Fall 2021 - MATH 1101 Discrete Structures
Lecture 1

- **Part 1. Introduction to the Course.**
- **Part 2. Sets, Functions, Cartesian Products, Multivariable Functions.**
- **Part 3. Set Operations. Algebra of Sets. Duality.**
- **Part 4. Equivalence of Sets. The Power of a Set.** (Mandatory for Math students)

Exercises. Set 1 (Solved Problems)

Exercises. Set 2 (Supplementary Problems)

PART 1. Introduction to the Course.

Discrete Structures arise both within mathematics itself and in its applications such as: computer sciences; electronics and electricity; all kinds of devices that form the basis for the functioning of computing systems; information converters; Internet of things; information coding and security issues and so on. Almost any research or application in information systems and technologies uses, by some way, discrete mathematical structures.

Examples of discrete structures are various finite algebraic systems, such as finite groups, finite graphs, Boolean algebras in various interpretations; some mathematical models of information converters; finite state machines, Turing machines, and so on.

The branch of mathematics that studies discrete structures is called **Discrete Mathematics**. Therefore, another equivalent name for our course is Discrete Mathematics.

So, **discrete mathematics is a part of mathematics devoted to the study of discrete objects.**

There is a huge variety of problems solved using discrete mathematics, for instance:

- How many ways are there to choose a valid password on a computer system?
- Is there a link between two computers in a network?
- How to detect spam in email messages?
- How to encrypt a message so that a random recipient cannot read it?
- What is the shortest path between two cities using a transportation system?
- How can a list of integers be sorted so that the integers are in increasing order?
- How many steps are required to do such a sorting?
- How can it be proved that a sorting algorithm correctly sorts a list?
- How many valid Internet addresses are there?
- etc.

In the proposed course, we will study the discrete structures and methods necessary to solve the above and many other problems.

More generally, discrete mathematics is used whenever objects are counted, when relationships between finite (or countable) sets are studied, and when processes involving a finite number of steps are analyzed. The main reason for the growing importance of discrete mathematics is that information is stored and processed by computers in a discrete fashion.

What the study of discrete structures (discrete mathematics) yields?

1. Through this course, you can develop your mathematical maturity, that is, the ability to understand and create mathematical arguments and reasoning. Without these skills, we cannot advance in the study of mathematical sciences.
2. Discrete mathematics is the gateway to more advanced mathematical courses, in particular: mathematical logic, set theory, number theory, advanced chapters of linear algebra, abstract algebra, combinatorics, graph theory, probability theory (the discrete part of the subject), etc. Discrete mathematics provides the mathematical background for most computer science courses, including data structures, algorithms, databases, automata theory and formal languages, compiler theory, computer security, and operating systems. It is difficult to master such courses without having the appropriate mathematical foundations from discrete mathematics.
3. Discrete mathematics contains the necessary mathematical background for solving problems in operations research (including many discrete optimization methods), chemistry, engineering, biology, and so on.

One of the primary goals of this course is to teach mathematical reasoning and problem solving. The exercises in this lecture notes are intended to reflect this goal. The material discussed in the lecture text provides the tools needed to solve these exercises, but the student's job is to successfully apply these tools using your own creativity.

Another goal of the course is to learn to analyze and develop approaches to solving problems. Unfortunately, learning how to solve only certain types of exercises is not enough to succeed in developing problem-solving skills needed in subsequent courses and professional work. *One of the objectives of the course is to help develop the skills necessary to master the additional material that will be needed in future courses.*

PART 2. Sets, Functions, Cartesian Products, Multivariable Functions.

Sets: definitions and basic terminology.

In Part 2, we study the fundamental discrete structure on which all other discrete structures are built, namely, the set. Sets are used to group objects together. Often, but not always, the objects in a set have similar properties. Indeed, our considerations in Part 2 are a brief introduction to the well-known terminology and definitions of naïve set theory to provide students with minimal tuple of tools and concepts.

We now provide a definition of a set. This definition is an intuitive one, which is not part of a formal (axiomatic) theory of sets. However, it will be enough for the purposes of our course.

