

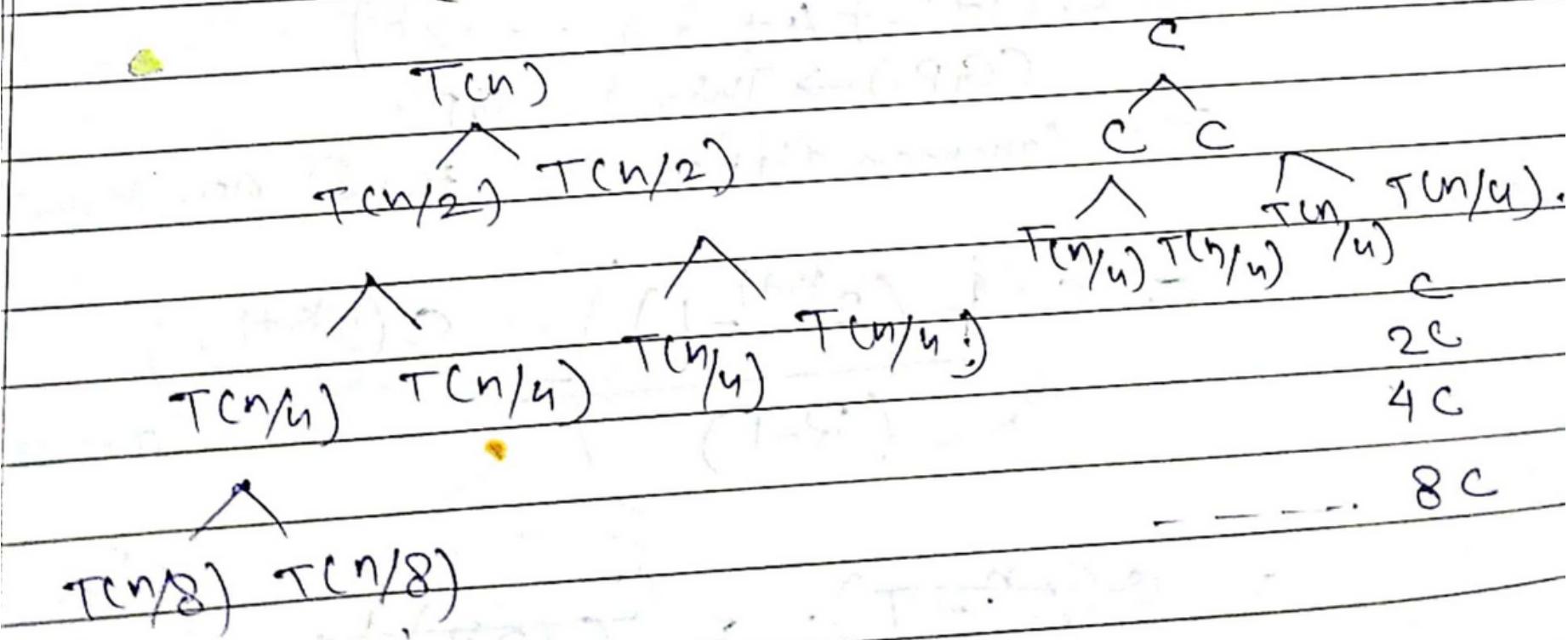
Recurrence / recursion Relation

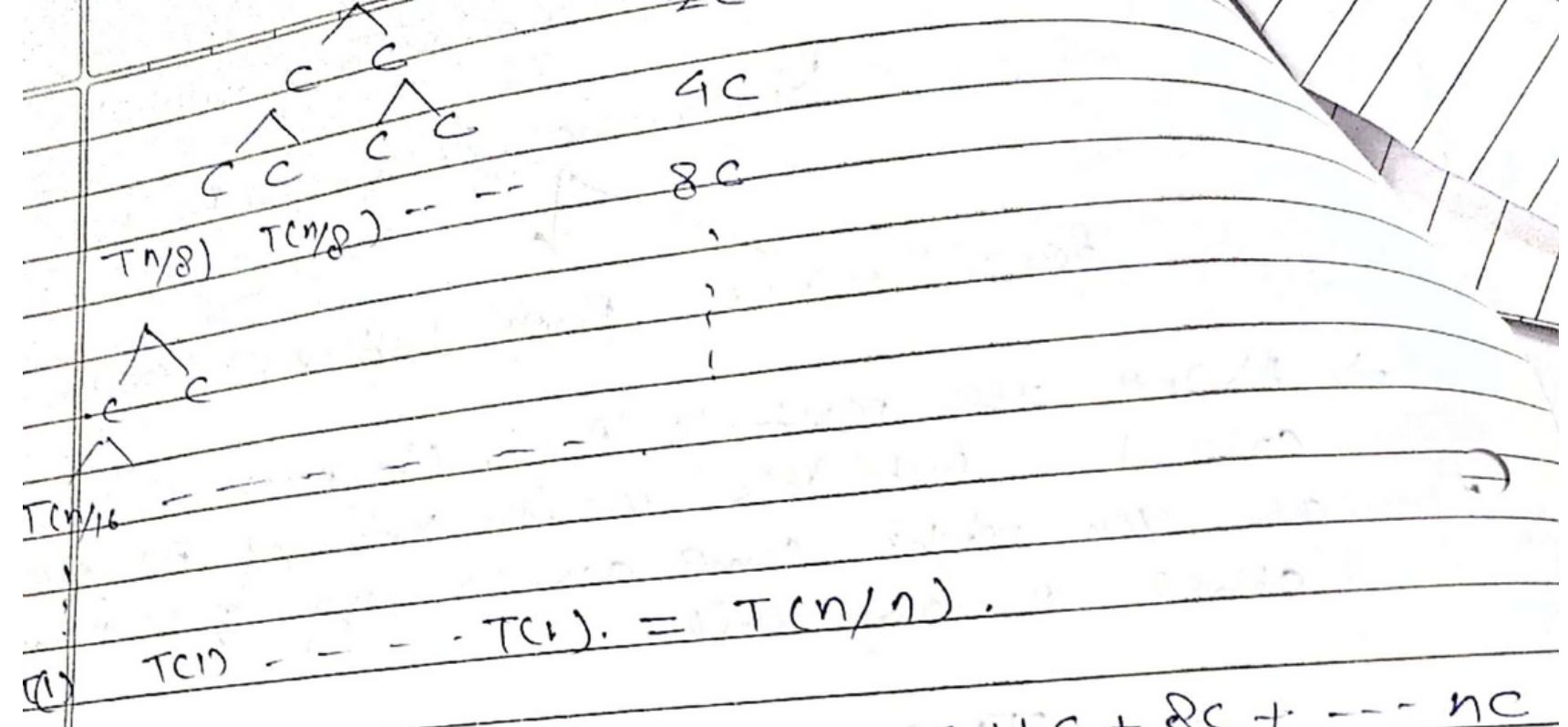
Recursion Tree Backsubstitution Master.

