Functions and Graphs

A function f is a rule that assigns a unique output value $f(x)$ to each input value x in the domain of the function. The domain is the set of input values for which the function is defined, and the range is the set of possible output values.
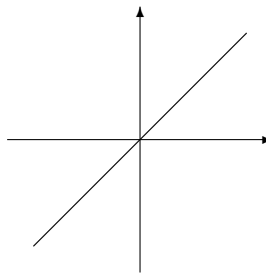
Functions can be represented in various ways, including:

- Algebraic expressions: $f(x) = x^2 + 2x - 3$

- Tables of values:

x	$f(x)$
0	3
1	4
2	7

- Or as a graph (y = x):

Operations on Functions

There are various operations that can be performed on functions:

1. **Vertical Stretch/Shrink**: $g(x) = af(x)$, where a is a constant. This operation stretches or shrinks the graph of $f(x)$ vertically by a factor of a.

2. **Horizontal Stretch/Shrink**: $h(x) = f(bx)$, where b is a constant. This operation stretches or shrinks the graph of $f(x)$ horizontally by a factor of $\frac{1}{b}$.

3. **Reflection**: $r(x) = -f(x)$ reflects the graph of $f(x)$ about the x-axis, and $s(x) = f(-x)$ reflects the graph about the y-axis.

4. **Translation**: $t(x) = f(x - c) + d$ translates the graph of $f(x)$ horizontally by c units and vertically by d units.

5. **Inverse Function**: If f is a one-to-one function, its inverse f^{-1} is defined such that $f^{-1}(f(x)) = x$ for all x in the domain of f. The inverse function "undoes" the operation of the original function.

6. **Composition**: $(f \circ g)(x) = f(g(x))$ is the composition of functions f and g, where the output of $g(x)$ becomes the input of f.

Function Composition

The composition of two functions f and g, denoted by $f \circ g$, is a new function defined by $(f \circ g)(x) = f(g(x))$. In other words, the output of $g(x)$ becomes the input of f.

For example, let $f(x) = x^2 + 1$ and $g(x) = 2x - 3$. Then, the composition $(f \circ g)(x)$ is given by:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x - 3) \\ &= (2x - 3)^2 + 1 \\ &= 4x^2 - 12x + 10\end{aligned}$$

Domain and Range

The domain of a function is the set of input values for which the function is defined, and the range is the set of possible output values.

For the composed function $(f \circ g)(x)$, the domain consists of all values of x for which both $g(x)$ and $f(g(x))$ are defined.

Consider the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 - 4$. The domain of $g(x)$ is $\mathbb{R}$ (all real numbers), but the domain of $f(x)$ is $x \geq 0$ (non-negative real numbers).

The graph of $(f \circ g)(x)$ is shown below, with the domain and range shaded:
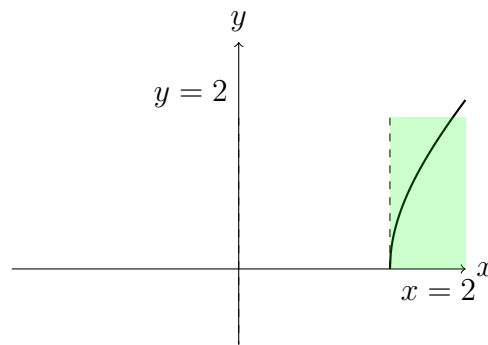
The domain of $(f \circ g)(x)$ is $x \geq 2$, and the range is $y \geq 0$.

Inverse Functions

The inverse of a function f, denoted by f^{-1}, is a function that "undoes" the operation of f. That is, $f^{-1}(f(x)) = x$ for all x in the domain of f.

To find the inverse of a function $f(x)$, we follow these steps:

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y.
4. Denote the resulting function as $f^{-1}(x)$.

For example, consider the function $f(x) = 2x + 3$.

1. Replace $f(x)$ with y: $y = 2x + 3$.
2. Interchange x and y: $x = 2y + 3$.
3. Solve for y: $2y = x - 3$, so $y = \frac{x-3}{2}$.
4. The inverse function is $f^{-1}(x) = \frac{x-3}{2}$.

To verify, we can check that $f^{-1}(f(x)) = x$:

$$f^{-1}(f(x)) = f^{-1}(2x+3)$$
$$= \frac{2x+3-3}{2}$$
$$= x$$

Hence, the inverse function correctly "undoes" the original function.

