On the analogy between curved space-time and non-impedance-matched anisotropic media

S. A. Mousavi[1], R. Roknizadeh[2*], SH. Dehdashti[3,4], and S. Sahebdivan[5]

[1] Department of Physics, Faculty of Science, University of Isfahan, Hezar Jerib, Isfahan, 81746-73441, Iran
[2] Department of Physics, Quantum Optics Group, Faculty of Science, University of Isfahan, Hezar Jerib, 81746-73441 Isfahan, Iran
[3] State Key Laboratory of Modern Optical Instrumentations, Zhejiang University, Hangzhou 310027, China
[4] The Electromagnetic Academy at Zhejiang University, Zhejiang University, Hangzhou 310027, China
[5] Quantum Optics, Quantum Nanophysics and Quantum Information, Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Wien
[*] Corresponding author: r.roknizadeh@gmail.com

Compiled January 6, 2016

An electromagnetic medium with gradient refractive index can resemble a geometrical analogy with an arbitrary curved space-time [?]. In this paper, we show that a non-impedance-matched medium with a varying optical axis can resemble the features of an optical metric. The medium with a varying optical axis is an engineered stratified slab of material, that in each layer, the orientation of optical axis would slightly change while the magnitude of refractive index remains constant. Instead of change in the refractive index, the inhomogeneity in such a medium is induced by the local anisotropy. Therefore the propagation of light depends on the local optical axis. We study the conditions that make the analogy between curved space-time and a medium with a varying optical axis. Extension of the transformation optics to the media with optical axis profile might ease some fabrication cumbersome of gradient refractive index materials for particular frequencies.

OCIS codes: *(160.1190) Anisotropic optical materials,(160.3918) Metamaterials, (260.2110) Electromagnetic optics*

http://dx.doi.org/10.1364/ao.XX.XXXXXX

1. INTRODUCTION

Transformation optics works based on the diffeomorphic map between a virtual space and physical space. For technical simplification, usually physical space is an isoropic electromagnetic medium with a position dependent refractive index profile, while the optical axis of the medium is fixed. This simplification restricts the diffeomorphic maps to the family of quasi-conformal maps. However, inspired by optical axes grating in liquid crystals [? ?], one can shows, to keep the transformation map between the virtual space and physical space, it is possible to vary the direction of optical axes instead of refractive index [?]. Medium with a variable optical axes is an inhomogeneous anisotropic material that its inhomogeneity induces by the change in the direction of the optical axes. The direction of the optical axes in these media depends on the position. These media can be realized in liquid crystal with applying the external field or internally charged particles [?]. Anisotropy and inhomogeneity in the medium is responsible for curving the light trajectories. In medium with a varying optical axis inho-

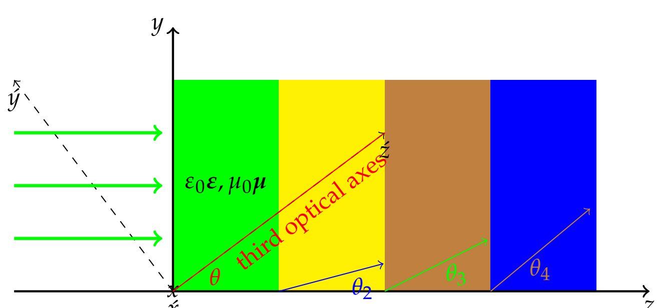

Fig. 1. Schematic of an anisotropic slab with ε and μ as a tensor, light incidents from left and incident plane is $y - z$ plane; two principle axes of media z' and y' do not lay on the z and y coordinate.

mogeneity is due to infinitesimal change of optical axis in each layer.

Anisotropic media are quite common in nature and also much cheaper to design artificially. Crystals are the best-known materials that perform electrical anisotropy based on their particular crystal structure and the symmetry of their space group. Many plastics also are birefringent, because their molecules are 'frozen' in a stretched conformation when the plastic is molded or extruded [?].