Definition 1. A *set* is an unordered collection of objects, called *elements* (or *members*) of the set. A set is said to *contain* its elements. Synonyms for "set" are "class", "collection", and "family". We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A. ■

Specifying Sets

There are essentially two ways to specify a set. One way is to list, if possible, its members separated by commas and contained in braces { }. Another way to describe a set is to use **set builder** notation which is the vertical line | and is read as "such that". We characterize all those elements in the set by stating the property or properties they must have to be members. And write down a property after set builder, |, as P(x), which means that element x has a property P. Examples illustrating these two ways are:

$A=\{2, 3, -1, 0, 15\}$ and

$B=\{x \mid x$ is an odd integer, $x > 0\}$ or B$==\{x \mid P(x)\}$,

here P means the property to be odd integer and positive one.

That is, A consists of the numbers 2, 3, -1, 0, 15. The second set, which reads: B is the set of x such that (**builder** notation=vertical line) x is an odd integer **and** x is greater than 0, denotes the set B whose elements are the positive odd integers. Note that a letter, usually x, is used to denote a typical member of the set; and the comma after builder notation is read as "and".

EXAMPLE 1.

(a). We cannot list all the elements of the above set B although frequently we specify the set by $B=\{1, 3, 5, \ldots\}$ where we assume that everyone knows what we mean. Observe that $1\in B$, but $4\notin B$.

(b). Let $E=\{x \mid x^2-3x+2=0\}$, $F=\{2, 1\}$ and $G=\{1, 2, 2, 1\}$. Then $E=F=G$. ■

We emphasize that a set does not depend on the way in which its elements are displayed. *A set remains the same if its elements are repeated or rearranged.*

Even if we can list the elements of a set, it may not be practical to do so. That is, we describe a set by listing its elements only if the set contains a few elements; otherwise we describe a set by the property which characterizes its elements.

Naive Set Theory

Note that the term *object* has been used in the definition of a set, Definition 1, without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated in 1895 by the German mathematician Georg Cantor. The theory that follows from this intuitive definition of a set (Definition 1), and the use of the intuitive idea that for any property there is a set consisting of objects with this property, leads to paradoxes or logical inconsistencies. This was shown by the English philosopher Bertrand Russell in 1902. These logical inconsistencies can be avoided by building set theory beginning with axioms. However, in our lecture notes we will use Cantor's original version of set theory, known as **naive set theory**, because all sets considered in here can be treated consistently using Cantor's original theory.

Subsets

Suppose every element in a set A is also an element of a set B, that is, suppose $a\in A$ implies $a\in B$. Then A is called a ***subset*** of B. We also say that A is ***contained*** in B or that B ***contains*** A. This relationship is written $A\subseteq B$ or $B\supseteq A$.

Two sets are equal if they consist of the same elements or, equivalently, if each of them is contained in the other. Clear that if $\boldsymbol{A=B}$ **then** $\boldsymbol{A\subseteq B}$ **and** $\boldsymbol{B\subseteq A}$. Inverse statement is also true (see Theorem 1 (ii)).

If A is not a subset of B, that is, if at least one element of A does not belong to B, we write $A\not\subset B$.

Property 1: The statement $A\subseteq B$ does not exclude the possibility that $A=B$. In fact, for every set A we have $A\subseteq A$ since, trivially, every element in A belongs to A. However, if $A\subseteq B$ and $A\neq B$, then we say A is ***a proper subset*** of B (sometimes written $A\subset B$).

Property 2: Suppose every element of a set A belongs to a set B and every element of B belongs to C, then $A\subseteq C$.

The above remarks yield the following theorem.

Theorem 1. Let *A*, *B*, *C* be any sets. Then:

(i) $A \subseteq A$

(ii) If $A \subseteq B$ and $B \subseteq A$, then $A=B$

(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Proof. Clear. ■

Special symbols

Some sets will occur frequently in our notes, and so we use special symbols for them. Some such symbols are:

$\mathbf{Z}_+$ **(or N)** = the set of *natural numbers* or positive integers: 1, 2, 3, …

Z = the set of all integers: ..., −2, −1, 0, 1, 2, ...

Q = the set of all rational numbers (= irreducible fractions p/q, where p∈**Z**, q∈**N)**. Equivalently, Q is the set of all finite decimal fractions or infinite periodic decimal fractions.

