Series and Applications

Series

A series is the sum of the terms of a sequence. Formally, if $\{a_n\}$ is a sequence, then the series is given by

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

Series Expansions

Exponential Series: e^x

The exponential function can be expanded as a Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Worked Example:

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = 1 + 2 + 2 + \frac{8}{6} + \cdots = 1 + 2 + 2 + 1.333 + \cdots$$

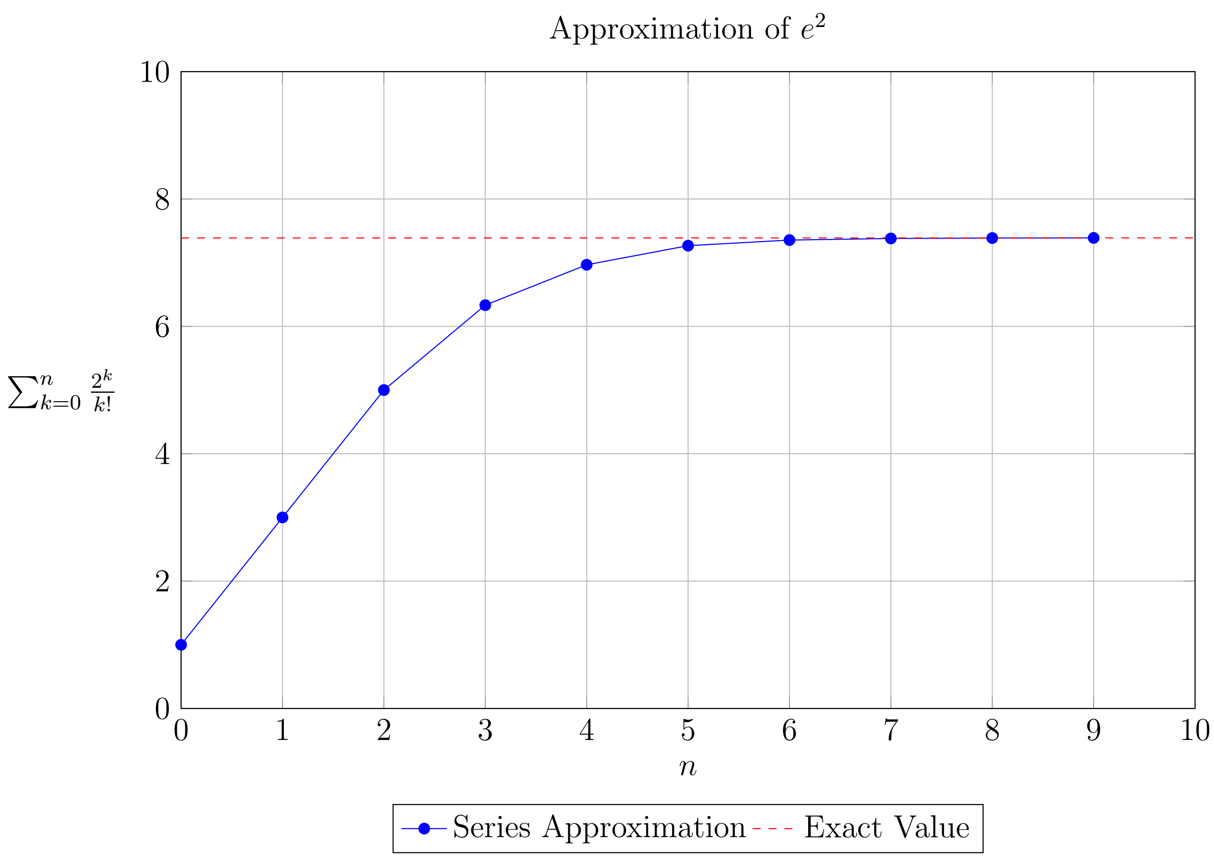

Figure 1: Approximation of e^2 using the Taylor series

Factorial Series

The factorial of n is defined as:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Worked Example:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Sine Series: $\sin x$

The sine function can be expanded as a Taylor series:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Worked Example:

$$\sin\left(\frac{\pi}{6}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!} = 0.5236 - 0.0239 + 0.0003 - \cdots \approx 0.5$$

