Functions

Functions, as we defined, are a special case of relations. Recall that a binary relation $R \subseteq S_1 \times S_2$ is a (partial) function if for each $x \in S_1$, there exists (at most) one $y \in S_2$ such that $(x, y) \in R$.

Predomain, codomain, domain and range.
For a functional relation $R_f \subseteq S_1 \times S_2$, the set S_1 is called the *pre-domain* and S_2 is called the *co-domain* of the function. We usually write $f : S_1 \to S_2$ instead of $R_f \subseteq S_1 \times S_2$ and write $y = f(x)$ when $(x, y) \in R_f$. We write $predom(f) = S_1$ and $codom(f) = S_2$; the relation R_f is often referred to as the "*extension*" or *graph* of the function *f*.

A partial function *need not* have a value $y = f(x)$ for each $x \in S_1$.
We say that function *f* is defined at $x \in S_1$ if there exists a $y \in S_2$ such that $(x, y) \in R_f$. The *domain* of a partial function $f : S_1 \to S_2$ is defined as $dom(f) = \{x \in S_1 \mid \exists y \in S_2 : y = f(x)\}$, i.e., the elements $x \in S_1$ for which $f(x) \in S_2$ is defined. Note that $dom(f) \subseteq S_1 = predom(f)$.
The *range* of a partial function $f : S_1 \to S_2$ is defined as $range(f) = \{y \in S_2 \mid \exists x \in S_1 : y = f(x)\}$, i.e., the elements $y \in S_2$ such that $y = f(x)$ for some $x \in S_1$. Note that $range(f) \subseteq S_2 = codom(f)$..

Total function. If the functional relation R_f is *total*, i.e., for each $x \in S_1$ there exists a $y \in S_2$ such that $y = f(x)$, we call *f* a total function and $dom(f) = S_1 = predom(f)$.

Mappings and notation. We will use the notation of mappings "$x \mapsto y$" to indicate that the element $x \in S_1$ is mapped by a function $f : S_1 \to S_2$ to the element $y \in S_2$. Note that the "mapping" arrow "$\mapsto$" tells us how *elements* are mapped by a function, and *notice the little vertical bar on the left of the arrow.* In contrast, $S_1 \to S_2$ is a *set-level construction* that depicts a functional relation between S_1, S_2.

Function Restriction. Suppose $f : S_1 \to S_2$ is a function. Let $A \subseteq S_1$. Then the restriction of function *f* to (pre)domain *A*, written as $f|_A$ is defined as the mapping that associates $f(x)$ to each element $x \in A$. ($f|_A(x) = f(x)$ for all $x \in A$). That is $f|_A$ behaves just as function *f* would but takes as its (pre)domain only the set *A*.

Onto. If a function f (namely, its functional relation R_f) is *onto*, i.e., for each $y \in S_2$ there exists an $x \in S_1$ such that $y = f(x)$, then we call *f* an *onto (surjective, epi)* function and $range(f) = S_2 = codom(f)$.

(1-1 or injection). A partial function $f : S_1 \to S_2$ is called *1-1 (injective, mono)* if whenever $f(x_1) = f(x_2) \in S_2$, then $x_1 = x_2 \in S_1$.

(Cardinality) The *cardinality* of set S_2 (written $|S_2|$) is at least as great as that of set S_1 (i.e., $|S_1| \le |S_2|$) if there exists a 1-1 total function $f : S_1 \to S_2$.

Exercise: If $S_1 \subseteq S_2$ then $|S_1| \leq |S_2|$.
Exercise. $|\mathbf{0}| \leq |S|$ for any set S.

(Bijection) A function $f : S_1 \to S_2$ is called a <u>*bijection*</u> if it is both *1-1 and onto*. Note that a bijection has a natural inverse *function*, written f^- or more commonly f^{-1}, whose graph is defined relationally as the symmetric inverse R_f^- of relation R_f.

Exercise: Show that the *identity* function on any set $id_S : S \to S$ defined by the mapping "$x \mapsto x$" is (i) total; (ii) 1-1; (iii) onto and (iv) a bijection.

Two sets S_1, S_2 are said to be in bijection with each other (written $S_1 \cong S_2$) if there exists a bijective function $f : S_1 \to S_2$. Note that if $S_1 \cong S_2$, then $|S_1| = |S_2|$

Exercise. Recall the set $\mathbf{1}$, with its canonical element written as ().
[Note: this is the empty tuple, and resembles writing a 0]

Show that $S \times \mathbf{1} \cong S \cong \mathbf{1} \times S$

Natural numbers and **Peano's Axioms:**
The set of Natural *numbers* $\mathbb{N}$ is an infinite set defined inductively in terms of two constructors: (zero) 0 and (successor) $s(\_)$. Natural numbers can be defined by the following Peano's Axioms.

