IX1501 HT23 Project 1

Course code: IX1501
Date: 2023-09-04

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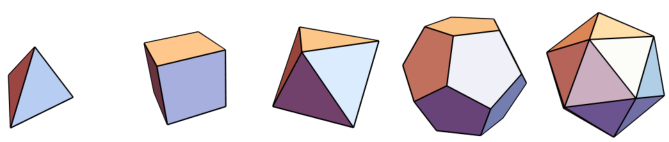
Probability distribution of a sum of independent random variables

Summary



In this project we analyzed the possible sum probabilities when playing a dice game where one throws 5 platonic dice and sums together that number on each dice. To this we used a convolution sum and created a probability distribution table.

We analyzed the probability of the win condition $P(S \leq 10 \text{ or } S \geq 45)$, where S is the sum of the 5 dice. We evaluated it mathematically to be $P_{\text{win}} = \frac{41}{3840} = 1.068\%$. We also used a Monte Carlo simulation to approximate the probability with 1000 throws, $P_{\text{win}} = 1.1\%$. Then we tried the Monte Carlo simulation for different number of trials and the conclusion of that is that the probability approximation gets more accurate as the number of trials increases.

We also tried to find the number of trials required so that the relative error between the simulated probability and real probability was less then 10%. We came to the conclusion that with 46080 trials, the number of possible outcomes in the dice game, we will with high probability get a relative error below 10%.

Method

Task 1: Determine the probability function of S

We will first define the probability distributions for each of the Platonic Solids. We will assume that each die is fair, meaning that each outcome is equally likely. Here are the probability distributions for each die:

- Tetrahedron (4-sided die)
  Probability of each outcome (1 to 4) is 1/4.
- Hexahedron (6-sided die)
  Probability of each outcome (1 to 6) is 1/6.
- Octahedron (8-sided die)
  Probability of each outcome (1 to 8) is 1/8.
- Dodecahedron (12-sided die)
  Probability of each outcome (1 to 12) is 1/12.
- Icosahedron (20-sided die)
  Probability of each outcome (1 to 20) is 1/20.

Now, we can calculate the convolution of these probability distributions to find the probability distribution of the sum S.

Task 2: Determine the probability of winning the game

To determine the probability of winning the game, we need to sum the probabilities of all values of S where S <= 10 or S >= 45.

Task 3: Obtain the probability of winning the game with 1000 trials (Monte Carlo simulation)

In this task, we will simulate throwing the five Platonic Solid dice 1000 times and calculate the proportion of simulations where we win. The steps are as follows:

- Simulate rolling each of the five dice in each trial and calculate the sum S for each trial.
- Count the number of trials where S satisfies the winning condition (S <= 10 or S >= 45).
- Calculate the probability of winning as the ratio of the number of winning trials to the total number of trials (1000 in this case).

Task 4: Discuss how the probability changes with the number of trials and provide a figure.

To observe how the probability changes with the number of trials, we can perform the simulation for various numbers of trials, ranging from a small number, 2, to a large number, 2^{18}. Then we can create a plot to visualize this change.

Task 5: Determine the number of trials needed to keep the relative error below 10%. Provide reasoning.

To determine the number of trials needed to achieve a relative error value below 10% between the exact probability and the simulation probability, we can perform Monte Carlo simulations with increasing numbers of trials until we get a relative error value that is less than 10%. Since the simulation result can vary, the aforementioned test is repeated a number of times so that we can calculate the average number of trials required to get a relative error value that is less than 10%.

We also analyze the probability of getting an error value less than 10% for different number of trials, this is done by picking a trial number to use for the Monte Carlo simulation and then repeating the simulation a certain amount of times while keeping track of the number of simulations that satisfies our condition. By doing this we approximate the probability of the relative error rate being less than 10% for different number of trials.

Result

- Determine the probability function of S. Express it as a form of a table.

