BHOS

Basic Mathematics

Mathematical Analysis

(Exercises and Solutions)

CE Department

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Contents

1 Limits of Functions

Exercise 1.1. *Find the limits:*

1. $\lim_{x\to -5}(2x+7)$,
2. $\lim_{x\to 4}(-x^2+2x+3)$,
3. $\lim_{x\to \frac{1}{2}} 2x^2(x+8)$,
4. $\lim_{x\to 3}\dfrac{3x+4}{3x^2+2x+1}$,
5. $\lim_{x\to 2}(3x^2+4)^{\frac{3}{4}}$.

Solution.

1. *Using Sum and Constant multiple Rules gives*

$$\lim_{x\to -5}(2x+7) = 2\lim_{x\to -5} x + \lim_{x\to -5} 7 = 2(-5)+7 = -3.$$

2. *Using Sum, Constant multiple and Power Rules gives*

$$\lim_{x\to 4}(-x^2+2x+3) = -(\lim_{x\to 4} x)^2 + 2\lim_{x\to 4} x + \lim_{x\to 4} 3 = -4^2 + 2\cdot 4 + 3 = -5.$$

3. $\lim_{x\to \frac{1}{2}} 2x^2(x+8) = 2\cdot\left(\frac{1}{2}\right)^2\cdot(\frac{1}{2}+8) = 2\cdot\frac{1}{4}\cdot\frac{17}{2} = \frac{17}{4}$.
4. $\lim_{x\to 3}\frac{3x+4}{3x^2+2x+1} = \frac{3\cdot 3+4}{3\cdot 3^2+2\cdot 3+1} = \frac{13}{34}$.
5. $\lim_{x\to 2}(3x^2+4)^{\frac{3}{4}} = (3\cdot 2^2+4)^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 8$.

Exercise 1.2. *Suppose* $\lim_{x\to c} f(x) = 7$ *and* $\lim_{x\to c} g(x) = 3$. *Find:*

1. $\lim_{x\to c} f(x)\cdot g(x)$,
2. $\lim_{x\to c}(3f(x)\cdot g(x)+1)$,
3. $\lim_{x\to c}(f(x)+3g(x))$,
4. $\lim_{x\to c}(2f(x)\cdot g^2(x))$,

5. $\lim_{x\to c}\dfrac{f(x)}{f(x)-g(x)}$.

Solution.

1. *Using Product Rule gives*

$$\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = 7\cdot 3 = 21.$$

2. *Using Constant multiple, Product and Sum Rules gives*

$$\lim_{x\to c}(3f(x)\cdot g(x)+1) = 3\lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) + \lim_{x\to c} 1 = 3\cdot 7\cdot 3+1 = 64.$$

3. *Using Sum and Constant multiple Rules gives*

$$\lim_{x\to c}(f(x)+3g(x)) = \lim_{x\to c} f(x) + 3\lim_{x\to c} g(x) = 7+3\cdot 3 = 16.$$

4. *Using Constant multiple, Product and Power Rules gives*

$$\lim_{x\to c}(2f(x)\cdot g^2(x)) = 2\lim_{x\to c} f(x)\cdot (\lim_{x\to c} g(x))^2 = 2\cdot 7\cdot 3^2 = 126.$$

5. *Using Quotient and Difference Rules gives*

$$\lim_{x\to c}\frac{f(x)}{f(x)-g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} f(x) - \lim_{x\to c} g(x)} = \frac{7}{7-3} = \frac{7}{4}.$$

Exercise 1.3. *Find the limits:*

1. $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$,
2. $\lim_{x\to 0}\dfrac{\sqrt{5x+4}-2}{x}$,
3. $\lim_{x\to 0}\dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$,
4. $\lim_{x\to 1}\dfrac{x^2+x-2}{x^2-1}$,
5. $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$,
6. $\lim_{x\to 1}\dfrac{x-1}{\sqrt{x+3}-2}$,

7. $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$.

Solution.

1. *We cannot substitute $x = 4$ because it makes the denominator zero. We test the numerator to see if it is zero at $x = 4$ too. It is, so it has a factor of $(x-4)$ in common with the denominator. Canceling the $(x-4)$'s gives a simpler fraction with the same values as the original for $x \neq 4$:*

$$\frac{x^2-16}{x-4} = \frac{(x+4)(x-4)}{x-4} = (x+4) \quad if \quad x \neq 4.$$

Using the simpler fraction, we find the limit of these values as $x \to 4$ by substitution:

$$\lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = \lim_{x\to 4}(x+4) = 4+4 = 8.$$

Note: *We will use this (Eliminating Zero Denominator method) for Exercise 3 and Exercise 4.*

2. *First we multiply the numerator and the denominator by the Conjugate fo the numerator. Then,*

$$\lim_{x\to 0} \frac{\sqrt{5x+4}-2}{x} = \lim_{x\to 0} \frac{(\sqrt{5x+4}-2)(\sqrt{5x+4}+2)}{x(\sqrt{5x+4}+2)} = \lim_{x\to 0} \frac{5x}{x(\sqrt{5x+4}+2)}$$
$$= \lim_{x\to 0} \frac{5}{(\sqrt{5x+4}+2)} = \frac{5}{2+2} = \frac{5}{4}.$$

3. $\lim\limits_{x\to 0} \dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim\limits_{x\to 0} \dfrac{\frac{x+1+x-1}{x^2-1}}{x} = \lim\limits_{x\to 0} \dfrac{2x}{x(x^2-1)} = \lim\limits_{x\to 0} \dfrac{2}{(x^2-1)} = \dfrac{2}{0^2-1} = -2.$

4. $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x+2)}{(x+1)(x-1)} = \lim\limits_{x\to 1} \dfrac{x+2}{x+1} = \dfrac{1+2}{1+1} = \dfrac{3}{2}.$

5. $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9} = \lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim\limits_{x\to 9} \dfrac{1}{\sqrt{x}+3} = \dfrac{1}{\sqrt{9}+3} = \dfrac{1}{6}.$

6.

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}$$
$$= \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x-1} = \lim_{x\to 1}(\sqrt{x+3}+2) = \sqrt{4}+2 = 4.$$

7.

$$\lim_{x\to-1}\frac{\sqrt{x^2+8}-3}{x+1}=\lim_{x\to-1}\frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(\sqrt{x^2+8}+3)(x+1)}=\lim_{x\to-1}\frac{x^2-1}{(\sqrt{x^2+8}+3)(x+1)}$$
$$=\lim_{x\to-1}\frac{x-1}{\sqrt{x^2+8}+3}=\frac{-1-1}{(-1)^2+3}=-\frac{2}{4}=-\frac{1}{2}.$$

Exercise 1.4. *Calculate the following limits:*

1. $\lim\limits_{x\to0}\dfrac{x^2-1}{2x^2-x-1}$,
2. $\lim\limits_{x\to1}\dfrac{x^2-1}{2x^2-x-1}$.

Solution.

1. $\lim\limits_{x\to0}\dfrac{x^2-1}{2x^2-x-1}=\dfrac{0^2-1}{2\cdot0^2-0-1}=1.$
2. $\lim\limits_{x\to1}\dfrac{x^2-1}{2x^2-x-1}=\lim\limits_{x\to1}\dfrac{(x-1)(x+1)}{2(x-1)(x+\frac{1}{2})}=\lim\limits_{x\to1}\dfrac{x+1}{2x+1}=\dfrac{1+1}{2+1}=\dfrac{2}{3}.$

Exercise 1.5. *Prove the limit statements:*

1. $\lim\limits_{x\to3}(x+4)=7$,
2. $\lim\limits_{x\to1}\dfrac{2}{x}=2$,
3. $\lim\limits_{x\to14}\sqrt{x-5}=3$.

Solution.

1. *Let $p=3$, $f(x)=x+4$ and $L=7$ in the definition of limit. For any given $\epsilon>0$ we have to find a suitable $\delta>0$ such that for all real $x\in R$, $0<|x-3|<\delta$ implies $|f(x)-7|<\epsilon$.*

   We find δ by working backward from the ϵ-inequality

   $$|f(x)-L|<\epsilon,$$
   $$|x+4-7|<\epsilon,$$
   $$|x-3|<\epsilon.$$

   We can take $\delta=\epsilon$. If $0<|x-3|<\delta=\epsilon$ then

   $$|x+4-7|=|x-3|<\epsilon,$$

   which proves that $\lim\limits_{x\to3}(x+4)=7$.

2. *Let* $p = 1$, $f(x) = \frac{2}{x}$ *and* $L = 2$. *Let us take* $\forall \epsilon > 0$. *We must find* $\delta > 0$ *such that for all real* x, *for which* $0 < |x - 1| < \delta$ $(x \neq 1,\ -\delta < x - 1 < \delta)$, $|f(x) - 2| < \epsilon$ *is true.*

   We solve the inequality
$$|f(x) - 2| < \epsilon.$$
   Then
$$\Big|\frac{2}{x} - 2\Big| < \epsilon,$$
$$-\epsilon < \frac{2}{x} - 2 < \epsilon,$$
$$2 - \epsilon < \frac{2}{x} < 2 + \epsilon,$$
   We can assume $\epsilon < 2$. *Then*
$$\frac{1}{2+\epsilon} < \frac{x}{2} < \frac{1}{2-\epsilon},$$
$$\frac{2}{2+\epsilon} < x < \frac{2}{2-\epsilon},$$
   We take δ *to be the distance from* $p = 1$ *to the nearer endpoint of* $\left(\frac{2}{2+\epsilon}, \frac{2}{2-\epsilon}\right)$. *In other words, we take*
$$\delta = \min\left\{1 - \frac{2}{2+\epsilon}, \frac{2}{2-\epsilon} - 1\right\} = \min\left\{\frac{\epsilon}{2+\epsilon}, \frac{\epsilon}{2-\epsilon}\right\} = \frac{\epsilon}{2+\epsilon}.$$
   Then for all x, $0 < |x - 1| < \delta = \dfrac{\epsilon}{2+\epsilon}$ *implies* $|f(x) - 2| < \epsilon$, $(\forall \epsilon < 2)$, *which proves that* $\lim\limits_{x\to 1} \frac{2}{x} = 2$.

3. *Let us take* $f(x) = \sqrt{x-5}$, $p = 14$ *and* $L = 3$ *be given. Let* $\epsilon > 0$. *We want to find a positive number* δ *such that for all* x $0 < |x - 4| < \delta$ $(x \neq 14,\ -\delta < x - 14 < \delta)$ *implies* $|f(x) - 3| < \epsilon$.

   We find δ *by working backward from the* ϵ*-inequality*
$$|f(x) - 3| = |\sqrt{x-5} - 3| = \Big|\frac{(\sqrt{x-5} - 3)(\sqrt{x-5} + 3)}{\sqrt{x-5} + 3}\Big|$$
$$= \Big|\frac{x - 14}{\sqrt{x-5} + 3}\Big| < \Big|\frac{x-14}{3}\Big| < \epsilon,$$
   Then,
$$|x - 14| < 3\epsilon.$$
   Thus, we take $\delta = 3\epsilon$.

   Then, whenever $|x - 14| < \delta = 3\epsilon$, *it is true that* $|f(x) - 3| < \Big|\dfrac{x-14}{3}\Big| < \dfrac{\delta}{3} = \dfrac{3\epsilon}{3} = \epsilon$, *Which proves that* $\lim\limits_{x\to 14} \sqrt{x-5} = 3$.

Exercise 1.6. *If* $3x^2 + 3 \leq f(x) \leq x^3 + 7$ *for* $0 \leq x \leq 5$, *find* $\lim_{x\to 2} f(x)$.

Solution. *Since*

$$\lim_{x\to 2}(3x^2+3) = 3\cdot 2^2 + 3 = 15 \quad \text{and} \quad \lim_{x\to 2}(x^3+7) = 2^3 + 7 = 15.$$

The Sandwich (Squeeze) Theorem implies $\lim_{x\to 2} f(x) = 15$.

Exercise 1.7. *It can be shown that the inequality*

$$1 - \frac{x^2}{6} \leq \frac{x\sin x}{2 - 2\cos x} \leq 1$$

holds for all values of x *close to zero. What, if anything, does this tell you about*

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}$$

Give reasons for your answer.

Solution. *Since*

$$\lim_{x\to 0}\left(1 - \frac{x^2}{6}\right) = 1 \quad \text{and} \quad \lim_{x\to 0} 1 = 1$$

According to The Sandwich (Squeeze) Theorem

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x} = 1.$$

Exercise 1.8. *If* $\lim_{x\to 5}\frac{f(x)-6}{x-3} = 4$, *find* $\lim_{x\to 5} f(x)$.

Solution.

$$f(x) - 6 = \frac{f(x)-6}{x-3}\cdot(x-3)$$

Because of this

$$\lim_{x\to 5}(f(x)-6) = \lim_{x\to 5}\frac{f(x)-6}{x-3}\cdot(x-3) = 4\cdot(5-3) = 8$$

Then,

$$\lim_{x\to 5} f(x) = 8 + 6 = 14.$$

Exercise 1.9. *If* $\lim_{x\to -1}\frac{f(x)}{x^2} = 5$, *find:*

1. $\lim\limits_{x\to -1} \frac{f(x)}{x}$,
2. $\lim\limits_{x\to -1} f(x)$.

Solution.

1. $\lim\limits_{x\to -1} \dfrac{f(x)}{x} = \lim\limits_{x\to -1} \left(\dfrac{f(x)}{x^2} \cdot x\right) = \lim\limits_{x\to -1} \dfrac{f(x)}{x^2} \cdot \lim\limits_{x\to -1} x = 5 \cdot (-1) = -5.$
2. $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \left(\dfrac{f(x)}{x^2} \cdot x^2\right) = \lim\limits_{x\to -1} \dfrac{f(x)}{x^2} \cdot \lim\limits_{x\to -1} x^2 = 5 \cdot (-1)^2 = 5.$

Exercise 1.10. *Find the one sided limits:*

1. $\lim\limits_{x\to 3^+} \dfrac{\sqrt{x-2}}{x+3}$,
2. $\lim\limits_{x\to 1^-} \dfrac{x^2+2x}{x-7}$,
3. $\lim\limits_{x\to 1^-} \left(\dfrac{x+1}{x}\right)\left(\dfrac{2-x^2}{3x}\right)$,
4. $\lim\limits_{x\to 3^+} \dfrac{x^3-27}{x-3}$,
5. $\lim\limits_{x\to 0^-} \dfrac{\sqrt{6}-\sqrt{5x^2+11x+6}}{x}$.

Solution.

1. $\lim\limits_{x\to 3^+} \dfrac{\sqrt{x-2}}{x+3} = \dfrac{\sqrt{3-2}}{3+3} = \dfrac{1}{6}.$
2. $\lim\limits_{x\to 1^-} \dfrac{x^2+2x}{x-7} = \dfrac{1^2+2\cdot 1}{1-7} = \dfrac{3}{-6} = -\dfrac{1}{2}.$
3. $\lim\limits_{x\to 1^-} \left(\dfrac{x+1}{x}\right)\left(\dfrac{2-x^2}{3x}\right) = \left(\dfrac{1+1}{1}\right)\left(\dfrac{2-1^2}{3\cdot 1}\right) = 2 \cdot \dfrac{1}{3} = \dfrac{2}{3}.$
4. $\lim\limits_{x\to 3^+} \dfrac{x^3-27}{x-3} = \lim\limits_{x\to 3^+} \dfrac{(x-3)(x^2+3x+9)}{x-3} = \lim\limits_{x\to 3^+} (x^2+3x+9) = 3^2+3\cdot 3+9 = 27.$
5. $$\begin{aligned}\lim_{x\to 0^-} \frac{\sqrt{6}-\sqrt{5x^2+11x+6}}{x} &= \lim_{x\to 0^-} \frac{6-(5x^2+11x+6)}{x(\sqrt{6}+\sqrt{5x^2+11x+6})} \\ &= \lim_{x\to 0^-} \frac{x(5x+11)}{x(\sqrt{6}+\sqrt{5x^2+11x+6})} = \frac{11}{2\sqrt{6}}.\end{aligned}$$

Exercise 1.11. *Evaluate the limits:*

1. $\lim\limits_{x\to -2^+}(x+3)\dfrac{|x+2|}{x+2}$,

2. $\lim\limits_{x\to -2^-}(x+3)\dfrac{|x+2|}{x+2}$.

Solution.

1. *If* $x\to -2^+$, *then* $x>2$,

   and $|x+2| = x+2$ *if* $x>2$. *Therefore,*

$$\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+}(x+3)\frac{(x+2)}{x+2} = \lim_{x\to -2^+}(x+3) = -2+3 = 1.$$

2. *If* $x\to -2^-$, *then* $x<2$,

   and $|x+2| = -(x+2)$ *if* $x<2$. *Therefore,*

$$\lim_{x\to -2^-}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^-}(x+3)\frac{-(x+2)}{x+2} = \lim_{x\to -2^-}(-x-3) = -(-2)-3 = -1.$$

Exercise 1.12. *Prove that:*

1. $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$,

2. $\lim\limits_{x\to 0}(x^2\sin\dfrac{1}{x} + 3x^2 + 2) = 2$.

Solution.

1. *From Trigonometry, you know* $-1 < \sin\dfrac{1}{x} < 1$.

$$\text{If } x>0 \quad \text{then} \quad -x < \sin\frac{1}{x} < x.$$

$$\text{If } x<0 \quad \text{then} \quad -x > \sin\frac{1}{x} > x.$$

$$\lim_{x\to 0^+} x = \lim_{x\to 0^+}(-x) = 0.$$

$$\lim_{x\to 0^-} x = \lim_{x\to 0^-}(-x) = 0.$$

Because of this, in both case, using the Sandwich theorem

$$\lim_{x\to 0^+} x\sin\frac{1}{x} = 0.$$

$$\lim_{x\to 0^-} x\sin\frac{1}{x} = 0.$$

Right sided and left sided limits are equal at $x=0$. *Hence,* $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$.

2. $-1 < \sin\frac{1}{x} < 1$, *therefore* $-x^2 < x^2\sin\frac{1}{x} < x^2$.

$$\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$$

According the Sandwich theorem

$$\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right) = 0.$$

Then,

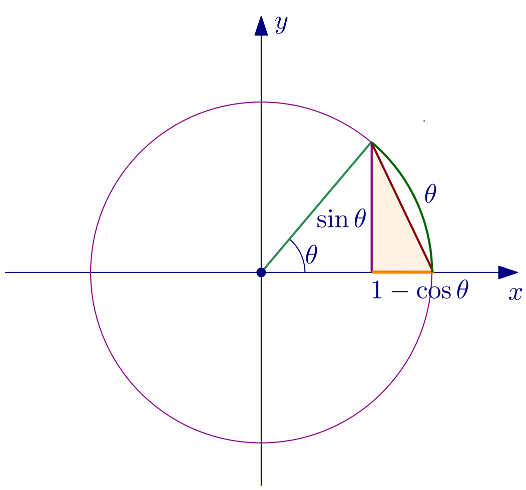
$$\lim_{x\to 0}\left(x^2\sin\frac{1}{x} + 3x^2 + 2\right) = \lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right) + \lim_{x\to 0}(3x^2+2) = 0 + 3\cdot 0^2 + 2 = 2.$$

Exercise 1.13. *Prove that:*

1. $\lim_{x\to 0}\sin x = 0$,
2. $\lim_{x\to 0}\cos x = 1$.

Solution.

First, let us prove that $-|x| \le \sin x \le |x|$ *and* $-|x| \le 1-\cos x \le |x|$ *are true.*
Take a circle with radius of 1 and θ *in the first quadrant.*



Using the Pythagorean Theorem gives,

$$\sin^2\theta + (1-\cos\theta)^2 \le \theta^2.$$

The terms on the left-hand side of inequality are both positive, so each is smaller than their sum and hence is less than or equal to θ^2*:*

$$\sin^2\theta \le \theta^2 \quad \textit{and} \quad (1-\cos\theta)^2 \le \theta^2$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \le |\theta| \quad \textit{and} \quad |1-\cos\theta| \le |\theta|$$

so

$$-|\theta| \le \sin\theta \le |\theta| \quad \textit{and} \quad |\theta| \le 1-\cos\theta \le |\theta|$$

To prove 1. and 2. we use these inequalities:

1. $-|x| \le \sin x \le |x|$, *and* $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0.$

   Then the Sandwich theorem expresses that $\lim_{x\to 0}\sin x = 0.$

2. $-|x| \le 1-\cos x \le |x|$, *and* $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0.$

   Then the Sandwich theorem expresses that $\lim_{x\to 0}(1-\cos x) = 0.$ *Then,*

$$\lim_{x\to 0}\cos x = 1.$$

Exercise 1.14. *Prove that,* $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$

Solution.

We begin with positive values of θ *less than* $\dfrac{\pi}{2}$*. According to accompanying figure,*

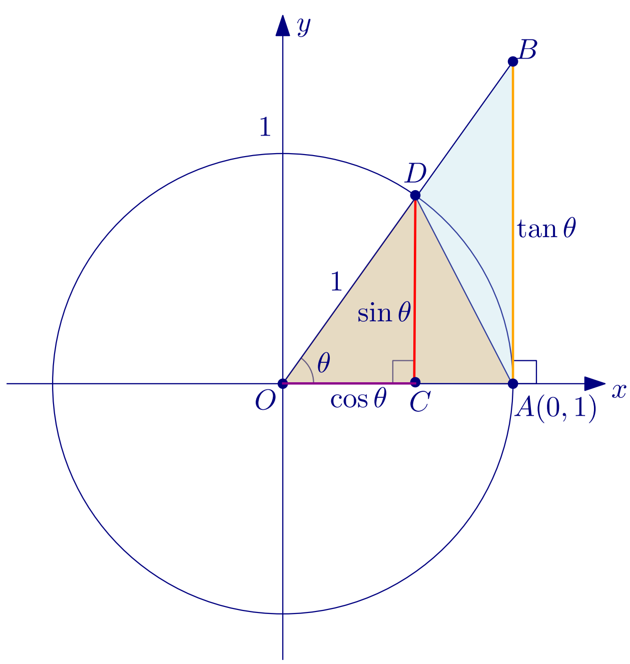
$$A_{\triangle OAD} < A_{\widehat{OAD}} < A_{\triangle OAB}$$

We can express these areas in terms of θ *as follows:*

$$A_{\triangle OAD} = \frac{1}{2}\cdot 1\cdot\sin\theta = \frac{1}{2}\sin\theta,$$

$$A_{\widehat{OAD}} = \frac{1}{2}r^2\theta = \frac{1}{2}\cdot 1^2\theta = \frac{\theta}{2},$$

$$A_{\triangle OAB} = \frac{1}{2} = \frac{1}{2}\cdot 1\cdot\tan\theta = \frac{1}{2}\tan\theta.$$



Thus,

$$\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta.$$

If we divide all three terms by the number $\frac{1}{2}\sin\theta$ *which is positive since* $0 < \theta < \frac{\pi}{2}$*:*

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocal reverses,

$$1 > \frac{\sin\theta}{\theta} > \cos\theta,$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$*, The Sandwich theorem gives*

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

Note: If f ***is even function and*** $\lim_{x\to 0^+} f(x) = L$ ***then***

$$\lim_{x\to 0} f(x) = L.$$

Using the previous note gives,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Exercise 1.15. *Find the following limits:*

1. $\lim_{x\to 0}\frac{\sin\sqrt{3}x}{\sqrt{3}x}$,
2. $\lim_{x\to 0}\frac{\tan 5x}{x}$,
3. $\lim_{x\to 0}\frac{x + x\cos x}{\sin x\cos x}$,
4. $\lim_{x\to 0}\frac{x^3 + x + \sin x}{2x}$,
5. $\lim_{x\to 0}\frac{\sin 2x\cot 4x}{x\cot 3x}$.

Solution.

