Numerical Methods

Newton-Raphson Method

The Newton-Raphson method is an iterative technique for finding the roots of a function $f(x) = 0$. The formula for the method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f'(x)$ is the derivative of $f(x)$.

Worked Example

Find the root of $f(x) = x^2 - 2$ using the Newton-Raphson method.

1. Start with an initial guess x_0. 2. Calculate the next approximation using the formula.

Given:

$$f(x) = x^2 - 2 \quad \text{and} \quad f'(x) = 2x$$

Starting with $x_0 = 1$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^2 - 2}{2 \cdot 1} = 1 - \frac{-1}{2} = 1.5$$

Next iteration:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{1.5^2 - 2}{2 \cdot 1.5} = 1.5 - \frac{0.25}{3} = 1.4167$$

Next iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.4167 - \frac{1.4167^2 - 2}{2 \cdot 1.4167} = 1.4167 - \frac{0.006944}{2.8334} \approx 1.4142$$

Thus, the root converges to approximately $\sqrt{2} \approx 1.4142$.

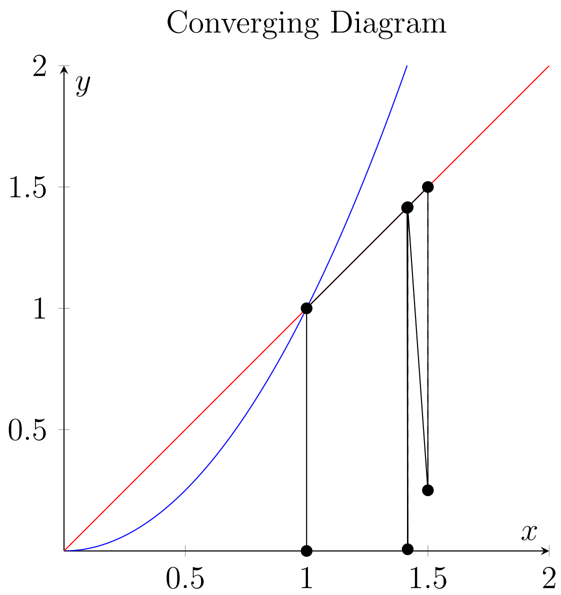
Spider Diagrams

These diagrams (below) are a visual way of representing how the method works and can be used in your working out.

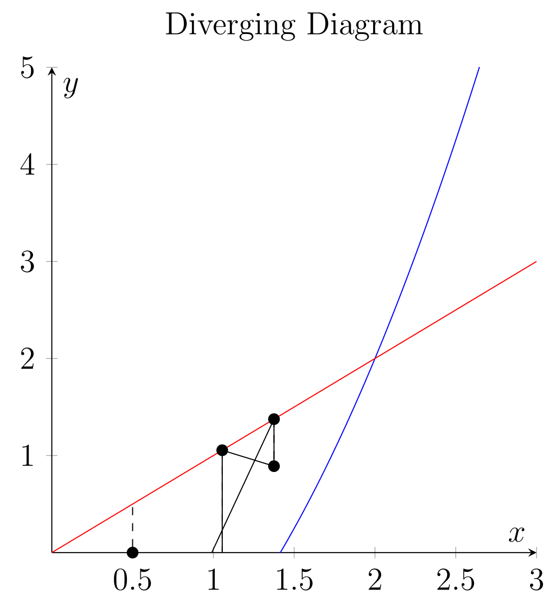
We will plot four types of diagrams:

1. Converging - We arrive at a solution.
2. Diverging - No solution with Newton-Raphson Method.
3. Cobweb - We arrive at a solution.
4. Staircase - No solution with Newton-Raphson Method.

