Calculus principles and applications

Definitions

- **Limit:** The value that a function (or sequence) approaches as the input (or index) approaches some value.
$$\lim_{x \to a} f(x) = L$$
means that as x approaches a, $f(x)$ approaches L.
- **Series:** The sum of the terms of a sequence. If $a_1, a_2, a_3, \ldots$ is a sequence, then the series is written as
$$\sum_{n=1}^{\infty} a_n$$
- **Derivative:** A measure of how a function changes as its input changes. It is the slope of the tangent line to the function at a given point.
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- **Convergent Series:** A series whose terms approach a specific value as more terms are added. Formally, a series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $S_N = \sum_{n=1}^{N} a_n$ approaches a limit as N approaches infinity.
- **Divergent Series:** A series that does not converge, meaning its terms do not approach a specific value as more terms are added.
- **Definite Integral:** The integral of a function over a specified interval. It represents the area under the curve of the function between two points a and b.
$$\int_a^b f(x)\,dx$$
- **Indefinite Integral:** The general form of the antiderivative of a function. It represents a family of functions whose derivative is the given function.
$$\int f(x)\,dx = F(x) + C$$
where $F(x)$ is the antiderivative of $f(x)$ and C is the constant of integration.
- **Volume of Revolution:** The volume of a solid formed by revolving a region around an axis. For a function $y = f(x)$ rotated about the x-axis from $x = a$ to $x = b$, the volume is given by
$$V = \pi \int_a^b [f(x)]^2\,dx$$
- **Arc Length:** The length of a curve described by a function $y = f(x)$ from $x = a$ to $x = b$. It is calculated as
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
- **Surface Area of Revolution:** The surface area of a solid formed by revolving a curve around an axis. For a function $y = f(x)$ rotated about the x-axis from $x = a$ to $x = b$, the surface area is given by
$$A = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

- **Centre of Mass:** The point at which the entire mass of an object can be considered to be concentrated. For a region R with density function $\rho(x, y)$, the coordinates of the center of mass $(\bar{x}, \bar{y})$ are given by

$$\bar{x} = \frac{\iint_R x\rho(x, y)\, dA}{\iint_R \rho(x, y)\, dA}, \quad \bar{y} = \frac{\iint_R y\rho(x, y)\, dA}{\iint_R \rho(x, y)\, dA}$$

- **Moments:** Measures of the distribution of mass within an object. The moment about the y-axis (first moment) is given by

$$M_y = \iint_R x\rho(x, y)\, dA$$

and the moment about the x-axis is

$$M_x = \iint_R y\rho(x, y)\, dA$$

Limits and Differentiation From First Principles

Limits

In calculus, a **limit** is the value that a function (or sequence) approaches as the input (or index) approaches some value. Limits are essential for defining both derivatives and integrals.

$$\lim_{x \to a} f(x) = L$$

This notation means that as x approaches a, the function $f(x)$ approaches the value L.

Differentiation from First Principles

The derivative of a function $f(x)$ at a point $x = a$ is defined as the limit of the average rate of change of the function over a small interval as the interval approaches zero. This is given by:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Here, h is a small increment in x.

Worked Example

Consider the function $f(x) = x^2$. We want to find the derivative of $f(x)$ at any point x using the definition.

First, calculate $f(x + h)$:

$$f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$$

Then, find the difference $f(x + h) - f(x)$:

$$f(x + h) - f(x) = (x^2 + 2xh + h^2) - x^2 = 2xh + h^2$$

Now, form the difference quotient:

