PARUL UNIVERSITY - FACULTY OF ENGINEERING & TECHNOLOGY
Department of Applied Science & Humanities
3rd Semester B. Tech (CSE, IT)
Discrete Mathematics (203191202)
UNIT-1 Sets, Relation & Functions

I. Introduction:

In everyday life, we often speak of collections of objects of a particular kind, such as, a pack of cards, a crowd of people, a cricket team, etc. In mathematics also, we come across collections, for example, of natural numbers, points, prime numbers, etc. More specially, we examine the following collections:

- **a.** Odd natural numbers less than 10, i.e., 1, 3, 5, 7, 9
- **b.** The rivers of India.
- **c.** The vowels in the English alphabet, namely, a, e, i, o, u
- **d.** Various kinds of triangles
- **e.** Prime factors of 210, namely, 2,3,5 and 7
- **f.** The solution of the equation: $x^2 - 5x + 6 = 0$, viz., x=2 and x=3.

We give below a few more examples of sets used particularly in mathematics
N: the set of all natural numbers
Z: the set of all integers
Q: the set of all rational numbers
R: the set of real numbers
Z^+: the set of positive integers
Q^+: the set of positive rational numbers
R^+: the set of positive real numbers.

Definition:
SET: A set is a well-defined collection of objects.

II. Representations of a set

There are two methods of representing a set:

a. Roster or tabular form

In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }.
For example:

- **i.** The set of all even positive integers less than 7 is described in roster form as {2, 4, 6}.

b. Set-builder form

In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set.

For example

1. In the set {a, e, i, o, u}, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property.

i.e. V = {x: x is a vowel in English alphabet}

III. Types of Sets:

i. The Empty Set

A set which does not contain any element is called the **empty set** or **the null set** or the **void set**. It is denoted by the symbol φ or { }.

For Example:

(i) **B** = { x : x is a student presently studying in both Classes X and XI }={}
A student cannot study simultaneously in both Classes X and XI. Thus, the set **B** contains no element at all.

ii. Finite and Infinite Sets

A set which is empty or consists of a definite number of elements is called **finite** otherwise, the set is called **infinite.**

Examples:

(i) Let W be the set of the days of the week. Then W is finite.

(ii) Let G be the set of points on a line. Then G is infinite.

iii. Equal Sets

Two sets A and B are said to be **equal** if they have exactly the same elements and we write A = B. Otherwise, the sets are said to be **unequal** and we write A ≠ B.

For examples:

(i) Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B.

iv. Subsets

A set A is said to be a **subset** of a set B if every element of A is also an element of B. A is subset of B is expressed in symbols as A ⊂ B.

$A \in B$ if $a \in A \Rightarrow a \in B$

NOTE:

1. Every set A is a subset of itself.
2. φ is a subset of every set.

v. Proper Subset and Superset

Let A and B be two sets. If $A \in B$ and A ≠ B, then A is called a proper subset of B and B is called superset of A.

For example:
A = {1, 2, 3} is a proper subset of B = {1, 2, 3, 4}.

vi. Singleton Set
If a set A has only one element, we call it a singleton set. A={a} is a singleton set.

Example 9: Consider the sets φ, A = {1, 3 }, B = {1, 5, 9}, C = {1, 3, 5, 7, 9}.

(i) φ ⊂ B (ii) A ⊄ B (iii) A ⊄ C (iv) B ⊂ C

Subsets of set of real numbers
The set of natural numbers **N** = {1, 2, 3, 4, 5 . . .}
The set of integers **Z** = {. . ., –3, –2, –1, 0, 1, 2, 3 . . .}
The set of rational numbers **Q** = {x: $x = \frac{p}{q}$, $p, q \in Z$, $q \neq 0$ }

