On the analogy between curved space-time and non-impedance-matched anisotropic media

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The geometrical analogy between an arbitrary curved space and an impedance-matched medium has introduced in [1]. In this paper, we study the condition that extends the analogy of a curved space to a special type of non-impedance-matched materials; namely the birefringent media. By applying the eigenvalues method, we derive the light path element for the normal incident light in two different media: a purely electric anisotropic medium and the impedance-matched medium. We show the two independent metrics exist, corresponding to the ordinary and the extraordinary modes of the birefringent medium. The optical line element in an impedance-matched medium is equal to the optical line element for the extraordinary path in the birefringent medium. Considering an optical multi-axes medium, the geometrical analogy between the extraordinary optical paths and curved space can be achieved. This possibility might ease some technical complexities regarding the fabrication of impedance-matched media and consequently provides more achievable means to control the light trajectory in the novel optical devices such as invisibility cloaks. Our results can consider as an extension of the transformation optics methods to the electric anisotropic media.

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1. INTRODUCTION

Transformation optics is a technique to control and manipulate the light by designing the optical medium's profile [2]. Mathematical tool for this purpose is solving the Maxwell equations in a general coordinate system (interpret as an empty curved space-time) and comparing with their solutions in a Cartesian coordinate filled with a medium. In this method, the geometry of the space-time appears as a constitutive relation of an impedance-matched magneto-electric anisotropic medium [1, 3, 4]. Today, this method is widely applied in theoretical and experimental researches such as analogue gravity [5–7], invisibility cloaks [8, 9] and other novel optical devices [10–15].

However, impedance-matched medium are very rare in nature, and they need to build and design artificially. Recent developments in metamaterial science [16] brought the hope that the novel adventures of transformation optics in designing the optical devices, or analogue media can realize in practice. Nevertheless, fabrication of artificial materials is in its infancy, and therefore realizing the optical devices or analogue media is not a trivial task. Thus, it would be of a great advantage if there were an alternative way to avoid the technical cumbersome of fabrication [17–23].

We show in this paper that it is possible to use the birefringent medium in transformation optics and restrict only to the electrical response of the medium instead of demanding the impedance-matched condition.

In the birefringent media, there are two normal modes; each of them feels a different refractive index with a different angle. A light ray in these anisotropic materials divides into the ordinary and extraordinary parts. This feature is called birefringence

[24–26].

We develop a mathematical structure, based on an eigenvalue wave equation for studying the analogy between curved space-time and anisotropic media. We obtain the optical line element in an inhomogeneous anisotropic slab of an electric medium and an anisotropic impedance-matched medium and construct their effective space-time metric respectively.

The optical line element in an anisotropic impedance-matched medium is equal to the optical line element for the extraordinary path in the electric anisotropic medium. For the extraordinary modes, the metrics depend on the principle values of the permittivity tensor and orientation of the optical axes. As an example, we derive the profile of a birefringent electric medium in which extraordinary modes feels the Rindler metric.

In addition, we show that if this anisotropic electrical slab were an impedance-matched medium, the normal incident light would not split. Then, the propagation direction would be identical to the extraordinary light direction in the electrical slab. Comparison between the two metrics for the extraordinary mode in electric medium and the normal mode in impedance-matched medium, points out the specific condition, under that the two metrics are equivalent. Electric medium appears for TM polarized light as good as impedance-matched medium for non-polarized light. By good we mean, conditions that make the geometrical optics exact. We can consider this important result as an extension of transformation optics to non-impedance-matched media.

Finally, we investigate the optical properties of media with variable optical axes, whether they can be constructed of a single type of anisotropic material [15] or realized in liquid crystal [27]. By using the ray-tracing method, we obtain the trajectory of light in these media for two arbitrary axes orientation. The light trajectory in this medium suggests, media with variable optical axes might consider as a curved space for light rays.

The organization of this paper is as follows. In section 2, we review the Maxwell equation in an inhomogeneous anisotropic medium in geometrical optics limit. Later on, we explain eigenvalue wave equation method for anisotropic media. Also, it will be yielded the optical line element in anisotropic media. In section 3, the eigenvalue wave equation method is applied for investigation of the electric and impedance-matched anisotropic slab. In section 4, we achieve optical path line elements for light in electric and impedance-matched media and compare these metrics. Also, we investigate the variable optical axes media. Finally in section 5, some results and remarks are summarized.

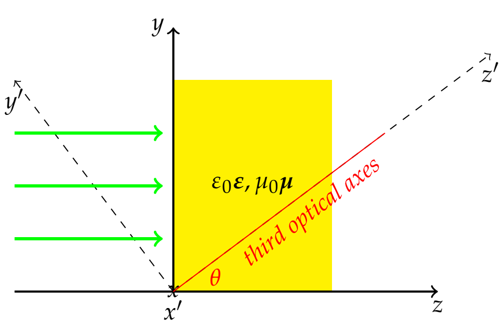

Fig. 1. Schematic of an anisotropic slab with ε and μ as a tensor, light incidents from left and incident plane is $y - z$ plane; two principle axes of media z' and y' do not lay on the z and y coordinate.