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Logarithmic Functions

The logarithmic function with base a is defined as:

$$\log_a(x) = y \quad \text{if and only if} \quad a^y = x$$

The natural logarithm (logarithm with base e) is denoted as $\ln(x)$:

$$\ln(x) = y \quad \text{if and only if} \quad e^y = x$$

Exponential Functions

The exponential function with base e is written as:

$$e^x$$

An exponential function with base a is written as:

$$a^x$$

Operations Involving Logarithmic and Exponential Functions

Addition:
$$\log_a(x) + \log_a(y) = \log_a(xy)$$
$$\ln(x) + \ln(y) = \ln(xy)$$

Subtraction:
$$\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$$
$$\ln(x) - \ln(y) = \ln\left(\frac{x}{y}\right)$$

Multiplication:
$$a \cdot \log_b(x)$$
$$a \cdot \ln(x)$$

Division:
$$\frac{\log_a(x)}{\log_a(y)} = \log_y(x)$$
$$\frac{\ln(x)}{\ln(y)}$$

Powers:
$$\log_a(x^y) = y \log_a(x)$$
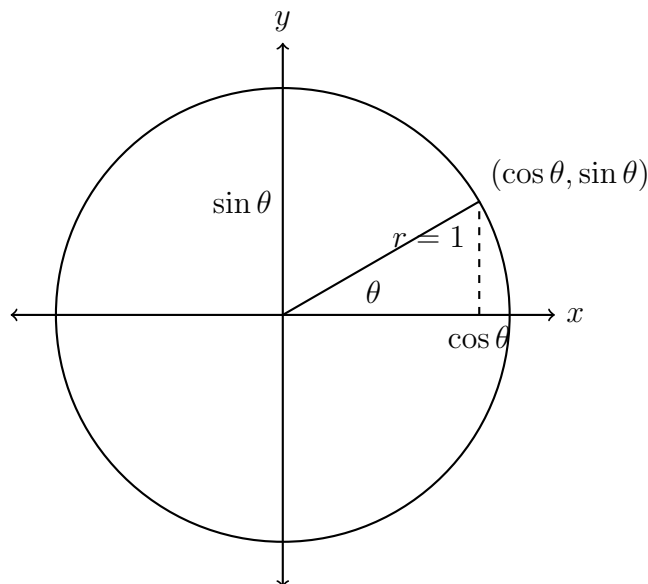$$\ln(x^y) = y \ln(x)$$

Below are the graphs of the natural logarithm function $\ln(x)$ and the exponential function e^x (which are the inverses of each other):
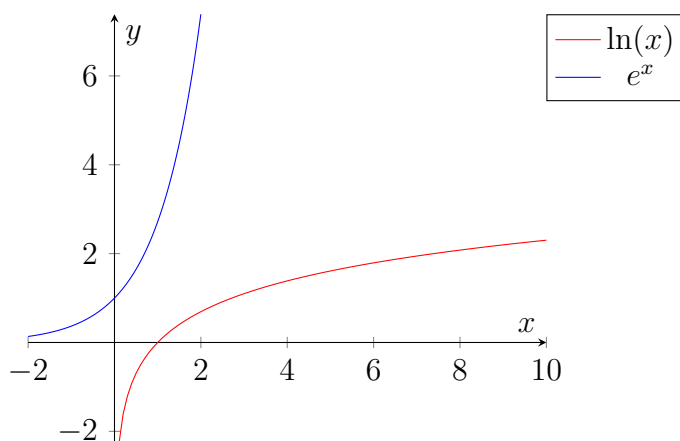
TRIGONOMETRIC FUNCTIONS
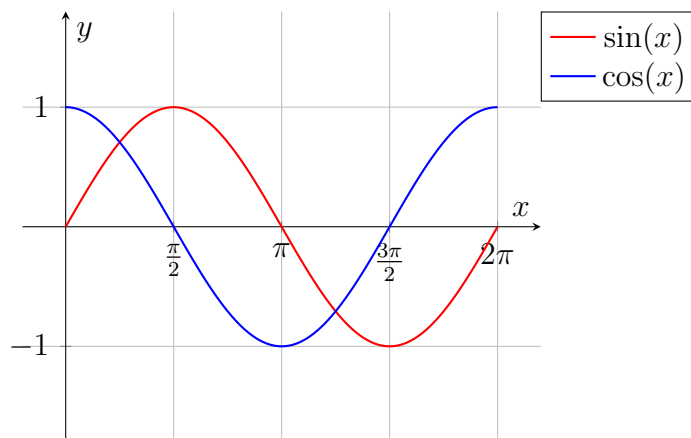
Derivation from a unit circle