R = the set of real numbers

C = the set of complex numbers

Below we bring a light to the structure some of them.

About Q. We say that Q is the set of irreducible fractions p/q, where p∈**Z**, q∈**N** to avoid repetitions. For instance, under such definition we choose the irreducible fraction $\frac{1}{2}$ (p=1, q=2) as rational number which represents the infinite family consisting of the equal numbers $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots\}$. Similarly, $(-\frac{1}{2})$ represents the infinite family consisting of equal numbers $\{\frac{-1}{2}, \frac{-2}{4}, \frac{-3}{6}, \ldots\}$. By the way, all fractions of the first family have a unique decimal representation, namely, 0.5. Regarding the second family, all elements are equal to the (-0.5).

On the other hand, the irreducible fraction $\frac{2}{3}$ which represents the family $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\right\}$ can be written uniquely as infinite but periodic decimal fraction 0,666… (period equal 6).

Starting with finite decimal fraction we can find uniquely defined irreducible fraction, for instance, 0.28=$\frac{28}{100} = \frac{7}{25}$, that is, irreducible fraction $\frac{7}{25}$ represents the same rational number as 0.25. Regarding infinite but periodic decimal numbers. For instance, 06777…=0.6(7)=

=$\frac{6}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots$

Starting from the element 7/100 the infinite sum (S) is the sum of geometric series with ratio r=(1/10)<1. From secondary school we know that S=$\frac{a_1}{1-r}$, here a_1=(7/100).

Hence 0.6(7)= $\frac{6}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots =) = \frac{6}{10} + \frac{\frac{7}{100}}{1-\frac{1}{10}} = \frac{6}{10} + \frac{7}{90} = \frac{61}{90}$.

The last number (61/90) is the irreducible fraction which represents the same rational number as 0.6(7).

About R-Q. From the secondary school program, you know that there are numbers which are not rational. For instance, $\sqrt{2}, \sqrt{3}, \sqrt{5}$ etc. Such numbers are called irrational numbers. An irrational number cannot be written as a fraction or finite (or infinite but periodic) decimal number. Any irrational number can be written as **infinite nonperiodic decimal fraction**. Set of all irrational numbers we denote as **IR.** So, we have:

Q – set of all rational numbers = set of all finite or infinite but periodic decimal fractions;

IR – set of all irrational numbers = set of all infinite NONperiodic decimal fractions.

From definition we have Q

We the set **IR=R-Q≠∅.**

About C (complex numbers). We specify C as a set of all ordered pairs (a, b) of real numbers (geometrically set of all points in the plane) with standard rule of addition operation:

$$(a, b)+(c, d)=(a+c, b+d)$$

and surprising multiplication operation:

$$(a, b)(c, d)=(ac-bd, ad+bc)$$

Under such definition of C and operations it is easy to observe that each real number can be viewed as a special case of complex number. Really, identify a real number a with pair $(a, 0)\in C$. Then we have

$a+b=(a, 0)+(b, 0)=(a+b, 0)=a+b.$

$(a, 0)(c, 0)=(ac-0, a0-0d)=(ac, 0)=ac$

So, considering real numbers as a particular case of complex numbers we observe that operations over real numbers come from operations over complex numbers (or, in other terminology, real numbers, as a particular case of complex numbers, are closed with respect to addition and multiplication of complex numbers). It is a reason why we say that $\mathbf{R}\subset\mathbf{C}$.

Finally, we have:

$$\mathbf{Z}_+\subset\mathbf{Z}\subset\mathbf{Q}\subset\mathbf{R}\subset\mathbf{C}.$$

Universal Set, Empty Set

All sets under investigation in any application of set theory are assumed to belong to some fixed large set called the ***universal set*** which we denote by **U** unless otherwise stated or implied. Given a universal set **U** and a property P, it may turn out that no element from **U** satisfies property P. For example, let **U**=**N** then the following set S has no elements:

$S=\{x \mid x\in\mathbf{U},\ 2x=5\}=\{x \mid P(x)\}$,

here P(x) is the property to be a positive integer satisfying the equation $2x=5$.

