Complex Numbers & Applications

Definitions

- **Complex Numbers**: A complex number is of the form $z = a + bi$, where a and b are real numbers, and i is the imaginary unit with the property $i^2 = -1$.
- **Real Numbers**: A real number is a value that represents a quantity along a continuous line. Real numbers can be positive, negative, or zero, and they include all the rational and irrational numbers.
- **Imaginary Numbers**: An imaginary number is a number of the form bi, where b is a real number and i is the imaginary unit with the property $i^2 = -1$. Imaginary numbers are a subset of complex numbers where the real part is zero.
- **Modulus**: The modulus of a complex number $z = a + bi$ is denoted by $|z|$ and is given by $|z| = \sqrt{a^2 + b^2}$.
- **Argument**: The argument of a complex number $z = a + bi$, denoted by $\arg(z)$, is the angle θ between the positive real axis and the line representing the complex number in the complex plane, measured in the counter-clockwise direction.
- **Conjugate**: The conjugate of a complex number $z = a+bi$ is denoted by $\overline{z}$ and is given by $\overline{z} = a-bi$.
- **Cartesian Form**: The Cartesian form of a complex number is expressed as $z = a + bi$, where a and b are real numbers.
- **Exponential Form**: The exponential form of a complex number uses Euler's formula and is given by $z = re^{i\theta}$, where $r = |z|$ is the modulus and $\theta = \arg(z)$ is the argument.

Complex number examples and methods

- Explanation of the method first, and then with examples
- **Addition**
  - **Real Numbers**: $a + b$
  - **Imaginary Numbers**: $bi + ci = (b + c)i$
  - **Complex Numbers**: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- **Subtraction**
  - **Real Numbers**: $a - b$
  - **Imaginary Numbers**: $bi - ci = (b - c)i$
  - **Complex Numbers**: $(a + bi) - (c + di) = (a - c) + (b - d)i$
- **Division**
  - **Real Numbers**: $\frac{a}{b}, \quad b \neq 0$
  - **Imaginary Numbers**: $\frac{bi}{ci} = \frac{b}{c}, \quad c \neq 0$
  - **Complex Numbers (Think of this like rationalising a denominator from GCSEs and A-Level)**:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}, \quad c + di \neq 0$$

- **Addition**
  - **Real Numbers**: $3 + 5 = 8$
  - **Imaginary Numbers**: $2i + 3i = (2 + 3)i = 5i$
  - **Complex Numbers**: $(2 + 3i) + (4 + 5i) = (2 + 4) + (3 + 5)i = 6 + 8i$
- **Subtraction**
  - **Real Numbers**: $7 - 2 = 5$
  - **Imaginary Numbers**: $6i - 4i = (6 - 4)i = 2i$
  - **Complex Numbers**: $(5 + 7i) - (3 + 2i) = (5 - 3) + (7 - 2)i = 2 + 5i$
- **Division**
  - **Real Numbers**: $\frac{10}{2} = 5$
  - **Imaginary Numbers**: $\frac{6i}{3i} = \frac{6}{3} = 2$
  - **Complex Numbers**:

$$\frac{2+3i}{1+4i} = \frac{(2+3i)(1-4i)}{(1+4i)(1-4i)} = \frac{2-8i+3i+12}{1+16} = \frac{14-5i}{17} = \frac{14}{17} - \frac{5}{17}i$$

- **Modulus**
  - The modulus of a complex number $z = a + bi$ is given by

$$|z| = \sqrt{a^2 + b^2}$$

  - Example: For $z = 3 + 4i$,

$$|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

- **Argument**
  - The argument of a complex number $z = a + bi$ is the angle θ such that $\tan\theta = \frac{b}{a}$. It is usually denoted as $\arg(z)$.
  - The correct quadrant for θ must be determined:
    * If $a > 0$ and $b \geq 0$, then $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
    * If $a > 0$ and $b < 0$, then $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
    * If $a < 0$, then $\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$
    * If $a = 0$ and $b > 0$, then $\theta = \frac{\pi}{2}$
    * If $a = 0$ and $b < 0$, then $\theta = -\frac{\pi}{2}$
  - Example: For $z = 1 + i$,

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

- **Conjugate**

- The conjugate of a complex number $z = a + bi$ is given by

$$\overline{z} = a - bi$$

- Example: For $z = 3 + 4i$,

$$\overline{z} = 3 - 4i$$

- **Cartesian to Modulus-Argument Form**
  - A complex number $z = a + bi$ can be converted to modulus-argument form $z = r(\cos\theta + i\sin\theta)$, where

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arg(z)$$

  - Example: For $z = 1 + i$,

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \theta = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

- **Modulus-Argument to Cartesian Form**
  - A complex number in modulus-argument form $z = r(\cos\theta + i\sin\theta)$ can be converted to Cartesian form $z = a + bi$, where

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

  - Example: For $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$,

$$a = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$$

$$b = 2\sin\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$z = 1 + \sqrt{3}i$$

- **Modulus-Argument to Exponential Form**
  - A complex number in modulus-argument form $z = r(\cos\theta + i\sin\theta)$ can be converted to exponential form $z = re^{i\theta}$ using Euler's formula.
  - Example: For $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$,

$$z = \sqrt{2}e^{i\frac{\pi}{4}}$$

- **Exponential to Modulus-Argument Form**
  - A complex number in exponential form $z = re^{i\theta}$ can be converted to modulus-argument form $z = r(\cos\theta + i\sin\theta)$ using Euler's formula.
  - Example: For $z = 2e^{i\frac{\pi}{6}}$,

$$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$$

Vector form with examples

(Will not be asked, but it is another way of understanding and representing complex numbers.)
A complex number $z = a + bi$ can be represented as a vector in the complex plane. The vector form of a complex number is written as:

$$\mathbf{z} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Here, a is the real part and b is the imaginary part of the complex number.

