

A Dynamic Programming Approach To Length-Limited Huffman Coding

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Abstract—The “state-of-the-art” in Length Limited Huffman Coding algorithms is the $\Theta(ND)$ -time, $\Theta(N)$ -space one of Hirschberg and Larmore, where $D \leq N$ is the length restriction on the code. This is a very clever, very problem specific, technique. In this note we show that there is a simple Dynamic-Programming (DP) method that solves the problem with the same time and space bounds. The fact that there was an $\Theta(ND)$ time DP algorithm was previously known; it is a straightforward DP with the *Monge* property (which permits an order of magnitude speedup). It was not interesting, though, because it also required $\Theta(ND)$ space.

The main result of this paper is the *technique* developed for reducing the space. It is quite simple and applicable to many other problems modeled by DPs with the Monge property. We illustrate this with examples from web-proxy design and wireless mobile paging.

Index Terms—Prefix-Free Codes, Huffman Coding, Dynamic Programming, Web-Proxies, Wireless Paging, the Monge property.

I. INTRODUCTION

Optimal prefix-free coding, or *Huffman coding*, is a standard compression technique. Given an *encoding alphabet* $\Sigma = \{\sigma_1, \dots, \sigma_r\}$, a *code* is just a set of words in Σ^* . Given n probabilities or nonnegative frequencies $\{p_i : 1 \leq i \leq n\}$, and associated code $\{w_1, w_2, \dots, w_n\}$ the *cost* of the code is $\sum_{i=1}^n p_i |w_i|$ where $|w_i|$ denotes the length of w_i . A code is *prefix-free* if no codeword w_i is a prefix of any other codeword w_j . An *optimal* prefix-free code for $\{p_i : 1 \leq i \leq n\}$ is a prefix-free code that minimizes its cost among all prefix-free codes.

In [1], Huffman gave the now classical $O(n \log n)$ time algorithm for solving this problem. If the p_i 's are given in sorted order, Huffman's algorithm can be improved to $O(n)$ time [2]. In this note we will always assume that the p_i 's are presorted and that $p_1 \leq p_2 \leq \dots \leq p_n$.

In some applications, it is desirable that the length of all code words are bounded by a constant, i.e., $|w_i| \leq D$ where D is given. The problem of finding the minimal cost prefix-free code among all codes satisfying this length constraint is the *length-limited Huffman coding* (LLHC) problem, which we will consider here. Fig. 1 gives an example of inputs for which the Huffman code is *not* the same as the length-limited Huffman code.

The first algorithm for LLHC was due to Karp [3] in 1961; his algorithm is based on integer linear programming (ILP),

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which, using standard ILP solving techniques, leads to an exponential time algorithm. Gilbert [4] in 1971 was interested in this problem because of the issue of inaccurately known sources; since the probabilities p_i 's are not known precisely, a set of codes with limited length will, in some sense, be “safe”. The algorithm presented in [4] was an enumeration one and therefore also runs in exponential time. In 1972 Hu and Tan [5] developed an $O(nD2^D)$ time Dynamic Programming (DP) algorithm. The first polynomial time algorithm, running in $O(n^2D)$ time and using $O(n^2D)$ space, was presented by Garey in 1974 [6]. Garey's algorithm was based on a DP formulation similar to that developed by Knuth for deriving optimal binary search trees in [7] and hence only works for binary encoding alphabets. A decade later, Larmore [8] gave an algorithm running in $O(n^{3/2}D \log^{1/2} n)$ time and using $O(n^{3/2}D \log^{-1/2} n)$ space. This algorithm is a hybrid of [5] and [6], and therefore also only works for the binary case. This was finally improved by Larmore and Hirschberg [9] who gave a totally different algorithm running in $O(nD)$ time and using $O(n)$ space. In that paper, the authors first transform the length-limited Huffman coding problem to the *Coin Collector's* problem, a special type of Knapsack problem, and then, solve the Coin Collector's problem by what they name the *Package-Merge* algorithm. Their result is a very clever special case algorithm developed for this specific problem.

Theoretically, Larmore and Hirschberg's result was later superseded for the case¹ $D = \omega(\log n)$ by two algorithms based on the *parametric search* paradigm [10]. The algorithm by Aggarwal, Schieber and Tokuyama [11] runs in $O(n\sqrt{D \log n} + n \log n)$ time and $O(n)$ space. A later improvement by Schieber [12] runs in $n2^{O(\sqrt{\log D \log \log n})}$ time and uses $O(n)$ space. These algorithms are very complicated, though, and even for $D = \omega(\log n)$, the Larmore-Hirschberg one is the one used in practice [13], [14]. For completeness, we point out that the algorithms of [9], [11], [12] are all only claimed for the binary ($r = 2$) case but they can be extended to work for the non-binary ($r > 2$) case using observations similar to those we provide in Appendix A for the derivation of a DP for the generic r -ary LLHC problem.

Shortly after [9] appeared, Larmore and Przytycka [15], [16], in the context of parallel programming, gave a simple dynamic programming formulation for the binary Huffman coding problem. Although their DP was for regular Huffman coding and not the LLHC problem, we will see that it is quite easy to modify their DP to model the LLHC problem. It is then straightforward to show that their formulation also permits

¹ $f(n) = \omega(g(n))$ if $\exists N, c > 0$ such that $\forall n > N$, $f(n) \geq g(n)$.

constructing the optimal tree in $\Theta(nD)$ time by constructing a size $\Theta(nD)$ DP table. This is done in Section II. This straight DP approach would not be as good as the Larmore-Hirschberg one, though, because, like many DP algorithms, it requires maintaining the entire DP table to permit backtracking to construct the solution, which would require $\Theta(nD)$ space. The main result of this note is the development of a simple technique (section III) that permits reducing the DP space consumption down to $O(n)$, thus matching the Larmore-Hirschberg performance with a straightforward DP model. Our technique is not restricted to Length-Limited coding. It can be used to reduce space from $O(nD)$ to $O(n+D)$ in a variety of $O(nD)$ time DPs in the literature. In Section IV we illustrate with examples from the D-median on a line problem (placing web proxies on a linear topology network) [17] and wireless paging [18].

II. THE DYNAMIC PROGRAMMING FORMULATION

Set $S_0 = 0$ and $S_m = \sum_{i=1}^m p_i$ for $1 \leq m \leq n$. Larmore and Przytycka [16] formulated the binary Huffman coding problem as a DP (1) where $H(0) = 0$ and for $0 < i < n$:

$$H(i) = \min_{\max\{0, 2i-n\} \leq j < i} (H(j) + S_{2i-j}). \quad (1)$$

In this DP, $H(n-1)$ is the cost of the optimal Huffman code. Another version of this DP, generalized for unequal-cost binary coding alphabets, also appeared in [19].

It is straightforward to modify (1) to model the binary LLHC problem. The resulting DP is

$$H(d, i) = \begin{cases} 0 & d = 0, i = 0 \\ \infty & d = 0, 0 < i < n \\ \min_{0 \leq j \leq i} (H(d-1, j) + c_{i,j}^{(d)}) & d > 0, 0 \leq i < n \end{cases} \quad (2)$$

where $H(D, n-1)$ will denote the cost of the optimal length-limited Huffman code and

$$c_{i,j}^{(d)} = \begin{cases} 0 & i = j = 0 \\ S_{2i-j} & \max\{0, 2i-n\} \leq j < i \\ \infty & \text{otherwise.} \end{cases} \quad (3)$$

In the next subsection we will see an interpretation of this DP (which also provides an interpretation of (1)). In order to make this note self-contained, a complete derivation of the DP for the r -ary alphabet case is provided in Appendix A.

As far as running time is concerned, (1) appears to a-priori require $O(n^2)$ time to fill in its corresponding DP table. [16] used the inherent concavity of S_m to reduce this time down to $O(n)$ by transforming the problem to an instance of the Concave Least Weight Subsequence (CLWS) problem and using one of the known $O(n)$ time algorithms, e.g., [20], for solving that problem.

Similarly, (2) appears to a-priori require $\Theta(n^2D)$ time to fill in its DP table. We will see that we may again use the concavity of S_m to reduce this down by an order of magnitude, to $O(nD)$ by using the SMAWK algorithm [21] for finding row-minima of matrices as a subroutine. Unlike the CLWS algorithms, the SMAWK one is very simple to code and very efficient implementations are available in different

packages, e.g., [22], [23]. In the conclusion to this note, after the application of the technique becomes understandable, we will explain why [16] needed to use the more complicated CLWS routine to solve the basic DP while we can use the simpler SMAWK one.