Always use master's algo. (in most of the cases), but it's worth knowing all the 3. as you might come across any qstn from other 2 as well.

$$T(n) = 2T(n/2) + c \quad ; \quad n > 1 \\ = c \quad ; \quad n = 1$$

If we do constant amount c of work, the program is divided into 2 sub parts - each no. of inputs $n/2$.
(elements)





$$\text{Total work done} = C + 2C + 4C + 8C + \dots + nC \\ = C(1 + 2 + 4 + 8 + \dots + n)$$

\Rightarrow let's assume $n = 2^k$.

Just to do the ans. in an easier way

There is no harm in that!

$$T = C(1 + 2 + 4 + 8 + \dots + 2^k)$$

(GP.) \rightarrow This is GP.

= Common diff. = 2, total no. terms = $k+1$

$$= C \cdot \frac{(2^{k+1} - 1)}{(2-1)} = C(2^{k+1} - 1) \quad n = 2^k$$

$$= C(2^n - 1) = O(2^n) \quad 2^k \cdot 2 - 1 \\ = O(n)$$

Divide & Conquer.



Break the prob.
into subprob.

solve these. using
recursion.

Ex. $T(n) = T(n/4) + T(n/2) + cn^2$.

⋮

$$\begin{array}{ccc} \text{Top } cn^2 & & \rightarrow cn^2 \\ & \nearrow & \\ T(n/4) & T(n/2) & n^2 \\ & \nearrow & \\ T\left(\frac{n}{4} \cdot \frac{1}{4}\right) & T\left(\frac{n}{2} \cdot \frac{1}{2}\right) & \frac{n^2}{4} \\ & \nearrow & \\ & & \frac{n \cdot 1}{2} \frac{n \cdot 1}{2} \left(\frac{n}{2}\right)^2 \end{array}$$

