LECTURE - 9

Logical Consequence

$$\Gamma \models \phi$$

- $\mathcal{M}(\Gamma) \subseteq \mathcal{M}(\phi)$

- Alternatively: For each ρ:

  if <u>for each</u> $\varphi' \in \Gamma$

  truth φ' ρ = true

  then

  truth ϕ ρ = true

<u>Fact</u>: If $\Gamma \models \phi$

then $\Gamma \cup \{\neg\phi\}$ is NOT SATISFIABLE

PROOF: Suppose $\mathfrak{M}(\Gamma) \subseteq \mathfrak{M}(\phi)$

$\therefore$ if $\rho \in \mathfrak{M}(\Gamma)$, $\rho \in \mathfrak{M}(\phi)$ ①

Suppose $\Gamma \cup \{\neg\phi\}$ SATISFIABLE

Let $\rho \in \mathfrak{M}(\Gamma \cup \{\neg\phi\})$

$\therefore$ $\rho \in \mathfrak{M}(\Gamma)$ and $\rho \in \mathfrak{M}(\neg\phi)$

But by ①: $\rho \in \mathfrak{M}(\phi)$

$\therefore$ $\rho \in \mathfrak{M}(\phi)$ <u>and</u> $\rho \in \mathfrak{M}(\neg\phi)$

CONTRADICTION!

PROOF STRATEGY:

TO PROVE $\Gamma \models \phi$,

SHOW $\Gamma \cup \{\neg p\}$ NOT SAT.

HOW? $\boxed{\text{RESOLUTION}}$

ASSUME : CONVERT
$\Gamma \cup \{\neg p\}$ to $\boxed{\text{CNF.}}$

(NOTE: THESE BECOME
– CONJUNCTIONS OF
→ $\boxed{\text{CLAUSES}}$ =
DISJUNCTION OF
$\boxed{\text{LITERALS}}$

A $\quad \neg A$
(+ve) (–ve)

NOTE: IF <u>ANY</u> CLAUSE IS NOT
SATISFIABLE, THE WHOLE
SET OF CLAUSES IS <u>NOT</u>
SATISFIABLE

Q: HOW IS A CLAUSE SATISFIED?

Q: SO WHICH CLAUSE IS <u>NOT</u>

SATISFIABLE?

HOWEVER: EACH CLAUSE MAY BE SATISFIABLE, BUT THE COLLECTION MAY NOT BE.

Eg: $\{\{A\}, \{\neg A\}\}$

Eg: $\{\{A, B\}, \{A, \neg B\},$
$\{\neg A, B\}, \{\neg A, \neg B\}\}$

RESOLUTION

Idea: If one clause contains literal A, and another contains ¬A, RESOLVE the two clauses.