The $O(nD)$ DP algorithm for solving the LLCH problem, while seemingly never explicitly stated in the literature, was known as folklore. Even though it is much simpler to implement than the $O(nD)$ Larmore and Hirschberg [9] Package-Merge algorithm it suffers from the drawback of requiring $\Theta(nD)$ space. The main contribution of this note is the observation that its space can be reduced down to $O(n+D)$ making it comparable with Package-Merge. Note that since, for the LLHC problem we may trivially assume $D \leq n$, this implies a space requirement of $O(n)$. Furthermore, our space improvement will work not only for the LLHC problem but for all DPs in form (2) where the $c_{i,j}^{(d)}$ satisfy a particular property.

A. The meaning of The DP

We quickly sketch the meaning of the DP (2) for the binary case. Figures 1 and 2 illustrate this sketch. We note that in order to stress the parts important to our analysis, our formalism is a bit different than [16], [19]. A complete derivation of the DP for the r -ary case with the appropriate general versions of the lemmas and observations stated below along with their proofs, is provided in Appendix A.

It is standard that there is a 1–1 correspondence between binary prefix-free code with n words and binary tree with n leaves. The set of edges from an internal node to its children are labeled by a **0** or **1**. Each leaf corresponds to a code word, which is the concatenation of the characters on the root-to-leaf path. The cost of the code equals the *weighted external path length* of the tree. So we are really interested in finding a binary tree with minimum weighted external path length.

Denote the height of the tree by h . The bottommost leaves are on level 0; the root on level h . Optimal assignments of the p_i 's to the leaves always assign smaller valued p_i 's to leaves at lower levels.

A node in a binary tree is *complete* if it has two children and a tree is *complete* if all of its internal nodes are complete. A min-cost tree must be complete, so we restrict ourselves to complete trees. A complete tree T of height h can be completely represented by a sequence (i_0, i_1, \dots, i_h) , where i_k denotes the number of internal nodes at levels $\leq k$. Note that, by definition, $i_0 = 0$, $i_h = n - 1$. Also note that every level must contain at least one internal node so $i_0 < i_1 < \dots < i_h$. Finally, it is straightforward (see Appendix A) to show that the total number of leaves on level $< k$ is $2i_k - i_{k-1}$, so $2i_k - i_{k-1} \leq n$ for all k . For technical reasons, because we will be dealing with trees having height *at most* h (but not necessarily *equal to* h), we allow initial padding of the sequence by **0**s so a sequence representing a tree will be of the form (i_0, i_1, \dots, i_h) that has the following properties

Definition 1: Sequence (i_0, i_1, \dots, i_h) is *valid* if

- $\exists t > 0$ such that $i_0 = i_1 = \dots = i_t = 0$,
- $0 < i_{t+1} < i_{t+2} < \dots < i_h \leq n - 1$
- $2i_k - i_{k-1} \leq n$ for all $1 \leq k \leq h$.

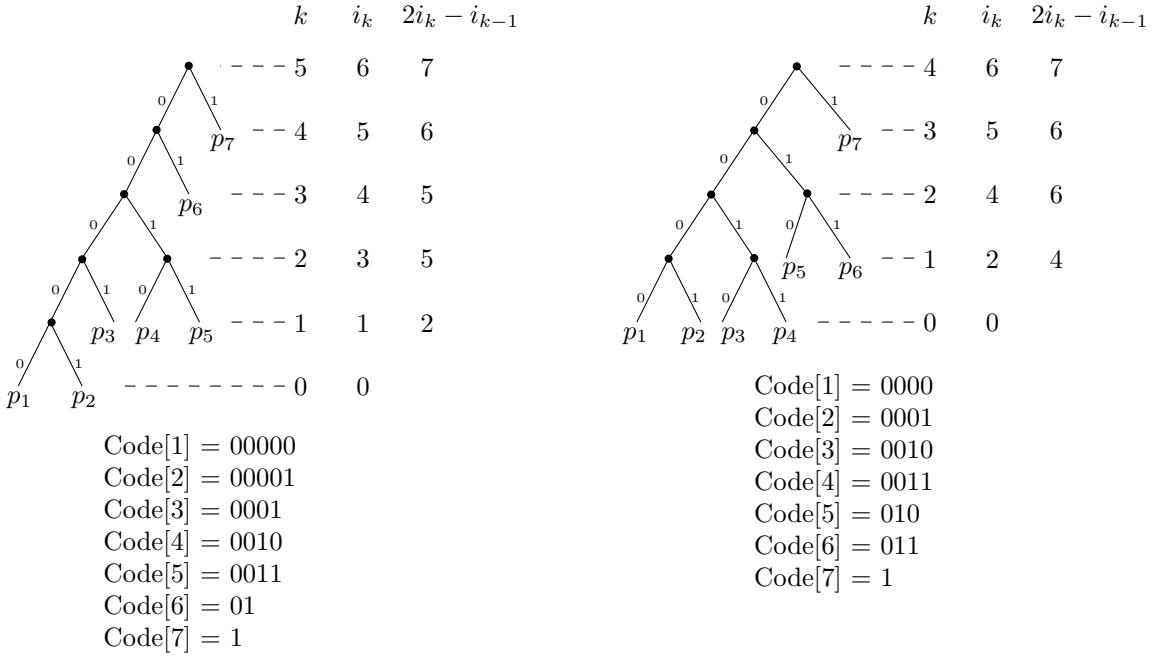


Fig. 1. Two trees and their corresponding sequences \mathcal{I} and codes. The left tree has sequence $\mathcal{I}_1 = (0, 1, 3, 4, 5, 6)$. The right tree has sequence $\mathcal{I}_2 = (0, 2, 4, 5, 6)$. Note that, for both trees, $2i_k - i_{k-1}$ is the number of leaves below level k . For input frequencies $(p_1, \dots, p_7) = (1, 1, 2, 2, 4, 5, 9)$. The left tree is an optimal Huffman code while the right tree is an optimal length-limited Huffman code for $D = 4$. Note that we allow padding sequences with initial **0**s, so the right tree could also be represented by sequences $(0, 0, 2, 4, 5, 6)$, $(0, 0, 0, 2, 4, 5, 6)$, etc..

$d \backslash i$	0	1	2	3	4	5	6
0	0	∞	∞	∞	∞	∞	∞
1	0	2	6	13	∞	∞	∞
2	0	2	6	10	19	35	∞
3	0	2	6	10	18	32	57
4	0	2	6	10	18	31	54

$d \backslash i$	0	1	2	3	4	5	6
0							
1	0	0	0	0	-	-	-
2	0	0	1	1	2	3	-
3	0	0	1	1	3	4	5
4	0	0	1	1	3	4	5

Fig. 2. Solving the DP in equation 2 for $(p_1, \dots, p_7) = (1, 1, 2, 2, 4, 5, 9)$ with $D = 4$. $H(d, i)$ is the value defined by (2); $J(d, i)$ is the index j for which the value $H(d, i)$ in (2) is achieved. The circled entries yield the sequence $(0, 2, 4, 5, 6)$ (the 6 comes from the fact that we are calculating $H(4, 6)$) which is exactly the sequence \mathcal{I}_2 from Figure 1. The righthand tree in Figure 1 is therefore an optimal length-limited Huffman code for $D = 4$.

A sequence is *complete* if it is valid and $i_h = n - 1$.

We can rewrite the cost function for a tree in terms of its complete sequence.

Lemma 1: If complete sequence (i_0, i_1, \dots, i_h) represents a tree, then the cost of the tree is $\sum_{k=1}^h S_{2i_k - i_{k-1}}$. (Note that padding complete sequences with initial **0**s does not change the cost of the sequence.)

We may mechanically extend this cost function to *all* valid sequences as follows.