2. PRELIMINARIES AND INTRODUCING THE METHOD

Maxwell's equations in an anisotropic source-free materials, $\rho = 0, \quad \mathbf{J} = 0$, are given by [28],

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \tag{1}$$

The constitutive equations for the anisotropic material are given as

$$\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon} \mathbf{E}, \qquad \mathbf{B} = \mu_0 \boldsymbol{\mu} \mathbf{H}, \tag{2}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the electric permittivity and the magnetic permeability tensors, respectively. These dielectric tensors for an arbitrary orientation of the crystal principle axes, can be obtained from following relations [29],

$$\boldsymbol{\varepsilon} = \mathbf{A}\boldsymbol{\varepsilon}'\mathbf{A}^T, \qquad \boldsymbol{\mu} = \mathbf{A}\boldsymbol{\mu}'\mathbf{A}^T, \tag{3}$$

where $\boldsymbol{\varepsilon}' = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\boldsymbol{\mu}' = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ are principle dielectric tensors and $\mathbf{A}$ is the rotation matrix, which describes the rotation of the coordinate axes with respect to the crystal principle axes and $\mathbf{A}^T$ is transpose of the rotation matrix $\mathbf{A}$. The wave equation, for the plane wave solution, i.e., $\mathbf{E} \propto \exp\left[i\mathbf{k}\cdot\mathbf{r} - i\omega t\right]$, can be written in k-space as [30],

$$\mathbf{k} \times \boldsymbol{\mu}^{-1}(\mathbf{k} \times \mathbf{E}) = -k_0^2 \boldsymbol{\varepsilon} \mathbf{E}. \tag{4}$$

A. Inhomogeneous anisotropic media

In inhomogeneous anisotropic media, one of the dielectric tensors, $\boldsymbol{\varepsilon}, \boldsymbol{\mu}$, atleast, vary with of position; then the wave vector in these media is a function of position. In this case, Maxwell equation can be solved by applying following general solutions [25, 27],

$$\{\mathbf{E}, \mathbf{H}\}(\mathbf{r}, t) = \{\mathbf{E}, \mathbf{H}\}(\mathbf{r}) e^{-i\varphi(\mathbf{r},t)}, \tag{5}$$
$$\varphi(\mathbf{r}, t) = k_0 \psi(\mathbf{r}) - \omega t, \tag{6}$$

where k_0 and $\psi(\mathbf{r})$ is the wave number in vacuum and the optical path function respectively. The wave vector $\mathbf{k}$ and the angular frequency ω are defined as

$$\mathbf{k} = \nabla\varphi(r, t) = k_0 \nabla\psi, \qquad \omega = -\frac{\partial\varphi(r, t)}{\partial t}. \tag{7}$$

Accordingly, the Maxwell's equations in inhomogeneous anisotropic media are given by,

$$\boldsymbol{\nabla}\psi \cdot \mathbf{D} = -\frac{1}{ik_0} \boldsymbol{\nabla} \cdot \mathbf{D},$$
$$\boldsymbol{\nabla}\psi \cdot \mathbf{B} = -\frac{1}{ik_0} \boldsymbol{\nabla} \cdot \mathbf{B},$$
$$\boldsymbol{\nabla}\psi \times \mathbf{E} - c\mathbf{B} = -\frac{1}{ik_0} \boldsymbol{\nabla} \times \mathbf{E},$$
$$\boldsymbol{\nabla}\psi \times \mathbf{H} + c\mathbf{D} = -\frac{1}{ik_0} \boldsymbol{\nabla} \times \mathbf{H}, \tag{8}$$

where $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

In geometrical optics regime, where $k_0 \gg 1$, the right hand sides of the above equations (8) can be neglected and these equations become similar to the Eq. (1). Therefore, the wave equation can be written as,

$$\nabla\psi \times \mu^{-1}(\nabla\psi \times \mathbf{E}) = -\varepsilon \mathbf{E}. \tag{9}$$

By using Eq. (7), the wave equation in inhomogeneous anisotropic media, similar to the Eq. (4), is given by

$$\mathbf{k} \times \xi(\mathbf{k} \times \mathbf{E}) = -k_0^2 \varepsilon \mathbf{E}, \tag{10}$$

where $\xi = \mu^{-1}$. We rewrite the equation (10) in a matrix equation form,

$$\mathbf{M}\mathbf{E} = -k_0^2 \varepsilon \mathbf{E},$$

in which the matrix elements of $\mathbf{M}$ are obtained as

$$\begin{aligned}
M_{11} &= 2\xi_{23}k_z k_y - (\xi_{33}k_y^2 + \xi_{22}k_z^2),\\
M_{12} &= \xi_{21}k_z^2 - \xi_{23}k_z k_x - \xi_{31}k_z k_y + \xi_{33}k_y k_x,\\
M_{13} &= \xi_{31}k_y^2 - \xi_{23}k_y k_x - \xi_{21}k_z k_y + \xi_{22}k_z k_x,\\
M_{22} &= 2\xi_{13}k_z k_x - (\xi_{33}k_x^2 + \xi_{11}k_z^2),\\
M_{23} &= \xi_{32}k_x^2 - \xi_{12}k_z k_x - \xi_{31}k_y k_x + \xi_{11}k_y k_z,\\
M_{33} &= 2\xi_{21}k_x k_y - (\xi_{22}k_x^2 + \xi_{11}k_y^2).
\end{aligned} \tag{11}$$

