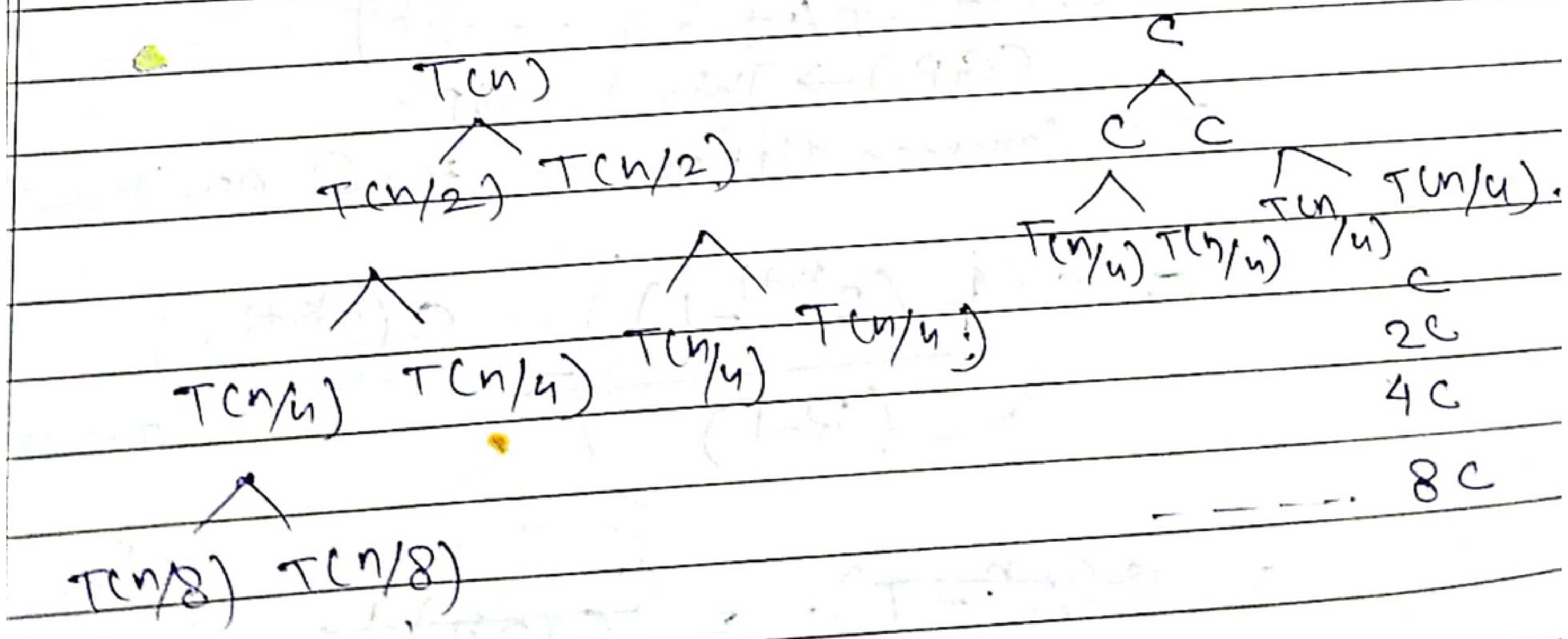
Recurrence / recursion Relatn

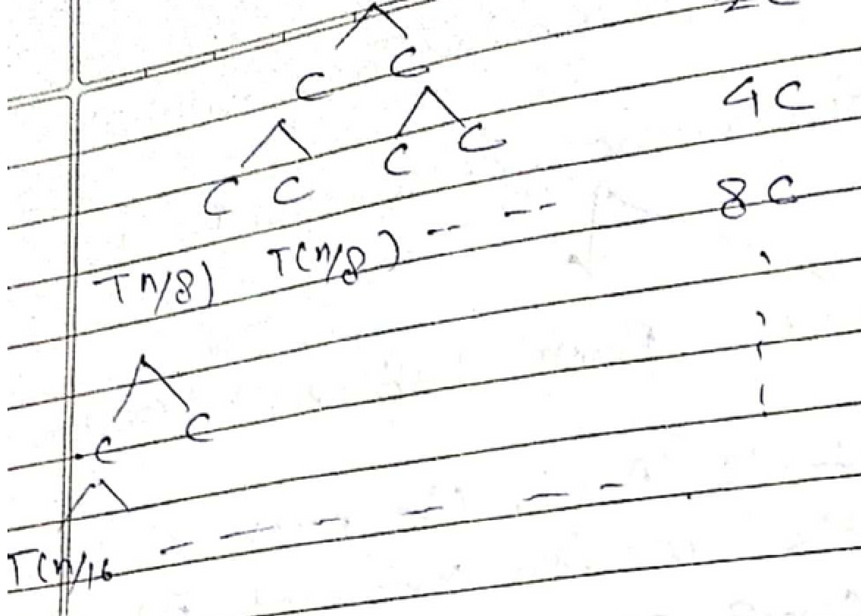
Recursion Tree — Back substitutn — Master.