Such **a set with no elements** is called the ***empty set*** or ***null set*** and is denoted by ∅.

There is only one empty set. That is, if S and T are both empty, then $S=T$, since they have the same elements, namely, none.

The empty set ∅ is also regarded as a subset of every other set. Thus, we have the following simple result which we state formally.

Theorem 2. For any set A, we have $\emptyset\subseteq A\subseteq\mathbf{U}$. ■

A set with one element is called a **singleton**. A common error is to confuse the empty set ∅ with the set {∅}, which is a singleton. The single element of the set {∅} is the empty set itself! A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Disjoint Sets

Two sets A and B are said to be ***disjoint*** if they have no elements in common. For example, suppose $A = \{1, 2\}$, $B= \{4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$. Then A and B are disjoint, and A and C are disjoint. But B and C are not disjoint since they have elements in common, e.g., 5 and 6. We note that if A and B are disjoint, then neither is a subset of the other (unless one is the empty set).

Classes of Sets

Given a set S, we might wish to talk about some of its subsets. Thus, we would be considering a *set of sets*. Whenever such a situation occurs, to avoid confusion, we will talk of a *class* of sets or *collection* of sets rather than a *set* of sets. If we wish to consider some of the sets in a given class of sets, then we talk of *subclass* or *subcollection*.

EXAMPLE 2. Suppose $S=\{1, 2, 3, 4\}$.

- Let A be the class of subsets of S which contain exactly three elements of S. Then
  $$A=\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$
  That is, the elements of A are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$. ■
- Let B be the class of subsets of S, each of which contains 2 and two other elements of S.
  Then $B=\{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}$
  The elements of B are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$. Thus, B is a subclass of A, since every element of B is also an element of A. ■

Finite and Infinite Sets.

The set of all vertices of a given polyhedron, the set of all prime numbers less than a given number, and the set of all residents of Ganja City (at a given time) have a certain property in common, namely, each set has a definite number of elements which can be found *in principle*, if not in practice. Accordingly, these sets are all said to be ***finite***. Clearly, we can be sure that a set is finite without knowing the number of elements in it.

On the other hand, the set of all positive integers, the set of all points on the line, and the set of all circles in the plane have a different property in common, namely, if we remove one element from each set, then remove two elements, three elements, and so on, there will still be elements left in the set at each stage. Accordingly, sets of this kind are said to be ***infnite***.

Given two finite sets, we can always decide whether or not they have the same number of elements, and if not, we can always determine which set has more elements than the other. It is natural to ask whether the same is true of infinite sets. In other words, does it make sense to ask, for example, whether there are more circles in the plane than rational points on the line, or more functions defined in the interval [0, 1] than lines in space? As will soon be apparent, questions of this kind can indeed be answered (see Part 4).

Power Sets (Case of Finite sets).

Given set S, we may introduce a class of **all subsets of *S***. This class, denoted by P(S), is called the *power set* of S. Note that the empty set ∅ and S itself belong to P(S) because both ∅ and S are subsets of S.

For a **finite** set S, let n(S) denotes number of elements in S.

Table below shows the pattern in the number of elements of power set, n(P(S)), related to the number of elements of the set S itself, n(S).

S	P(S)	n(S)	n(P(S))
$\emptyset$	$\{\emptyset\}$	0	1
$\{a\}$	$\{\emptyset, \{a\}\}$	1	2
$\{a, b\}$	$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$	2	4

{a, b, c}	**{**∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}**}**	3	8
…	…	…	…
$\{a_1, a_2, \ldots, a_k\}$	**{**∅, $\{a_1\}, \{a_2\},\ldots, \{a_k\}$, $\{(a_i, a_j) \mid i, j=1, 2, \ldots n;\ i<j\}$, $\{(a_i, a_j, a_k, \mid i, j, k=1, 2, \ldots, n;\ i<j<k\}, \ldots$ **}**	k	2^k

Statement 1. If *S* is finite, then so is P(S) and $n(P(S))=2^{n(S)}$

For this reason, the power set of S is sometimes denoted by 2^S, instead of P(S).

Proof. Two distinct proofs of the statement will be suggested later in this course (by Mathematical Induction and by combinatorial method).