Definition 2: For valid $\mathcal{I} = (i_0, i_1, \dots, i_h)$, set

$$\text{cost}(\mathcal{I}) = \sum_{k=1}^h S_{2i_k - i_{k-1}}.$$

\mathcal{I} is *optimal* if $\text{cost}(\mathcal{I}) = \min_{\mathcal{I}'} \text{cost}(\mathcal{I}')$ where the minimum is taken over all length h sequences $\mathcal{I}' = (i'_0, i'_1, \dots, i'_h)$ with $i'_h = i_h$, i.e., all sequences of the same length that end with the same value.

Our goal is to find optimal trees by using the DP to optimize over valid sequences. An immediate issue is that not all complete sequences represent trees, e.g., $\mathcal{I} = (0, 3, 4, 5)$ is complete for $n = 6$ but, by observation, does not represent a tree. The saving fact is that even though not all complete sequences represent trees, all *optimal* complete sequences represent trees.

Lemma 2: An optimal valid sequence ending in $i_h = n - 1$ always represents a tree.

Thus, to solve the LLHC problem of finding an optimal tree of height $\leq D$, we only need to find an optimal valid sequence of length $h = D$ ending with $i_D = n - 1$ (reconstructing the tree from the sequence can be done in $O(n)$ time). In the DP defined by equations (2) and (3), $H(d, j)$ clearly models the recurrence for finding an optimal valid sequence (i_0, i_1, \dots, i_d) of length d with $i_d = j$ so this DP solves the problem.

Note that, a-priori, filling in the DP table $H(\cdot, \cdot)$ one

entry at a time seems to require $O(n^2D)$ time. We will now sketch the standard way of reducing this time down to $O(nD)$. Before doing so we must distinguish between the *value problem* and the *construction problem*. The value problem would be to calculate the value of $H(D, n - 1)$. The construction problem would be to construct an optimal valid sequence $\mathcal{I} = (I_1, I_2, \dots, I_D)$ with $I_D = n - 1$ and $\text{cost}(\mathcal{I}) = H(D, n - 1)$. This would require backtracking through the DP table by setting $I_0 = 0$, $I_D = n - 1$ and finding I_1, I_2, \dots, I_{D-1} such that

$$\forall 0 < d \leq D, \quad H(d, I_d) = H(d - 1, I_{d-1}) + c_{I_d, I_{d-1}}^{(d)}. \quad (4)$$

B. Solving the Value problem in $O(nD)$ time

Definition 3: An $n \times m$ matrix M is *Monge*² if for $0 \leq i < n - 1$ and $0 \leq j < m - 1$

$$M_{i,j} + M_{i+1,j+1} \leq M_{i+1,j} + M_{i,j+1} \quad (5)$$

The Monge property can be thought of as a discrete version of concavity. It appears implicitly in many optimization problems for which it permits speeding up their solutions ([24]) provides a nice survey). One of the classic techniques used is the SMAWK algorithm for finding row-minima.

Given an $n \times m$ matrix M , the *minimum* of row i , $i = 1, \dots, n$ is the entry of row i that has the smallest value; in case of ties, we take the rightmost entry. Thus, a solution of the row-minima problem is a collection of indices $j(i)$, $i = 1, \dots, n$ such that

$$M_{i,j(i)} = \min_{0 \leq j < m} M_{i,j} \text{ and } j(i) = \max\{j : M_{i,j} = M_{i,j(i)}\}. \quad (6)$$

Figure 3 gives four examples of Monge matrices and their row minima.

At first glance it seems that we would have to examine all of the mn entries in M to find the row minima but, [21] proved³

Lemma 3: (The SMAWK algorithm [21])

Let M be a $n \times m$ Monge matrix such that entry $M_{i,j}$ can be calculated in $O(1)$ time. Then the row minima problem on M can be solved in $O(n + m)$ time.

The constant hidden by the $O()$ is very small, around 2, and the algorithm is easy to code, so it is quite practical to use.

Note that the SMAWK algorithm doesn't have the time available to build the entire $n \times m$ matrix. Instead, it searches through the matrix in a clever way, constructing entries as needed. One standard use of the SMAWK algorithm is in the speedup of dynamic programs that have Monge properties.

Definition 4: A DP in the form (2) is *Monge* if, for all $1 \leq d \leq D$ and $0 \leq j \leq i < n$,

$$c_{i,j}^{(d)} + c_{i+1,j+1}^{(d)} \leq c_{i+1,j}^{(d)} + c_{i,j+1}^{(d)} \quad (7)$$

Note: In many DP applications, it is possible that for some i, j , $c_{i,j}^{(d)} = \infty$. The inequality in (7) treats ∞ in the natural way, e.g.,

²This property is sometimes alternatively defined by: for $0 \leq i < i' < n$ and $0 \leq j < j' < m$ $M_{i,j} + M_{i',j'} \leq M_{i',j} + M_{i,j'}$ but it is well known, see, e.g., [24], that this is equivalent to (5).

³Technically, [21] proved their result for a larger class, the *totally-monotone matrices*. But all applications in the literature seem to be for Monge matrices.

for any constant c ; $c < \infty$ and $c + \infty = \infty$. Also, $\infty + \infty = \infty$. The SMAWK algorithm permits the use of ∞ in this way.

Now suppose that a DP defined by (2) is Monge. For $d = 1, 2, \dots, D$ define matrix $M^{(d)}$ by

$$M_{i,j}^{(d)} = \begin{cases} H(d - 1, j) + c_{i,j}^{(d)} & \text{if } 0 \leq j \leq i < n \\ \infty & \text{otherwise} \end{cases}$$

Then, from (7), we have

$$\begin{aligned} M_{i,j}^{(d)} + M_{i+1,j+1}^{(d)} &= H(d - 1, j) + H(d - 1, j + 1) + c_{i,j}^{(d)} + c_{i+1,j+1}^{(d)} \\ &\leq H(d - 1, j) + H(d - 1, j + 1) + c_{i+1,j}^{(d)} + c_{i,j+1}^{(d)} \\ &= M_{i+1,j}^{(d)} + M_{i,j+1}^{(d)} \end{aligned}$$

and $M^{(d)}$ is Monge. Note that

$$\begin{aligned} H(d, i) &= \min_{0 \leq j \leq i} (H(d - 1, j) + c_{i,j}^{(d)}) \\ &= \min_{0 \leq j \leq i} M_{i,j}^{(d)} = \min_{0 \leq j \leq N} M_{i,j}^{(d)}. \end{aligned}$$

So, $H(d, i)$ are just the row-minima of $M^{(d)}$. See Figure 3. Since $M^{(d)}$ is Monge, we can use the SMAWK algorithm to, in $O(n)$ time, find *all* of its row minima at one time. More specifically, let $J(d, i)$ and $M_{i,J(d,i)}^{(d)}$ be the corresponding values (6) returned when running SMAWK($M^{(d)}$). Then the algorithm for filling in the table is just to iteratively run down the rows of the table, using SMAWK to fill in each row by using knowledge of the previous row:

Fill_Table
For $d = 1$ to $D - 1$

SMAWK ($M^{(d)}$)

$$\forall 0 \leq i < n \text{ set } H(d, i) = M_{i,J(d,i)}^{(d)}$$

Fig. 4. The $O(nD)$ algorithm for the value problem.

Note that this algorithm uses $\Theta(nD)$ time, since, for each fixed d , the SMAWK algorithm only uses $O(n)$ time. Also note that if we're only interested in the final row, then the algorithm uses only $O(n)$ space, since once row d has been calculated, the values from row $d - 1$ can be thrown away.

We now return to the LLHC problem and show that it can be plugged into the above machinery.

Lemma 4: The $c_{i,j}^{(d)}$ defined in (3) satisfy Monge property (7).

Proof: If $i = j = 0$ the righthand side of (7) is ∞ , so (7) is satisfied.

If $j + 1 = i$ or $2(i + 1) - n > j$, the righthand side of (7) is ∞ , so (7) is satisfied.