$N \subset Z \subset Q$,
$Q \subset R$

vii. Power Set:
The collection of all subsets of a set A is called the power set of A. It is denoted by P(A).
In P(A), every element is a set.
Thus, as in above, if A = { 1, 2 }, then P(A) = { φ,{ 1 }, { 2 }, { 1,2 }}
Also, note that n[P(A)]=4= 2^2
NOTE: If A is a set with n (A) =m, then it can be shown that n [P (A)] = 2^m.

viii. Universal Set:
While studying the system of numbers, we are interested in the set of natural numbers and its subsets such as the set of all prime numbers, the set of all even numbers, and so forth. This basic set is called the **"Universal Set".** The universal set is usually denoted by U and all its subsets by the letters A, B, C, etc.

IV. Venn Diagrams:
Most of the relationships between sets can be represented by means of diagrams which are known as Venn diagrams. Venn diagrams are named after the English logician, John Venn (1834-1883). These diagrams consist of rectangles and closed curves usually circles. The universal set is represented usually by a rectangle and its subsets by circles.
In Venn diagrams, the elements of the sets are written in their respective circles.

Example: U = {1,2,3, ..., 10} is the universal set of which A = {2,4,6,8,10} is a subset.

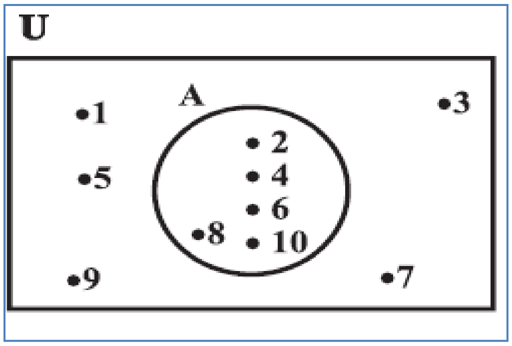


Figure:1-1

Example: U = {1,2,3, ..., 10} is the universal set of which A = {2,4,6,8,10} and B = {4, 6} are subsets, and also $B \subset A$.

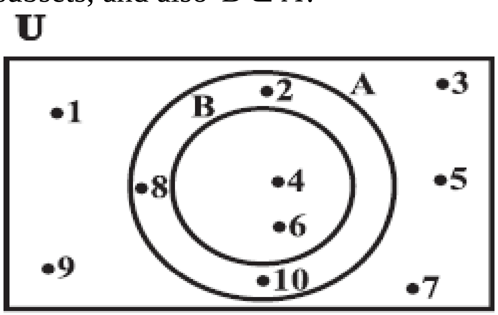


Figure:1-2

V. Operations on Sets

1. Union of sets

The union of two sets A and B is the set which consists of all those elements which are either in A or in B (including those which are in both).

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

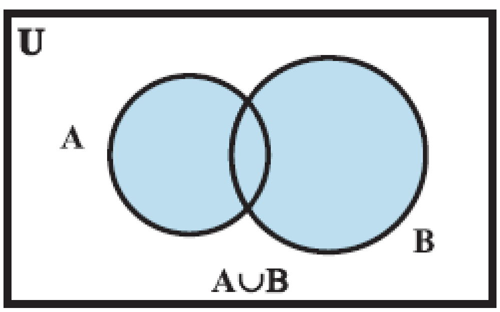


Figure:1-3

Some Properties of the Operation of Union

(i) $A \cup B = B \cup A$ (Commutative law)

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)

(iii) $A \cup \varphi = A$ (Law of identity element, φ is the identity of ∪)

(iv) $A \cup A = A$ (Idempotent law)

(v) $U \cup A = U$ (Law of U)

2. Intersection of sets

The intersection of two sets A and B is the set of all those elements which belong to bothA and B.A ∩ B = {x : $x \in A$ and $x \in B$}