We conclude that a medium with the variable optical axes can consider as a curved space for light rays. They show the capacity for being used in transformation optics designs when impedance-matched materials are costly. More elaborated theory of transformation optics with variable optical axes profile would appear shortly in the further publications.

The organization of the paper is as follows: In section 2 we briefly describe our material and methods used in this research. In section 3, we explain a technique, eigenvalue wave equation method, to solve the Maxwell equations in an anisotropic medium. In section 4, above method is applied to a purely electric and an impedance-matched anisotropic slab respectively. In section 5, The idea of the medium with variable optical axes is introduced and the effective metric of the propagation plane in purely electric media and impedance-matched media are achieved and their corresponding metrics are compared. Finally in section 6, some results and remarks are summarized.

2. METHOD

A. Maxwell equations in anisotropic media

Maxwell's equations in an homogeneous anisotropic source-free medium, $\rho = 0, \quad \mathbf{J} = 0$, are given by [?]:

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \tag{1}$$

While the constitutive relations are hold as:

$$\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon} \mathbf{E}, \qquad \mathbf{B} = \mu_0 \boldsymbol{\mu} \mathbf{H}, \tag{2}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are respectively the electric permittivity and the magnetic permeability tensors for an arbitrary optical axis.

Explicitly assuming electric and magnetic anisotropy, wave equation can be written as

$$\nabla \times \boldsymbol{\mu}^{-\mathbf{1}} (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial^2}{\partial t^2} \mathbf{D}. \tag{3}$$

In a homogeneous anisotropic medium, the plane waves, $\mathbf{E} \propto \exp\left[i\mathbf{k} \cdot \mathbf{r} - i\omega t\right]$, is usually assumed as the general solution of the Maxwell wave equation [?]. Using this assumption, Eq. (3) becomes:

$$\mathbf{k} \times \boldsymbol{\xi} (\mathbf{k} \times \mathbf{E}) = -k_0^2 \boldsymbol{\varepsilon} \mathbf{E}, \tag{4}$$

where $\boldsymbol{\xi} = \boldsymbol{\mu}^{-1}$. We rewrite the equation (4) in a matrix equation form,

$$\mathbf{M}\mathbf{E} = -k_0^2 \boldsymbol{\varepsilon} \mathbf{E},$$

in which the matrix elements of $\mathbf{M}$ are obtained as

$$\begin{aligned}
M_{11} &= 2\xi_{23} k_z k_y - (\xi_{33} k_y^2 + \xi_{22} k_z^2), \\
M_{12} &= \xi_{21} k_z^2 - \xi_{23} k_z k_x - \xi_{31} k_z k_y + \xi_{33} k_y k_x, \\
M_{13} &= \xi_{31} k_y^2 - \xi_{23} k_y k_x - \xi_{21} k_z k_y + \xi_{22} k_z k_x, \\
M_{22} &= 2\xi_{13} k_z k_x - (\xi_{33} k_x^2 + \xi_{11} k_z^2), \\
M_{23} &= \xi_{32} k_x^2 - \xi_{12} k_z k_x - \xi_{31} k_y k_x + \xi_{11} k_y k_z, \\
M_{33} &= 2\xi_{21} k_x k_y - (\xi_{22} k_x^2 + \xi_{11} k_y^2).
\end{aligned} \tag{5}$$

B. Eigenvalue equation; method

In an anisotropic material, because the anisotropy, the electric and magnetic field are not necessarily perpendicular on the wave vector, they can vary in the medium. But the electric and and magnetic induction, through the relation $\nabla \cdot \mathbf{D} = 0$ and $\nabla \cdot \mathbf{B} = 0$, are always perpendicular on the wave vector. So, it is useful to write the wave equation in the form of eigenvalue equation for $\mathbf{D}$ [?]. By defining the phase refractive index as a retio between wave number in medium and in vacuum, $n = k/k_0$ and substituting $\boldsymbol{k} = nk_0\hat{U}$ in Eq. (4), we get eigenvalue equation,

$$\hat{U} \times \boldsymbol{\xi} (\hat{U} \times \boldsymbol{\eta} \mathbf{D}) = -\frac{1}{n^2} \mathbf{D}, \tag{6}$$

