

# Constructing Pseudo-Triangle from Visibility Graph

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**Abstract.** The visibility graph of a simple polygon represents visibility relations between its vertices. Knowing the correct order of the vertices around the boundary of a polygon and its visibility graph, it is an open problem to locate the vertices in a plane in such a way that it will be consistent with this visibility graph. This problem has been solved for special cases when we know that the target polygon is a *tower* or a *spiral*. Knowing that a given visibility graph belongs to a simple polygon with at most three concave chains on its boundary, a *pseudo-triangle*, we propose a linear time algorithm for reconstructing one of its corresponding polygons. Moreover, we introduce a set of necessary and sufficient properties for characterizing visibility graphs of pseudo-triangles and propose polynomial algorithms for checking these properties.

**Keywords:** computational geometry, visibility graph, polygon reconstruction, pseudo-triangle.

## 1 Introduction

The *visibility graph* of a simple polygon  $\mathcal{P}$  is a graph  $\mathcal{G}(V, E)$  where  $V$  is the vertices of  $\mathcal{P}$  and an edge  $(u, v)$  exists in  $E$  if and only if the line segment  $uv$  lies completely inside  $\mathcal{P}$ . In the visibility graph reconstruction problem, the goal is to build a polygon whose visibility graph is isomorphic to a given graph. *Everett* showed that this problem is in PSPACE[1], and this is the only result known for general polygons. This problem has been solved only for special cases of *spiral* and *tower* polygons when other than the visibility graph, the correct order of the vertices on the boundary of the polygon is known as well. These results are obtained by *Everett* and *Corneil* [2] for spiral polygons and by *Colley et al.* [3] for tower polygons. In spiral polygons there is at most one concave chain (Fig. 1a) and the boundary of a tower polygon is composed of two concave chains and a single edge (Fig. 1b).

Although there is a bit progress on this type of reconstruction problem, there has been plenty of studies on characterizing visibility graphs [2–7]. In 1988, *Ghosh* introduced three necessary conditions for visibility graphs and conjectured their suffi-

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ciency[5]. In 1990, *Everett* proposed a counter-example graph rejecting *Ghosh's* conjecture[1]. She also refined the third *Ghosh's* necessary condition to a new stronger condition[8]. In 1992, *Abello et al.* built a graph satisfying *Ghosh's* conditions and the stronger version of the third condition which was not the visibility graph of any simple polygon rejecting the sufficiency of these conditions[9]. In 1997, *Ghosh* added his forth necessary condition and conjectured that this condition along with his first two conditions and the stronger version of the third condition are sufficient for a graph to be a visibility graph. Finally, in 2005 *Streinu* proposed a counter example for this conjecture[10].

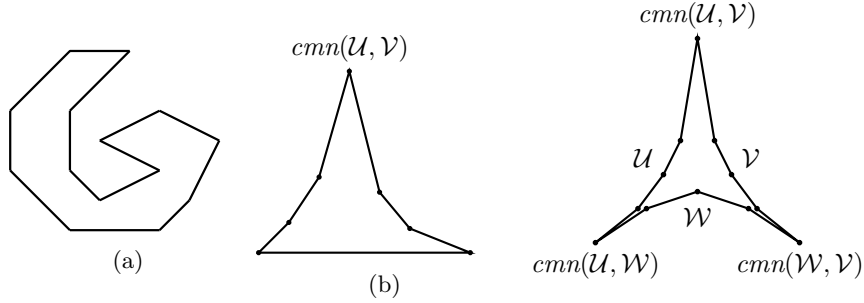


Fig. 1: (a) A Spiral polygon, (b) a tower polygon, and (c) a pseudo-triangle.

In this paper, we solve the reconstruction problem for pseudo-triangles. A *pseudo-triangle* is a simple polygon consisting of three concave side-chains each pair shares one convex vertex (called a corner). Let  $\mathcal{P}$  be a pseudo-triangle formed by the concave chains  $\mathcal{U} = [cmn(\mathcal{V}, \mathcal{U}), \dots, cmn(\mathcal{U}, \mathcal{W})]$ ,  $\mathcal{V} = [cmn(\mathcal{U}, \mathcal{V}), \dots, cmn(\mathcal{V}, \mathcal{W})]$ , and  $\mathcal{W} = [cmn(\mathcal{U}, \mathcal{W}), \dots, cmn(\mathcal{W}, \mathcal{V})]$  where  $cmn(\mathcal{V}, \mathcal{U}) = cmn(\mathcal{U}, \mathcal{V})$ ,  $cmn(\mathcal{U}, \mathcal{W}) = cmn(\mathcal{W}, \mathcal{U})$ , and  $cmn(\mathcal{W}, \mathcal{V}) = cmn(\mathcal{V}, \mathcal{W})$  are the corners (Fig. 1c). We use  $cmn(\cdot, \cdot)$  as an abbreviation for the common vertex of the two side-chains. Let  $\mathcal{H} = \langle cmn(\mathcal{V}, \mathcal{U}), \dots, cmn(\mathcal{U}, \mathcal{W}), \dots, cmn(\mathcal{W}, \mathcal{V}), \dots, cmn(\mathcal{V}, \mathcal{U}) \rangle$  be the Hamiltonian cycle of the visibility graph of  $\mathcal{P}$  which indicates the order of vertices on the boundary of  $\mathcal{P}$ . Having  $\mathcal{H}$  for a visibility graph  $\mathcal{G}(V, E)$  which is known to belong to a pseudo-triangle, we propose a linear-time algorithm for reconstructing a pseudo-triangle  $\mathcal{P} = \langle cmn(\mathcal{V}, \mathcal{U}), \dots, cmn(\mathcal{U}, \mathcal{W}), \dots, cmn(\mathcal{W}, \mathcal{V}), \dots, cmn(\mathcal{V}, \mathcal{U}) \rangle$  with  $\mathcal{G}(V, E)$  as its visibility graph.

In Section 2, we briefly describe the *Colley's* algorithm for reconstructing tower polygons which is used as a sub-routine in our algorithm. In Section 3, we define notation and initialization step of our algorithm and in Section 4, we propose the reconstruction algorithm. Our reconstruction algorithm uses a set of properties the visibility graph of a pseudo-triangle must support. To save space, we have moved all these properties to Appendix 1.

## 2 Reconstructing Tower Polygons

A *strong ordering* on a bipartite graph  $\mathcal{G}(V, E)$  with partitions  $U$  and  $W$  is a pair of  $<_U$  and  $<_W$  orderings on respectively (*resp.*)  $U$  and  $W$  such that if  $u <_U u'$ ,  $w <_W w'$ , and there are edges  $(u, w')$  and  $(u', w)$  in  $E$ , the edges  $(u, w)$  and  $(u', w')$  also exist in  $E$ .

The following theorem by *Colley et al.* [3] indicates the main property of the visibility graph of a tower polygon and guarantees the existence of a tower polygon consistent with such a visibility graph.

**Theorem 1.** [3] *Removing the edges of the reflex chains from the visibility graph of a tower gives an isolated vertex plus a connected bipartite graph for which the ordering of the vertices in the partitions provides a strong ordering. Conversely, any connected bipartite graph with strong ordering belongs to some tower. Furthermore, such a tower can be constructed in linear time in terms of the number of vertices.*

The outline of the *Colley's* algorithm is as follows. As input, it takes a connected bipartite graph  $\mathcal{G}(V, E)$  with vertices partitioned into two independent sets  $\mathcal{U} = \{u_0, \dots, u_m\}$  and  $\mathcal{V} = \{v_0, \dots, v_n\}$  having strong ordering. In the first step, the position of the corner  $u_0 = v_0$  and the vertices  $u_1$  and  $v_1$  are determined as in Fig. 2. In a middle step, suppose that the positions of the vertices  $cmn(\mathcal{V}, \mathcal{U}), \dots, u_{j-1}$  and  $cmn(\mathcal{U}, \mathcal{V}), \dots, v_{k-1}$  are known and it has been determined the half-lines from  $u_{j-1}$  and  $v_{k-1}$  which contain  $u_j$  and  $v_k$ , respectively, where  $(u_j, v_k) \in E$ . To complete such a middle step, the position of the vertex  $u_j$  and the half-line from  $u_j$  which contains  $u_{j+1}$  (where  $u_{j+1}$  is visible from  $v_k$ ) must be determined. For this purpose,  $u_j$  is located somewhere on its containing half-line horizontally below the vertex  $v_c$  which  $v_c$  has the minimum index among vertices of  $\mathcal{V}$  visible from  $u_{j+1}$ . Then, the containing half-line of  $u_{j+1}$  lies on the supporting line of  $u_j$  and  $s_{j+1}$  downward from  $u_j$ . Here,  $s_{j+1}$  is a point on  $v_{c-1}v_c$  with an  $\epsilon$  distance below  $s_j$ , when  $s_j$  lies on  $v_{c-1}v_c$ . If  $s_j$  does not lie on  $v_{c-1}v_c$ , then  $s_{j+1}$  is a point on  $v_{c-1}v_c$  with an  $\epsilon$  distance below  $v_{c-1}$ . According to this construction,  $s_j$  is the intersection of  $\mathcal{V}$  and the supporting line of  $u_j$  and  $u_{j-1}$ . Similarly,  $r_i$  is the intersection of  $\mathcal{U}$  and the supporting line of  $v_j$  and  $v_{j-1}$  (Fig. 2).

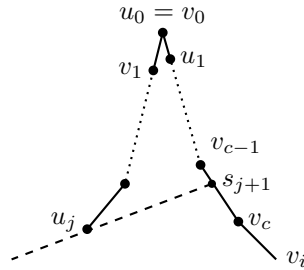


Fig. 2: Constructing a tower polygon.

### 3 Identifying Side-Chains

In this section, we describe a method for identifying the vertices of side-chains of a pseudo-triangle corresponding to a pair of  $\mathcal{H}$  and  $\mathcal{G}$  as its Hamiltonian cycle and visibility graph, respectively.