$$\frac{f(x + h) - f(x)}{h} = \frac{2xh + h^2}{h} = 2x + h$$

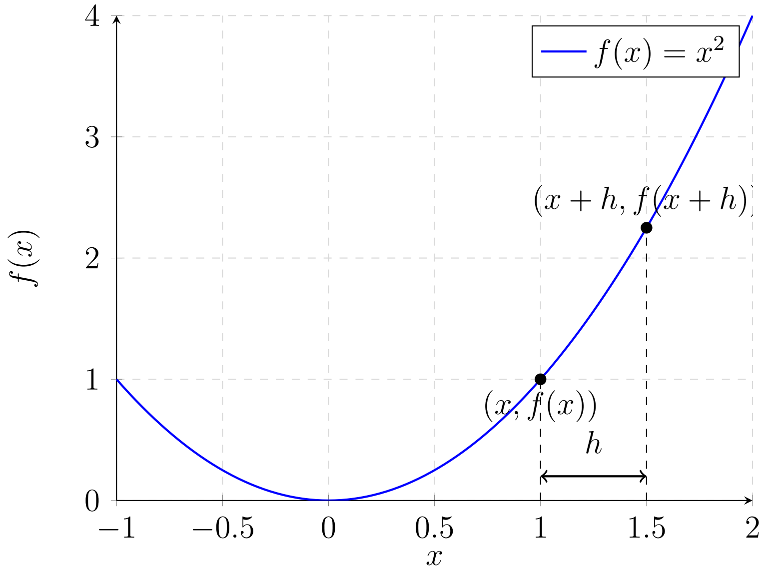
Finally, take the limit as h approaches 0:

$$f'(x) = \lim_{h \to 0}(2x + h) = 2x$$

So, the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Graphical Representation of Δx

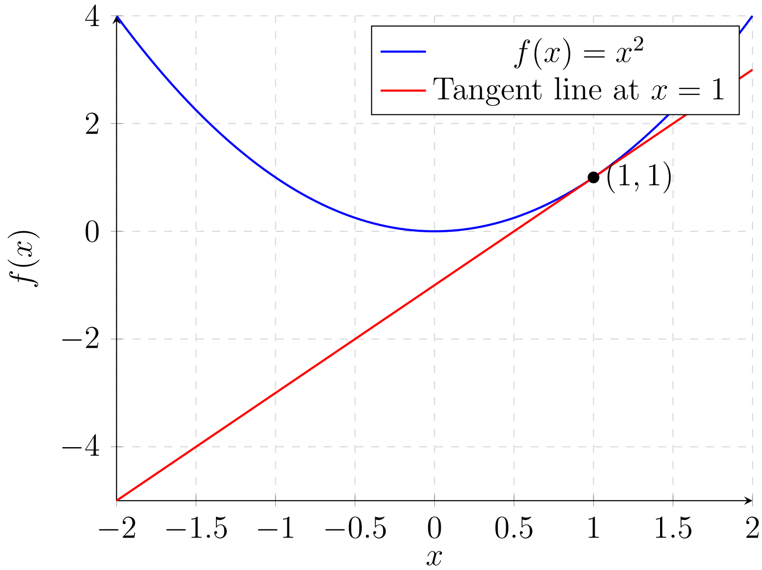
To visualize the concept of Δx (or h) as a small increment added to x, consider the following graph:



In this graph, we can see the function $f(x) = x^2$ and two points: $(x, f(x))$ and $(x + h, f(x + h))$. The distance between x and $x + h$ is the increment h, which we think of as a small increment added to x.

Graphical Representation of the Tangent Line

To further visualize differentiation, consider the function $f(x) = x^2$ and its derivative $f'(x) = 2x$. The graph below shows the function $f(x) = x^2$ and the tangent line at $x = 1$, where the slope of the tangent line is the derivative at that point.



The blue curve represents the function $f(x) = x^2$, and the red line represents the tangent at $x = 1$. The slope of this tangent line is $f'(1) = 2 \times 1 = 2$, which matches our derivative calculation.

Volumes of revolution, arc-length, surface area of revolutions

The volume of a solid of revolution is calculated by rotating a region around an axis. The formulas differ depending on whether the rotation is about the x-axis or the y-axis.