where $\hat{U} = \mathbf{k}/k$ is a direction of the wavre number and $\boldsymbol{\eta} = \boldsymbol{\varepsilon}^{-1}$. Therefore, according to eigenvalue equation, we find two directions of $\mathbf{D}$ for each direction of the wave vector.
The Eq. (6) can be written in operator form,

$$\mathbf{L}\mathbf{D} = -\frac{1}{n^2} \mathbf{D}. \tag{7}$$

In a general coordinate, matrix L might has a complicated form. Nevertheless, knowing that the light propagates in a plane, we can establish a fixed coordinate system such that the propagation plane coincides with one of the principal planes of the coordinate system. As Fig. 1 shows, we choose the $y - z$ as the propagation plane, so the x component of the wave vector vanishes. Thus, matrix L simplifies to the following non-zero components:

$$\begin{aligned}
L_{1i} &= (\xi_{31}\eta_{3i} - \xi_{33}\eta_{1i}) u_2^2 + (-\xi_{22}\eta_{1i} + \xi_{21}\eta_{2i}) u_3^2 \\
&\quad + (2\xi_{23}\eta_{1i} - \xi_{31}\eta_{2i} - \xi_{21}\eta_{3i}) u_2 u_3, \\
L_{2i} &= (\xi_{12}\eta_{1i} - \xi_{11}\eta_{2i}) u_3^2 + (-\xi_{13}\eta_{1i} + \xi_{11}\eta_{3i}) u_2 u_3, \\
L_{3i} &= (\xi_{13}\eta_{1i} - \xi_{11}\eta_{3i}) u_2^2 + (-\xi_{12}\eta_{1i} + \xi_{11}\eta_{2i}) u_2 u_3,
\end{aligned} \tag{8}$$

where $i = 1, 2, 3$.

One can obtain electrical displacement $\mathbf{D}$ by solving Eq. (7) through matrix algebra:

$$\det(\mathbf{L} + \frac{1}{n^2}\mathbf{I}) = 0, \tag{9}$$

Having the components of $\mathbf{D}$, other fields and the Poynting vector can be calculated easily [?]

$$\mathbf{E} = \frac{\boldsymbol{\eta}}{\varepsilon_0}\mathbf{D}, \qquad \mathbf{B} = n\sqrt{\mu_0\varepsilon_0}\,\hat{U} \times \mathbf{E}, \qquad \mathbf{H} = \frac{\boldsymbol{\xi}}{\mu_0}\mathbf{B}, \qquad \mathbf{S} = \mathbf{E} \times \mathbf{H}. \tag{10}$$

Finally, the direction of the field propagation for each mode is determined from the Poynting vector as:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} = \frac{\mathbf{S}}{S}. \tag{11}$$

3. NORMAL INCIDENT OF LIGHT ON A SINGLE LAYER

In this section, we study the behavior of the perpendicular incident light on a single anisotropic slab of material. ?????? and then extend it to an arbitrary angle of incidence. ?????????? It is well known that in the normal incidence, the direction of the wave vector does not change. We assume $y-z$ plane as the propagation plane, while, the light comes parallel to the z axis. Therefore:

$$|\mathbf{k}| = k_z \quad \rightarrow \quad n_p^2 = n_{zz}^2. \tag{12}$$

Further, we assume that two principal axes y' and z' of the slab lay in this $y-z$ plane, Fig. 1. In this case the dielectric tensor can be obtained from following relation [?],

$$\boldsymbol{\varepsilon} = \mathbf{A}\boldsymbol{\varepsilon}'\mathbf{A}^T, \tag{13}$$

where $\boldsymbol{\varepsilon}' = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are principle permittivity tensors and $\mathbf{A}$ is the rotation matrix, which describes the rotation of the coordinate axes with respect to the crystal principle axes and $\mathbf{A}^T$ is transpose of the rotation matrix $\mathbf{A}$.