B. Eigenvalue wave equation method

In an inhomogeneous anisotropic material, direction of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{H}$, can vary inside the medium. But direction of the displacement $\mathbf{D}$ and induction field $\mathbf{B}$ would be constant. Therefore, we can use the eigenvalue equation instead, which is an equation in term of $\mathbf{D}$ [24]. By substituting $\nabla\psi = n\hat{U}$ in wave equation (4), we get eigenvalue form of the wave equation,

$$\hat{U} \times \xi(\hat{U} \times \eta\mathbf{D}) = -\frac{1}{n^2}\mathbf{D}, \tag{12}$$

where $\eta = \varepsilon^{-1}$. For each direction of the wave vector we find two directions for $\mathbf{D}$.
The equation (12) can be written in alternative form,

$$\mathbf{L}\mathbf{D} = -\frac{1}{n^2}\mathbf{D}. \tag{13}$$

In the general, L-matrix in Eq. (13) can be too complicated. Nevertheless, knowing the fact that the light propagates in a plane, we can establish a fixed coordinate system so that the propagation plane lays in one of the principal planes of the coordinate system. As is shown in figure 1, we choose $y-z$ plane as the propagation plane, so the x component of the wave vector vanishes. In this case, we simply obtain the components of the L-matrix:

$$\begin{aligned}
L_{1i} &= (\xi_{31}\eta_{3i} - \xi_{33}\eta_{1i})u_2^2 + (-\xi_{22}\eta_{1i} + \xi_{21}\eta_{2i})u_3^2\\
&\quad + (2\xi_{23}\eta_{1i} - \xi_{31}\eta_{2i} - \xi_{21}\eta_{3i})u_2u_3,\\
L_{2i} &= (\xi_{12}\eta_{1i} - \xi_{11}\eta_{2i})u_3^2 + (-\xi_{13}\eta_{1i} + \xi_{11}\eta_{3i})u_2u_3,\\
L_{3i} &= (\xi_{13}\eta_{1i} - \xi_{11}\eta_{3i})u_2^2 + (-\xi_{12}\eta_{1i} + \xi_{11}\eta_{2i})u_2u_3,
\end{aligned} \tag{14}$$

where $i = 1, 2, 3$. The eigenvalues of the Eq. (13) can be achieved from

$$\det(\mathbf{L} - \frac{1}{n^2}\mathbf{I}) = 0, \tag{15}$$

and then by using Eq. (13) the component of $\mathbf{D}$ are achieved. Other fields and the Poynting vector can be obtained by [28]

$$\mathbf{E} = \frac{\eta}{\varepsilon_0}\mathbf{D}, \qquad \mathbf{B} = n\sqrt{\mu_0\varepsilon_0}\hat{U} \times \mathbf{E}, \qquad \mathbf{H} = \frac{\xi}{\mu_0}\mathbf{B}, \qquad \mathbf{S} = \mathbf{E} \times \mathbf{H}. \tag{16}$$

Finally, the direction of the field propagation for each mode is achieved from the Poynting vector as

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} = \frac{\mathbf{S}}{S}. \tag{17}$$

C. Line element in anisotropic media

The equiphase surfaces of the the fields can be defined by,

$$\mathrm{d}\varphi(r,t) = 0. \tag{18}$$

Using Eqs. (6) and (7), Eq. (18) can be written as,

$$\nabla\varphi(r,t) \cdot \mathrm{d}\mathbf{r} - \omega \mathrm{d}t = 0. \tag{19}$$

During the time interval $\mathrm{d}t$, the optical length is given by

$$\mathrm{d}r_v = \nabla\psi(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}, \tag{20}$$

in vacuum, where index v stands for vacuum. Therefore, we can write the line element in space-time as

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r_v^2. \tag{21}$$

According to Eq. (20), we write $\mathrm{d}r_v^2$ as

$$\mathrm{d}r_v^2 = \mathrm{d}\mathbf{r}^T \left(\nabla\psi(\mathbf{r}) \otimes \nabla\psi(\mathbf{r})\right) \mathrm{d}\mathbf{r}. \tag{22}$$

So, optical path length (OPL), by using (22), can be written as

$$OPL \equiv \int \sqrt{\mathrm{d}r_v^2} = \int \sqrt{\mathrm{d}\mathbf{r}^T \mathbf{n}_p^2 \mathrm{d}\mathbf{r}}, \tag{23}$$

in which,

$$\mathbf{n}_p^2 = \nabla\psi(\mathbf{r}) \otimes \nabla\psi(\mathbf{r}). \tag{24}$$