About the x-Axis

For a function $y = f(x)$ rotated about the x-axis from $x = a$ to $x = b$, the volume is given by

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

Worked Example

Find the volume of the solid obtained by rotating $y = x^2$ about the x-axis from $x = 0$ to $x = 1$.

$$V = \pi \int_0^1 (x^2)^2 \, dx = \pi \int_0^1 x^4 \, dx = \pi \left[\frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{5} - 0 \right) = \frac{\pi}{5}$$

About the y-Axis

For a function $x = g(y)$ rotated about the y-axis from $y = c$ to $y = d$, the volume is given by

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

Worked Example

Find the volume of the solid obtained by rotating $x = \sqrt{y}$ about the y-axis from $y = 0$ to $y = 1$.

$$V = \pi \int_0^1 (\sqrt{y})^2 \, dy = \pi \int_0^1 y \, dy = \pi \left[\frac{y^2}{2} \right]_0^1 = \pi \left(\frac{1}{2} - 0 \right) = \frac{\pi}{2}$$

Graphs

Arc Length

The arc length of a curve described by a function $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx$$

Worked Example

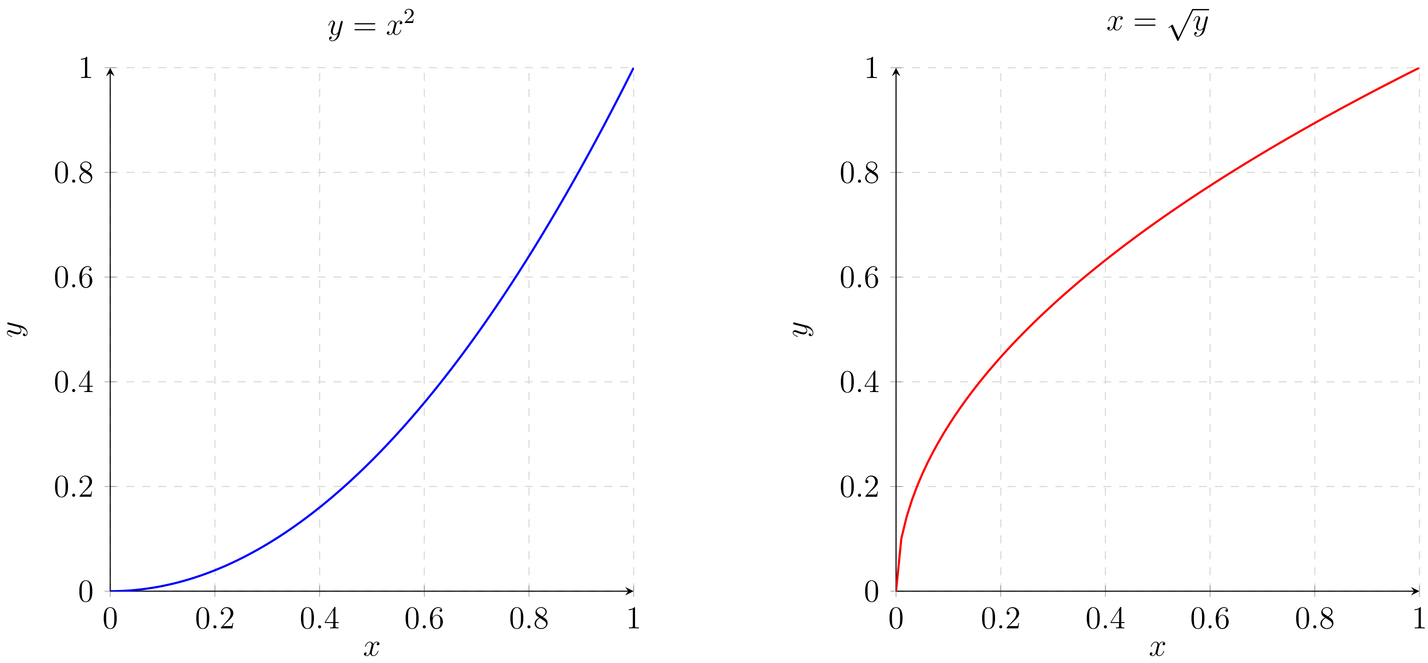
Find the arc length of the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