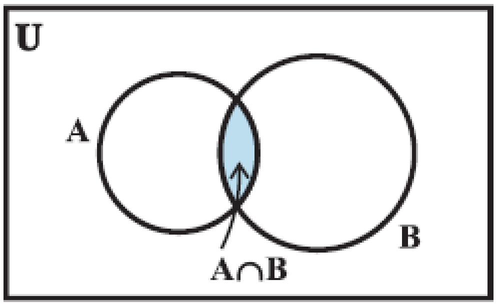


Figure:1-4

3. **Disjoint Sets**: If A and B are two sets such that A ∩ B = φ, then A and B are called **disjoint sets.**

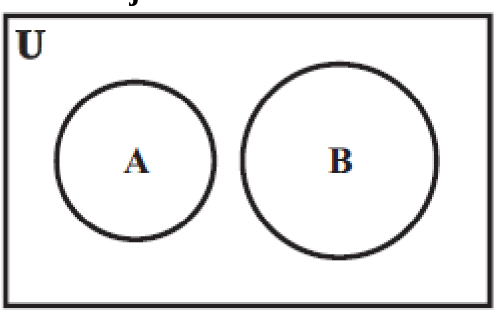


Figure:1-5

Some Properties of Operation of Intersection

(i) A ∩ B = B ∩ A (Commutative law).
(ii) (A ∩ B) ∩ C = A ∩ (B ∩ C) (Associative law).
(iii) φ ∩ A = φ, U ∩ A = A (Law of φ and U).
(iv) A ∩ A = A (Idempotent law)
(v) A ∩ ($B \cap C$) = (A ∩ B) $\cup$ (A ∩ C) (Distributive law)i. e., ∩ distributes over $\cup$