Suppose that two principal axes y' and z' of the slab lay in this plane, Fig. 1, then the rotation matrix is given by

$$\mathbf{A} = \mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{14}$$

where θ is a angle between z and z'. From Eq. (13), we can obtain the rotated permittivity tensor,

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta & -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ 0 & -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta & \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \end{pmatrix}. \tag{15}$$

and also the inverse permittivity tensor,

$$\boldsymbol{\eta} = \frac{1}{\varepsilon_2\varepsilon_3}\begin{pmatrix} \frac{\varepsilon_2\varepsilon_3}{\varepsilon_1} & 0 & 0 \\ 0 & \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta & (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ 0 & (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta & \varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta \end{pmatrix}. \tag{16}$$

From Maxwell equations we have $\boldsymbol{\nabla}\cdot\mathbf{D} = 0$, hence, for the normal incident it reads as $D_z = 0$. For other components of the $\mathbf{D}$ by replacing relation (16) in (6), we would have:

$$\begin{pmatrix} -\eta_{11}\xi_{22} & \xi_{21}\eta_{22} \\ \eta_{11}\xi_{12} & -\xi_{11}\eta_{22} \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix} = -\frac{1}{n^2}\begin{pmatrix} D_x \\ D_y \end{pmatrix}. \tag{17}$$

Applying the condition (9), the eigenvalues of the Eq. (17) are obtained from the following equation:

$$\left(\frac{1}{n^2} - \eta_{11}\xi_{22}\right)\left(\frac{1}{n^2} - \xi_{11}\eta_{22}\right) - \xi_{21}^2\eta_{22}\eta_{11} = 0 \tag{18}$$

This is a quadratic equation in terms of $1/n^2$ which has two eigenvalues;

$$\frac{1}{n^2} = \frac{1}{2}\left(\eta_{11}\xi_{22} + \xi_{11}\eta_{22}\right) \pm \sqrt{\left(\eta_{11}\xi_{22} - \xi_{11}\eta_{22}\right)^2 + 4\xi_{21}^2\eta_{11}\eta_{22}}. \tag{19}$$

Corresponding eigenvectors would determine the physical components of the field.

In the rest of this section, we investigate two special examples: first the anisotropic electric slab, with $\boldsymbol{\mu} = 1$, and then the anisotropic impedance-matched slab, with $\boldsymbol{\mu} = \boldsymbol{\varepsilon}$.

A. Anisotropic electric medium

For the purely electric medium, the refractive indices (19) are given by

$$n_o^2 = \varepsilon_1, \tag{20}$$

$$n^2(\theta) = \frac{\varepsilon_2\varepsilon_3}{\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta}. \tag{21}$$

Relation (21) shows the refractive index depends on principle values of the permittivity tensor, i.e, ε_2 and ε_3, and, θ, the angle between the direction of the wave vector and third principle axis. Whereas the refractive index (20) only depends on ε_1.

For a specific angle of incident, in this example perpendicular incident, there is two modes associated to the above refractive indices profiles.

For refractive index (20), we obtain $\mathbf{D}$ by solving the eigenvalue equation (17):

$$\mathbf{D} = D_x\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T. \tag{22}$$

and for the refractive index (21) we have:

$$\mathbf{D} = D_y\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T. \tag{23}$$

By applying the normalization condition, $\mathbf{E}\cdot\mathbf{E} = 1$, we can determine the components D_x and D_y. For the first normal mode (22), electric and magnetic fields are written in the from (10),

$$\mathbf{E} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}}(\varepsilon_1)^{\frac{1}{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{24}$$

We can see that the normal mode is TE polarized light. Electrical component is lies in the propagating plane $y-z$. For second normal mode (23), $\mathbf{E}$ and $\mathbf{H}$ are given by,

$$\mathbf{E} = \left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta\right)^{-\frac{1}{2}}\begin{pmatrix} 0 \\ \varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta \\ (\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \end{pmatrix}, \tag{25}$$

$$\mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}}\left(\frac{\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta\right)}{\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}\right)^{\frac{1}{2}}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{26}$$