First, find the derivative:

$$\frac{dy}{dx} = x$$

Then, compute the arc length:

$$L = \int_0^1 \sqrt{1 + x^2} \, dx$$



(a) Rotating about the x-axis

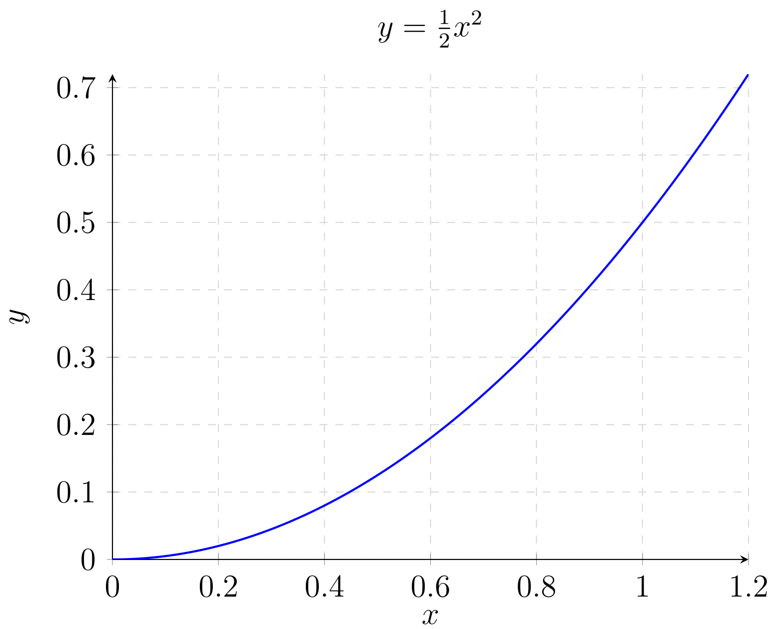
(b) Rotating about the y-axis

Figure 1: Graphs of the functions rotated to find volume

To solve this integral, we can use a trigonometric substitution, $x = \sinh(u)$, but here we'll provide the final result:

$$L = \left[\frac{1}{2}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|)\right]_0^1 = \frac{1}{2}\left(1\cdot\sqrt{2} + \ln(1+\sqrt{2})\right) - 0 = \frac{\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2}$$

Graph



Surface Area of Revolution

The surface area of a solid formed by revolving a curve around an axis is calculated as follows:

About the x-Axis

For a function $y = f(x)$ rotated about the x-axis from $x = a$ to $x = b$, the surface area is given by

$$A = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Worked Example

Find the surface area of the solid obtained by rotating $y = x^2$ about the x-axis from $x = 0$ to $x = 1$.

First, find the derivative:

$$\frac{dy}{dx} = 2x$$

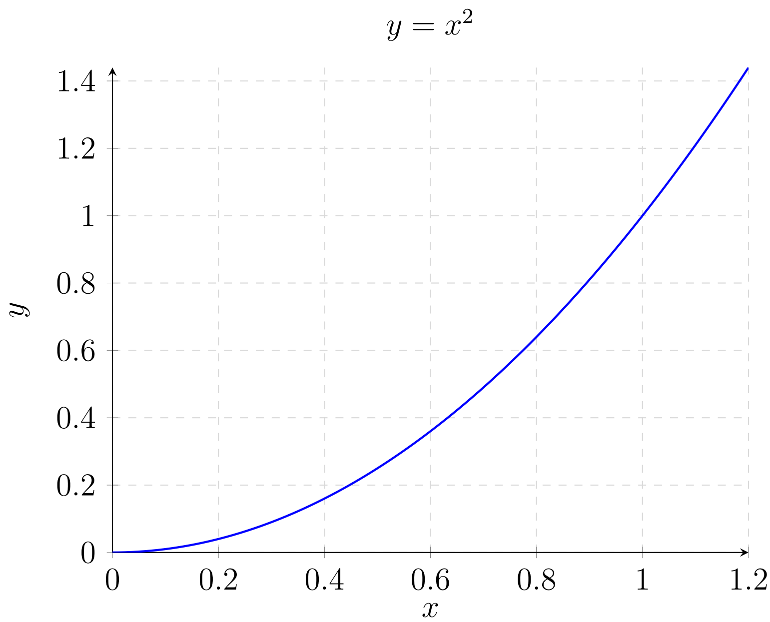
Then, compute the surface area:

$$A = 2\pi \int_0^1 x^2\sqrt{1 + (2x)^2}\, dx = 2\pi \int_0^1 x^2\sqrt{1 + 4x^2}\, dx$$