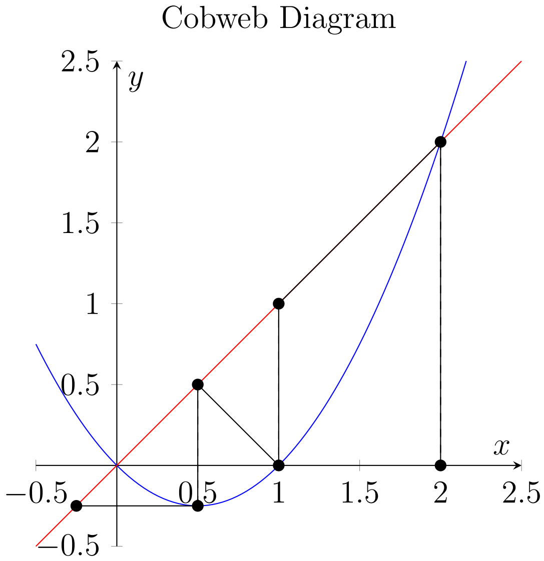
Converging Diagram



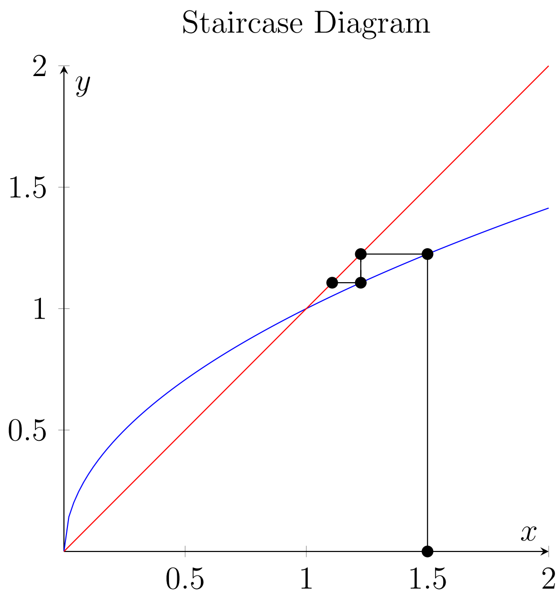
Diverging Diagram



Cobweb Diagram



Staircase Diagram



Numerical Integration

Numerical integration is a method to approximate the definite integral of a function when it is difficult or impossible to find the integral analytically. Two common methods are the Trapezium Rule and Simpson's Rule.

Trapezium Rule

The Trapezium Rule approximates the integral of a function $f(x)$ over the interval $[a, b]$ by dividing the area under the curve into trapezoids.

The formula for the Trapezium Rule is:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$$

where n is the number of subintervals and $x_i = a + i \cdot \frac{b-a}{n}$.

Worked Example

Approximate the integral of $f(x) = x^2$ from 0 to 2 using the Trapezium Rule with $n = 4$.

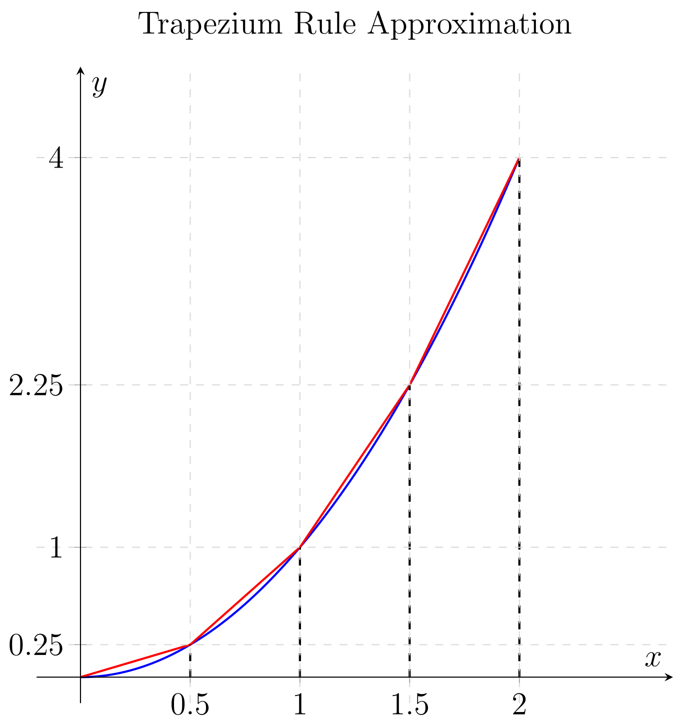
$$a = 0, \quad b = 2, \quad n = 4, \quad h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

$$x_i = 0, 0.5, 1, 1.5, 2$$

$$f(x_i) = 0^2, 0.5^2, 1^2, 1.5^2, 2^2 = 0, 0.25, 1, 2.25, 4$$

$$\int_0^2 x^2\,dx \approx \frac{0.5}{2}\left(0 + 2(0.25 + 1 + 2.25) + 4\right) = 0.25\left(0 + 2(3.5) + 4\right) = 0.25\left(0 + 7 + 4\right) = 0.25 \times 11 = 2.75$$