It is worth mentioning that, while the TE polarized field is propagating along the electric flux density, modes with TM polarization do not. The Poynting vectors of TE and TM polarizations can derive as

$$\mathbf{S}_{TE} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} (\varepsilon_1)^{\frac{1}{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{27}$$

$$\mathbf{S}_{TM} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_2\varepsilon_3)^{\frac{1}{2}}(\varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta)^{\frac{1}{2}}}{\varepsilon_2^2 \sin^2\theta + \varepsilon_3^2 \cos^2\theta} \times \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta \end{pmatrix}. \tag{28}$$

The ray direction is given by the angle of the ray with respect to the z-axis, determined by the relations (27), (28) and (11). For the TE mode, we can write:

$$\tan\phi = \frac{S_y}{S_z} = 0, \qquad \frac{d\mathbf{r}}{dl} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{29}$$

While for the TM mode, we achieve:

$$\tan\phi = \frac{-(\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta}{\varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta}, \quad \frac{d\mathbf{r}}{dl} = \frac{1}{\sqrt{\varepsilon_2^2 \sin^2\theta + \varepsilon_3^2 \cos^2\theta}} \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta \end{pmatrix}. \tag{30}$$

Since the wave vector is along the z axis, ϕ is equivalent with the deviation angle of the light from the wave vector direction. In the relation (30), the propagation direction of TM polarization is not identical with the wave vector. Therefore it represents extraordinary ray, whereas the ray direction of TE polarization is along the wave vector, and, therefore, it is the ordinary ray. In other terminology geometrical optics is exact for TE modes but it is not exact for TM components.

B. Impedance-matched medium

Suppose an anisotropic medium that satisfies the impedance-matched condition, i.e., $\xi_{ij} = \eta_{ij}$, refractive index profile reduces to the relation:

$$n_{imp}^2(\theta) = \frac{\varepsilon_1\varepsilon_2\varepsilon_3}{\varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta}. \tag{31}$$

Therefore, alike the birefringent media, for the impedance-matched medium we have one eigenvalue, i.e. the impedance matched media do not birefringence. For more investigation we compared this medium with the electric medium result achieved in previous subsection. By comparing the refractive indices (31) with (21), we find that,

$$n_{imp}(\theta) = \sqrt{\varepsilon_1} n_{el}(\theta), \tag{32}$$

where index el (imp) stands for electric (impedance-matched). ε_1 govern the electrical responses of the ordinary mode in birefringent medium,

$$n_{imp}^2(\theta) = n_o^2 n_{el}^2(\theta). \tag{33}$$

Also, we investigate the behavior of the birefringent medium normal mode, TE and TM polarization, in the impedance-matched medium.

Following the Eq. (10), for the TM mode, the electric and magnetic field are given by,

$$\mathbf{E} = \left(\varepsilon_2^2 \sin^2\theta + \varepsilon_3^2 \cos^2\theta\right)^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta \\ (\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta \end{pmatrix}, \tag{34}$$

$$\mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}(\varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta)^{\frac{1}{2}}}{(\varepsilon_2^2 \sin^2\theta + \varepsilon_3^2 \cos^2\theta)^{\frac{1}{2}}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{35}$$

While for the TE polarization the electric and magnetic field became,

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \tag{36}$$

$$\mathbf{H} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}}{(\varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta)^{\frac{1}{2}}} \frac{1}{\varepsilon_2\varepsilon_3} \times \begin{pmatrix} 0 \\ \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta \\ (\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta \end{pmatrix}. \tag{37}$$

Expressions (36) and (37) show that for the TE mode, the electric field **E** and displacement field **D**, are in the same direction but the magnetic field **H** is not along the induction **B**, whereas for the TM mode it is the opposite.

