Espace Étalé of a Presheaf

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Remark: This is actually given as a exercises question in Hartshorne.

Let X be the topological space and $\mathcal{F}$ be the presheaf on X.
Define topological space $Sp\acute{e}(\mathcal{F})$, called espace étalé of $\mathcal{F}$ as follows:
As a set $Sp\acute{e}(\mathcal{F}) = \bigcup_{p \in X} \mathcal{F}_p$ and define projection map $\pi : Sp\acute{e}(\mathcal{F}) \longrightarrow X$ given by $s_p \mapsto p$. For each open set $U \subseteq X$ and each sectiom $S \in \mathcal{F}(U)$ we obtain $\bar{s} : U \longrightarrow Sp\acute{e}(\mathcal{F})$ given by $p \mapsto s_p = [U, s] \in \mathcal{F}_p$. Then we have $\pi \circ \bar{s} = id_U$.We make $Sp\acute{e}(\mathcal{F})$ into topological space by giving it the strongest topology such that $\bar{s} : U \longrightarrow Sp\acute{e}(\mathcal{F})$ for all U. and all $s \in \mathcal{F}(U)$ are continuous.
Denote $\mathcal{O}(X)$ to be the set of all open subsets of X then $\bar{s}(U)$ form the basic open subset of $Sp\acute{e}(\mathcal{F})$ for $U \in \mathcal{O}(X)$: fix $x \in X$ and $s_x \in Sp\acute{e}(\mathcal{F})$ then $s_x \in \mathcal{F}$ so there exists $U \subseteq X$ such that $x \in U$ and $s \in \mathcal{F}(U) \implies s_x \in \bar{s}$. Therefore $\{\bar{s}(U)\}_{U \in \mathcal{O}(X)}$ cover $Sp\acute{e}(\mathcal{F})$. Now, take $u_x \in \bar{s}(U) \bigcap \bar{t}(V)$ which will imply that $u_x = s_x$ and $u_x = t_x$ and hence there will exist $W \subset U \bigcap V$ with $x \in W$ and $s|_W = u|_W = t|_W \in \mathcal{F}(W)$, So we have the following maps $\overline{s|_W} : W \longrightarrow Sp\acute{e}(\mathcal{F})$ and similarly maps $\overline{u|_W}$ and $\overline{t|_W}$. From above we have $\overline{s|_W}(W) = \overline{u|_W}(W) = \overline{t|_W}(W)$. For $x \in W$ and $u|_W \in \mathcal{F}(W)$, $(u|_W)_x \in \overline{u|_W}(W)$. Therefore $u_x = (u|_W)_x \in \overline{u|_W}(W) = \overline{t|_W}(W) = \overline{s|_W}(W) \subset \bar{s}(U)$ and similarly
$u_x = (u|_W)_x \in \overline{u|_W}(W) = \overline{s|_W}(W) = \overline{t|_W}(W) \subset \bar{t}(v)$.Hence
$u_x \in \overline{u|_W}(W) \subseteq \bar{s}(U) \bigcap \bar{t}(V)$. Also, $\bar{s} : U \longrightarrow Sp\acute{e}(\mathcal{F})$ is continuous since for $\bar{t}(V)$ a basic open subset of $Sp\acute{e}(\mathcal{F})$ we have $\bar{s}^{-1}(\bar{t}(V)) = \{p \in U | \bar{s}(p) \in \bar{t}(V)\} = \{p \in U \bigcap V | s_p = t_p\}$ which is open: let $x \in \bar{s}^{-1}(\bar{t}(V))$ then $s_p = t_p$ and by definition there exists $W \subseteq U \bigcap V$ with $x \in W$ and $s|_W = t|_W$. Therefore $W \subseteq \bar{s}^{-1}(\bar{t}(V))$.Inclusion,$i$, of $\bar{s}(U)$ in $Sp\acute{e}(\mathcal{F})$ is continous and π is continuous so $\pi \circ i$ is continuous.

Definition 1 (Local homeomorphism) *$f : X \longrightarrow Y$ is continuous then for all $x \in X$ there exists open set U containing x such that $f(U)$ is open in Y and $f|_U : U \longrightarrow f(U)$ is homeomorphism.*

Now we show that $\pi : Sp\acute{e}(\mathcal{F}) \longrightarrow X$ is local homeomorphism. Let $s_p \in Sp\acute{e}(\mathcal{F})$ then there exists $U \in \mathcal{O}(X)$ and $s \in \mathcal{F}(U)$. By definition $s_p = [U, s]$ with $p \in U$ and $s \in \mathcal{F}(U)$ and $\pi(\bar{s}(U)) = U$ is open .
To show: $\pi|_{\bar{s}(U)} : \bar{s}(U) \longrightarrow \pi(\bar{s}(U))$ is homeomorphism.
$\pi|_{\bar{s}(U)} : \bar{s}(U) \longrightarrow U$ is continuous and $\bar{s} : U \longrightarrow \bar{s}(U)$ is continuous.Therefore $\pi_{\bar{s}(U)}$ has continuous inverse given by $\bar{s}$. Hence π is local homeomorphism and $\bar{s}$ is a section of $Sp\acute{e}(\mathcal{F})$.

Now, $\mathcal{F}^+(U) = \{\bar{s} : U \longrightarrow Sp\acute{e}(\mathcal{F}) | \bar{s}(x) = s_x$ for all $s \in \mathcal{F}(U)\}$.
For open subset U define $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{F}^+(U)$ by $s \mapsto \bar{s}$.
Consider the following construction,let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $X.\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be morphism of presheaves. If $\sigma \in \mathcal{F}^+(U)$ and x is point in U then $\sigma(x) \in \mathcal{F}_x$, that is $\sigma(x) = [V, s]$ for $s \in \mathcal{V}$. Define $\sigma'(x) = (\psi_V(s))_x$ then $\sigma' : U \longrightarrow Sp\acute{e}(\mathcal{G})$ is well defined and continuos such that $\pi \circ \sigma' = id_U$. Define $\psi_U^+(\sigma) = \sigma'$. This gives $\psi_U^+ : \mathcal{F}^+(U) \longrightarrow \mathcal{G}^+(U)$ morphism of sheaves and ψ is unique since sheafification does not alter stalks.If $\mathcal{G}$ is a sheaf then we get that $\mathcal{F}^+$ is the sheafification of $\mathcal{F}$.