S	P(S=s)
5	$\frac{1}{46\,080}$
6	$\frac{1}{9216}$
7	$\frac{1}{3072}$
8	$\frac{7}{9216}$
9	$\frac{23}{15\,360}$
10	$\frac{121}{46\,080}$
11	$\frac{97}{23\,040}$
12	$\frac{29}{4608}$
13	$\frac{409}{46\,080}$
14	$\frac{61}{5120}$
15	$\frac{707}{46\,080}$
16	$\frac{293}{15\,360}$
17	$\frac{53}{2304}$
18	$\frac{311}{11\,520}$
19	$\frac{95}{3072}$
20	$\frac{1597}{46\,080}$
21	$\frac{39}{1024}$
22	$\frac{379}{9216}$
23	$\frac{1007}{23\,040}$
24	$\frac{211}{4608}$
25	$\frac{1091}{23\,040}$
26	$\frac{223}{4608}$
27	$\frac{1127}{23\,040}$
28	$1127/$[illegible]

23 040

29	$\frac{223}{4608}$
30	$\frac{1091}{23\,040}$
31	$\frac{211}{4608}$
32	$\frac{1007}{23\,040}$
33	$\frac{379}{9216}$
34	$\frac{39}{1024}$
35	$\frac{1597}{46\,080}$
36	$\frac{95}{3072}$
37	$\frac{311}{11\,520}$
38	$\frac{53}{2304}$
39	$\frac{293}{15\,360}$
40	$\frac{707}{46\,080}$
41	$\frac{61}{5120}$
42	$\frac{409}{46\,080}$
43	$\frac{29}{4608}$
44	$\frac{97}{23\,040}$
45	$\frac{121}{46\,080}$
46	$\frac{23}{15\,360}$
47	$\frac{7}{9216}$
48	$\frac{1}{3072}$
49	$\frac{1}{9216}$
50	$\frac{1}{46\,080}$

- Determine the probability of winning the game.

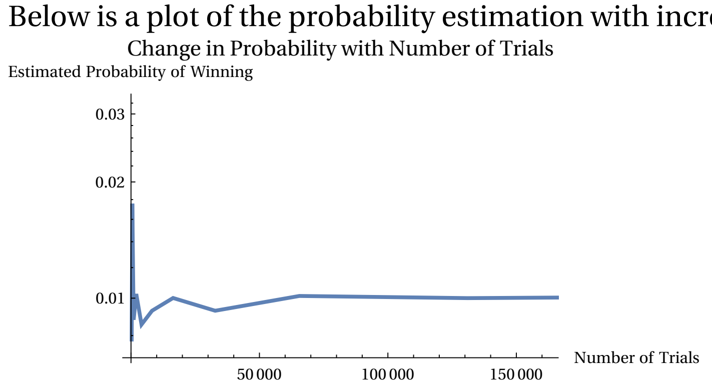
  $\frac{41}{3840}$

- Obtain the probability of winning the game with 1000 trials.

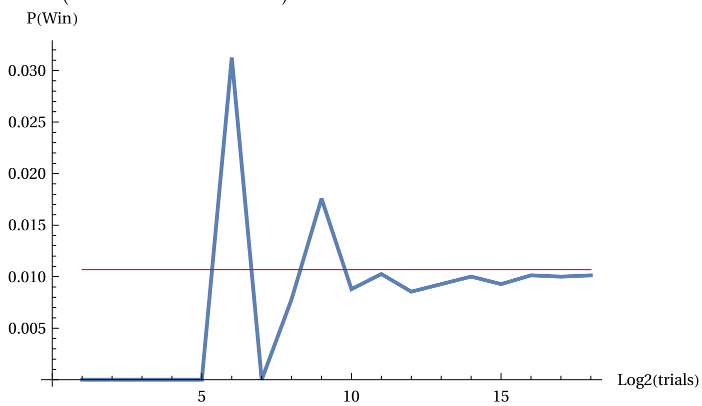
  $\frac{11}{1000}$

- Discuss how the probability changes with the number of trials and provide a figure(Results).

  Below is a plot of the probability estimation with increasing number of trials.



Below is another plot. But it is instead of how the probability estimation changes with $\log_2$(number of trials). The red line is the actual probability value as calculated in task 2.



- Determine the number of trials needed to keep the relative error below 10%. Provide reasoning.

Average number of trials across multiple Monte Carlo simulations before reaching a relative error less than 10% is 2228 trials.