Any sequence  $\langle v_i, \dots, v_j \rangle$  on the Hamiltonian cycle is called a chain and denoted by  $[v_i, \dots, v_j]$ . A vertex  $v_a$  on a chain  $[v_i, \dots, v_j]$  is a *blocking vertex* for the invisible pair  $(v_i, v_j)$  if there is no visible pair of vertices  $v_l$  on  $[v_i, \dots, v_{a-1}]$

and  $v_k$  on  $[v_{a+1}, \dots, v_j]$ . *Ghosh* showed that for every invisible pair of vertices  $(u, v)$  in a visibility graph, there is at least one blocking vertex in  $[u, \dots, v]$  or  $[v, \dots, u]$ . Furthermore, every vertex on the shortest Euclidean path between  $u$  and  $v$  is a blocking vertex for this pair[5]. Note that in a pseudo-triangle the shortest Euclidean path between two invisible vertices turns in only one direction (e.i. clockwise or counterclockwise).

Let  $\mathcal{U}$ ,  $\mathcal{V}$ , and  $\mathcal{W}$  be the side-chains of a pseudo-triangle. The order of vertices in these chains is defined with respect to one of their common vertices. Precisely, for a vertex  $u \in \mathcal{U}$ ,  $u_{cmn(\mathcal{U}, \mathcal{V})}^{-k}$  is a vertex in the subchain  $[cmn(\mathcal{U}, \mathcal{V}), \dots, u]$  where the length of the subchain  $[u_{cmn(\mathcal{U}, \mathcal{V})}^{-k}, \dots, u]$  is  $k$ . Similarly,  $u_{cmn(\mathcal{U}, \mathcal{V})}^{+l}$  is a vertex in  $[u, \dots, cmn(\mathcal{U}, \mathcal{W})]$  where the length of  $[u, \dots, u_{cmn(\mathcal{U}, \mathcal{V})}^{+l}]$  is  $l$ . In addition, we use  $u_{cmn(\mathcal{U}, \mathcal{V})}(x)$  and  $U_{cmn(\mathcal{U}, \mathcal{V})}(x)$  to respectively denote the closest and the farthest vertices of  $\mathcal{U}$  to  $cmn(\mathcal{U}, \mathcal{V})$  that are visible to  $x$  where  $x$  lies on a side-chain other than  $\mathcal{U}$ . If  $u' = u_{cmn(\mathcal{U}, \mathcal{V})}(x)$ , then  $u_{cmn(\mathcal{U}, \mathcal{V})}^{+i}(x)$  and  $u_{cmn(\mathcal{U}, \mathcal{V})}^{-i}(x)$  notations are used for respectively  $u_{cmn(\mathcal{U}, \mathcal{V})}^{'+i}$  and  $u_{cmn(\mathcal{U}, \mathcal{V})}^{'-i}$ . Similarly, if  $u' = U_{cmn(\mathcal{U}, \mathcal{V})}(x)$ , then  $U_{cmn(\mathcal{U}, \mathcal{V})}^{+i}(x)$  and  $U_{cmn(\mathcal{U}, \mathcal{V})}^{-i}(x)$  notations are used for respectively  $U_{cmn(\mathcal{U}, \mathcal{V})}^{'+i}$  and  $U_{cmn(\mathcal{U}, \mathcal{V})}^{'-i}$ . Fig. 3a depicts these notations.

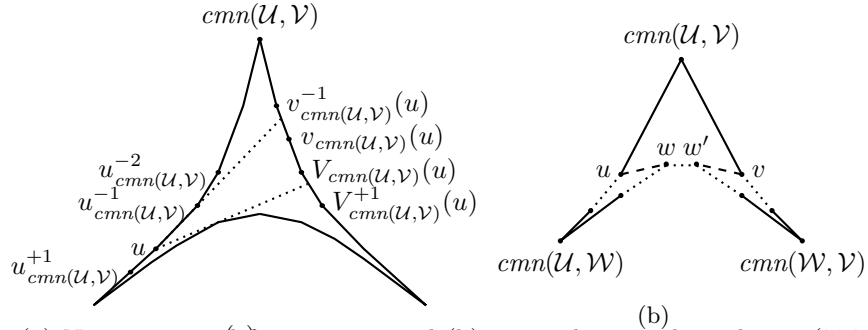


Fig. 3: (a) Notations used for vertices, and (b) a pseudo-triangle with  $cmn(\mathcal{U}, \mathcal{W})$  and  $cmn(\mathcal{W}, \mathcal{V})$  as its detectable corners.

**Lemma 1.** *It is always possible to identify at least two corners of a pseudo-triangle  $\mathcal{P}$  from its corresponding Hamiltonian cycle and visibility graph.*

*Proof.* Since a corner is a convex vertex, it cannot be a blocking vertex for its neighbors. Therefore, in the Hamiltonian cycle of a pseudo-triangle, there are at most three vertices whose adjacent vertices are visible pairs. By traversing the Hamiltonian cycle, these visible pairs and so the corresponding corners can be identified.

Suppose that this method does not identify all three corners. Without loss of generality (w.l.o.g.), assume that  $cmn(\mathcal{U}, \mathcal{V})$  is an unidentified corner. This means that  $u = cmn(\mathcal{V}, \mathcal{U})_{cmn(\mathcal{U}, \mathcal{V})}^{+1}$  and  $v = cmn(\mathcal{U}, \mathcal{V})_{cmn(\mathcal{U}, \mathcal{V})}^{+1}$  do not see each other and there must be a blocking vertex for this invisible pair. Due to their concavity, this blocking vertex cannot belong to the side-chains  $\mathcal{U}$  and  $\mathcal{V}$ . Consider the Shortest Euclidean path between  $u$  and  $v$  (Fig. 3b). It is clear that this path is a subchain of

$\mathcal{W}$ , saying  $[w, \dots, w']$  where  $w' = w_{cmn(\mathcal{U}, \mathcal{W})}^{+l}$  and the edges  $(u, w)$  and  $(w', v)$  belong to the visibility graph. The polygon formed by  $\langle u, \dots, cmn(\mathcal{U}, \mathcal{W}), \dots, w \rangle$  is a tower polygon with base  $(u, w)$  and corner  $cmn(\mathcal{U}, \mathcal{W})$ . The corner of this tower is the isolated vertex obtained by removing the edges of its Hamiltonian cycle from its visibility graph. Therefore, the corner  $cmn(\mathcal{U}, \mathcal{W})$  is detectable. The same argument holds for the tower polygon formed by  $\langle w', \dots, cmn(\mathcal{W}, \mathcal{V}), \dots, v \rangle$  from which the corner  $cmn(\mathcal{W}, \mathcal{V})$  can be identified. This means that if  $cmn(\mathcal{U}, \mathcal{V})$  cannot be identified from the visibility graph, the other two corners will be detectable.  $\square$

Consider a pseudo-triangle  $\mathcal{P}$  with side-chains  $\mathcal{U}$ ,  $\mathcal{V}$ , and  $\mathcal{W}$ , and  $\mathcal{G}$  and  $\mathcal{H}$  as its visibility graph and Hamiltonian cycle, respectively. Assume that the method described in Lemma 1, identifying only two corners of  $\mathcal{P}$ . *W.l.o.g.*, assume that  $cmn(\mathcal{U}, \mathcal{V})$  is the unidentified one. This means that there is a subchain on  $\mathcal{W}$  which blocks the visibility of  $u$  and  $v$ . Then, there is no visibility edge between a vertex from  $\mathcal{U}$  and a vertex on  $\mathcal{V}$ . By removing the edges of the Hamiltonian cycle from the visibility graph, two isolated vertices  $cmn(\mathcal{U}, \mathcal{W})$  and  $cmn(\mathcal{W}, \mathcal{V})$  and a connected bipartite graph is obtained. By adding one of the adjacent edges ( $e$ ) of these isolated vertices to this bipartite graph, we will have a single isolated vertex and a graph with strong ordering which according to Theorem 1 corresponds to a tower polygon with base  $e$  and  $\mathcal{G}$  and  $\mathcal{H}$  as its visibility graph and Hamiltonian cycle, respectively.

Therefore, we have the following property about the pair of  $\mathcal{H}$  and  $\mathcal{G}$  of a pseudo-triangle.

**Property 1.** If  $\mathcal{H}$  and  $\mathcal{G}$  are respectively the Hamiltonian cycle and visibility graph of a pseudo-triangle  $\mathcal{P}$ , at least two corners of  $\mathcal{P}$  can be identified. Furthermore, if only two corners are detectable, the given  $\mathcal{H}$  and  $\mathcal{G}$  belong to a pseudo-triangle if and only if there is a tower polygon with  $\mathcal{H}$  and  $\mathcal{G}$  as its Hamiltonian cycle and visibility graph, respectively.

So, in the remainder of this paper we assume that the method described in the proof of Lemma 1 identifies all three corners.

## 4 Pseudo-Triangle Reconstruction

In this section,  $\mathcal{G}(V, E)$  denotes the visibility graph of a pseudo-triangle  $\mathcal{P}$  with  $\mathcal{U}$ ,  $\mathcal{V}$ , and  $\mathcal{W}$  side-chains and  $\mathcal{H} = \langle cmn(\mathcal{V}, \mathcal{U}), \dots, cmn(\mathcal{U}, \mathcal{W}), \dots, cmn(\mathcal{W}, \mathcal{V}), \dots, cmn(\mathcal{V}, \mathcal{U}) \rangle$  is the Hamiltonian cycle in  $\mathcal{G}$  denoting the order of vertices in the boundary of  $\mathcal{P}$ . We propose an algorithm for reconstructing a pseudo-triangle corresponding to the given pair of  $\mathcal{G}$  and  $\mathcal{H}$ . This algorithm is based on a set of properties hold on the input  $\mathcal{G}$  and  $\mathcal{H}$ . Because of the space limitation, we have moved these properties to Appendix 1.