$$T(n/16) \quad T(n/8) \quad - \quad - \quad -$$

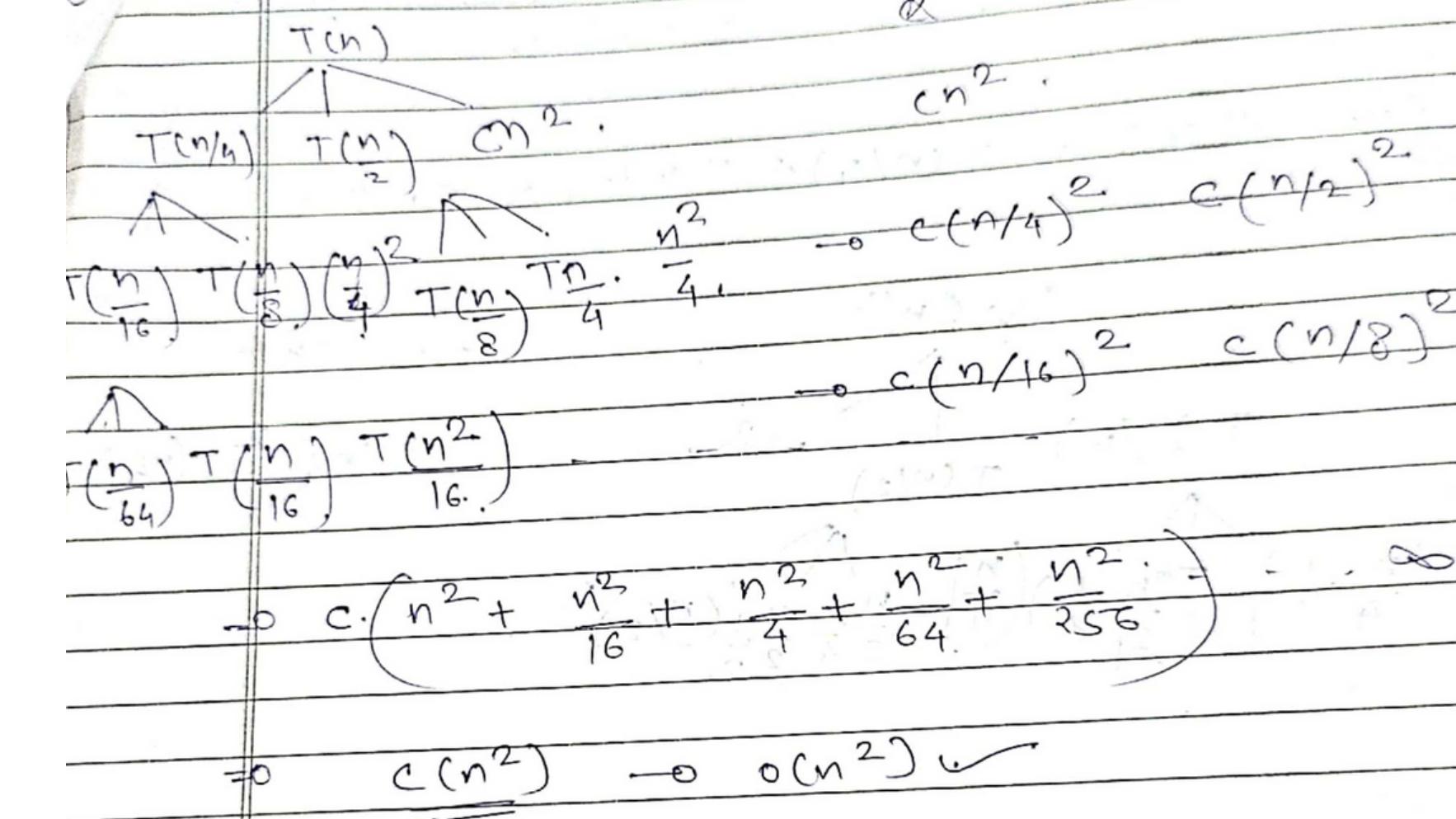
⋮
⋮
⋮

$$T(1)$$

$$\Rightarrow T(n) = c(n^2 + c(n^2)) + c \frac{n^2}{4} + \dots$$

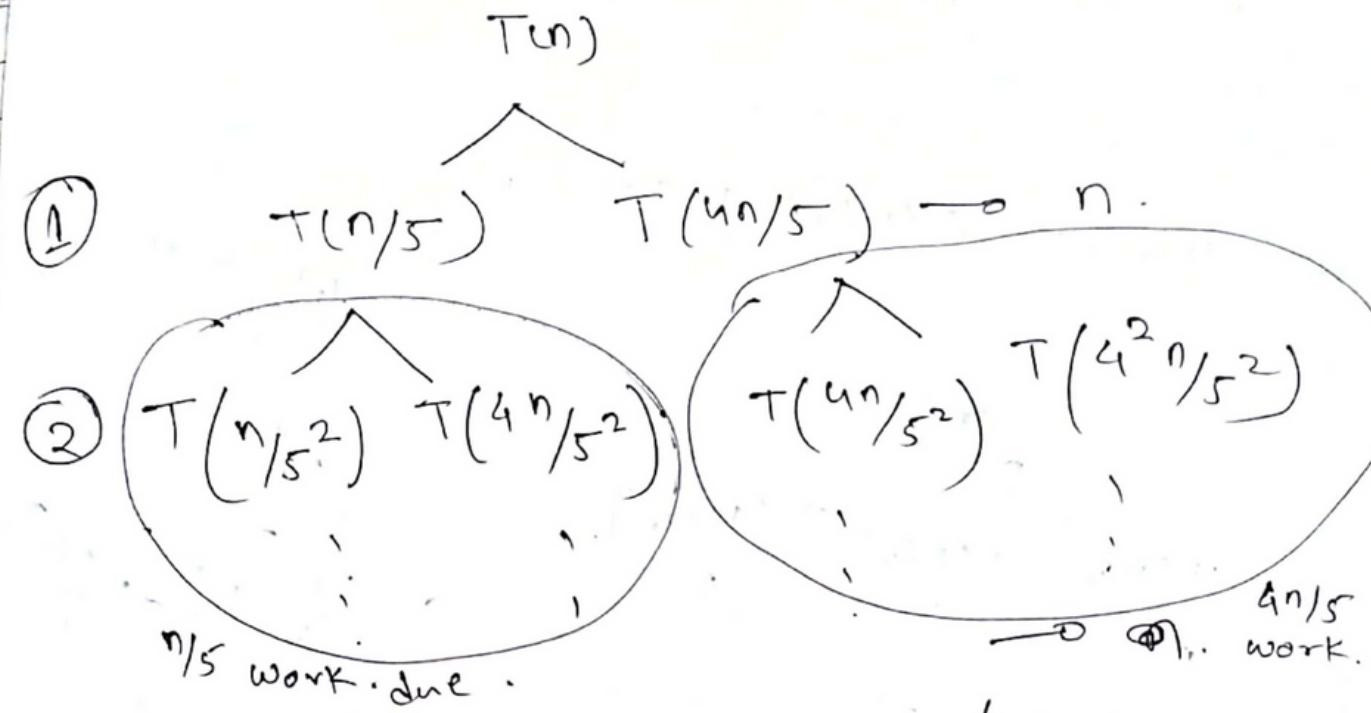
$$= c\left(\frac{n^2}{16} + \frac{n^2}{4}\right) + \frac{n^2}{4}$$

$$T(n) = T(n/4) + T(n/2) + cn^2.$$



$$T(n) = T(n/5) + T(4n/5) + n.$$

= 1. if $n=1$.



