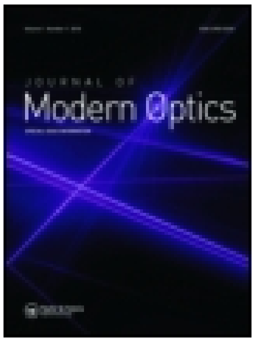

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Capacity of an optical channel in the presence of noise

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SUMMARY. — *An optical instrument can be considered as a transmission channel. The capacity of the channel is evaluated by applying standard results of information theory. The instrument is assumed to be free from aberrations and colour is not taken into account.*

SOMMAIRE. — *Un instrument d'optique peut être considéré comme un canal de transmission. La capacité de ce conduit est évaluée en appliquant les résultats classiques de la théorie de l'information.*

ZUSAMMENFASSUNG. — *Ein optisches Instrument kann als Ubertragungskanal im Sinne der Informationstheorie angesehen werden. Dann lassen sich die Ergebnisse dieser Theorie zur Berechnung der Leistungsfähigkeit der optischen Instrumente benutzen.*

Introduction. — Information theory has been mainly developed in the field of electric communications. Only in recent times a few workers have made attempts at applying the theory to optics [1, 2, 3, 4]. This can be done from different points of view. Some of them are of purely theoretical or academic interest, while some others may lead to practical results only in very particular cases.

It is difficult to anticipate at present whether or not information theory will be able to bring about in the long run any material improvement to applied optics. It seems however that the best way to reach a conclusion in the near or far future is to try and apply to optics the central problem of communication theory. In order to do this one should not indulge too much in dealing with coherent or semi-coherent illumination, frequency analysis and all straightforward translations of well-known radio communication results into optical terminology. Such topics have been perhaps a little overstressed in optics. The eye is not the ear and any interpretation of vision in terms of frequencies is really too far-fetched (1).

Now, what has been called above the central problem of communication theory may be stated as follows : Given an information source and a transmission channel to find the way to transmit the greatest possible amount of information in the least time. The solution of this problem depends mainly on two factors : 1) the statistics of the source and 2) the capacity of the channel.

(1) This does not mean, of course, that FOURIER analysis cannot be a very useful or even necessary tool in some intermediate steps, as is well known at least since Lord RAYLEIGH. Frequency analysis can also be important whenever the channel includes a radio-transmission stage, as in television. However this paper will only be concerned with a purely optical channel.

The statistics of the source represents in optics a fascinating problem which will be dealt with in a forthcoming paper. The present paper will only be concerned with the capacity of an optical instrument when considered as a transmission channel.

It is customary to define the capacity of a channel as the maximum number of bits that the channel can transmit per unit time. However it seems advisable slightly to change this definition in optics. For most optical instruments the time of observation is to be considered as practically unlimited. It will therefore be reasonable to define the capacity of an optical instrument as the number of bits that the instrument can transmit per single image or as the greatest possible number of bits obtainable from an image formed by the instrument.

In order to make the argument very clear and to keep the number of free parameters as low as possible only the black and white case will be discussed in the present paper. Color vision may be included in a later refinement of the theory. Further the instrument will be considered to be free from aberrations.

However, contrary to what is done in some wrong applications of information theory, the consideration of noise cannot be dispensed with and is essential to the argument. Indeed it is well known that, according to a correct definition, the entropy of a set of continuous probabilities, as are those to be considered here, would turn out to be infinite [5]. It is therefore only the difference of two entropies which can have a real significance. This difference is made when noise is taken into account.

Degrees of Freedom of an Optical Image. — First the following problem will be solved: How many numbers are necessary to specify completely an optical image ?

To begin with consider a perfect optical instrument, having a one-dimensional pupil of width a. The illumination I_p in the image of a point source will have the form.

$$I_p = \text{sinc}^2\, \alpha \frac{a}{\lambda} \tag{1}$$

where α represents the angular position coordinate and sinc x, following WOODWARD's notation stands for the function $(\sin \pi x)/\pi x$. If $I_g(\alpha)$ indicates the illumination of the image of an extended object which would result from purely geometrical optics and $I_w(\alpha)$ the illumination of the same image when wave optics is taken into account, the following relation will obviously hold

$$I_w(\alpha) = I_g(\alpha) * \text{sinc}^2\, \alpha \frac{a}{\lambda} \tag{2}$$

where P*Q indicates the convolution of P and Q.