Definitions

On the unit circle, for an angle θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = y$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = x$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Graphs of $\sin(x)$ and $\cos(x)$

Trigonometric Identities

Pythagorean Identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

Quotient Identities:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Reciprocal Identities:

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

Differentiation and Integration of Trigonometric Functions

Differentiation:

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

Integration:

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = -\ln|\cos x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

Inverse Functions

The inverse sine function $\sin^{-1}(x)$ (or $\arcsin(x)$) is derived as follows:

$$y = \sin^{-1}(x) \implies \sin(y) = x$$

Differentiating implicitly:

$$\cos(y)\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Using $\cos^2(y) = 1 - \sin^2(y)$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Thus,

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

The inverse cosine function $\cos^{-1}(x)$ (or $\arccos(x)$) is derived similarly:

$$y = \cos^{-1}(x) \implies \cos(y) = x$$

Differentiating implicitly:

$$-\sin(y)\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = -\frac{1}{\sin(y)}$$

Using $\sin^2(y) = 1 - \cos^2(y)$:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Thus,

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

Transformations of Trigonometric Functions

Consider the function $y = \sin(x)$. The following transformations can be applied:

1. Vertical Translation:

$$y = \sin(x) + k$$

Shifts the graph vertically by k units.

2. Horizontal Translation:

$$y = \sin(x - h)$$

Shifts the graph horizontally by h units.

3. Vertical Stretch/Compression:
$$y = A \sin(x)$$

Stretches the graph vertically by a factor of $|A|$. If $|A| > 1$, it is a stretch; if $0 < |A| < 1$, it is a compression.

4. Horizontal Stretch/Compression:
$$y = \sin(Bx)$$

Compresses the graph horizontally by a factor of $|B|$. If $|B| > 1$, it is a compression; if $0 < |B| < 1$, it is a stretch.

5. Reflection:
$$y = -\sin(x)$$

Reflects the graph across the x-axis.

6. Phase Shift (Horizontal Translation):
$$y = \sin(x - C)$$

Shifts the graph horizontally by C units.

7. Composite Functions:
$$y = \sin(f(x))$$

Applies a function $f(x)$ to the argument of the sine function, modifying its frequency, phase, or amplitude.

Example of Transformations

The function $y = 2\sin(3x - \frac{\pi}{4}) + 1$ includes several transformations: - Amplitude is 2 (vertical stretch). - Period is $\frac{2\pi}{3}$ (horizontal compression by factor of 3). - Phase shift is $\frac{\pi}{4}$ units to the right. - Vertical shift is 1 unit up.
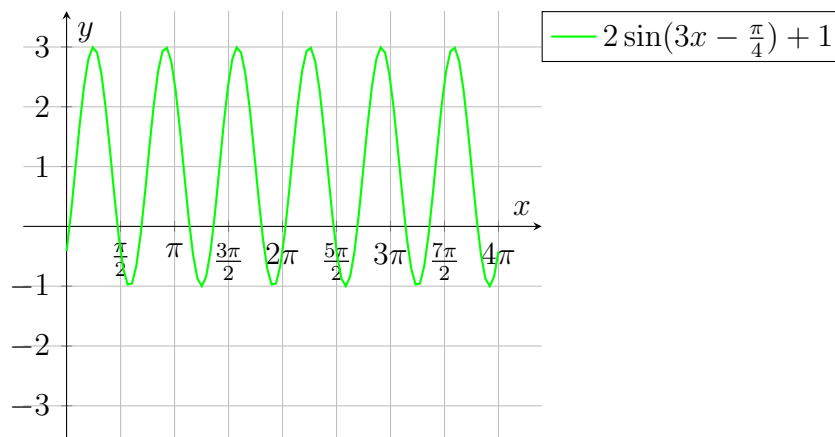
Even and Odd Functions

Definitions

A function $f(x)$ is called **even** if:
$$f(-x) = f(x) \quad \text{for all } x \text{ in the domain of } f.$$

Even functions are symmetric about the y-axis.

A function $f(x)$ is called **odd** if:
$$f(-x) = -f(x) \quad \text{for all } x \text{ in the domain of } f.$$

Odd functions are symmetric about the origin.