At Each level $\rightarrow n$ work done /

$$\textcircled{2} \quad \frac{n}{15} + \frac{4n}{5} = n$$

$\textcircled{3}$ select longest Path for time com^x .
 i.e. ~~4n/5~~ $(4n/5)$ one.
 \rightarrow At least height / no. of levels $\rightarrow \log_{5/4} n$.

~~AS.~~

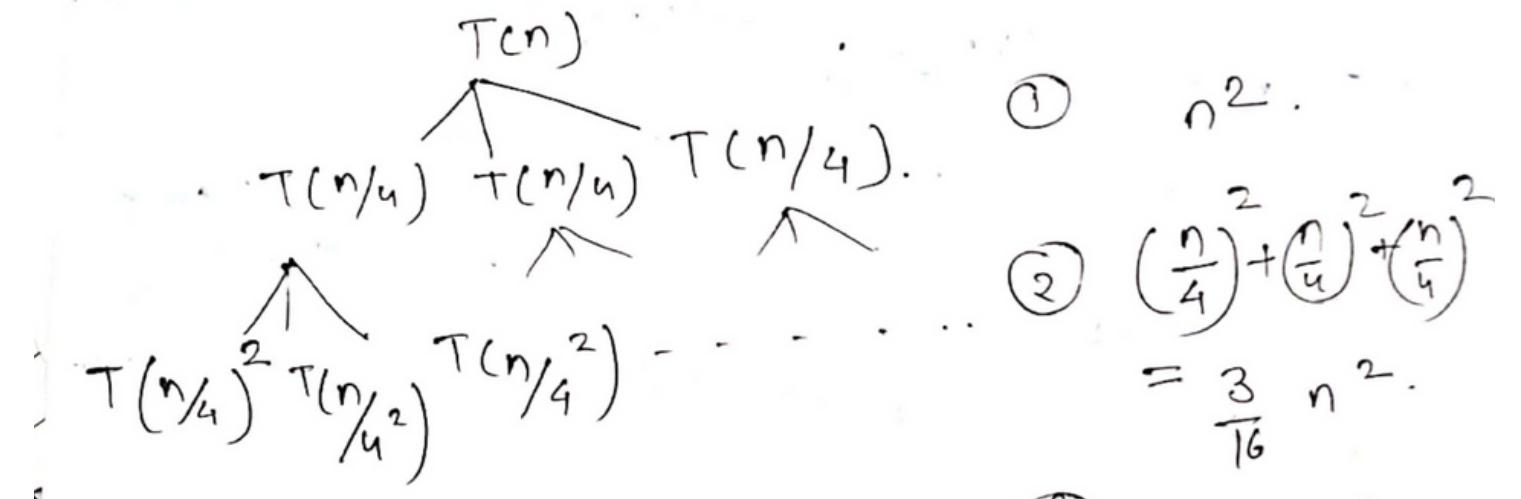
$$\Rightarrow \left(\frac{4}{5}\right)^k \cdot n = 1 \Rightarrow \left(\frac{5}{4}\right)^k = \underline{\underline{\log n}}.$$

$$\Rightarrow k = \underline{\underline{\log_{5/4} n}}$$

$$\begin{aligned} &\Rightarrow \text{work done at each level} \times \text{no. of levels} \\ &\Rightarrow (n \times \log_{\frac{1}{4}} n) \checkmark \end{aligned}$$

$\therefore T(n) = 3T(n/4) + cn^2.$

Work Done.



Next ③ level. $\frac{9}{16} n^2$

To stop, $\frac{n}{4^k} = 1.$ No limit condtn given.