Approximation of probabilities of a relative error rate of less than 10% for different number of trials:

trials	P(relative error < 10)
1000	27%
2228	41%
5000	53%
10 000	62%
20 000	85%
46 080	99%

Discussion

Task 1

The problem statement can be described as:

$$S = X_4 + X_6 + X_8 + X_{12} + X_{20}$$

Where $X_i =$ The numbered side we get when throwing an i sided platonic dice.
To get the probability distribution of S we convolve the probability distributions of each X_i.

Task 2

To determine the probability of $P_s(S \leq 10 \cup S \geq 45)$ we realize that the set of elements in the sample space which leads to $S(u) = s$ is unique for each possible s. Basically all the events in the sample space are disjoint.

And by Kolmogorov’s axiom the probability is essentially this sum of probability distributions:

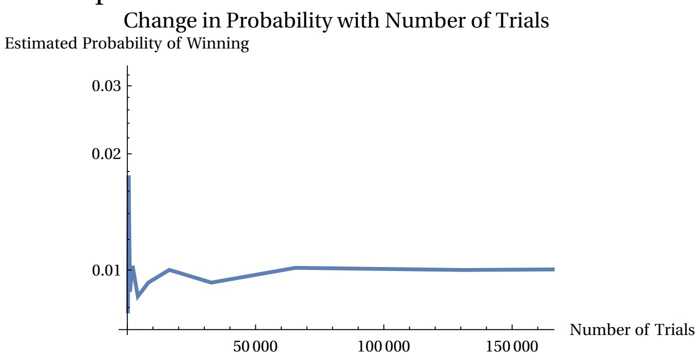
$$P_s\left(S \leq 10 \cup S \geq 45\right) = P_s(S \leq 10) + P_s(S \geq 45)$$

Task 3 & 4

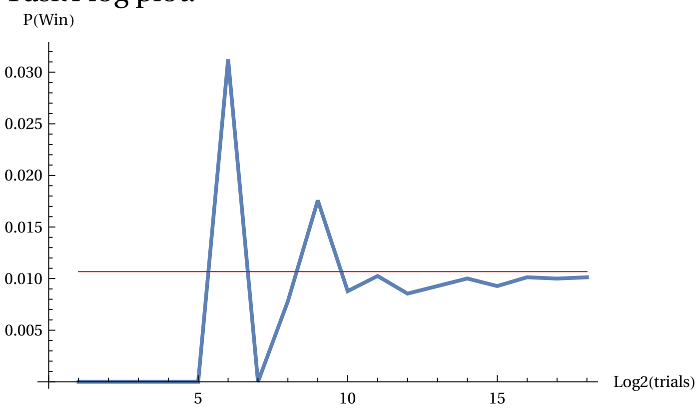
The idea behind a Monte Carlo simulation is to approximate the probability of a certain event rather than mathematically evaluate it. This can be useful if the evaluation is complex or difficult, or if one simply does not know how to evaluate it. The accuracy of the probability approximation increases as the number of trials increases. And as the number of trials become arbitrarily large $N \to \infty$ then $P_{\text{montecarlo}} = P$, the approximation will equal the actual probability.

For task 4 to investigate how the probability changes with different numbers of trials, we can repeat Task 3 with varying numbers of trials , 2 to 2^{18} and observe the results.

Task 4 plot:



Task4 log plot:



As can be seen by the plots in Task 4 that with increasing number of trials we get an increasingly accurate probability estimation, As expected by a Monte Carlo simulation.

Task 5

The Monte Carlo principle can be described as:

If N is the size of the sample space, number of simulations, and m is the number of simulations that satisfy a certain condition C. And if we know the true probability of C. Then $P(C) \approx m / N$ and if $N \to \infty$ then $P(C) = m / N$.

The relative error value can be described as: $E = \left| P(C) - m / N \right| \big/ P(C)$. This implies that if $N \to \infty$ then $E = \left| P(C) - P(C) \right| / P(C) = 0$. Thus the relative error decreases as the number of simulations increases.

To find an exact number of trials which will satisfy that the relative error is less than 10% is impossible, the number of trials will vary since the trials are random. Only scenario where you can guarantee it is less than 10% is when $N \to \infty$. However we can try to find the number of trials which will with high probability give us a relative error rate that is less than 10%.