Always use master's Algo. (in most of the cases), but it's worth knowing All the 3. as you might come across any qstn from other 2 as well.

$$T(n) = 2T(n/2) + c \quad ; \quad n > 1$$
$$= c \quad ; \quad n = 1.$$

If we do constant amount c of work, the program is divided into 2 sub parts — each no. of inputs n/2.
(elements)





T(1) - - - - T(1). = T(n/n).

Total work work done = c + 2c + 4c + 8c + - - - nc

= c(1+2+8+ - - - n)

⇒ let's assume $n = 2^k$.

just to do the ans. in an easier way

There is no harm in that.

$T = c(1+2+4+8+ - - - 2^k)$

(GP.) → This is GP.

= Common diff. = 2, total no. terms = k+1

$= c\left(\frac{1(2^{k+1}-1)}{(2-1)}\right) = c(2^{k+1}-1)$

$n = 2^k$

2^k.

$c(2^k \cdot 2 - 1)$

$= c(2n-1). = O(2n)$

O(n).

Divide & Conquer.

Break the prob. into sub prob. $\rightarrow$ solve these using recursion.

Ex. $T(n) = T(n/4) + T(n/2) + cn^2.$

$T(n)\ cn^2 \rightarrow cn^2$

$T(n/4) \quad T(n/2) \quad n^2. \quad c(n/4)^2 \quad c(n/2)^2$

$T\left(\frac{n}{4}\cdot\frac{1}{4}\right) \quad T\left(\frac{n}{4}\cdot\frac{1}{2}\right) \quad \left(\frac{n^2}{4}\right) \quad \frac{n}{2}\cdot\frac{1}{4} \quad \frac{n}{2}\cdot\frac{1}{2} \quad (n/2)^2 \ldots$

$T(n/16) \quad T(n/8) \ldots$

$T(1) \ldots$

$\Rightarrow T(n) = c(n^2 + c\left(\frac{n^2}{16}\right) + c\frac{n^2}{4} + \ldots$

$= c\left(n^2 + \frac{n^2}{16} + \frac{n^2}{4}\right.$

$T(n) = T(n/4) + T(n/2) + cn^2$.

$T(n)$ $\qquad$ α

$T(n/4)$ $\quad$ $T\left(\frac{n}{2}\right)$ $\quad$ cn^2. $\qquad$ cn^2.

$T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{8}\right)$ $\left(\frac{n}{4}\right)^2$ $T\left(\frac{n}{8}\right)$ $T\frac{n}{4}$. $\frac{n^2}{4}$. $\rightarrow c(n/4)^2$ $\quad$ $c(n/2)^2$