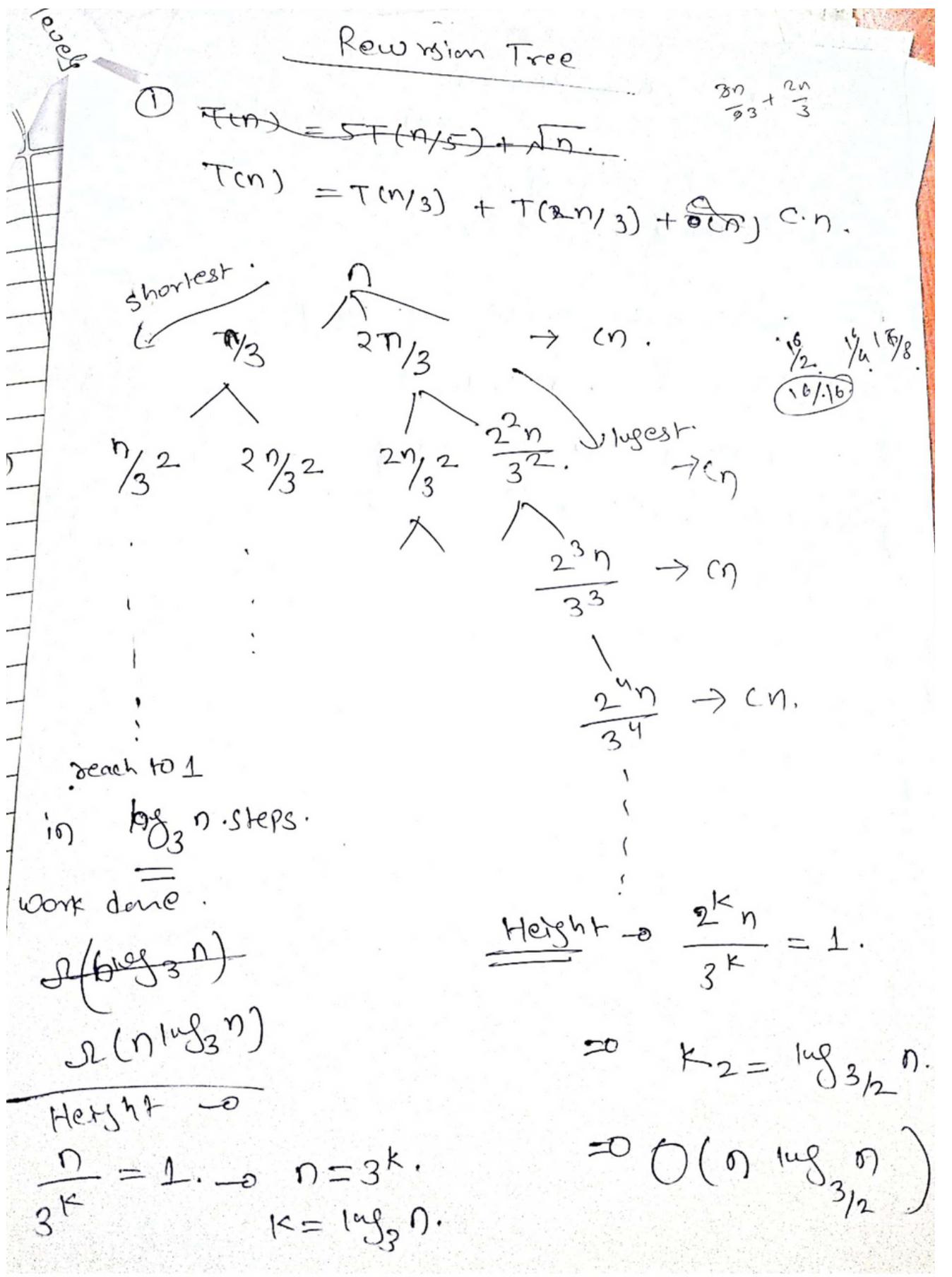
$$\begin{aligned} &n = 4^k. \\ &k = \log_4 n. \end{aligned}$$

no. of levels, but
not sure, as no
limiting condtn given.

$$\Rightarrow \text{Work done Total} = \left(n^2 + \frac{3}{16} n^2 + \frac{9}{16} n^2 + \dots \right)$$

~~$\log_4 n$~~ .

$$\begin{aligned} &= n^2 \left(1 + \frac{3}{16} + \frac{9}{16} + \dots \right) \\ &\qquad \qquad \qquad \text{O.G.P.} \\ &= O(n^2) \checkmark \end{aligned}$$



~~Ques.~~ $f(n) = n$, $g(n) = \frac{n}{1 + \sin n}$.
Sol. $f(n) = n$, $g(n) = n$
 n is a positive integer.

- (1) $f(n) = O(g(n))$
(2) $f(n) = \Omega(g(n))$?

Ans. $\sin(n)$ values range -1 to 1.

$g(n)$ can have n^2 , n , 1.

\therefore If, $g(n) = 1$, (1) is wrong

If, $g(n) = n^2$ (2) is wrong.

MASTERS THEOREM

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$$T(n) = aT(n/b) + n^k \log^p n.$$

$a, b \geq 1, k \geq 0, p \rightarrow \text{real no.}$

① $a > b^k, T(n) = \Theta(n^{\log_b a})$

② $a = b^k,$

$$P > -1 \quad T(n) = \Theta(n^{\log_b a \log^{p+1} n})$$

$$P = -1 \quad T(n) = \Theta(n^{\log_b a \log \log n})$$

$$P < -1 \quad T(n) = \Theta(n^{\log_b a})$$

③ $a < b^k$

$$P \geq 0 \quad T(n) = \Theta(n^k \log^p n)$$

$$P < 0 \quad T(n) = \Theta(n^k).$$

Aus. $T(n) = 3T(n/2) + n^2.$
 $\Rightarrow a=3, b=2, k=2, P=0$

① $a(3) 2^2 (b^k), \Rightarrow 3 < 4.$
 $a < b^k.$

$$T(n) = \Theta(n^k \log^p n) = \Theta(n^2 \log^0 n) \\ = \Theta(n^2)$$

-
- 1) If $a > b^k$, then $T(n) = \Theta(n^{\log_b^a})$
 - 2) If $a = b^k$
 - a. If $p > -1$, then $T(n) = \Theta(n^{\log_b^a} \log^{p+1} n)$
 - b. If $p = -1$, then $T(n) = \Theta(n^{\log_b^a} \log \log n)$
 - c. If $p < -1$, then $T(n) = \Theta(n^{\log_b^a})$
 - 3) If $a < b^k$
 - a. If $p \geq 0$, then $T(n) = \Theta(n^k \log^p n)$
 - b. If $p < 0$, then $T(n) = O(n^k)$
-

$$\textcircled{1} \quad T(n) = 3T(n/2) + n^2.$$

$a=3$ $b=2$ $k=2$ $p=0$.