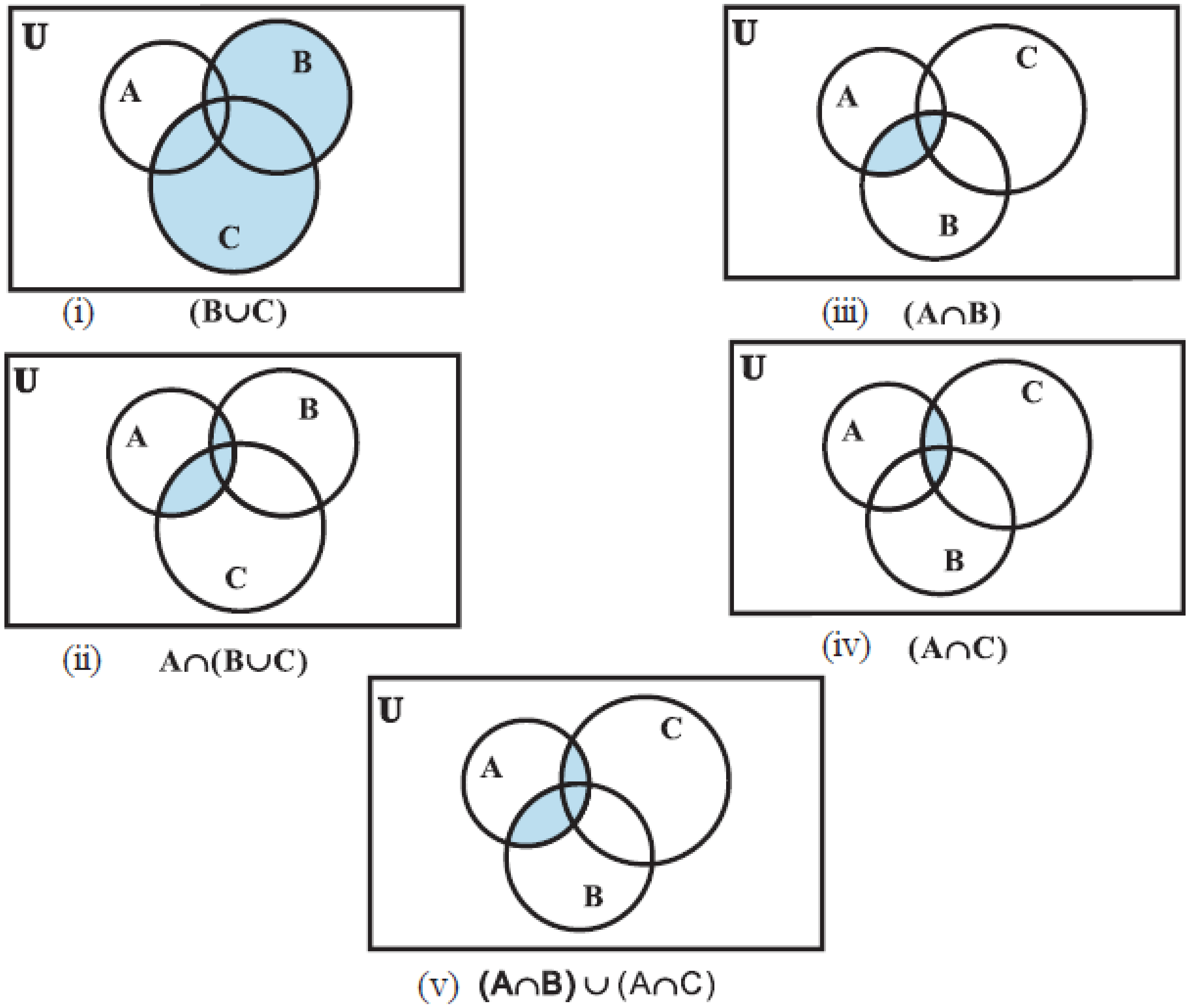


Figure:1-6

4. Difference of sets

The difference of the sets A and B is the set of elements which belong to A but not to B.

$A - B = \{ x : x \in A \text{ and } x \notin B \}$

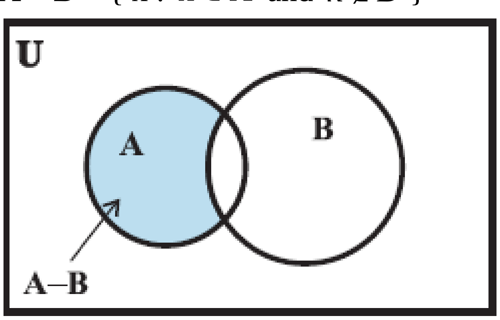


Figure:1-7

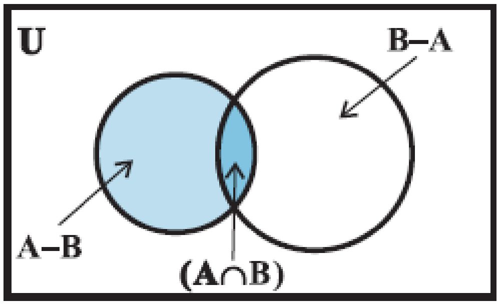
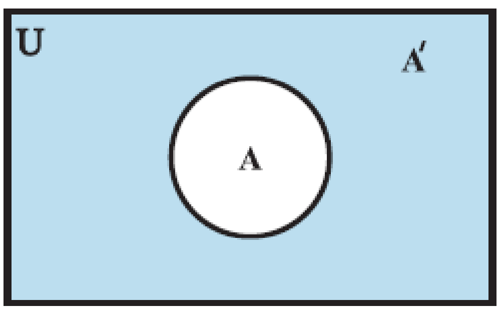


Figure:1-8

5. **Complement of a Set**

Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A.

A′ = {x: $x \in U$ and $x \notin A$ }



A'=U-A

Figure:1-9

Some Properties of Complement Sets

1. Complement laws: (i) $A \cup A'$ = U (ii) $A \cap A'$ = φ

2. Law of double complementation: (A′)′ = A

3. Laws of empty set and universal set φ′ = U and U′ = φ.

Note: If A is a subset of the universal set U, then its complement A′ is also a subset of U.