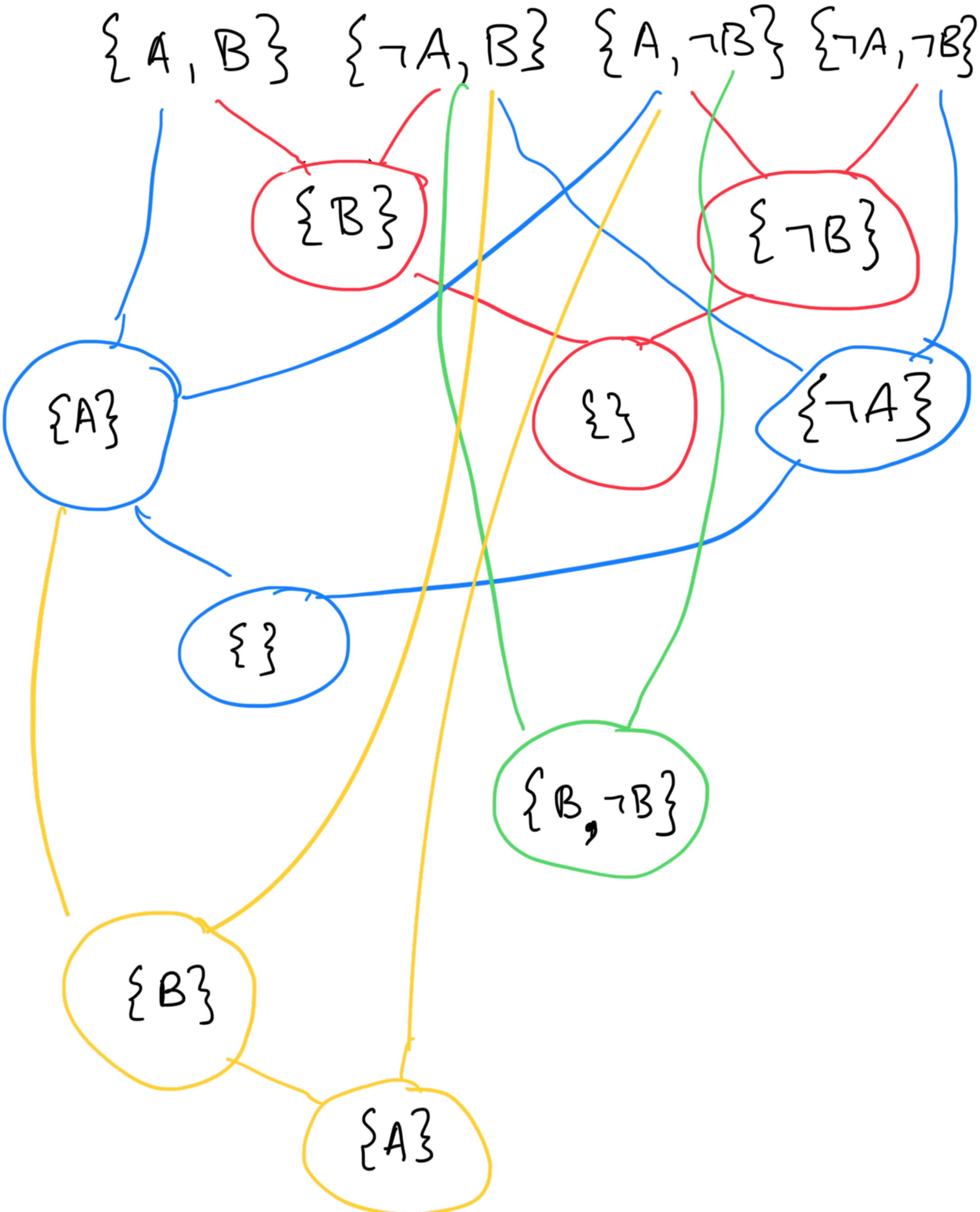
$$C_i = \{l_{i1}, l_{i2}, \dots l_{ik}\}$$

$$C_j = \{l_{j1}, l_{j2}, \dots l_{jm}\}$$

w.l.o.g. $l_{i1} = \neg l_{j1}$

$$\boxed{C^{res}_{i,j} = \{l_{i2},\dots,l_{ik}, l_{j2},\dots,l_{jm}\}}$$

RESOLVENT ON l_{i1}.



LEMMA: SUPPOSE A SET OF CLAUSES

$$S = \{C_1, \ldots, C_n\}$$

IS SATISFIED BY ρ

SUPPOSE THERE ARE CLAUSES

$$C_i = \{l_{i1}, l_{i2}, \ldots l_{ik}\}$$

$$C_j = \{l_{j1}, l_{j2}, \ldots l_{jm}\}$$

s. t. $\quad l_{i1} = \neg l_{j1}$

Let $C^{res}_{i,j} = \{l_{i2}, \ldots, l_{ik}, l_{j2}, \ldots, l_{jm}\}$

THEN ρ satisfies $S \cup \{C^{res}_{i,j}\}$

PROOF: ρ satisfies S

$\therefore$ ρ satisfies $C_i = \{l_{i1}, l_{i2}, \ldots l_{ik}\}$

and ρ satisfies $C_j = \{l_{j1}, l_{j2}, \ldots l_{jm}\}$

and $l_{i1} = \neg l_{j1}$ (w.l.o.g)

Claim: ρ satisfies $C^{res}_{i,j} = \{l_{i2}, \ldots l_{ik}, l_{j2}, \ldots l_{jm}\}$

- Suppose $\text{truth } l_{i1}\, \rho = \text{true}$

  $\therefore$ $\text{truth } l_{j1}\, \rho = \text{false}$

  So since ρ satisfies C_j,

  $\text{truth } l\, \rho = \text{true}$ for at least one $l \in \{ \boxed{l_{j2}, \dots, l_{jm}} \}$

  So ρ satisfies $C^{res}_{i,j}$.