check $\rightarrow a < b^k$
 $3 < 4$.

$$\textcircled{1\frac{1}{2}} \quad \text{case} \rightarrow p \geq 0 \quad n^k \log^p n.$$

$$\Rightarrow n^2 \log^0 n = \Theta(n^2)$$

$$\textcircled{2} \quad T(n) = 4T(n/2) + n^2.$$

$a=4$, $b=2$, $k=2$, $p=0$.

check $a = b^k$.

$$\textcircled{1\frac{1}{2}} \quad \text{case} : - p > -1, \quad n^{\log_b a} \log^{p+1} n.$$

$$\cdot \Theta(n^{\log_2 4} \log n)$$

$$= \Theta(n^2 \log n)$$

$$\textcircled{3} \quad T(n) = T(n/2) + n^2.$$

$a=1$, $b=2$, $k=2$, $p=0$.

check. $a < b^k$.

$$\textcircled{1\frac{1}{2}} \quad \text{case} : \quad p \geq 0 \quad \rightarrow \Theta(n^k \log^p n)$$

$$\rightarrow \Theta(n^2)$$

(4) $T(n) = 2^n T(n/2) + n^n \cdot X.$

Master's doesn't apply here
as a is not constant.

X (5) $2T(n/2) + n/\log n.$

$\Rightarrow 2T(n/2) + n \cdot \log^{-1} n.$
 $a=2, b=2, k=1, p=-1.$

check $a=b^k.$

(II) case. $\frac{p=-1}{B case.} \Theta(n^{\log_b a \cdot \log \log n}).$

$\Rightarrow \Theta(n \cdot \log \log n).$ ✓

O. $T(n) = 0.5 T(n/2) + \gamma n.$
not applied as a, b must be $\geq 1.$

O. $T(n) = 16 T(n/4) + n!.$

$a=16, b=4,$

check. Here $(n!)$ is dominating term here.
so no need to check. $\Rightarrow \Theta(n^n)$ ✓

$$T(n) = 2T(\sqrt{n}) + \log n.$$

let $n = 2^k$.

$$\sqrt{n} = 2^{k/2}.$$

~~$$T(n) = 2T(n^{k/2}) + \log 2^k.$$~~

~~$$\text{let } T(n^{k/2}) = s(k)$$~~

~~$$T(n) s(k) = 2$$~~

$$T(2^k) = 2T(2^{k/2}) + \log 2^k.$$

let $T(2^k) = s(k)$.

$$s(k) = 2s(k/2) + \cancel{\log 2^k}.$$

\Rightarrow Master's $a=2, b=2, k=1, p=0$
II case(A) $\Theta(n^{\log_b a \log^{p+1} n})$
 $\Rightarrow \Theta(k \log k)$
 $\Rightarrow \Theta(k \log k)$

$\Rightarrow \boxed{k = \log n}$

$$\Theta(\log n \log(\log n)) \checkmark$$

$T(n) = 2T(n/2) + n \lg n.$
 $a=2, b=2, k=1, p=1$

① Compare $a, b^k \rightarrow 2, 2$.
 $a = b^k \rightarrow$ ② condition.
 ↓
 ① condition $p > -1$.

$T(n) = \Theta(n^{\log_2 2} \lg^{p+1} n)$
 $= \Theta(n^{\lg 2^2} \lg^2 n)$
 ~~$= (n \lg^2 n)$~~

✗

② $T(n) = 2T(n/2) + (\overline{n \lg n})$

$T(n) = 2T(n/2) + n \lg^{-1} n.$
 $\Rightarrow a=2, b=2, k=1, p=-1$.

① condition \rightarrow ② condition $p = -1$.
 $T(n) = \Theta(n^{\lg_2 2} \lg \lg n)$
 $= \Theta(n^{\lg 2^2} \lg \lg n). = (n(\lg \lg n))$

✗

③ $T(n) = 2T(n/4) + n^{0.51}$
 $a=2, b=4, k=0.51, p=0$.

Ans. $a < b^k$. ③ Case - $T(n) = \Theta(n^k \lg^p n)$
 ~~$= \Theta(n^{0.51} \lg^0 n)$~~
 $= (n^{0.51})$

~~★~~ $T(n) = 0.5 T(n/2) + \gamma n$.
 for the master's theorem α has to be ≥ 1
 which is not. so this is not valid.
 As per the master's theorem.

~~②~~ $T(n) = 6T(n/3) + n^2 \log n$,
 $a=6, b=3, k=2, p=1$.