If $j + 1 < i$ and $2(i + 1) - n \leq j$, (7) can be rewritten as

$$S_{2i-j} + S_{2(i+1)-(j+1)} \leq S_{2i-(j+1)} + S_{2(i+1)-j} \quad (8)$$

It is easy to verify

$$\begin{aligned} S_{2i-j} + S_{2(i+1)-(j+1)} - S_{2i-(j+1)} - S_{2(i+1)-j} \\ = p_{2i-j} - p_{2i-j+2} \leq 0 \end{aligned}$$

Hence, (8) holds. ■

Thus, from the discussion above, we can find all of the $H(d, i)$ in $\Theta(nD)$ time. In particular, $H(D, n - 1)$ will be the

$M^{(1)}$						
0						
2	∞					
6	∞	∞				
13	∞	∞	∞			
∞	∞	∞	∞	∞		
∞	∞	∞	∞	∞	∞	
∞	∞	∞	∞	∞	∞	∞

$M^{(2)}$						
0						
2	∞					
6	6	∞				
13	10	12	∞			
∞	24	19	21	∞		
∞	∞	∞	35	∞	∞	
∞	∞	∞	∞	32	32	∞
∞	∞	∞	∞	∞	∞	57

$M^{(3)}$						
0						
2	∞					
6	6	∞				
13	10	12	∞			
∞	24	19	18	∞		
∞	∞	∞	32	32	∞	
∞	∞	∞	∞	∞	57	∞

$M^{(4)}$						
0						
2	∞					
6	6	∞				
13	10	12	∞			
∞	24	19	18	∞		
∞	∞	∞	32	31	∞	
∞	∞	∞	∞	∞	54	∞

Fig. 3. The matrices used for calculating the DP tables in Fig. 2. The shaded entries are the row minima. The row minima for $M^{(i)}$ are exactly the row entries in the $H(d, i)$ table in Fig. 2. The column indices of the corresponding row minima are the $J(d, i)$ entries.

cost of the optimal tree with height *at most* D which is the required cost of the optimum D -limited code.

We have thus seen how to solve the value problem in $O(nD)$ time. The difficulty is that *constructing* the optimal tree associated with $H(D, n - 1)$ would require finding the associated optimal valid sequence with $i_D = n - 1$. This would require solving the construction problem by finding all indices I_d in (4). The standard way of solving this problem is to maintain an array storing the $J(d, i)$ values returned by the algorithm. Starting from $H(D, n - 1)$ and backtrack through the $j(\cdot, \cdot)$ array, constructing the corresponding sequence by setting $I_D = n - 1$ and $I_{d-1} = j(d, I_d)$. Unfortunately, this requires maintaining a size $\Theta(nD)$ auxiliary array, which requires too much space.

III. SOLVING THE CONSTRUCTION PROBLEM IN $O(nD)$ TIME AND $O(n + D)$ SPACE

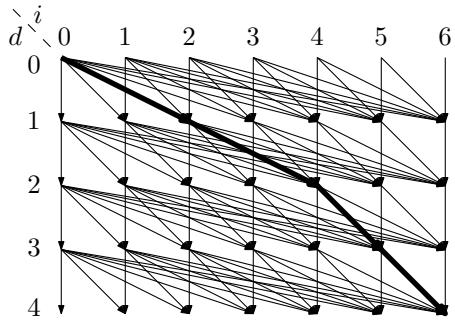


Fig. 5. The dropping-level graph associated with the example from Figures 2 and 3. The bold edges are the minimum cost path from $(0, 0)$ to $(6, 4)$. Note that the i coordinates of the path are $(0, 2, 4, 5, 6)$ which is exactly the sequence of $J(d, i)$'s corresponding to optimal solution of the problem, which is also the sequence corresponding to the optimal tree.

Let V be the grid nodes (d, i) with $0 \leq d \leq D$ and $0 \leq i < n$. Consider the directed graph $G = (V, E)$ in which (d, i) points to all nodes immediately below it and to its right, i.e.,

$$E = \{((d, j), (d+1, i)) \mid (d, j) \in V, d < D, j \leq i\}$$

See Figure 5. Such graphs are sometimes called *dropping level-graphs* [25]. Now assign edge $((d-1, j), (d, i))$ the weight $c_{i,j}^{(d)}$. The length of a path in G will just be the sum of the weights of the edges in the path. The important observation

is that $H(d, i)$ in DP (2) is simply the length of the min-cost path from $(0, 0)$ to (d, i) in this weighted G . More specifically, the value problem is to find the *length of a shortest path* and the construction problem is to find an *actual shortest path*.

A-priori, finding such a path seems to require $O(nD)$ space. There are two different algorithms in the literature for reducing the space down to $O(n + D)$ in related problems.

The first was for finding a maximum common subsequence of two sequences. This reduced down to the problem of finding a *max-length* path in something very similar to a dropping level-graph in which each vertex has bounded indegree and bounded outdegree. Hirschberg [26] developed an $\Theta(nD)$ time, $\Theta(n + D)$ space algorithm for this problem. His algorithm was very influential in the bioinformatics community and its technique is incorporated into many later algorithms e.g [27], [28]. The techniques's performance is very dependent upon the bounded degree of the vertices, which is not true in our case.

The second, due to Munro and Ramirez [25], was exactly for the problem of constructing min-cost paths in full dropping level-graphs. Their algorithm ran in $\Theta(n^2D)$ time and $\Theta(n + D)$ space. Their $\Theta(n^2D)$ time is too expensive for us. We will now see how to reduce this down to $\Theta(nD)$ using the Monge speedup while still maintaining the $\Theta(n + D)$ space.

The general problem will be to construct an optimal $u-w$ path in G where $u = (d_u, i_u)$ is above and not to the left of $w = (d_w, i_w)$, i.e., $d_u < d_w$ and $i_u \leq i_w$. Let $G(u, w)$ be the subgrid with upper-left corner u and lower-right corner w (with associated induced edges from G). First note that, because G is a dropping level-graph, any optimal (min or max cost) $u-w$ path in G must lie completely in $G(u, w)$. Both algorithms [26], [25] start from the same observation, which is to build the path *recursively* i.e., by first (a) finding a point $v = (\bar{d}, \bar{i})$ halfway (by link distance) on the optimal $u-w$ path in $G(u, w)$ and then (b) output the recursively constructed optimal $u-v$ path in $G(u, v)$ and optimal $v-w$ path in $G(v, w)$.

For dropping level-graphs, if $u = (d_1, i_1)$ and $w = (d_2, i_2)$ then the midlevel must be $\bar{d} = \lfloor (d_1 + d_2)/2 \rfloor$. Suppose that we had an algorithm $Mid(u, w)$ that returned a point $v = (\bar{d}, \bar{i})$ on a shortest $u-w$ path in $G(u, w)$. Then, translated into our notation and with appropriate termination conditions the

construction algorithm can be written as:

```
Path( $u, w$ )
1. If  $u = (\bar{d}, j)$  and  $w = (d + 1, i)$  then
2.   output edge  $(u, w)$ 
3. Else if  $u = (d, i)$  and  $w = (d', i)$  then
4.   Output vertical path from  $u$  to  $w$ 
5. Else
6.   set  $v = \text{Mid}(u, w)$ 
7.   Path( $u, v$ ); Path( $v, w$ )
```

Fig. 6. The algorithm for constructing a min-cost u - w path.

(Figure 7 illustrates this idea.) To solve the original problem we just call $\text{Path}(u_0, w_0)$ where $u_0 = (0, 0)$ and $w_0 = (D, n - 1)$. Correctness follows from the fact that at each recursive call, the vertical distance $d_w - d_u$ decreases so the recursion must terminate. Furthermore, when the recursion terminates, either (i) $u = (\bar{d}, j)$ and $w = (d + 1, j)$ so the *only* u - w path in $G(u, w)$ is the edge (u, w) or (ii) $u = (d, i)$ and $w = (d', i)$ so the *only* u - w path in $G(u, w)$ is the vertical path going down from u to w .

The efficiency of the resulting algorithm, both in time and space, will depend upon how efficiently $v = \text{Mid}(u, w)$ can be found. Note that with the exception of the calls of type $\text{Mid}(u, w)$, the rest of the execution of $\text{Path}(u_0, w_0)$ (including all recursive calls) only requires a total of $O(D)$ space, since each recursive call uses only $O(1)$ space and there are at most $O(D)$ such calls. Thus, if $\text{Mid}(u, w)$ can be found using $O(n + D)$ space, then the entire procedure requires only $O(n + D)$ space. This is actually how both [26], [25] achieve their space bounds. The two algorithms differ in how they calculate v . Although both their approaches can be used for our problem, we will work with a modified version of that of [25], since it will be simpler to explain.