Functions

Considering sets we face two questions:

1. How to establish a relationship between them.
2. How to formalize "equivalence" of sets.

Relationship between sets.

Processes in nature, in everyday life, in biological, physical, and other systems describe certain relationships between sets of inputs and outputs.
In mathematical language such relationships are called a functional relationship or simply functions.
A technical definition of a function is: *a relation from a set of inputs to a set of possible outputs where each input is related to exactly one output.*

Definition 2. Let X and Y be sets. A *function (mapping) from* X *to* Y is a *rule, (*say *f)* that assigns to every element $x \in X$ exactly one element $y \in Y$. The set X is called the *domain* (or set of inputs) of the function, and the set Y is called the *codomain* (or set of outputs). ■

The notion of a function is easily understood using the metaphor of a function machine that takes in an object for its input and based on that input, spits out another object as its output.

Denotations and terminology.

Functions are ordinarily denoted by symbols. For a function, say *f*, from X to Y we write *f*: X→Y which is read: "*f* is a function from X to Y," or "*f* takes (or maps) X to Y."

If *x*∈X, then *f*(*x*) (read: "*f* of *x*") denotes the **unique** element of Y which *f* assigns to *x*; it is called the *image* of *x* under *f*, or the *value* of *f* at *x*. The set X is also called *domain* of function *f*. The set of all image values (for all *x*∈X) is called the *range* or *image* of *f*. The image of *f* : X →Y is denoted by Range(*f*), Im(*f*) or *f*(*X*).

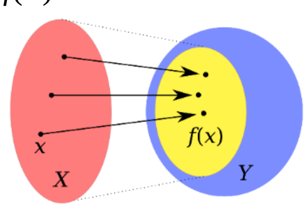


Figure 1. Yellow region is the range (or image) of the domain X (red region) under *f*.

For every $y \in Y$ the set $\{x \in X \mid f(x)=y\}$, denoted as $f^{-1}(y)$, that is, $f^{-1}(y)=\{x \in X \mid f(x)=y\}$, is called *preimage* of y. Clear that $f^{-1}(y)=\emptyset$ if and only if $y \notin \mathrm{Im}(f)$.

The function $id_X: X \to X$ defined by formula $id_X(x)=x$ is called *identity* function on X.

If $f: X \to Y$ *and* $g: Y \to Z$ *are arbitrary functions then the function* $(gf): X \to Z$ defined by $(gf)(x)=g(f(x))$ is called *composite* or *superposition* of f and g.

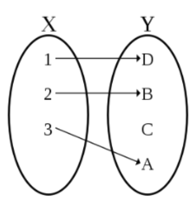
A function $f: X \to Y$ is called *mono or injective* function if it maps distinct elements to distinct elements; that is, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ or, equivalently, (logically is called contrapositive) $f(x_1)=f(x_2)$ implies $x_1=x_2$, or, equivalently, f is injective if for any $y \in Y$ the *preimage* $f^{-1}(y)$ contains at most one element.

For an injective function $f: X \to Y$ define the function $f^{-1}: f(X) \to X$ (here $f(X) \subseteq Y$) which is called *inverse* of f, as following: for every $y \in f(X)$ there exist *only one* x, such that $f(x)=y$. We define $f^{-1}(y)=x$. It is easy to see that $(f^{-1}f)=id_X: X \to X$ and $(ff^{-1})=id_{f(X)}: f(X) \to f(X)$.

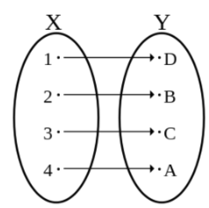
If $f(X)=Y$ then f is called *onto or surjective* function. In other words, the preimage $f^{-1}(y)$ of every $y \in Y$ is nonempty.

The function f is called a *bijective* function (or *bijection*, or *one-to-one correspondence*) if it is *injective and surjective* at the same time. That is, $f: X \to Y$ is bijective if any $y \in Y$ **the preimage $f^{-1}(y)$ contains exactly one element**.

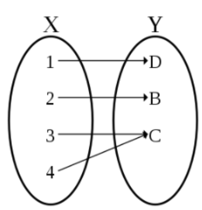
Figure 2. Different types of functions.