1. *We use the substution* $\sqrt{3}x = t$*,*

   then $x \to 0 \iff t \to 0$*.*

   Because of this,

$$\lim_{x\to 0}\frac{\sin\sqrt{3}x}{\sqrt{3}x} = \lim_{t\to 0}\frac{\sin t}{t} = 1.$$

Note: *We will apply this substitution method for following limits.*

2. $\lim\limits_{x\to 0}\dfrac{\tan 5x}{x}=\lim\limits_{x\to 0}\dfrac{5\sin 5x}{5x\cos 5x}=5\cdot\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos 5x}=5\cdot 1\cdot 1=5.$

3. $\lim\limits_{x\to 0}\dfrac{x+x\cos x}{\sin x\cos x}=\lim\limits_{x\to 0}\dfrac{x(1+\cos x)}{\sin x\cos x}=\lim\limits_{x\to 0}\dfrac{1+\cos x}{\frac{\sin x}{x}\cdot\cos x}=\dfrac{1+1}{1\cdot 1}=2.$

4. $\lim\limits_{x\to 0}\dfrac{x^3+x+\sin x}{2x}=\lim\limits_{x\to 0}\left(\dfrac{x^2}{2}+\dfrac{1}{2}+\dfrac{\sin x}{2x}\right)=1.$

5. $\lim\limits_{x\to 0}\dfrac{\sin 2x\cot 4x}{x\cot 3x}=\lim\limits_{x\to 0}\dfrac{2\sin 2x}{2x}\cdot\dfrac{4x\cdot\cos 4x}{4\sin 4x}\cdot\dfrac{3\sin 3x}{3x\cdot\cos 3x}=2\cdot\dfrac{1}{4}\cdot 3=\dfrac{3}{2}.$

Exercise 1.16. *Find the limits:*

1. $\lim\limits_{x\to 0}(3\sin x-2)$

2. $\lim\limits_{x\to 0}(x^2-2)(\cos^2 x-2)$

3. $\lim\limits_{x\to -\pi}\sqrt{x+4}\cos(x+\pi)$

Solution. *Find the limits:*

1. $\lim\limits_{x\to 0}(3\sin x-2)=3\lim\limits_{x\to 0}(\sin x)-\lim\limits_{x\to 0}2=-2$

2. $\lim\limits_{x\to 0}(x^2-2)(\cos^2 x-2)=(0^2-2)(1^2-2)=2.$

3. *Let us make substitution $x+\pi=t$, then $x\to-\pi\iff t\to 0$.*
   $\lim\limits_{x\to -\pi}(\sqrt{x+4}\cos(x+\pi))=\lim\limits_{t\to 0}(\sqrt{t-\pi+4}\cos t)=\sqrt{0-\pi+4}=\sqrt{4-\pi}.$

Exercise 1.17. *Let f be a function defined by*

$$f(x)=\begin{cases}\dfrac{1}{x}, & x<-2\\ x^2-5 & -2<x\le 3\\ \sqrt{x+13}, & x>3\end{cases}$$

Find:

1. $\lim\limits_{x\to -2}f(x)$,

2. $\lim\limits_{x\to 0}f(x)$.

3. $\lim\limits_{x\to 3}f(x)$.

Solution. *We will determine the stated two-sided limit by first considering the corresponding one-sided limits.*

1. $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x} = -\frac{1}{2}$,

   $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x^2-5) = (-2)^2 - 5 = -1$.

   From which it follows that $\lim_{x\to -2} f(x)$ *does not exist.*

2. $f(x) = x^2 - 5$ *on both sides of* 0, *therefore,*

   $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2-5) = 0^2 - 5 = -5$

3. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-5) = 3^2 - 5 = 4$,

   $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x+13} = \sqrt{3+13} = 4$.

   Since left-sided and right sided limits are equal, 4, *we have* $\lim_{x\to 3} f(x) = 4$.

Exercise 1.18. *Let* f *be a function defined by the expression*

$$f(x) = \begin{cases} x-1, & x \leq 3 \\ 3x-7, & x > 3 \end{cases}$$

Find:

1. $\lim_{x\to 3^-} f(x)$,
2. $\lim_{x\to 3^+} f(x)$,
3. $\lim_{x\to 3} f(x)$.

Solution.

1. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x-1) = 3 - 1 = 2$,
2. $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3x-7) = 3\cdot 3 - 7 = 2$.
3. *Since one-sided limits are equal, we have* $\lim_{x\to 3} f(x) = 2$.

Exercise 1.19. *Let* f *be a function defined by*

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq 3 \\ k, & x = -3 \end{cases}$$

Find

1. *Find* k *so that* $f(-3) = \lim_{x\to -3} f(x)$

2. *With* k *assigned the value* $\lim_{x\to -3} f(x)$, *show that* f *can be expressed as a polynomial.*

Solution.

1. $k = f(-3) = \lim_{x\to -3} f(x) = \lim_{x\to -3}(x-3) = -3-3 = -6$

2. $$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq 3 \\ -6, & x = -3 \end{cases}$$

   That is, $f(x) = \dfrac{x^2-9}{x+3} = x-3$ *if* $x \neq 3$ *and* $f(x) = -6$ *if* $x = 3$. *So,* f *is equivalent to the polynomial* $g(x) = x-3$.

Exercise 1.20. *Prove that*

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exist.

Solution. *First we calculate one sided limits,*

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0)} 1 = 1 \qquad (x > 0)$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{(-x)}{x} = \lim_{x\to 0^-} (-1) = -1 \qquad (x < 0)$$

Since the one-sided limits are different, $\lim_{x\to 0} \frac{|x|}{x}$ *does not exist.*

Exercise 1.21. *At what points are the functions defined by following expressions continuous?*

1. $f(x) = \dfrac{x+3}{x^2-3x+2}$

2. $g(x) = \dfrac{3x}{(x+7)^2} + 5$

3. $h(x) = |x-2| + \sin x$

4. $l(x) = \dfrac{3x}{|x|-6} + x^2 + 4$

Solution.

1. *Since any rational function is continuous on its domain, The function f defined by $f(x) = \dfrac{x+3}{x^2-3x+2}$ is continuous any points at which $x^2-3x+2=0$. Let us solve $x^2-3x+2=0$*

$$x^2-3x+2=0 \iff (x-2)(x-1)=0 \implies x=2,\ x=1.$$

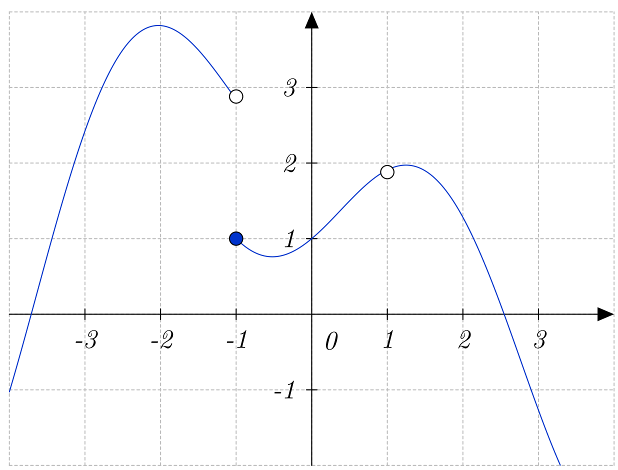
   Therefore, f is continuous on $\mathbb{R}\setminus\{1,2\}$.

2. *A function defined by $m(x) = \dfrac{3x}{(x+7)^2}$ is continuous on the set of real numbers and a function defined by $n(x)=5$ (constant function) is also continuous on $\mathbb{R}$, therefore g will be continuous on $\mathbb{R}/\{-7\}$.*

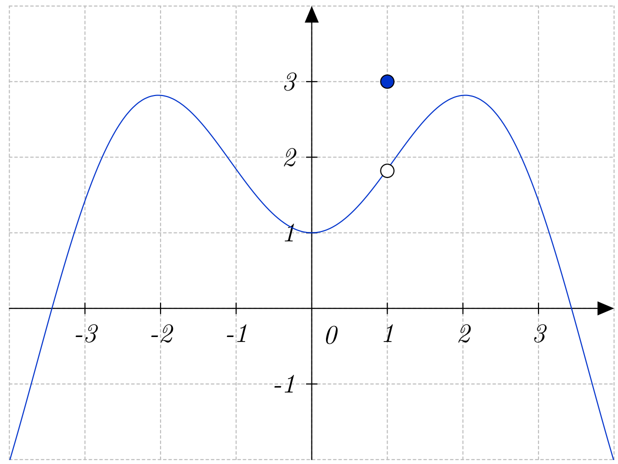
3. *Absolute value function and trigonometric function are continuous on their domains, because of this h is also continuous on $\mathbb{R}$*

4. *The function l is continuous on $\mathbb{R}\setminus\{-6,6\}$.*

Exercise 1.22. *Say whether the functions represented by a graph are continuous on $[-2,3]$ If not, where do them fail to be continuous and why?*



(a)



(b)

Solution.

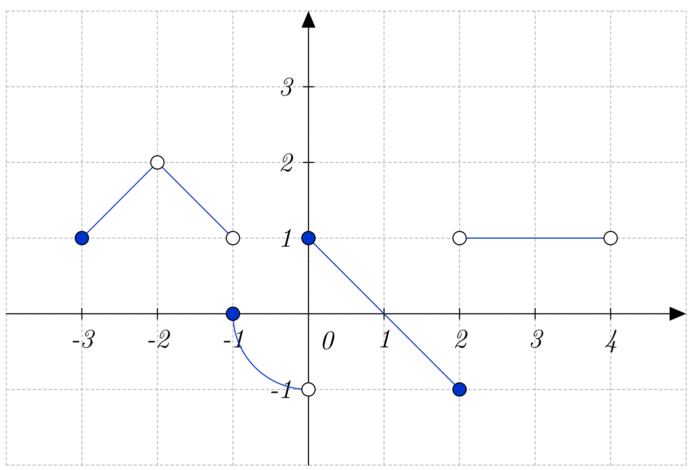
Geometric explanation: *Any function whose graph can be sketched over its domain in one continuous motion without lifting the pencil is a continuous function. Then the function represented in (a) is discontinuous. It is discontinuous at* $x = -1$ *and* $x = 1$.

The function represented in (b) is discontinuous. It is discontinuous at $x = 1$.

Exercise 1.23. *The function* f *given by*

$$f(x) = \begin{cases} x+4, & -3 \le x < -2 \\ -x, & -2 < x < -1 \\ -\sqrt{1-x^2}, & -1 \le x < 0 \\ -x+1, & 0 \le x \le 2 \\ 1, & 2 < x < 4 \end{cases}$$

is graphed in the accompanying figure.



1. *Does* $f(-3)$ *exist?*

2. *Does* $\lim_{x\to -2} f(x)$ *exist?*

3. *Does* $\lim_{x\to 2} f(x)$ *exist?*

4. *Does* $\lim_{x\to 2^-} f(x) = f(-1)$*?*

5. *Does* $\lim_{x\to -1^-} f(x)$ *exist?*

6. *Does* $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$ *?*

7. *Does* $\lim_{x\to -1^-} f(x) = f(-1)$*?*

8. *Define the points at which* f *is discontinuous.*

Solution.

Note: Geometric explanation.

1. $f(-3)$ *exists.*

2. *The limit of a function does not depend on how the function is defined at the point being approached. If we approach* -2 *from both sides, results will be the same. Because of this* $\lim_{x\to -2} f(x)$ *exists*

3. *If we approach* 2 *from both sides, results will be the different. Because of this* $\lim_{x\to 2} f(x)$ *doesn't exist.*

4. *According to the graph* $\lim_{x\to 2^-} f(x) = 1$ *but* $f(-1) = 0$*. So* $\lim_{x\to 2^-} f(x) \neq f(-1)$

5. $\lim_{x\to -1^-} f(x)$ *exist*

6. $\lim_{x\to -1^-} f(x) = 1$ *and* $\lim_{x\to -1^+} f(x) = 0$*. Therefore* $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$

7. *Does* $\lim_{x\to -1^-} f(x) = 1$ *and* $f(-1) = 0$*. So* $\lim_{x\to -1^-} f(x) \neq f(-1)$

8. $f(x)$ *is discontinuous at* $x = -2, -1, 0, 2, 4$*.*

Exercise 1.24. *At what points is the function* f *defined by the following expression continuous?*

$$f(x) = \begin{cases} \dfrac{x^2 - 6x + 8}{x - 4}, & x \neq 4 \\ 2, & x = 4 \end{cases}$$

Solution. $f(x) = \dfrac{x^2-6x+8}{x-4} = \dfrac{(x-2)(x-4)}{x-4} = x-2$ *if* $x \neq 4$. *Since every polynomial is continuous everywhere,* $f(x)$ *is continuous on* $\mathbb{R} \setminus \{4\}$. *Lets consider* $x = 4$. *Since*

$$\lim_{x\to 4} \frac{x^2-6x+8}{x-4} = \lim_{x\to 4}(x-2) = 2,$$

$$f(4) = 2$$

$f(x)$ *is continuous at* $x = 4$. *Therefore,* $f(x)$ *is continuous everywhere.*

Exercise 1.25. *At what points is the function* f *defined by*

$$f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9}, & x \neq -3 \ \ and \ \ x \neq 3 \\ 9, & x = 3 \\ 1, & x = -3 \end{cases}$$

continuous?

Solution.

$$f(x) = \frac{x^3-27}{x^2-9} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \frac{x^2+3x+9}{x+3}$$

if $x \neq -3$ *and* $x \neq 3$. *So* f *is continuous for* $x \neq -3$ *and* $x \neq 3$.

Consider $x = -3$ *and* $x = 3$.

If $x = -3$,

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x^3-27}{x^2-9} = \lim_{x\to -3} \frac{x^2+3x+9}{x+3}$$

does not exist. Therefore, f *is not continuous at* $x = -3$.

If $x = 3$,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2+3x+9}{x+3} = \frac{3^2+3\cdot 3+9}{3+3} = \frac{27}{6} = \frac{9}{2} = 4.5 \neq 1 = f(-3)$$

Because of this f *is not continuous at* $x = 3$. *Finally,* f *is continuous on* $\mathbb{R} \setminus \{-3, 3\}$.

Exercise 1.26. *Let* g *be a function defined by* $g(x) = \dfrac{x^2-25}{x-5}$. *Define* $g(5)$ *in a way that extends* g *to be continuous at* $x = 5$.

Solution. *If* $\lim\limits_{x\to c} f(x) = f(c)$ *then it is continuous at* c*. Hence, since*

$$\lim_{x\to 5} g(x) = \lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5}(x+5) = 10$$

we define $g(5) = 10$*, that extends* g *to be continuous at* $x = 5$*.*

Exercise 1.27. *Let* g *be a function defined by* $h(x) = \dfrac{x^2+3x-2}{x-2}$ *Define* h *in a way that extends* h *to be continuous at* $x = 2$*.*

Solution. *Since*

$$\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2+3x-2}{x-2} = \lim_{x\to 2}(x+1) = 3$$

we define $h(2) = 3$*, that extends* h *to be continuous at* $x = 2$*.*

Exercise 1.28. *Let* f *be a function defined by*

$$f(x) = \begin{cases} x^2-3, & x < 3 \\ 4ax, & x \geq 3 \end{cases}$$

For what value of a *is the function* f *continuous at every* x*?*

Solution. f *is polynomial if* $x \neq 3$*, so it is continuous for* $x \neq 3$*.*

Consider $x = 3$*.*

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-3) = 6,$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 4ax = 12a,$$
$$f(3) = 4a\cdot 3 = 12a.$$

Since f *is continuous if and only if it is right continuous and left continuous,*

$$\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x).$$

Then, $6 = 12a = 12a$*, and* $a = 1/2$*.*

Exercise 1.29. *Let* g *be a function defined by*

$$g(x) = \begin{cases} \dfrac{x-b}{b+1}, & x \leq 0 \\ x^2+b, & x > 0 \end{cases}$$

For what value of b *is the function* g
continuous at every x*?*

Solution. *gis polynomial if $x \neq 0$, because of this it is continuous at $x \neq 0$.*
Consider $x = 0$.

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{x-b}{b+1} = \frac{-b}{b+1},$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x^2+b) = b,$$
$$g(0) = \frac{-b}{b+1}.$$
$$\lim_{x\to 0^-} g(x) = g(0) = \lim_{x\to 0^+} g(x).$$

Then, $\frac{-b}{b+1} = b = \frac{-b}{b+1}$, hence, $b = 0$ or $b = 2$.

Exercise 1.30. *Let f be a function defined by*

$$f(x) = \begin{cases} ax^3+2b, & x \leq 0 \\ x^2+3a-b, & 0 < x \leq 2 \\ 3x-5, & x > 2 \end{cases}$$

For what values of a and b is f
continuous at every x?

Solution. *$f$is polynomial if $x \neq 0$ and $x \neq 2$, because of this it is continuous at $x \neq 0$ and $x \neq 2$.*
Let us consider $x = 0$ and $x = 2$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (ax^3+2b) = 2b,$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2+3a-b) = 3a-b,$$
$$f(0) = 2b.$$

If $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$. then, f is continuous at $x = 0$. Hence, $2b = 2b = 3a-b$, finally, $a = b$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2+3a-b) = 4+3a-b,$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-5) = 1,$$
$$f(2) = 4+3a-b.$$

$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$. *Then, $4+3a-b = 4+3a-b = 1$, so $b-3a = 3$.*
We get the system of equations

$$\begin{cases} a = b \\ b-3a = 3 \end{cases}$$

Therefore $f(x)$ is continuous if $a = -3/2$ and $b = -3/2$.

Exercise 1.31. *Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function defined by*

$$f(x) = \begin{cases} 1, & if \;\; x \;\; is \; rational \\ 0, & if \;\; x \;\; is \; irrational \end{cases}$$

is discontinuous at every point.

Solution.

Suppose p is rational, then $f(p) = 1$. Let us choose $\epsilon = \frac{1}{3}$. For any δ there is an irrational number x in the interval $(p - \delta, p + \delta)$ for which $f(x) = 0$. Then for this x, $0 < |x - p| < \delta$ but $|f(x) - f(p)| = 1 > \frac{1}{3}$ so $\lim_{x \to p} f(x) \neq f(p)$. Hence f is discontinuous at p rational.

If p irrational, it is proved in the similar method.

Exercise 1.32. *Explain why the equation $\cos x = x$ has at least one solution.*

Solution. *Let us take the function f defined by $f(x) = \cos x - x$. A zero of f is the root of $\cos x = x$.*

If $x = -\frac{\pi}{2}$, $f(-\frac{\pi}{2}) = \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} > 0$.

If $x = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = -\frac{\pi}{2} < 0$.

Since f is continuous over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the Intermediate Value Theorem implies that there is some c on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that $f(c) = 0$. Therefore c is the solution of $\cos x = x$.

Exercise 1.33. *Show that there is a root of the equation $x^3 + 3x^2 - x = 1$.*

Solution. *Let f be a function defined by $f(x) = x^3 + 3x^2 - x - 1$*

If $x = 0$, $f(0) = -1 < 0$.

If $x = 1$, $f(1) = 1^3 + 3 \cdot 1^2 - 1 - 1 = 2 > 0$.

Since f is continuous over the interval $[0, 1]$, the Intermediate Value Theorem implies that $f(c) = 0$ for some $c \in [0, 1]$. Therefore c is the solution of $x^3 + 3x^2 - x = 1$.

Exercise 1.34. ***A fixed point theorem*** *Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a fixed point of f).*

Solution. *Let us define $F(x) = f(x) - x$, that is continuous on $[0, 1]$.*

Since $F(0) = f(0) - 0 = f(0) \geq 0$ and $F(1) = f(1) - 1 \leq 0$ according the IVT $F(c) = 0$ for some $c \in [0, 1]$. Hence, $f(c) = c$.

Exercise 1.35. *Prove that the equation* $8\sqrt{x} - \sqrt{1-x} = 5$ *has at least one solution.*

Solution. *Let* f *be a function defined by* $f(x) = 8\sqrt{x} - \sqrt{1-x} - 5$

If $x = 0,\ f(0) = -6 < 0.$

If $x = 1,\ f(1) = 3 > 0.$
Since f *is continuous over the interval* $[0,1]$*, the Intermediate Value Theorem implies that* $f(c) = 0$ *for some* $c \in [0,1]$*. Therefore* c *is the solution of* $8\sqrt{x} - \sqrt{1-x} = 5.$

Exercise 1.36. *Show that:*

1. $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0,$
2. $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0.$

Solution.

1. *Let* $\epsilon > 0$ *be given. We must find a number* M *such that for all* x

$$x > M \quad \implies \quad \left|\frac{1}{x} - 0\right| < \epsilon$$

We find M *by working backward from the* ϵ*-inequality:*

$$\left|\frac{1}{x} - 0\right| < \epsilon \quad \iff \quad \left|\frac{1}{x}\right| < \epsilon \quad \iff \quad -\epsilon < \frac{1}{x} < \epsilon \quad \implies \quad x > \frac{1}{\epsilon}$$

We take all x *which satisfy* $\epsilon > \dfrac{1}{x} > 0$ *for which* ϵ*-inequality is true. Hence* $x > \dfrac{1}{\epsilon}$

Then, if we take $M = \dfrac{1}{\epsilon}$*, for all* x

$$x > M = \frac{1}{\epsilon} \quad \implies \quad \left|\frac{1}{x} - 0\right| < \epsilon$$

which yields that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0.$

2. *Let* $\epsilon > 0$ *be given. We must find a number* N *such that for all* x

$$x < N \quad \implies \quad \left|\frac{1}{x} - 0\right| < \epsilon$$

We find N *by working backward from the* ϵ*-inequality:*

$$\left|\frac{1}{x} - 0\right| < \epsilon \iff \left|\frac{1}{x}\right| < \epsilon \iff -\epsilon < \frac{1}{x} < \epsilon$$

We take all x which satisfies $-\epsilon < \frac{1}{x} < 0$ for which ϵ-inequality is true. Hence $x < -\frac{1}{\epsilon}$

Then, if we take $N = -\frac{1}{\epsilon}$, for all x

$$x < N = -\frac{1}{\epsilon} \implies \left|\frac{1}{x} - 0\right| < \epsilon$$

which yields that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Exercise 1.37. *Find the following limits:*

1. $\lim_{x\to\infty} \left(3 + \frac{1}{x}\right)$,
2. $\lim_{x\to\infty} \left(\frac{1}{x^3} + \frac{2}{x^2}\right)$.

Solution.

1. $\lim_{x\to\infty} \left(3 + \frac{1}{x}\right) = \lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{1}{x} = 3 + 0 = 3.$
2. $\lim_{x\to\infty} \left(\frac{1}{x^3} + \frac{2}{x^2}\right) = 0.$

Exercise 1.38. *Find the following limits:*

1. $\lim_{x\to\infty} \frac{3x^2 + 2x - 1}{x^2 + 7}$,
2. $\lim_{x\to\infty} \frac{5x^3 + 4x - 8}{2x^4 + 3x^3 - 5x}$,
3. $\lim_{x\to-\infty} \frac{-x^2 - 6x + 1}{2x^4 - 3x^2 - 6}$,
4. $\lim_{x\to-\infty} \frac{10x^5}{-2x^5 + x^4}$,
5. $\lim_{x\to\infty} \sqrt{x - 5} - \sqrt{x - 7}$,
6. $\lim_{x\to\infty} \sqrt{2x^2 - 2x} - \sqrt{2x^2 + 3x}$,

7. $\lim\limits_{x\to-\infty} \sqrt{x^2+8}+x.$

Solution.