VI. De Morgan's Laws

The complement of the union of two sets is the intersection of their complements and the complement of the intersection of two sets is the union of their complements. These are called De Morgan's laws.

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Applications:

1. Let A and B be finite sets. If A ∩ B = φ, then n(A∪B)=n(A)+n(B)
2. In general, if A and B are finite sets, then n(A∪B)=n(A)+n(B)–n(A∩B)
3. The sets A – B, A ∩ B and B – A are disjoint and their union is A ∪ B

n(A ∪ B)=n(A–B)+n(A∩B)+n(B–A)

$= n(A–B)+n(A∩B)+n(B–A)+n(A∩B)–n(A∩B)$
$= n(A)+n(B)–n(A∩B)$

4. If A, B and C are finite sets, then
$n(A∪B∪C)=n(A)+n(B)+n(C)–n(A∩B)–n(B∩C)–n(A∩C)+n(A∩B∩C)$

Example: If X and Y are two sets such that $X \cup Y$ has 50 elements, X has 28 elements and Y has 32 elements, how many elements does $X \cap Y$ have?
Solution: Given that $n(X\cup Y)=50$, $n(X)=28$, $n(Y)=32$, $n(X\cap Y) = ?$
By using the formula
$n (X\cup Y) = n (X) + n (Y) – n (X \cap Y)$,

$n (X \cap Y) = n (X) + n (Y) – n (X \cup Y)$
$= 28 + 32 – 50 = 10$

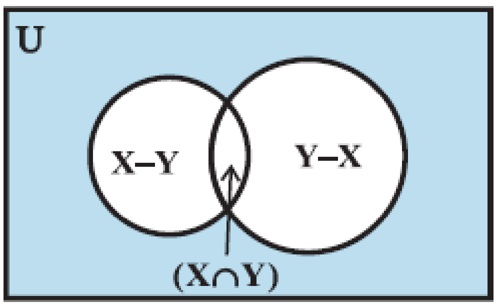


Figure:1-10

Example:

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 5, 7, 9}, B={2, 4, 5, 8, 9, 10}, C={1, 5, 9}. Find
(i) A', B', C'
(ii) $A\cup B$
(iii) $A\cap B$
(iii) Verify DeMorgan's Laws
$(A\cap B)' = (A'\cup B')$ and $(A\cup B)' = (A'\cap B')$

Application:

The time required to manipulate information in a database depends on how this information is stored. The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database. Because of the importance of these operations, various methods for representing databases have been developed. One of these methods, called the **relational data model** is based on the concept of a relation.

The database query language SQL (short for Structured Query Language) can be used to carry out the operations we have described in this section. Example 12 illustrates how SQL commands are related to operations on *n*-ary relations.

CARTESIAN PRODUCTS OF SETS

The ***ordered n-tuple*** is the ordered collection a1,a2,…,an that has a1 as its first element, a2 as its second element,…, and an as its *n*th element.
Two ordered *n*-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, a1,a2,…,an=(b1,b2,…,bn) if and only if ai=bi for i = 1, 2, . . . , n.

Definition

Let ***P*** and ***Q*** be two sets. The Cartesian product ***P*** × ***Q*** is the set of all ordered pairs p,q**, where** p∈P **and** q∈Q

i.e. P×Q=p,q:p∈P, q∈Q

Note the following:

- The *Cartesian product* of the sets A1,A2,…,An denoted by A1A2An is the set of ordered *n*-tuples a1,a2,…,an, where ai belongs to Ai for i = 1, 2, . . . , n.
  In other words, A1A2An=aiAi, for i=1,2,…,nThe ordered pairs (a, b) and c, d are equal if and only if a = c and b = d.
- If either ***P*** or ***Q*** is the null set, then ***P*** × ***Q*** will also be empty set.
- If ***A*** and ***B*** are non-empty sets and either ***A*** or ***B*** is an infinite set, then so is ***A*** × ***B*** .
- If there are **p** elements in ***A*** and **q** elements in ***B*,** then there will be **pq** elements in ***A*** × ***B***, i.e., if ***n(A) = p*** and ***n(B) = q*,** then ***n(A × B) = pq*.**
- An=A×A××A=ai∈A, for i=1,2,…,n

Illustration

Consider the two sets: A={DL, MP, KA}, where DL, MP, KA represent Delhi, Madhya Pradesh and Karnataka, respectively and B = 01,02, 03 representing codes for the licence plates of vehicles issued by DL, MP and KA .