Examples of Even and Odd Functions

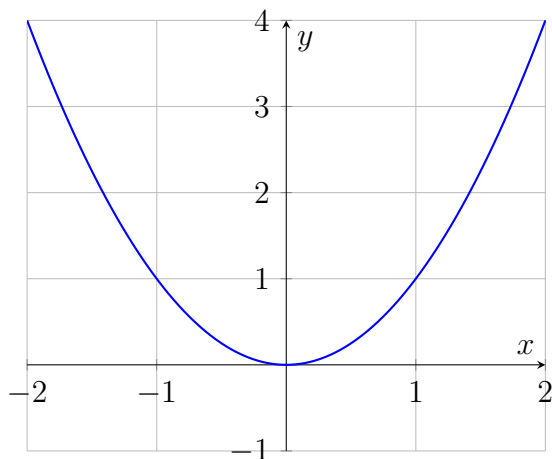Examples of even functions include:

$$f(x) = x^2, \quad f(x) = \cos(x)$$
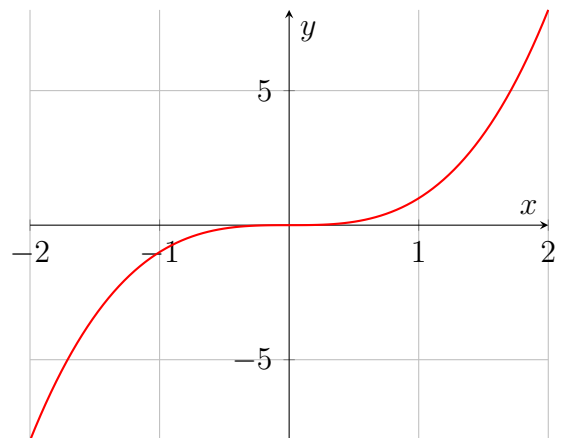
Examples of odd functions include:

$$f(x) = x^3, \quad f(x) = \sin(x)$$
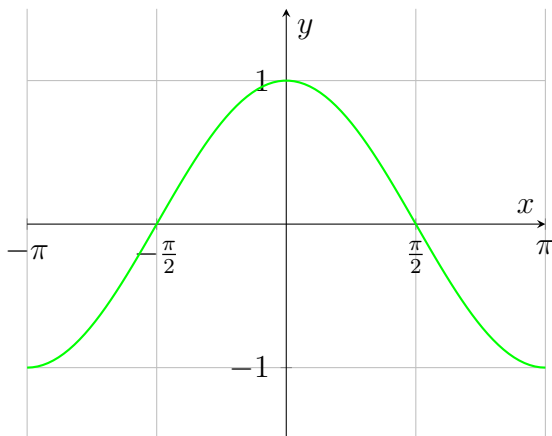
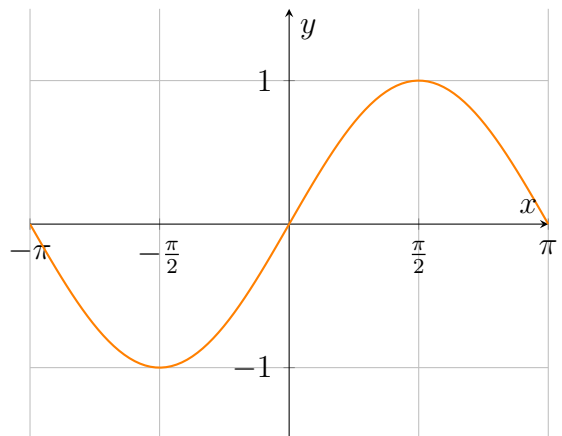## Graphs of Even and Odd Functions

Even Function: $f(x) = x^2$

Odd Function: $f(x) = x^3$

Even Function: $f(x) = \cos(x)$

Odd Function: $f(x) = \sin(x)$

GRAPH SKETCHING

1. Finding Intercepts

Finding x-intercepts

To find the x-intercepts, set $y = 0$ and solve for x.

Finding y-intercepts

To find the y-intercepts, set $x = 0$ and solve for y.