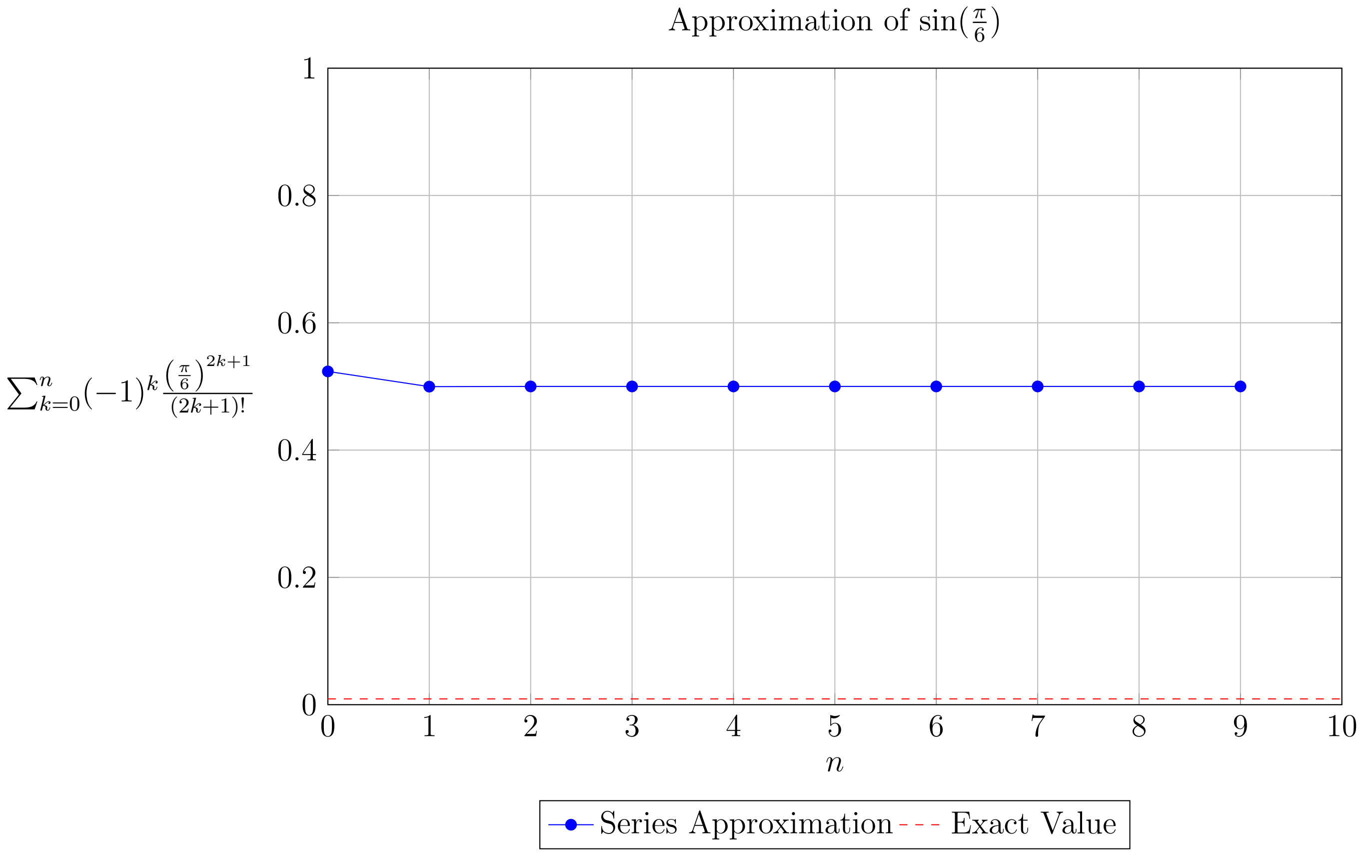

Figure 2: Approximation of $\sin(\frac{\pi}{6})$ using the Taylor series

Cosine Series: $\cos x$

The cosine function can be expanded as a Taylor series:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Worked Example:

$$\cos\left(\frac{\pi}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = 1 - 0.5483 + 0.0501 - \cdots \approx 0.5$$