From relation (11), we obtain the Poynting vector for the TM and TE modes in impedance-matched medium

$$\mathbf{S}_{TM} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}(\varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta)^{\frac{1}{2}}}{\varepsilon_2^2 \sin^2\theta + \varepsilon_3^2 \cos^2\theta} \times \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta \end{pmatrix}, \tag{38}$$

$$\mathbf{S}_{TE} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{(\varepsilon_1\varepsilon_2\varepsilon_3)^{\frac{1}{2}}}{(\varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta)^{\frac{1}{2}}} \frac{1}{\varepsilon_2\varepsilon_3} \times \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta \end{pmatrix}. \tag{39}$$

It is clear that both TE and TM modes have the same Poynting vector. Consequently, light in impedance-matched media, is not divided into ordinary and extra-ordinary rays as it is expected. The angle between ray and z-axis can be written as

$$\tan\phi_{(TE)} = \tan\phi_{(TM)} = \frac{-(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta}{\varepsilon_2\sin^2\theta+\varepsilon_3\cos^2\theta}. \tag{40}$$

Also, we can write the ray direction in the impedance-matched slab as

$$\frac{\mathbf{dr}}{dl} = \frac{1}{\sqrt{\varepsilon_2^2\sin^2\theta+\varepsilon_3^2\cos^2\theta}}\begin{pmatrix}0\\ -(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta\\ \varepsilon_2\sin^2\theta+\varepsilon_3\cos^2\theta\end{pmatrix}. \tag{41}$$

Relation (41) shows important fact about the impedance=matched media; In these media although the birefringent effect do not occur, but the ray direction in this medium is the extraordinary ray. The ray direction in the impedance-matched medium do not necessarily a long the wave vector. Also comparison between (30) and (41) shows the direction of this extraordinary ray is equal to the extraordinary ray direction in the electric slab.

4. MEDIUM WITH VARIABLE OPTICAL AXES

In this section we investigate the effective space-time perceived by light ray in a medium that its optical axes orientation depends only on z: $\theta = \theta(z)$; We can simplify this inhomogeneity by considering this in homogeneous medium as a Fig. 1, the array of the anisotropic slabs in the z direction that the principle axes of each slab is different according the slab position in the $y-z$ plane.

According to the Fermat principle, light in a medium follows the geodesics. By studying the properties of the light geodesics in the medium, we might be able to associate a geometry to the medium. Collective properties of the light geodesics give us information about the curvature of the analogues space-time that the light perceive through the medium.

We apply the ray-tracing method to investigate the behavior of the light in two examples of planar media with variable optical axes. First, a medium that its optical axes orientation varies with z as $\theta = z$; And in second, $\theta = sqrtz$. We assume the principle values of permittivity are equal to $\varepsilon_2 = 2.75$ and $\varepsilon_3 = 2.21$, which is associated to the calcite crystal principal permittivities [?]. According to the relation 30 we can plot the trajectory of the extraordinary light in layered anisotropic media, which is equivalent with the light trajectory in impedance-matched media. In Fig. 2, the plotted trajectories of the family of extraordinary rays are shown. Left and right diagrams are corresponding to the $\theta = z$ and $\theta = sqrtz$ respectively, which indicates the normal incidence of light.

Fig. 2. Ray tracing in anisotropic media with variable optical axes, $\theta = z$ (left) and $\theta = sqrtz$ (right), the green arrows indicate orientations of the third principle axis and the red lines indicate ray trajectories in the media.

As we can see in the Fig. 2, the light geodesics through these media are not straight lines. Also, it seems that these surface are under tensions. Therefore, we expect non-zero Riemann tensor for them. we can construct the metric of this two dimensional distorted surface, Fig 2, by choosing two bases of this space as

$$e_1 = c_1\begin{pmatrix}1\\0\end{pmatrix} \qquad e_2 = c_2\begin{pmatrix}0\\1\end{pmatrix} \tag{42}$$

where coefficients c_1 and c_2 are scalar functions provides null geodesics condition, $ds^2 = 0$. Using these bases (42), the metric components [?] given by

$$g_{ij} = e_i \cdot e_j. \tag{43}$$

Therefore, we can write the space-time metric in matrix form as,

$$\mathbf{g} = \begin{pmatrix}-1 & 0 & 0\\ 0 & c_1 & 0\\ 0 & 0 & c_2\end{pmatrix} \tag{44}$$

or equivalently in the form of Riemann line element as

$$ds^2 = -c^2dt^2 + c_1^2dy^2 + c_2^2dz^2 \tag{45}$$

The null geodesics condition,$ds^2 = 0$, results,

$$c_1^2dy^2 + c_2^2dz^2 - c^2dt^2 = 0 \tag{46}$$

We can assume $c_1 = c_2$, so we achieve

$$c_1^2 = c_2^2 = \frac{c^2dt^2}{dl^2} \tag{47}$$

where $dl^2 = dy^2 + dz^2$.