In relation (24), $\mathbf{n}_p^2$ is a square of the phase refractive index, which appears as a tensor. Also, according to relations (7) and (20), $\nabla\psi(\mathbf{r})$ is phase refractive vector, $\mathbf{n}_p = \nabla\psi(\mathbf{r}) = \mathbf{k}/k_0$ which is normal to the wave front. So we can write $\mathbf{k} = k_0\nabla\psi(\mathbf{r}) = k_0 n_p \hat{U}$. Therefore, in eigenvalue wave equation (12), n refers to the phase refractive index [25].
Moreover, base on the fact the ray direction is not along the wave vector, it is appropriate to define the ray refractive index, n_{ray}, as the ratio between optical line element, dr_v and the line element in medium, dr, [25]

$$n_{ray}^2 = \frac{\mathrm{d}r_v^2}{\mathrm{d}l^2}. \tag{25}$$

Using relation (22), we can write ray refractive index as

$$n_{ray}^2 = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l}^T \left(\nabla\psi(\mathbf{r}) \otimes \nabla\psi(\mathbf{r})\right) \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l}. \tag{26}$$

Hence, we can write space-time line element in term of phase refractive index or equivalently in term of ray refractive index,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{r}^T \mathbf{n}_p^2 \mathrm{d}\mathbf{r}, \tag{27}$$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + n_{ray}^2 \mathrm{d}l^2. \tag{28}$$

On the other hand, the line element in a Riemann geometry is given by, [9],

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{29}$$

where $g_{\mu\nu}$ is the metric tensor. By comparing the equations (27) and (28) with (29) we can achieve the metric tensor for light in medium.

3. NORMAL INCIDENT ON ANISOTROPIC SLAB

In this section, we investigate behavior of the normal incident light on an anisotropic slab of material. It is well known that in the normal incidence, the direction of the wave vector does not change. We assume $y - z$ plane as incident plane and the light comes parallel to z axis.Therefore:

$$|\mathbf{k}| = k_z \quad \rightarrow \quad n_p^2 = n_{zz}^2. \tag{30}$$

Suppose that two principal axes y' and z' of the slab lay in this plane, Fig. 1, then the rotation matrix is given by

$$\mathbf{A} = \mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{31}$$

where θ is a angle between z and z'. From Eq. (3), we can obtain the rotated permittivity tensor,

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta & -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ 0 & -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta & \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \end{pmatrix}. \tag{32}$$

and also the inverse permittivity tensor,

$$\boldsymbol{\eta} = \frac{1}{\varepsilon_2\varepsilon_3} \begin{pmatrix} \frac{\varepsilon_2\varepsilon_3}{\varepsilon_1} & 0 & 0 \\ 0 & \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta & (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ 0 & (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta & \varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta \end{pmatrix}. \tag{33}$$

According to the Maxwell's equation, $\boldsymbol{\nabla}\cdot\mathbf{D} = 0$, for the normal incident, we have $D_z = 0$. By replacing relation (33) in (12), the eigenvalue equation for $\mathbf{D}$ takes the form,

$$\begin{pmatrix} -\eta_{11}\xi_{22} & \xi_{21}\eta_{22} \\ \eta_{11}\xi_{12} & -\xi_{11}\eta_{22} \end{pmatrix} \begin{pmatrix} D_x \\ D_y \end{pmatrix} = -\frac{1}{n^2} \begin{pmatrix} D_x \\ D_y \end{pmatrix}. \tag{34}$$

Applying the condition (15), the eigenvalues of the Eq. (34) are obtained from

$$\left(\frac{1}{n^2} - \eta_{11}\xi_{22}\right)\left(\frac{1}{n^2} - \xi_{11}\eta_{22}\right) - \xi_{21}^2\eta_{22}\eta_{11} = 0 \tag{35}$$

This is a quadratic equation in terms of $1/n^2$ and have two response as

$$\frac{1}{n^2} = \frac{1}{2}(\eta_{11}\xi_{22} + \xi_{11}\eta_{22}) \pm \sqrt{(\eta_{11}\xi_{22} - \xi_{11}\eta_{22})^2 + 4\xi_{21}^2\eta_{11}\eta_{22}} \tag{36}$$

In the following, We investigate two special cases: the electric slab, with $\boldsymbol{\mu} = 1$, and the impedance-matched slab, with $\boldsymbol{\mu} = \boldsymbol{\varepsilon}$.

A. Electric slab

For the purely electric medium, the refractive indices is given by

$$n_o^2 = \varepsilon_1, \tag{37}$$

$$n^2(\theta) = \frac{\varepsilon_2\varepsilon_3}{\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta}. \tag{38}$$

The relation (38) shows that, the refractive index depends on two principle values of the permittivity tensor, i.e, ε_2 and ε_3, as well as the angle between the direction of the wave vector and third principle axis denotes by θ. Whereas the refractive index (37) depends only on ε_1. We obtain normal modes of these two refractive indices profiles by solving the eigenvalue equation (34).
For refractive index (37), we obtain $\mathbf{D}$ as

$$\mathbf{D} = D_x \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T. \tag{39}$$

and for the refractive index (38),

$$\mathbf{D} = D_y \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T. \tag{40}$$

By introducing the normalization condition, $\mathbf{E}\cdot\mathbf{E} = 1$, we can determine the components D_x and D_y. For the first normal mode (39), electric and magnetic fields can be obtained from (16),

$$\mathbf{E} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} (\varepsilon_1)^{\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{41}$$