Diagram



Simpson's Rule

Simpson's Rule approximates the integral of a function $f(x)$ over the interval $[a, b]$ by dividing the area under the curve into parabolic segments.

The formula for Simpson's Rule is:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left(f(x_0) + 4\sum_{i=1,\text{ odd}}^{n-1} f(x_i) + 2\sum_{i=2,\text{ even}}^{n-2} f(x_i) + f(x_n)\right)$$

where n is the number of subintervals (must be even) and $x_i = a + i \cdot \frac{b-a}{n}$.

Worked Example

Approximate the integral of $f(x) = x^2$ from 0 to 2 using Simpson's Rule with $n = 4$.

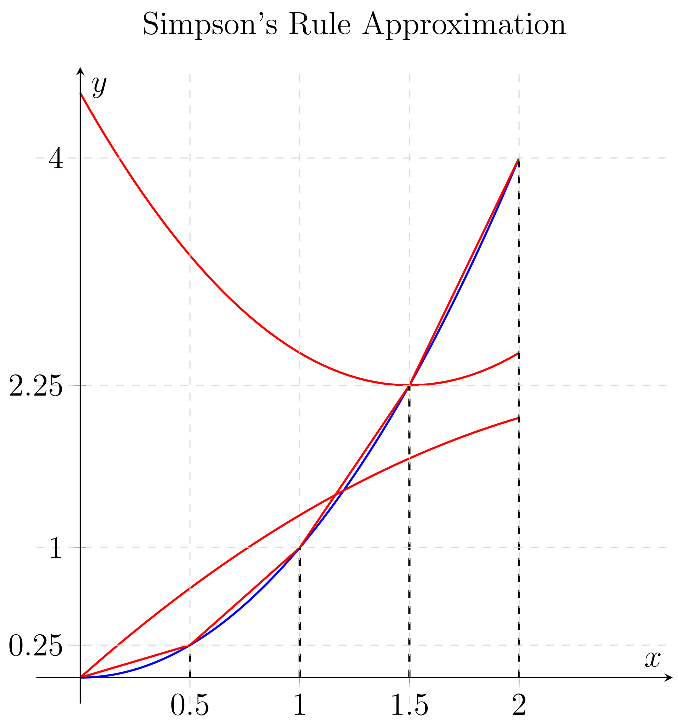
$$a = 0, \quad b = 2, \quad n = 4, \quad h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

$$x_i = 0, 0.5, 1, 1.5, 2$$

$$f(x_i) = 0^2, 0.5^2, 1^2, 1.5^2, 2^2 = 0, 0.25, 1, 2.25, 4$$

$$\begin{aligned}
\int_0^2 x^2\,dx &\approx \frac{0.5}{3}\left(0 + 4(0.25 + 2.25) + 2(1) + 4\right) \\
&= \frac{1}{6}\left(0 + 4 \cdot 2.5 + 2 \cdot 1 + 4\right) \\
&= \frac{1}{6}\left(0 + 10 + 2 + 4\right) \\
&= \frac{1}{6} \times 16 \\
&= \frac{16}{6} \\
&= \frac{8}{3} \approx 2.67
\end{aligned}$$

Diagram



Relation to Integration

Both the Trapezium Rule and Simpson's Rule are methods of numerical integration used to approximate the value of a definite integral. These methods are particularly useful when the function does not have an elementary antiderivative or when an analytical approach is impractical. The Trapezium Rule approximates the area under the curve using trapezoids, while Simpson's Rule uses parabolic segments, providing a more accurate approximation for smooth functions.