We approximated the on average number of trials required to satisfy the error rate condition, that number is 2228. This tells us that on average across multiple Monte Carlo simulations, it will require about 2228 trials to get an error rate condition less than 10%.

Then we approximated the probabilities of a relative error rate of less than 10% for different number of trials. Below are the results:

trials	P(relative error < 10)
1000	27%
2228	41%
5000	53%
10 000	62%
20 000	85%
46 080	99%

As we can see the on average 2228 trials required before a Monte Carlo simulation has a relative error rate of less than 10% has a quite poor probability of having an error rate of less than 10% at 41%. The number of required trials to get a satisfying error rate with high probability ($\geq$ 99%) is 46080, which is interesting since that is the number of possible outcomes when throwing 5 platonic solids.

Code

Task 1: Determine the probability function of S

Define the probability distributions for each die

In[∘]:=
```
probabilitiesTetrahedron = Table[1/4, {i, 1, 4}];
probabilitiesHexahedron = Table[1/6, {i, 1, 6}];
probabilitiesOctahedron = Table[1/8, {i, 1, 8}];
probabilitiesDodecahedron = Table[1/12, {i, 1, 12}];
probabilitiesIcosahedron = Table[1/20, {i, 1, 20}];
```

Convolve the probability distributions to get the probability distribution of S

In[∘]:=
```
convolvedDistribution1 =
  ListConvolve[probabilitiesTetrahedron, probabilitiesHexahedron, {1, -1}, 0];
convolvedDistribution2 =
  ListConvolve[probabilitiesOctahedron, probabilitiesDodecahedron, {1, -1}, 0];
convolvedDistributionComb =
  ListConvolve[convolvedDistribution1, convolvedDistribution2, {1, -1}, 0];
totalPdf =
  ListConvolve[convolvedDistributionComb, probabilitiesIcosahedron, {1, -1}, 0];
```

Create a table of probabilities for each possible sum S

In[∘]:=
```
probabilityTable = Table[{s, totalPdf[[s - 4]]}, {s, 5, 50}];
```

Display the probability table

In[∘]:=
```
TableForm[probabilityTable, TableHeadings → {None, {"S", "P(S=s)"}}]
```

Out[∘]//TableForm=

S	P(S=s)
5	$\frac{1}{46080}$
6	$\frac{1}{9216}$
7	$\frac{1}{3072}$
8	$\frac{7}{9216}$
9	$\frac{23}{15360}$
10	$\frac{121}{46080}$
11	$\frac{97}{23040}$
12	$\frac{29}{4608}$
13	$\frac{409}{46080}$
14	$\frac{61}{5120}$
15	$\frac{707}{46080}$
16	$\frac{293}{15360}$
17	$\frac{53}{2304}$
18	$\frac{311}{11520}$
19	$\frac{95}{3072}$

20	$\frac{1597}{46080}$
21	$\frac{39}{1024}$
22	$\frac{379}{9216}$
23	$\frac{1007}{23040}$
24	$\frac{211}{4608}$
25	$\frac{1091}{23040}$
26	$\frac{223}{4608}$
27	$\frac{1127}{23040}$
28	$\frac{1127}{23040}$
29	$\frac{223}{4608}$
30	$\frac{1091}{23040}$
31	$\frac{211}{4608}$
32	$\frac{1007}{23040}$
33	$\frac{379}{9216}$
34	$\frac{39}{1024}$
35	$\frac{1597}{46080}$
36	$\frac{95}{3072}$
37	$\frac{311}{11520}$
38	$\frac{53}{2304}$
39	$\frac{293}{15360}$
40	$\frac{707}{46080}$
41	$\frac{61}{5120}$
42	$\frac{409}{46080}$
43	$\frac{29}{4608}$
44	$\frac{97}{23040}$
45	$\frac{121}{46080}$
46	$\frac{23}{15360}$
47	$\frac{7}{9216}$
48	$\frac{1}{3072}$
49	$\frac{1}{9216}$
50	$\frac{1}{46080}$

Task 2: Determine the probability of winning the game

Calculate the probability of winning the game

In[∘]:=
```
winningProbability = Total[Select[probabilityTable, #[[1]] ≤ 10 || #[[1]] ≥ 45 &][[All, 2]]];
```

The variable `winningProbability` now contains the probability of winning the game. We can evaluate this code to get the result.

