COL 765: Introduction to Logic and Functional Programming
Quiz 5, 05.09.2024

Name: _____ Entry No. _____

Consider the signature $\sum = \{ 0, 1, +, * \}$, with *arity*(0) = 0, *arity* (1) = 0, *arity*(+) = 2, and *arity* (*) = 2.

Consider two standard $\sum$-algebras

- $\mathscr{A}$ with carrier set $\mathbb{N}$ of the natural numbers (rendered in OCaml as the type `int`) with 0 interpreted as *zero*, 1 as *one*, + as *addition on the naturals* and * as *multiplication on the naturals*.
- $\mathscr{B}$ with carrier set $\mathbb{B}$, *i.e.* the standard booleans (and rendered in OCaml as the type `bool`) with 0 interpreted as `false`, 1 as `true`, + as *xor* (exclusive or) on the *booleans* and * as (binary) *conjunction* on *booleans*.

Q1 [4] Show from the definition of $\sum$-homomorphism that the function *odd*: $\mathbb{N} \rightarrow \mathbb{B}$, which returns `true` if a given natural number is odd and `false` if it is even, is a $\sum$-homomorphism from $\mathscr{A}$ to $\mathscr{B}$. Provide explanations for the interpretations of + and *.

odd($0_{\mathscr{A}}$) = `false`

odd($1_{\mathscr{A}}$) = `true`

odd($+_{\mathscr{A}}(x, y)$) = *odd(x) xor odd(y)* — because the sum of two odd natural numbers is even, the sum of two even natural numbers is even, but the sum of an odd and an even natural number is odd.

odd($*_{\mathscr{A}}(x, y)$) = *odd(x)* `&&` *odd(y)* — because the product of two odd numbers is odd only if both are odd; if even one of two numbers is not odd (i,e., even), then their product is even.

Q2. [3] The function *even*: $\mathbb{N} \rightarrow \mathbb{B}$, which returns `true` if a given natural number is even and `false` if it is odd, is also a $\sum$-homomorphism from $\mathscr{A}$ but not to $\mathscr{B}$, but to another $\sum$-algebra, say $\mathscr{B}'$.
What is the interpretation of symbols of signature $\sum$ in the $\sum$-algebra $\mathscr{B}'$? (Explain for)

0 is interpreted as `true`

1 is interpreted as `false`

+ is interpreted as *equals (i.e., iff)* because the sum of two even numbers is even, and the sum of two non-even (i.e., odd) numbers is even, but the sum of an even and an odd number is odd.

* is interpreted as *or* — because the product of two numbers is even only if at least one of them is even; the product is not even (i.e., odd) only if both numbers are odd.

Q3. [3] Show that the function *iszero*: $\mathbb{N} \to \mathbb{B}$, , which returns `true` if a given natural number is *zero* and `false` otherwise, is also a $\sum$-homomorphism from $\mathscr{A}$ to another particular $\sum$-algebra $\mathscr{B}''$ with the carrier set $\mathbb{B}$. What is the interpretation of the symbols of signature $\sum$ in $\sum$-algebra $\mathscr{B}''$? (Explain for + and *)

0 is interpreted as `true`

1 is interpreted as `false`

+ is interpreted as *and (i.e.,* `&&`*)* — since the sum of two natural numbers is *zero* only if both are *zero*

* is interpreted as *or (i.e.,* `||`*)* — since the product of two natural numbers is *zero* if either or both are *zero*