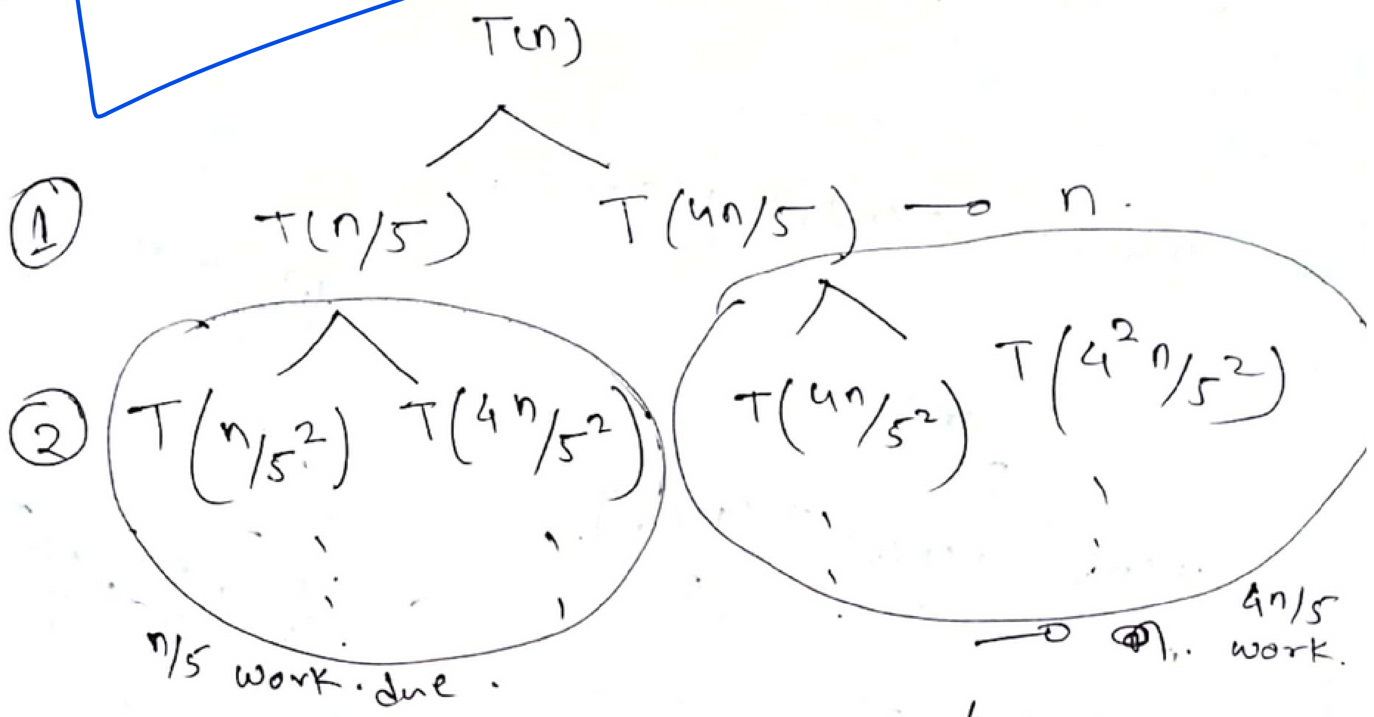
$\rightarrow c(n/16)^2$ $\quad$ $c(n/8)^2$

$T\left(\frac{n}{64}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n^2}{16}\right)$

$$\rightarrow c.\left(n^2 + \frac{n^2}{16} + \frac{n^2}{4} + \frac{n^2}{64} + \frac{n^2}{256}\right) \cdots \infty$$

$$\rightarrow c(n^2) \rightarrow O(n^2)$$

$$T(n) = T(n/5) + T(4n/5) + n.$$
$$= 1. \quad \text{if } n = 1.$$



At Each level → n work done /

(2) $\frac{n}{5} + \frac{4n}{5} = n.$

(3) select longest Path for time Comx.
i.e. $(4n/5)$ one.
→ ~~At least~~ Height / no. of levels → $\log_{5/4} n$.
→ As. ~~$\frac{4n}{5} = 1$~~ →

$$\Rightarrow \left(\frac{4}{5}\right)^k \cdot n = 1 \Rightarrow \left(\frac{5}{4}\right)^k = \log n.$$
$$\Rightarrow k = \log_{5/4} n.$$

$\Rightarrow$ n work done at each level $\times$ No. of levels

$\Rightarrow \left(n \times \log_{5/4} n \right)$ ✓

* **Q.** $T(n) = 3T(n/4) + cn^2.$

Work done.

$T(n)$ ① n^2

$T(n/4) \quad T(n/4) \quad T(n/4)$ ② $\left(\frac{n}{4}\right)^2 + \left(\frac{n}{4}\right)^2 + \left(\frac{n}{4}\right)^2$

$T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \; \ldots\ldots$ $= \frac{3}{16} n^2$

Next level ③ $9/16\, n^2$

To stop, $\frac{n}{4^k} = 1$. No limit condition given.

$n = 4^k$.

$k = \log_4 n$ no. of levels. but not sure. as no limiting condition given.

$\Rightarrow$ work done Total $= \left(n^2 + \frac{3}{16} n^2 + \frac{9}{16} n^2 + \ldots\right)$

~~$\log_4 n$~~.

$= n^2 \left(1 + 3/16 + 9/16 + \ldots\ldots \right)$

∞ G.P.