It is clear that in this case, the normal mode is TE polarized light. For second normal mode (40), $\mathbf{E}$ and $\mathbf{H}$ are given by,

$$\mathbf{E} = \left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta\right)^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \\ (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \end{pmatrix}, \tag{42}$$

$$\mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \left(\frac{\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta\right)}{\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}\right)^{\frac{1}{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{43}$$

It is worth mentioning that, while the TE polarized field is propagating along the electric flux density, modes with TM polarization do not. The Poynting vectors of TE and TM polarizations can derive as

$$\mathbf{S}_{TE} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} (\varepsilon_1)^{\frac{1}{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{44}$$

$$\mathbf{S}_{TM} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_2\varepsilon_3)^{\frac{1}{2}}(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta)^{\frac{1}{2}}}{\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta} \times \begin{pmatrix} 0 \\ -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \end{pmatrix}. \tag{45}$$

The ray direction is given by the angle of the ray with respect to the z-axis, determined by the relations (44), (45) and (17). For the TE mode, we can write:

$$\tan\phi = \frac{S_y}{S_z} = 0, \qquad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{46}$$

Also, for the TM mode, we achieve:

$$\tan\phi = \frac{-(\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta}{\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta},$$
$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} = \frac{1}{\sqrt{\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}} \begin{pmatrix} 0 \\ -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \end{pmatrix}. \tag{47}$$

Since the wave vector is along the z axis, ϕ is equivalent with the deviation angle of the light from the wave vector direction. In the relation (47), the propagation direction of TM polarization is not identical with the wave vector, therefore it represents extraordinary ray, whereas the ray direction of TE polarization is along the wave vector and therefore it is ordinary ray.

B. Anisotropic impedance-matched media

In this section, we suppose anisotropic media which is impedance-matched, i.e., $\xi_{ij} = \eta_{ij}$. In this case, both refractive indexes in (36) are equal,

$$n_{imp}^2(\theta) = \frac{\varepsilon_1\varepsilon_2\varepsilon_3}{\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta}. \tag{48}$$

By comparison between the refractive indices (48) and (38), it is clear that,

$$n_{imp}(\theta) = \sqrt{\varepsilon_1}\, n_{el}(\theta), \tag{49}$$

where index el (imp) stands for electric (impedance match) slab. We now investigate the behavior of the TE and TM polarization. By using relation (16), for TM mode, the electric and magnetic field read as,

$$\mathbf{E} = \left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta\right)^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \\ (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \end{pmatrix}, \tag{50}$$

$$\mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta)^{\frac{1}{2}}}{(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta)^{\frac{1}{2}}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{51}$$

Also, for the TE polarization, Eq. (39), the electric and magnetic field are given by

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \tag{52}$$

$$\mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}}{(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta)^{\frac{1}{2}}} \frac{1}{\varepsilon_2\varepsilon_3} \times \begin{pmatrix} 0 \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \\ (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \end{pmatrix}. \tag{53}$$

From expersions (52) and (53), one can see, that for the TE mode, the electric field**E** and displacement field **D**, are in the same direction but the magnetic field **H** is not along the induction **B**, whereas in TM mode it is other way around.

From relation (17), we obtain the Poynting vector for TM and TE modes in impedance-matched medium

$$\mathbf{S}_{TM} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta)^{\frac{1}{2}}}{\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta} \times \begin{pmatrix} 0 \\ -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \end{pmatrix}, \tag{54}$$

$$\mathbf{S}_{TE} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}}{(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta)^{\frac{1}{2}}} \frac{1}{\varepsilon_2\varepsilon_3} \times \begin{pmatrix} 0 \\ -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \end{pmatrix}. \tag{55}$$

It is clear both TE and TM mode have the same Poynting vector. Consequently, light in impedance-matched media, is not divided into ordinary and extra-ordinary rays. The angle between ray and z-axis can be written as

$$\tan\phi_{(TE)} = \tan\phi_{(TM)} = \frac{-(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta}{\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta}. \tag{56}$$

However, we can write the ray direction in the impedance-matched slab as

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} = \frac{1}{\sqrt{\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}} \begin{pmatrix} 0 \\ -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \end{pmatrix}. \tag{57}$$

Comparison between(47) and (57) shows that the extraordinary ray direction in the electric slab is equal to the ray direction in impedance matched ones.

4. EFFECTIVE GEOMETRY FOR THE LIGHT IN ANISOTROPIC MEDIA

In this section, we achieve the equivalent metric perceived by light in the electric and impedance-matched media. As we seen in the section 3, the only non zero component of the $\mathbf{n}^2$ tensor is the n_{zz}, (30). So, we can write the line element (27), as

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + n_p^2\mathrm{d}z, \tag{58}$$

This line element (58) gives the phase velocity of the light as c/n_p. On the other hand, the line element (28) can be written as

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + n_{zz}^2\cos^2\phi\,\mathrm{d}l^2, \tag{59}$$

The line element, (58), describe light in (1+1) space-time, whereas, in previous section, we saw that the light ray, unlike the wave vector, propagates in the $y-z$ plane. Therefore the line element (59) is appropriate for description of the light in the anisotropic media. In the following parts, by using this line element, (59), we investigate non-impedance-matched and impedance-matched analogue space-time.