In order to reconstruct the pseudo-triangle  $\mathcal{P}$ , we divide  $\mathcal{P}$  into four sub-polygons  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{Z}$ , and  $\mathcal{Z}'$  as follows and reconstruct each one separately. For the sake of brevity,  $u_i = cmn(\mathcal{U}, \mathcal{V})_{cmn(\mathcal{U}, \mathcal{V})}^{+i}$ ,  $v_j = cmn(\mathcal{U}, \mathcal{V})_{cmn(\mathcal{U}, \mathcal{V})}^{+j}$ , and  $w_k = cmn(\mathcal{U}, \mathcal{W})_{cmn(\mathcal{U}, \mathcal{W})}^{+k}$  where  $i, j, k \geq 0$ . We assume that  $\mathcal{U}$  and  $\mathcal{V}$  has respectively  $\alpha + 1$  and  $\delta + 1$  vertices.

The subpolygon  $\mathcal{X}$  is formed by subchains  $[cmn(\mathcal{V}, \mathcal{U}), \dots, u_\nu]$  and  $[cmn(\mathcal{U}, \mathcal{V}), \dots, v_\mu]$  and edge  $(u_\nu, v_\mu)$  where  $V_{cmn(\mathcal{U}, \mathcal{V})}(u_\nu) = v_\mu$  and  $U_{cmn(\mathcal{U}, \mathcal{V})}(v_\mu) = u_\nu$ . The

subpolygon  $\mathcal{X}$  is a tower polygon with strong ordering in its visibility graph. Note that  $u_{\nu+1}$  or  $v_{\mu+1}$  exists only when the side-chain  $\mathcal{W}$  has more than one edge, otherwise, two identified adjacent corners  $u_\nu$  and  $v_\mu$  compose the base of a tower polygon which can be constructed by the *Colley's* algorithm. So, we assume that  $\mathcal{W}$  has more than one edge.

The subpolygon  $\mathcal{Y}$  is identified as follows: Let  $[w_i, \dots, w_j]$  be the maximum subchain of  $\mathcal{W}$  visible from both  $u_\nu$  and  $v_\mu$ . According to Property 6 (see Appendix 1), this chain is nonempty and continuous. Let  $W_{cmn(\mathcal{U}, \mathcal{W})}(u_{\nu+1}) = w_k$  and  $w_{cmn(\mathcal{U}, \mathcal{W})}(v_{\mu+1}) = w_l$ . From Property 8,  $k \leq j$  and  $l \geq i$  and from Property 9,  $k \leq l$ . We define  $M$  and  $N$  as  $\max(k, i)$  and  $\min(l, j)$ , respectively. It is clear that chain  $[w_M, \dots, w_N]$  contains at least one vertex. Then,  $\mathcal{Y}$  is defined to be the polygon with  $\langle u_\nu, w_M, \dots, w_N, v_\mu \rangle$  as its boundary.

The subpolygon  $\mathcal{Z}$  is formed by subchains  $[u_\nu, \dots, cmn(\mathcal{U}, \mathcal{W})]$  and  $[cmn(\mathcal{U}, \mathcal{W}), \dots, w_M]$  and edge  $(u_\nu, w_M)$ . Similarly, subchains  $[v_\mu, \dots, cmn(\mathcal{V}, \mathcal{W})]$  and  $[cmn(\mathcal{V}, \mathcal{W}), \dots, w_N]$  and edge  $(v_\mu, w_N)$  specify the subpolygon  $\mathcal{Z}'$ . It is clear that  $\mathcal{P}$  is the union of  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{Z}$ , and  $\mathcal{Z}'$ .

Our reconstruction algorithm first builds  $\mathcal{X}$  using the *Colley's* algorithm in such a way that vertices of  $\mathcal{U}$  lie to the left of vertices of  $\mathcal{V}$ . According to this algorithm, positions of vertices  $cmn(\mathcal{V}, \mathcal{U}), \dots, u_{\nu-1}$  and  $cmn(\mathcal{U}, \mathcal{V}), \dots, v_{\mu-1}$  are obtained and directions of the edges  $(u_{\nu-1}, u_\nu)$  and  $(v_{\mu-1}, v_\mu)$  are determined. In order to specify the actual position of  $u_\nu$ , choose a point on its half-line that is below the horizontal line passing through  $v_{cmn(\mathcal{U}, \mathcal{V})}(u_\nu)$ . The position of  $v_\mu$  is determined similarly. Then, we extend this polygon to build  $\mathcal{Y}$  (Section 4.1) and build and attach  $\mathcal{Z}$  and  $\mathcal{Z}'$  parts to this polygon (Section 4.2) to complete the construction procedure.

#### 4.1 Reconstructing $\mathcal{Y}$

In this step, we build the subpolygon  $\mathcal{Y} = \langle u_\nu, w_M, \dots, w_N, v_\mu \rangle$ . We know the position of vertices  $u_\nu$  and  $v_\mu$  from the previous step, which are also on the boundary of  $\mathcal{Y}$ . To locate positions of other vertices, we show that there are nonempty regions in which these vertices can be placed.

For an arbitrary vertex  $w_j$  from  $\mathcal{Y}$  which  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_j) = u_i$  and  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_j) = v_l$ , we define a region  $\mathcal{W}_{i,l}^j$  from which each point sees all vertices in the subchains  $[u_i, \dots, u_\nu]$  and  $[v_l, \dots, v_\mu]$ . Therefore,  $w_j$  can be placed in  $\mathcal{W}_{i,l}^j$  supporting the visibility constraints between  $w_i$  and vertices of  $\mathcal{X}$ . We use  $\mathcal{W}^j$  instead of  $\mathcal{W}_{i,l}^j$  whenever  $i$  and  $l$  indices are not important. The region  $\mathcal{W}^j$  is determined as follows: While  $w_j$  sees  $u_\nu$  and  $v_\mu$ , the vertices  $u_i$  and  $v_l$  always exist and are well-defined. If  $u_i$  and  $v_l$  are identical, then  $i = l = 0$  and the region  $\mathcal{W}^j = \mathcal{W}_{0,0}^j$  is defined to be the part of the cone formed by the lines through  $(cmn(\mathcal{V}, \mathcal{U}), u_1)$  and  $(cmn(\mathcal{V}, \mathcal{U}), v_1)$  restricted to the underneath of the line through  $u_\nu v_\mu$ . Trivially, each point of  $\mathcal{W}^j$  sees all vertices  $u_\nu, \dots, cmn(\mathcal{V}, \mathcal{U}) = cmn(\mathcal{U}, \mathcal{V}), \dots, v_\mu$ .

Let  $\mathcal{F}_z(x, y)$  be the ' $z$ ' half-plane defined by the line through  $x$  and  $y$  where ' $z$ ' is 'b' (bottom), 'r' (right), or 'l' (left). If  $u_i$  and  $v_l$  are distinct vertices, according to Property 11, at least one of the pairs  $(u_{i+1}, v_{l-1})$  and  $(u_{i-1}, v_{l+1})$  does not see each other. The invisible pair is determined by applying Corollary 5 and Property 11.

W.l.o.g., assume that  $(u_{i+1}, v_{l-1})$  is the invisible pair. Then,  $\mathcal{W}_{i,l}^j$  is defined to be  $\mathcal{F}_r(s_{i+1}, u_i) \cap \mathcal{F}_r(v_l, u_i) \cap \mathcal{F}_l(u_{i-1}, u_i) \cap \mathcal{F}_l(v_{l-1}, u_i) \cap \mathcal{F}_b(v_\mu, u_\nu)$  (Fig. 4). It is simple to see that all points in this region satisfy the visibility constraints of  $w_j$  with respect to the vertices  $u_\nu, \dots, \text{cmn}(\mathcal{U}, \mathcal{V}), \dots, v_\mu$ .

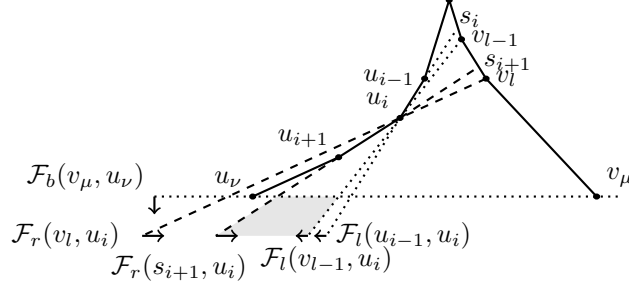


Fig. 4:  $\mathcal{W}_{i,l}^j$  is the shaded region.

From concavity of  $\mathcal{U}$  and  $\mathcal{V}$ ,  $\mathcal{F}_r(s_{i+1}, u_i) \cap \mathcal{F}_l(u_{i-1}, u_i)$  and  $\mathcal{F}_r(v_l, u_i) \cap \mathcal{F}_l(v_{l-1}, u_i)$  are not empty. Therefore,  $\mathcal{W}_{i,l}^j$  will be empty only when  $\mathcal{F}_r(s_{i+1}, u_i) \cap \mathcal{F}_l(v_{l-1}, u_i)$  is empty or  $\mathcal{F}_r(v_l, u_i) \cap \mathcal{F}_l(u_{i-1}, u_i)$  is empty. The first case is impossible, because otherwise,  $u_{i+1}$  must be visible from  $v_{l-1}$  which is in contradiction with invisibility assumption of  $(u_{i+1}, v_{l-1})$ . The second case is also impossible, because then, the pair  $u_i$  and  $v_l$  must be invisible. But, according to Property 7,  $u_i$  and  $v_l$  are visible from each other. Therefore, the region  $\mathcal{F}_r(s_{i+1}, u_i) \cap \mathcal{F}_r(v_l, u_i) \cap \mathcal{F}_l(u_{i-1}, u_i) \cap \mathcal{F}_l(v_{l-1}, u_i)$  is nonempty and some part of this intersection lies in half-plane  $\mathcal{F}_b(v_\mu, u_\nu)$ .