- Suppose $\text{truth } l_{i1}\, \rho = \text{false}$

  $\therefore$ $\text{truth } l_{j1}\, \rho = \text{true}$

  Since ρ satisfies C_i,

  $\text{truth } l\, \rho = \text{true}$ for at least one $l \in \{ \boxed{l_{i2}, \dots l_{ik}} \}$

  $\therefore$ ρ satisfies $C^{res}_{i,j}$.

__COROLLARY__: If we start with a set S of clauses, and perform a series of resolution steps

$S_0 = S$

$\vdots$

$S_{i+1} = S_i \cup \{ C'_i \}$

$\vdots$

If ever the i^{th} resolvent $C'_i = \{ \}$

then S was UNSATISFIABLE.

QUESTION: HOW DO WE CHOOSE THE "RIGHT" SEQUENCE?

NOTE:
- CAN DO USELESS RESOLUTIONS
- CAN GET INTO CYCLES

HORN CLAUSES

AT MOST ONE +ve LITERAL

- 0 +ve, 0 -ve: $\{\}$
- 1 +ve, 0 -ve: A "fact"
- 1 +ve, $k > 0$ -ve: $A \vee \neg B_1 \vee \ldots \vee \neg B_k$ "rule"

$$(B_1 \wedge \ldots \wedge B_k) \to A$$

(by De Morgan)

- 0 +ve, $k > 0$ -ve: "goal"

$$\neg B_1 \vee \ldots \vee \neg B_k$$

$$\rightsquigarrow \neg (B_1 \wedge \ldots \wedge B_k)$$

Suppose

Γ — only facts & rules
(each clause has exactly $\underline{1}$ +ve literal)

ϕ - $A_1 \wedge A_2 \ldots \wedge A_k$

To "prove" $\Gamma \models \phi$

take $\Gamma \cup \{\neg\phi\}$

and perform RESOLUTION.

Note: $\neg\phi$ is a "GOAL"

$$\boxed{\text{\underline{ALWAYS} RESOLVE GOAL WITH A fact / rule.}}$$

$\rightarrow$ WILL ALWAYS GET A <u>NEW</u> <u>GOAL</u>.