Euler's Method to solve differential equations

Euler's method is a numerical technique used to approximate solutions to ordinary differential equations (ODEs). Given an initial value problem of the form:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Euler's method iteratively estimates $y(x)$ at discrete points by using the slope of the tangent line at each step.

The general formula for Euler's method is:

$$y_{i+1} = y_i + hf(x_i, y_i)$$

where h is the step size, x_i is the current x-value, and y_i is the current y-value.

Worked Example

Consider the differential equation:

$$\frac{dy}{dx} = y, \quad y(0) = 1$$

using Euler's method with a step size $h = 0.2$.

Solution

Using Euler's method, we have:

$$y_{i+1} = y_i + hf(x_i, y_i) = y_i + hy_i = (1 + h)y_i$$

Starting with the initial condition $y(0) = 1$:

$$x_0 = 0, \quad y_0 = 1$$

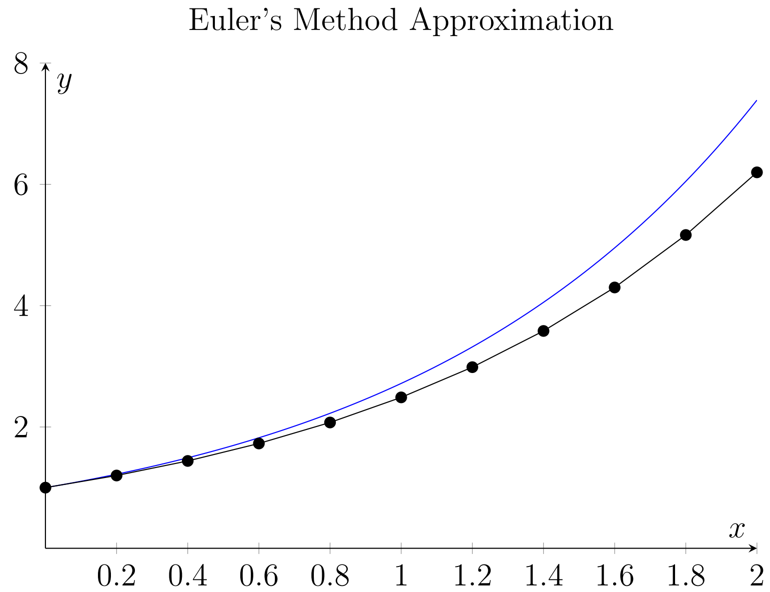
At $x = 0$, $y = 1$, so $y_1 = (1 + 0.2) \times 1 = 1.2$.

At $x = 0.2$, $y = 1.2$, so $y_2 = (1 + 0.2) \times 1.2 = 1.44$.

Continuing this process:

$$\begin{aligned}
x_3 = 0.4, &\quad y_3 = (1 + 0.2) \times 1.44 = 1.728 \\
x_4 = 0.6, &\quad y_4 = (1 + 0.2) \times 1.728 = 2.074 \\
x_5 = 0.8, &\quad y_5 = (1 + 0.2) \times 2.074 = 2.488 \\
x_6 = 1.0, &\quad y_6 = (1 + 0.2) \times 2.488 = 2.986 \\
x_7 = 1.2, &\quad y_7 = (1 + 0.2) \times 2.986 = 3.583 \\
x_8 = 1.4, &\quad y_8 = (1 + 0.2) \times 3.583 = 4.3 \\
x_9 = 1.6, &\quad y_9 = (1 + 0.2) \times 4.3 = 5.165 \\
x_{10} = 1.8, &\quad y_{10} = (1 + 0.2) \times 5.165 = 6.198
\end{aligned}$$

Graph of the Solution



The blue curve represents the exact solution to the differential equation, and the points marked with asterisks represent the approximate solution obtained using Euler's method.

Steps to Solving a Differential Equation using Euler's Method

To solve a differential equation using Euler's method, follow these steps:

1. Identify the differential equation and initial conditions.
2. Choose a step size (h).
3. Start with the initial condition and compute the next approximation using Euler's formula: $y_{i+1} = y_i + hf(x_i, y_i)$.
4. Repeat step 3 until you reach the desired x-value.
5. Plot the approximate points to visualize the solution.