According to the above discussion,  $\mathcal{W}^j$  is defined by  $\mathcal{F}_b(v_\mu, u_\nu)$  and two half-planes of  $\{\mathcal{F}_r(s_{i+1}, u_i), \mathcal{F}_r(v_l, u_i), \mathcal{F}_l(u_{i-1}, u_i), \mathcal{F}_l(v_{l-1}, u_i)\}$ . The apex of  $\mathcal{W}^j$  is defined to be the intersection of the corresponding lines of these two half-planes. Note that if the apex of  $\mathcal{W}^j$  lies on  $\mathcal{U}$ , the apex of  $\mathcal{W}^{j-1}$  will lie on  $\mathcal{U}$ , as well. Furthermore,  $\mathcal{W}^{j-1}$  is either completely coinciding  $\mathcal{W}^j$  or is completely on its left. Similarly, if the apex of  $\mathcal{W}^j$  lies on  $\mathcal{V}$ , then the apex of  $\mathcal{W}^{j+1}$  lies on  $\mathcal{V}$  as well, and  $\mathcal{W}^{j+1}$  is either coinciding  $\mathcal{W}^j$  or is completely on its right.

Then, we can place the vertices  $w_M, \dots, w_N$  of  $\mathcal{V}$  on an arbitrary concave chain inside  $\mathcal{F}_b(v_\mu, u_\nu)$  in such a way that  $w_j \in \mathcal{W}^j$ . This placement satisfies the visibility constraints for  $\mathcal{X}$  and  $\mathcal{Y}$ . However, to guarantee the reconstruction of  $\mathcal{Z}$  and  $\mathcal{Z}'$ , we define some constraints on this concave chain which is described in the rest of this section.

Let  $s'_i$  ( $i > \nu$ ) be the intersection of  $\mathcal{V}$  and the line through  $u_i$  and  $W_{\text{cmn}(\mathcal{U}, \mathcal{V})}(u_i)$ ,  $r'_k$  ( $k > \mu$ ) be the intersection of  $\mathcal{U}$  and the line through  $v_k$  and  $w_{\text{cmn}(\mathcal{U}, \mathcal{V})}(v_k)$ ,  $t'_j$  ( $j < M$ ) be the intersection of  $\mathcal{V}$  and the line through  $w_j$  and  $w_{j+1}$ , and  $t'_j$  ( $j > N$ ) be the intersection of  $\mathcal{U}$  and the line through  $w_j$  and  $w_{j-1}$  (see Figure. 5).

Note that although we have not yet determined positions of vertices defining  $s'_i$ ,  $r'_k$ , and  $t'_j$ , we can determine their containing edges from the visibility information as follows: For  $i > \nu$ , if  $u_i$  sees at least one vertex from  $\mathcal{V}$ ,  $s_i$  lies on  $(v_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1}(u_i), v_{\text{cmn}(\mathcal{U}, \mathcal{V})}(u_i))$  and  $s'_i$  lies on  $(V_{\text{cmn}(\mathcal{U}, \mathcal{V})}(u_i), V_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{+1}(u_i))$ . On the other hand, if  $u_i$  sees no vertex from  $\mathcal{V}$ , then for  $k \geq i$ , both  $s_k$  and  $s'_k$  lie on  $(V_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1}(u_j), V_{\text{cmn}(\mathcal{U}, \mathcal{V})}(u_j))$  where  $u_j$  has the highest index among the vertices

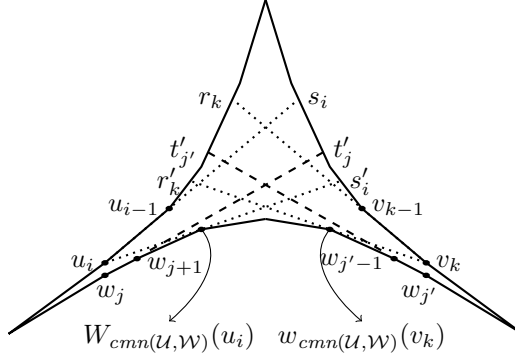


Fig. 5: Points  $s(\cdot)$ ,  $s'(\cdot)$ ,  $r(\cdot)$ ,  $r'(\cdot)$ , and  $t'(\cdot)$ .

of  $\mathcal{U}$  that see at least one vertex from  $\mathcal{V}$ . However, this has an exception when  $i = \nu + 1$  and  $w_{M-1}$  is visible to both  $u_\nu$  and  $v_\mu$ , for which both  $s_k$  and  $s'_k$  for  $k \geq i$  lie on  $(v_\mu, v_{\mu+1})$ . The same situation happens for  $r_l$  and  $r'_l$  when  $l > \mu$ .

The containing edge of  $t'_j$  for  $j < M$  is determined as follows: If  $w_j$  sees at least one vertex from  $\mathcal{V}$ , then  $t'_j$  lies on  $(V_{cmn(\mathcal{U}, \mathcal{V})}(w_j), V_{cmn(\mathcal{U}, \mathcal{V})}^{+1}(w_j))$ , otherwise, it lies on the containing edge of  $s'_\alpha$  (Note that according to our assumption at the beginning of Section 4,  $\alpha$  and  $\delta$  are respectively the greatest indices of vertices  $u_i$  and  $v_j$  on  $\mathcal{U}$  and  $\mathcal{V}$  side-chains.). Similarly, for  $j > N$ , if  $w_j$  sees at least one vertex from  $\mathcal{U}$ , then  $t'_j$  lies on  $(U_{cmn(\mathcal{U}, \mathcal{V})}(w_j), U_{cmn(\mathcal{U}, \mathcal{V})}^{+1}(w_j))$ , and otherwise, it lies on the containing edge of  $r'_\delta$ .

The containing edges of  $s'_\alpha$  and  $r'_\delta$  are respectively called “the floating edge in  $\mathcal{V}$ ” and “the floating edge in  $\mathcal{U}$ ”. We call these edges floating because we increase their lengths, and reposition their underneath vertices to enforce the concavity in building  $\mathcal{Z}$  and  $\mathcal{Z}'$ .

We define the vertices  $w_{M^*}$  and  $w_{N^*}$  as follows: If  $\mathcal{W}$  has two edges, then  $w_M$  and  $w_N$  are both equal to  $w_1$  (the middle vertex of  $\mathcal{W}$ ), and  $w_{M^*}$  and  $w_{N^*}$  are also defined to be  $w_1$ . When  $\mathcal{W}$  has more than two edges,  $M^*$  is defined to be  $M$  when the apex of  $\mathcal{W}^M$  does not lie on a vertex of  $\mathcal{V}$  below its floating edge. Otherwise,  $M^*$  is defined to be  $j$  where  $j$  is the maximum index for which the apex of  $\mathcal{W}^j$  lies above the floating edge of  $\mathcal{V}$  (this edge may lie on  $\mathcal{U}$ ). If the index of  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_{M^*})$  is greater than  $\nu$ , the apex of  $\mathcal{W}^{M^*}$  is temporarily assumed to be  $u_\nu$  and  $\mathcal{W}^{M^*}$  is defined to lie between  $\mathcal{F}_r(s_{i+1}, u_\nu)$  and  $\mathcal{F}_l(u_{\nu-1}, u_\nu)$ . The index  $N^*$  is defined similarly. It is clear that at least  $w_{M^*} = w_M$  or  $w_{N^*} = w_N$ .

We use  $\mathcal{R}(x, y)$  to denote the ray from  $x$  towards  $y$ . In addition,  $\mathcal{R}_a(x, y)$  denotes the ray from  $a$  and parallel to  $\mathcal{R}(x, y)$  (Fig. 6).

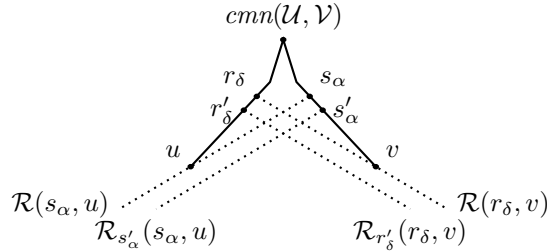


Fig. 6: The rays  $\mathcal{R}(s_\alpha, u)$ ,  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ ,  $\mathcal{R}(r_\delta, v)$ , and  $\mathcal{R}_{r'_\delta}(r_\delta, v)$ .



Despite our definition of the regions  $\mathcal{W}^i$  for all vertices  $w_i \in \mathcal{W}$ , we refine this definition for  $\mathcal{W}^{N^*}$  (resp.  $\mathcal{W}^{M^*}$ ) when  $N^* \neq N$  (resp.  $M^* \neq M$ ) or the floating edge of  $\mathcal{V}$  (resp.  $\mathcal{U}$ ) lies under the line through  $u_\nu$  and  $d_\mu$ . It is simple to see that at most one of the floating edges lies under  $\mathcal{R}(u_\nu, d_\mu)$ . Let  $v$  be a point on  $\mathcal{R}_{v_\mu}(r_{\mu+1}, v_\mu)$  when the floating edge of  $\mathcal{V}$  lies under  $\mathcal{R}(u_\nu, v_\mu)$ , or be  $v_\mu$  otherwise. Similarly,  $u$  is defined to be either  $u_\nu$  or a point on  $\mathcal{R}_{u_\nu}(s_{\nu+1}, u_\nu)$ . The regions  $\mathcal{W}^{N^*}$  and  $\mathcal{W}^{M^*}$  are restricted to lie under the line through  $u$  and  $v$ . Moreover, we know that at most one of the indices  $M^*$  and  $N^*$  is not equal to its corresponding index  $M$  or  $N$ . *W.l.o.g.*, assume that  $N^* \neq N$ . Then, we additionally restrict the region  $\mathcal{W}^{N^*}$  as follows (this restriction is not applied when we reconstruct  $\mathcal{Z}$  or  $\mathcal{Z}'$ ). Let  $p$  be a point inside the intersection of  $\mathcal{W}^N$  and  $\mathcal{F}_b(u, v)$  and with an arbitrary positive distance from  $\mathcal{R}(u, v)$ . We determine  $t'_{N^*}$  on its edge and with  $\epsilon l$  distance above the lower endpoint of this edge where  $\epsilon > 0$  and  $l$  is the number of vertices in  $\mathcal{V}$  and  $\mathcal{W}$  whose  $r'_{(\cdot)}$ 's and  $t'_{(\cdot)}$ 's lie on this edge. The region  $\mathcal{W}^{N^*}$  is restricted to lie under the line through  $t'_{N^*}$  and  $p$  (see Figure. 7).