① $A, B \rightarrow C.$

② $B, C \rightarrow D.$

③ $A, D \rightarrow E.$

④ $C, E \rightarrow F.$

⑤ $C, D \rightarrow G$

$\{①\ldots⑦\} \overset{?}{\models} F \wedge G$

5 C, D → G.

6 A.

7 B.

$\{\neg F, \neg G\}$ 4

1

$\{\neg C, \neg E, \neg G\}$

$\{\neg A, \neg B, \neg E, \neg G\}$ 6

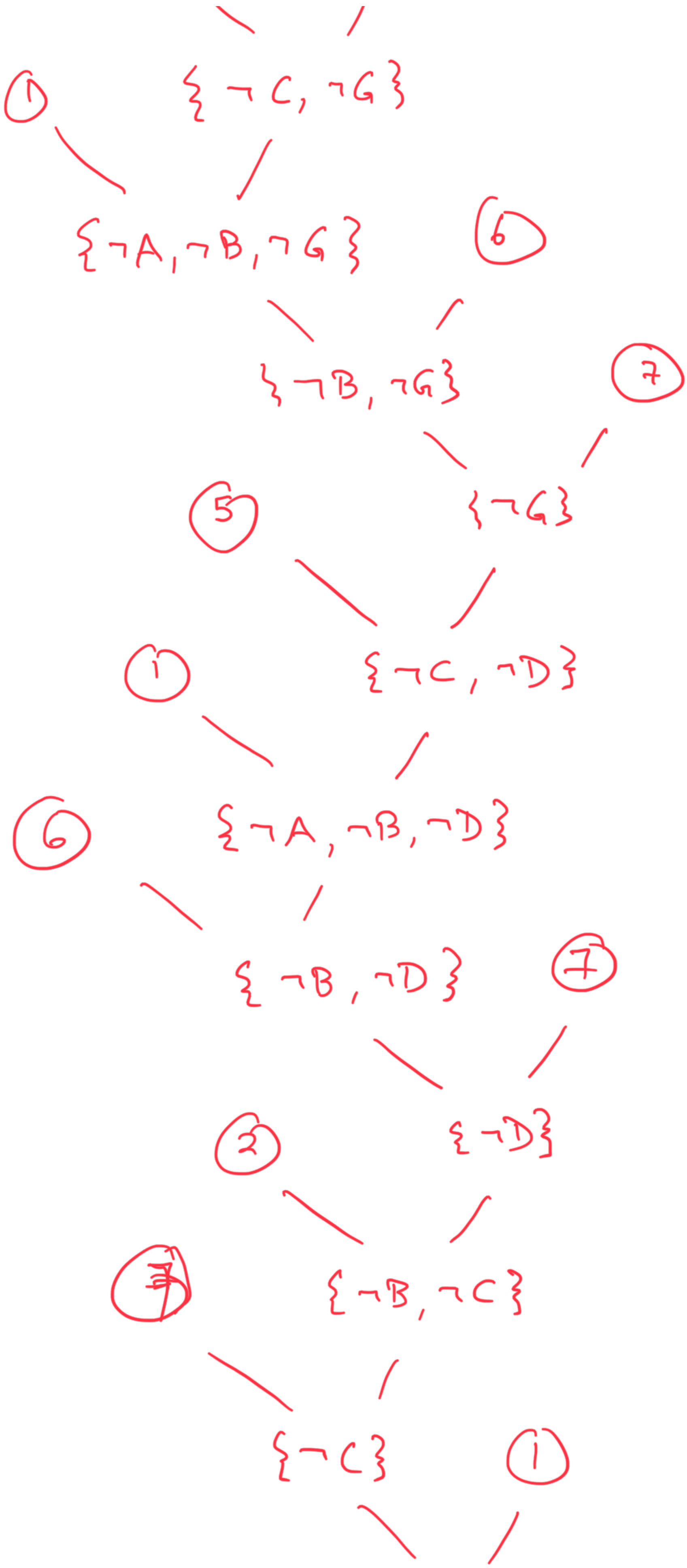
7 $\{\neg B, \neg E, \neg G\}$

3 $\{\neg E, \neg G\}$

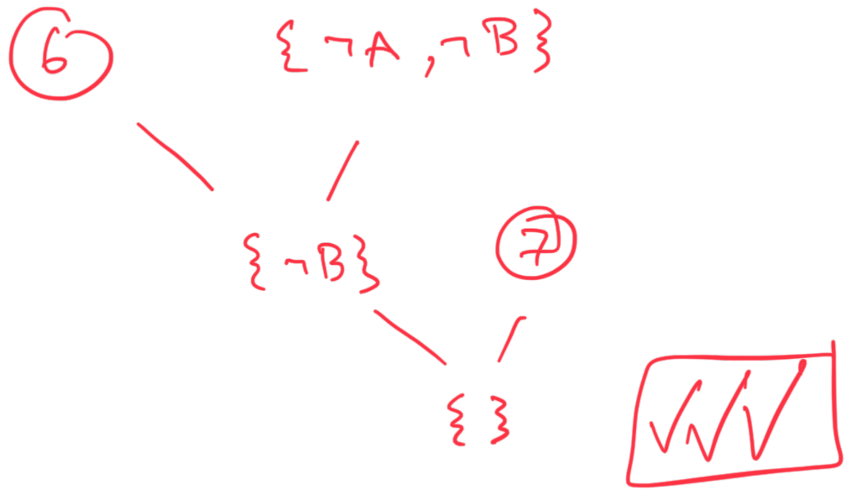
6 $\{\neg A, \neg D, \neg G\}$

$\{\neg D, \neg G\}$ 2

7 $\{\neg B, \neg C, \neg G\}$



{¬C, ¬G}
1
{¬A, ¬B, ¬G}
6
{¬B, ¬G}
7
5
{¬G}
1
{¬C, ¬D}
6
{¬A, ¬B, ¬D}
{¬B, ¬D}
7
2
{¬D}
3
{¬B, ¬C}
{¬C}
1



Q: DOES RULE/FACT ORDER MATTER?

Q: DOES ORDER OF LITERALS IN GOAL/ RULE BODY MATTER?