$a < b^k$ (3 condition) $\Rightarrow T(n) = \Theta(n^k \log^p n)$
 $6 < 3^2$.
 $= \Theta(n^2 \log^2 n)$ ✓

~~★~~ $T(n) = \sqrt{2} T(n/2) + \log n$.
 $a = \sqrt{2}, b = 2, k = 0, p = 1$

$a > b^k$. (1 condition) $\Rightarrow T(n) = \Theta(n^{\log_b a})$

$\sqrt{2} > 2^0$
 $\therefore T(n) = \Theta(n^{\log_{\sqrt{2}} \sqrt{2}}) = \Theta(n^{\frac{1}{2}}) = \boxed{\sqrt{n}}$ ✓

$\log_{\sqrt{2}} \sqrt{2} = 1$
 $\log_{\sqrt{2}} \sqrt{2} = \log_2 2 = 1$

$\therefore T(n) = \Theta(n^1) = \Theta(n)$

~~★~~ $T(n) = 2T(n/2) + \sqrt{n}$

$a=2, b=2, k=1/2, p=0$

(1) $a > b^k$ (1 case) $\Rightarrow T(n) = \Theta(n^{\log_b a})$
 $2 > 2^{1/2}$
 $= \Theta(n^{\log_2 2})$
 $= \Theta(n)$

① Apply masters

$$T(n) = 4T(\sqrt{n}) + \log^5 n.$$

let $n = 2^k$.

$$k = \log n.$$

$$T(2^k) = 4T(2^{k/2}) + k^5.$$

let $T(2^k) = S(k)$.

$$S(k) = 4S(k/2) + k^5.$$

$$a=4, b=2, k=5, P=0.$$

$a < b^k$. III case.

$$\frac{P \geq 0}{\therefore} \rightarrow \underline{\underline{n^k \log^P n}}.$$

$$\Rightarrow \underline{\underline{\Theta(k^5)}} \quad \checkmark$$

② Big O

$$T(n) = T(n-1) + \sqrt{n}, \quad n \geq 1 \quad [n-k=0]$$
$$= 0 \quad \text{else.}$$

$$\therefore T(n) = T(n-3) + \sqrt{n-2} + \sqrt{n-1} + \sqrt{n}.$$

$$= T(n-k) + \sqrt{n-(k-1)} + \dots + \sqrt{n}.$$

$$= T(n-k) + \sqrt{1} + \sqrt{2} + \dots + \sqrt{n}.$$

$$= 0 + \boxed{}$$

$$= \sum n^x = \int_1^n \sqrt{x} dx.$$

in Φ

$$\approx \frac{2}{3}(n^{3/2} - 1) = \underline{\underline{O(n^{3/2})}}$$

put ①: functn which grows slowest (means min. time).

$$\textcircled{1} 2^{\lg n} \quad \textcircled{2} n^{10} \quad \textcircled{3} (\sqrt{\lg n})^{\lg^2 n} \quad \textcircled{4} \lg n^{\sqrt{\lg n}}$$

$$\textcircled{5} 2^{2\sqrt{\lg n \lg n}}.$$

Ans. $\textcircled{1} 2^{\lg 2} = n^{\lg 2}.$
 $\textcircled{2} n^{10}. \quad \textcircled{1} \ll \textcircled{2}.$

$$\textcircled{3} (\sqrt{\lg n})^{\lg^2 n} = \underline{\underline{(\lg n)^{\frac{\lg^2 n}{2}}}} \Rightarrow \\ \textcircled{4} \underline{\underline{\lg n^{\lg n^{1/2}}}} \quad \left((\lg n)^{1/2} (\lg n)^2 \right) \text{ for large value of } n,$$

$$\textcircled{4} \ll \textcircled{3}$$

$$(\lg 2^{300}) (\lg 2^{\frac{300}{2}})$$

$$\leq \frac{(300)^2}{2}.$$

check b/w ①, ④ $(\lg n \sqrt{\lg n}) \leq \underline{\underline{2^{\lg 2} (\lg n)^{\sqrt{\lg n}}}}$

$\cancel{\textcircled{1}} 2^{\lg n}.$ compare $\lg n \quad \lg(\lg n^{\sqrt{\lg n}})$
 $f_1 \gg f_4. \quad \lg n > \sqrt{\lg n} \lg \lg n$

$$\therefore T(n) = \sqrt{n} T(\sqrt{n}) + n, \quad n > 2. \quad \left. \begin{array}{l} \\ \\ \text{else} \end{array} \right\}$$

let $n = 2^k$, $\sqrt{n} = 2^{k/2}$. $k = \log n$.

$$T(2^k) = 2^{k/2} (T(2^{k/2})) + 2^k.$$

Divide by 2^k

$$\frac{T(2^k)}{2^k} = \frac{2^{k/2} T(2^{k/2})}{2^k} + 1.$$

$$\therefore \frac{T(2^k)}{2^k} = \underbrace{T(2^{k/2})}_{2^{k/2}} + 1.$$

Let it be $y(k)$

$$\begin{aligned} \frac{2^{k/2}}{2^k} &= \frac{2^{k/2} \cdot \frac{1}{2} - 1}{2^k} \\ &= \frac{2^{k/2} - 2}{2^k} \\ &= \frac{1}{2} \cdot 2^{k/2} - \frac{2}{2^k} \end{aligned}$$

$$y(k) = y(k/2) + 1.$$

Master. $a=1$ $b=2$ $n=0, p=0$.