A. Metric for light in electric slab

We have found that the TE and TM are the ordinary and extraordinary normal modes, respectively. For TE polarization, $\tan\phi = 0$ and $n_{33}^2 = \varepsilon_1$, the line element (59) can be written as

$$ds^2 = -c^2dt^2 + \varepsilon_1 dz^2. \tag{60}$$

As is shown, the line element (60) is equivalent to the one in isotropic media, which has a refractive index $n_1 = \sqrt{\varepsilon_1}$. Also, it is conformal to the metric,

$$ds^2 = -\frac{1}{\varepsilon_1}c^2dt^2 + dz^2. \tag{61}$$

If ε_1 is a function of z coordinate as

$$\varepsilon_1(z) \propto \frac{1}{z^2}, \tag{62}$$

the line element (61) can be obtained as a Rindler metric [31],

$$ds^2 = -az^2c^2dt^2 + dz^2. \tag{63}$$

Hence, a non-homogeneous anisotropic media, which is equivalent to the stratified anisotropic slabs, with the ε_1 as a function of z coordinate of the form (62), can be appeared as a Rindler space-time for ordinary light. In addition, an isotropic layered medium may be used for the same analogy [19].

Also, for the TM polarization, we can write the line element (59) as

$$ds^2 = -c^2dt^2 + \frac{\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}(dy^2 + dz^2). \tag{64}$$

The metric (64) depends on ε_2 and ε_3 and also on θ. An important point to be mentioned is that the extraordinary ray feels a curved space-time in the array of anisotropic media if their principle axes be variable.

B. Metric for light in anisotropic impedance-matched media

In the previous section, we found that the ray directions in impedance-matched and electric slab are identical, $\tan\phi_{imp} = \tan\phi_{el}$. Also, for their refractive indices, we achieve the relation (49), $n_{imp} = \sqrt{\varepsilon_1}n_e$. Therefore, using relation (59), the metric for light in anisotropic impedance-matched media can be written as

$$ds^2 = -c^2dt^2 + \frac{\varepsilon_1\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}(dy^2 + dz^2). \tag{65}$$

On the other hand, in transformation optics [9], space metric for impedance-matched media is given by

$$\mathbf{g} = (\det\boldsymbol{\varepsilon})\boldsymbol{\varepsilon}^{-1}. \tag{66}$$

This metric, for our case, Fig. 1, can be written in the form

$$\mathbf{g}_{imp}(y-z) = \begin{pmatrix} \varepsilon_1(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta) & \varepsilon_1(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_1(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta & \varepsilon_1(\varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta) \end{pmatrix}. \tag{67}$$

At first glance, the metric (67) is different from the metric (65), but , by using following relation:

$$\begin{aligned}\varepsilon_2\varepsilon_3 = {} & (\varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta)(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta) \\ & - \left[(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta\right]^2,\end{aligned} \tag{68}$$

we obtain that the metrics (65) and (67) are equivalent.

C. Transformation optics and non impedance matched media

In this part, we would like to show how transformation optics would apply to non-impedance-matched media. For this aim, we consider the section 3, the direction of the extraordinary ray (47) and the direction of the light in impedance-matched media (57) are the same. But, their refractive indexes only differ on one factor, (49). Therefore, we expect an analogous in their optical path length if $\varepsilon_1 = constant$. Mathematically, we consider the metrics (65) and (64); Time part of these metrics are equal but their space part is not the same. For better comparison between them, we can write following metric, (69) as a conformal form of the metric (64),

$$ds_{el}^2 = -\varepsilon_1c^2dt^2 + \frac{\varepsilon_1\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}dl^2. \tag{69}$$

As we can see,in this form, the space part of this metric, (69), is the same with the metric (65),

$$ds_{imp}^2 = -c^2dt^2 + \frac{\varepsilon_1\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}dl^2. \tag{70}$$

The comparison between two above metrics, (69 and (70), shows that, if $\varepsilon_1 = costant$, two metrics are equivalent. Therefore, an electric medium, which is a non-impedance-matched material, appears for the TM polarized light, like the impedance-matched medium for the unpolarized light. We can write metric for the extraordinary light in electric media in terms of the impedance -matched media metric,

$$\mathbf{g}_{el}(2+1) = \begin{pmatrix} -\varepsilon_1 & 0 \\ 0 & \mathbf{g}_{imp}(y-z) \end{pmatrix}. \tag{71}$$

As the result, for properly polarized light, in the case of normal incident, we can use electric medium instead of the impedance-match one to make the geometry exact.

D. Analogy between variable optical axes media and curved-space

Variable optical axes media are inhomogeneous anisotropic media that their inhomogeneity are due to changing on the orientation of their optical axes, i.e, optical axes direction in these media depend on position and principle permittivities $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are constant. These media, despite the most of the meta-materials which are constructed from two kind of materials, can be artificially constructed from one kind of homogeneous anisotropic materials [15]. Also these media can be realized in liquid crystal with applying the external field or internal charged particles [27].