First we divide the numerator and numerator by the highest power of the denominator:

1.
$$\lim_{x\to\infty}\frac{3x^2+2x-1}{x^2+7}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{7}{x^2}}=\lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{1}{x^2}}{1+\frac{7}{x^2}}=\frac{3+0}{1+0}=3.$$

2.
$$\lim_{x\to\infty}\frac{5x^3+4x-8}{2x^4+3x^3-5x}=\lim_{x\to\infty}\frac{\frac{5x^3}{x^4}+\frac{4x}{x^4}-\frac{8}{x^4}}{\frac{2x^4}{x^4}+\frac{3x^3}{x^4}-\frac{5x}{x^4}}=\lim_{x\to\infty}\frac{\frac{5}{x}+\frac{4}{x^3}-\frac{8}{x^4}}{2+\frac{3}{x}-\frac{5}{x^3}}=\frac{0-0+0}{2-0-0}=0.$$

3.
$$\lim_{x\to-\infty}\frac{-x^2-6x+1}{2x^4-3x^2-6}=\lim_{x\to-\infty}\frac{-\frac{x^2}{x^4}-\frac{6x}{x^4}+\frac{1}{x^4}}{\frac{2x^4}{x^4}-\frac{3x^2}{x^4}-\frac{6}{x^4}}=\lim_{x\to-\infty}\frac{-\frac{1}{x^2}-\frac{6}{x^3}+\frac{1}{x^4}}{2-\frac{3}{x^2}-\frac{6}{x^4}}=\frac{0}{2}=0.$$

4.
$$\lim_{x\to-\infty}\frac{10x^5}{-2x^5+x^4}=\lim_{x\to-\infty}\frac{\frac{10x^5}{x^5}}{\frac{-2x^5}{x^5}+\frac{x^4}{x^5}}=\lim_{x\to-\infty}\frac{10}{-2+\frac{1}{x}}=\frac{10}{-2}=-5.$$

5.
$$\lim_{x\to\infty}\sqrt{x-5}-\sqrt{x-7}=\lim_{x\to\infty}\frac{(\sqrt{x-5}-\sqrt{x-7})(\sqrt{x-5}+\sqrt{x-7})}{\sqrt{x-5}+\sqrt{x-7}}$$
$$=\lim_{x\to\infty}\frac{x-5-x+7}{\sqrt{x-5}+\sqrt{x-7}}=\lim_{x\to\infty}\frac{2}{\sqrt{x-5}+\sqrt{x-7}}=0.$$

6.
$$\lim_{x\to\infty}\sqrt{2x^2-2x}-\sqrt{2x^2+3x}=\lim_{x\to\infty}\frac{(\sqrt{2x^2-2x}-\sqrt{2x^2+3x})(\sqrt{2x^2-2x}+\sqrt{2x^2+3x})}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}$$
$$=\lim_{x\to\infty}\frac{2x^2-2x-2x^2-3x}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}=\lim_{x\to\infty}\frac{-5x}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}$$
$$=\lim_{x\to\infty}\frac{-5}{\sqrt{2-\frac{2}{x}}+\sqrt{2+\frac{3}{x}}}=\frac{-5}{2+2}=-\frac{5}{4}.$$

7.

$$\lim_{x\to-\infty}\sqrt{x^2+8}+x=\lim_{x\to-\infty}\frac{(\sqrt{x^2+8}+x)(\sqrt{x^2+8}-x)}{\sqrt{x^2+8}-x}=\lim_{x\to-\infty}\frac{x^2+8-x^2}{\sqrt{x^2+8}-x}$$
$$=\lim_{x\to-\infty}\frac{8}{|x|\sqrt{1+\frac{8}{x^2}}-x}=\lim_{x\to-\infty}\frac{8}{-x\sqrt{1+\frac{8}{x^2}}-x}=\lim_{x\to-\infty}\frac{8}{-x\left(\sqrt{1+\frac{8}{x^2}}+1\right)}=0.$$

Exercise 1.39. *Find the limits:*

1. $\lim\limits_{x\to0^+}\frac{2}{5x}$,
2. $\lim\limits_{x\to0^-}\frac{2}{5x}$,
3. $\lim\limits_{x\to6^+}\frac{1}{x-6}$,
4. $\lim\limits_{x\to0^-}\frac{1}{x-6}$.

Solution.

1. $\lim\limits_{x\to0^+}\frac{2}{5x}=+\infty.$ *(since the numerator and denominator are positive)*
2. $\lim\limits_{x\to0^-}\frac{2}{5x}=-\infty.$ *(since the numerator is positive and the denominator is negative)*
3. $\lim\limits_{x\to6^+}\frac{1}{x-6}=+\infty.$ *(since the numerator and denominator are positive)*
4. $\lim\limits_{x\to0^-}\frac{1}{x-6}=-\infty.$ *(since the numerator is positive and the denominator is negative)*

Exercise 1.40. *Find the limits:*

1. $\lim\limits_{x\to-3^+}\frac{2x}{5x+15}$,
2. $\lim\limits_{x\to-3^-}\frac{2x}{5x+15}$,
3. $\lim\limits_{x\to-2}\frac{1}{(x+2)^2}$,
4. $\lim\limits_{x\to7}\frac{2x}{(x-7)^2}$.

Solution.

1. $\lim_{x\to -3^+} \frac{2x}{5x+15} = \lim_{x\to -3^+} \frac{2x}{5(x+3)} = -\infty.$ *(since the numerator is negative and the denominator is positive)*

2. $\lim_{x\to -3^-} \frac{2x}{5x+15} = \lim_{x\to -3^-} \frac{2x}{5(x+3)} = \infty.$ *(since the numerator and denominator are negative)*

3. $\lim_{x\to -2} \frac{1}{(x+2)^2} = \infty.$ *(since the left and right hand limits are the same, ∞)*

4. $\lim_{x\to 7} \frac{2x}{(x-7)^2} = \infty.$ *(since the left and right hand limits are the same, ∞)*

Exercise 1.41. *Find the limit of f defined by* $f(x) = \frac{x^2}{4} - \frac{3}{x}$,

1. *as* $x \to 0^+$,

2. *as* $x \to 0^-$,

3. *as* $x \to 5$,

4. *as* $x \to -2$,

Solution.

1. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\frac{x^2}{4} - \frac{3}{x}\right) = \lim_{x\to 0^+} \frac{x^2}{4} - \lim_{x\to 0^+} \frac{3}{x} = -\infty.$ *(since $0 - \infty = -\infty$)*

2. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(\frac{x^2}{4} - \frac{3}{x}\right) = \lim_{x\to 0^-} \frac{x^2}{4} - \lim_{x\to 0^-} \frac{3}{x} = \infty.$ *(since $0 - (-\infty) = \infty$)*

3. $\lim_{x\to 5} f(x) = \lim_{x\to 5} \left(\frac{x^2}{4} - \frac{3}{x}\right) = \frac{5^2}{4} - \frac{3}{5} = 5.65.$

4. $\lim_{x\to -2} f(x) = \lim_{x\to -2} \left(\frac{x^2}{4} - \frac{3}{x}\right) = \frac{(-2)^2}{4} - \frac{3}{-2} = 2.5.$

Exercise 1.42. *Find the limit of f given by* $f(x) = \frac{x^2 - 5x + 6}{x^3 - 9x}$,

1. *as* $x \to -3^+$,

2. *as* $x \to -3^-$,

3. *as* $x \to -3$,

4. *as* $x \to 0^-$,

Solution.

1. $\lim\limits_{x\to -3^+} f(x) = \lim\limits_{x\to -3^+} \dfrac{x^2-5x+6}{x^3-9x} = \lim\limits_{x\to -3^+} \dfrac{(x-2)(x-3)}{x(x-3)(x+3)} = \lim\limits_{x\to -3^+} \dfrac{(x-2)}{x(x+3)} = \infty.$

2. $\lim\limits_{x\to -3^-} f(x) = \lim\limits_{x\to -3^-} \dfrac{x^2-5x+6}{x^3-9x} = \lim\limits_{x\to -3^-} \dfrac{(x-2)}{x(x+3)} = -\infty.$

3. $\lim\limits_{x\to -3} f(x) = \lim\limits_{x\to -3} \dfrac{x^2-5x+6}{x^3-9x}$ *does not exist, because one sided limits are different.*

4. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{x^2-5x+6}{x^3-9x} = \lim\limits_{x\to 0^-} \dfrac{(x-2)}{x(x+3)} = \infty.$

Exercise 1.43. *Find the limit of* f *which is defined by* $f(x) = \dfrac{1}{x^{2/3}} + \dfrac{1}{x-1}$,

1. *as* $x \to 0^+$,
2. *as* $x \to 0^-$,
3. *as* $x \to 1^+$,
4. *as* $x \to 1^-$.

Solution.

1. $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \left(\dfrac{1}{x^{2/3}} + \dfrac{1}{x-1}\right) = \lim\limits_{x\to 0^+} \dfrac{1}{x^{2/3}} + \lim\limits_{x\to 0^+} \dfrac{1}{x-1} = \infty.$

2. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \left(\dfrac{1}{x^{2/3}} + \dfrac{1}{x-1}\right) = \lim\limits_{x\to 0^-} \dfrac{1}{x^{2/3}} + \lim\limits_{x\to 0^-} \dfrac{1}{x-1} = \infty.$

3. $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \left(\dfrac{1}{x^{2/3}} + \dfrac{1}{x-1}\right) = \lim\limits_{x\to 1^+} \dfrac{1}{x^{2/3}} + \lim\limits_{x\to 1^+} \dfrac{1}{x-1} = \infty.$

4. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \left(\dfrac{1}{x^{2/3}} + \dfrac{1}{x-1}\right) = \lim\limits_{x\to 1^-} \dfrac{1}{x^{2/3}} + \lim\limits_{x\to 1^-} \dfrac{1}{x-1} = -\infty.$

Exercise 1.44. *Let* f *be defined as follows. Find the horizontal asymptotes of the graph of* f.

1. $f(x) = \dfrac{4x^3+2x+1}{x^3+3x^2}$,

2. $f(x) = \dfrac{x^2}{-2x^2+6x+10}$,

3. $f(x)=\dfrac{|x|^3-3x}{2x^3+3x^2-1}$,

4. $f(x)=\dfrac{x^3+x^2-4x-6}{x+3}$.

Solution.

1.

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{4x^3+2x+1}{x^3+3x^2}=\lim_{x\to-\infty}\frac{4+\dfrac{2}{x^2}+\dfrac{1}{x^3}}{1+\dfrac{3}{x}}=4,$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{4x^3+2x+1}{x^3+3x^2}=\lim_{x\to\infty}\frac{4+\dfrac{2}{x^2}+\dfrac{1}{x^3}}{1+\dfrac{3}{x}}=4.$$

These limits imply that the line of $y=4$ is the horizontal asymptote of the graph of f on both the right and the left.

2.

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x^2}{-2x^2+6x+10}=-\frac{1}{2},$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^2}{-2x^2+6x+10}=-\frac{1}{2}.$$

Then the line of $y=-\frac{1}{2}$ is the horizontal asymptote of the graph of f on both the right and the left (or at $-\infty$ and $-\infty$).

3.

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{|x|^3-3x}{2x^3+3x^2-1}=\lim_{x\to-\infty}\frac{-x^3-3x}{2x^3+3x^2-1}\quad (\textit{since}\quad x<0)$$
$$=\lim_{x\to-\infty}\frac{-1-\dfrac{3}{x^2}}{2+\dfrac{3}{x}-\dfrac{1}{x^3}}=-\frac{1}{2}.$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{|x|^3-3x}{2x^3+3x^2-1}=\lim_{x\to\infty}\frac{x^3-3x}{2x^3+3x^2-1}\quad (\textit{since}\quad x>0)$$
$$=\lim_{x\to\infty}\frac{1-\dfrac{3}{x^2}}{2+\dfrac{3}{x}-\dfrac{1}{x^3}}=\frac{1}{2}.$$

Theare are the horizontal asymptotes of $y=-\frac{1}{2}$ and $y=\frac{1}{2}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3+x^2-4x-6}{x+3} = \lim_{x\to\infty} \frac{x^2+x-4-\dfrac{6}{x}}{1+\dfrac{3}{x}} = \infty.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3+x^2-4x-6}{x+3} = \lim_{x\to-\infty} \frac{x^2+x-4-\dfrac{6}{x}}{1+\dfrac{3}{x}} = \infty.$$

The graph of f has no horizontal asymptote.

Exercise 1.45. *Find the horizontal asymptotes of the graph of f given by:*

1. $f(x) = 3 \cdot e^{2x}$,
2. $f(x) = x - \sqrt{x^2+9}$.

Solution.

1. *Since*

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (3 \cdot e^{2x}) = 0,$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (3 \cdot e^{2x}) = \infty.$$

the graph of f has a horizontal asymptote of $y = 0$ only at $-\infty$.

2.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x - \sqrt{x^2+9}) = \lim_{x\to-\infty} \left(x - |x|\sqrt{1+\frac{9}{x^2}}\right)$$
$$= \lim_{x\to-\infty} \left(x + x\sqrt{1+\frac{9}{x^2}}\right) = \lim_{x\to-\infty} x\left(1 + \sqrt{1+\frac{9}{x^2}}\right) = -\infty,$$

This limit implies that the curve of f has no horizontal asymptote at $-\infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x - \sqrt{x^2+9}) = \lim_{x\to\infty} \frac{(x-\sqrt{x^2+9})(x+\sqrt{x^2+9})}{(x+\sqrt{x^2+9})}$$
$$= \lim_{x\to\infty} \frac{x^2-x^2-9}{x+\sqrt{x^2+9}} = \lim_{x\to\infty} \frac{-9}{x+|x|\sqrt{1+\dfrac{9}{x^2}}}$$
$$= \lim_{x\to\infty} \frac{-9}{x+x\sqrt{1+\dfrac{9}{x^2}}} = \lim_{x\to-\infty} \frac{-9}{x\left(1+\sqrt{1+\dfrac{9}{x^2}}\right)} = 0.$$

Then the line of $y = 0$ will be the horizontal asymptote of f at ∞.

Exercise 1.46. *Find the horizontal asymptotes of the graph of f defined by*

1. $f(x) = \cos\dfrac{1}{x}$,
2. $f(x) = \dfrac{1}{x^2}\cos x$,
3. $f(x) = x\sin\dfrac{1}{x}$.

Solution.

1.
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \cos\frac{1}{x} = 0,$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \cos\frac{1}{x} = 0,$$
Therefore the line of $y = 0$ is the horizontal asymptote of the curve f at both $-\infty$ and $+\infty$.

2.
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(\frac{1}{x^2}\cos x\right) = 0, \quad (\textit{from the Sandwich theorem})$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(\frac{1}{x^2}\cos x\right) = 0.$$
Hence, the line of $y = 0$ is the horizontal asymptote of the curve f at both $-\infty$ and $+\infty$.

3.
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(x\sin\frac{1}{x}\right) = 0,$$
Let us make the substitution $t = \dfrac{1}{x}$, then $t \to -\infty \iff x \to 0^-$, hence,
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(x\sin\frac{1}{x}\right) = \lim_{t\to0^-} \left(\frac{1}{t}\sin t\right) = 1,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(x\sin\frac{1}{x}\right) = 0,$$
We substitute $t = \dfrac{1}{x}$, then $t \to +\infty \iff x \to 0^+$, hence,
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(x\sin\frac{1}{x}\right) = \lim_{t\to0^+} \left(\frac{1}{t}\sin t\right) = 1.$$
Therefore, the line of $y = 1$ is the horizontal asymptote of the curve f on both the right and the left (or at $-\infty$ and $-\infty$).

Exercise 1.47. *Find the vertical asymptotes of the graph of f defined by*

1. $f(x) = \dfrac{1}{x-2}$,
2. $f(x) = \dfrac{x^2-3x+2}{x^2-2x}$,
3. $f(x) = \sec x, \quad x \in \Big[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\Big]$,
4. $f(x) = \tan x$.

Solution.

1. *We consider the point 2, such that* $\lim_{x\to 2}\left|\dfrac{1}{x-2}\right| = +\infty$. *Since*

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\frac{1}{x-2} = -\infty,$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{1}{x-2} = +\infty,$$

the line of the equation $x = 2$ is a vertical asymptote of f both from the right and from the left.

2. $f(x) = \dfrac{x^2-3x+2}{x^2-2x} = \dfrac{(x-2)(x-1)}{x(x-2)}$.

Since,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0^-}\frac{(x-2)(x-1)}{x(x-2)} = \lim_{x\to 0^-}\frac{x-1}{x} = -\infty,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{1}{x-2}\lim_{x\to 0^+}\frac{(x-2)(x-1)}{x(x-2)} = \lim_{x\to 0^+}\frac{x-1}{x} = +\infty,$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2^-}\frac{x-1}{x} = \frac{1}{2},$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2^+}\frac{x-1}{x} = \frac{1}{2}.$$

The line of $x = 0$ is a vertical asymptote of f both from the right and from the left.

3. $f(x) = \sec x = \dfrac{1}{\cos x}$, *and* $\cos x = 0$ *if* $x = -\dfrac{\pi}{2}$ *and* $x = \frac{\pi}{2}$, *so we find,*

$$\lim_{x\to -\frac{\pi}{2}^+} f(x) = \lim_{x\to -\frac{\pi}{2}^+}\sec = \lim_{x\to -\frac{\pi}{2}^+}\frac{1}{\cos x} = +\infty,$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} \sec = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x} = +\infty,$$

Because of these the lines of the equations $x = -\frac{\pi}{2}$ *is vertical asymptote of the graph of* f *from the left,* $x = -\frac{\pi}{2}$ *is vertical asymptotes of the graph of* f *from the right.*

4. $f(x) = \tan x = \frac{\sin x}{\cos x}$*, and* $\cos x = 0$ *if* $x = \frac{\pi}{2} + \pi k$ *where* $k \in \mathbb{Z}$*, so we find,*

$$\lim_{x\to(\frac{\pi}{2}+\pi k)^+} f(x) = \lim_{x\to(\frac{\pi}{2}+\pi k)^+} \sec = \lim_{x\to(\frac{\pi}{2}+\pi k)^+} \frac{1}{cosx} = -\infty$$

$$\lim_{x\to(\frac{\pi}{2}+\pi k)^-} f(x) = \lim_{x\to(\frac{\pi}{2}+\pi k)^-} \sec = \lim_{x\to(\frac{\pi}{2}+\pi k)^-} \frac{1}{cosx} = +\infty$$

Because of this the lines of the equations $x = \frac{\pi}{2} + \pi k$*,* $k \in \mathbb{Z}$ *are vertical asymptotes of the graph of* f *both from the right and from the left.*

Exercise 1.48. *Find the oblique asymptote of the graph of* f *defined by*

1. $f(x) = \frac{2x^2 + 3x - 1}{x - 7}$,
2. $f(x) = \frac{x^3 + 3x^2 - 3}{x^2 - 2}$,
3. $f(x) = \sqrt{x^2 + 3x - 1} - x$,
4. $f(x) = \frac{x^{3/2} + 2x - 4}{\sqrt{x} - 1}$.

Solution.

1. *If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asymptote. We find an equation for the asymptote by dividing numerator by denominator to express it as a linear function plus a remainder that goes to zero as* $x \to \pm\infty$ *(this method is only for rational functions)*

$$f(x) = \frac{2x^2 + 3x - 1}{x - 7} = (2x + 17) + \frac{118}{x - 7},$$

where $\frac{118}{x-7} \to 0$ *as* $\quad x \to \pm\infty$.

Becaise of this the slant line defined by $y = 2x + 17$ *is the oblique asymptote of the graph of* f.

2. *(General method) for the oblique asymptote which is defined by the expression* $y = ax+b$ *we define the slope as* $a = \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x}$ *and* b *as* $b = \lim\limits_{x\to\pm\infty}(f(x) - ax)$

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\dfrac{x^3+3x^2-3}{x^2-2}}{x} = \lim_{x\to-\infty} \frac{x^3+3x^2-3}{x^3-2x} = 1,$$

$$\lim_{x\to-\infty} \left(\frac{x^3+3x^2-3}{x^2-2} - x\right) = \lim_{x\to-\infty} \frac{3x^2+2x-3}{x^2-2} = 3,$$

so the slant line of the equation $y = x-3$ *is the oblique asymptote to the curve at* $-\infty$.

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\dfrac{x^3+3x^2-3}{x^2-2}}{x} = \lim_{x\to+\infty} \frac{x^3+3x^2-3}{x^3-2x} = 1,$$

$$\lim_{x\to+\infty} \left(\frac{x^3+3x^2-3}{x^2-2} - x\right) = \lim_{x\to+\infty} \frac{3x^2+2x-3}{x^2-2} = 3,$$

then the line defined by $y = x-3$ *is also the oblique asymptote of the curve at* $+\infty$.

3.

$$\begin{aligned} a &= \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+3x-1} - x}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x}{x} \\ &= \lim_{x\to-\infty} \frac{-x\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x}{x} = \lim_{x\to-\infty} \frac{-x\left(\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} + 1\right)}{x} \\ &= \lim_{x\to-\infty} -\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}} + 1\right) = -2. \end{aligned}$$

$$\begin{aligned} b &= \lim_{x\to-\infty}(f(x) - ax) = \lim_{x\to-\infty}(\sqrt{x^2+3x-1} - x - (-2x)) = \lim_{x\to-\infty}(\sqrt{x^2+3x-1} + x) \\ &= \lim_{x\to-\infty} \frac{(\sqrt{x^2+3x-1}+x)(\sqrt{x^2+3x-1}-x)}{\sqrt{x^2+3x-1}-x} = \lim_{x\to-\infty} \frac{x^2+3x-1-x^2}{\sqrt{x^2+3x-1}-x} \\ &= \lim_{x\to-\infty} \frac{3x-1}{\sqrt{x^2+3x-1}-x} = \lim_{x\to-\infty} \frac{3x-1}{|x|\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x} = \lim_{x\to-\infty} \frac{3x-1}{-x\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x} \\ &= \lim_{x\to-\infty} \frac{3x-1}{-x\left(\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} + 1\right)} = \lim_{x\to-\infty} \frac{-3+\frac{1}{x}}{\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} + 1} = -\frac{3}{2}. \end{aligned}$$

Then, the slant line of $y=-2x-\frac{3}{2}$ will be the oblique asymptote to f at $-\infty$.

$$a=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\sqrt{x^2+3x-1}-x}{x}=\lim_{x\to+\infty}\frac{|x|\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}-x}{x}$$
$$=\lim_{x\to+\infty}\frac{x\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}-x}{x}=\lim_{x\to+\infty}\frac{x\Big(\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}-1\Big)}{x}$$
$$=\lim_{x\to+\infty}\Big(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-1\Big)=0.$$

Since $a=0$, the curve has no oblique asymptote.

$$b=\lim_{x\to+\infty}(f(x)-ax)=\lim_{x\to+\infty}(\sqrt{x^2+3x-1}-x-0)=\lim_{x\to+\infty}(\sqrt{x^2+3x-1}-x)$$
$$=\lim_{x\to+\infty}\frac{(\sqrt{x^2+3x-1}-x)(\sqrt{x^2+3x-1}+x)}{\sqrt{x^2+3x-1}+x}=\lim_{x\to+\infty}\frac{x^2+3x-1-x^2}{\sqrt{x^2+3x-1}+x}$$
$$=\lim_{x\to+\infty}\frac{3x-1}{\sqrt{x^2+3x-1}+x}=\lim_{x\to+\infty}\frac{3x-1}{|x|\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}+x}=\lim_{x\to+\infty}\frac{3x-1}{x\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}+x}$$
$$=\lim_{x\to+\infty}\frac{3x-1}{x\Big(\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}+1\Big)}=\lim_{x\to+\infty}\frac{3-\frac{1}{x}}{\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}+1}=\frac{3}{2}.$$

Therefore, it has a horizontal asymptote defined by $y=\dfrac{3}{2}$ at $+\infty$.

4. The function is defined on for all $x\geq 0$, because of this we investigate a limit at $+\infty$

$$a=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\dfrac{x^{3/2}+2x-4}{\sqrt{x}-1}}{x}=\lim_{x\to+\infty}\frac{x^{3/2}+2x-4}{x^{3/2}-x}=1.$$

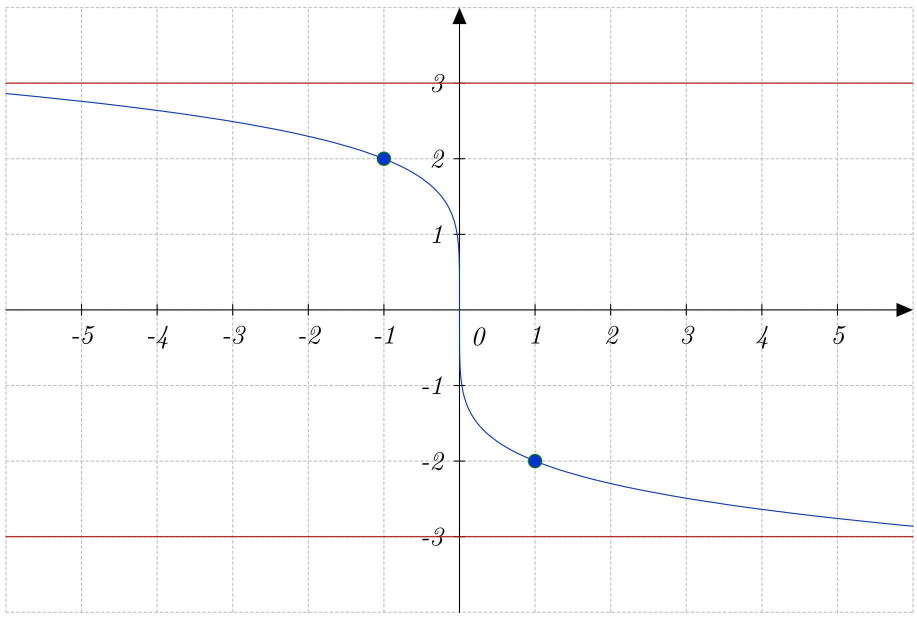
$$b=\lim_{x\to+\infty}(f(x)-ax)=\lim_{x\to+\infty}\Big(\frac{x^{3/2}+2x-4}{\sqrt{x}-1}-x\Big)$$
$$=\lim_{x\to+\infty}\frac{3x-4}{\sqrt{x}-1}=\lim_{x\to+\infty}\frac{3\sqrt{x}-\dfrac{4}{\sqrt{x}}}{1-\dfrac{1}{\sqrt{x}}}=+\infty.$$

Since $b=+\infty$, the curve does not an oblique asymptote.