We now describe how to use the SMAWK algorithm to find $\text{Mid}(u_0, w_0)$ in $O(nD)$ time and $O(n)$ space. The extension to general $\text{Mid}(u, w)$ will follow later. Recall that the procedure **Fill_Table** from Figure 4 used the fact that $H(\cdot, \cdot)$ was Monge and the SMAWK algorithm to iteratively fill in the rows $H(d, \cdot)$, for $d = 1, 2, \dots, D$. Given row $H(d - 1, \cdot)$, the procedure calculated $H(d, \cdot)$ in $O(n)$ time using SMAWK, and then threw away $H(d - 1, \cdot)$.

Consider an arbitrary node (d, i) on level $d > \bar{d}$. The shortest path from u_0 to (d, i) must pass through *some* node on level \bar{d} . We now modify **Fill_Table** to “remember” this node. More specifically, our algorithm will calculate auxiliary data $\text{pred}(d, i)$.

- For $d < \bar{d}$, $\text{pred}(d, i)$ will be undefined.
- For $d \geq \bar{d}$, $\text{pred}(d, i)$ will be an index j such that node (\bar{d}, j) appears on some shortest path from u_0 to (d, i) .

So, when the procedure terminates, $v = (\bar{d}, \text{pred}(\bar{d}, n - 1))$ will be $\text{Mid}(u_0, w_0)$.

By definition, on level \bar{d} , we have $\text{pred}(\bar{d}, i) = i$.

For $d > \bar{d}$ suppose $(d - 1, j')$ is the immediate predecessor of (d, i) on the shortest path from u_0 to (d, i) . Then (i) a shortest path from u_0 to $(d - 1, j')$ followed by (ii) the edge from $(d - 1, j')$ to (d, i) is (iii) a shortest path from u_0 to (d, i) ; we may therefore set $\text{pred}(d, i) = \text{pred}(d - 1, j')$.

We can use this observation to modify **Fill_Table** to calculate the $\text{pred}(d, \cdot)$ information.

```
Mid( $u_0, w_0$ )
For  $d = 1$  to  $\bar{d}$ 
  SMAWK ( $M^{(d)}$ )
     $\forall 0 \leq i < n$  set  $H(d, i) = M_{i, J(d, i)}^{(d)}$ 
     $\forall 0 \leq i < n$  set  $\text{pred}(\bar{d}, i) = i$ ;
  For  $d = \bar{d} + 1$  to  $D$ 
    SMAWK ( $M^{(d)}$ )
     $\forall 0 \leq i < n$ , set  $H(d, i) = M_{i, J(d, i)}^{(d)}$ 
     $\forall 0 \leq i < n$ , set  $\text{pred}(d, i) = \text{pred}(\bar{d} + 1, j(d, i))$ 
```

Fig. 8. Returns the midpoint, by link distance, on min-cost u_0 - w_0 path.

Note that $\text{Mid}(u_0, w_0)$ can throw away all of the values $\text{pred}(d - 1, \cdot)$ and $H(d - 1, \cdot)$ after the values $\text{pred}(d, \cdot)$ and $H(d, \cdot)$ have been calculated, so it only uses $O(n)$ space. Similarly to the analysis of **Fill_Table**, it uses only $O(nD)$ time since each call to the SMAWK algorithm uses only $O(n)$ time.

So far, we have only shown how to find $v = \text{Mid}(u_0, w_0)$. Note that the *only assumptions* we used were that $H(\cdot, \cdot)$ satisfies DP (2) and is Monge, i.e., the $c_{i,j}^{(d)}$ satisfy (7).

Now suppose that we are given

$$u = (d_u, i_u), w = (d_w, i_w) \text{ with } d_u < d_w \text{ and } i_u \leq i_w.$$

$G(u, w)$ is a dropping level-graph on its own nodes so the cost of the shortest path from u to any node $(d_u + d, i_u + i) \in G(u, w)$ is $\tilde{H}(d, i)$ defined by

$$\tilde{H}(d, i) = \begin{cases} 0 & \text{if } d = 0, i = 0 \\ \infty & \text{if } d = 0, 0 < i < N \\ \min_{0 \leq j \leq i} (\tilde{H}(d - 1, j) + \tilde{c}_{i,j}^{(d)}) & \text{if } d > 0, 0 < N \end{cases} \quad (9)$$

where $N = i_w - i_u + 1$ and $\tilde{c}_{i,j}^{(d)} = c_{i_u+i, i_u+j}^{(d)}$. Note that this new DP is exactly in the same form as (2), just with a different n and shifted $c_{i,j}^{(d)}$. Since the original $c_{i,j}^{(d)}$ satisfy (7), so do the $\tilde{c}_{i,j}^{(d)}$. Thus (9) with the $\tilde{c}_{i,j}^{(d)}$ is Monge as well.

Therefore, we can run exactly the same algorithm written in Figure 8 to find the midpoint $v = (\bar{d}, \bar{i}) = \text{Mid}(u, w)$, of the min-cost u - w path in $O((d_w - d_u)N)$ time and $O((d_w - d_u) + N) = O(D + n)$ space.

As discussed previously, if $\text{Mid}(u, w)$ only requires $O(n + D)$ space, then $\text{Path}(u, w)$ only requires $O(n + D)$ space, so we have completed the space analysis.

It remains to analyze running time. Set

$$\text{Area}(u, w) = (N - 1)(d_w - d_u)$$

to be the “area” of $G(u, w)$. Recall that line 3 of **Path**(u, w) implies that $d_u \neq d_w$ when $\text{Mid}(u, w)$ is called. Therefore $N \geq 1$ and the running time of $\text{Mid}(u, w)$ is

$$O((d_w - d_u)N) = O(\text{Area}(u, w)).$$

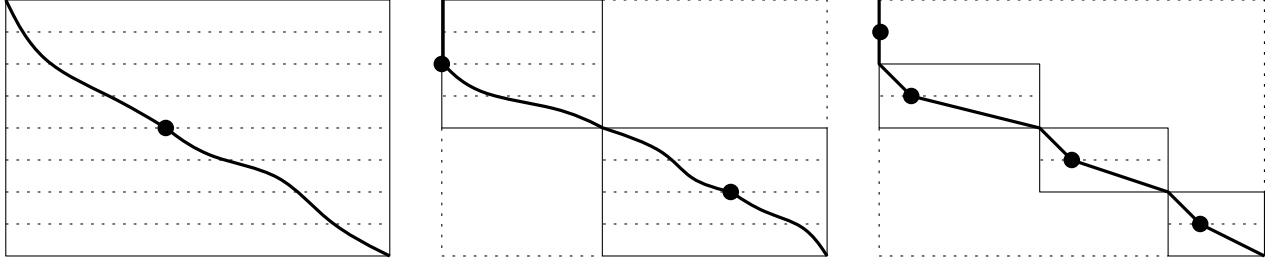


Fig. 7. An illustration for finding the optimal path. Here, $D = 8$ and there are 3 levels of recursions. The solid circles are the intermediate nodes found by the $\text{Mid}(u, v)$ procedures. The first level of recursion finds the midpoint on level 4; the second level, the midpoints on levels 2 and 6; the third the midpoints on levels 1, 3, 5, 7. At that point all subproblems are of height one and easily solvable. Note that each recursive call splits a problem on a box of height 2^i into two problems on disjoint boxes of height 2^{i-1} .

We now analyze the running time of $\text{Path}(u_0, w_0)$. First consider the recursive calls when lines 1-4 occur, i.e., the recursion terminates. The *total* work performed by such calls is the total number of edges outputted. Since an edge is outputted only once and the total path contains D edges, the total work performed is $O(D)$.

Next consider the calls when line 5-7 occur. Since each such call returns a vertex v on the path, there are only $D - 1$ such calls so lines 6 and 7 are only called $O(D)$ times and their total work, with the exception of the call to $\text{Mid}(u, v)$, is $O(D)$.

Finally consider the work performed by the $\text{Mid}(u, w)$ calls. Partition the calls into levels.

- Level 1 is the original call $\text{Mid}(u_0, w_0)$.
- Level 2 contains the recursive calls directly made by the level-1 call.
- In general, level i contains the recursive calls directly made by the level- $(i - 1)$ calls.