1. (Base constant Zero) $0 \in \mathbb{N}$.
2. (Equality-Reflexive) For all $x \in \mathbb{N}$, $x = x$.
3. (Equality-Symmetric) For all $x, y \in \mathbb{N}$, if $x = y$ then $y = x$
4. (Equality-Transitive) For all $x, y, z \in \mathbb{N}$, if $x = y$ and $y = z$ then $x = z$
5. (Closure-Equality) For all x, y, if $x \in \mathbb{N}$, and $x = y$ then $y \in \mathbb{N}$.
6. (Closure-Successor) For all $x \in \mathbb{N}$, $s(x) \in \mathbb{N}$.
7. (Successor-Injective) For all $x, y \in \mathbb{N}$, if $s(x) = s(y)$ then $x = y$.
8. (Zero-not-Successor) For all $x \in \mathbb{N}$, $s(x) \neq 0$.
9. (Induction-Set) Let $A \subseteq \mathbb{N}$. **If** (i) $0 \in A$; and (ii) for all $x \in \mathbb{N}$: if $x \in A$ then $s(x) \in A$; **then** $A = \mathbb{N}$.

(9') (Induction-Pred) For *any* unary predicate P: **If** (i) $P(0)$; and (ii) for all $x \in \mathbb{N}$: if $P(x)$ implies $P(s(x))$; **then** for all $x \in \mathbb{N}$: $P(x)$.

Axioms 1 and 6 define the constructors, Axioms 2-5 characterise equality and closure of the set with respect to it, and Axiom 7 and 8 define (non)equality properties of the constructors. Axiom 9 is an induction scheme. (Axiom 9') is an alternative way to state Axiom 9, on predicates rather than on sets.

Finite and Denumerable Sets.

A prefix of cardinality n of the natural numbers $\mathbb{N}$ is defined as the set $\mathbb{N}_n = \{0, 1, \ldots (n-1)\}$.

Let $N' \subseteq \mathbb{N}$. A set A is called <u>*denumerable*</u> or *countable* if there exists a 1-1 total function $f : A \to N'$, i.e., there is an injective mapping *from all of A to* a *subset* of the *natural numbers*.

A *finite* set is a set whose cardinality is equal to some $n \in \mathbb{N}$. Clearly every finite set of size $n \in \mathbb{N}$ has a 1-1 total function mapping to $\mathbb{N}_n$.

> Note that *every* natural number is <u>finite</u>, but the *set of natural numbers* $\mathbb{N}$ is denumerably infinite (countably infinite). $\mathbb{N}$ is not in bijection with any $\mathbb{N}_n$ for $n \in \mathbb{N}$.

Every finite set is trivially denumerable.

Finite-domain functions. A function $f : S_1 \to S_2$ is called a *finite-domain function* if its domain is *finite*. That is $dom(f)$ is a finite subset of S_1. We write $f : S_1 \to_{fin} S_2$ to highlight $dom(f)$ is finite.

Function composition is a special case of relational composition where the relations involved are (partial) functions: Suppose $f : S_1 \to S_2$, and $g : S_2 \to S_3$ are (partial) functions. Then their composition, which we will write as $f; g : S_1 \to S_3$ is defined as the mapping $x \mapsto g(f(x))$ provided element $g(f(x)) \in S_3$ is defined (that is, there must exist $y = f(x) \in S_2$, and $z = g(y) \in S_3$).

Exercise: Show that function composition is associative, i.e., if $f : S_1 \to S_2$, $g : S_2 \to S_3$ and $h : S_3 \to S_4$, then $f; (g; h) = (f; g); h : S_1 \to S_4$.

Exercise: Show that $dom(f; g) = \{x \in dom(f) \mid f(x) \in dom(g)\}$ and $range(f; g) = \{g(y) \in range(g) \mid y \in range(f) \cap dom(g)\}$.

Exercise: Show that total functions are closed under composition. That is, if $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are total functions, then so is $f; g : S_1 \to S_3$.

Exercise: Show that 1-1 functions are closed under composition. That is, if $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are 1-1 functions, then so is $f; g : S_1 \to S_3$.

Exercise: Show that onto functions are closed under composition. That is, if $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are onto functions, then so is $f; g : S_1 \to S_3$.

Exercise: Show that bijections are closed under composition. That is, if $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are bijections, then so is $f; g : S_1 \to S_3$.

Exercise: Show that finite-domain functions are closed under composition. That is, if $f : S_1 \to_{fin} S_2$ and $g : S_2 \to_{fin} S_3$ are finite-domain functions, then so is $f; g : S_1 \to_{fin} S_3$.

Some denumerable sets

Proposition: (i) The set of even natural numbers is denumerable but not finite.
(ii) The set of odd natural numbers is denumerable but not finite.
(iii) The cardinalities of the even naturals (respectively, odd naturals) is the same as that of $\mathbb{N}$.

Proof hint: (Trivial) Define a total 1-1 function from the even (respectively odd) natural numbers to a subset of the naturals. (More interesting) Also show that the cardinality of these sets is at least that of the natural numbers by showing a total 1-1 function *from* the naturals to each of them.

Proposition: The set of integers $\mathbb{Z}$ is denumerable (countably infinite).
Proof hint: Define a total 1-1 function from $\mathbb{Z}$ to (a subset of) $\mathbb{N}$.