Exercise 1.49. *Sketch the graph of a function defined by the expression $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)*

$$f(0) = 1, \quad f(1) = -2, \;\; f(-1) = 2, \quad \lim_{x\to-\infty} f(x) = 3, \quad \lim_{x\to+\infty} f(x) = -3.$$

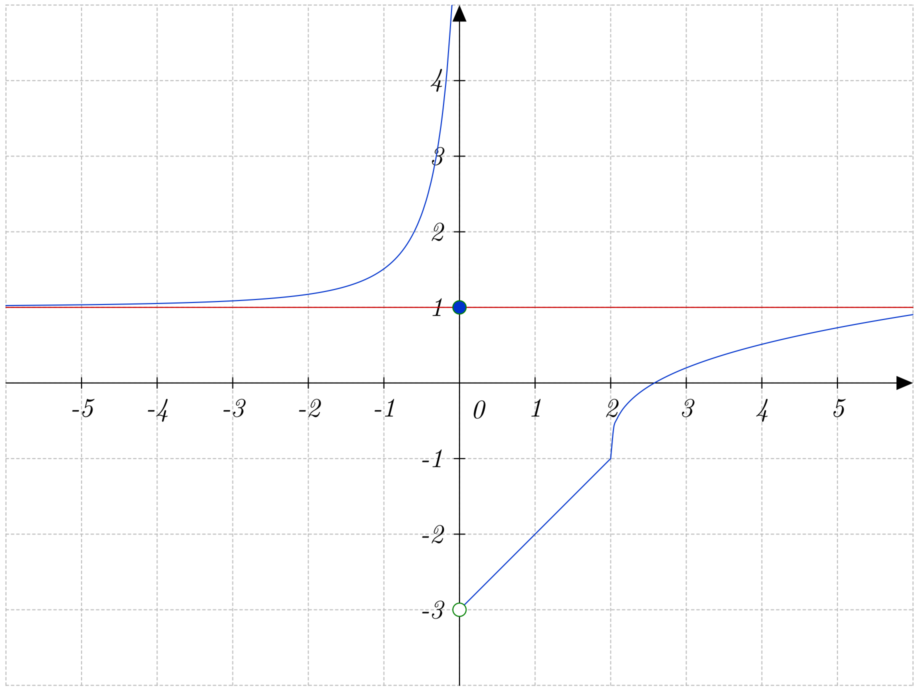
Solution. *One of the choices for the graph of the function can be as follows*



Exercise 1.50. *Sketch the graph of a function defined by $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)*

$$f(0) = 1, \quad \lim_{x\to-\infty} f(x) = 1, \quad \lim_{x\to+\infty} f(x) = 1 \quad \lim_{x\to0^+} f(x) = -3, \quad \lim_{x\to0^-} f(x) = +\infty.$$

Solution. *One of the choices is as follows*



Exercise 1.51. *Let f is defined by $f(x) = \sqrt{x^2 - 7x}$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f =]-\infty, 0[\cup]7, +\infty[$.

(a) *H.A.*

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{x^2 - 7x} = \lim_{x\to-\infty} |x|\sqrt{1 - \frac{7}{x}} = \lim_{x\to-\infty} -x\sqrt{1 - \frac{7}{x}} = -\infty,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty} \sqrt{x^2 - 7x} = \lim_{x\to+\infty} |x|\sqrt{1 - \frac{7}{x}} = \lim_{x\to+\infty} x\sqrt{1 - \frac{7}{x}} = +\infty.$$

Therefore, the curve of f has no horizontal asymptote at $\pm\infty$.

(b) *V.A.*

Since there is no point a such that $\lim_{x\to a} |f(x)| = +\infty$ the graph of f has no vertical asymptote.

(c) *O.A.*

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2 - 7x}}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{1 - \frac{7}{x}}}{x}$$

$$= \lim_{x\to-\infty} \frac{-x\sqrt{1 - \frac{7}{x}}}{x} = \lim_{x\to-\infty} \left(-\sqrt{1 - \frac{7}{x}}\right) = -1,$$

$$\begin{aligned}
b &= \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(f(x)-ax) = \lim_{x\to-\infty}(\sqrt{x^2-7x}+x) \\
&= \lim_{x\to-\infty}\frac{(\sqrt{x^2-7x}+x)(\sqrt{x^2-7x}-x)}{\sqrt{x^2-7x}-x} = \lim_{x\to-\infty}\frac{x^2-7x-x^2}{\sqrt{x^2-7x}-x} \\
&= \lim_{x\to-\infty}\frac{-7x}{|x|\sqrt{1-\dfrac{7}{x}}-x} = \lim_{x\to-\infty}\frac{-7x}{-x\sqrt{1-\dfrac{7}{x}}-x} \\
&= \lim_{x\to-\infty}\frac{-7x}{-x\Big(\sqrt{1-\dfrac{7}{x}}+1\Big)} = \lim_{x\to-\infty}\frac{7}{\sqrt{1-\dfrac{7}{x}}+1} = \frac{7}{2}.
\end{aligned}$$

Hence, the graph of f has an oblique asymptote of $y=-x+\dfrac{7}{2}$ at $-\infty$.

Position of the curve of f with respect to the asymptote of $y=-x+\dfrac{7}{2}$ at $-\infty$:

$$\begin{aligned}
&\lim_{x\to-\infty}(f(x)-y) = \lim_{x\to-\infty}\Big(\sqrt{x^2-7x}-\Big(-x+\frac{7}{2}\Big)\Big) \\
&= \lim_{x\to-\infty}\Big(\sqrt{x^2-7x}+x-\frac{7}{2}\Big) = \lim_{x\to-\infty}\Big(\sqrt{x^2-7x}+x\Big)-\frac{7}{2} \\
&= \lim_{x\to-\infty}\frac{7}{\sqrt{1-\dfrac{7}{x}}+1}-\frac{7}{2} = 0^-.
\end{aligned}$$

Hence, The graph of f is below the oblique asymptote of $y=-x+\dfrac{7}{2}$ at $-\infty$.

$$\begin{aligned}
a &= \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\sqrt{x^2-7x}}{x} = \lim_{x\to+\infty}\frac{|x|\sqrt{1-\dfrac{7}{x}}}{x} \\
&= \lim_{x\to+\infty}\frac{x\sqrt{1-\dfrac{7}{x}}}{x} = \lim_{x\to+\infty}\sqrt{1-\frac{7}{x}} = 1.
\end{aligned}$$

$$b = \lim_{x\to+\infty} f(x) = \lim_{x\to-\infty}(f(x) - ax) = \lim_{x\to+\infty}(\sqrt{x^2-7x} - x)$$
$$= \lim_{x\to+\infty} \frac{(\sqrt{x^2-7x}-x)(\sqrt{x^2-7x}+x)}{\sqrt{x^2-7x}+x} = \lim_{x\to+\infty} \frac{x^2-7x-x^2}{\sqrt{x^2-7x}+x}$$
$$= \lim_{x\to+\infty} \frac{-7x}{|x|\sqrt{1-\frac{7}{x}}+x} = \lim_{x\to+\infty} \frac{-7x}{x\sqrt{1-\frac{7}{x}}+x}$$
$$= \lim_{x\to+\infty} \frac{-7x}{x\left(\sqrt{1-\frac{7}{x}}+1\right)} = \lim_{x\to+\infty} \frac{-7}{\sqrt{1-\frac{7}{x}}+1} = -\frac{7}{2}$$

Therefore, the graph of f has an oblique asymptote of $y = x - \frac{7}{2}$ at $+\infty$.

Position of the curve of f with respect to the asymptote of $y = x - \frac{7}{2}$ at $+\infty$:

$$\lim_{x\to+\infty}(f(x) - y) = \lim_{x\to+\infty}\left(\sqrt{x^2-7x} - \left(x - \frac{7}{2}\right)\right)$$
$$= \lim_{x\to+\infty}\left(\sqrt{x^2-7x} - x + \frac{7}{2}\right) = \lim_{x\to+\infty}\left(\sqrt{x^2-7x} - x\right) + \frac{7}{2}$$
$$= \lim_{x\to+\infty} \frac{-7}{\sqrt{1-\frac{7}{x}}+1} + \frac{7}{2} = 0^-$$

The graph of f is also below the oblique asymptote of $y = x - \frac{7}{2}$ at $+\infty$.

Exercise 1.52. *Let f is defined by $f(x) = \frac{x^2-3x+7}{x+3}$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f = \mathbb{R} \setminus \{-3\}$.

(a) H.A.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-3x+7}{x+3} = \lim_{x\to-\infty} \frac{x-3+\frac{7}{x}}{1+\frac{3}{x}} = -\infty,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2-3x+7}{x+3} = \lim_{x\to+\infty} \frac{x-3+\frac{7}{x}}{1+\frac{3}{x}} = +\infty.$$

Therefore, the curve of f has no horizontal asymptote.

(b) *V.A.*

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{x^2-3x+7}{x+3} = -\infty$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{x^2-3x+7}{x+3} = +\infty$$

Therefore the line of $x=-3$ is the vertical asymptote of the the graph of f. The graph of f is on the left of the horizontal asymptote of $x=-3$ as $y\to-\infty$, on the right of the horizontal asymptote of $x=-3$ as $y\to+\infty$.

(c) *O.A.*

Note: *At $f(x)=\dfrac{x^2-3x+7}{x+3}$ the degree of the numerator in only one greater then the degre of the denominator, becauce of this the curve of f has an oblique asymptote.*

$$f(x) = \frac{x^2-3x+7}{x+3} = x-6+\frac{25}{x+3},$$

where $\frac{25}{x+3}\to 0$ as $x\to\pm+\infty$.
Hence, the line of $y=x-6$ is the oblique asymptote of the graph of f at $\pm+\infty$.
Since

$$\lim_{x\to+\infty}(f(x)-y) = \lim_{x\to+\infty}\Big(\frac{x^2-3x+7}{x+3}-(x-6)\Big) = \lim_{x\to+\infty}\frac{25}{x+3} = 0^+$$

the curve of f is above the asymptote of $y=x-6$ at $+\infty$.

$$\lim_{x\to-\infty}(f(x)-y) = \lim_{x\to-\infty}\Big(\frac{x^2-3x+7}{x+3}-(x-6)\Big) = \lim_{x\to-\infty}\frac{25}{x+3} = 0^-$$

the curve of f is below the asymptote of $y=x-6$ at $-\infty$.

Exercise 1.53. *Let f is defined by $f(x)=\dfrac{3x^2+2}{x^2+7}$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f=\mathbb{R}$.

(a) *H.A. Since*

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{3x^2+2}{x^2+7} = 3,$$

and

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty}\frac{3x^2+2}{x^2+7} = 3,$$

the curve of f has the horizontal asymptote of $y=3$ at both $-\infty$ and $+\infty$.

$$\lim_{x\to-\infty}(f(x)-y)=\lim_{x\to-\infty}\left(\frac{3x^2+2}{x^2+7}-3\right)=\lim_{x\to-\infty}\frac{3x^2+2-3x^2-21}{x^2+7}=\lim_{x\to-\infty}\frac{-19}{x^2+7}=0^-$$

Therefore, the graph of f is below the line of $y=3$ at $-\infty$.

$$\lim_{x\to+\infty}(f(x)-y)=\lim_{x\to+\infty}\left(\frac{3x^2+2}{x^2+7}-3\right)=\lim_{x\to+\infty}\frac{3x^2+2-3x^2-21}{x^2+7}=\lim_{x\to+\infty}\frac{-19}{x^2+7}=0^-$$

Therefore, the graph of f is below the line of $y=3$ at $+\infty$ as well.

(b) V.A.

There is no point p such that $\lim_{x\to p}|f(x)|=+\infty$, so the curve of f has no vertical asymptote.

(c) O.A. Since the degree of the numerator and denominator are equal in $f(x)$ which defines the function f, f the curve of f has no oblique asymptote.

*(**Another explanation:** Since $a=\lim_{x\to\pm+\infty}\frac{f(x)}{x}=0$, (slope is 0), the curve of f has no oblique asymptote.)*

Exercise 1.54. *Let f is defined by $f(x)=\frac{x^2}{x-2}+|x-1|$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f=\mathbb{R}\setminus\{2\}$.

(a) H.A. Since

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{x^2}{x-2}+|x-1|=\lim_{x\to-\infty}\left(\frac{x^2}{x-2}-x+1\right)$$
$$=\lim_{x\to-\infty}\frac{x^2-x^2+2x+x-2}{x-2}=\lim_{x\to-\infty}\frac{3x-2}{x-2}=\lim_{x\to-\infty}\frac{3-\frac{2}{x}}{1-\frac{2}{x}}=3,$$

the curve of f has the horizontal asymptote of $y=3$ at $-\infty$.

$$\lim_{x\to-\infty}(f(x)-y)=\lim_{x\to-\infty}\left(\frac{x^2}{x-2}+|x-1|-3\right)=\lim_{x\to-\infty}\left(\frac{3-\frac{2}{x}}{1-\frac{2}{x}}-3\right)=0^-.$$

Hence, the curve is below of the horizontal asymptote of $y=3$ at $-\infty$.

Since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{x-2} + |x-1| = \lim_{x\to+\infty} \left(\frac{x^2}{x-2} + x - 1\right)$$
$$= \lim_{x\to+\infty} \frac{x^2 + x^2 - 2x - x + 2}{x-2} = \lim_{x\to+\infty} \frac{2x^2 - 3x + 2}{x-2} = +\infty,$$

the curve of f does not have a horizontal asymptote at $+\infty$.

(b) V.A.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2}{x-2} + |x-1| = -\infty,$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2}{x-2} + |x-1| = +\infty.$$

Therefore the line of $x = 2$ is the vertical asymptote of the the graph of f. The graph of f is on the left of the vertical asymptote of $x = 2$ as $y \to -\infty$, on the right of the horizontal asymptote of $x = 2$ as $y \to +\infty$.

(c) O.A. Since the curve of f has horizontal asymptote at $-\infty$, we will consider the limit of f at only $+\infty$.

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\frac{x^2}{x-2} + |x-1|}{x} = \lim_{x\to+\infty} \frac{\frac{2x^2-3x+2}{x-2}}{x}$$
$$= \lim_{x\to+\infty} \frac{2x^2 - 3x + 2}{x^2 - 2x} = 2,$$

$$b = \lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} (f(x) - ax) = \lim_{x\to+\infty} \left(\frac{x^2}{x-2} + |x-1| - 2x\right)$$
$$= \lim_{x\to+\infty} \left(\frac{2x^2 - 3x + 2}{x-2} - 2x\right) = \lim_{x\to+\infty} \frac{2x^2 - 3x + 2 - 2x^2 + 4x}{x-2}$$
$$= \lim_{x\to+\infty} \frac{x+2}{x-2} = 1.$$

Then, the curve of f has the oblique asymptote of $y = 2x + 1$ at $+\infty$.

$$\lim_{x\to+\infty} (f(x) - y) = \lim_{x\to+\infty} \left(\frac{x^2}{x-2} + |x-1| - 2x - 1\right)$$
$$= \lim_{x\to+\infty} \left(\left(\frac{x^2}{x-2} + |x-1| - 2x\right) - 1\right) = \lim_{x\to+\infty} \left(\frac{x+2}{x-2} - 1\right)$$
$$= \lim_{x\to+\infty} \frac{x + 2 - x + 2}{x-2} = \lim_{x\to+\infty} \frac{4}{x-2} = 0^+.$$

Therefore, the curve of f is above the oblique asymptote of $y = 2x + 1$ at $+\infty$.

2 Differentiability

2.1 Important Tools

Exercise 2.1. *Let f be defined by*

1. $f(x) = (x+3)^2(2x-4)$,
2. $f(x) = |x+3|\sqrt{x-2}$,
3. $f(x) = \sqrt{x^2+5x+6}$,
4. $f(x) = \dfrac{2x^2}{x-2} + |x|$.

Define the sign table of f.

Solution.

1. *We find the points which are the zeros of f or at which f is not defined: $x = -3$ and $x = 2$.*

x	$-\infty$		-3		2		$+\infty$
$(x+3)^2$		$+$	0	$+$		$+$	
$2x-4$		$-$		$-$	0	$+$	
$(x+3)^2(2x-4$		$-$	0	$-$	0	$+$	

2.

x	$-\infty$		-3		2		$+\infty$
$\|x+3\|$		$+$	0	$+$		$+$	
$x-2$		$-$		$-$	0	$+$	
$\|x+3\|\sqrt{x-2}$					0	$+$	

3.

x	$-\infty$		-3		-2		$+\infty$
$x+3$		$-$	0	$+$		$+$	
$x+2$		$-$		$-$	0	$+$	
$\sqrt{(x+3)(x+2)}$		$+$	0		0	$+$	

4. If $x \geq 0$, *then* $f(x) = \dfrac{2x^2}{x-2} + x = \dfrac{3x^2-2x}{x-2} = \dfrac{x(3x-2)}{x-2}$.

x	0		$\frac{2}{3}$		2		$+\infty$
x	0	$+$		$+$		$+$	
$3x-2$		$-$	0	$+$		$+$	
$x-2$		$-$		$-$	0	$+$	
$\dfrac{x(3x-2)}{x-2}$	0	$+$	0	$-$	$\Vert$	$+$	

If $x < 0$, *then* $f(x) = \dfrac{2x^2}{x-2} - x = \dfrac{x^2+2x}{x-2} = \dfrac{x(x+2)}{x-2}$.

x	$-\infty$		-2		0
x		$-$		$-$	$\Vert$
$x+2$		$-$	0	$+$	$\Vert$
$x-2$		$-$		$-$	$\Vert$
$\dfrac{x(x+2)}{x-2}$		$-$	0	$+$	$\Vert$

Exercise 2.2. *Using the table of variation of a function f sketch its graph. No formulas are required - just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)*

1.

x	-2		-1		2		3
Variations of f and values $f(x)$	7	$\searrow$	-1	$\nearrow$	4	$\searrow$	1

2.

x	-2		-1		1		4
Variations of f and values $f(x)$	5	$\nearrow$	-2	$\searrow$	$\Vert$	$\nearrow$	2

3.

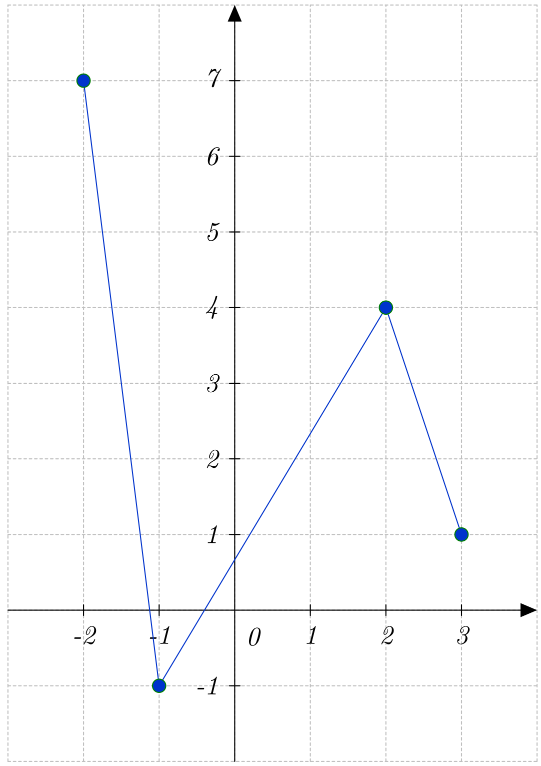
x	-4		0		2		4
Variations of f and values $f(x)$	6	$\searrow$	$\Vert$	$\nearrow$	$\Vert$	$\nearrow$	0

4.

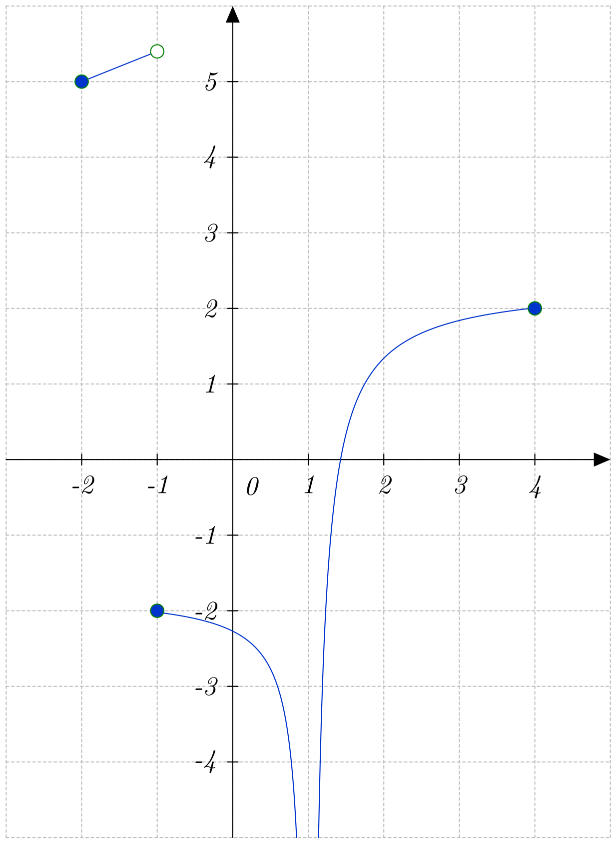
x	-3		-1		1		3
Variations of f and values $f(x)$	$\Vert$	$\nearrow$	6	$\searrow$	$\Vert$	$\searrow$	0

Solution.

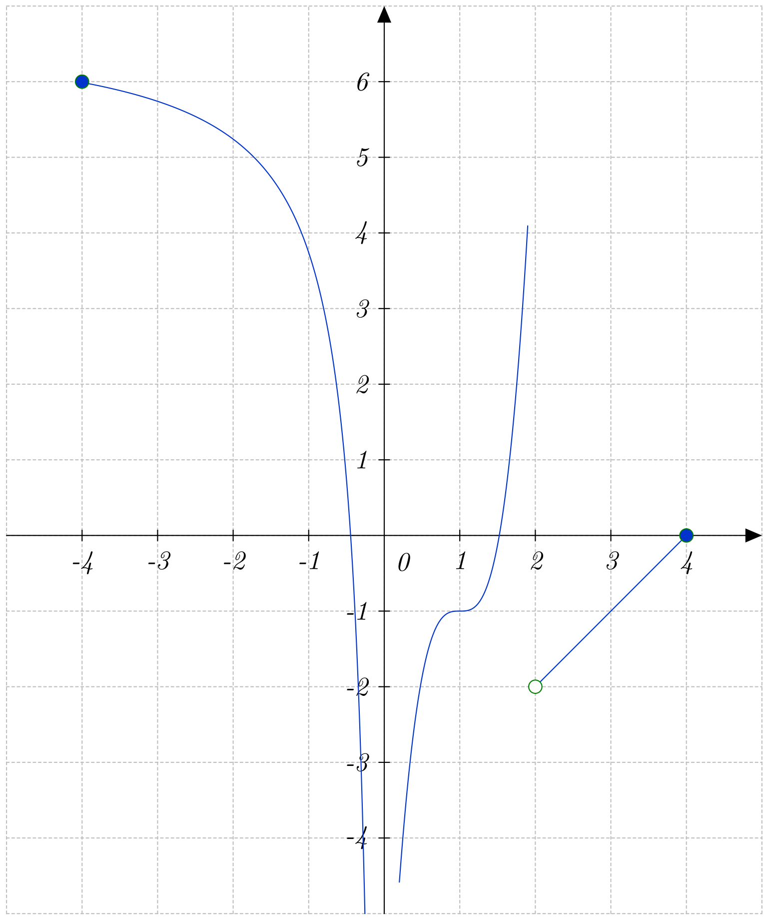
1.