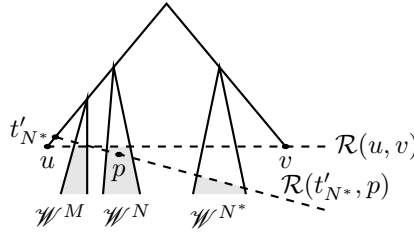


Fig. 7: Restricting  $\mathcal{W}^{N^*}$ .

Let  $s_\alpha$  be a point on its edge and with  $\epsilon k$  distance below the upper endpoint of this edge where  $\epsilon > 0$  and  $k$  is the number of vertices in  $\mathcal{U}$  whose  $s_{(\cdot)}$ 's lie on this edge. Similarly, let  $s'_\alpha$  be a point on its edge and with  $\epsilon m$  distance above the lower endpoint of this edge where  $\epsilon > 0$  and  $m$  is the number of vertices in  $\mathcal{U}$  and  $\mathcal{W}$  whose  $s'_{(\cdot)}$ 's and  $t'_{(\cdot)}$ 's lie on this edge. The value of  $\epsilon$  is small enough such that  $s_\alpha$  lies above  $s'_\alpha$ . The points  $r_\delta$  and  $r'_\delta$  are defined similarly.

Let  $\mathcal{S}$  (resp.  $\mathcal{T}$ ) be the strip defined by the supporting lines of  $\mathcal{R}(s_\alpha, u)$  and  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$  (resp.  $\mathcal{R}(r_\delta, v)$  and  $\mathcal{R}_{r'_\delta}(r_\delta, v)$ ).

**Lemma 2.** *It is always possible to enlarge the floating edges of  $\mathcal{V}$  (resp.  $\mathcal{U}$ ) such that the intersection of  $\mathcal{W}^{M^*}$  (resp.  $\mathcal{W}^{N^*}$ ) and  $\mathcal{S}$  (resp.  $\mathcal{T}$ ) is not empty.*

*Proof.* Assume that the intersection of  $\mathcal{W}^{M^*}$  and  $\mathcal{S}$  is empty. According to the definition of  $M^*$ , the apex of  $\mathcal{W}^{M^*}$  either lies above the floating edge of  $\mathcal{V}$  or lies on  $\mathcal{U}$ . This implies that enlarging the floating edge of  $\mathcal{V}$  only affects the upper half-plane that defines  $\mathcal{W}^{M^*}$ . Then, we can enlarge the floating edge of  $\mathcal{V}$  in such a way that the lower defining ray of  $\mathcal{S}$  and the upper defining half-plane of  $\mathcal{W}^{M^*}$  intersect inside  $\mathcal{W}^{M^*}$  which means that the intersection of  $\mathcal{W}^{M^*}$  and  $\mathcal{S}$  is not empty. Moreover, when this intersection is not empty, this extension will just increase the intersection. To complete the proof, it is simple to see that extending the floating edge of  $\mathcal{U}$  will again increase the intersection of  $\mathcal{W}^{M^*}$  and  $\mathcal{S}$ .

The proof for  $w_{N^*}$  is analogously the same.  $\square$

After, locating the position of vertices in  $\mathcal{X}$  (by possibly extending the floating edges), we place the vertices of  $\mathcal{Y}$  as follows: If  $N^* \neq N$ , then we set  $p$  as  $w_N$  and place  $w_{M^*} = w_M$  inside the intersection of  $\mathcal{W}^M$  and  $\mathcal{S}$  in such a way that both  $w_M$  and  $w_N$  be visible to  $u$  and  $v$ ; neither  $w_M$  blocks the visibility of  $w_N$ , nor  $w_N$  blocks the visibility of  $w_M$ . When  $M^* \neq M$ ,  $w_M$  and  $w_N$  are positioned analogously. Finally, if  $M^* = M$  and  $N^* = N$ , we select a point from  $\mathcal{S} \cap \mathcal{W}^M$  as  $w_M$  and a point from  $\mathcal{T} \cap \mathcal{W}^N$  as  $w_N$ , again, in such a way that both see  $u$  and  $v$ . Then, we put the vertices  $w_{M+1}, \dots, w_{N-1}$  on a slightly concave chain from  $w_M$  to  $w_N$  in such a way that each  $w_j$  ( $M \leq j \leq N$ ) lies inside  $\mathcal{W}^j$  and sees  $u$  and  $v$ . It is simple to check that this setting is compatible with the visibility graph restricted to the vertices of  $\mathcal{X}$  and  $\mathcal{Y}$ .

## 4.2 Reconstructing $\mathcal{Z}$ and $\mathcal{Z}'$

In this step, we place the vertices of  $\mathcal{Z}$  and  $\mathcal{Z}'$  to complete the reconstruction procedure. As said before,  $\mathcal{Z}$  and  $\mathcal{Z}'$  are parts of the target pseudo-triangle with respectively  $\langle u_\nu, u_{\nu+1}, \dots, cmn(\mathcal{U}, \mathcal{W}), w_1, \dots, w_M \rangle$  and  $\langle v_\mu, v_{\mu+1}, \dots, cmn(\mathcal{V}, \mathcal{W}), w_1, \dots, w_N \rangle$  boundaries. Here, we only describe how to build  $\mathcal{Z}$ . The construction of  $\mathcal{Z}'$  is the same.

Location of a vertex  $u_i \in \mathcal{Z}$  is determined by the intersection point of the rays  $\mathcal{R}(s_i, u_{i-1})$  and  $\mathcal{R}(s'_i, W_{cmn(\mathcal{U}, \mathcal{W})}(u_i))$  and location of a vertex  $w_h \in \mathcal{Z}$  is an arbitrary point on  $\mathcal{R}(t'_h, w_{h+1})$  inside the region  $\mathcal{W}^h$ . Therefore, to construct  $\mathcal{Z}$  we start from  $u_{\nu+1}$  and  $w_{M-1}$ , and in each step, we determine the position of one of the vertices and go forward to the next vertex. This is done by incrementally determining direction of the rays  $\mathcal{R}(s_i, u_{i-1})$ ,  $\mathcal{R}(s'_i, W_{cmn(\mathcal{U}, \mathcal{W})}(u_i))$ , and  $\mathcal{R}(t'_h, w_{h+1})$  as well as  $\mathcal{W}^h$  regions.

Consider the edges of the pseudo-triangle on which the points  $s_i, s'_i, r_j, r'_j$ , and  $t'_l$  for  $i > \nu, j > \mu$ , and  $l < M$  and  $l > N$  lie. Keep an upper point and a lower point for each edge. Initialize the upper point with the upper endpoint of that edge or the latest located  $s_{(\cdot)}$  or  $r_{(\cdot)}$  on this edge. Initialize the lower point with the lower endpoint of the edge. Position of each  $s_{(\cdot)}, r_{(\cdot)}, s'_{(\cdot)}, r'_{(\cdot)}$ , and  $t'_{(\cdot)}$  is determined whenever we need the rays passing through them. Place the points  $s'_{(\cdot)}, r'_{(\cdot)}$ , and  $t'_{(\cdot)}$ , with  $\epsilon > 0$  distance above the current lower point of their edges and place the points  $s_{(\cdot)}$  and  $r_{(\cdot)}$ , with  $\epsilon > 0$  distance below the upper point of their edges. Whenever a new  $s_{(\cdot)}, r_{(\cdot)}, s'_{(\cdot)}, r'_{(\cdot)}$ , or  $t'_{(\cdot)}$  point is located on an edge, the upper or lower point of that edge is updated properly.

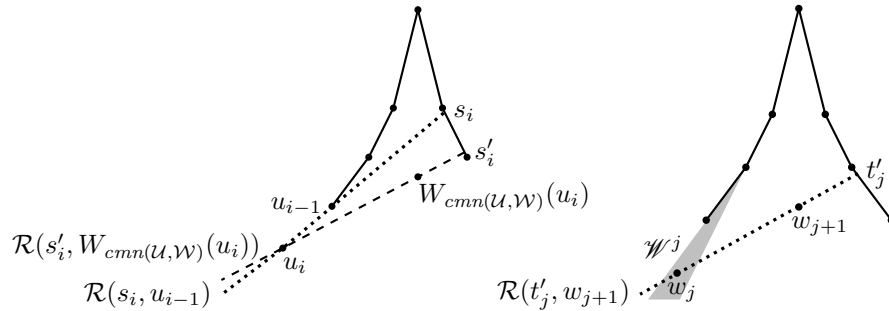


Fig. 8: (a) Determination of  $u_i$ , and (b)  $w_j$

More precisely, assume that we have already determined positions of  $u_\nu, u_{\nu+1}, \dots, u_{i-1}$  ( $i > \nu$ ) as well as the vertices  $w_M, w_{M-1}, \dots, w_{j+1}$  ( $j < M$ ). To determine position of one of the vertices  $u_i$  and  $w_j$  we do as follows: Let  $w_k$  be  $W_{cmn(\mathcal{U}, \mathcal{W})}(u_i)$ . If  $k < j$ , then we have already located the position of  $w_k$ , and directions of the rays  $\mathcal{R}(s_i, u_{i-1})$  and  $\mathcal{R}(s'_i, W_{cmn(\mathcal{U}, \mathcal{W})}(u_i))$  are known. Lemma 3 shows that these rays intersect. So,  $u_i$  is located on the intersection point of these rays (Fig. 8a). Otherwise, we must first determine position of  $w_j$  which lies on  $\mathcal{R}(t'_j, w_{j+1})$  and inside  $\mathcal{W}^j$  (Fig. 8b). The position of  $w_{j+1}$  is already known and  $t'_{j+1}$  is determined according to the above paragraph. From these two points the direction of  $\mathcal{R}(t'_j, w_{j+1})$  is obtained. The region  $\mathcal{W}^j$  is determined as follows: Suppose that  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_j) = u_k$  and  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_j) = v_l$ . We define  $\mathcal{W}^j$  as in the previous section with the exception that it may be possible that only one of the vertices  $u_k$  and  $v_l$  exists. By Corollary 4, for  $j < M$ ,  $u_j$  always exists. If  $w_j$  sees no vertex from  $\mathcal{V}$ , then it would see a part of the floating edge of  $\mathcal{V}$ . Thus, we consider the upper endpoint of this edge as  $v_{l-1}$ . From properties 7 and 11 we know that  $\mathcal{W}^j$  is not empty and lies to the left of  $\mathcal{W}^{j+1}$ . Moreover, it is shown in Lemma 3 that  $\mathcal{R}(t'_j, w_{j+1})$  intersects  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ . Since  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$  passes through all  $\mathcal{W}^{(\cdot)}$ ,  $\mathcal{R}(t'_j, w_{j+1})$  passes through  $\mathcal{W}^j$ . Therefore, we can determine the position of  $w_j$ . According to the definition of  $\mathcal{R}(s_i, u_{i-1})$  and  $\mathcal{R}(s'_i, W_{cmn(\mathcal{U}, \mathcal{W})}(u_i))$  for  $u_i$  and  $\mathcal{R}(t'_j, w_{j+1})$  and  $\mathcal{W}^j$  for  $w_j$ , in both cases (locating  $u_i$  or  $w_j$ ), visibility of the newly located vertex is exactly the same as its visibility in the visibility graph (restricted to the vertices of  $\mathcal{X}$ ,  $\mathcal{V}$ , and the constructed part of  $\mathcal{Z}$ ).