In[∘]:=
```
winningProbability
```

Out[∘]= $\frac{41}{3840}$

Task 3: Obtain the probability of winning the game with 1000 trials (Monte Carlo simulation)

For this task, we'll use Monte Carlo simulation to estimate the probability of winning the game with 1000 trials. We'll simulate the rolling of five dice and count how many times we win.

In[∘]:=
```
numTrials = 1000;
winCount = 0;

For[i = 1, i ≤ numTrials, i++,
 diceRolls = RandomInteger[{1, 4}, 1]~Join~RandomInteger[{1, 6}, 1]~Join~
   RandomInteger[{1, 8}, 1]~Join~RandomInteger[{1, 12}, 1]~Join~RandomInteger[{1, 20}, 1];
 totalSum = Total[diceRolls];
 If[totalSum ≤ 10 || totalSum ≥ 45, winCount++];]

estimatedProbability = winCount / numTrials
```

Out[∘]= $\frac{1}{100}$

expectedInvestment=expectedWin*2 //N

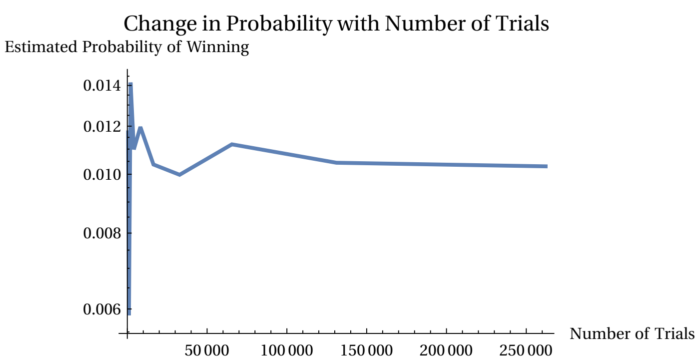
Task 4: Discuss how the probability changes with the number of trials and provide a figure.

To analyze how the probability changes with the number of trials, we can perform simulations for various trial numbers such as 2 to 2^18 and observe the trend. We'll create a plot to visualize this change.

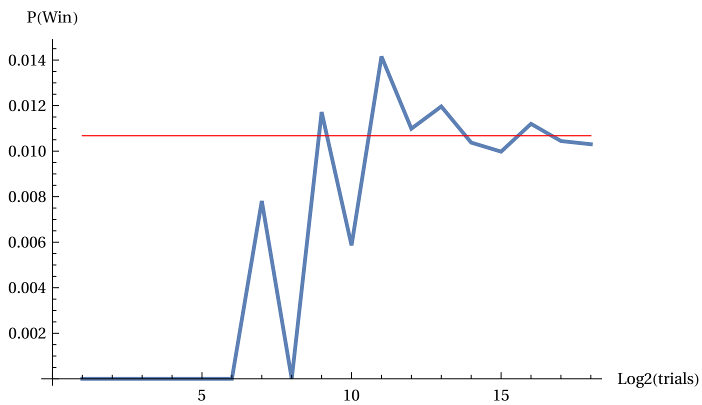
In[∘]:=

```
randSum := Total[#] & /@ {RandomInteger[{1, #}] & /@ {4, 6, 8, 12, 20}};

Nwins[samples_] :=
  Module[{s = samples, wins = 0, curr = 0}, For[i = 1, i ≤ s, i++, curr = randSum[[1]];
    If[curr ≤ 10 || curr ≥ 45, wins++]];
   Return[<|"wins" → wins, "probability" → N[wins / samples]|>]];

trialCounts = Table[2 ^ i, {i, 1, 18}];
estimatedProbabilities = Nwins[#][["probability"]] & /@ trialCounts;

listlog = ListLogPlot[Transpose[{trialCounts, estimatedProbabilities}], Joined → True,
  AxesLabel → {"Number of Trials", "Estimated Probability of Winning"},
  PlotLabel → "Change in Probability with Number of Trials"]

probabilityplot =
  Plot[winningProbability, {i, 1, 18}, PlotStyle → {Red, Thickness[0.002]}];

log2Transform = {Log2[#[[1]]], #[[2]]} & /@
   Table[{trialCounts[[i]], estimatedProbabilities[[i]]}, {i, 1, Length[trialCounts]}];
log2plot = ListPlot[log2Transform, Joined → True, AxesLabel → {"Log2(trials)", "P(Win)"}];

Show[log2plot, probabilityplot]
```

Out[∘]=



Out[ ]=



In[ ]:=

Task 5: Determine the number of trials needed to keep the relative error below 10%. Provide reasoning.