Addition and Subtraction

Addition and subtraction of complex numbers in vector form can be done component-wise.

- **Addition**: If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then

$$\mathbf{z}_1 + \mathbf{z}_2 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$

- **Subtraction**: If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then

$$\mathbf{z}_1 - \mathbf{z}_2 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} - \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix}$$

Example: Let $z_1 = 3 + 4i$ and $z_2 = 1 + 2i$.

Addition:

$$\mathbf{z}_1 + \mathbf{z}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Subtraction:

$$\mathbf{z}_1 - \mathbf{z}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Scalar Multiplication

Scalar multiplication involves multiplying each component of the vector by the scalar.

$$c \cdot \mathbf{z} = c \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$$

Example: Let $z = 2 + 3i$ and $c = 2$.

$$2 \cdot \mathbf{z} = 2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Multiplication of Complex Numbers

Multiplication of complex numbers in vector form is more involved. If $z_1 = a + bi$ and $z_2 = c + di$, the product is given by:

$$z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

In vector form, this corresponds to:

$$\mathbf{z}_1 \cdot \mathbf{z}_2 = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$$

Example: Let $z_1 = 1 + 2i$ and $z_2 = 3 + 4i$.

$$z_1 z_2 = (1 + 2i)(3 + 4i) = (1 \cdot 3 - 2 \cdot 4) + (1 \cdot 4 + 2 \cdot 3)i = (3 - 8) + (4 + 6)i = -5 + 10i$$

In vector form:

$$\mathbf{z}_1 \cdot \mathbf{z}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 - 2 \cdot 4 \\ 1 \cdot 4 + 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$$

Division of Complex Numbers

Division of complex numbers in vector form is also more involved. If $z_1 = a + bi$ and $z_2 = c + di$, the division is given by:

$$\frac{z_1}{z_2} = \frac{(a + bi)(c - di)}{c^2 + d^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

In vector form, this corresponds to:

$$\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{1}{c^2 + d^2} \begin{pmatrix} ac + bd \\ bc - ad \end{pmatrix}$$

Example: Let $z_1 = 5 + 6i$ and $z_2 = 3 + 4i$.

$$\frac{z_1}{z_2} = \frac{(5 + 6i)(3 - 4i)}{3^2 + 4^2} = \frac{(15 + 24) + (-20 + 18)i}{9 + 16} = \frac{39 - 2i}{25} = \frac{39}{25} - \frac{2}{25}i$$

In vector form:

$$\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{1}{25} \begin{pmatrix} 39 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{39}{25} \\ -\frac{2}{25} \end{pmatrix}$$

Finding Roots of a complex number

Square Roots

To find the square roots of a complex number $z = a + bi$:

1. Convert z to polar form: $z = re^{i\theta}$. 2. The square roots are given by:

$$\sqrt{z} = \sqrt{r} e^{i \frac{\theta + 2k\pi}{2}}, \quad k = 0, 1$$

Example: Find the square roots of $z = 3 + 4i$.

1. Convert to polar form:

$$r = |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\theta = \arg(z) = \tan^{-1}\left(\frac{4}{3}\right)$$

Thus, $z = 5e^{i\theta}$.

2. Find the square roots:

$$\sqrt{z} = \sqrt{5}e^{i\frac{\theta}{2}}, \quad \sqrt{5}e^{i\frac{\theta+2\pi}{2}}$$

Therefore, the square roots are:

$$\sqrt{5}e^{i\frac{\theta}{2}}, \quad \sqrt{5}e^{i\left(\frac{\theta}{2}+\pi\right)}$$

Cube Roots

To find the cube roots of a complex number $z = a + bi$:

1. Convert z to polar form: $z = re^{i\theta}$. 2. The cube roots are given by:

$$\sqrt[3]{z} = \sqrt[3]{r}e^{i\frac{\theta+2k\pi}{3}}, \quad k = 0, 1, 2$$

Example: Find the cube roots of $z = 8$.

1. Convert to polar form:

$$r = |z| = 8, \quad \theta = 0$$

Thus, $z = 8e^{i\cdot 0}$.

2. Find the cube roots:

$$\sqrt[3]{z} = 2e^{i\frac{0}{3}}, \quad 2e^{i\frac{2\pi}{3}}, \quad 2e^{i\frac{4\pi}{3}}$$

Therefore, the cube roots are:

$$2, \quad 2e^{i\frac{2\pi}{3}}, \quad 2e^{i\frac{4\pi}{3}}$$

Fourth Roots

To find the fourth roots of a complex number $z = a + bi$:

1. Convert z to polar form: $z = re^{i\theta}$. 2. The fourth roots are given by:

$$\sqrt[4]{z} = \sqrt[4]{r}e^{i\frac{\theta+2k\pi}{4}}, \quad k = 0, 1, 2, 3$$

Example: Find the fourth roots of $z = 16$.

1. Convert to polar form:

$$r = |z| = 16, \quad \theta = 0$$

Thus, $z = 16e^{i\cdot 0}$.

2. Find the fourth roots:

$$\sqrt[4]{z} = 2e^{i\frac{0}{4}}, \quad 2e^{i\frac{2\pi}{4}}, \quad 2e^{i\frac{4\pi}{4}}, \quad 2e^{i\frac{6\pi}{4}}$$

Therefore, the fourth roots are:

$$2, \quad 2e^{i\frac{\pi}{2}}, \quad 2e^{i\pi}, \quad 2e^{i\frac{3\pi}{2}}$$

Conclusion

To conclude, you will need to know how to add, subtract, divide and find the roots of complex numbers. This topic is important as a lot of the imaginary number notation will be used in ENG1003 and ENG1004 in first year.