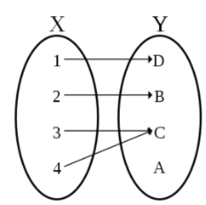
(a) An injective non-surjective function (injection, not a bijection)



(b) An injective surjective function (bijection)



(c) A non-injective surjective function (surjection, not a bijection)



(d) A non-injective non-surjective function (also not a bijection)

Equivalences of sets.

Definition 3. Two sets X and Y are called **equivalent**, $X \approx Y$, if there is a **bijection** $f: X \to Y$. ■

- Clear that $X \approx X$ ($id_X: X \to X$ is the bijection). Such a property is called **reflexivity.**
- If $f: X \to Y$ is a bijection then $f^{-1}: Y \to X$ is also the bijection, that is, if $X \approx Y$, then $Y \approx X$. (**symmetry** property).
- If $X \approx Y$ ($f: X \to Y$ is a bijection) and $Y \approx Z$ ($g: Y \to Z$ is a bijection) then $X \approx Z$ (composition $gf: X \to Z$ is a bijection) – **transitivity** property.

EXAMPLE 3. Let X and Y consist of five chairs and five students, respectively. Then in the set theory X and Y are considered as equivalent sets $X \approx Y$, because we can obviously build up a bijective mapping from X to Y.

EXAMPLE 4. Let X be the open interval, $X=(-\pi/2, \pi/2)$, and $Y=\mathbf{R}$. Define $f : X \to Y$ as $f(x)=\tan(x)$. Clear that f is a **bijection** (this fact is known from the high school program), because for each $a \in \mathbf{R}$ the horizontal line y=a intersects the curve y=tan(x) exactly in one point, that is, for each $a \in \mathbf{R}$ the preimage $f^{-1}(y)$ contains exactly one element.

This example is a surprising one because it establishes equivalences (the same "number" of elements) between infinite set $Y=\mathbf{R}$ and its proper subset $X=(-\pi/2, \pi/2)$. This unexpectable result will be discussed later in Part 4 (additional readings) as a special property of "infinite" sets. ■

EXAMPLE 5. Let f: $\mathbf{R} \to \mathbf{R}$, $f(x)=\sin x$. Clear that f is not injective ($0 \neq \pi$ but $\sin 0=\sin\pi$). f is not onto because $f^{-1}(2)$ is empty. ■

EXAMPLE 6. Let $f(x)=\sin x$ is considered as a mapping: $\mathbf{R} \to [-1, 1]$. Then f is onto. It means that property of a function $f: X \to Y$ to be "onto" properly depends on Y. ■

In mathematics, mostly in logic and different branches of discrete mathematics, a special kind of functions which are called propositional functions are frequently used for many reasons. A propositional function P with domain X is defined as rule P which assigns to each $x \in X$ some statement, say P(x), which is true or false but not both.

We will discuss this type of functions in details later in Lecture Notes dedicated to Logic.

It is easy to check that a function $f : X \to Y$ is bijective if and only if it admits an inverse function, that is, a function $g: Y \to X$ such that $gf=id_X$: and $fg=id_Y$.

Cartesian Product

The Cartesian product $X_1 \times \ldots \times X_n$ o f sets $X_1, \ldots, X_n$ is the set of all n-tuples $(x_1, \ldots, x_n)$ such that $x_i \in X_i$ for every i with $1 \leq i \leq n$. That is:

$$X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i, 1 \leq i \leq n\}.$$

EXAMPLE 7.

- Let X={1, 2, 3}, Y={a, b}. Then X×Y={(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}. ■
- Let X=[0, 1], Y=[0,1], here [0, 1] is the unit closed interval on the set **R** of real numbers, , that is, $[0, 1]=\{a \in \mathbf{R} \mid 0 \leq a \leq 1\}$. Then X×Y=[0, 1]×[0, 1] geometrically is the square S on the plane $\mathbf{R}^2$. $S=\{(a, b) \in \mathbf{R}^2 \mid 0 \leq a \leq 1, 0 \leq b \leq 1\}$. ■
- Let X=[0, 1], Y=[-1, 5], Z=[-2, 3]. Then X×Y×Z=[0, 1]×[-1, 5]×[-2, 3] geometrically is the rectangular box S in $\mathbf{R}^3$, $S=\{(a, b, c) \in \mathbf{R}^2 \mid 0 \leq a \leq 1, -1 \leq b \leq 5, -2 \leq c \leq 3\}$. ■

