COL 765: Introduction to Logic and Functional Programming
Quiz 2, 01.08.2024
(Monotonicity of Closures)

Name: ____________SOLUTION_________________________ Entry No. ____XXXX_

Q1 [4] Let $R_1 \subseteq R_2$. Prove Monotonicity of Symmetric Closure: Show from the definition or properties of symmetric closure (the smallest symmetric relation containing the given relation) that $R_1^{\leftrightarrow} \subseteq R_2^{\leftrightarrow}$.

Proof Let $R_1 \subseteq R_2$.

So by Def of subset, if $(x, y) \in R_1$ then $(x, y) \in R_2$.
But $(x, y) \in R_1$ if and only if $(y, x) \in R_1^{-}$. (Def of relational inverse & its involution)
So $(y, x) \in R_2^{-}$, (since $(x, y) \in R_2$.)
Therefore $R_1^{-} \subseteq R_2^{-}$. (This establishes monotonicity of relational inverse).
So $R_1 \cup R_1^{-} \subseteq R_2 \cup R_2^{-}$ (monotonicity of union; this can also be proved easily)
That is, by Def of symmetric closure (or a proved fact about it), $R_1^{\leftrightarrow} \subseteq R_2^{\leftrightarrow}$.

Q2 [6] Let $R_1 \subseteq R_2$. Prove Monotonicity of Transitive Closure: Show from the definition or properties of transitive closure (the smallest transitive relation containing the given relation) that $R_1^{+} \subseteq R_2^{+}$.

Proof Let $R_1 \subseteq R_2$.

We show *by induction* that for each $i > 0 \in \mathbb{N}$, $R_1^i \subseteq R_2^i$.
Base case: $(i = 1)$. $R_1^1 = R_1 \subseteq R_2 = R_2^1$ (definition of R^1)
Induction Hypothesis: *Assume* that for $i = n \in \mathbb{N}$, $R_1^n \subseteq R_2^n$.
Induction Step: Let $i = 1 + n$

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\begin{aligned}
R_1^{1+n} &= R_1 \circ R_1^n && \text{(definition of } R^{1+n}) \\
&\subseteq R_2 \circ R_2^n && \text{(monotonicity of relational composition } \circ\text{; prove this)} \\
&= R_2^{1+n} && \text{(definition of } R^{1+n})
\end{aligned}
$$

So, by the *principle of simple induction*, for each $i > 0 \in \mathbb{N}$, $R_1^i \subseteq R_2^i$.
So, by monotonicity of denumerable union (prove this),

$$\bigcup_{i>0\in\mathbb{N}} R_1^i \subseteq \bigcup_{i>0\in\mathbb{N}} R_2^i$$

That is, by Def of transitive closure (or a proved result about it), $R_1^{+} \subseteq R_2^{+}$.