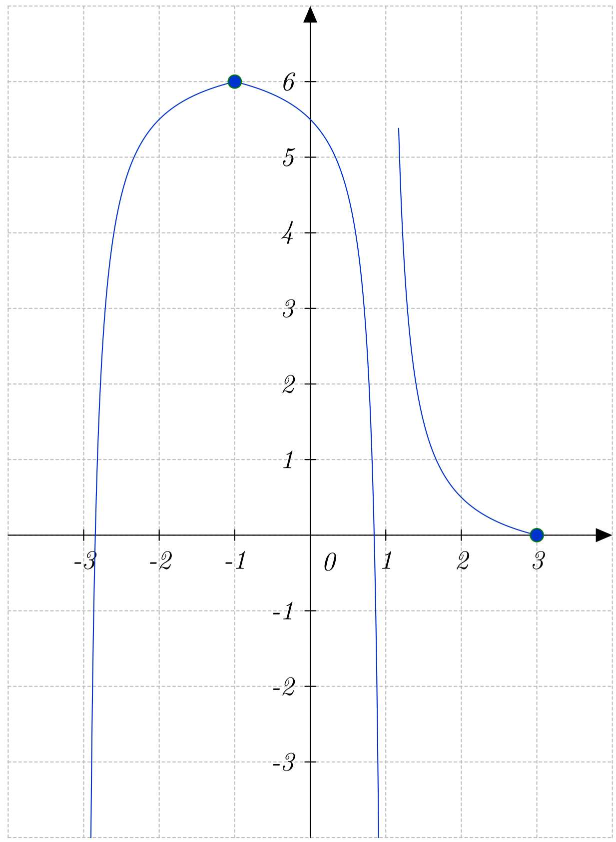
2.



3.



4.



Exercise 2.3. *Let a function f be defined by $f(x) = x^2 + 3$. Using the definition of the slope of the curve (tangent line) find the slope of the curve (tangent line) and an equation for the tangent line to the curve at the point $(-2, 7)$.*

Solution.

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(-2 + h) - f(-2)}{h} = \lim_{h \to 0} \frac{(-2 + h)^2 + 3 - 7}{h}$$
$$= \lim_{h \to 0} \frac{4 - 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{h(-4 + h)}{h} = \lim_{h \to 0} (-4 + h) = -4.$$

Hence, the slope of the curve (tangent line) is -4.
an equation for a line with a slope m, that passes through a point $(x_0, f(x_0))$ is

$$y = m(x - x_0) + f(x_0)$$

Because of this, an equation for the tangent line to the curve at the point $(-2, 7)$ is

$$y = -4(x - (-2)) + 7 = -4x - 1.$$

Exercise 2.4. *Let a function g be defined by $g(x) = \dfrac{1}{x^2} + \dfrac{8}{9}$. Using the definition of the slope of the curve (tangent line) find the slope of the curve (tangent line) and an equation for the tangent line to the curve at the point $(3, 1)$.*

Solution.

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(3+h)^2}+\dfrac{8}{9}-1}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{1}{9+6h+h^2}-\dfrac{1}{9}}{h} = \lim_{h\to 0} \frac{9-9-6h-h^2}{9h(9+6h+h^2)} = \lim_{h\to 0} \frac{-6-h}{9(9+6h+h^2)} = -\frac{2}{27}.$$

Hence, the slope of the curve (tangent line) is $-\dfrac{2}{27}$.

An equation for the tangent line to the curve at the point $(3, 1)$ is

$$y = -\frac{2}{27}(x-3)+1.$$

Exercise 2.5. *Let a function f be defined by $f(x) = \sin x$. Using the definition of the slope of the curve (tangent line) find the slope of the curve (tangent line) and an equation for the tangent line to the curve at the point $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{2}\right)$.*

Solution.

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{f\left(\frac{\pi}{3}+h\right)-f\left(\frac{\pi}{3}\right)}{h} = \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+h\right)-\sin\frac{\pi}{3}}{h}$$
$$= \lim_{h\to 0} \frac{\sin\frac{\pi}{3}\cdot\cos h+\cos\frac{\pi}{3}\cdot\sin h-\sin\frac{\pi}{3}}{h} = \lim_{h\to 0} \frac{\sin\frac{\pi}{3}(\cos h-1)+\cos\frac{\pi}{3}\cdot\sin h}{h} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

Hence, the slope of the curve (tangent line) is $\dfrac{1}{2}$.

An equation for the tangent line to the curve at the point $(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{2})$ is

$$y = \frac{1}{2}(x-\frac{\pi}{3})+\frac{\sqrt{3}}{2} = \frac{1}{2}x-\frac{\pi}{6}+\frac{\sqrt{3}}{2}.$$

Exercise 2.6. *Let a function f be defined by*

$$f(x) = \begin{cases} xe^x+2, & x \neq 0 \\ 2 & x = 0 \end{cases}.$$

Does the graph of f have a tangent at the origin? Give reasons for your answer.

Solution.

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h)e^{0+h}+2-2}{h}$$
$$= \lim_{h\to 0} \frac{he^h}{h} = \lim_{h\to 0} e^h = 1.$$

Then, the slope of the curve is $m = 1$, so ve define the equation for the tangent. Hence, the curve has a tangent line at the origin.

Exercise 2.7. *Let a function g be defined by*

$$g(x) = \begin{cases} 2, & x > 0 \\ 0, & x = 0 \\ -2, & x < 0 \end{cases}.$$

Does the graph of f have a tangent at the origin? Give reasons for your answer.

Solution. *The function is discontinuous at the origin, because of this it is not differentiable at the origin.*

Exercise 2.8. *An object is released from rest from The Maiden Tower at a height 30m above street level. The height of the object can be modeled by the position function defined by $s(x) = 30 - 16t^2$,*

(a) Verify that the object is still falling at $t = 1s$.

(b) Find the object's instantaneous velocity at time $t = 1s$.

Solution.

(a) $s(1) = 30 - 16 \cdot 1^2 = 14$, and $14 < 30$. Therefore the object is still falling at $t = 1s$.

(b)

$$v_{ins} = \lim_{h\to 0} \frac{s(1+h)-s(1)}{h} = \lim_{h\to 0} \frac{30-16(1+h)^2-14}{h} =$$
$$= \lim_{h\to 0} \frac{30-16-32h-16h^2-14}{h} = \lim_{h\to 0} \frac{-32h-16h^2}{h} = \lim_{h\to 0}(-32-16h) = -32.$$

Exercise 2.9. *If a ball is thrown into the air with a velocity of 20 ft/s, its height (in feet) after t seconds is given by $s(t) = 20t - 16t^2$. Find the velocity when $t = 2$.*

Solution.

$$v_{ins} = \lim_{h\to 0} \frac{s(2+h)-s(2)}{h} = \lim_{h\to 0} \frac{20(2+h)-16(2+h)^2-(20\cdot 2-16\cdot 2^2)}{h} =$$
$$= \lim_{h\to 0} \frac{40+20h-64-64h-16h^2-40+64}{h} = \lim_{h\to 0} \frac{-44h-16h^2}{h} = -44.$$

2.2 Differentiability

Exercise 2.10. *Let f be defined by $f(x) = \sqrt{2x+1}$. Using the definition of the derivative, calculate the derivative of the function f at $x = 3$.*

Solution. *Using the definition of the derivative at the given point gives,*

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{2(3+h)+1} - \sqrt{2\cdot 3+1}}{h} = \lim_{h\to 0} \frac{\sqrt{7+2h} - \sqrt{7}}{h}$$
$$= \lim_{h\to 0} \frac{2h}{h(\sqrt{7+2h} + \sqrt{7})} = \lim_{h\to 0} \frac{2}{\sqrt{7+2h} + \sqrt{7}} = \frac{2}{\sqrt{7}+\sqrt{7}} = \frac{1}{\sqrt{7}}.$$

Exercise 2.11. *Let g be defined by $g(t) = \dfrac{1-t}{t}$. Using the definition of the derivative, calculate the derivative of the function g at $t = 2$.*

Solution.

$$g'(2) = \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h\to 0} \frac{\dfrac{1-(2+h)}{2+h} - \dfrac{1-2}{2}}{h} = \lim_{h\to 0} \frac{\dfrac{-1-h}{2+h} + \dfrac{1}{2}}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{-2-2h+2+h}{(2+h)\cdot 2}}{h} = \lim_{h\to 0} \frac{-1}{(2+h)\cdot 2} = -\frac{1}{4}.$$

Exercise 2.12. *Let f be defined by $f(x) = \dfrac{1}{\sqrt{x+1}}$. Using the definition of the derivative, calculate the derivative of the function f at $x = 4$.*

Solution.

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{(4+h)+1}} - \dfrac{1}{\sqrt{4+1}}}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{5+h}} - \dfrac{1}{\sqrt{5}}}{h} =$$
$$= \lim_{h\to 0} \frac{\sqrt{5} - \sqrt{5+h}}{\sqrt{5+h}\cdot\sqrt{5}\cdot h} = \lim_{h\to 0} \frac{-h}{(\sqrt{5}+\sqrt{5+h})\sqrt{5+h}\cdot\sqrt{5}\cdot h} = -\frac{1}{10\sqrt{5}}.$$

Exercise 2.13. *Let f be defined by*

$$f(x) = \begin{cases} x^2+2, & x \le 2 \\ 3x, & x > 2 \end{cases}$$

Using the definitions of the continuity and differentiability show that f is continuous but not differentiable at $x = 2$. Sketch the graph of f.

Solution. *Since*

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2+2) = 6,$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x) = 6,$$

$$f(2) = 2^2 + 2 = 6,$$

the function f is continuous at $x = 2$

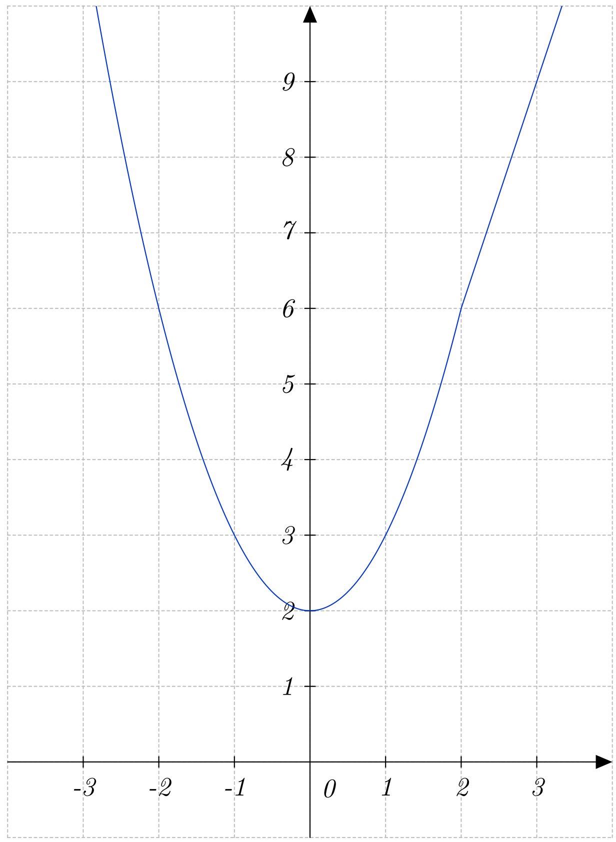
$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

We divide this limit into two parts - left hand side limit and right hand side limit:

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{(2+h)^2+2-(2^2+2)}{h} = \lim_{h\to 0^-} \frac{(2+h)^2+2-(2^2+2)}{h}$$
$$= \lim_{h\to 0^-} \frac{4+4h+h^2+2-6}{h} = \lim_{h\to 0^-} \frac{4h+h^2}{h} = 4.$$

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{3(2+h)-(2^2+2)}{h} = \lim_{h\to 0^+} \frac{6+3h-6}{h} = 3.$$

Therefore, $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$ *does not exist, so the function f has no derivative at $x = 2$.*



Exercise 2.14. *Let f be defined by*

$$f(x) = \begin{cases} x^2+1, & x \le 1 \\ 2x, & x > 1 \end{cases}$$

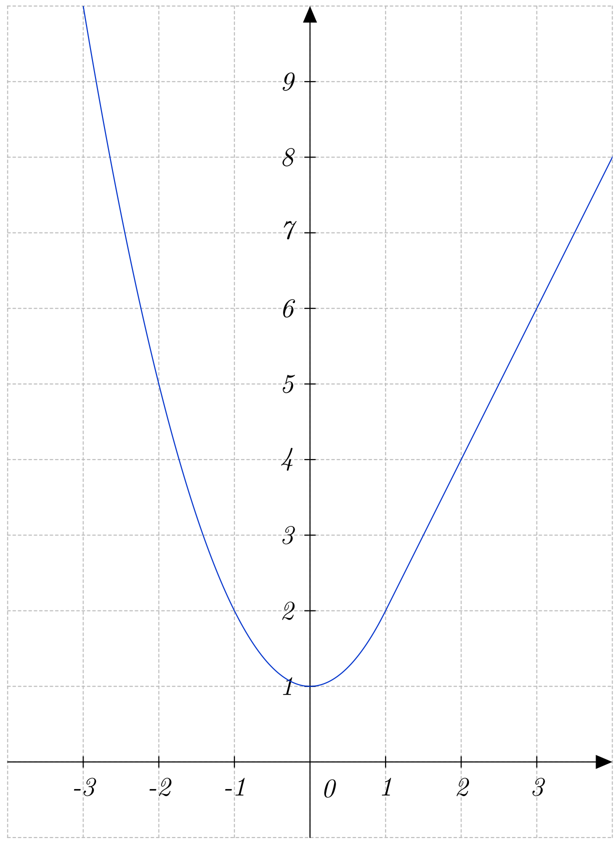
Using the definitions of the continuity and differentiability show that f is continuous and differentiable at $x = 1$. Sketch the graph of f.

Solution.

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^2+1-(1^2+1)}{h} = \lim_{h\to 0^-} \frac{1+2h+h^2-1}{h} = 2.$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h)-(1^2+1)}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Hence, $f'(1) = 2$, The function f is differentiable at $x = 1$.
If a function differentiable at a given point then it is also continuous at this point.
Since f is differentiable at $x = 1$, it is also continuous at $x = 1$.



Exercise 2.15. *Let f be defined by $f(x) = 2x^2+3x-1$. Using the definition of the derivative, calculate the derivative of f with respect to the variable x.*

Solution.

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{2(x+h)^2+3(x+h)-1-(2x^2+3x-1)}{h}\\ &= \lim_{h\to 0}\frac{2x^2+4xh+h^2+3x+3h-1-2x^2-3x+1}{h}\\ &= \lim_{h\to 0}\frac{4xh+h^2+3h}{h} = \lim_{h\to 0}(4x+h+3) = 4x+3\end{aligned}$$

Exercise 2.16. *Let f be defined by $f(x) = \sqrt{x^2+1}$. Using the definition of the derivative, calculate the derivative of f with respect to the variable x.*

Solution.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{x^2+2xh+h^2+1}-\sqrt{x^2+1}}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{(\sqrt{x^2+2xh+h^2+1}+\sqrt{x^2+1})h}$$
$$=\lim_{h\to 0}\frac{2xh+h^2}{(\sqrt{x^2+2xh+h^2+1}+\sqrt{x^2+1})h}=\lim_{h\to 0}\frac{2x+h}{\sqrt{x^2+2xh+h^2+1}+\sqrt{x^2+1}}$$
$$=\frac{2x}{\sqrt{x^2+1}+\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}.$$

Exercise 2.17. *Let f be defined by $f(t)=t-\dfrac{2}{t}$. Using the definition of the derivative, calculate the derivative of f with respect to the variable t.*

Solution.

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}=\lim_{h\to 0}\frac{t+h-\dfrac{2}{t+h}-\left(t-\dfrac{2}{t}\right)}{h}$$
$$=\lim_{h\to 0}\frac{t+h-\dfrac{2}{t+h}-t+\dfrac{2}{t}}{h}=\lim_{h\to 0}\frac{t^3+2t^2h+th^2-2t-t^3-t^2h+2t+2h}{(t+h)th}$$
$$=\lim_{h\to 0}\frac{t^2h+th^2+2h}{(t+h)th}=\lim_{h\to 0}\frac{t^2+th+2}{(t+h)t}=\frac{t^2+2}{t^2}$$

Exercise 2.18. *Find the derivatives of the functions defined by*

1. $f(x)=3x^2-\sqrt{x}+2e^x$,
2. $f(x)=(2\sqrt{x}+1)(x^2-x)$,
3. $f(x)=\dfrac{x}{x^3+1}$,
4. $f(x)=\dfrac{1}{2}\,x^{-2}+\dfrac{x^2-1}{x^3}$.

Solution.

1.
$$f'(x)=6x+\frac{1}{2\sqrt{x}}+2e^x,$$

2.

$$f'(x) = \frac{1}{\sqrt{x}}(x^2 - x) + (2\sqrt{x} + 1)(2x - 1)$$
$$= x\sqrt{x} - \sqrt{x} + 4x\sqrt{x} - 2\sqrt{x} + 2x - 1 = 5x\sqrt{x} - 3\sqrt{x} + 2x - 1.$$

3.

$$f'(x) = \frac{1 \cdot (x^3 + 1) - x \cdot 3x^2}{(x^3 + 1)^2} = \frac{x^3 + 1 - 3x^3}{(x^3 + 1)^2} = \frac{-2x^3 + 1}{(x^3 + 1)^2}.$$

4.

$$f(x) = \frac{1}{2}\ x^{-2} + \frac{x^2 - 1}{x^3} = \frac{1}{2}\ x^{-2} + x^{-1} - x^{-3}.$$

Then,

$$f'(x) = -x^{-3} - x^{-2} + 3x^{-4} = -\frac{1}{x^3} - \frac{1}{x^2} + \frac{3}{x^4}.$$

Exercise 2.19. *Find the derivatives of the functions defined by*

1. $f(x) = 3\cos x + 2\sin x$,
2. $f(x) = \sec x \cdot \tan x$,
3. $f(x) = \dfrac{\csc x}{1 + x\tan x}$,
4. $f(x) = \dfrac{1}{2}\ x^2 \cot x + \dfrac{x}{\sec x}$.

Solution.

1.

$$f'(x) = -3\sin x + 2\cos x,$$

2.

$$f'(x) = \sec x \cdot \tan x \cdot \tan x + \sec x \cdot \sec^2 x = \sec x \cdot \tan^2 x + \sec^3 x,$$

3.

$$f'(x) = \frac{-\csc x\cot x(1 + x\tan x) - \csc x(\tan x + x\sec^2 x)}{(1 + x\tan x)^2}$$

4.

$$f(x) = \frac{1}{2}\ x^2 \cot x + \frac{x}{\sec x} = \frac{1}{2}\ x^2 \cot x + x \cos .$$

Then,

$$f(x) = \frac{1}{2}(2x \cot x - x^2 \csc^2 x) + \cos x - x \sin x = x \cot x - \frac{1}{2}x^2 \csc^2 x + \cos x - x \sin x.$$

Exercise 2.20. *Find the derivatives of the functions defined by*

1. $f(x) = \tan(x^2 + 3)$,
2. $f(x) = (1 + x^5 \cot x)^{-4}$,
3. $f(x) = \dfrac{e^{x^2}}{\sqrt{1 + x + x^2}}$,
4. $f(x) = \cos^2 \sqrt{\left(\dfrac{x^3}{3x + 1}\right)}$,
5. $f(x) = \dfrac{\ln x}{x^2} - x^2 \ln x$,
6. $f(x) = 6^{2x} + \ln\left(\dfrac{x}{2}\right) + \log_4 x^2$.

Solution.

1.

$$f'(x) = \sec(x^2 + 3) \cdot 2x = 2x \sec^2(x^2 + 3),$$

2.

$$\begin{aligned} f'(x) &= -4(1 + x^5 \cot x)^{-3} \cdot (5x^4 \cot x + x^5(-\csc x)) \\ &= (-20x^4 \cot x + 4x^5 \csc x)(1 + x^5 \cot x)^{-3}, \end{aligned}$$

3.

$$f'(x) = \frac{2xe^{x^2}\sqrt{1 + x + x^2} - e^{x^2}\dfrac{1 + 2x}{\sqrt{1 + x + x^2}}}{1 + x + x^2}$$

$$= \frac{4xe^{x^2}(1 + x + x^2) - 2xe^{x^2} - e^{x^2}}{2(1 + x + x^2)\sqrt{1 + x + x^2}} = \frac{(-1 + 2x + 4x^2 + 4x^3)e^{x^2}}{2(1 + x + x^2)\sqrt{1 + x + x^2}}.$$

4.

$$f'(x) = 2\cos\sqrt{\frac{x^3}{3x+1}} \cdot \sin\sqrt{\frac{x^3}{3x+1}} \cdot \frac{1}{2\sqrt{\left(\frac{x^3}{3x+1}\right)}}$$

$$\times\frac{3x^2(3x+1)-3x^3}{(3x+1)^2} = \frac{9x^3\sin\sqrt{\frac{x^3}{3x+1}}\cos\sqrt{\left(\frac{x^3}{3x+1}\right)}}{2(3x+1)^2\sqrt{\left(\frac{x^3}{3x+1}\right)}}.$$

5.

$$f'(x) = \frac{\frac{1}{x}\cdot\frac{1}{x^2} - 2x\ln x}{x^4} - 2x\ln x - x^2\cdot\frac{1}{x} = \frac{1}{x^7} - \frac{2\ln x}{x^3} - 2x\ln x - x.$$

6.

$$f'(x) = 6^{2x}\ln 6\cdot 2 + \frac{1}{\frac{x}{2}}\cdot\frac{1}{2} + \frac{1}{x^2\ln 4}\cdot 2x$$

$$= 2\cdot\ln 6\cdot 6^{2x} + \frac{1}{x} + \frac{2}{x\ln 4}.$$

Exercise 2.21. *The following equations define an implicit relation between the variables x and y.*

1. $x^3 - y^3 = 12xy$,
2. $\frac{1}{y}\cos(\pi - y) + xy = 2y^2$,
3. $(x^2+y^3)^2 = y^2 - 3x$.

Find $\frac{dy}{dx}$.

Solution.

1.

$$3x^2 - 3y^2\frac{dy}{dx} = 12y + 12x\frac{dy}{dx},$$

$$(3y+12x)\frac{dy}{dx} = 3x^2 - 12y,$$

$$\frac{dy}{dx} = \frac{3x^2-12y}{3y+12x},$$

2.

$$-y^{-2}y'\cos(\pi-y)+\frac{1}{y}(-\sin(\pi-y))(-y')+y+xy'=4yy',$$

$$-\frac{1}{y^2}y'\cos(\pi-y)+\frac{1}{y}\sin(\pi-y)y'+y+xy'=4yy',$$

$$-\frac{1}{y^2}y'\cos(y-\pi)-\frac{1}{y}\sin(y-\pi)y'+xy'-4yy'=-y,$$

$$y'=\frac{y}{\frac{1}{y^2}\cos(y-\pi)-\frac{1}{y}\sin(y-\pi)-x+4y}.$$

3.

$$(x^2+y^3)^2=y^2-3x,$$

$$2(x^2+y^3)(2x+3y^2y')=2yy'-3,$$

$$4x^3+4xy^3+6x^2y^2y'+6y^5y'=2yy'-3,$$

$$6x^2y^2y'+6y^5y'-2yy'=-4x^3-4xy^3-3,$$

$$y'(6x^2y^2+6y^5-2y)=-4x^3-4xy^3-3,$$

$$y'=\frac{-4x^3-4xy^3-3}{6x^2y^2+6y^5-2y}.$$

Exercise 2.22. *The following equations define an implicit relation between the variables x and y.*

1. $x^2-2x=y$,
2. $\frac{x}{y}+2y=x$,

Find $\frac{d^2y}{dx^2}$.

Solution.