This integral is solved using substitution or numerical methods. Here, we'll provide the final result:

$$A \approx 2\pi \left(\frac{1}{12} + \frac{\sinh^{-1}(2)}{8}\right)$$

Graph



Centre Of Mass

Centre of Mass in One Dimension

The center of mass (or centroid) of a system of particles in one dimension is given by:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

where m_i is the mass of the i-th particle and x_i is the position of the i-th particle.

For a continuous mass distribution along a line, the center of mass is given by:

$$\bar{x} = \frac{\int x\, dm}{\int dm}$$

Centre of Mass of a Rectangle

Consider a rectangle with uniform density, width w, and height h.

Steps to Find the Centre of Mass

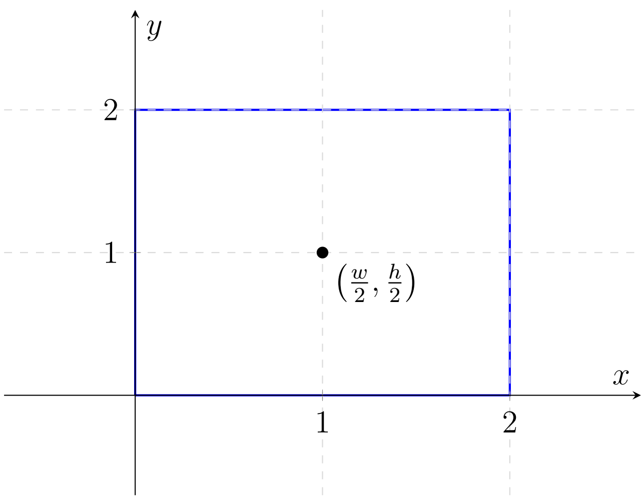
1. Set up the coordinate system with the origin at the bottom-left corner of the rectangle. 2. The area element dA is $dx\,dy$. 3. Integrate to find the center of mass:

$$\bar{x} = \frac{\int_0^w \int_0^h x\,dy\,dx}{\int_0^w \int_0^h dy\,dx} = \frac{\int_0^w xh\,dx}{wh} = \frac{h\left[\frac{x^2}{2}\right]_0^w}{wh} = \frac{h\frac{w^2}{2}}{wh} = \frac{w}{2}$$

$$\bar{y} = \frac{\int_0^w \int_0^h y\,dy\,dx}{\int_0^w \int_0^h dy\,dx} = \frac{\int_0^h yw\,dy}{wh} = \frac{w\left[\frac{y^2}{2}\right]_0^h}{wh} = \frac{w\frac{h^2}{2}}{wh} = \frac{h}{2}$$

Diagram

Centre of Mass of a Rectangle



Centre of Mass of a Circle

Consider a circle with radius R and uniform density.