$= O(n^2)$ ✓

Recursion Tree

$\frac{2n}{3} + \frac{2n}{3}$

1. ~~$T(n) = 5T(n/5) + \sqrt{n}$.~~

$T(n) = T(n/3) + T(2n/3) + \cancel{O(n)}\ c \cdot n.$

shortest

n

$n/3 \quad 2n/3 \quad \rightarrow cn.$

$\frac{16}{2} \quad \frac{16}{4} \quad \frac{16}{8}$

16/16

$n/3^2 \quad 2n/3^2 \quad 2n/3^2 \quad \frac{2^2 n}{3^2}$ longest $\rightarrow cn$

$\frac{2^3 n}{3^3} \rightarrow cn$

$\frac{2^4 n}{3^4} \rightarrow cn.$

reach to 1

in $\log_3 n$ steps.

work done.

~~$\Omega(\log_3 n)$~~

$\Omega(n \log_3 n)$

Height $\rightarrow$

$\frac{n}{3^k} = 1. \rightarrow n = 3^k.$

$k = \log_3 n.$

Height $\rightarrow \frac{2^k n}{3^k} = 1.$

$\Rightarrow k_2 = \log_{3/2} n.$

$\Rightarrow O(n \log_{3/2} n)$

Q. $f(n) = n$, $g(n) = n^{1+\sin n}$.

n is +ve integer.

(1) $f(n) = O(g(n))$

(2) $f(n) = \Omega\, g(n)$?

Ans. $\sin(n)$ values range -1 to 1.

$g(n)$ can have n^2, n, 1.

$\therefore$ If, $g(n) = 1$. (1) is wrong

If $g(n) = n^2$ (II) is wrong.

MASTERS THEOREM

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$T(n) = aT(n/b) + n^k \log^p n$.

$a, b \geq 1$, $k \geq 0$, $p \rightarrow$ real no.

(1) $a > b^k$, $T(n) = \Theta(n^{\log_b a})$

(2) $a = b^k$,

$p > -1$ $\quad T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$

$p = -1$ $\quad T(n) = \Theta(n^{\log_b a} \log \log p)$

$p < -1$. $\quad T(n) = \Theta(n^{\log_b a})$

(3) $a < b^k$

$p \geq 0$ $\quad T(n) = \Theta(n^k \log^p n)$

$p < 0$. $\quad T(n) = \Theta(n^k)$.

$T(n) = 3T(n/2) + n^2$.

Ans. $a = 3$, $b = 2$, $k = 2$, $p = 0$

(1) $a(3)$ $\quad 2^2 (b^k)$. $\Rightarrow 3 < 4$.

$a < b^k$.

$T(n) = \Theta(n^k \log^p n) = \Theta(n^2 \log^0 n)$

$= \Theta(n^2)$

1) If a > b^k, then $T(n) = \Theta\left(n^{log_b^a}\right)$
2) If a= b^k
    a. If $p > -1$, then $T(n) = \Theta\left(n^{log_b^a} log^{p+1} n\right)$
    b. If $p = -1$, then $T(n) = \Theta\left(n^{log_b^a} loglogn\right)$
    c. If $p < -1$, then $T(n) = \Theta\left(n^{log_b^a}\right)$
3) If $a < b^k$
    a. If $p \geq 0$, then $T(n) = \Theta(n^k log^p n)$
    b. If $p < 0$, then $T(n) = O(n^k)$

① $T(n) = 3T(n/2) + n^2.$

$a = 3 \quad b = 2 \quad k = 2 \quad p = 0.$

<u>check</u> $\rightarrow \quad a < b^k$

$3 \quad 4.$

(III) case $\rightarrow p \geq 0 \qquad n^k \log^p n.$

$\Rightarrow n^2 \log^0 n = \Theta(n^2)$

② $T(n) = 4T(n/2) + n^2.$

$a = 4, \quad b = 2, \quad k = 2, \quad p = 0.$

<u>check</u> $\quad a = b^k.$

(I) case :- $p > -1, \qquad n^{\log_b a} \log^{p+1} n.$

$\Theta(n^{\log_2 4} \log n.)$

$\Rightarrow \Theta(n^2 \log n)$

③ $T(n) = T(n/2) + n^2.$

$a = 1, \quad b = 2, \quad k = 2, \quad p = 0.$

<u>check.</u> $\quad a < b^k.$

(III) case. $\qquad p \geq 0 \qquad \rightarrow \Theta(n^k \log^p n.)$

$\rightarrow \Theta(n^2).$

A.