To determine the number of trials needed to keep the relative error below 10%, we can iterate through different trial numbers and stop when the relative error is less than 10%. We can than repeat this a certain amount of times, 1000 in our case, to get an average of the number of trials needed to get a relative error value below 10%.

In[*]:=
```
desiredRelativeError = 0.1; (*Desired relative error threshold*)
numTrials = 2; (*Initial number of trials*)

(*Initialize a variable to store the relative error*)
relativeError = 1.0;

relativeErr[Pm_] := Abs[winningProbability - Pm] / winningProbability;

repeats = 1000;
totalTrials = 0;
totalRelativeError = 0;
For[j = 1, j ≤ repeats, j++,
  numTrials = 2;
  relativeError = 1.0;
  While[relativeError > desiredRelativeError,
       res = Nwins[numTrials];
       relativeError = relativeErr[res[["probability"]]];
       numTrials *= 2
  ];

  totalTrials = (totalTrials + (numTrials / 2));
  totalRelativeError = (totalRelativeError + relativeError);

 ];
avgTrials = totalTrials / repeats;
avgRelativeError = totalRelativeError / repeats;
```

The above code snippet will keep doubling the number of trials until the relative error falls below 10%.

In[*]:=
```
numTrials
relativeError
N[avgTrials]
avgRelativeError
```

Out[*]= 2048

Out[*]= 0.00609756

Out[*]= 2228.22

Out[*]= 0.0752308

In[◦]:=
```
nTrials = Floor[avgTrials]
```
Out[◦]=
```
2228
```

In[◦]:=
```
Nwins[nTrials]
```
Out[◦]=
```
<|wins → 28, probability → 0.0125673|>
```

This function performs the monte Carlo simulation for a certain number of trials and then checks if the relative error is less than 10%. It does so repeatedly a certain amount of times, 100, and keeps track of the number of simulations which satisfies our condition. Using this data we can approximate the probability of getting a relative error less than 10% for a certain number of trials in the Monte Carlo simulation.

In[◦]:=
```
Pf[trials_, iters_] := Module[{max = iters, prob0 = 0, err = 0, condMet = 0},
   For[j = 1, j ≤ max, j++,
    prob0 = Nwins[trials][["probability"]];
    err = relativeErr[prob0];
    If[err < desiredRelativeError, condMet++];
   ]; Return [condMet / max]];
```

We use the Pf function on a few number of possible trials to see probability for different number of trials to have a relative error less than 10%.

In[◦]:=
```
plist = {#, Pf[#, 100]} & /@ {1000, nTrials, 5000, 10 000, 20 000, 4 * 6 * 8 * 12 * 20}
```
Out[◦]=
$$\left\{\left\{1000, \frac{27}{100}\right\}, \left\{2228, \frac{41}{100}\right\}, \left\{5000, \frac{53}{100}\right\}, \left\{10\,000, \frac{31}{50}\right\}, \left\{20\,000, \frac{17}{20}\right\}, \left\{46\,080, \frac{99}{100}\right\}\right\}$$

In[◦]:=
```
plistfiltered = ({#[[1]], PercentForm[N[#[[2]]]]}) & /@ plist
```
Out[◦]=
```
{{1000, 27%}, {2228, 41%}, {5000, 53%}, {10 000, 62%}, {20 000, 85%}, {46 080, 99%}}
```

In[◦]:=
```
Grid[Prepend[plistfiltered, {"trials", "P(relative error < 10)"}], Frame → All]
```
Out[◦]=

trials	P(relative error < 10)
1000	27%
2228	41%
5000	53%
10 000	62%
20 000	85%
46 080	99%

The final value of numTrials will give you an estimate of how many trials are needed to achieve this level of accuracy.