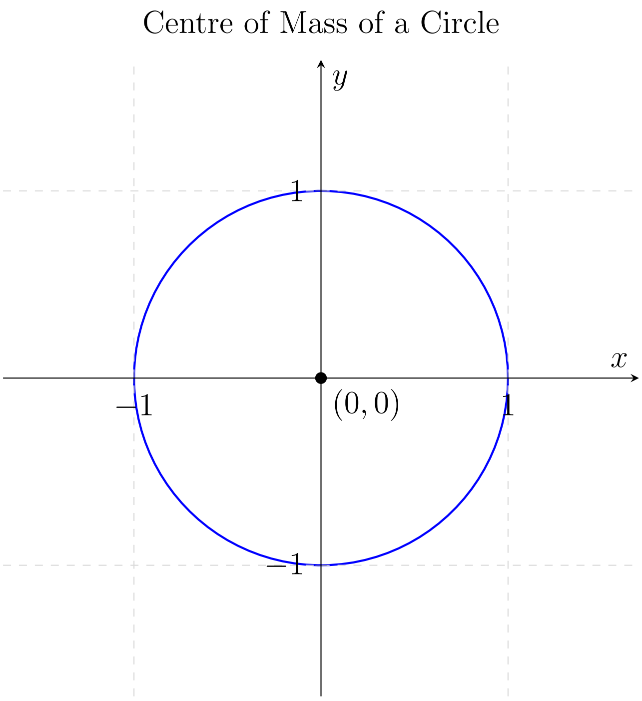
Steps to Find the Centre of Mass

1. Set up the coordinate system with the origin at the center of the circle. 2. Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. 3. Integrate to find the center of mass:

$$\bar{x} = \frac{\int_0^{2\pi} \int_0^R r\cos\theta\, r\,dr\,d\theta}{\int_0^{2\pi} \int_0^R r\,dr\,d\theta} = 0 \quad \text{(by symmetry)}$$

$$\bar{y} = \frac{\int_0^{2\pi} \int_0^R r\sin\theta\, r\,dr\,d\theta}{\int_0^{2\pi} \int_0^R r\,dr\,d\theta} = 0 \quad \text{(by symmetry)}$$

Diagram



Centre of Mass of a Triangle

Consider a triangle with vertices at $(0,0)$, $(a,0)$, and $(0,h)$ with uniform density.

Steps to Find the Centre of Mass

1. Set up the coordinate system with the origin at the bottom-left corner of the triangle. 2. The area element dA is $dx\,dy$. 3. The area of the triangle is $\frac{1}{2}ah$. 4. Integrate to find the center of mass:

$$\bar{x} = \frac{1}{\text{Area}} \int_{\text{Area}} x\,dA$$

Given the triangle's linear boundaries, this becomes:

$$\bar{x} = \frac{1}{\frac{1}{2}ah} \int_0^a \int_0^{h\left(1-\frac{x}{a}\right)} x\,dy\,dx$$

$$\begin{aligned}
\bar{x} &= \frac{2}{ah} \int_0^a [xy]_0^{h\left(1-\frac{x}{a}\right)}\,dx = \frac{2}{ah} \int_0^a xh\left(1-\frac{x}{a}\right)\,dx = \frac{2h}{ah} \int_0^a \left(x - \frac{x^2}{a}\right)\,dx \\
&= \frac{2}{a}\left[\frac{x^2}{2} - \frac{x^3}{3a}\right]_0^a = \frac{2}{a}\left(\frac{a^2}{2} - \frac{a^3}{3a}\right) = \frac{2}{a}\left(\frac{a^2}{2} - \frac{a^2}{3}\right) = \frac{2}{a} \cdot \frac{a^2}{6} = \frac{2a}{6} = \frac{a}{3}
\end{aligned}$$

$$\bar{y} = \frac{1}{\frac{1}{2}ah} \int_0^a \int_0^{h\left(1-\frac{x}{a}\right)} y\,dy\,dx = \frac{2}{ah} \int_0^a \left[\frac{y^2}{2}\right]_0^{h\left(1-\frac{x}{a}\right)}\,dx = \frac{2}{ah} \int_0^a \frac{h^2}{2}\left(1-\frac{x}{a}\right)^2\,dx$$

$$= \frac{h^2}{ah} \int_0^a \left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right)\,dx = \frac{h}{a}\left[x - \frac{x^2}{a} + \frac{x^3}{3a^2}\right]_0^a = \frac{h}{a}\left(a - \frac{a^2}{a} + \frac{a^3}{3a^2}\right) = \frac{h}{a}\left(a - a + \frac{a}{3}\right)$$

$$= \frac{h}{a} \cdot \frac{a}{3} = \frac{h}{3}$$

Diagram

Centre of Mass of a Triangle