④ $T(n) = 2^n T(n/2) + n^n$. X.

Master's doesn't Apply here as a is not constant.

⑤ $2T(n/2) + n/\log n$.

$\Rightarrow 2T(n/2) + n \cdot \log^{-1} n$.

$a = 2,\ b = 2,\ k = 1 \quad p = -1$.

check $a = b^k$.

(II) case. $\dfrac{p = -1.}{\text{B case.}}$ $\theta(n^{\log_b a} \log \log n)$

$\Rightarrow \theta(n \cdot \log \log n)$ ✓

Q. $T(n) = 0.5\, T(n/2) + 1/n$.

X not applied as a, b Must be ≥ 1.

Q. $T(n) = 16\, T(n/4) + n!$.

$a = 16,\ b = 4,$

check. Here $(n!)$ is dominating term here. so no need to check. $\Rightarrow \theta(n^n)$ ✓

$\theta(n!)$

$T(n) = 2T(\sqrt{n}) + \log n.$

let $n = 2^k.$

$\sqrt{n} = 2^{k/2}.$

~~$T(n) = 2T(n^{k/2}) + \log 2^k.$~~

~~let $T(n^{k/2}) = S(k)$~~

~~$f(n)$~~ $S(k) = 2$

$T(2^k) = 2T(2^{k/2}) + \log 2^k.$

let $T(2^k) = S(k).$

$S(k) = 2S(k/2) + \cancel{\log 2^k}\; k.$

$\Rightarrow$ Masters's. $a=2,\ b=2,\ k=1,\ p=0$

II case (A) $\Theta(n^{\log_b a} \log^{p+1} n)$

$\Rightarrow \Theta(k \log k)$

$\Rightarrow \Theta(k \log k)$

$\Rightarrow$ $\boxed{k = \log n.}$ $\Theta(\log n \log(\log n))$

$T(n) = 2T(n/2) + n\log n.$

$a=2,\ b=2,\ K=1,\ P=1$

(1) Compare $a,\ b^k \rightarrow 2,\ 2.$

$a = b^k \longrightarrow$ (2) condition.

$\downarrow$

(1) condition $P > -1.$

$T(n) = \Theta(n^{\log_b a} \log^{P+1} n)$

$= \Theta(n^{\log_2 2} \log^2 n)$

$= (n \log^2 n)$

(2) $\boxed{T(n) = 2T(n/2) + (n/\log n)}$

$T(n) = 2T(n/2) + n\log^{-1} n.$

$\Rightarrow a=2,\ b=2,\ K=1,\ P=-1.$

(A) 1 condition $\longrightarrow$ (2) conditn $P=-1.$

$T(n) = \Theta(n^{\log_b a} \log\log n)$

$= \Theta(n^{\log_2 2} \log\log n).$ $= (n(\log\log n))$

(4) $T(n) = 2T(n/4) + n^{0.51}$

$a=2,\ b=4,\ K=0.51,\ P=0.$

Ans. $a < b^K.$ (3) Case - $T(n) = \Theta(n^K \log^P n)$

$= \Theta(n^{0.51} \log^0 n)$

$= (n^{0.51})$

$T(n) = 0.5\, T(n/2) + 1/n.$

for the master's theorem a has to be ≥ 1 which's not. so this is not valid. As per the master's theorem.

$T(n) = 6T(n/3) + n^2 \log n.$

$a = 6,\ b = 3,\ k = 2,\ p = 1.$

$a < b^k$ (3) conditn $\rightarrow$ $T(n) = \theta(n^k \log^p n)$

$6 < 3^2.$

$= \theta(n^2 \log^2 n)$

$T(n) = \sqrt{2}\, T(n/2) + \log n.$

$a = \sqrt{2},\ b = 2,\ k = 0,\ p = 1$

$a > b^k.$ (1) conditn $T(n) = \theta(n^{\log_b a})$

$\sqrt{2} > 2^0.$

$= n^{\log_2 \sqrt{2}}$ $\boxed{\sqrt{n}}$

$= n^{\log_2 2^{1/2}}$

$= n^{1/2}.$

$T(n) = 2T(n/2) + \sqrt{n}$

$a = 2,\ b = 2,\ k = 1/2,\ p = 0.$

(1) $a \quad b^k$ (1) case $- T(n) = \theta(n^{\log_b a})$

$2 > 2^{1/2}$

$= \theta(n^{\log_2 2})$

$= \theta(n).$

① Apply Masters

$$T(n) = 4T(\sqrt{n}) + \log^5 n.$$

let $n = 2^k$.