**Lemma 3.** *The rays  $\mathcal{R}(s_i, u_{i-1})$  and  $\mathcal{R}(s'_i, W_{cmn(\mathcal{U}, \mathcal{W})}(u_i))$  for  $i > \nu$  are convergent inside  $\mathcal{S}$ .*

*Proof.* Remember that  $\mathcal{S}$  is the strip defined by the supporting lines of  $\mathcal{R}(s_\alpha, u)$  and  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ . By Corollary 1 and Corollary 3, we know that  $s_i$  lies above the strip  $\mathcal{S}$  and  $s'_i$  lies below this strip. Then, it is enough to show that for  $i > \nu$ ,  $\mathcal{R}(s'_i, W_{cmn(\mathcal{U}, \mathcal{W})}(u_i))$  crosses  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$  and  $\mathcal{R}(s_i, u_{i-1})$  crosses  $\mathcal{R}(s_\alpha, u)$ . We first prove that  $\mathcal{R}(s'_i, W_{cmn(\mathcal{U}, \mathcal{W})}(u_i))$  intersects  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ . Let  $W_{cmn(\mathcal{U}, \mathcal{W})}(u_i) = w_h$ . For  $M^* \leq h \leq M$ , it can be easily shown by induction that  $w_h$  is located above  $\mathcal{R}(t'_{M^*}, w_M)$ . Moreover, it is simple to see that  $s'_i$  must lie below  $t'_{M^*}$ . Then, knowing that  $\mathcal{R}(t'_{M^*}, w_M)$  crosses  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$  implies that  $\mathcal{R}(s'_i, w_h)$  intersects  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$  as well. From the fact that  $w_{M^*}$  lies inside  $\mathcal{S}$ , it can also be shown by induction that  $w_h$  for  $h < M^*$  lies inside  $\mathcal{S}$  which means that  $\mathcal{R}(s'_i, w_h)$  crosses  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ .

To complete the proof, we prove by induction on  $i$  that  $\mathcal{R}(s_i, u_{i-1})$  crosses  $\mathcal{R}(s_\alpha, u)$ . It is clear that  $s_{\nu+1}$  is located above  $s_\alpha$  which means that  $\mathcal{R}(s_{\nu+1}, u_\nu)$  intersects  $\mathcal{R}(s_\alpha, u)$ . From the previous paragraph we know that  $\mathcal{R}(s'_{\nu+1}, W_{cmn(\mathcal{U}, \mathcal{W})}(u_{\nu+1}))$  intersects  $\mathcal{R}_{s'_\alpha}(s_\alpha, u)$ . Therefore,  $\mathcal{R}(s_{\nu+1}, u_\nu)$  and  $\mathcal{R}(s'_{\nu+1}, W_{cmn(\mathcal{U}, \mathcal{W})}(u_{\nu+1}))$  will intersect at a point within  $\mathcal{S}$ . Since we put  $u_{\nu+1}$  at this intersection point, as the induction step, assume that  $u_{i-1}$  lies inside  $\mathcal{S}$  where  $i > \nu + 1$ . Then,  $\mathcal{R}(s_i, u_{i-1})$  intersects  $\mathcal{R}(s_\alpha, u)$ .  $\square$

### 4.3 Analysis

In this section, we compute the running time of the reconstruction algorithm presented in this section. Assume that  $\mathcal{G}$  satisfies all of the properties introduced

in Section 3 and Appendix 1 and we know the visible subchains from each vertex according to their order in  $\mathcal{H}$ . The side-chains of the target pseudo-triangle are identified in linear time according to the algorithm described in the proof of Lemma 1. Reconstructing  $\mathcal{X}$  is done using *Colley's* algorithm that its running time is  $\mathcal{O}(|\mathcal{X}|)$ . To reconstruct  $\mathcal{Y}$ , the algorithm needs to determine the floating edges of  $\mathcal{U}$  and  $\mathcal{V}$  which can be done in constant time. Computing the  $\mathcal{W}$ -type regions (for each vertex  $w_i \in \mathcal{W}$ ) and determining the vertices  $w_{N^*}$  and  $w_{M^*}$  needs  $\mathcal{O}(n)$  time. If the conditions of Lemma 3 are not satisfied, the floating edges of  $\mathcal{U}$  and  $\mathcal{V}$  must be extended which is done in  $\mathcal{O}(1)$ : A lower bound for the increase in floating edges can be computed by using *Thales' theorem* and trigonometric functions. Locating each vertex of  $\mathcal{Y}$  is also done in constant time. Finally, placing each vertex of  $\mathcal{Z}$  and  $\mathcal{Z}'$  takes constant time. Therefore, the total running time of the algorithm is  $\mathcal{O}(|E|)$ .

## 5 Conclusion

In this paper, we propose an algorithm to reconstruct a pseudo-triangular polygon from a given visibility graph which is assumed to belong to some unknown pseudo-triangle. This reconstructing problem in its general case has a long history and it seems that there is still a long way to be completed for all polygons.

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## Appendix 1

An *interval* of a side-chain with endpoints  $p$  and  $q$  is the set of points on this side-chain connecting  $p$  to  $q$ . Note that it is not necessary for an endpoint of an interval to be a vertex of the side-chain.

**Property 2.** Every non-corner vertex of a side-chain sees a single nonempty interval from each of the other side-chains.

*Proof.* The inner angle of such a vertex is more than  $\pi$  and its inner visibility region cannot be bounded by a single concave chain. Therefore, it will see some parts from any of the other side-chains. The continuity of this visible parts on each side-chain is proved by contradiction. Assume that a vertex  $u \in \mathcal{U}$  sees two disjoint intervals  $[v_i, \dots, v_j]$  and  $[v_k, \dots, v_l]$  from  $\mathcal{V}$  meaning that the interval  $(v_j, \dots, v_k)$  is not visible from  $u$ . Consider an invisible point  $v'$  in  $(v_j, \dots, v_k)$ . There must be a blocking vertex for the invisible pair  $(u, v')$ . This blocking vertex must lie on the third side-chain which will also blocks either the visibility of  $u$  and  $v_j$  or  $u$  and  $v_k$ .  $\square$

**Property 3.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a pair of vertices  $(u, v)$  where  $u \in \mathcal{U}$ ,  $v \in \mathcal{V}$ ,  $v \neq \text{cmn}(\mathcal{U}, \mathcal{V})$ , and  $u = u_{\text{cmn}(\mathcal{U}, \mathcal{V})}(v)$ , then  $(v_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1}, u) \in E$ .

*Proof.* Consider the subpolygon  $\langle u, u_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1}, \dots, \text{cmn}(\mathcal{U}, \mathcal{V}), \dots, v_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1}, v \rangle$ . If we triangulate this polygon, there is no internal diagonal connected to  $v$  which means that  $\langle u, v, v_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1} \rangle$  must be a triangle in any triangulation. Therefore, the edge  $(u, v_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1})$  is a diagonal and this edge must exist in the visibility graph.  $\square$

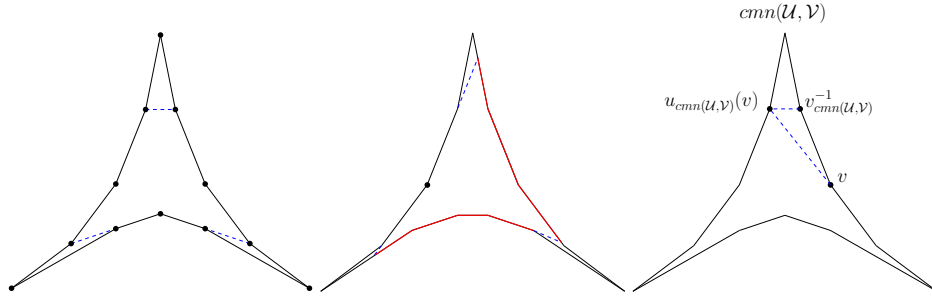


Fig. 9: (a) Property 1, (b) Property 2, and (c) Property 3

**Corollary 1.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a vertex  $v \in \mathcal{V}$ , if  $u_{\text{cmn}(\mathcal{U}, \mathcal{V})}(v) = u_j$  and  $u_{\text{cmn}(\mathcal{U}, \mathcal{V})}(v_{\text{cmn}(\mathcal{U}, \mathcal{V})}^{-1}) = u_k$ , then  $u_k$  is not closer than  $u_j$  to  $\text{cmn}(\mathcal{U}, \mathcal{V})$ .

**Corollary 2.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a vertex  $v \in \mathcal{V}$ , if  $v$  does not see any vertex from  $\mathcal{U}$ , then  $v_{cmn(\mathcal{U}, \mathcal{V})}^{+1}$  does not see any vertex of  $\mathcal{U}$  as well.