1.

$$x^2-2x=y,$$

$$2x-2=y',$$

$$y''=2.$$

2.
$$\frac{y-xy'}{y^2}+2y'=1,$$
$$y-xy'+2y'y^2=y^2, \qquad (y\neq 0)$$
$$y'(-x+2y^2)=y^2-y, \qquad (y\neq 0)$$
$$y'=\frac{y-y^2}{x-2y^2}, \qquad (y\neq 0)$$
$$y''=\frac{(y'-2yy')(x-2y^2)-(y-y^2)(1-4yy')}{(x-2y^2)^2}, \qquad (y\neq 0)$$
$$y''=\frac{y'(x-2xy+2y^2-6y^3)-y-y^2}{(x-2y^2)^2}, \qquad (y\neq 0)$$
$$y''=\frac{\frac{y-y^2}{x-2y^2}(x-2xy+2y^2-6y^3)-y-y^2}{(x-2y^2)^2}, \qquad (y\neq 0)$$
$$y''=\frac{(y-y^2)(x-2xy+2y^2-6y^3)-(x-2y^2)(y+y^2)}{(x-2y^2)^3}. \qquad (y\neq 0)$$

Exercise 2.23. *Let functions f and g be defined by $f(x)=x^3+2$ and $g(x)=\sqrt[3]{x-2}$ appropriately.*

1. *Show that f and g are inverses of one another.*
2. *Find the derivative of f at $x=3$.*
3. *Find the derivative of g at $x=29$.*
4. *Define the relation between $f'(3)$ and $g'(29)$.*

Solution.

1. *Since*
$$f(g(x))=g^3(x)+3=(\sqrt[3]{x-2})^3+2=x \qquad \text{and} \qquad D_f=R_g$$
and
$$g(f(x))=\sqrt[3]{f(x)-2}=\sqrt[3]{x^3+2-2}=x \qquad \text{and} \qquad D_g=R_f$$
The functions f and g are inverses of each other.

2.
$$f'(x)=3x^2.$$
Then,
$$f'(3)=3\cdot 3^2=27.$$

3.

$$g'(x) = \frac{1}{3 \cdot (x-2)^{\frac{2}{3}}}.$$

Hence,

$$g'(29) = \frac{1}{3 \cdot (29-2)^{\frac{2}{3}}} = \frac{1}{27}.$$

4. $f'(3) = \dfrac{1}{g'(29)}$.

Exercise 2.24. *Let f be defined by $f(x) = x^3 + 3x^2 - 3$. Find the value of $\dfrac{df^{-1}}{dx}$ at the point $x = 17 = f(2)$.*

Solution.

$$\frac{df(x)}{dx} = 3x^2 + 6x$$

Since

$$\frac{df^{-1}(x)}{dx} = \frac{1}{\frac{df(f^{-1}(x))}{dx}},$$

$$\frac{df^{-1}(17)}{dx} = \frac{1}{\frac{df(f^{-1}(17))}{dx}} = \frac{1}{\frac{df(2)}{dx}} = \frac{1}{3 \cdot 2^2 + 6 \cdot 2} = \frac{1}{24}.$$

Exercise 2.25. *Let f be defined by $f(x) = 2x^2 + 5x$ and $D_f = [0, +\infty[$. Find the value of $\dfrac{df^{-1}}{dx}$ at the point $x = 3$.*

Solution.

$$2x^2 + 5x = 3$$

$$2x^2 + 5x - 3 = 0$$

Then, $x = \frac{1}{2}$, $x = -3$.

$\frac{1}{2} \in [0, +\infty[$. *Because of this we consider the point* $x = \frac{1}{2}$.

Since $f\left(\frac{1}{2}\right) = 3$ *and* $f'(x) = 4x + 5$,

$$\frac{df^{-1}(3)}{dx} = \frac{1}{\frac{df(f^{-1}(3))}{dx}} = \frac{1}{\frac{df\left(\frac{1}{2}\right)}{dx}} = \frac{1}{4 \cdot \frac{1}{2} + 5} = \frac{1}{7}.$$

Exercise 2.26. *Find the derivatives of the functions defined by*

1. $f(x) = 4^x$,
2. $f(x) = 2^{x^2+1}$,
3. $f(x) = x^{x+1}$,
4. $f(x) = x^{\cos x}$.

Solution.

1. $f'(x) = 4^x \ln 4$,
2. $f'(x) = 2x \ln 2 \cdot 2^{x^2+1}$
3. $$f(x) = x^{x+1} = e^{\ln x^{x+1}} = e^{(x+1)\ln x}.$$
   Then,
   $$f'(x) = \Big(\ln x + 1 + \frac{1}{x}\Big)e^{(x+1)\ln x} = \Big(\ln x + 1 + \frac{1}{x}\Big)e^{\ln x^{x+1}} = \Big(\ln x + 1 + \frac{1}{x}\Big)x^{x+1}.$$
4. $$f(x) = x^{\cos x} = e^{\ln x^{\cos x}} = e^{\cos x \ln x},$$
   $$f'(x) = \Big(-\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}\Big)e^{\cos x \ln x} = \Big(-\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}\Big)x^{\cos x}.$$

Exercise 2.27.

1. *Find an equation for the line perpendicular to the tangent to the curve of f defined by $f(x) = 2x^3 - 3x + 7$ at the point $(2, 17)$.*
2. *What is the smallest slope of the curve? At what point on the curve does the curve have this slope?*

Solution.

1. *If the slope of the tangent to the at $(2, 17)$ is m, The slope of the line that is perpendicular to the tangent to the curve at $(2, 17)$ is k. Then*
   $$m \cdot k = -1.$$
   $$f'(x) = 6x^2 - 3$$
   Because of this, $m = f(2) = 6 \cdot 2^2 - 3 = 21$. Then $k = -\dfrac{1}{21}$.
   Hence, an equation for the line perpendicular to the tangent to the curve of f is
   $$y = -\frac{1}{21}(x - 2) + 17.$$

2. *the minimum value of*

$$f'(x) = 6x^2 - 3$$

is -3*, so the smallest slope of the curve is* $m = -3$*.* f' *takes its minimum value at* $x = 0$*.*

Exercise 2.28. *Suppose that a function* f *satisfies the following conditions for all real values of* x *and* y*:*

1. $f(x + y) = f(x) \cdot f(y)$,
2. $f(x) = 1 + xg(x)$ *where* $\lim_{x\to 0} g(x) = 1$.

Show that the derivative function f' *exists at every value of* x *and that* $f'(x) = f(x)$

Solution. *From the conditions we have* $f(x + h) = f(x) \cdot f(h)$, $f(h) - 1 = hg(h)$ *and* $\lim_{h\to 0} g(h) = 1$*. Therefore,*

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) \cdot f(h) - f(x)}{h}$$
$$= \lim_{h\to 0} f(x) \cdot \left(\frac{f(h) - 1}{h}\right) = f(x) \lim_{h\to 0} g(h) = f(x) \cdot 1 = f(x).$$

Because of this, $f'(x) = f(x)$ *and* f' *exists at every value of* x*.*

Exercise 2.29. *Let* f *be defined by*

$$f(x) = \begin{cases} x^2 + x + 1, & x \leq 1 \\ 3x, & x > 1 \end{cases}$$

Show that f *is continuous at* $x = 1$*. Determine whether* f *is differentiable at* $x = 1$*. If so, Find the value of the derivative there.*

Solution. *Since* $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 + x + 1) = 3$ *and* $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x) = 3$,

$$\lim_{x\to 1} f(x) = 3.$$

$f(1) = 1^2 + 1 + 1 = 3$*, therefore* $\lim_{x\to 1} f(x) = f(1)$*. This expresses that* f *is continuous at* $x = 1$*.*

Theorem 2.1. *Let* f *be continuous at* p *and suppose that* $\lim_{x\to p} f'(x)$ *exists. Then* f *is differentiable at* p *and* $f'(p) = \lim_{x\to p} f'(x)$*.*

Since

$$f'(x) = \begin{cases} 2x+1, & x < 1 \\ 3, & x > 1 \end{cases}$$

$\lim_{x\to 1} f'(x) = 3$ *which exists. and the limit is* 3.

f *is continuous at* $x = 1$ *and* $\lim_{x\to 1} f'(x) = 3$. *Then using the Theorem gives* f *is differentiable at* $x = 1$ *and its derivative at* $x = 1$ *is* 3.

Exercise 2.30. *Let* f *be defined by*

$$f(x) = |5x-2| + x^2 + \sin x$$

Find the points where f *fails to be differentiable.*

Solution.

1. *if* $x > \frac{2}{5}$, *then* $f(x) = 5x - 2 + x^2 + \sin x$, *which is differentiable for all* $x > \frac{2}{5}$
2. *if* $x < \frac{2}{5}$, *then* $f(x) = -5x + 2 + x^2 + \sin x$, *which is differentiable for all* $x < \frac{2}{5}$
3. *Let us consider* $x = \frac{2}{5}$.

   At $x = \frac{2}{5}$ *a function* g *defined by* $g(x) = |5x-2|$ *is not differentiable. Because of this* f *fails to be differentiable at* $x = \frac{2}{5}$.

Exercise 2.31. *Let* f *be defined by*

$$f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < \frac{1}{2} \\ \frac{3}{4}x^2, & x \geq \frac{1}{2} \end{cases}$$

Determine whether f *is differentiable at* $x = \frac{1}{2}$. *If so, Find the value of the derivative there.*

Solution.

$$\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to \frac{1}{2}^-} \left(x^3 + \frac{1}{16}\right) = \frac{1}{8} + \frac{1}{16} = \frac{3}{16}$$

and

$$\lim_{x\to \frac{1}{2}^+} f(x) = \lim_{x\to \frac{1}{2}^+} \frac{3}{4}x^2 = \frac{3}{16}.$$

Hence $\lim_{x\to\frac{1}{2}} f(x) = \frac{3}{16}$.

$f\left(\frac{1}{2}\right) = \frac{3}{16}$. *Thus,* $\lim_{x\to\frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$. *Therefore f is continuous at* $x = \frac{1}{2}$.

$$f'(x) = \begin{cases} 3x^2, & x < \frac{1}{2} \\ \frac{3}{2}x, & x > \frac{1}{2} \end{cases}$$

$\lim_{x\to\frac{1}{2}} f'(x) = \frac{3}{4}$. *Because of this f is differentiable at* $\frac{1}{2}$ *and* $f'\left(\frac{1}{2}\right) = \frac{3}{4}$.

Exercise 2.32. *Let a function f be defined by*

1. $f(x) = x\ln(x+1)$,
2. $f(x) = x\sqrt{9-x^2}$,
3. $f(x) = \sqrt{-3x^2+13x-14}$,
4. $f(x) = \begin{cases} x^2-4x+3, & x \geq \frac{\pi}{3} \\ \sin x, & -\pi \leq x < \frac{\pi}{3} \end{cases}$.

Determine D_f and find the critical points and the table of variations of f.

Solution.

1. $f(x) = x\ln(x+1)$. *Then the domain of f is* $D_f = \{x \in \mathbb{R} : \ x > -1\}$.

$$f'(x) = \ln(x+1) + \frac{x}{x+1}.$$

We find the points at which the value of f' is zero or not defined.

f' is not defined if $x < -1$.

$$f'(x) = 0,$$
$$\ln(x+1) + \frac{x}{x+1} = 0$$

$$x = 0$$

Then, at $x = 0$ the value of f' is 0.

Since 0 is the interior point of D_f and $x : x < -1$ is not the interior point of D_f. Only 0 is the critical point of f.

x	-1		0		$+\infty$
$f'(x)$	$\Vert$	$-$	0	$+$	
Variations of f and values $f(x)$	$\Vert$	$\searrow$		$\nearrow$	

2. $f(x) = x\sqrt{9-x^2}$. *Then the domain of* f *is* $D_f = [-3, 3]$.

$$f'(x) = \sqrt{9-x^2} - \frac{x \cdot 2x}{2\sqrt{9-x^2}} = \sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}} = \frac{9-2x^2}{\sqrt{9-x^2}}.$$

f' *is not defined if* $x < -3$ *or* $x > 3$.

Since

$$f'(x) = 0,$$

$$\frac{9-2x^2}{\sqrt{9-x^2}} = 0,$$

$$9 - 2x^2 = 0,$$

$$x = \pm\frac{3}{\sqrt{2}},$$

f' *takes the value* 0 *at* $x = \pm\frac{3}{\sqrt{2}}$.

$\pm\frac{3}{\sqrt{2}}$ *is the interior point of* D_f. $x : x < -3$ *or* $x > 3$ *is not the interior point of* D_f. *Hence according to the definition of a critical point* $\pm\frac{3}{\sqrt{2}}$ *is the critical point of* f.

x	-3		$-\frac{3}{\sqrt{2}}$		$\frac{3}{\sqrt{2}}$		3
$3-\sqrt{2}x$		$+$		$+$	0	$-$	
$3+\sqrt{2}x$		$+$	0	$-$		$-$	
$\sqrt{9-x^2}$	0	$+$		$+$		$+$	0
$f'(x)$	$\Vert$	$+$	0	$-$	0	$+$	$\Vert$
Variations of f and values $f(x)$		$\nearrow$		$\searrow$		$\nearrow$	

3. $f(x)=\sqrt{-3x^2+13x-14}$.

$$-3x^2+13x-14=0,$$
$$\Delta=13^2-4(-3)(-14)=1.$$

Then,

$$x=\frac{-13\pm 1}{2(-3)}$$

So,

$$x_1=\frac{-13-1}{2(-3)}=\frac{7}{3},\quad x_2=\frac{-13+1}{2(-3)}=2.$$

Then, using the sign table gives that, the domain of f is $D_f=\left[2,\frac{7}{3}\right]$.

$$f'(x)=\frac{-6x+13}{2\sqrt{-3x^2+13x-14}}$$

f' *is not defined if* $x>\frac{7}{3}$ *or* $x<2$.

Since

$$f'(x)=0,$$
$$\frac{-6x+13}{2\sqrt{-3x^2+13x-14}}=0,$$

$$-6x+13=0,$$

$$x=\frac{13}{6},$$

$f'(x) = 0, \text{ if } x = \dfrac{13}{6}.$

$\dfrac{13}{6} \in]-3, 3[$. $x : x > \dfrac{7}{3}$ *or* $x < 2$ *is not the interior point of* D_f. *Hence according to the definition of a critical point,* $x = \dfrac{13}{6}$ *is the critical point of* f.

x	2		$\frac{13}{6}$		$\frac{7}{3}$
$-6x + 13$		$+$	0	$-$	
$\sqrt{-3x^2 + 13x - 14}$		$+$		$+$	
$f'(x)$		$+$	0	$-$	
Variations of f and values f(x)		$\nearrow$		$\searrow$	

4. $f(x) = \begin{cases} x^2 - 4x + 3, & x \geq \dfrac{\pi}{3} \\ \sin x, & -\pi \leq x < \dfrac{\pi}{3} \end{cases}$. *Hence,* $D_f = \Big[-\pi, +\infty\Big[$.

We find the points at which the value of f' *is zero or not defined.*

Consider $x \neq \dfrac{\pi}{3}$, *then*

$$f'(x) = \begin{cases} 2x - 4, & x > \dfrac{\pi}{3} \\ \cos x, & -\pi \leq x < \dfrac{\pi}{3} \end{cases}$$

which is defined for $x > \dfrac{\pi}{3}$ *or* $-\pi \leq x < \dfrac{\pi}{3}$. *So* f *is differentiable on* $\Big[\pi, \dfrac{\pi}{3}\Big[\cup \Big]\dfrac{\pi}{3}, +\infty\Big[$.

We find the points at which $f'(x) = 0$.

$f'(x) = 0$ *if* $x = 2$ *or* $x = -\dfrac{\pi}{2}$.

Consider $x = \dfrac{\pi}{3}$,

$$\lim_{x \to \frac{\pi}{3}^-} f(x) = \lim_{x \to \frac{\pi}{3}^-} x^2 - 4x + 3 = \left(\frac{\pi}{3}\right)^2 - 4 \cdot \frac{\pi}{3} + 3.$$

$$\lim_{x \to \frac{\pi}{3}^+} f(x) = \lim_{x \to \frac{\pi}{3}^+} \sin x = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

So, f has no limit at $\frac{\pi}{3}$, that is it is discontinuous at this point. Hence f is not differentiable at $\frac{\pi}{3}$.

Finally, over the interval $[-\pi, +\infty[$ f is not differentiable at $\frac{\pi}{3}$ and takes 0, at 2 or $-\frac{\pi}{2}$.

Since $2, \ -\frac{\pi}{2}, \ \frac{\pi}{3} \in (-\pi, +\infty)$, $\quad 2, \ -\frac{\pi}{2}, \ \frac{\pi}{3}$ are critical points of f.

x	$-\pi$		$-\frac{\pi}{2}$		$\frac{\pi}{3}$		2		$+\infty$
$f'(x)$		$-$	0	$+$	$\Vert$	$-$	0	$+$	
Variations of f and values $f(x)$		↘		↗		↘		↗	

Exercise 2.33. *A function f is defined by $f(x) = \dfrac{x^2}{x-1} + |x|$.*

(a) Determine D_f,

(b) Find f',

(c) Deduce the critical points and the table of variations of f,

(d) Find its local extreme values.

(e) Find its global (absolute) extreme values.

Solution.

(a) $D_f = \{x \in \mathbb{R} : x \neq 1\}$.

(b)

$$f(x) = \begin{cases} \dfrac{x^2}{x-1} + x, & x \geq 0 \\ \dfrac{x^2}{x-1} - x, & x < 0 \end{cases} = \begin{cases} \dfrac{2x^2 - x}{x-1}, & x \geq 0 \\ \dfrac{x}{x-1}, & x < 0 \end{cases}$$

Then,

$$f'(x)=\begin{cases}\dfrac{(4x-1)(x-1)-(2x^2-x)\cdot 1}{(x-1)^2}, & x\geq 0\\ \dfrac{1\cdot(x-1)-x\cdot 1}{(x-1)^2}, & x<0\end{cases}=\begin{cases}\dfrac{4x^2-x-4x+1-2x^2+x}{(x-1)^2}, & x\geq 0\\ \dfrac{x-1-x}{(x-1)^2}, & x<0\end{cases}$$
$$=\begin{cases}\dfrac{2x^2-4x+1}{(x-1)^2}, & x\geq 0\\ \dfrac{-1}{(x-1)^2}, & x<0\end{cases}$$

which is not defined at $x=0$ and $x=1$.

(c) From (b), $f'(x)=0$ if $x=1\pm\dfrac{\sqrt{2}}{2}$ and f' is not defined at $x=0$ and $x=1$.

Finally, f' gets a value 0 at $x=1\pm\dfrac{\sqrt{2}}{2}$ and is not defined at $x=0$ and $x=1$.

$1-\dfrac{\sqrt{2}}{2}$, 0 and $1+\dfrac{\sqrt{2}}{2}$ are interior points of D_f. 1 is not an interior point of D_f. Hence, $1-\dfrac{\sqrt{2}}{2}$, 0 and $1+\dfrac{\sqrt{2}}{2}$ are critical points of f.

To deduce the variation table of f we need to form the sign table of f'

In addiction at the next step we will need local and global extreme values, therefore we have to know the values of f at critical points and endpoints of f. If the domain of f is not closed interval, in this case we find the limits of f at open ends.

If $x\geq 0$

$$f(0)=0$$

$$f\left(1-\frac{\sqrt{2}}{2}\right)=-2\sqrt{2}+3$$

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}\left(\frac{2x^2-x}{x-1}\right)=-\infty$$

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}\left(\frac{2x^2-x}{x-1}\right)=+\infty$$

$$f\left(1+\frac{\sqrt{2}}{2}\right)=2\sqrt{2}+3$$

$$\lim_{x\to +\infty}f(x)=\lim_{x\to +\infty}\left(\frac{2x^2-x}{x-1}\right)=+\infty$$

x	0		$1-\frac{\sqrt{2}}{2}$		1		$1+\frac{\sqrt{2}}{2}$		$+\infty$
$x-1+\frac{\sqrt{2}}{2}$		$-$	0	$+$		$+$		$+$	
$x-1-\frac{\sqrt{2}}{2}$		$-$		$-$		$-$	0	$+$	
$(x-1)^2$		$+$		$+$	0	$+$		$+$	
$f'(x)$	$\Vert$	$+$	0	$-$	$\Vert$	$-$	0	$+$	
Variations of f and values $f(x)$	0	$\nearrow$	$-2\sqrt{2}+3$	$\searrow$	$-\infty \Vert +\infty$	$\searrow$	$2\sqrt{2}+3$	$\nearrow$	$+\infty$

If $x<0$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{x}{x-1}\right) = 1$$

x	$-\infty$		0
$(x-1)^2$		$+$	$\Vert$
$f'(x)$		$-$	$\Vert$
Variations of f and values $f(x)$	1	$\searrow$	$\Vert$

(d) *To define local extreme values we consider endpoints and critical points,*

1. $f(0)=0$ *is a local minimum value of* f.
2. $f\left(1-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}+3$ *is a local maximum value of* f.
3. $f\left(1+\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}+3$ *is a local minimum value of* f.

(e) *Using (c) gives,*

$$\lim_{x\to+\infty} f(x) = +\infty$$

which is greater than any numbers. Therefore f has no global (absolute) maximum.

$$\lim_{x \to 1^-} f(x) = -\infty$$

which is less than any numbers. This implies that f has no global (absolute) minimum value.

Exercise 2.34. *Let a function f is defined by $f(x) = e^{\sqrt{x}}$.*

(a) Determine D_f,

(b) Find f',

(c) Deduce the critical points and the table of variations of f,

(d) Find its local extreme values.

(e) Find its global (absolute) extreme values.

Solution.

(a) $D_f = \{x \in \mathbb{R} : x \geq 0\}$.

(b)

$$f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}.$$

which is not defined at $x : x \leq 0$.

(c) From (b), $f'(x)$ is not defined at $x : x \leq 0$ and it is different from 0 on its domain.

$x : x \leq 0$ is not an interior point of D_f. So f has no critical point.

To deduce the variation table of f first we need to draw the sign table of f'

In addiction at the next step we will need local and global extreme values, therefore we have to know the values of f at critical points and endpoints of f. If the domain of f is not closed interval, in this case we find the limits of f at open ends.

$$f(0) = 1$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^{\sqrt{x}} = +\infty$$

(d) 1. $f(0) = 1$ is a local minimum value of f.

2. f has no local maximum.

(e)
$$\lim_{x\to+\infty} f(x) = +\infty$$
which is greater than any numbers. Therefore f has no global (absolute) maximum. Global (absolute) minimum value of f is 1.

Exercise 2.35. *A function f is defined by $f(x) = \sin x - \cos x$.*

(a) Determine D_f,

(b) Find f',

(c) Deduce the critical points and the table of variations of f,

(d) Find its local extreme values.

(e) Find its global (absolute) extreme values.

Solution.

(a) $D_f = \mathbb{R}$.

(b)
$$f'(x) = \cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right).$$

(c) $f'(x)$ is defined everywhere.

Since
$$f'(x) = 0$$
$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 0$$
$$x + \frac{\pi}{4} = \pi k, \quad \textit{where} \quad k \in \mathbb{Z}$$
$$x = -\frac{\pi}{4} + \pi k, \quad \textit{where} \quad k \in \mathbb{Z}$$

at $x = -\frac{\pi}{4} + \pi k$, where $k \in \mathbb{Z}$ the value of f' is 0.

$x = -\frac{\pi}{4} + \pi k$, $(k \in \mathbb{Z})$ are within $\mathbb{R}$ (is interior point of the domain) 0 is not an interior point of D_f. That is why the critical points of f are $x = -\frac{\pi}{4} + \pi k$, $(k \in \mathbb{Z})$.

To deduce the variation table of f first we need to draw the sign table of f'

In addiction at the next step we will need local and global extreme values, therefore we have to know the values of f at critical points and endpoints of f. If the domain of f is not closed interval, in this case we find the limits of f at open ends.

$$f(-\frac{\pi}{4}+\pi k)=\sin\Big(-\frac{\pi}{4}+\pi k\Big)-\cos\Big(-\frac{\pi}{4}+\pi k\Big)$$
$$=\sqrt{2}\sin\Big(-\frac{\pi}{2}+\pi k\Big)=\begin{cases}-\sqrt{2} & if \quad k=2n, \qquad n\in\mathbb{Z}\\ \sqrt{2} & if \quad k=2n+1, \quad n\in\mathbb{Z}\end{cases}$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\Big(\sin x-\cos x\Big)=\lim_{x\to+\infty}\sqrt{2}\sin\Big(x-\frac{\pi}{4}\Big) \quad doesn't \ \ exist,$$

but the values of f are between $-\sqrt{2}$ and $\sqrt{2}$.

x	$-\infty$	$\cdots$	$-\frac{5\pi}{4}$		$-\frac{\pi}{4}$		$\frac{3\pi}{4}$		$\frac{7\pi}{4}$	$\cdots$	$+\infty$
$\sqrt{2}\sin\Big(x+\frac{\pi}{4}\Big)$			0	$-$	0	$+$	0	$-$	0		
$f'(x)$			0	$-$	0	$+$	0	$-$	0		
Variations of f and values $f(x)$		$\nearrow$	$\sqrt{2}$	$\searrow$	$-\sqrt{2}$	$\nearrow$	$\sqrt{2}$	$\searrow$	$-\sqrt{2}$		

(d) 1. $f(-\frac{\pi}{4}+2\pi n)=-\sqrt{2}$ *is a local minimum value of f.*

2. $f(-\frac{\pi}{4}+2\pi n+1)=\sqrt{2}$ *has no local maximumis a local maximum value of f.*

(e) *The values of f are between $-\sqrt{2}$ and $\sqrt{2}$ and f has these values. So $-\sqrt{2}$ is global minimum, $\sqrt{2}$ is global maximum of f.*

Exercise 2.36. *Let a function f is defined by $f(x)=(x-2)^2(x+3)$.*

(a) *Determine the domain of f*

(b) *Determine f',*

(c) *Deduce the critical points and the table of variations of f,*

(d) *Find its local extreme values.*

(e) *Find its global (absolute) extreme values.*

Solution.