If the three states, Delhi, Madhya Pradesh and Karnataka were making codes for the licence plates of vehicles, with the restriction that the code begins with an element from set ***A***, which are the pairs available from these sets and how many such pairs will there be ?

The available pairs are:

(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03) and the product of set ***A*** and set ***B*** is given by

A × ***B*** = {(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03)}.

It can easily be seen that there will be 9 such pairs in the Cartesian product, since there are 3 elements in each of the sets A and B. This gives us 9 possible codes.

Also note that the order in which these elements are paired is crucial.

RELATIONS AND THEIR PROPERTIES

Definition

A relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$.

The second element is called the image of the first element.

For example, $R = \{(a,0),(a,1),(a,3),(b,1),(b,2),(c,0),(c,3)\}$ is a relation from the set $\{a,b,c\}$ to the set $\{0,1,2,3\}$.

A relation from a set A to itself is called a relation on A.

Definition

The set of all first elements of the ordered pairs in a relation ***R*** from a set ***A*** to a set ***B*** is called the **domain** of the relation ***R***.

Definition

The set of all second elements in a relation ***R*** from a set ***A*** to a set ***B*** is called the ***range*** of the relation ***R***. The whole set ***B*** is called the **codomain** of the relation R.

Note the following:

- Range ⊆ Codomain.
- If $(a,b) \in R$, then we say that $a\ is\ related\ to\ b$, which can also be written as aRb.
- The total number of relations that can be defined from a set ***A*** to a set ***B*** is the number of possible subsets of ***A × B***.
- If **n(*A*) = p** and **n(*B*) = *q***, then **n (*A* × *B*) = pq** and **the total number of relations =** $\mathbf{2^{pq}}$
- A relation ***R*** in a set ***A*** is called **empty relation (void relation)**, if no element of A is related to any element of A, i.e., ***R* = φ⊂*A* × *A***.
- A relation ***R*** in a set ***A*** is called **universal relation**, if each element of ***A*** is related to every element of ***A***, i.e., ***R* = *A* × *A***.
- Both the empty relation and the universal relation are sometimes called **trivial relations**.
- A relation ***R*** in a set ***A*** is called **identity relation**, if each element of ***A*** is related to itself only. i.e., $\boldsymbol{R = \{(a,a) | a \in A\}}$
- In case of relations from a set A to a set B, $A \times B$ is considered as the universal relation. The complement relation of a relation R is denoted and given as $R' = (A \times B) - R$.
- If $R = \{(a,b) | a \in A, b \in B\}$ then the inverse relation is denoted and given as $R^{-1} = \{(b,a) | a \in A, b \in B\}$.
- Union, intersection, difference and other operations of sets are all applicable to relations as they are for sets.

Problem.1

What is the largest possible relation from the set $A = \{1,2,3,4,5\}$ to the set $B = \{1,2,3\}$?

