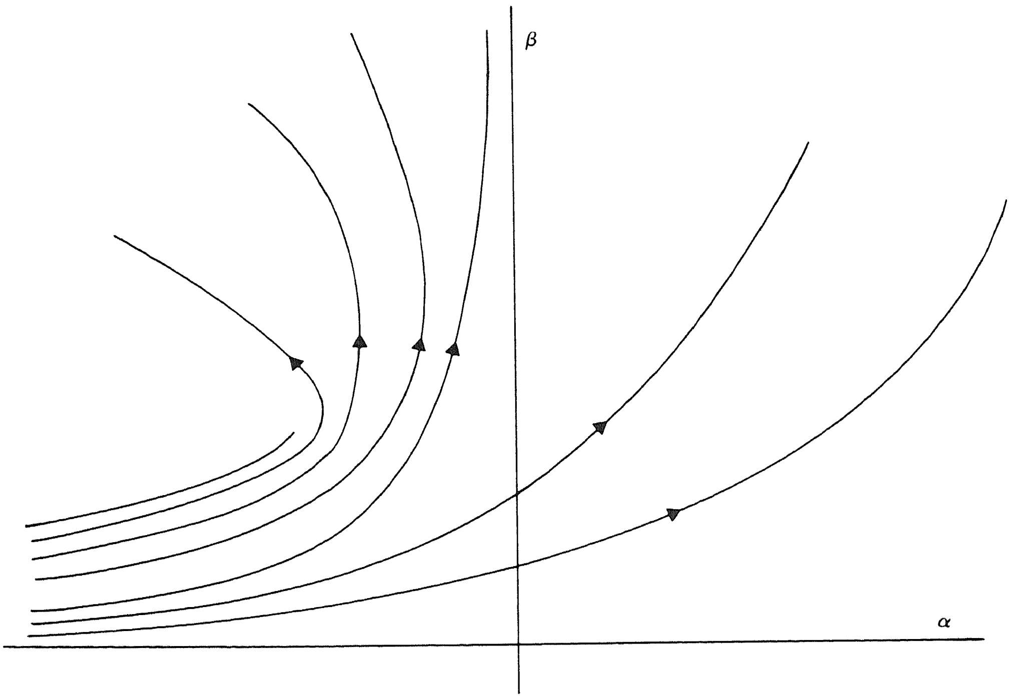

FIG. 2

The shown directions correspond to increasing values of ρ on l_ϕ. The half-lines $l_{\pm 2\pi}$ are mapped on part of the hyperbolae $\alpha\beta = \mp 2\pi$. The points $\eta^{\pm} = e^{\pm 3\pi i/4} 2\sqrt{\pi}$ are singular points of the mapping. Other singular points are located in other Riemann sheets of the η-plane. Convenient branch-cuts for the function $\lambda(\eta)$ are the parts of the hyperbolae $\alpha\beta = \pm 2\pi$ with $\alpha \leqq -\sqrt{2\pi}$. With the η-plane cut along these curves, lines l_ϕ with the values of ϕ outside the interval $[-2\pi, 2\pi]$ can be traced, but for our problem this is superfluous.

It is concluded that any point in the finite η-plane (not on the branch-cuts), corresponds to a point in the λ-plane with $|\arg \lambda| < 2\pi$. Consequently, if we integrate the second integral of (2.8) along a path that avoids the branch-cuts in the η-plane, the function $\mu(\eta) = \lambda(\eta) - 1$ is holomorphic. The conditions for allowing values of arg a in $(-\pi, \pi)$ are amply satisfied, since admissable directions in the η-plane can be found in the sector $-\pi < \arg \eta < \pi$.

Remark. Singular points of the mapping $\eta \to \lambda(\eta)$ can also be found by considering the derivative $d\lambda/d\eta = \lambda\eta/(\lambda - 1)$; $\lambda = 1$ gives a regular point but $\lambda = e^{2\pi i n}$ $(n = \pm 1, \pm 2, \cdots)$ gives (due to the many-valuedness of the logarithm in (5.1)) singular points η_n satisfying $\frac{1}{2}\eta_n^2 = -2\pi i n$, $n = \pm 1, \pm 2, \cdots$.

The integration by parts procedure leads eventually to (2.9) and (2.11). From the properties of the coefficients c_k and by taking appropriate contours in (2.11) it follows that for $N = 0, 1, 2, \cdots$,

$$e^{-(1/2)a\eta^2} G_N(\eta; a) = O(1), \qquad a \to \infty$$

uniformly in $|\arg a| \leqq \pi - \varepsilon_1$, $|\arg \lambda| \leqq 2\pi - \varepsilon_2$.

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