**Property 4.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a pair of vertices  $(u, v)$  where  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$  and  $k, l > 0$ , if both  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{-k}, v)$  and  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{-l})$  exist in  $E$ , then  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{-k}, v_{cmn(\mathcal{U}, \mathcal{V})}^{-l})$  also exists in  $E$ .

*Proof.* If  $u_{cmn(\mathcal{U}, \mathcal{V})}^{-k}$  and  $v_{cmn(\mathcal{U}, \mathcal{V})}^{-l}$  are not visible, then the blocking vertex cannot belong to the third side-chain. If this blocking vertex lies on  $\mathcal{V}$ , it will also block the visibility of  $u_{cmn(\mathcal{U}, \mathcal{V})}^{-k}$  and  $v$ . Similarly, it cannot belong to  $\mathcal{U}$ .  $\square$

**Property 5.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a pair of vertices  $(u, v)$  where  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$ , if both  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+k}, v_{cmn(\mathcal{U}, \mathcal{V})}^{+l})$  and  $(u, v)$  exist in  $E$  where  $l, k > 0$ , then at least one of the edges  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  and  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  exists in  $E$ .

*Proof.* It follows from triangulating the subpolygon formed by the boundary  $< u_{cmn(\mathcal{U}, \mathcal{V})}^{+k}, \dots, u, v, \dots, v_{cmn(\mathcal{U}, \mathcal{V})}^{+l} >$ .  $\square$

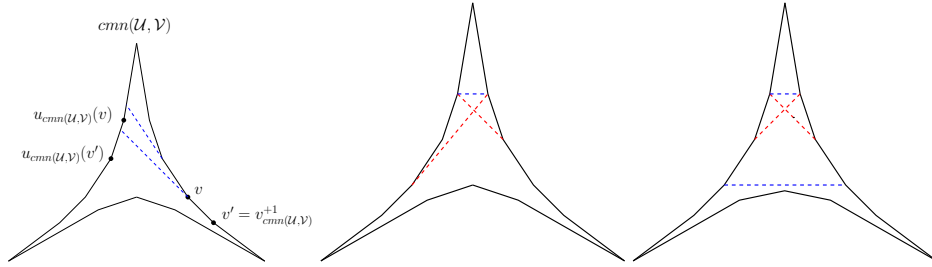


Fig. 10: (a) Corollary 1, (b) Property 4, and (c) Property 5

**Corollary 3.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a pair of vertices  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$ , where  $(u, v) \in E$  and none of the edges  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  and  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  exists in  $E$ , all visible vertices  $v' \in \mathcal{V}$  from  $u_{cmn(\mathcal{U}, \mathcal{V})}^{+k}$  are also visible from  $u_{cmn(\mathcal{U}, \mathcal{V})}^{+(k-1)}$  (for any  $k > 0$ ). This implies that  $V_{cmn(\mathcal{U}, \mathcal{V})}(u_{cmn(\mathcal{U}, \mathcal{V})}^{+k})$  must lie above  $v$ .

*Proof.* The vertex  $v'$  must belong to  $[cmn(\mathcal{U}, \mathcal{V}), \dots, v_{cmn(\mathcal{U}, \mathcal{V})}^{-1}]$ . Otherwise, according to Property 5 either  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  or  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  must exist. According to Corollary 1,  $v(u)$  is closer to  $cmn(\mathcal{U}, \mathcal{V})$  than  $v(u_{cmn(\mathcal{U}, \mathcal{V})}^{+k})$ , and because of the continuity of the chain visible from  $u$  (Property 2),  $v'$  will be visible from  $u$ . This implies that  $v'$  is visible from vertices of the chain  $[u, \dots, u_{cmn(\mathcal{U}, \mathcal{V})}^{+k}]$ .  $\square$

It is clear that every diagonal edge  $(u, v)$  in the visibility graph of a pseudo-triangle specifies a tower formed by the boundary vertices  $< u, \dots, cmn(\mathcal{U}, \mathcal{V}), \dots, v >$ .

The vertices of this tower satisfy the strong ordering defined earlier. However, we do not define this as a formal property, because it can be derived from properties 2, 4, and 5.

**Property 6.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a pair of vertices  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$ , where  $(u, v) \in E$  and none of the edges  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  and  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  exists in  $E$ , there is a nonempty subchain of the third side-chain  $\mathcal{W}$  which is visible from both  $u$  and  $v$ .

*Proof.* Triangulating  $\mathcal{P}$  using the edge  $(u, v)$ , the adjacent triangle of this edge in the opposite side of  $cmn(\mathcal{U}, \mathcal{V})$  must have its third vertex on  $\mathcal{W}$ . This is due to the invisibility of  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  and  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  pairs. Therefore, this chain contains at least one vertex. From Property 2 we know that the visible part of  $\mathcal{W}$  from any one of  $u$  and  $v$  vertices is continuous and their intersection will be continuous as well.  $\square$

**Corollary 4.** For any side-chain  $\mathcal{W}$ , there exists at least one vertex  $w \in \mathcal{W}$  that sees some vertices from both of the other side-chains. Furthermore, every vertex  $w_{cmn(\mathcal{U}, \mathcal{W})}^{-k}$  where  $k > 0$ , sees at least one vertex from  $\mathcal{U}$ .

*Proof.* The first part follows from Property 6 and the second part follows from Property 3.  $\square$

**Property 7.** For any side-chain  $\mathcal{W}$  and a vertex  $w \in \mathcal{W}$ ,  $u = u_{cmn(\mathcal{U}, \mathcal{V})}(w)$  and  $v = v_{cmn(\mathcal{U}, \mathcal{V})}(w)$  are visible from each other.

*Proof.* Let  $\mathcal{P}'$  be the subpolygon with boundary  $\langle cmn(\mathcal{V}, \mathcal{U}), \dots, u, w, v, \dots, cmn(\mathcal{V}, \mathcal{U}) \rangle$ . The vertex  $w$  does not see any other vertex of  $\mathcal{P}'$  which means that the diagonal  $uv$  must be used to triangulate  $\mathcal{P}'$ . This means that  $u$  and  $v$  must be visible from each other.  $\square$

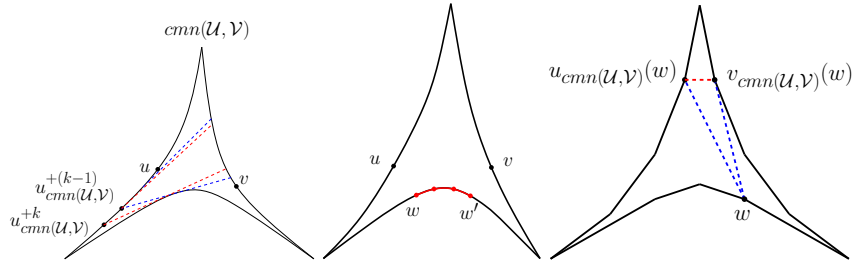


Fig. 11: (a) Corollary 3, (b) Property 6, and (c) Property 7

**Property 8.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a pair of vertices  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$ , where  $(u, v) \in E$  and none of the edges  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  and  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  exists in  $E$ , let  $[w, \dots, w']$  be the maximum subchain of  $\mathcal{W}$  visible to both  $u$  and  $v$  where  $w' = w_{cmn(\mathcal{U}, \mathcal{W})}^{+l}$ ,  $l \geq 0$ . Then,  $w'$  is not closer to  $cmn(\mathcal{U}, \mathcal{W})$  than  $w_{cmn(\mathcal{U}, \mathcal{W})}(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$ .

*Proof.* According to Property 6, the subchain  $[w, \dots, w']$  is nonempty. For the sake of a contradiction, assume that  $w'' = W_{cmn(\mathcal{U}, \mathcal{W})}(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  is farther from  $cmn(\mathcal{U}, \mathcal{W})$  than  $w'$ . Then, the edges  $(w'', u_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  and  $(w', v)$  intersect within the pseudo-triangle. Let  $p$  be this intersection point. The subpolygon formed by the boundary vertices  $< u, u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, p, v >$  is a convex polygon which completely lies inside the pseudo-triangle. So, the diagonal edge  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  must exist in  $E$  which is a contradiction.  $\square$

**Property 9.** For any pair of side-chains  $\mathcal{U}$  and  $\mathcal{V}$  and a pair of vertices  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$ , where  $(u, v) \in E$  and none of the edges  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v)$  and  $(u, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  exists in  $E$ ,  $w_{cmn(\mathcal{U}, \mathcal{W})}(v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  is not closer to  $cmn(\mathcal{U}, \mathcal{W})$  than  $W_{cmn(\mathcal{U}, \mathcal{W})}(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$ .

*Proof.* Let  $v'$  be  $v_{cmn(\mathcal{U}, \mathcal{V})}^{+1}$  and  $u'$  be  $u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}$ . For the sake of contradiction, assume that  $w_{cmn(\mathcal{U}, \mathcal{W})}(v')$  is closer to  $cmn(\mathcal{U}, \mathcal{W})$  than  $W_{cmn(\mathcal{U}, \mathcal{W})}(u')$ . Therefore, the edges  $(v', w_{cmn(\mathcal{U}, \mathcal{W})}(v'))$  and  $(u', W_{cmn(\mathcal{U}, \mathcal{W})}(u'))$  intersect within the pseudo-triangle. Let  $p$  be this intersection point. The subpolygon formed by the boundary vertices  $< u, v, v', p, u' >$  is a convex polygon which completely lies inside the pseudo-triangle. So, all diagonal edges  $(u, v')$ ,  $(u', v)$ , and  $(u', v')$  must exist in  $E$  which is a contradiction.  $\square$

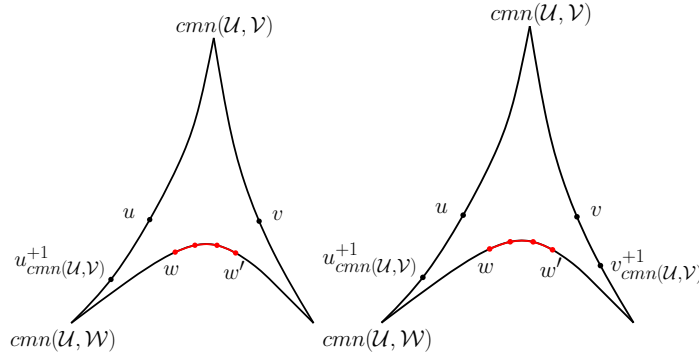


Fig. 12: (a) Property 8, and (b) Property 9

**Property 10.** For any side-chain  $\mathcal{W}$ , let  $u$  and  $v$  be respectively the closest vertices on  $\mathcal{U}$  and  $\mathcal{V}$  to  $cmn(\mathcal{U}, \mathcal{V})$  that are visible from some vertex (not necessarily the same) of  $\mathcal{W}$ . Then, there exists a nonempty subchain  $[w, \dots, w']$  in  $\mathcal{W}$ ,  $w' = w_{cmn(\mathcal{U}, \mathcal{W})}^{+l}$  and  $l \geq 0$ , that either all vertices of this subchain are visible from both  $u$  and  $v$ , or,  $(w, w')$  is an edge of  $\mathcal{W}$  and  $w$  sees  $v$  and  $w'$  sees  $u$ .