(a) $D_f = \mathbb{R}$.

(b)
$$f'(x) = 2(x-2)(x+3) + (x-2)^2 = (x-2)(3x+4).$$

(c) From (b), $f'(x)$ is defined everywhere.

$$f'(x) = 0,$$
$$(x-2)(3x+4) = 0,$$

from here,
$$x = 2, \quad x = -\frac{4}{3}.$$

So at $x = 2, \quad x = -\dfrac{4}{3}$.

$x = 2, \quad x = -\dfrac{4}{3}$ *are interior points of D_f, therefore they are critical points of f.*

To deduce the variation table of f first we need to draw the sign table of f'

In addiction at the next step we will need local and global extreme values, therefore we have to know the values of f at critical points and endpoints of f. If the domain of f is not closed interval, in this case we find the limits of f at open ends.

$$f(2) = (2)^2 \cdot 3 = 12,$$

$$f\Big(-\frac{4}{3}\Big) = \frac{500}{27},$$

$$\lim_{x\to+\infty} f(x) = +\infty,$$

$$\lim_{x\to-\infty} f(x) = -\infty.$$

x	$-\infty$		$-\frac{4}{3}$		2		$+\infty$
$3x+4$		$-$	0	$+$		$+$	
$x-2$		$-$		$-$	0	$+$	
$f'(x)$		$+$	0	$-$	0	$+$	
Variations of f and values $f(x)$	$-\infty$	$\nearrow$	$\frac{500}{27}$	$\searrow$	0	$\nearrow$	$+\infty$

(d) 1. $\frac{500}{27}$ is a local maximum value of f.

2. 0 is a local minimum of f.

(e) Since

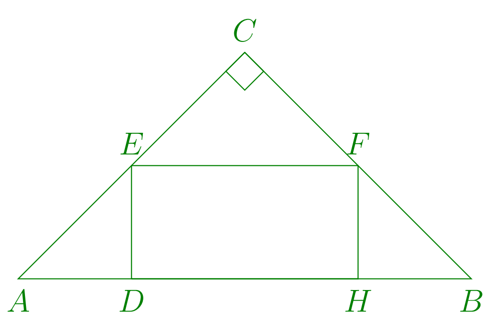
$$\lim_{x\to+\infty} f(x) = +\infty,$$

$$\lim_{x\to-\infty} f(x) = -\infty,$$

f has no global (absolute) extreme values.

Exercise 2.37. *Find the maximum area of a rectangle inscribed in an isosceles right triangle whose hypotenuse is 20 cm long.*

Solution. *Let ΔABC be an isosceles right triangle with $AB = 20$ and $DEFH$ be rectangle inscribed in ΔABC.*



Suppose $AD = x$, then $DH = 20 - 2x$, $ED = AD = x$. Hence, the area of $DEFH$ is given by a function A defined by

$$A(x) = x \cdot (20 - 2x) = 20x - 2x^2,$$

which is defined on $(0, 10)$.
We find the maximum value of the function A:

$$A'(x) = 20 - 4x$$

at $x = 5$ *the value of* A' *is* 0.

x	0	5	10
$A'(x)$	$\parallel$ +	0	− $\parallel$

From here, the maximum value of A (Area) is $A(5) = 20 \cdot 5 - 2 \cdot 5^2 = 50$.

Exercise 2.38. *The vertical posts, with heights of* $7m$ *and* $13m$*, are secured by a rope going from the top of one post to a point on the ground between the posts and then to the top of the other post. The distance between two posts is 25 m. Where should the point at which the rope touches the ground be located so that the least amount of rope is used?*

Solution. *The function L for the length of the rope is*

$$L(x) = \sqrt{13^2 + (25 - x)^2} + \sqrt{x^2 + 7^2}, \qquad \textit{where} \quad 0 < x < 25,$$

$$L(x) = \sqrt{13^2 + (25 - x)^2} + \sqrt{x^2 + 7^2},$$

$$L(x) = \sqrt{794 - 50x + x^2} + \sqrt{x^2 + 49},$$

Then,

$$L'(x) = \frac{2x - 50}{\sqrt{794 - 50x + x^2}} + \frac{2x}{\sqrt{x^2 + 49}}.$$

$$\frac{2x - 50}{\sqrt{794 - 50x + x^2}} + \frac{2x}{\sqrt{x^2 + 49}} = 0,$$

$$\frac{(2x - 50)\sqrt{x^2 + 49} + 2x\sqrt{794 - 50x + x^2}}{\sqrt{794 - 50x + x^2}\sqrt{x^2 + 49}} = 0,$$

$$x = 8,75$$

x	0	8.5	25
$L'(x)$	−	0	+

Therefore the least amount will be $L(8.5) = 32.099$.

Exercise 2.39. *Let f be defined by $f(x) = x - \sin x$ on the closed interval $[0, 2\pi]$*

(a) Determine $f''(x)$

(b) Find the points of inflection of f,

(c) Deduce the concavity of the curve of f.

Solution.

(a)

$$f'(x) = 1 - \cos x$$

$$f''(x) = \sin x$$

(b) At a point of inflection $(p, (p))$, either $f''(c) = 0$ or $f''(c)$ fails to exist. So we look for an inflection point between the points which satisfy bottom conditions.

The second derivative is defined everywhere on $[0, 2\pi]$ and is 0 if,

$$\sin x = 0$$

$$x = 0,\ \ \pi,\ \ 2\pi.$$

x	0		π		2π
$f''(x)$	0	$+$	0	$-$	0
Concavity					

Only at $x = \pi$ concavity changes and $f(\pi) = 1 - \cos\pi = 2$. Therefore $(\pi, 2)$ is a point of inflection.

(c) The curve of f is concave down on $(0, \pi)$, concave up on $(\pi, 2\pi)$.

Exercise 2.40. *Let f be defined by $f(x) = \dfrac{x^2 - x + 2}{x}$.*

(a) Determine $f''(x)$

(b) Find the points of inflection of f,

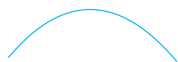
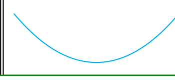
(c) Deduce the concavity of the curve of f.

Solution.

(a)

$$f(x) = \frac{x^2 - x + 2}{x} = x - 1 + 2x^{-1}.$$
$$f'(x) = 1 - 2x^{-2},$$
$$f''(x) = 4x^{-3} = \frac{4}{x^3},$$

(b) f'' *is not defined at* $x = 0$

x	$-\infty$	0	$+\infty$
$f''(x)$	$-$	‖	$+$
Concavity		‖	

At $x = 0$ *concavity changes, but* f *is not defined at this point. So* f *has no point of inflection.*

(c) The curve of f *is concave down on* $(-\infty, 0)$*, concave up on* $(0, +\infty)$*.*

Exercise 2.41. *Let* f *be defined by* $f(x) = \dfrac{x^2+1}{2x}$.

1. *Determine the domain of* f*,*
2. *Find the points of intersection of the curve of* f *with the* x*- and* y*-axis,*
3. *Determine if* f *even or odd and the symmetry of its graph,*
4. *Give the sign table of* f*,*
5. *Find all the asymptotes to the curve of* f *and determine the position of the curve of* f *with respect to its asymptotes,*
6. *Determine* $f'(x)$ *and the critical points of* f*,*
7. *Deduce the table of variations of* f *and its extreme values (Hint: to define absolute extrema use asymptotes),*
8. *Determine* $f''(x)$
9. *Deduce the concavity of the curve of* f*, its points of inflection*

10. *Graph the curve of f.*

Solution.

1. $D_f = \mathbb{R}/\{0\}$.
2. *The intersection with the x-axis:*
$$\frac{x^2+1}{2x} = 0$$
$$x^2+1=0$$
which has no solution. So the curve of f doesn't intersect the x-axis.

   The intersection with the y-axis:
   f is not defined at 0. Because of this the curve of f doesn't intersect the x-axis.
3. $f(-x) = \dfrac{(-x)^2+1}{2\cdot(-x)} = -\dfrac{x^2+1}{2\cdot x} = -f(x)$. *Therefore f is an odd function.*
4. $f(x) = \dfrac{x^2+1}{2x}$. *Then,*

x	$-\infty$	0	$+\infty$
x^2+1	$+$		$+$
$2x$	$-$	‖	$+$
Sign of $f(x)$	$-$	‖	$+$

5. (a) *H.A.*
Since
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2+1}{2x} = +\infty$$
and
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+1}{2x} = -\infty$$
the curve of f has no horizontal asymptote at $\pm\infty$.

(b) *V.A.*

Since

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+1}{2x} = -\infty$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+1}{2x} = +\infty$$

The line of $x = 0$ *is the vertical asymptote of the curve of* f *and the curve of* f *is on the left of the asymptote as* $y \to -\infty$, *on the right of the asymptote as* $y \to +\infty$.

(c) *O.A.*

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2+1}{2x^2} = \frac{1}{2}.$$

$$b = \lim_{x\to+\infty} (f(x) - ax) = \lim_{x\to+\infty} \left(\frac{x^2+1}{2x} - \frac{x}{2}\right) = \lim_{x\to+\infty} \frac{1}{2x} = 0.$$

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{x^2+1}{2x^2} = \frac{1}{2}.$$

$$b = \lim_{x\to-\infty} (f(x) - ax) = \lim_{x\to-\infty} \left(\frac{x^2+1}{2x} - \frac{x}{2}\right) = \lim_{x\to-\infty} \frac{1}{2x} = 0.$$

That is why $y = \frac{x}{2}$ *is the oblique asymptote of* C_f *at* $\pm\infty$. *Since*

$$\lim_{x\to+\infty} (f(x) - y) = \lim_{x\to+\infty} \left(\frac{x^2+1}{2x} - \frac{x}{2}\right) = \lim_{x\to+\infty} \frac{1}{2x} = 0^+.$$

the curve of f *is above the asymptote of* $y = \frac{x}{2}$ *at* $+\infty$.

Since

$$\lim_{x\to-\infty} (f(x) - y) = \lim_{x\to-\infty} \left(\frac{x^2+1}{2x} - \frac{x}{2}\right) = \lim_{x\to-\infty} \frac{1}{2x} = 0^-.$$

the curve of f *is below the asymptote of* $y = \frac{x}{2}$ *at* $-\infty$.

6.

$$f'(x) = \frac{2x \cdot 2x - (x^2+1)\cdot 2}{4x^2} = \frac{x^2-1}{2x^2} = \frac{(x-1)(x+1)}{2x^2}$$

which is not defined at $x = 0$.

f' *is different from* 0 *for all* $x \in D_f$ *and is not defined at* 0, *but* 0 *is not an interior point of* D_f. *So,* f *has no critical point.*

7. *The sign table of* f' *and the variation table of* f *is*

x	$-\infty$		-1		0		1		$+\infty$
$x-1$		$-$		$-$		$-$	0	$+$	
$x+1$		$-$	0	$+$		$+$		$+$	
$2x^2$		$+$		$+$	0	$+$		$+$	
$f'(x)$		$+$	0	$-$	$\Vert$	$-$	0	$+$	
Variations of f and values $f(x)$	$-\infty$	$\nearrow$	-1	$\searrow$	$-\infty \Vert +\infty$	$\searrow$	1	$\nearrow$	$+\infty$

-1 is a local maximum value of f and 1 is a local minimum of f. f has no global extreme values.

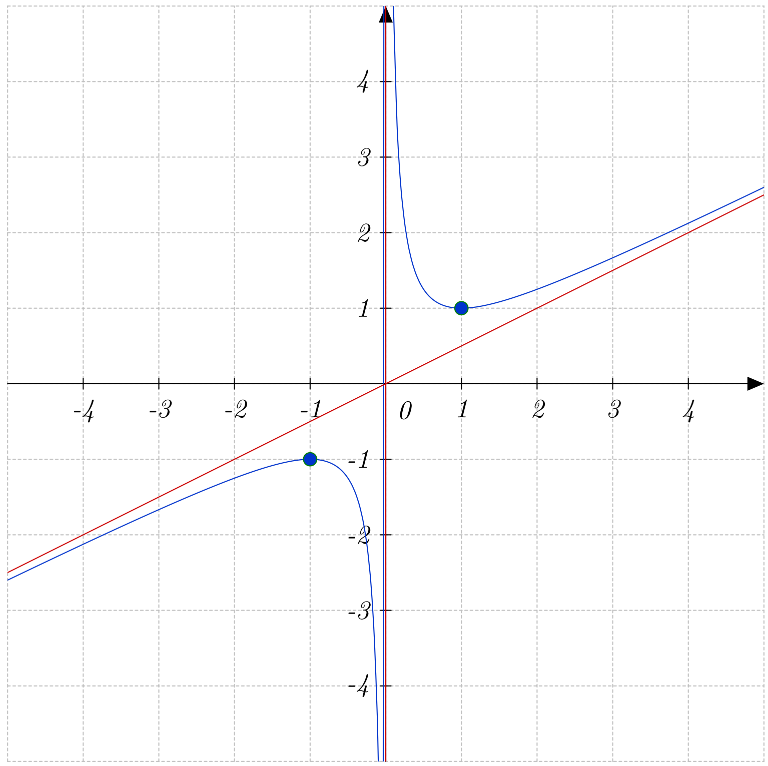
8.
$$f''(x) = \frac{2x \cdot 2x^2 - (x^2-1) \cdot 4x}{4x^4} = \frac{1}{x^3}.$$

9. *The concavity table of the curve of f is*

x	$-\infty$	0	$+\infty$
$f''(x)$	$-$	$\Vert$	$+$
Concavity	*concave down*	$\Vert$	*concave up*

We look for the inflection points which is in the domain among the points at which the second derivative of the function either is zero or the first derivative of f is not defined. Here 0 is not in D_f so, there is no point of inflection.

10. *Graph the curve of f is*



Exercise 2.42. *Let f be defined by $f(x) = x^5 - 6x^4$.*

1. *Determine the domain of f,*
2. *Find the points of intersection of the curve of f with the x- and y-axis,*
3. *Determine if f even or odd and the symmetry of its graph,*
4. *Give the sign table of f,*
5. *Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes,*
6. *Determine $f'(x)$ and the critical points of f,*
7. *Deduce the table of variations of f and its extreme values (Hint: to define absolute extrema use asymptotes),*
8. *Determine $f''(x)$*
9. *Deduce the concavity of the curve of f, its points of inflection*
10. *Graph the curve of f.*

Solution.

1. $D_f = \mathbb{R}$.

2. *The intersection with the x-axis:*

$$x^5 - 6x^4 = 0,$$

$$x^4(x-6) = 0,$$

$$x = 0, \quad x = 6.$$

So, the curve of f intersects the x-axis at $(0,0)$ and $(6,0)$.

The intersection with the y-axis:

$f(0) = 0^5 - 6 \cdot 0^4 = 0$. *Because of this the curve of f intersects the y-axis at $(0,0)$.*

3.
$$f(-1) = (-1)^5 - 6 \cdot (-1)^4 = -1 - 6 = -7$$

and
$$f(1) = 1^5 - 6 \cdot (-1)^4 = 1 - 6 = -5.$$

Hence, $f(-1) \neq -f(1)$ and $f(-1) \neq f(1)$.

So $f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$ for $x = 1$. Therefore f is neither even nor odd.

4. $f(x) = x^5 - 6x^4 = x^4(x-6)$. *Then,*

x	$-\infty$		0		6		$+\infty$
x^4		$+$	0	$+$		$+$	
$x-6$		$-$		$-$	0	$+$	
Sign of $f(x)$		$-$	0	$-$	0	$+$	

5. (a) *H.A.*

Since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (x^5 - 6x^4) = \lim_{x\to+\infty} x^4(x-6) = +\infty$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x^5-6x^4) = \lim_{x\to-\infty} x^4(x-6) = -\infty$$

the curve of f has no horizontal asymptote at $\pm\infty$.

(b) *V.A.*

Since there is no point p such that $\lim_{x\to p}|f(x)| = +\infty$, the graph of f has no vertical asymptote.

(c) *O.A.*

$$a = \lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{x^5-6x^4}{x} = \lim_{x\to\pm\infty} x^4-6x^3 = +\infty$$

Because of this, the curve of f does not an oblique asymptote at $\pm\infty$.

6.

$$f'(x) = 5x^4 - 24x^3 = x^3(5x-24)$$

which is defined everywhere and 0 at $x=0$ and $x=4.8$.

$x=0$ and $x=4.8$ are interior points of D_f. So, they are critical points of f.

7. *The sign table of f' and the variation table of f is*

x	$-\infty$		0		4.8		$+\infty$
x^3		$-$	0	$+$		$+$	
$5x-24$		$-$		$-$	0	$+$	
$f'(x)$		$+$	0	$-$	0	$+$	
Variations of f and values $f(x)$	$-\infty$	$\nearrow$	0	$\searrow$	-637.00992	$\nearrow$	$+\infty$

0 is a local maximum value of f and -637.00992 is a local minimum of f. f has no global extreme values.

8.

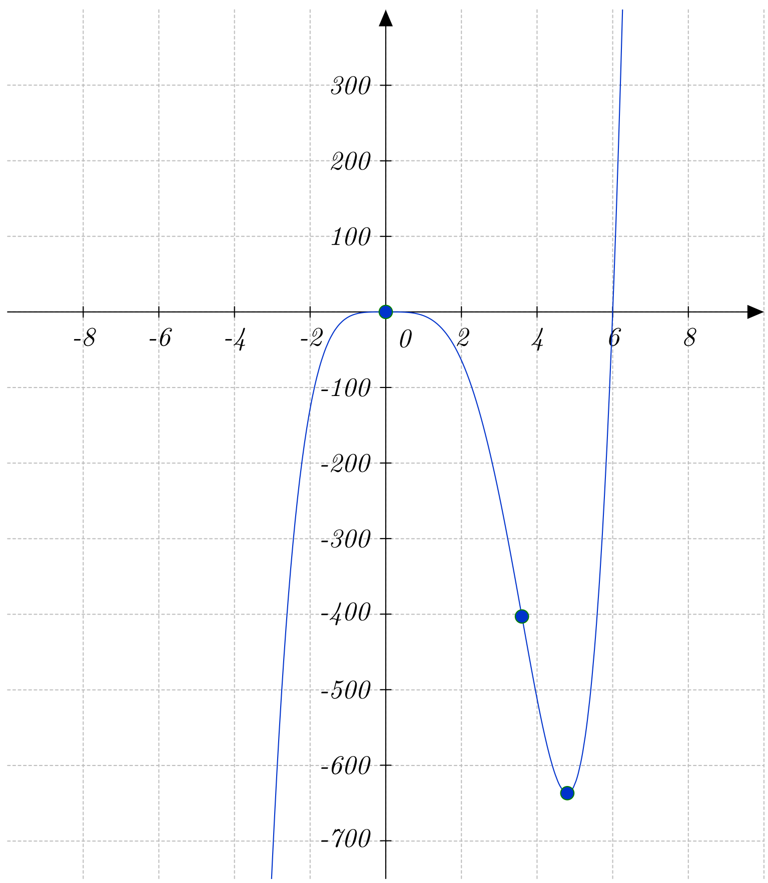
$$f''(x) = 20x^3 - 72x^2 = x^2(20x-72).$$

9. *The concavity table of the curve of f is*

x	$-\infty$		0		3.6		$+\infty$
x^2		$+$	0	$+$		$+$	
$20x-72$		$-$		$-$	0	$+$	
$f''(x)$		$-$		$-$	0	$+$	
Concavity			0		-403.1		

We look for the inflection points which is in the domain among the points at which the second derivative of the function either is zero or the first derivative of f is not defined. Here f is defined at 3.6 and at $(3.6, -403.1)$ concavity changes. Therefore $(3.6, -403.1)$ is the point of inflection.

10. Graph the curve of f is



Exercise 2.43. *Let f be defined by* $f(x) = \dfrac{x^2}{x+5} - |x|$.

1. *Determine the domain of f,*
2. *Find the points of intersection of the curve of f with the x- and y-axis,*
3. *Determine if f even or odd and the symmetry of its graph,*
4. *Give the sign table of f,*
5. *Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes,*
6. *Determine $f'(x)$ and the critical points of f,*
7. *Deduce the table of variations of f and its extreme values (Hint: to define absolute extrema use asymptotes),*
8. *Determine $f''(x)$*
9. *Deduce the concavity of the curve of f, its points of inflection*
10. *Graph the curve of f.*

Solution.

1. $D_f = \mathbb{R}$.
2. $$f(x) = \frac{x^2}{x+5} - |x| = \begin{cases} \dfrac{x^2}{x+5} - x, & x \geq 0 \\ \dfrac{x^2}{x+5} + x, & x < 0 \end{cases} = \begin{cases} \dfrac{-5x}{x+5}, & x \geq 0 \\ \dfrac{2x^2+5x}{x+5}, & x < 0 \end{cases}$$

   The intersection with the x-axis:

   If $x \geq 0$, then

   $$\frac{-5x}{x+5} = 0.$$

   Hence,

   $$x = 0.$$

   If $x < 0$, then

   $$\frac{2x^2+5x}{x+5} = 0.$$

Hence,

$$x = -2.5$$

Therefore the curve of f intersects the x-axis at $(0,0)$ and $(-2.5,0)$.

The intersection with the y-axis:

$$f(0) = \frac{0^2}{0+5} - |0| = 0.$$

It expresses that The curve of f intersects the y-axis at $(0,0)$.

3.

$$f(-1) = \frac{(-1)^2}{-1+5} - |-1| = \frac{1}{4} - 1 = -\frac{3}{4}.$$

and

$$f(1) = \frac{1^2}{1+5} - |1| = \frac{1}{6} - 1 = -\frac{5}{6}$$

Hence, $f(-1) \neq -f(1)$ and $f(-1) \neq f(1)$.

So $f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$ for $x = 1$. Therefore f is neither even nor odd.

4. $f(x) = \begin{cases} \dfrac{-5x}{x+5}, & x \geq 0 \\ \dfrac{2x^2+5x}{x+5}, & x < 0 \end{cases} = \begin{cases} \dfrac{-5x}{x+5}, & x \geq 0 \\ \dfrac{2x(x+2.5)}{x+5}, & x < 0 \end{cases}$ *. Then,*

x	0		$+\infty$
$-5x$	0	$-$	
$x+5$		$+$	
Sign of $f(x)$	0	$-$	

and

x	$-\infty$		-5		-2.5		0
$2x$		$-$		$-$		$-$	‖
$x+2.5$		$-$		$-$	0	$+$	‖
$x+5$		$-$	0	$+$		$+$	‖
Sign of $f(x)$		$-$	‖	$+$	0	$-$	‖

Finally,

x	$-\infty$		-5		-2.5		0		$+\infty$
Sign of $f(x)$		$-$	‖	$+$	0	$-$	0	$-$	

5. (a) *H.A.*

Since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{-5x}{x+5} = \lim_{x\to+\infty} \frac{-5}{1+\dfrac{5}{x}} = -5^{+}$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2+5x}{x+5} = -\infty$$

the curve of f has a horizontal asymptote of $y=-5$ at $+\infty$ and it is above the horizontal asymptot, and it does not have a horizontal asymptote at $-\infty$.

(b) *V.A. Since*

$$\lim_{x\to-5^-} f(x) = \lim_{x\to-5^-} \left(\frac{x^2}{x+5} - |x|\right) = -\infty$$

and

$$\lim_{x\to-5^+} f(x) = \lim_{x\to-5^+} \left(\frac{x^2}{x+5} - |x|\right) = +\infty$$

The line of $x=-5$ is the vertical asymptote of the curve of f and the curve of f is on the left of the asymptote as $y\to-\infty$, on the right of the asymptote as $y\to+\infty$.