Write the relations from A to B in each of the following cases when

1) a is related to b if and only if $a \geq b$
2) a is related to b if and only if $a > b$
3) a is related to b if and only if $a < b$
4) aRb if and only if $a + b > 4$
5) $(a,b) \in R$ if and only if $a\ is\ a\ devisor\ of\ b$

Solution:

The largest possible relation from A to B is

$$A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),$$
$$(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$$

1) $R = \{(a,b) | a \geq b, a \in A, b \in B\}$
$= \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$
2) $R = \{(a,b) | a > b, a \in A, b \in B\} = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$
3) $R = \{(a,b) | a < b, a \in A, b \in B\} = \{(1,2),(1,3),(2,3)\}$
4) $R = \{(a,b) | a + b > 4, a \in A, b \in B\} =$
$\{(2,3),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$
5) $R = \{(a,b) | a \text{ is a devisor of } b, a \in A, b \in B\} = \{(1,1),(1,2),(1,3),(2,2),(3,3)\}$

Problem.2

Let $A = \{1, 2, 5\}$ and $B = \{3, 5, 7\}$ and let $R = \{(a,b) | 7 \leq a + b < 10, a \in A, b \in B\}$.

1) Write all the elements of R and R' 2) Write the inverse relation of R.

3) Find the Domain and Range of R and R^{-1} .

Solution:

1) $R = \{(1,7),(2,5),(2,7),(5,3)\}$
$R' = \{(1,3),(1,5),(2,3),(5,5),(5,7)\}$
2) $R^{-1} = \{(7,1),(5,2),(7,2),(3,5)\}$
3) $Dom(R) = \{1,2,5\}, Range(R) = \{3,7,5\}$
$Dom(R^{-1}) = \{3,7,5\}, Range(R) = \{1,2,5\}$

Problem.3

Consider these relations on the set of integers:

$R1 = \{(a,b) \mid a \leq b\}$,
$R2 = \{(a,b) \mid a > b\}$,
$R3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R4 = \{(a,b) \mid a = b\}$,
$R5 = \{(a,b) \mid a = b + 1\}$,
$R6 = \{(a,b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,−1), and (2,2)?

Solution:

The pair (1,1) is in R_1, R_3, R_4, and R_6; (1,2) is in R_1 and R_6; (2,1) is in R_2, R_5, and R_6; (1,−1) is in R_2, R_3, and R_6; and finally, (2,2) is in R_1, R_3, and R_4.

Exercise

1. Let $A = \{1, 2, 3, \ldots, 14\}$. Let a relation R on A be defined as
$R = \{(x,y) : 3x - y = 0, \text{where } x, y \in A\}$. Write down its domain, codomain and range.
2. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A to B. Which of the following is not a relation from A to ? Justify your answer.
   (i) $\{(x, 1), (y, 2), (z, 3)\}$
   (ii) $\{(x, 1), (x, 2)\}$
   (iii) $\{(x, 2), (y, 2), (z, 2)\}$
   (iv) $\{(1, x), (2, x)\}$
3. If $R = \{(1,2),(2,4),(3,3)\}$ and $S = \{(1,3),(2,4),(4,2)\}$ represents some relations on some sets then what is 1) $R \cup S$ 2) $R \cap S$ 3) $R - S$ 4) $S - R$ 5) $R \oplus S$
   Also verify if (i) $\text{Domain of } (R \cup S) = (\text{Domain of } R) \cup (\text{Domain of } S)$
   and (ii) $Range(R \cap S) \subseteq Range(R) \cap Range(S)$

DIFFERENCE BETWEEN RELATION AND FUNCTION

Function

We can, visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as ‘map’ or ‘mapping’ used to denote a function.

Definition

A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.

In other words,

A relation f is a function from a non-empty set A to a non-empty set B if

(i) the domain of f is A

(ii) no two distinct ordered pairs in f have the same first elements.

Note:

If f is a function from A to B and $(a, b) \in f$, then we write $f(a) = b$, where b is called the image of a under f and a is called the preimage of b under f.

Problem.1

Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not on the given domain?

- **(i)** R = {(2,1),(3,1), (4,2)}, Domain= {1,2,3,4}
- **(ii)** R = {(2,2),(2,4),(3,3), (4,4)} , Domain= {2,3,4}
- **(iii)** R = {(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)}, Domain= {1,2,3,4,5,7}