$k = \log n$.

$$T(2^k) = 4T(2^{k/2}) + k^5.$$

let $T(2^k) = S(k)$.

$$S(k) = 4S(k/2) + k^5.$$

$a = 4,\ b = 2,\ k = 5.\ p = 0.$

$a < b^k$. III case.

$\underline{p \geq 0.}$ → $\underline{n^k \log^p n.}$

$$\Rightarrow \underline{\underline{\theta(k^5)}}$$

② Big (O) →.

$$T(n) = T(n-1) + \sqrt{n}. \quad n \geq 1 \quad [n-k=0]$$
$$= 0 \quad \text{else}.$$

$$T(n) = T(n-3) + \sqrt{n-2} + \sqrt{n-1} + \sqrt{n}.$$

⋮ k.

$$\Rightarrow T(n) = T(n-k) + \sqrt{n-(k-1)} + \cdots\cdots \sqrt{n}.$$
$$= T(n-k) + \underbrace{\sqrt{1} + \sqrt{2} + \cdots\cdots \sqrt{n}.}$$
$$= 0 +$$

$= \Sigma \sqrt{n} = \int_1^n \sqrt{x}\,dx$

$= \frac{2}{3}(n^{3/2} - 1) = O(n^{3/2})$

f4 $

f5

put

Q. functn which grows the slowest (means min. time).

① $2^{\log n}$ ② n^{10} ③ $(\sqrt{\log n})^{\log^2 n}$ ④ $\log n^{\sqrt{\log n}}$

⑤ $2^{2\sqrt{\log\log n}}$

Ans. ① $2^{\log_2 n} = n^{\log 2}$.

② n^{10}. ① << ②.

③ $(\sqrt{\log n})^{\log^2 n} = (\log n)^{\frac{\log^2 n}{2}} \Rightarrow$

④ $\log n^{\log n^{1/2}}$ $\left((\log n)^{1/2} \quad (\log n)^2\right)$ for large value of n,

$(\log 2^{300})$ $(\log 2^{300})^2 / 2$.

④ << ③ 300 $\leq \frac{(300)^2}{2}$.

check b/w ①, ④ $(\log n^{\sqrt{\log n}})^n = 2^{\log_2 (\log n^{\sqrt{\log n}})}$

$ ① $2^{\log n}$. compare → $\log n$ $\log(\log^{\sqrt{\log n}})$

$\log n > \sqrt{\log n} \log\log n$

$f_1 >> f_4$. >

Q. $T(n) = \sqrt{n}\, T(\sqrt{n}) + n, \quad n > 2$
$\quad\; = 2 \quad$ else.

let $n = 2^k$, $\sqrt{n} = 2^{k/2}$. $k = \log n$.

$T(2^k) = 2^{k/2}(T(2^{k/2})) + 2^k$.

$\frac{2^{k/2}}{2^k} = 2^{-k \cdot \frac{1}{2} - 1}$
$= 2^{-k/2} \cdot \frac{1-2}{2}$
$= \frac{1}{2^{k/2}}$.

Divide by 2^k

$$\frac{T(2^k)}{2^k} = \frac{2^{k/2}\, T(2^{k/2})}{2^k} + 1.$$

$$\Rightarrow \frac{T(2^k)}{2^k} = \frac{T(2^{k/2})}{2^{k/2}} + 1.$$

let it be $y(k)$

$y(k) = y(k/2) + 1.$

Master. $a = 1$ $b = 2$ $n = 0, p = 0.$

$y(k) = n^{\log_b a} \log^{p+1} n.$

$= \log k$

$\because \frac{T(2^k)}{2^k} = y(k) \Rightarrow T(2^k) = 2^k \log k.$

as $\Rightarrow n = 2^k, k = \log n \rightarrow T(n) = \theta(n \log \log n.)$

Ans.

Q. $n^2 + o(n^2) = \Theta(n^2)$ why.

In this Ex. $\rightarrow$ $o(n^2)$ means terms can have value lower than n^2, like constant, linear function of n, ... but not more than they.

n^2 means. at least n^2 must be there.