(c) *O.A. Since the curve of f has the horizontal asymptote at $+\infty$ we will just consider $-\infty$.*

$$a = \lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\dfrac{2x^2+5x}{x+5}}{x} = \frac{2x+5}{x+5} = 2.$$

$$b = \lim_{x\to-\infty}(f(x)-ax) = \lim_{x\to-\infty}\left(\frac{2x^2+5x}{x+5}-2x\right) = \lim_{x\to-\infty}\frac{-5x}{x+5} = -5.$$

Therefore, the line of $y = 2x-5$ is the oblique asymptote of the curve of f at $-\infty$. Since

$$\lim_{x\to-\infty}(f(x)-(ax+b)) = \lim_{x\to-\infty}\left(\frac{2x^2+5x}{x+5}-(2x-5)\right) = \lim_{x\to-\infty}\frac{25}{x+5} = 0^-,$$

the curve of f is below the oblique asymptote at $-\infty$.

6.

$$f'(x) = \begin{cases} \dfrac{-5(x+5)-(-5x)}{(x+5)^2}, & x \geq 0 \\ \dfrac{(4x+5)(x+5)-(2x^2+5x)}{(x+5)^2}, & x < 0 \end{cases} = \begin{cases} \dfrac{-25}{(x+5)^2}, & x \geq 0 \\ \dfrac{2x^2+20x+25}{(x+5)^2}, & x < 0 \end{cases}$$

which is not defined at $x = 0$ and $x = -5$ and 0 at $x = -5 \pm 2.5\sqrt{2}$.
0 and $-5 \pm 2.5\sqrt{2}$ are an interior points of D_f. So, they are critical points of f.

7. *If $x \geq 0$, then $f'(x) = \dfrac{25}{(x+5)^2}$.*

The sign table of f' and the variation table of f is

x	0		$+\infty$
$(x+5)^2$		$+$	
$f'(x)$	$\Vert$	$-$	
Variations of f and values $f(x)$	0	$\searrow$	-5

If $x < 0$, *then* $f'(x) = \dfrac{2(x+5+2.5\sqrt{2})(x+5-2.5\sqrt{2})}{(x+5)^2}$.

The sign table of f' *and the variation table of* f *is*

x	$-\infty$		$-5-2.5\sqrt{2}$		-5		$-5+2.5\sqrt{2}$		0
$x+5+2.5\sqrt{2}$		$-$	0	$+$		$+$		$+$	$\Vert$
$x+5-2.5\sqrt{2}$		$-$		$-$		$-$	0	$+$	$\Vert$
$(x+5)^2$		$+$		$+$	0	$+$		$+$	$\Vert$
$f'(x)$		$+$	0	$-$	$\Vert$	$-$	0	$+$	$\Vert$
Variations of f *and values* $f(x)$	$-\infty$	$\nearrow$	-29.142	$\searrow$	$-\infty$ $\Vert$ $+\infty$	$\searrow$	-0.858	$\nearrow$	$\Vert$

Finally, The sign table of f' *and the variation table of* f *over the* D_f *is*

x	$-\infty$		$-5-2.5\sqrt{2}$		-5		$-5+2.5\sqrt{2}$		0		$+\infty$
$f'(x)$		$+$	0	$-$	$\Vert$	$-$	0	$+$	$\Vert$	$-$	
Variations of f *and values* $f(x)$	$-\infty$	$\nearrow$	-29.142	$\searrow$	$-\infty$ $\Vert$ $+\infty$	$\searrow$	-0.858	$\nearrow$	0	$\searrow$	-5

-29.142 *is a local maximum value of* f *and* -0.858 *is a local minimum of* f. f *has no global extreme values.*

8.

$$f'(x) = \begin{cases} \dfrac{-25}{(x+5)^2}, & x \geq 0 \\ \dfrac{2x^2+20x+25}{(x+5)^2}, & x < 0 \end{cases}$$

Hence,

$$f''(x) = \begin{cases} \dfrac{50}{(x+5)^3}, & x \geq 0 \\ \dfrac{50}{(x+5)^3}, & x < 0 \end{cases} = \dfrac{50}{(x+5)^3}$$

9. *At $x=-5$ and $x=0$, f' is not defined.*

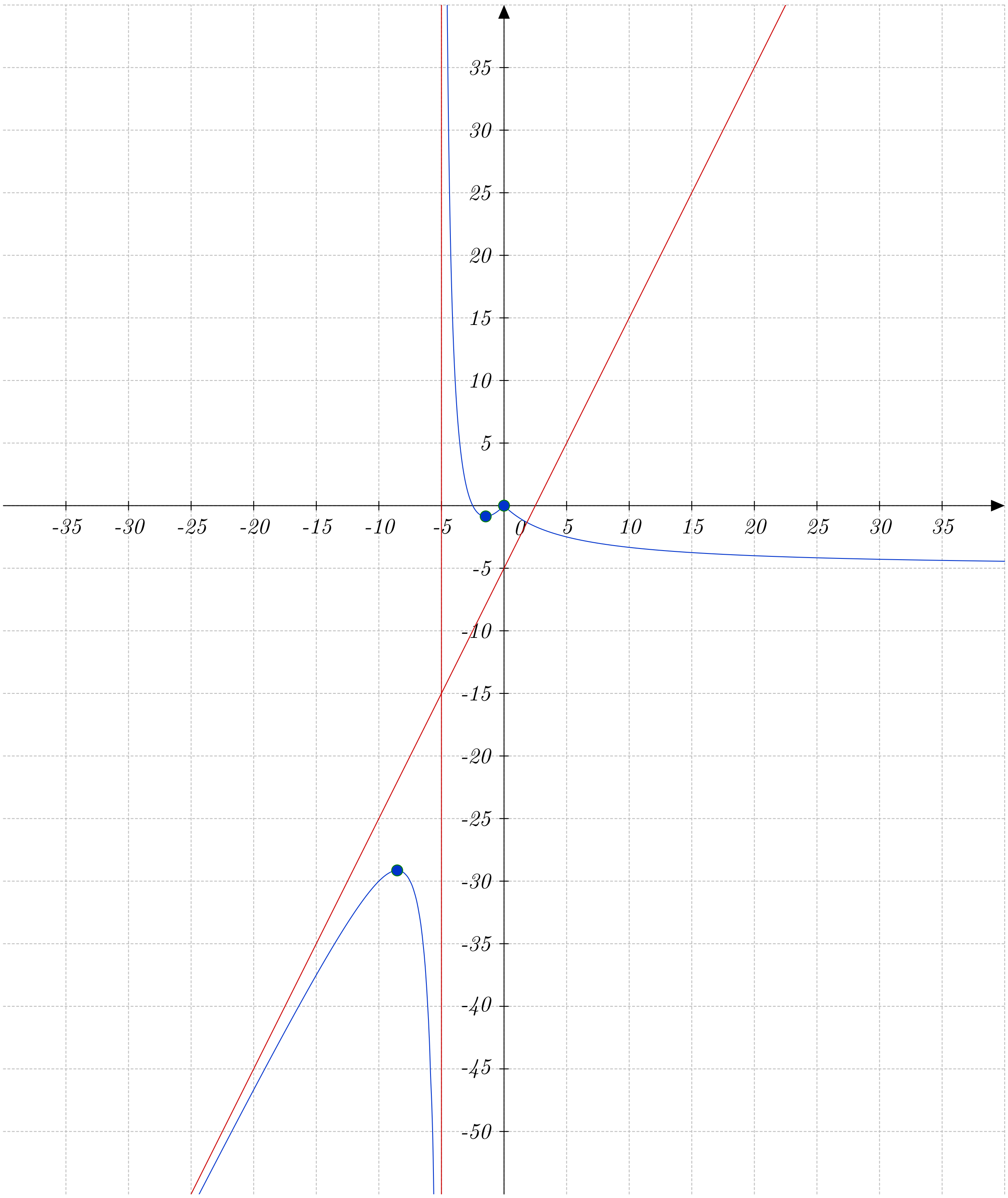
   Then, the concavity table of the curve of f is

x	$-\infty$		-5		0		$+\infty$
$(x+5)^3$		$-$	0	$+$		$+$	
$f''(x)$		$-$	$\|$	$+$	0	$+$	
Concavity		$\frown$	$\|$	$\smile$	0	$\smile$	

   We look for the inflection points which is in the domain among the points at which the second derivative of the function either is zero or the first derivative of f is not defined.

   Therefore is no inflection point.

10. *Graph the curve of f is*



35
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35

3 Inverse functions, Exponential and Logarithm

3.1 Inverse functions

Exercise 3.1. *Let f be defined by*

1. $f(x) = 2x^2 + 2$, $x \geq 0$ *and* $a = 4$,
2. $f(x) = x^3 + 1$ *and* $a = 2$.

(a) Find the inverse of f.

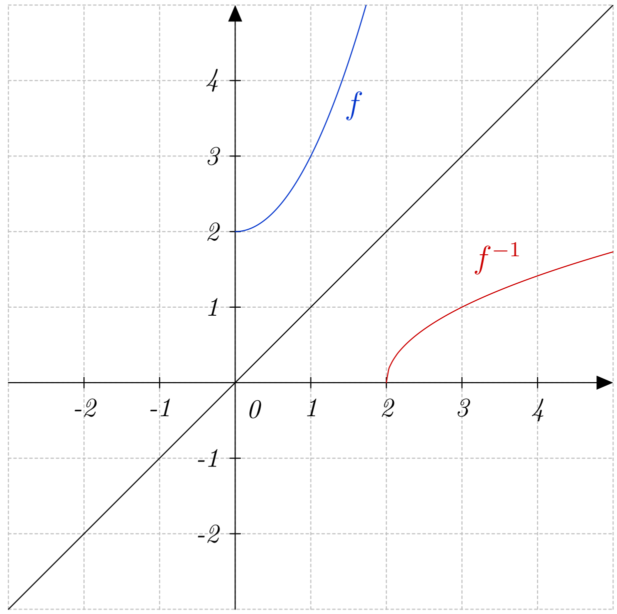
(b) Graph f and f^{-1} together.

(c) Evaluate $\dfrac{df}{dx}$ at $x = a$, and $\dfrac{df^{-1}}{dx}$ at $x = f(a)$ to show that at these points derivatives are reciprocal.

Solution.

1. (a)
$$f(x) = 2x^2 + 2,$$
Write y instead of $f(x)$:
$$y = 2x^2 + 2,$$
Solve for x in terms of y:
$$2x^2 = y - 2,$$
$$x = \sqrt{\frac{y-2}{2}}.$$
Interchange x and y:
$$y = \sqrt{\frac{x-2}{2}}.$$
Then the inverse of f is the function f^{-1} defined by $f^{-1}(x) = \sqrt{\dfrac{x-2}{2}}$.

   (b) *Graphs of f and f^{-1} are*



(c) $\dfrac{df(x)}{dx} = 4x$, then $\dfrac{df(x)}{dx}\Big|_{x=a} = 4a$.

$$\frac{df^{-1}(x)}{dx} = \frac{\frac{1}{2}}{2\sqrt{\frac{x-2}{2}}} = \frac{1}{4\sqrt{\frac{x-2}{2}}},$$

then $\dfrac{df^{-1}(x)}{dx}\Big|_{x=f(a)} = \dfrac{df^{-1}(x)}{dx}\Big|_{x=2a^2+2} = \dfrac{1}{4\sqrt{\dfrac{2a^2+2-2}{2}}} = \dfrac{1}{4a}$. from here, the derivatives are reciprocal at these points.

2.

(a)

$$f(x) = x^3 + 1,$$

Write y instead of $f(x)$:

$$y = x^3 + 1,$$

Solve for x in terms of y:

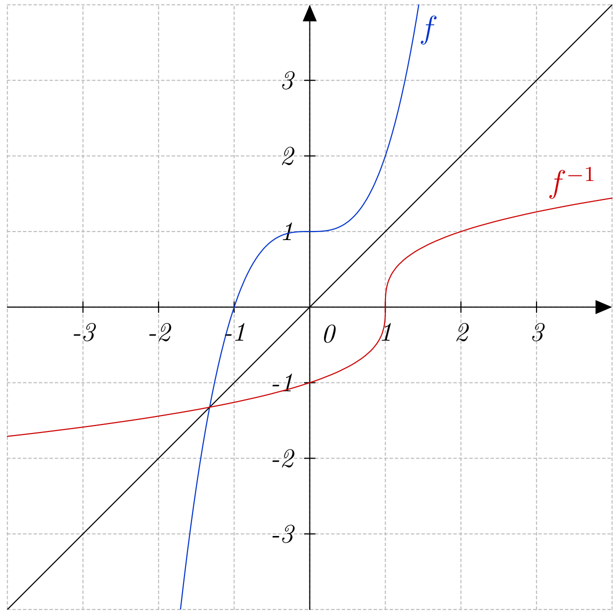
$$x^3 = y - 1,$$
$$x = \sqrt[3]{y-1}.$$

Interchange x and y:

$$y = \sqrt[3]{x-1}.$$

Then the inverse of f is the function f^{-1} defined by $f^{-1}(x) = \sqrt[3]{x-1}$.

(b) Graphs of f and f^{-1} are



(c) $\dfrac{df(x)}{dx} = 4x$, *then* $\dfrac{df(x)}{dx}\Big|_{x=a} = 4a.$

$$\frac{df^{-1}(x)}{dx} = \frac{\frac{1}{2}}{2\sqrt{\frac{x-2}{2}}} = \frac{1}{4\sqrt{\frac{x-2}{2}}},$$

then $\dfrac{df^{-1}(x)}{dx}\Big|_{x=f(a)} = \dfrac{df^{-1}(x)}{dx}\Big|_{x=2a^2+2} = \dfrac{1}{4\sqrt{\frac{2a^2+2-2}{2}}} = \dfrac{1}{4a}$. *from here, the derivatives are reciprocal at these points.*

Exercise 3.2. *Let f and g be defined by $f(x) = \dfrac{x^3}{4}$ and $g(x) = (4x)^{\frac{1}{3}}$ respectively.*

(a) Show that f and g are inverses of one another.

(b) Graph the curves of f and g to show the curves intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line of $y = x$.

(c) Find the slopes of the tangents to the graphs of f and g at $(2, 2)$ and $(-2, -2)$.

Solution.

(a) Since,

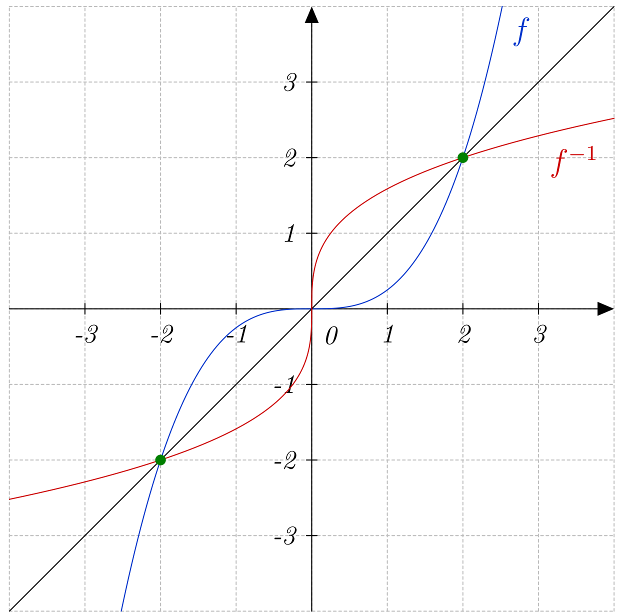
$$f(g(x)) = f((4x)^{\frac{1}{3}}) = \frac{((4x)^{\frac{1}{3}})^3}{4} = x$$

and

$$g(f(x)) = g\left(\frac{x^3}{4}\right) = \left(4 \cdot \frac{x^3}{4}\right)^{\frac{1}{3}} = x$$

f and g are inverses of one another.

(b) The curves of f and g



Exercise 3.3. *Let f be defined by $f(x) = x^2 - 4x - 4$, $x > 2$. Find the value of $\frac{f^{-1}}{dx}$ at $x = 1$.*

Solution. *Since*

$$f(x) = x^2 - 4x - 4 = 1,$$
$$x^2 - 4x - 5 = 0$$

if

$$x = -1 \quad \text{and} \quad x = 5,$$

$f(5) = 1$.

$$\frac{df(x)}{dx} = 2x - 4 \qquad \text{and} \qquad \frac{df(x)}{dx}\Big|_{x=5} = 2 \cdot 5 - 4 = 6.$$

From here,

$$\frac{df^{-1}(x)}{dx}\Big|_{x=1} \frac{1}{\frac{df(x)}{dx}\Big|_{x=5}} = \frac{1}{6}$$

.

3.2 Exponential and Logarithm

Exercise 3.4. *Using the derivative of the natural exponential function prove that it is one-to-one.*

(Note: Since the natural exponential function one-to-one it has its inverse-the natural logarithm function.)

Solution. *The natural exponential function is defined by* $f(x) = e^x$.

$$f'(x) = e^x$$

which is positive on $\mathbb{R}$. *Because of this* f *is increasing on* $\mathbb{R}$. *Since every increasing function is one-to-one,* f *is one-to-one on* $\mathbb{R}$.

Exercise 3.5. *Let* g *be defined by* $g(x) = \ln x$ *(the natural logarithm function). Prove that* $g'(x) = \dfrac{1}{x}$.

(Hint: Use the derivative of the natural exponential function.)

Solution.

$$g(x) = \ln x,$$

then,

$$e^{g(x)} = e^{\ln x} = x$$

We take the derivative of both side,

$$e^{g(x)} g'(x) = 1,$$

from here,

$$e^{\ln x} g'(x) = 1.$$

Therefore,

$$g'(x) = \frac{1}{x}.$$

Exercise 3.6. *Let a function* g *be defined by* $g(x) = a^x$. *Prove that the derivative of* g *is the function defined by* $g'(x) = a^x \ln a$.

Solution.

$$g(x) = a^x = e^{\ln a^x} = e^{x \ln a}.$$

Then,

$$g'(x) = e^{x \ln a} \ln a = a^x \ln a.$$

Exercise 3.7. *Let a function f be defined by $f(x) = \log_a x$. Prove that $f'(x) = \dfrac{1}{x \ln a}$.*

Solution.

$$f(x) = \frac{1}{x \ln a} = \frac{\ln x}{\ln a}.$$

Because of this,

$$f'(x) = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}.$$

Exercise 3.8. *Find the derivatives of the functions defined by*

1. $f(x) = \ln x^3$,
2. $f(x) = \sqrt{\ln(x^2 + 1)}$,
3. $f(x) = \log_2(x^3 + 2x)$,
4. $f(x) = 2a^{\sin x}$.

Solution.

1. (a) *The first method: f is the composite of the functions g and h that are defined by $g(u) = \ln u$ and $u = h(x) = x^3$. Therefore*

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{dh}{dx}.$$

Then,

$$\frac{dg}{du} = \frac{1}{u},$$
$$\frac{dh}{dx} = 3x^2.$$

Hence,

$$\frac{df}{dx} = \frac{1}{u} \cdot 3x^2 = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}$$

(b) *The second method:*

$$f'(x) = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}.$$

2. (a) *The first method: f is the composite of the functions g, h and l: $f(x) = g(h(l(x)))$ that are defined by $g(u) = \sqrt{u}$, $u = h(v) = \ln v$ and $v = l(x) = x^2 + 1$. Therefore*

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{dh}{dv} \cdot \frac{dl}{dx}.$$

Then,

$$\frac{dg}{du} = \frac{1}{2\sqrt{u}},$$
$$\frac{dh}{dv} = \frac{1}{v},$$
$$\frac{dl}{dx} = 2x$$

Hence,

$$\frac{df}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{1}{v} \cdot 2x = \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot 2x = \frac{x}{(x^2+1)\sqrt{\ln(x^2+1)}}.$$

(b) *The second method:*

$$f'(x) = \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot 2x = \frac{x}{(x^2+1)\sqrt{\ln(x^2+1)}}.$$

(a) *The first method: f is the composite of the functions g and h which are defined by $g(u) = \log_2 u$ and $u = h(x) = (x^3 + 2x)\ln 2$. Therefore*

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{dh}{dx}.$$

Then,

$$\frac{dg}{du} = \frac{1}{u \ln 2},$$
$$\frac{dh}{dv} = 3x^2 + 2.$$

Hence,

$$\frac{df}{dx} = \frac{1}{u\ln 2} \cdot (3x^2+2) = \frac{1}{(x^3+2x)\ln 2} \cdot (3x^2+2) = \frac{3x^2+2}{(x^3+2x)\ln 2}.$$

(b) *The second method:*

$$f'(x) = \frac{1}{(x^3+2x)\ln 2} \cdot (3x^2+2) = \frac{3x^2+2}{(x^3+2x)\ln 2}.$$

3. (a) *The first method: f is the composite of the functions g and h which are defined by $g(u) = 2a^u$ and $u = h(x) = \sin x$. Therefore*

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{dh}{dx}.$$

Then,

$$\frac{dg}{du} = 2a^u \ln a,$$
$$\frac{dh}{dv} = \cos x.$$

Hence,

$$\frac{df}{dx} = 2a^u \ln a \cdot \cos x = 2a^{\sin x} \ln a \cdot \cos x.$$

(b) *The second method:*

$$f'(x) = 2a^{\sin x} \ln a \cdot \cos x.$$

Exercise 3.9. *Find the derivatives of the functions defined by*

1. $f(x) = x^{x+1}$,
2. $f(x) = (\sin x)^x$,
3. $f(x) = x^{\sin x}$.

Solution.

1.

$$f(x) = x^{x+1},$$

Then,

$$\ln f(x) = \ln x^{x+1} = (x+1)\ln x,$$

Therefore,

$$\frac{1}{f(x)} f'(x) = \ln x + \frac{x+1}{x},$$
$$f'(x) = x^{x+1}\Big(\ln x + \frac{x+1}{x}\Big).$$

2.

$$f(x) = (\sin x)^x = e^{\ln{(\sin x)^x}} = e^{x \ln(\sin x)}.$$

Then, the derivative of f is

$$f'(x) = e^{x\ln(\sin x)}\Big(\ln(\sin x) + x \cdot \frac{\cos x}{sinx}\Big) = (\sin x)^x\Big(\ln(\sin x) + x \cdot \cot x\Big).$$

3.

$$f(x) = x^{\sin x},$$

then,

$$\ln f(x) = \sin x \ln x.$$

We take the derivatives of both sides,

$$\frac{1}{f(x)} f'(x) = \cos x \ln x + \frac{\sin x}{x}.$$

Therefore,

$$f'(x) = x^{\sin x}\Big(\cos x \ln x + \frac{\sin x}{x}\Big).$$

Exercise 3.10. *Use logarithmic differentiation to find the derivative of f with respect to the given independent variable. If*

1. $f(x) = \sqrt{x(x-1)}$,
2. $f(t) = \sqrt{\dfrac{t}{t+2}}$,

Solution.

1.

$$\ln f(x) = \ln\sqrt{x(x-1)} = \frac{1}{2}\ln x(x-1) = \frac{1}{2}\Big(\ln x + \ln(x-1)\Big).$$

Hence,

$$\ln f(x) = \frac{1}{2}\Big(\ln x + \ln(x-1)\Big).$$

Then, we take the derivatives of both sides,

$$\frac{1}{f(x)} f'(x) = \frac{1}{2}\Big(\frac{1}{x} + \frac{1}{x-1}\Big),$$

Finally,

$$f'(x) = \frac{(2x-1)\sqrt{x(x-1)}}{2x^2-2x}$$

2.

$$f(t)=\sqrt{\frac{t}{t+2}}.$$

Then,

$$\ln f(t)=\frac{1}{2}\cdot\frac{t}{t+2}=\frac{1}{2}\Big(\ln t-\ln(t+2)\Big).$$

Then,

$$\frac{1}{f(x)}f'(t)=\frac{1}{2}\Big(\frac{1}{t}-\frac{1}{t+2}\Big).$$

Finally,

$$f'(t)=\frac{1}{2}\sqrt{\frac{t}{t+2}}\Big(\frac{1}{t}-\frac{1}{t+2}\Big)=\sqrt{\frac{t}{t+2}}\cdot\frac{1}{t^2+2t}.$$