Note that if $\text{Mid}(u, w)$ is a level i call with $u = (d_u, i_u)$ and $w = (d_w, i_w)$ then

$$\frac{D}{2^i} \leq d_w - d_u < \frac{D}{2^i} + 1.$$

Furthermore, by induction, if $\text{Mid}(u, w)$ and $\text{Mid}(u', w')$ are two different level i calls, then horizontal ranges $[d_u, d_w]$ and $[d_{u'}, d_{w'}]$ are *disjoint* except for possibly $d_w = d_{u'}$ or $d_u = d_{w'}$.

Fix i . Let (u_j, w_j) $j = 1, \dots, t$ be the calls at level i . The facts that each grid $G(u_j, w_j)$ has height $\leq \frac{D}{2^i} + 1$ and that the horizontal ranges of the grids are disjoint implies

$$\sum_{j=1}^t \text{Area}(u_j, w_j) \leq n \left(\frac{D}{2^i} + 1 \right).$$

Thus the total of all level- i calls is $O(n(\frac{D}{2^i} + 1))$. Summing over the $\lceil \log D \rceil$ levels we get that the total work performed by all of the $\text{Mid}(u, w)$ calls on line 6 is

$$O\left(\sum_i n \left(\frac{D}{2^i} + 1 \right)\right) = O(nD).$$

Thus, the total work performed by $\text{Path}(u_0, w_0)$ is $O(nD)$ and we are finished.

IV. FURTHER APPLICATIONS

We just saw how, in $\Theta(nD)$ time and $\Theta(n + D)$ space, to solve the construction problem for any DP in form (2) that satisfies the Monge property (7). $\Theta(nD)$ time was known previously; the $\Theta(n + D)$ space bound, is the new improvement. There are many other DP problems besides the binary LLHC that satisfy (7) and whose space can thus be improved. We illustrate with three examples.

The r -ary LLHC problem:

We have discussed the binary LLHC problem in which $|\Sigma| = 2$. The general r -ary alphabet case with N probabilities is still modeled by a DP in form (2) but with $n = \frac{N-1}{r-1} + 1$. The only difference is that (3) is replaced by

$$c_{i,j}^{(d)} = \begin{cases} S_{ri-j} & \text{if } \max\{0, ri - N\} \leq j < i \\ \infty & \text{otherwise.} \end{cases} \quad (10)$$

A full derivation of this DP is given in Appendix A. The proof that the $c_{i,j}^{(d)}$ satisfy the Monge property (7) is similar to the proof of Lemma 4. Thus, we can construct a solution to the r -ary LLHC problem in $\Theta(ND)$ time and $\Theta(N)$ space as well.

D medians on a line:

We are given $n - 1$ customers located on the positive real line; customer i is at location v_i . Without loss of generality, assume $v_1 < v_2 < \dots < v_{n-1}$. There are $D \leq n$ service centers located on the line and a customer is serviced by the closest service center to its left (thus we always assume a service center at $v_0 = 0$). Each customer has a service request $w_i > 0$. The cost of servicing customer i is w_i times the distance to its service center. In [17], motivated by the application of optimally placing web proxies on a linear topology network, Woeginger showed that this problem could be modeled by a DP in form (2) where

$$c_{i,j}^{(d)} = \sum_{l=j+1}^i w_l(v_l - v_{j+1})$$

and proved that these $c_{i,j}^{(d)}$ satisfy Monge property (7). He then used the SMAWK algorithm to construct a solution in $O(nD)$ time and $O(nD)$ space. Using the technique we just described, this can be reduced to $O(nD)$ time and $O(n)$ space.

We also mention that there is an undirected variant of this problem in which a node is serviced by its *closest* service

center looking both left and right. There are many algorithms in the literature that (explicitly or implicitly) use concavity to construct solutions for this problem in $O(nD)$ time using $O(n)$ space, e.g., [29], [30], [31]. [31] does this by using a DP formulation that is in the DP form (2) and satisfies the Monge property (7) so the technique in this paper can reduce the space for this problem down to $O(n)$ as well.

Wireless Paging:

The third application comes from wireless mobile paging. A user can be in one of N different cells. We are given a probability distribution in which p_i denotes the probability that a user will be in cell i and want to minimize the bandwidth needed to send paging requests to identify the cell in which the user resides. This problem was originally conjectured to be NP-complete, but [32] developed a DP algorithm for it. The input of the problem is the n probabilities $p_1 \geq p_2 \geq \dots \geq p_n$ and an integer $D \leq n$ (corresponding to the number of paging rounds used). The DP developed by [32] is exactly in our DP form (2) with

$$c_{i,j}^{(d)} = \begin{cases} i \left(\sum_{\ell=j+1}^i p_\ell \right) & \text{if } d-1 \leq j < i \\ \infty & \text{otherwise.} \end{cases} \quad (11)$$

The goal is to compute $H(D, n)$, which will be the minimum expected bandwidth needed. Solving the construction version of this DP permits constructing the actual paging protocol that yields this minimum bandwidth.

[32] used the naive algorithm to solve the DP in $\Theta(n^2D)$ time and $\Theta(nD)$ space. [33] proved that the $c_{i,j}^{(d)}$ defined by (11) satisfy the the Monge property (7) and thus reduced the time to $\Theta(nD)$, but still required $\Theta(nD)$ space. The algorithm in this paper permits improving the space complexity of constructing the protocol down to $\Theta(n)$.

V. CONCLUSION

The standard approach to solving the Length-Limited Huffman Coding (LLHC) problem is via the special purpose Package-Merge algorithm of Hirschberg and Larmore [9] which runs in $O(nD)$ time and $O(n)$ space, where n is the number of codewords and D is the length-limit on the code.

In this note we point out that this problem can be solved in the same time and space using a straightforward Dynamic Programming formulation. We started by noting that it was known that the LLHC problem could be modeled using a DP in the form

$$H(d, i) = \begin{cases} 0 & \text{if } d = 0, i = 0 \\ \infty & \text{if } d = 0, 0 < i < n \\ \min_{0 \leq j \leq i} (H(d-1, j) + c_{i,j}^{(d)}) & \text{if } d > 0, 0 < i < n \end{cases} \quad (12)$$

where $H(d, n)$ will denote the minimum cost of a code with longest word at most d and the $c_{i,j}^{(d)}$ are easily calculable constants. This implies an $O(n^2D)$ time $O(nD)$ space algorithm. We then note that, using standard DP speedup techniques, e.g., the SMAWK algorithm, the time could be reduced down to $O(nD)$. The main contribution of this paper is to note that, once the problem is expressed in this formulation, the space can be reduced down to $O(n)$ while maintaining the time at

$O(nD)$. The space reduction developed for this problem was also shown to apply to other problems in the literature that previously had been thought to require $\Theta(nD)$ space.

We conclude by noting that if we're only interested in solving the standard Huffman coding problem and not the LLHC one then DP (12) with $c_{i,j}^{(d)}$ defined by (10) collapses down to

$$H(i) = \min_{\max\{0, ri-N\} \leq j < i} H(j) + S_{ri-j}. \quad (13)$$

where $H(i)$ denotes the minimum cost of a “valid sequence” ending in i . $H\left(\frac{N-1}{r-1}\right)$ will be the cost of an optimal complete sequence and solving the construction problem for this DP will give this optimal sequence. We can construct the code from this optimal sequence in $O(N)$ time.

There is a subtle point here which should be mentioned. The matrix M defined by

$$M_{i,j} = \begin{cases} H(j) + S_{ri-j} & \text{if } \max\{0, ri - N\} \leq j < i \\ \infty & \text{otherwise} \end{cases}$$

is Monge (the proof is similar to that of Lemma 4). We can *not* use the SMAWK algorithm to find its row minima and solve the problem, though. The reason is that, as stated in Lemma 3, the SMAWK algorithm requires being able to calculate any arbitrary requested entry $M_{i,j}$ in $O(1)$ time. In our current DP, though, the $M_{i,j}$ are dependent upon the values $H(j)$ which are the row-minima of other rows in the same matrix! Thus, we have no way of calculating $M_{i,j}$ in $O(1)$ time when required and the SMAWK algorithm can not be applied. This is the reason why Larmore and Przytycka [16] needed to use the more sophisticated CLWS algorithm of [20] to solve the binary ($r = 2$) version of this problem. Other algorithms for more generalized versions of the CLWS have since appeared, e.g., [34], that could also be used to solve this problem in $O(n)$ time, but they are also quite complicated. To summarize, by transforming r -ary Huffman coding into a DP and using sophisticated tools such as [20] or [34] we can solve the problem in $O(n)$ time. This is not of practical interest, though, since the simple, greedy, Huffman encoding algorithm is just as fast. Where the DP formulation helps is in the LLHC problem, exactly where the greedy procedure fails. In that case we have the added practical benefit of being able to use the simple SMAWK algorithm rather than the more complicated [20] or [34].