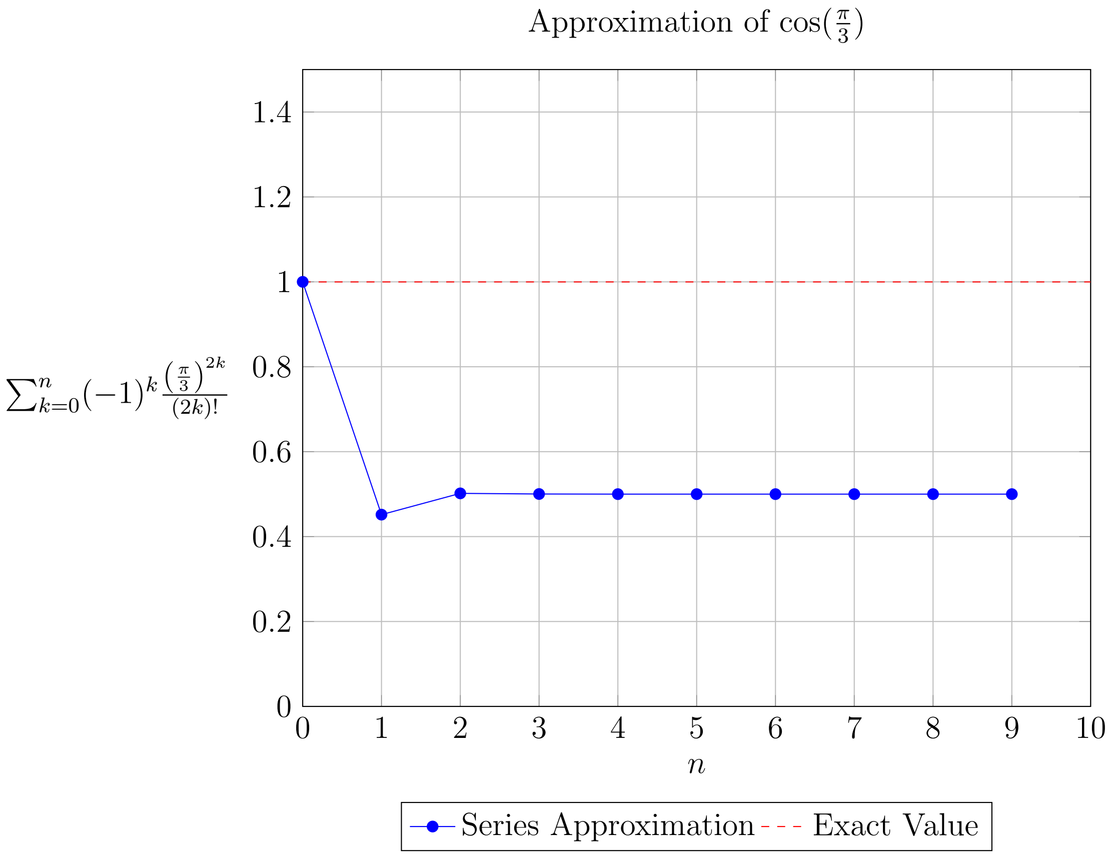

Figure 3: Approximation of $\cos(\frac{\pi}{3})$ using the Taylor series

Natural Logarithm Series: $\ln(1 + x)$

The natural logarithm can be expanded as a Taylor series:

$$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Worked Example:

$$\ln(1 + 0.5) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{0.5^n}{n} = 0.5 - 0.125 + 0.0417 - \cdots \approx 0.405$$

Geometric Series

A geometric series is a series of the form:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

where a is the first term and r is the common ratio. For $|r| < 1$, the series sums to:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$$

Worked Example:

$$\sum_{n=0}^{\infty} 2 \left(\frac{1}{2}\right)^n = 2 + 1 + 0.5 + 0.25 + \cdots = \frac{2}{1 - \frac{1}{2}} = \frac{2}{0.5} = 4$$