Example:

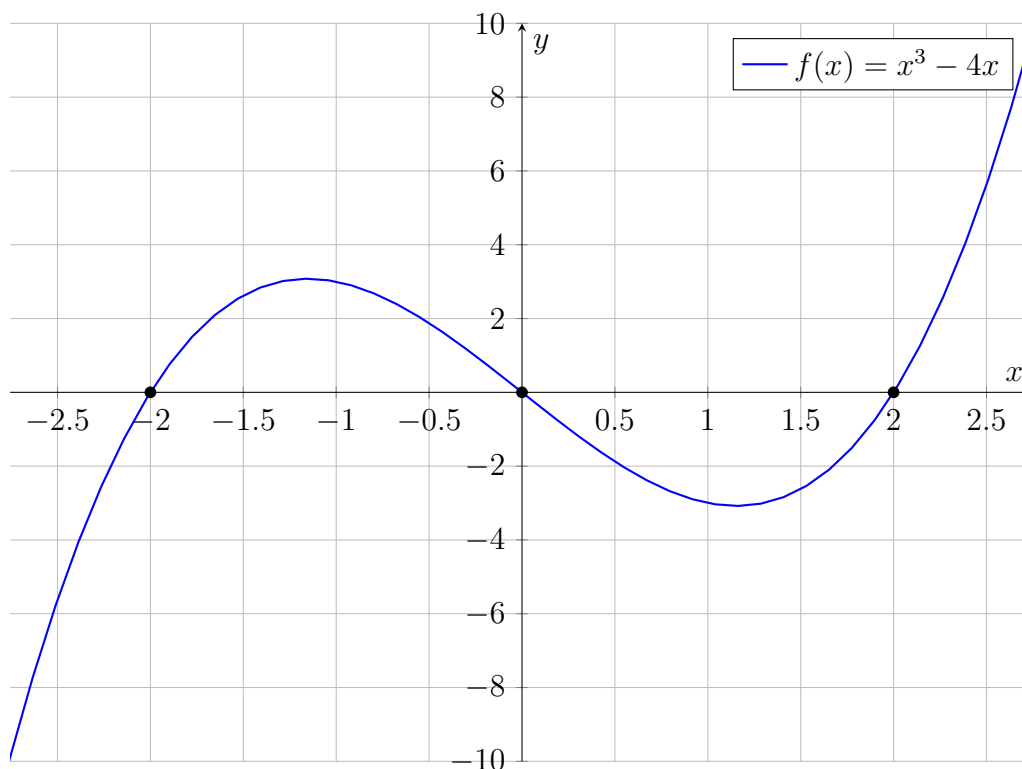Given the function $f(x) = x^3 - 4x$:

y-intercept:

$$f(0) = 0^3 - 4 \cdot 0 = 0$$

So, the y-intercept is $(0, 0)$.

x-intercepts:

$$x^3 - 4x = 0 \implies x(x^2 - 4) = 0 \implies x(x - 2)(x + 2) = 0$$

So, the x-intercepts are $x = 0, 2, -2$.

2. Finding Stationary Points

To find stationary points, differentiate the function and set the derivative to zero.

$$f'(x) = \frac{d}{dx}(x^3 - 4x) = 3x^2 - 4$$

$$3x^2 - 4 = 0 \implies x^2 = \frac{4}{3} \implies x = \pm\frac{2}{\sqrt{3}}$$

To find the corresponding y-values:

$$f\left(\frac{2}{\sqrt{3}}\right) = \left(\frac{2}{\sqrt{3}}\right)^3 - 4 \cdot \frac{2}{\sqrt{3}} = \frac{8}{3\sqrt{3}} - \frac{8\sqrt{3}}{3} = \frac{8 - 8\sqrt{3}}{3\sqrt{3}}$$

$$f\left(-\frac{2}{\sqrt{3}}\right) = \left(-\frac{2}{\sqrt{3}}\right)^3 - 4 \cdot \left(-\frac{2}{\sqrt{3}}\right) = -\frac{8}{3\sqrt{3}} + \frac{8\sqrt{3}}{3} = \frac{-8 + 8\sqrt{3}}{3\sqrt{3}}$$

So, the stationary points are $\left(\frac{2}{\sqrt{3}}, \frac{8-8\sqrt{3}}{3\sqrt{3}}\right)$ and $\left(-\frac{2}{\sqrt{3}}, \frac{-8+8\sqrt{3}}{3\sqrt{3}}\right)$.