*Proof.* It is simple to show that  $(u, v) \in E$ . Assume that there is no vertex on  $\mathcal{W}$  that sees both vertices  $u$  and  $v$ . Then, we show that there is a pair of vertices  $w$  and  $w' = w_{cmn(\mathcal{U}, \mathcal{W})}^{+l}$  where  $w$  sees  $v$  and  $w'$  sees  $u$ . Let  $w$  be  $w_{cmn(\mathcal{W}, \mathcal{V})}(v)$  and  $w'$  be



$w_{cmn(\mathcal{U}, \mathcal{W})}(u)$ . Trivially,  $w \neq w'$  and  $w$  is closer to  $cmn(\mathcal{U}, \mathcal{W})$  than  $w'$  (otherwise,  $u$  and  $v$  will be visible to both  $w$  and  $w'$ ). To complete the proof, it is enough to show that  $w' = w_{cmn(\mathcal{U}, \mathcal{W})}^{+1}$ . This is done by showing that any vertex  $w''$  between  $w$  and  $w'$  on  $\mathcal{W}$  must see at least one of the vertices  $u$  and  $v$  which contradicts the definition of  $w$  and  $w'$ .

Assume that there is a vertex  $w''$  between  $w$  and  $w'$  and it sees neither  $u$  nor  $v$ . In the tower polygon formed by boundary  $\langle u, \dots, cmn(\mathcal{U}, \mathcal{W}), \dots, w'', w' \rangle$ , the blocking vertex for the invisible pair  $(w'', u)$  must lie on  $\mathcal{U}$ . Similarly, in the tower formed by boundary  $\langle w, w'', \dots, cmn(\mathcal{W}, \mathcal{V}), \dots, v \rangle$ , the blocking vertex for the invisible pair  $(w'', v)$  must lie on  $\mathcal{V}$ . Therefore, at least one of the side-chains  $\mathcal{U}$  and  $\mathcal{V}$  must be convex which is a contradiction. So,  $w''$  must see at least one of the vertices  $u$  and  $v$ .  $\square$

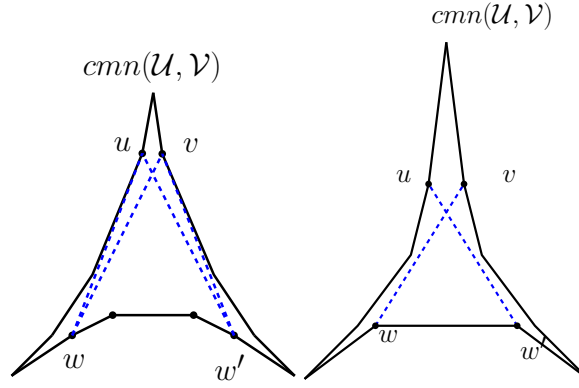


Fig. 13: Different cases of Property 10

**Corollary 5.** *If  $w$  and  $w'$  satisfy the conditions of Property 10, then for  $k > 0$ :*

- $u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-k})$  is not closer to  $cmn(\mathcal{U}, \mathcal{V})$  than  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-(k-1)})$ .
- $v_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}^{-k})$  (if exists) is not closer to  $cmn(\mathcal{U}, \mathcal{V})$  than  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}^{-(k-1)})$ .

*Proof.* For the sake of contradiction, assume that  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-k})$  is closer to  $cmn(\mathcal{U}, \mathcal{V})$  than  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-(k-1)})$ . The diagonal edge  $(w', u_{cmn(\mathcal{U}, \mathcal{V})}(w'))$  forms a tower polygon with boundary  $\langle w', \dots, cmn(\mathcal{W}, \mathcal{U}), \dots, u_{cmn(\mathcal{U}, \mathcal{V})}(w') \rangle$  which contains the vertices  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-k})$  and  $u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-(k-1)})$ , and satisfies strong ordering. Having the edges  $(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-k}, u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-k}))$  and  $(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-(k-1)}, u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-(k-1)}))$ , the edge  $(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-(k-1)}, u_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}'^{-k}))$  must also exist in  $E$ .

We prove the second part by contradiction. Let  $w_{cmn(\mathcal{U}, \mathcal{W})}^{-l}$  be the closest vertex of  $\mathcal{W}$  to  $cmn(\mathcal{U}, \mathcal{W})$  which sees at least one vertex from  $\mathcal{V}$  ( $l \geq 0$ ). For  $l \geq k > 0$ , assume that  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}^{-k})$  is closer to  $cmn(\mathcal{U}, \mathcal{V})$  than  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}^{-(k-1)})$ . Since  $v_{cmn(\mathcal{U}, \mathcal{V})}(w)$  is not farther from  $cmn(\mathcal{U}, \mathcal{V})$  than  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}^{-k})$ , Property 4 implies that  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}^{-k})$  sees  $w$ . According to Property 2,  $v_{cmn(\mathcal{U}, \mathcal{V})}(w_{cmn(\mathcal{U}, \mathcal{W})}^{-k})$  is also visible from  $w_{cmn(\mathcal{U}, \mathcal{W})}^{-(k-1)}$  which is a contradiction.  $\square$

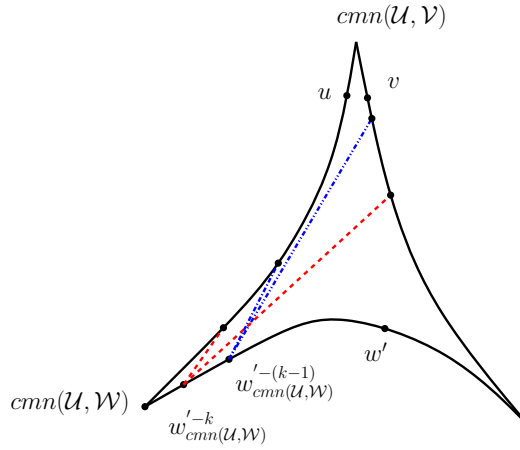


Fig. 14: Corollary 5

**Property 11.** For any vertex  $w \in \mathcal{W}$ , if  $u = u_{cmn(\mathcal{U}, \mathcal{V})}(w)$  and  $v = v_{cmn(\mathcal{U}, \mathcal{V})}(w)$  are the closest vertices to  $cmn(\mathcal{U}, \mathcal{V})$  which are visible to  $w$ , then:

- If  $w$  belongs to the subchain of  $\mathcal{W}$  satisfying Property 10, then at least one of the pairs  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$  and  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{-1}, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  are invisible.
- Otherwise, if  $w$  is closer to  $cmn(\mathcal{U}, \mathcal{W})$  than any vertex  $w'$  satisfying the conditions of Corollary 5, then  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$  is an invisible pair.

*Proof.* Consider the subpoygon  $\mathcal{P}'$  formed by the boundary  $\langle w, v, \dots, cmn(\mathcal{U}, \mathcal{V}), \dots, u \rangle$ . The pairs  $(w, v_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$  and  $(w, u_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$  are invisible. These pairs share the same blocking vertex. If  $u$  is the blocking vertex, then  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$  is an invisible pair, and if  $v$  is the blocking vertex, then the pair  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{-1}, v_{cmn(\mathcal{U}, \mathcal{V})}^{+1})$  is invisible.

Now, we show the correctness of the second part. It is clear that at least one of the vertices  $u_{cmn(\mathcal{U}, \mathcal{V})}(w')$  and  $v_{cmn(\mathcal{U}, \mathcal{V})}(w')$  is farther from  $cmn(\mathcal{U}, \mathcal{V})$  than  $u$  and  $v$ . For the sake of contradiction, assume that  $(u_{cmn(\mathcal{U}, \mathcal{V})}^{+1}, v_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$  is a visible pair. Then, considering  $\mathcal{P}' = \langle w, v, \dots, cmn(\mathcal{U}, \mathcal{V}), \dots, u \rangle$ ,  $v$  must be the blocking vertex for the pairs  $(w, v_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$  and  $(w, u_{cmn(\mathcal{U}, \mathcal{V})}^{-1})$ . This vertex

also blocks the pairs  $(w_{cmn(\mathcal{U}, \mathcal{W})}^{+l}, u_{cmn(\mathcal{U}, \mathcal{V})}^{-i})$  and  $(w_{cmn(\mathcal{U}, \mathcal{W})}^{+l}, v_{cmn(\mathcal{U}, \mathcal{V})}^{-j})$ . But, for some  $l > 0$  and  $i$  and  $j \geq 0$ ,  $w_{cmn(\mathcal{U}, \mathcal{W})}^{+l} = w'$ ,  $u_{cmn(\mathcal{U}, \mathcal{V})}^{-i} = u_{cmn(\mathcal{U}, \mathcal{V})}(w')$ , and  $v_{cmn(\mathcal{U}, \mathcal{V})}^{-j} = v_{cmn(\mathcal{U}, \mathcal{V})}(w')$  which contradicts the definition of  $w'$ .  $\square$

As it has been mentioned earlier, *Ghosh* introduced four necessary conditions for a visibility graph of a simple polygon. It can be shown that these properties are derived from the properties described in this section.