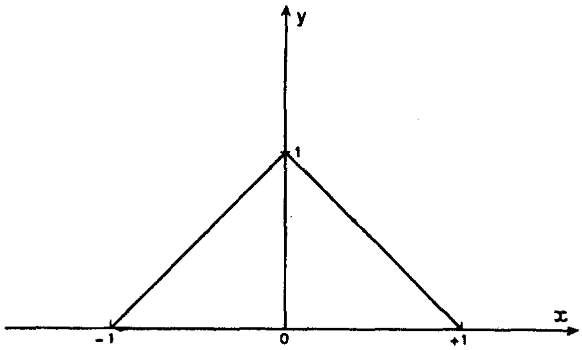

FIG. 1. — The function y = triang x.

According to a well-known theorem, the FOURIER transform of the left side of (2) equals the product of the FOURIER transforms of the two factors of the right side. Now it can be proved by standard methods that the transform of sinc² $(\alpha a/\lambda)$ is given by (λ/a) triang $(f\,\lambda/a)$, where f stands for the frequency and the function y = triang x is that plotted in figure 1. As a result it can be written

$$i_w(f) = \frac{\lambda}{a}\, i_g(f)\, \text{triang}\, f\, \frac{\lambda}{a} \tag{3}$$

where $i_w(f)$ and $i_g(f)$ represent the transforms of $I_w(\alpha)$ and $I_g(\alpha)$ respectively. It is evident from (3) and from figure 1 that $i_w(f)$ vanishes for $|f| > a/\lambda$. Thus $I_w(\alpha)$ contains no frequency exceeding a/λ. An application of the so-called sampling theorem [6] leads to the conclusion that $I_w(\alpha)$ is completely determined by giving its values at a series of discrete points (sampling points) spaced $\lambda/2\,a$ apart. If $\bar{\alpha}$ represents the total field angle of the instrument, the total number of sampling points will be $2\,\bar{\alpha}\,a/\lambda$.

Consider now a two-dimensional pupil, say a rectangle ab. The image of a point source will be represented by

$$I_p = \text{sinc}^2\, \alpha \frac{a}{\lambda}\ \text{sinc}^2\, \beta \frac{b}{\lambda}. \tag{4}$$

By a simple extension of the foregoing argument one would reach the conclusion that the number of sampling points which are necessary completely to specify the image I_w is given by $4\,\bar{\alpha}\,\bar{\beta}\,ab/\lambda^2$. If Ω designates the total solid angle of the field and S the pupil area this number can be written as

$$N = 4\,\Omega \frac{S}{\lambda^2} \tag{5}$$

One may say that N represents the number of degrees of freedom of the image. However, it should be noted that this number represents rather an order of magnitude than a precise figure. The sampling values cannot be chosen completely at will, because the illumination can never be negative. This question will be discussed in a forthcoming paper.

Entropy of the Image. — As a rule the illumination I at each sampling point will be comprised between 0 and a value I_{max} depending on the object. Maximum entropy will be obtained when all values of I inside the range $0 < I < I_{max}$ have equal probability. According to the usual definition, the entropy is then found to be equal to $\log I_{max}$.

The instrument has been assumed to be perfect from the geometrical standpoint. However, even in a perfect instrument, there is always some stray light. The distribution of stray light in the field may very well be irregular in some extreme cases. However, in the great majority of practical cases, stray light will be regularly diffused so as to form a uniform veil superimposed on the image. The corresponding illumination will be designated by I_d.

The resultant illumination at a general sampling point of the image will be termed the objective illumination and will be designated by I_0. The probability distribution for I_0 is uniform inside the range $I_d < I_0 < I_{max} + I_d$. The entropy of this distribution is still given by $\log I_{max}$.

Since the illuminations at different sampling points can be assumed to be independent of one another, the total entropy of the image will be given by $H = N \log I_{max}$, N being the number of degrees of freedom defined by (1).

It may be noted that the value of I_d will in most cases be proportional to the mean brightness of the object, or what amounts to the same, to the highlights of the object. To simplify matters it will be assumed that

$$I_d = p\, I_{max} \tag{6}$$

p being a numerical constant depending on the instrument. Thus the range of I_0 will be defined by $pI_{max} < I_0 < (1 + p)\, I_{max}$.