On the other hand, The surface of equal phases for the propagating light fields define as the solutions of:

$$d\varphi(r,t) = 0. \tag{48}$$

This condition (48) results

$$\mathbf{k}\cdot\mathbf{dr} - \omega dt = 0. \tag{49}$$

our equivalently

$$cdt = n\hat{U}\cdot\mathbf{dr} \tag{50}$$

Now, using equation (41) and (30) we can write

$$cdt = \frac{\varepsilon_2\sin^2\theta+\varepsilon_3\cos^2\theta}{\sqrt{\varepsilon_2^2\sin^2\theta+\varepsilon_3^2\cos^2\theta}}dl \tag{51}$$

Substituting (51) in relation (47) the coefeicint c_1 and c_2 are easily determined,

$$c_1^2 = c_2^2 = n\frac{\left(\varepsilon_2\sin^2\theta+\varepsilon_3\cos^2\theta\right)^2}{\varepsilon_2^2\sin^2\theta+\varepsilon_3^2\cos^2\theta} \tag{52}$$

$$ds^2 = -c^2dt^2 + n\frac{\left(\varepsilon_2\sin^2\theta+\varepsilon_3\cos^2\theta\right)^2}{\varepsilon_2^2\sin^2\theta+\varepsilon_3^2\cos^2\theta}dl^2 \tag{53}$$

According to the previous section the refraction index of the extraordinary ray is differ from the refractive index of the light in the impedance-matched media. Therefore despite the same trajectory of the ray in these media the space time metrics of them is different. The electric media construct the following metric for the extraordinary ray

$$ds^2 = -c^2dt^2 + \frac{\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}(dy^2+dz^2). \quad \textbf{(54)}$$

wherease the impedance matched media appear as following metric for the light ray

$$ds^2 = -c^2dt^2 + \frac{\varepsilon_1\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}(dy^2+dz^2). \quad \textbf{(55)}$$

On the other hand, using transformation optics [?], we can find the space metric for impedance-matched media by

$$\mathbf{g} = (\det\varepsilon)\varepsilon^{-1}. \quad \textbf{(56)}$$

This metric, for our case, Fig. 1, can be written in the form

$$\mathbf{g}_{imp}(y-z) = \begin{pmatrix} \varepsilon_1(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta) & \varepsilon_1(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta \\ \varepsilon_1(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta & \varepsilon_1(\varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta) \end{pmatrix}. \quad \textbf{(57)}$$

At first glance, the metric (57) is different from the metric (55), but , by using following relation:

$$\begin{aligned} \varepsilon_2\varepsilon_3 = (\varepsilon_2\cos^2\theta + \varepsilon_3\sin^2\theta)(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\theta) \\ - \left[(\varepsilon_2-\varepsilon_3)\sin\theta\cos\theta\right]^2, \end{aligned} \quad \textbf{(58)}$$

we obtain that the metrics (55) and (57) are equivalent. this can be considered as a verification of our method.

