# Continuity

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BHOS

Calculus

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**Theorem (Intermediate Value Theorem (IVT) )**

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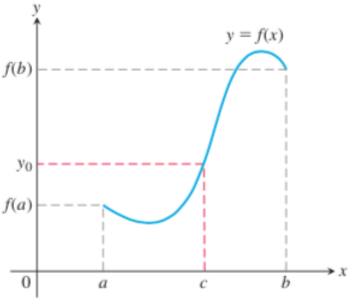

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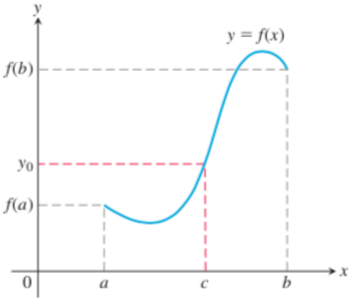


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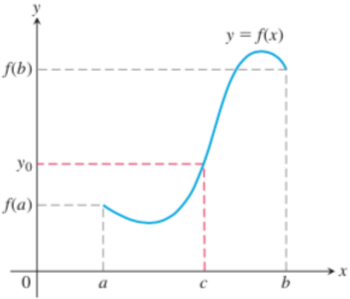


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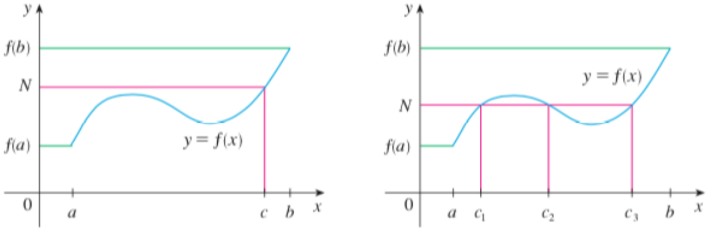
Note that the value $N$ can be taken on once or more than once




(a) (b)

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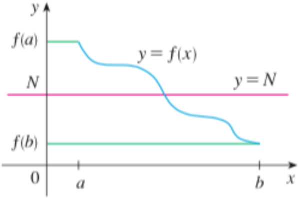

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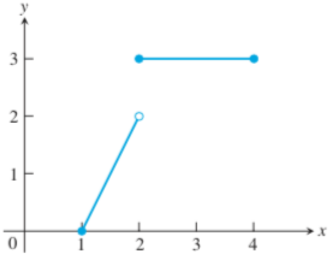


$f$ does not take any value between 2 and 3.

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**Example:** Show that there is a root of the equation

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In other words, $4x^3 - 3x^2 + 2x - 1 = 0$ has at least one root $c$ in (0,1).

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$F(x)$ is continuous at $x = 0$ because $\lim_{x\to 0} F(x) = F(0)$.

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The function $F(x)$ is continuous at $x = c$ and is called **continuous extension** of $f$.

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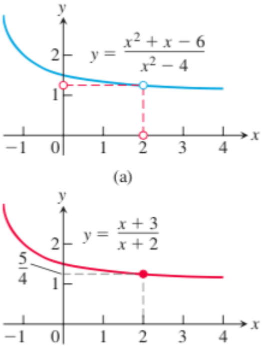
Thus, $F$ is continuous extension of $f$ to $x = 2$.



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Thus, $F$ is continuous extension of $f$ to $x = 2$.