The Noise.— It has been remarked earlier that an expression of the entropy like that found in the preceding section is extremely conventional and has no meaning unless it is considered only as a first step in the evaluation of the channel capacity. A result of practical value can be obtained only when noise is taken into account.

Let x and y represent the transmitted and received signals respectively in the case of a general communication system. Then the mean transfer of information can be represented by the expression [5]

$$H(y) - Av_x \; H_x(y) \tag{7}$$

where $H(y)$ stands for the entropy of y, $H_x(y)$ for the same entropy when the transmitted signal x is known and Av_x indicates the average over all possible values of x. In making the difference (7) the infinite constants of two entropies cancel and the expression acquires a definite meaning.

It would seem quite natural to assume that in an optical channel noise be represented by stray light. However this is a little too theoretical, for in practice, apart from a few not very important cases, stray light does not show any randomness. The value of the entropy found in the previous section is independent of the stray light percentage.

In order to find out a reasonable cause of random noise to use in the theory somebody has taken into account such phenomena as photon fluctuation or fluctuations in the illumination of the object. Now it is evident that at the present stage, when a universally accepted theory of optical information is still to be developed, these refinements are better left out of the treatment. They may present some interest, if any, only in some particular cases to be investigated much later.

If one considers solely what happens in real or every day optics one cannot fail to reach the conclusion that the only cause of randomness and uncertainty which is worth mentioning is represented by the receptor. Optical noise (2) is mainly receptor noise. The portion of the channel preceding the receptor is practically noiseless.

In order to go further it is necessary to specify the receptor. It is quite natural to refer to the eye. On the other hand, since the capacity is defined as the maximum information obtainable from the image, the eye will be supposed to work in the best possible conditions. The brightness level and the magnification will be assumed to be sufficient for the resolving power to be limited by the instrument and not by the eye.

The conclusions which will be reached for the eye will also be valid for the photographic plate, provided that the grain be much finer than the resolving power.

It is well known that when a portion of the retina is illuminated with an objective illumination I_0, the subjective illumination I_s, i. e. the illumination perceived by the observer is in general somewhat different from I_0. There is an uncertainty in the observer's judgement, which is closely related to the differential threshold. This uncertainty represents noise and will be denoted by I_n. Thus one can write

$$I_s = I_0 + I_n. \tag{8}$$

The probability distribution for I_n may be assumed to be the distribution holding in general for experimental uncertainties, i. e. the Gaussian one. If $p(I_n)\mathrm{d}I_n$ indicates the probability that the noise be comprised between I_n and $I_n + \mathrm{d}I_n$, one may write

$$p(I_n) = \frac{1}{I_N\sqrt{2\pi}} \exp\left(-\frac{I_n^2}{2I_N^2}\right), \tag{9}$$

I_N being the standard deviation.

On the other hand it is known from experiment that, inside the most favorable range of illuminations, I_N is practically proportional to I_0 (WEBER's psychophysical law). Thus

$$I_N = \varepsilon\, I_0 \tag{10}$$

where ε is a numerical constant derived from experiment.

Capacity of the Optical Channel. — It is now possible to evaluate the capacity of the channel by means of the general expression (7).

The received signal y is, strictly speaking, represented by I_s. If I_s has a uniform probability, it is seen from (8) and (9) that the probability distribution for I_0 is not uniform. However this does not matter at all. Whatever the resultant distribution for I_0 may be, the capacity must be evaluated with that probability distribution for y which maximises $H(y)$. This would precisely be a uniform distribution, provided that I_s were limited to a well-determined range. The last condition is not strictly true in the case of I_s for both the upper and lower limits are slightly blurred by the psychophysical uncertainty. However it is clear that only a very small error will be committed if it is assumed that the two limits are fixed and still represented by pI_{max} and $(1 + p)\, I_{max}$ respectively. In conclusion, for each degree of freedom $H(y)$ will be given by

$$H(y) = \log I_{max}. \tag{11}$$

The conditional entropy $H_x(y)$ is the entropy of I_s when I_0 is known. It is clear from (8) that $H_x(y)$ is simply the entropy of I_n, i. e. of the Gaussian distribution (9). Thus by recalling a standard result [6] it follows that

$$H_x(y) = \log\left(\sqrt{2\pi e}\; I_N\right). \tag{12}$$

Next the standard deviation I_N will be replaced by its expression (10) in terms of I_0 and the average will be made over all values of I_0. By carrying out the integration one obtains

(2) *Optical noise* is certainly a very odd expression. However a more appropriate one would hardly be likely to be widely accepted.