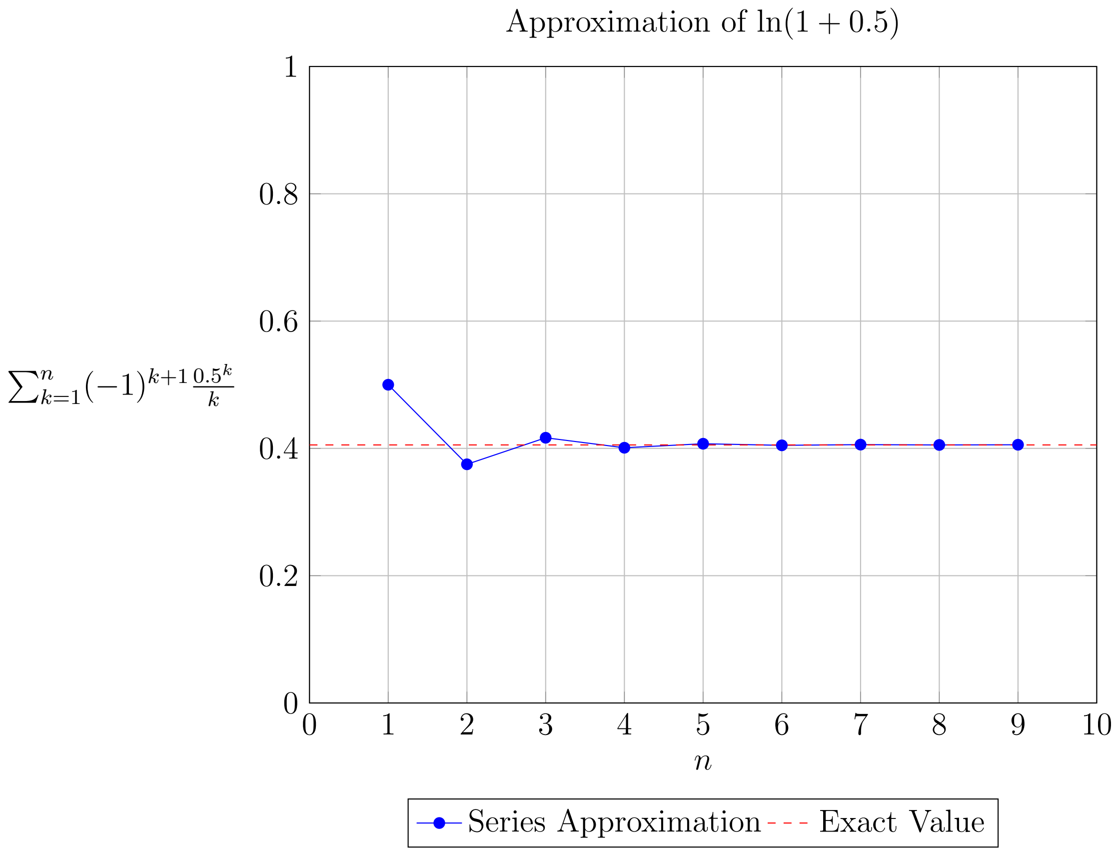

Figure 4: Approximation of $\ln(1 + 0.5)$ using the Taylor series

Summation of a Geometric Series

Summation notation is used to represent the sum of a sequence. The summation of a geometric series is:

$$\sum_{n=0}^{N} ar^n = a + ar + ar^2 + \cdots + ar^N = a\frac{1 - r^{N+1}}{1 - r} \text{ for } r \neq 1$$

Worked Example:

$$\sum_{n=0}^{4} 2\left(\frac{1}{2}\right)^n = 2 + 1 + 0.5 + 0.25 + 0.125 = 3.875$$

Summation

Summation notation is used to represent the sum of a sequence. For example:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Worked Example:

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = \frac{5(5+1)}{2} = \frac{30}{2} = 15$$

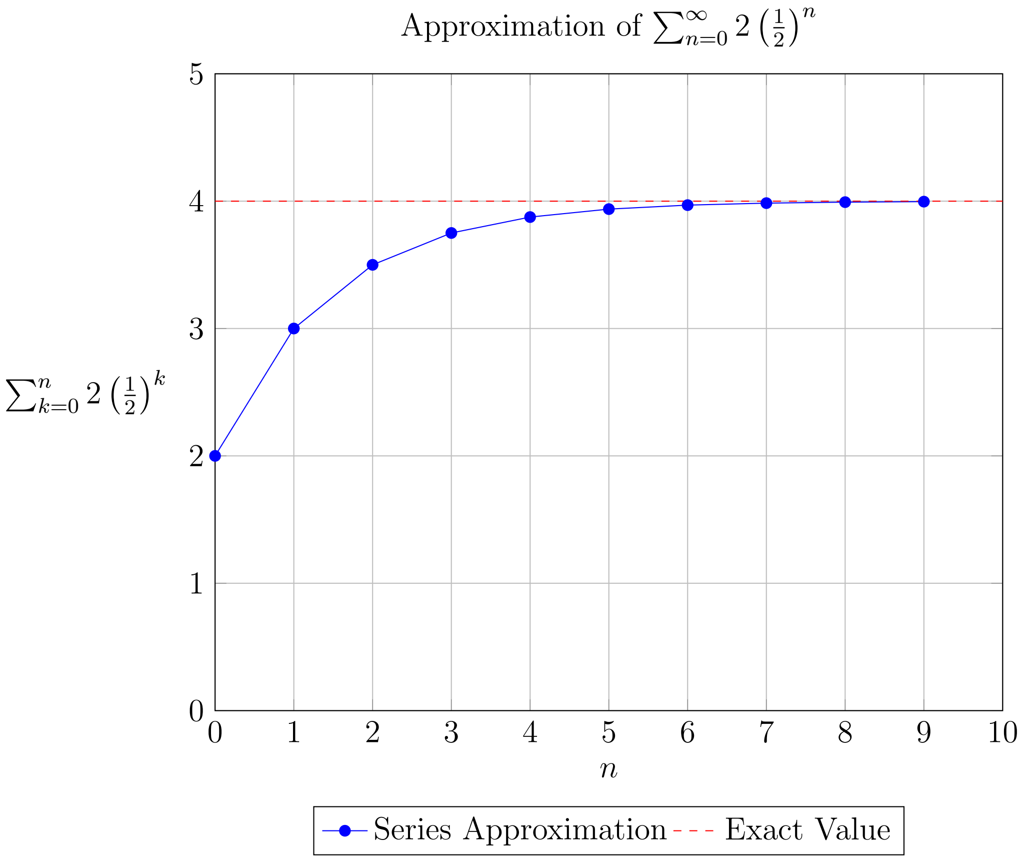

Figure 5: Approximation of the geometric series using partial sums