Q. $f(n) = \Omega(n), g(n) = O(n), h(n) = \Theta(n).$

$f(n) \cdot g(n) + h(n) = ?$

S. $\Omega(n) \cdot O(n).$

T

```
for (int i=0 i<n i++)
for (int j=i j<i^2 j++)
if (j%i==0)
{
   for (int k=0 k<j k++)
      print (        )
}
```

Ans. $i=2$, for $J = 2,4$. inner loop will execute.

$J=2$ → inner loop execute 2 times.

$J=4$. —" ———— 4 times.

⇒ Total 6 times.

$i=3$. $J = 3,6,9$ the inner loop will execute.

for $J=3$, for $K = 3$ times

$J=6$, $K = 6$ times

$J=9$, 9 times.

Pattern is → for $i=m$, inner loop (k) will execute $3m$ times. for value of $J = m, 2m, 3m$ --

A.P. → $m+2m+3m+ \cdots m^2$.

Total $= m/2\,(m+m^2)$.

$$\sum_{m=1}^{n} m^2/2 + m^3/2 = O(n^4).$$

Q. which one grows fastest?

Ans

$T(n) = 4T(n/2) + 10n.$

$T(n) = 8T(n/3) + 24n^2.$

$T(n) = 16\,T(n/4) + 10n^2.$

$T(n) = 25\,T(n/5) + 20\,(n \log n)^{1.9}$

(a) $T(n) = \Theta(n^2)$ (b) $\Theta(n^2)$ (c) $\Theta(n^2 \log n)$ (d) $\Theta(n^2)$

Ans.

DL

Q.

(A) $2^{2\sqrt{\log\log n}} < 2^{\sqrt{\log n}} < n$

(B) $2^{\sqrt{\log n}} < n < 2^{2\sqrt{\log\log n}}.$

(C) $n < 2^{\sqrt{\log n}} < 2^{2\sqrt{\log\log n}}$

let $n = 2^{2^k}.$

$2^{\sqrt{\log 2^{(2^k)}}} = 2^{\sqrt{2^k}}$ ✓

$2^{2^{\sqrt{\log 2^k}}} = \log 2^{(2^{\sqrt{k}})}$ → check powers.

$\Rightarrow$ Take log → $\sqrt{2^k}$ $\quad 2^{\sqrt{k}}$ $\quad$ let $k = 256.$

$\frac{256}{2}$ 128.

→ $\sqrt{2^{256}}$ $\quad 2^{16}.$

$(2^{128}.\ 2^{128})$

2^{128} $\quad 2^{16}.$

$n = 2^{(2^{2^8})}$

$\Rightarrow 2^{2^{\sqrt{\log\log n}}} \longrightarrow \log(2^{2^8}) \quad 2^8 \Rightarrow 2^{2^{\sqrt{2^8}}}$

$\Rightarrow \sqrt{256}$

$2^{2^{\cdot}} \Rightarrow \boxed{2^{2^{16}}}$

$\Rightarrow 2^{\sqrt{2^{2^8}}} = 2^{\sqrt{2^{256}}} = 2^{2^{256/2}} = 2^{2^{128}}$

B) . A < B < C   Option A Correct.

Q. $T(n) = 4T(n/2) + n^2 \cdot \sqrt{2}$, $T(1) = 1$.

$n/2^k = 1$

$n = 2^k$.

$T(n/2) = 4T(n/4) + (n/2)^2 \sqrt{2}$

$T(n/4) = 4 \cdot T(n/8) + (n/4)^2 \sqrt{2}$

$\Rightarrow T(n) = 4^3 T(n/2^3) + \cancel{4^2 T(n/2^2)}\ 4^2 (n/2^2)^2 \sqrt{2}$

$+ 4 (n/2)^2 \sqrt{2} + 4^0 (n/2^0)^2 \sqrt{2}$.

$\Rightarrow T(n) =$

$\left(\frac{n}{2^{k-1}}\right)^2 = \left(\frac{n}{2^k} \cdot 2\right)^2 = (2)^2$

$\Rightarrow = 4^k T(n/2^k) +$

$\Rightarrow 4^k + \ldots\ldots$

$\Rightarrow \sqrt{2}\,(n^2 + n^2 + n^2 \ldots) K$

$\Rightarrow O(n^2 \log n)$.