In this subsection, we apply ray-tracing method to investigate the behavior of the light in two examples of planar media with variable optical axes. First, a medium that its optical axes orientation depends only on z: $\theta = z$; And second, a medium that its optical axes orientation depend on both y and z: $\tan\theta = y/z$. We assume the principle values of permittivity equal to that $\varepsilon_2 = 2.75$ and $\varepsilon_3 = 2.21$, which describes the calcite crystal principal permittivities [26]. In Fig. 2, the plotted trajectories of the family of extraordinary rays are shown. Left and right diagrams are corresponding to the $\theta = z$ and $\tan\theta = y/z$ respectively, which indicates the normal incidence of light.

According to the Fermat principle, the trajectories of the light rays in matter are geodesic. Therefore, by studying the behavior of these geodesics, we can get some information about the analogue geometry of medium and its curvature. As it is shown in

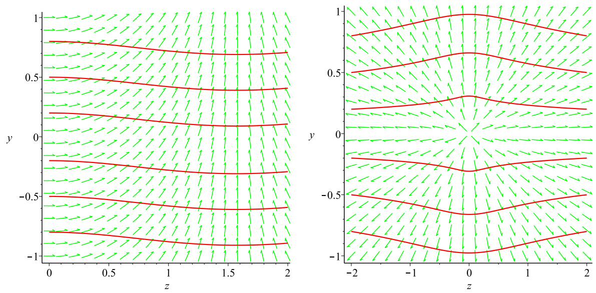

Fig. 2. Ray tracing in anisotropic media with variable optical axes, $\theta = z$ (left) and $\tan\theta = y/z$ (right), the green arrows indicate orientations of the third principle axis and the red lines indicate ray trajectories in the media.

the Fig. 2, geodesics of these surfaces do not straight line. Also it seems that these surfaces are under tensions. Therefore, we expect non zero Riemann tensor for them. For better investigation of the effective geometry of these media, in the following, we obtain the metric and Riemann curvature tensor of these surfaces.

The metric of these two dimensional distorted surfaces, Fig 2, can be constructed by choosing the two bases as

$$e_1 = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{72}$$

where coefficient c_1, c_2 is a scalar function that provide null geodesics condition, $ds^2 = 0$, for light in the space-time. Using these bases(72), we can achieve the metric components by [9]:

$$g_{ij} = e_i \cdot e_j. \tag{73}$$

Therefore, the space-time metric for the extraordinary light is achieved,

$$\mathbf{g}(2+1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \dfrac{\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)} & 0 \\ 0 & 0 & \dfrac{\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)} \end{pmatrix}. \tag{74}$$

This metric is in agreement with the metric (59), which we achieve for the extraordinary ray.

On the other hand, Riemann curvature tensor given by [9]

$$R^i_{jkl} \equiv \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk} \tag{75}$$

where Γ^i_{jk} is Christoffel symbol and the comma notation "," refers to partial differentiation. The Christoffel symbol can be expressed in terms of the metric component as

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}) \tag{76}$$

For the metric (74), which is conformally two dimensional flat spaces, $g_{ij} = n^2\delta_{ij}$, we can obtain Christoffel symbol as

$$\Gamma^i_{ij} = \frac{1}{2n^2} n^2_{,j} \tag{77}$$

Using relation (77), Riemann curvature tensor, (75) can be achieved as

$$R_{11} = R_{22} = \frac{1}{2n^4}\{(\frac{\partial}{\partial z}n^2)^2 + (\frac{\partial}{\partial y}n^2)^2\} - \frac{1}{2n^2}\{(\frac{\partial}{\partial z}\frac{\partial}{\partial z}n^2) + (\frac{\partial}{\partial y}\frac{\partial}{\partial y}n^2)\} \tag{78}$$

We calculate this tensor for the two above example of the variable axes media, $\tan\theta = y/z$ and $\theta = z$ and find that

$$R_{ii} \neq 0. \tag{79}$$

Consequently, the medium with $\tan\theta = y/z$ and $\theta = z$ have a non zero Riemann curvature tensor and appears as a curved space for the extraordinary light. Therefore we can use variable optical axes media for realizing curved space time in the laboratory. As final point, according to the figure 2 left, the space has stretched in such a way that extra ordinary light can not pass through the center of the plane; if we put an object in this area, the object is hidden. Hence we can also use these media for cloaking devices.

5. CONCLUSION

In this paper, we have studied the effective geometry of the impedance-matched and non-impedance-matched anisotropic media for normal incident light. We obtain the effective geometry by calculating the optical lines element for each kind, based on eigenvalue equation method. In the impedance-matched media, light rays do not split; there is a single direction of propagation for each angle of incidence. This direction coincides with the extraordinary ray's direction in the electric media. In our paper, the metric for the light in the impedance-matched media is obtained, which is the same as one might get from transformation optics method. In the non-impedance-matched media, for the ordinary and the extraordinary modes in birefringent medium, we associate two general space-time metrics. For the extraordinary mode, we have showed the metric depends on the principle values of the permittivity tensor as well as the orientation of the optical axes. A comparison between the two metrics, one from the impedance-matched medium and the other, from the extraordinary mode in electric medium, shows they are equivalent. As an example of realization of curved space-time in a non-impedance-matched medium, we obtained the Rindler space-time, by adopting a specific value of ε_1 for ordinary mode. In the end, we introduce the family of media with variable optical axes and indicate the role of the directional variability of the optical axes in forming the effective metric. We show if the optical axes orientation varies, would results in emergence of effective geometry in the medium. Our finding is a new method for manipulation of electromagnetic waves.