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APPENDIX A DERIVATION OF THE LLHC DYNAMIC PROGRAM

In order to make this note self-contained we provide a brief derivation of the DP that models the LLHC. To the best of our knowledge, the derivation for the general r -ary case has never been written down before (although it is known as "folklore").

A set of n prefix-free codes in an r -ary alphabet can be represented by an r -ary tree with n leaves. The i^{th} edge from an internal node to its children is labeled by σ_i . Each leaf corresponds to a code word, which is the concatenation of the characters on the root-to-leaf path. Then, the expected code length equals the weighted external path length of the tree.

Denote the height of the tree by h . The lowest leaves are on level 0; the root is at level h . Optimal (min weighted external path-length) assignments of the probability p_i 's to the leaves always assign smaller probabilities to leaves at lower levels. Since the probabilities are given in sorted order, this assignment can be done in $O(n)$ time for a given tree. The cost of a tree is its weighted external path length w.r.t. an optimal assignment.

Define the *degree* of a node to be the number of its children. A node is *complete* if it is of degree r , and a tree is *complete* if all its internal nodes are complete. The following properties are easy to prove

Property 1: In an optimal tree, the internal nodes at levels ≥ 2 are complete.

Property 2: There is an optimal tree that has at most one incomplete internal node, and if this node exists, it is at level 1. Furthermore, the degree of this incomplete node is ≥ 2 .

These properties imply that the optimal tree is almost complete and has $\lceil \frac{n-1}{r-1} \rceil$ internal nodes. If $n-1$ is divisible by $r-1$, the tree is complete. Otherwise, we can add

$$n - 1 - \left\lfloor \frac{n-1}{r-1} \right\rfloor (r-1) \leq r-2$$

dummy leaves to make it complete. We assign dummy p_i 's with zero values to these dummy leaves. It is easy to see that the new tree with these dummy leaves is precisely an optimal tree for the probabilities with the added zero-valued dummy p_i 's. So, finding an optimal tree for probabilities with these dummy p_i 's is equivalent to the original problem. Therefore, w.l.o.g., we assume in the original problem, the optimal tree is a complete tree, i.e., we assume $n-1$ is always a multiple of $r-1$. In this way we transform the r -ary Huffman coding problem to the problem of finding an optimal complete r -ary tree with n leaves.

A complete tree of height h can be fully represented by a sequence (i_0, i_1, \dots, i_h) , where i_k denotes the number of internal nodes at levels $\leq k$. Note that from this sequence we can calculate $I_k = i_k - i_{k-1}$, the number of internal nodes on level k and with that information we can reconstruct the tree in $O(n)$ time as follows:

Create

1. For $k = 1$ to h
2. Create I_k nodes $V_k = \{v_1, \dots, v_{I_k}\}$ on level k ;
3. Create $rI_k - I_{k-1}$ leaves on level $k-1$;
4. Make $\{v_1, \dots, v_{I_k}\}$ the parents of the rI_k nodes on level $k-1$.

We will now see how to rewrite the cost of a tree using its representative sequence:

Lemma 5: If $\mathcal{I} = (i_0, i_1, \dots, i_h)$ represents tree T , then T has $ri_k - i_{k-1}$ leaves on levels $< k$.

Proof: Consider the forest which is the portion of T on or below level k . It is composed of $I_k = i_k - i_{k-1}$ trees with roots on level k ,

In total, the forest contains i_k internal nodes.

If T' is a complete r -ary tree with m internal nodes then T' has $(r-1)m+1$ leaves so our forest must contain $(r-1)i_k + I_k = ri_k - i_{k-1}$ leaves. ■

Recall that $S_m = \sum_{i=1}^m p_i$ for $1 \leq m \leq n$. Using the lemma above, we have

Lemma 6: If the sequence (i_0, i_1, \dots, i_h) represents a tree, then the cost of the tree is $\sum_{k=1}^h S_{ri_k - i_{k-1}}$.

Proof: Recall from Lemma 5 that $ri_k - i_{k-1}$ is the number of leaves at levels $< k$. So

Cost of the tree

$$\begin{aligned} &= \sum_{\ell=0}^{h-1} (\text{sum of weights of leaves at level } \ell) \cdot (h-\ell) \\ &= \sum_{\ell=0}^{h-1} (\text{sum of weights of leaves at level } \ell) \cdot \sum_{k=\ell+1}^h 1 \\ &= \sum_{k=1}^h \sum_{\ell=0}^{k-1} (\text{sum of weights of leaves at level } \ell) \\ &= \sum_{k=1}^h (\text{sum of the weights of leaves at levels } < k) \\ &= \sum_{k=1}^h S_{ri_k - i_{k-1}} \end{aligned}$$

For a complete r -ary tree with n leaves, we have $0 = i_0 < i_1 < \dots < i_h = \frac{n-1}{r-1}$ and, from Lemma 5, $ri_k - i_{k-1} \leq n$ for all $1 \leq k \leq h$.

For technical reasons, because we will be dealing with trees having height *at most* (but not necessarily equal to) h , we allow initial padding of the sequence by **0s** so that a sequence representing a tree will be of the form (i_0, i_1, \dots, i_h) that has the following properties

Definition 5: A sequence (i_0, i_1, \dots, i_h) is a valid (n, r) -sequence, if

- $\exists t$ such that $i_0 = i_1 = \dots = i_t = 0$.
- $0 < i_t < \dots < i_h \leq \frac{n-1}{r-1}$
- $ri_k - i_{k-1} \leq n$ for all $1 \leq k \leq h$.

A sequence is *complete* if it is valid and $i_h = \frac{n-1}{r-1}$.

It is straightforward to see that padding the sequence representing a tree with initial **0s**, does not change the tree built by the *Create* procedure or the validity of Lemmas 5 and 6.

We can now extend our cost function to *all* valid (n, r) -sequences sequences, not just the ones representing trees.

Definition 6: For valid (n, r) -sequence $\mathcal{I} = (i_0, i_1, \dots, i_h)$ define

$$\text{cost}(\mathcal{I}) = \sum_{k=1}^h S_{ri_k - i_{k-1}}.$$

\mathcal{I} is *optimal* if $\text{cost}(\mathcal{I}) = \min_{\mathcal{I}'} \text{cost}(\mathcal{I}')$ where the minimum is taken over all valid length h (n, r) -sequences $\mathcal{I}' = (i'_0, i'_1, \dots, i'_h)$ with $i'_h = i_h$, i.e., all sequences of the same length that end with the same value.

Note: padding a sequence with initial **0s** doesn't change its completeness or cost. Furthermore, if \mathcal{I} is created by padding the sequence corresponding to tree T with initial **0s**, then procedure *Create* will still recreate T from \mathcal{I} .

It follows from the definitions that for fixed (n, r) we can calculate $H(d, j)$, the cost of an optimal (n, r) -sequence $(0, i_1, i_2, \dots, i_d)$ with $i_d = j$ using the DP (2) with

$$c_{i,j}^{(d)} = \begin{cases} 0 & \text{if } i=j=0 \\ S_{ri-j} & \text{if } \max\{0, ri-n\} \leq j < i \\ \infty & \text{otherwise.} \end{cases} \quad (14)$$

The subtle issue is that not all complete sequences correspond to trees, e.g., $(0, 3, 4, 5)$ is a complete $(6, 2)$ sequence that does not represent any binary tree. Thus, a-priori, finding an optimal complete sequence might not help us find an optimal tree. We are saved by the next lemma.

Lemma 7: An optimal complete (n, r) -sequence always represents a tree.

Thus, we can find an optimal tree by first solving the construction problem for DP (2) with conditions (14) to get an optimal complete (n, r) -sequence \mathcal{I} and then building the tree that corresponds to \mathcal{I} .

Before proving Lemma 7 we will need to extend our definitions from trees to forests. See Figure 9(a).