$$(13)\quad A v_x H_x(y) = \frac{1}{I_{\max}} \int_{pI_{\max}}^{(1+p)\, I_{\max}} \log\left(\sqrt{2\pi e}\ \varepsilon\ I_0\right) dI_0 = $$

$$= \log\left[\sqrt{\frac{2\pi}{e}}\ \varepsilon\ \frac{(1+p)^{1+p}}{p^p}\right] + \log I_{\max}.$$

Finally by inserting (11) and (13) into (7) the capacity of the channel per degree of freedom is found to be

$$(14)\quad C = \log\left[\sqrt{\frac{e}{2\pi}}\ \frac{1}{\varepsilon}\ \frac{p^p}{(1+p)^{1+p}}\right]$$

and by (5) the total capacity of the instrument is

$$(15)\quad C = 4\,\Omega\,\frac{S}{\lambda^2} \log\left[\sqrt{\frac{e}{2\pi}}\ \frac{1}{\varepsilon}\ \frac{p^p}{(1+p)^{1+p}}\right].$$

As was to be expected from a sound theory, the value of $I_{\max}$ does not appear in the expression for the capacity. The capacity depends solely on the aperture and field of the instrument, on the differential threshold of the receptor and on the stray light percentage.

The above expressions for the capacity are of practical interest, for they lend themselves immediately to the derivation of numerical results. For instance putting $\varepsilon = .1$, as is reasonable to assume for the differential threshold of very close points and applying (14), one obtains for C the values which are plotted in figure 2 against the values of p. ; C is expressed in bits and p as a percentage.

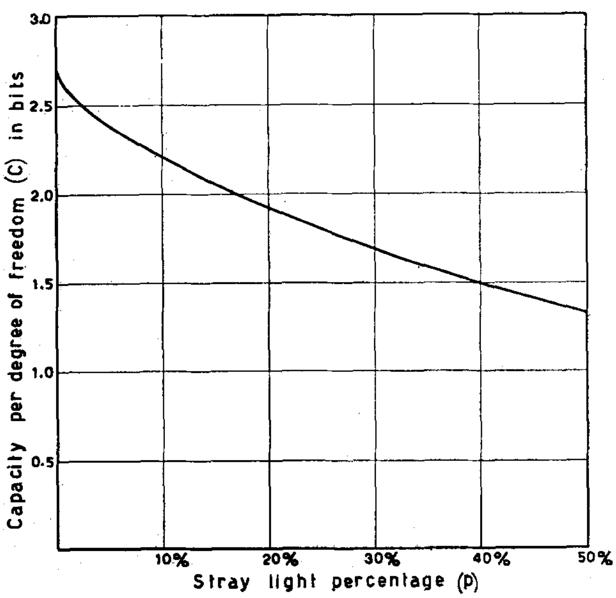

FIG. 2. — Capacity of an optical channel per degree of freedom as a function of stray light percentage.

If the value of ε is somewhat changed, the result, owing to the logarithm, does not change very rapidly. As a consequence it will never be very far from the truth to state that *a good optical instrument has a capacity of about* 2.5 *bits per degree of freedom.* Stray light reduces this capacity according to the universal law represented in figure 2.

Other numerical results which can be derived from (15) are the following : The human eye, when kept fixed, has a foveal capacity of about 10^5 bits. If the observer is allowed to turn his head in all possible directions the capacity becomes about 10^9 bits. A good telescope having a pupil of 100 mm diameter and 1° of field would have a capacity of about 10^8 bits.

Conclusion. — The author is well aware that not all of the assumptions made in the present paper have an absolute and general validity. However it is believed that a greater rigour, while terribly complicating the argument, would not alter the end results very much.

If information theory is to be of any use in optics and bring about some material progress, one must begin with a very simple and clear theory, discarding a great number of parameters wich are only of a secondary importance.

The theory developed above takes into account only the essential factors (excepting colour) of vision through a good optical instrument. The rule of thumb of 2.5 bits per degree of freedom is easy to remember. The dependence of capacity upon stray light is represented by the universal function plotted in figure 2. A change of the value adopted for the differential threshold merely brings about a shift along the ordinates.

All necessary elements are now at hand for investigating the redundancy of an optical system when employed to observe objects of given optical statistics. Starting from this point it will be interesting to try to eliminate the redundancy by suitably encoding the input data.

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