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REFERENCES

1. U. Leonhardt and T. G. Philbin, "General relativity in electrical engineering," New Journal of Physics **8**, 247 (2006).

2. J. B. Pendry, D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," science **312**, 1780–1782 (2006).
3. K. Yao, H. Chen, Y. Liu, and X. Jiang, "An analogy strategy for transformation optics," New Journal of Physics **16**, 063008 (2014).
4. R. T. Thompson, S. A. Cummer, and J. Frauendiener, "A completely covariant approach to transformation optics," Journal of Optics **13**, 024008 (2011).
5. E. Bittencourt, V. De Lorenci, R. Klippert, M. Novello, and J. Salim, "The light rays analogue of a static black hole," in "PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS 2014 (ICNAAM-2014)," , vol. 1648 (AIP Publishing, 2015), vol. 1648, p. 290002.
6. H. Chen, C. Chan, and P. Sheng, "Transformation optics and metamaterials," Nature materials **9**, 387–396 (2010).
7. C. Sheng, H. Liu, Y. Wang, S. Zhu, and D. Genov, "Trapping light by mimicking gravitational lensing," Nature Photonics **7**, 902–906 (2013).
8. M. McCall, "Transformation optics and cloaking," Contemporary Physics **54**, 273–286 (2013).
9. U. Leonhardt and T. Philbin, *Geometry and light: the science of invisibility* (Courier Corporation, 2012).
10. Y. Lai, J. Ng, H. Chen, D. Han, J. Xiao, Z.-Q. Zhang, and C. Chan, "Illusion optics: the optical transformation of an object into another object," Physical review letters **102**, 253902 (2009).
11. J. Courtial, S. Oxburgh, and T. Tyc, "Direct stigmatic imaging with curved surfaces," JOSA A **32**, 478–481 (2015).
12. D. Roberts, N. Kundtz, and D. Smith, "Optical lens compression via transformation optics," Optics express **17**, 16535–16542 (2009).
13. M. Yan, W. Yan, and M. Qiu, "Cylindrical superlens by a coordinate transformation," Physical Review B **78**, 125113 (2008).
14. Y. G. Ma, C. Ong, T. Tyc, and U. Leonhardt, "An omnidirectional retroreflector based on the transmutation of dielectric singularities," Nature Materials **8**, 639–642 (2009).
15. Z. Liang, X. Jiang, F. Miao, S. Guenneau, and J. Li, "Transformation media with variable optical axes," New Journal of Physics **14**, 103042 (2012).
16. W. Cai and V. M. Shalaev, *Optical metamaterials* (Springer, 2010).
17. E. E. Narimanov and A. V. Kildishev, "Optical black hole: Broadband omnidirectional light absorber," Applied Physics Letters **95**, 041106 (2009).
18. Y. Lee, E. Kang, K. Jung, J. Lee, and D. Ahn, "Elliptic cylindrical pseudo-optical black hole for omnidirectional light absorber," JOSA B **31**, 1948–1953 (2014).
19. S. Dehdashti, R. Roknizadeh, and A. Mahdifar, "Analogue special and general relativity by optical multilayer thin films: the rindler space case," Journal of Modern Optics **60**, 233–239 (2013).
20. M. Visser, "Survey of analogue spacetimes," in "Analogue Gravity Phenomenology," (Springer, 2013), pp. 31–50.
21. S. J. Robertson, "The theory of hawking radiation in laboratory analogues," Journal of Physics B: Atomic, Molecular and Optical Physics **45**, 163001 (2012).
22. H.-W. Wang and L.-W. Chen, "A cylindrical optical black hole using graded index photonic crystals," Journal of Applied Physics **109**, 103104 (2011).
23. Q. Bai, J. Chen, N.-H. Shen, C. Cheng, and H.-T. Wang, "Controllable optical black hole in left-handed materials," Optics express **18**, 2106–2115 (2010).
24. B. E. Saleh, M. C. Teich, and B. E. Saleh, *Fundamentals of photonics*, vol. 22 (Wiley New York, 1991).
25. M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light* (Cambridge university press, 1999).
26. A. Yariv and P. Yeh, *Optical waves in crystals*, vol. 10 (Wiley, New York, 1984).
27. M. Sluijter, *Ray-optics analysis of inhomogeneous optically anisotropic media* (TU Delft, Delft University of Technology, 2010).
28. J. D. Jackson and J. D. Jackson, *Classical electrodynamics*, vol. 3 (Wiley New York etc., 1962).
29. P. Yeh, "Optics of anisotropic layered media: a new 4×4 matrix algebra," Surface Science **96**, 41–53 (1980).
30. J. Hao and L. Zhou, "Electromagnetic wave scatterings by anisotropic metamaterials: Generalized 4×4 transfer-matrix method," Physical Review B **77**, 094201 (2008).
31. S. M. Carroll, *Spacetime and geometry. An introduction to general relativity*, vol. 1 (2004).