Definition 7: A legal (n, r) -forest, or *forest*, is a collection of complete r -ary trees that together contain at most n leaves, all of whose roots are at the same height.

Given $p_1 \leq p_2 \leq \dots, p_n$ we can assign the p_i to the leaves of forest F from bottom to top of tree and define the cost of F (with respect to the p_i) to be the sum of the costs of its component trees. Note that a tree with n leaves is a forest and its cost as a forest will be the same as its cost as a tree.

Now, for forest F let i_k be the number of internal nodes it has at level $\leq k$. Then, we can talk about the sequence $\mathcal{I} = (i_0, i_1, \dots, i_h)$ associated with the forest. Reviewing the proofs of Lemmas 5 and 6 we see that they were actually statements about forests and not trees so F has $ri_k - i_{k-1}$ leaves on levels $< k$ and $\text{cost}(F) = \text{cost}(\mathcal{I})$.

We will prove

Lemma 8: An optimal (n, r) -sequence $\mathcal{I} = (i_0, i_1, \dots, i_h)$ always represents a forest.

Note that this will immediately imply Lemma 7 because if \mathcal{I} is complete then $i_h = \frac{n-1}{r-1}$ and, by validity, $ri_h - i_{h-1} \leq n$, implying $i_{h-1} = i_h - 1$. Thus the forest corresponding to \mathcal{I} is composed of exactly $i_h - i_{h-1} = 1$ trees at level h and is therefore a tree itself.

Proof: (of Lemma 8)

Without loss of generality assume that $i_0 = 0 < i_1$. Our proof will be by induction on h .

First note that if $h = 1$, then $\mathcal{I} = (0, i_1)$ for some $i_1 > 0$ and this represents the forest composed of i_1 complete trees each of height 1 so the lemma is trivially correct.

Now let $h > 1$. Set $I_h = i_h - i_{h-1}$ and $I_{h-1} = I_{h-2} - I_{h-1}$.

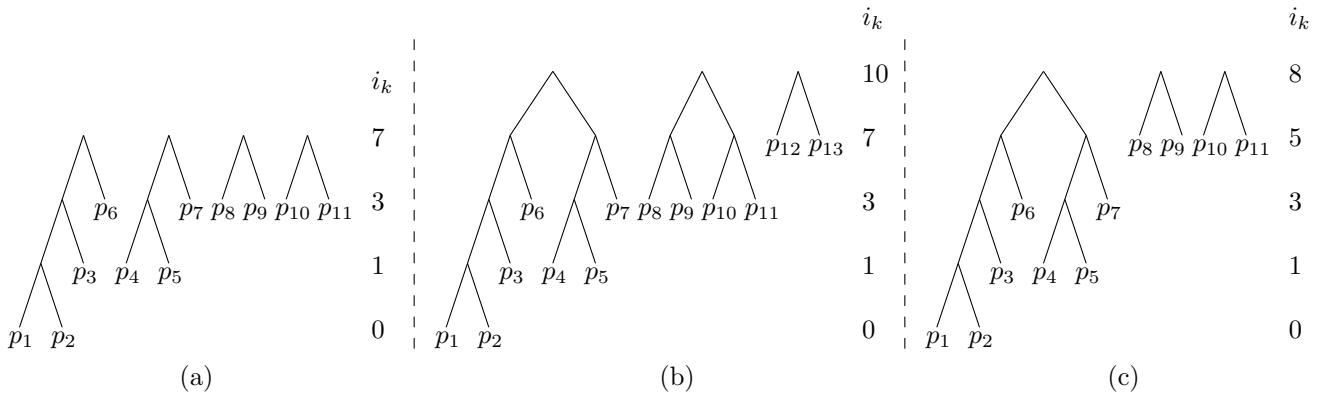


Fig. 9. Illustration of the two cases in the proof of Lemma 8. Here, $r = 2$ and $h = 4$. (a) is the forest F' corresponding to the old sequence $\mathcal{I}' = (0, 1, 3, 7)$. (b) illustrates case 1: if $i_h = 10$ then $I_h = 3$ and $2I_h = 6 \geq 4 = I_{h-1}$ so we can create a forest corresponding to the new sequence $(0, 1, 3, 7, 10)$. (c) illustrates case 2: if $i_h = 8$ then $I_h = 1$ and $2I_h = 2 < 4 = I_{h-1}$. In this case the sequence $\bar{\mathcal{I}} = (0, 1, 3, 5, 8)$ (corresponding to the forest pictured) has cost $S_2 + S_5 + S_7 + S_{11}$. This is cheaper than the cost $S_2 + S_5 + S_{11} + S_9$ of the sequence $\mathcal{I} = (0, 1, 3, 7, 8)$. As noted in the proof, $\bar{\mathcal{I}}$ is constructed by lifting two subtrees in the forest in (a) and then writing down the corresponding sequence.

Define $\mathcal{I}' = (i_0, i_1, \dots, i_{h-1})$. Since \mathcal{I}' is optimal, by induction, \mathcal{I}' represents a forest F' with I_{h-1} roots at level $h-1$ and a total of $L_{h-1} = ri_{h-1} - i_{h-2}$ leaves. There are now two cases: see Figure 9.

Case 1: $ri_h \geq I_{h-1}$:

Then \mathcal{I} represents a forest with I_h roots whose ri_h children are exactly the I_{h-1} roots from F' and another $ri_h - I_{h-1} \geq 0$ leaves. So the Lemma is correct.

Case 2: $ri_h < I_{h-1}$:

We will show that this contradicts the optimality of \mathcal{I} and is therefore impossible. Thus Case 1 will be the only possible case and the Lemma correct.

Assume now that $ri_h < I_{h-1}$ and set $s = I_h - 1 - ri_h > 0$. This can be rewritten as $r(i_h - i_{h-1}) + s = r(i_{h-1} - i_{h-2})$ so

$$ri_h - i_{h-1} = ri_{h-1} - i_{h-2} - s = L_{h-1} - s.$$

Now consider F as being labeled with the L_{h-1} smallest p_i and construct a new forest \bar{F} as follows. Choose s trees from \bar{F} containing the s largest weights in the forest, i.e., p_j , $j = L_{h-1}, L_{h-1}-1, \dots, L_{h-1}-(s-1)$. Move those s forests up one level so their roots are now at height h and not $h-1$. Now add I_h new nodes to level h . Make them the parents of the remaining ri_h nodes on level $h-1$. This forest is a legal forest. Call its representative sequence $\bar{\mathcal{I}} = (\bar{i}_0, \bar{i}_1, \dots, \bar{i}_h)$.

We now observe

(a) $\bar{i}_{h-1} = i_{h-1} - s$ so

$$\bar{i}_h = \bar{i}_{h-1} + s + I_h = i_{h-1} + I_h + s = i_h.$$

(b) Thus $r\bar{i}_h - \bar{i}_{h-1} = ri_h - (i_{h-1} - s) = L_{h-1}$ and

$$S_{r\bar{i}_h - \bar{i}_{h-1}} = S_{L_{h-1}} = S_{ri_h - i_{h-1}} + \sum_{j=L_{h-1}-s+1}^{L_{h-1}} p_j$$

(c) Let \bar{F}' be levels 0- $(h-1)$ of \bar{F} . Since every complete tree contains at least r nodes, the s trees raised contain at least the s nodes p_j where $L_{h-1} - s < j \leq L_{h-1}$ and one other node.

Since every such node was raised one level,

$$\begin{aligned} \sum_{m=1}^{h-1} S_{r\bar{i}_m - \bar{i}_{m-1}} &= \text{cost}(\bar{F}') \\ &< \text{cost}(F') - \sum_{j=L_{h-1}-s+1}^{L_{h-1}} p_j \\ &= \left(\sum_{m=1}^{h-1} S_{ri_m - i_{m-1}} \right) - \sum_{j=L_{h-1}-s+1}^{L_{h-1}} p_j \end{aligned}$$

Combining (b) and (c) shows that $\text{cost}(\bar{\mathcal{I}}) < \text{cost}(\mathcal{I})$. This is a contradiction since both \mathcal{I} and $\bar{\mathcal{I}}$ are valid sequences of length h that end with the same value i_h and \mathcal{I} is optimal. Thus the case $ri_h < I_{h-1}$ can not happen and we are finished. \blacksquare