Fourier Series

Fourier Series Forms and Calculations

The Fourier series is a way to represent a periodic function as the sum of simple sine and cosine functions.

Definition

A periodic function $f(x)$ with period $2L$ can be represented by a Fourier series as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where the coefficients a_0, a_n, and b_n are given by:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

Complex Form

The Fourier series can also be expressed in complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where c_n are the Fourier coefficients given by:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}}\, dx$$

General Interval and Half Range Forms

For a function $f(x)$ defined on an interval (a, b), we can extend it to a periodic function with period $2L = b - a$, and the Fourier series becomes:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_0 = \frac{1}{L} \int_{a}^{b} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{a}^{b} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L} \int_{a}^{b} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

For a function $f(x)$ defined on $[0, L]$, the Fourier series can be represented as a half-range series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

Worked Example

Let's find the Fourier series of the function $f(x) = x$ defined on $[-\pi, \pi]$.

Step 1: Calculate a_0

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi} = 0$$

Step 2: Calculate a_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\, dx = 0$$

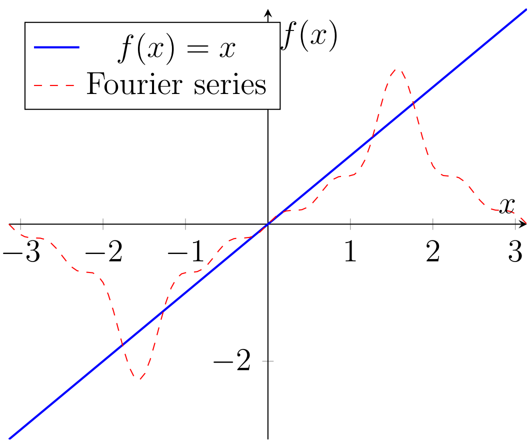
Step 3: Calculate b_n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{2}{\pi} \left[-\frac{x \cos(nx)}{n} \right]_{-\pi}^{\pi} = \frac{2}{\pi} \left(\frac{(-1)^{n+1} - (-1)^{n+1}}{n} \right) = \frac{4}{n\pi}(-1)^{n+1}$$

Step 4: Construct the Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1} \sin(nx)$$

Now, let's plot the function and its Fourier series.



As you can see by the curves, the Fourier Series approximation doesn't work particularly well outside of the range [-1:1].

Complex Form of Fourier Series Example

The complex form of the Fourier series represents a periodic function as the sum of complex exponential functions.

Definition

The complex form of the Fourier series of a periodic function $f(x)$ with period $2L$ is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where the coefficients c_n are given by:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}\,dx$$

Worked Example

Let's find the complex form of the Fourier series of the function $f(x) = \begin{cases} 1, & \text{if } -\pi < x < \pi \\ 0, & \text{otherwise} \end{cases}$.

Step 1: Calculate c_n

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\,dx$$

Using the result of the Fourier series for $e^{i\theta}$:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we have:

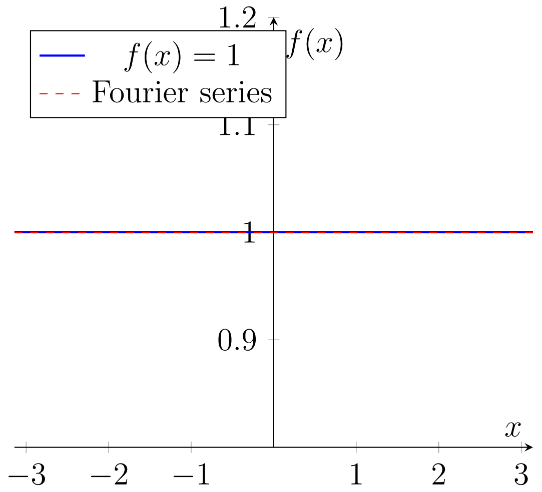
$$c_n = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{e^{in\pi} - e^{-in\pi}}{-in}\right) = \frac{1}{2\pi}\left(\frac{(-1)^n - (-1)^n}{-in}\right) = \frac{1}{2\pi in}(1-1) = 0$$

for all $n \neq 0$. For $n = 0$, $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx = \frac{1}{2\pi}(2\pi) = 1$.

So, the complex form of the Fourier series is:

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} = 1$$

Now, let's plot the function and its Fourier series.



This concludes the worked example and explanation of the complex form of Fourier series.