$$y(k) = n^{\log_b a} \underbrace{\log^{p+1} n}_{\text{Ans.}}.$$

$$\therefore \frac{T(2^k)}{2^k} = y(k) \Rightarrow T(2^k) = 2^k \log k.$$

as $\Rightarrow n=2^k, k=\log n \rightarrow T(n) = \underbrace{n^{\log \log n}}_{\text{Ans.}}$

$$\textcircled{2}: n^2 + O(n^2) = \Theta(n^2) \text{ why.}$$

In this ex. $\rightarrow O(n^2)$ means terms can have value lower than n^2 , like constant, linear function of n , ... but not more than this.

n^2 means. at least n^2 must be there.

$$\textcircled{3}: f(n) = \Omega(n), g(n) = O(n), h(n) = \Theta(n).$$

$$f(n) \cdot g(n) + h(n) = ?$$

$$3: \Omega(n) \cdot O(n)$$

```

    for (int i=0 i<n i++)
        for (int j=i j<i^2 j++)
            if (j/i == 0)
                for (int k=0 k<j k++)
                    print ();
    }

```

Ans. $i=2$, for $j = 2, 4$, inner loop will execute.

$j=2 \rightarrow$ inner loop execute 2 times.

$j=4$. " " 4 times.

\Rightarrow Total 6 times.

$i=3$. $j=3, 6, 9$, inner loop will execute.

for $j=3$, for $k=3$ times

$j=6$, $k=6$ times

$j=9$, 9 times.

Pattern is for $i=m$, inner loop (k) will execute 3 times.
for value of $j = m, 2m, 3m \dots$

A.P. $\rightarrow m + 2m + 3m + \dots m^2$.

Total = $m/2 (m+m^2)$.

$$\sum_{m=1}^n \frac{m^2}{2} + \frac{m^3}{2} = O(n^4)$$

Q. which one grows faster?

Ans

$$T(n) = 4T(n/2) + 10n.$$

$$T(n) = 8T(n/3) + 21n^2.$$

$$T(n) = 16T(n/4) + 10n^2.$$

$$T(n) = 25T(n/5) + 20(n \log n)^{1.5}$$

- (a) $T(n) = O(n^2)$ (d) $O(n^2)$
 (b) $O(n^2)$ (c) $\underline{O(n^2 \log n)}$
- J/ Ans.

Q. (A) $2^{2\sqrt{\log n}} < 2^{\sqrt{\log n}} < n$

(B) $2^{\sqrt{\log n}} < n < 2^{2\sqrt{\log \log n}}$.

(C) $n < 2^{\sqrt{\log n}} < 2^{2\sqrt{\log \log n}}$

Let $n = 2^{2^k}.$

~~(A)~~ $2^{\sqrt{\log 2^{2^k}}} = 2^{\sqrt{2^k}} \checkmark \rightarrow \text{check powers.}$

$$2^{2^{\sqrt{\log 2^k}}} = \log_2(2^{\sqrt{k}}).$$

To take log $\rightarrow \sqrt{2^k}$

$\rightarrow \sqrt{256} \quad 2^{\sqrt{k}}$

$(2^{128}, 2^{128})$

$\cancel{2^{128}} \quad \cancel{2^{128}}$

Let $k = \frac{6}{256} \cancel{256} \quad 12.8.$

$$\begin{aligned}
 & \text{Given } n = 2^{(2^8)} \\
 \Rightarrow & 2^{\frac{\sqrt{\log \log n}}{2}} = 2^{\frac{\log(2^8)}{2}} = 2^{\frac{8}{2}} = 2^4 = 16 \\
 & \Rightarrow \boxed{16} \\
 \Rightarrow & 2^{\frac{\sqrt{2^8}}{2}} = 2^{\frac{2^4}{2}} = 2^{\frac{16}{2}} = 2^8 = 256 \\
 \Rightarrow & \boxed{256} \\
 \Rightarrow & A < B < C \quad \text{option A correct.}
 \end{aligned}$$

$$\begin{aligned}
 & T(n) = 4T(n/2) + n^2 \cdot \sqrt{2}, \quad T(1) = 1 \\
 & T(n/2) = 4T(n/4) + (n/2)^2 \cdot \sqrt{2} \\
 & T(n/4) = 4 \cdot (T(n/8) + (n/4)^2 \cdot \sqrt{2}) \\
 \Rightarrow & T(n) = 4^3 T(n/2^3) + 4^3 \cdot (n/2^3)^2 \cdot \sqrt{2} \\
 & \quad + 4 \cdot (n/2^2)^2 \cdot \sqrt{2} + 4^0 \cdot (n/2^0)^2 \cdot \sqrt{2} \\
 \Rightarrow & T(n) = 4^k T(n/2^k) + \\
 & = 4^k T\left(\frac{n}{2^k}\right) + \\
 & = 4^k + \dots \\
 & = \sqrt{2} (n^2 + n^2 + n^2 + \dots) k \\
 & = \Theta(n^2 \log n).
 \end{aligned}$$