Also, Using differential geometry relations, we can investigate the curvature of the above conformal flat spaces. Riemann curvature tensor given by [?]

$$R^i_{jkl} \equiv \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk} \quad \textbf{(59)}$$

where Γ^i_{jk} is Christoffel symbol and the comma notation "," refers to partial differentiation. The Christoffel symbol can be expressed in terms of the metric component as

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}) \quad \textbf{(60)}$$

For the metric (**??**), which is conformally two dimensional flat spaces, $g_{ij} = n^2\delta_{ij}$, we can obtain Christoffel symbol from relation

$$\Gamma^i_{ij} = \frac{1}{2n^2}n^2_{,j}. \quad \textbf{(61)}$$

Using this relation, (61), Riemann curvature tensor, (59) can be achieved as

$$\begin{aligned} &R_{11} = R_{22} = \\ &\frac{1}{2n^4}\{(\frac{\partial}{\partial z}n^2)^2 + (\frac{\partial}{\partial y}n^2)^2\} - \frac{1}{2n^2}\{(\frac{\partial}{\partial z}\frac{\partial}{\partial z}n^2) + (\frac{\partial}{\partial y}\frac{\partial}{\partial y}n^2)\} \end{aligned} \quad \textbf{(62)}$$

We calculated this tensor for the above example of the variable axes media, $\theta = z$ and find that

$$R_{ii} \neq 0. \quad \textbf{(63)}$$

Consequently, the medium with $\theta = z$ have a non-zero Riemann curvature tensor and appears as a curved space for the extraordinary light. Therefore, we can use variable optical axes media for realizing curved space time in the laboratory.

A. Transformation optics for the non-impedance matched media

In this part, we show how transformation optics would apply to the non-impedance-matched media. Remember from 3, the direction of the extraordinary ray (30) and the direction of the light in impedance-matched media (41) are the same. But, their refractive indexes only differ on one factor, (32). Therefore, we expect an analogous in their optical path length if $\varepsilon_1 = constant$ such as Fig. 2 Mathematically, we consider the metrics (55) and (**??**); Temporal parts of these metrics are equal but their spatial part is not the same. For better comparison between them, we can write following metric, (64) as a conformal form of the metric (**??**),

$$ds^2_{el} = -\varepsilon_1c^2dt^2 + \frac{\varepsilon_1\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}dl^2. \quad \textbf{(64)}$$

As we can see,in this form, the space part of this metric, (64), is the same with the metric (55),

$$ds^2_{imp} = -c^2dt^2 + \frac{\varepsilon_1\varepsilon_2\varepsilon_3\left(\varepsilon_2\sin^2\theta + \varepsilon_3\cos^2\right)}{\left(\varepsilon_2^2\sin^2\theta + \varepsilon_3^2\cos^2\right)}dl^2. \quad \textbf{(65)}$$

The comparison between two above metrics, (64 and (65), shows that, if $\varepsilon_1 = costant$, two metrics are equivalent. Therefore, an electric medium, which is a non-impedance-matched material, appears for the TM polarized light, like the impedance-matched medium for the unpolarized light. We can write metric for the extraordinary light in electric media in terms of the impedance -matched media metric,

$$\mathbf{g}_{el}(2+1) = \begin{pmatrix} -\varepsilon_1 & 0 \\ 0 & \mathbf{g}_{imp}(y-z) \end{pmatrix}. \quad \textbf{(66)}$$

As the result, for suitably polarized light, we can use electric medium instead of the impedance-match one to make the geometry exact.

5. CONCLUSION

In this paper, we have studied the the impedance-matched and non-impedance-matched anisotropic media for normal incident light. In the non-impedance-matched media, we have the ordinary and the extraordinary modes that for the extraordinary mode, we have shown the ray direction depends on the principle values of the permittivity tensor as well as the orientation of the optical axes. In the impedance-matched media, light rays do not split; there is a single direction of propagation for each angle of incidence. This direction coincides with the extraordinary ray's direction in the electric media. Consequently we introduce the family of media with variable optical axes and indicate the role of the directional variability of the optical axes in forming the effective metric. We show if the optical axes orientation varies, would results in the emergence of effective geometry in the medium. In our paper, the metric for the light in the impedance-matched media is obtained, which is the same as one might get from transformation optics method. A comparison between the two metrics, one from the impedance-matched medium and the other, from the extraordinary mode in electric medium, shows that they are equivalent. Our finding is a new method for manipulation of electromagnetic waves.

ACKNOWLEDGMENT

S.A.M, R.R wish to thank the office of graduate studies of the university of Isfahan for their support.