3. Finding Inflection Points

To find inflection points, differentiate the function twice and set the second derivative to zero.

$$f''(x) = \frac{d^2}{dx^2}(x^3 - 4x) = \frac{d}{dx}(3x^2 - 4) = 6x$$
$$6x = 0 \implies x = 0$$

To find the corresponding y-value:
$$f(0) = 0^3 - 4 \cdot 0 = 0$$

So, the inflection point is $(0, 0)$.

4. Asymptotes

Vertical Asymptotes

Vertical asymptotes occur where the function approaches infinity as x approaches a certain value.

Horizontal Asymptotes

Horizontal asymptotes occur where the function approaches a constant value as x approaches infinity.

Slant Asymptotes

Slant (oblique) asymptotes occur when the polynomial in the numerator is of a higher degree than the polynomial in the denominator.

Example:

Given the function $f(x) = \frac{x^2+1}{x-1}$:

Vertical Asymptotes:

Set the denominator to zero:
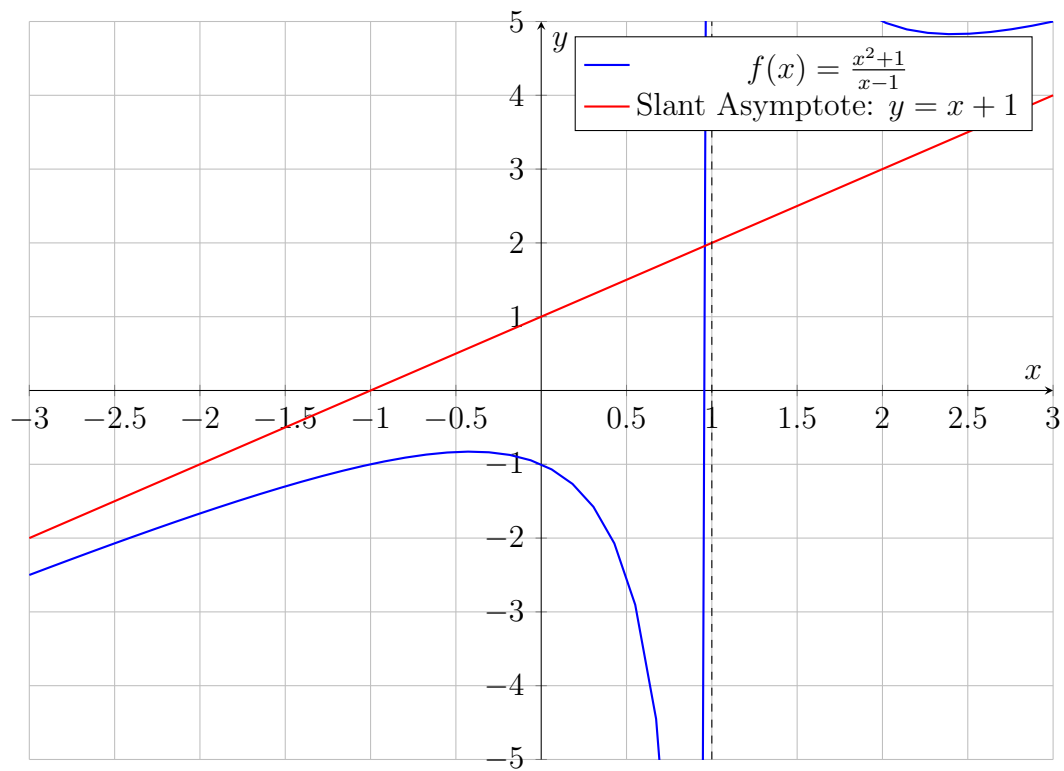$$x - 1 = 0 \implies x = 1$$

Horizontal Asymptotes:

Compare the degrees of the numerator and the denominator. Since the degree of the numerator is 2 and the degree of the denominator is 1, there is no horizontal asymptote.

Slant Asymptotes:

Perform polynomial division of $x^2 + 1$ by $x - 1$:

$$\frac{x^2 + 1}{x - 1} = x + 1 + \frac{2}{x - 1}$$

So, the slant asymptote is $y = x + 1$.

Summary of Steps for Graph Sketching

1. Find the intercepts.
2. Find the stationary points by differentiating and setting the derivative to zero.
3. Find the inflection points by differentiating twice and setting the second derivative to zero.
4. Identify asymptotes:
- Vertical asymptotes: Set the denominator to zero.
- Horizontal asymptotes: Compare the degrees of the numerator and denominator.
- Slant asymptotes: Perform polynomial division if the degree of the numerator is higher than that of the denominator.