Proposition: (First diagonal argument — Cauchy) The set of pairs of natural numbers $\mathbb{N} \times \mathbb{N}$ is denumerable (countably infinite).
Proof hint: Define a total 1-1 function from $\mathbb{N} \times \mathbb{N}$ to (a subset of) $\mathbb{N}$.

Proposition: (Corollary) The set of pairs of integers $\mathbb{Z} \times \mathbb{Z}$ is denumerable (countably infinite).

Proposition: (Corollary) The set of rational numbers $\mathbb{Q}$ is denumerable (countably infinite).
Proof: $\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z} \wedge n > 0 \wedge m, n \text{ co-prime}\}$.
Consider $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ defined by the mapping (for $q \in \mathbb{Q} = m/n$), $q \mapsto (m, n)$.

Proposition: (Corollary) Let A be a denumerable set. Let us define

$$A^0 \cong \mathbf{1}$$
$$A^{s(n)} \cong A \times A^n$$

For each $n \in N$, the set A^n of lists over A of length n is a denumerable set.
Proof Hint: Proof by induction on $n \in N$.

Theorem (Cauchy): The denumerable union of denumerable sets is denumerable.

Proposition: (Corollary) Let A be a denumerable set. Let us define

$$A^* = \bigcup_{n \geq 0} A^n$$

Then the set A^* of all lists (sequences) built over elements from A is a denumerable set.

Proposition: The cardinality $|\mathbb{R}|$ of the set of real numbers $\mathbb{R}$ is the same as that of any open interval (a, b), in particular $(0,1)$ or $(-\pi/2,\ \pi/2)$.

Theorem (Cantor: Second Diagonal Argument): The set of real numbers $\mathbb{R}$ is *not* denumerable.
Proof by Cantordiction.
We rely on the fact that if any subset of a $\mathbb{R}$ is *not* denumerable, then $\mathbb{R}$ is *not* denumerable.

(1) Assume $(0,1)$ is denumerable.

(2) Consider the set D of *all* non-terminating decimals in $(0,1)$. D is a subset of $(0,1)$, and so must be denumerable (a subset of a denumerable set is also denumerable). That is, there is a total 1-1 function $e : D \to \mathbb{N}$.
Therefore we can write down *all* the non-terminating decimals that are elements of $(0,1)$, i.e., the elements of D, in *some* enumerated order as a series of "rows":

$$\begin{array}{l} r_0 = d_{00}\ d_{01}\ldots d_{0j}\ldots \\ r_1 = d_{10}\ d_{11}\ldots d_{1j}\ldots \\ \vdots\ \vdots\ \vdots \quad \vdots \ldots \vdots \ldots \\ r_i = d_{i0}\ d_{i1}\ldots\ d_{ij}\ldots \\ \vdots\ \vdots\ \vdots \quad \vdots \ldots \vdots \ldots \end{array}$$

where each of the rows (r_i) represents a non-terminating decimal of the form $0.d_{i0}d_{i1}\ldots\ d_{ij}\ldots$ [Note $r_i \in (0,1)$, and if $e(x) = i$, then x is represented as the i^{th} row r_i.]

(Note that all finite decimals, which have finitely many digits followed by an infinite number of 0's at the end can instead be represented by an indistinguishable form "in the limit"— with one less in the last non-zero digit and then an infinite series of 9's. For example, instead of 0.5 let us write $0.49999\ldots$)

(Note: we don't know — _and don't care_ — what the order of enumeration is. It is not at all likely to be the usual < ordering. We only assume that a suitable total 1-1 function $e : D \to \mathbb{N}$ exists, but know no other properties about it.)

Now consider a number represented as $r' = r'_0 r'_1\ldots r'_i\ldots$ such that for *each* $j \in \mathbb{N}$: $r'_j \neq d_{jj} \wedge r'_j \neq 0$. (The second conjunct is to ensure we do not get a terminating decimal). That is, r'_j — j^{th} digit of r' — is different from the "diagonal digit" d_{jj} (the j^{th} digit of the element r_j.)
[Note: There are many many such choices for r'.]

Claim: $r' \in D$. By construction, $r' = r'_0 r'_1\ldots r'_i\ldots$ is a non-terminating decimal in $(0,1)$.

Therefore by (2), $r' = r_k$ for some $k \in \mathbb{N}$, since D is assumed denumerable, and so $e(r') = k$ for some $k \in \mathbb{N}$.

But $r' \neq r_k$ because its k^{th} digit $r'_k \neq d_{kk}$, which is k^{th} digit of the r_k. (Note that by construction, for each $j \in \mathbb{N}$, the j^{th} digit $r'_j \neq d_{jj}$).
Therefore $r' \in D$ but $e(r') \notin \mathbb{N}$. So e is either not total, or not 1-1.

So D is *not denumerable*. This contradicts (2), and thus contradicts Assumption (1). So $(0,1)$ and therefore $\mathbb{R}$ are not denumerable. $\square$