Limits at Infinity

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BHOS

Calculus

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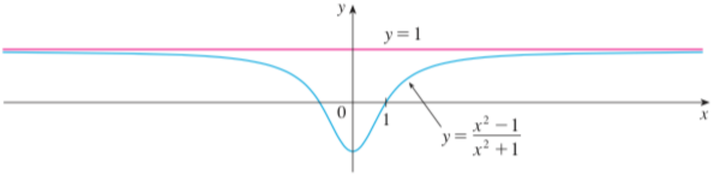

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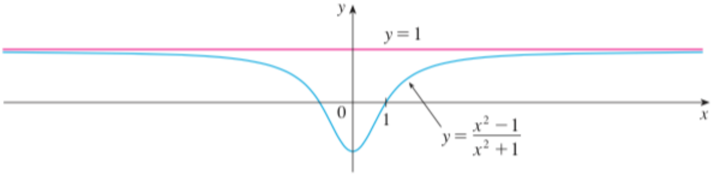

As x grows larger and larger, the values of $f(x)$ get closer and closer to 1.

Let us see some values in table:

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
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Mathematically,

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1.$$

Definitions:

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

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if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

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Solution: Let $\epsilon > 0$ be given. We must find a number M such that

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If we choose $M = \frac{1}{\epsilon}$, then we get

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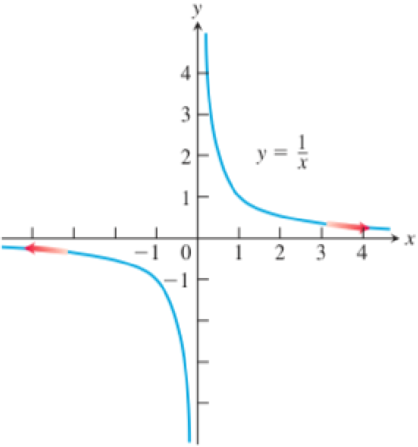
Infinite Limits



Theorem:

All the Limit Laws are true when we replace $\lim_{x\to x_0}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$ that is, the variable x may approach a finite number x_0 or $\pm\infty$

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Example: Find $\lim_{x \to \pm\infty} \frac{x^3+3}{|x|^3-2}$

Solution:

1. For $x \geq 0$:

$$\lim_{x \to \infty} \frac{x^3+3}{|x|^3-2} = \lim_{x \to \infty} \frac{x^3+3}{x^3-2} = \lim_{x \to \infty} \frac{1+3/x^3}{1-2/x^3} = 1.$$

2. For $x < 0$:

$$\lim_{x \to \infty} \frac{x^3+3}{|x|^3-2} = \lim_{x \to \infty} \frac{x^3+3}{-x^3-2} = \lim_{x \to \infty} \frac{1+3/x^3}{-1-2/x^3} = -1.$$

Limits at Infinity of Rational Functions

Let $P(x) = a_p x^p + a_{p-1}x^{p-1} + \cdots + a_0$ and $Q(x) = b_q x^q + b_{q-1}x^{q-1} + \cdots + b_0$. Then,to find the limit of $\frac{P(x)}{Q(x)}$ as $x \to \pm\infty$ we first divide the denominator and numerator by the highest power of x in the denominator.

Example: Find $\lim_{x\to-\infty} \frac{x^3+2x^2+1}{x-1}$.

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Example: Find $\lim_{x\to-\infty} \frac{x^3+2x^2+1}{x-1}$.

Solution: Divide numerator and denominator by x

$$\lim_{x\to-\infty} \frac{x^3+2x^2+1}{x-1} = \lim_{x\to-\infty} \frac{x(x+2)+1/x}{1-1/x} = \infty$$

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Limits at Infinity of Rational Functions

1. If $p > q$, then $\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \infty$ (+ or - depending on the sign)
2. If $p = q$, then $\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \frac{a_p}{b_q}$
3. If $p < q$, then $\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = 0$.

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$y = b$ is called a horizontal asymptote of the graph of $y = f(x)$ if

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The line $y = 5/3$ is a horizontal asymptote on both right nd left.

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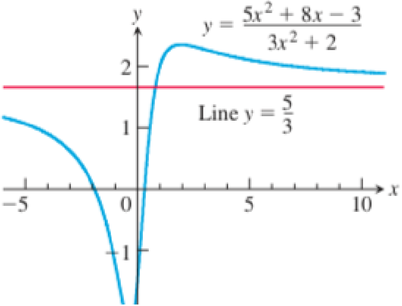

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For $x < 0$: $$\lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

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The lines $y = 1$ and $y = -1$ are horizontal asymptotes.

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