Any subset of the Cartesian product X×Y is called *a relation* from X to Y. We will discuss relations later in details. Now we provide an equivalent definition of a function as a special case of a relation. **A relation (subset) $f \subset X \times Y$ is called *a function from X to Y* if for each $x \in X$ there exist exactly one element $y \in Y$ of the form $(x, y) \in f$.** Frequently we use the form $y=f(x)$ instead of $(x, y) \in f$. Sometimes we use another terminology for a function defined as a relation (=subset of X×Y), namely, instead of function we use "graph of function", denoted as *grf*, to denote the same subset: $grf=\{(x, y) \in X \times Y \mid y=f(x)\}$.

Multivariable functions and operations.

A function from Definition 2 is also called as a single-variable function. Any function $F: X_1 \times \ldots \times X_n \rightarrow Y$ is called *multivariable function* or *function of n variables,* or *n-variable function*. Let $X_1 = \ldots = X_n = X$. Then any function $F: X \times \ldots \times X \rightarrow X$ is called *n-ary operation on X*. For n=1, 2, 3 operation is called unary, binary, and ternary, respectively.

If $f: X_1 \times \ldots \times X_n \rightarrow Y$ is a function of *n* variables then "graph of function", denoted as *grf*, is defined as a following subset of $X_1 \times \ldots \times X_n \times Y$

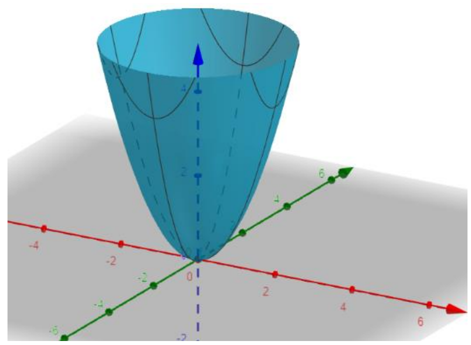
$$grf = \{(x_1, x_2, \ldots, x_n, y) \in X_1 \times \ldots \times X_n \times Y \mid y = f(x_1, x_2, \ldots, x_n,)\}.$$

Below we provide simple examples of multivariable functions.

EXAMPLE.

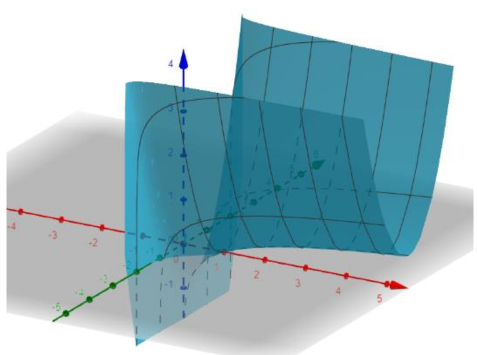
- Consider the function f: $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ given by formula $f(x, y) = x^2 + y^2$.

grf is the blue paraboloid in $\mathbf{R}^3$



- Let f: $(\mathbf{0}, \infty) \times \mathbf{R} \rightarrow \mathbf{R}$ is defined as $f(x, y) = \log(x) + y^2$.

Then *grf* is the presented below



EXAMPLE 8.

Below we provide an example of function which will be intensively used as a tool in each exam.

Let *a* be an integer, ($a \in \mathbf{Z}$) and $\boldsymbol{m}$ is a positive integer ($\boldsymbol{m} \in \mathbf{N}$). From high school algebra it is known (Euclidean algorithm) that there exist **uniquely defined** integers *q* and *r*, $0 \le r < \boldsymbol{m}$ such that

$$a = q\boldsymbol{m} + r \qquad (\#_1)$$

The number *q* is called the *quotient* while *r* is called the *remainder*.

For instance,

If a=13, $\boldsymbol{m}$=3, then q=4, r=1

If a=-13, $\boldsymbol{m}$=3, then q=-5, r=2

We say ***m* divides** *a* if r=0.

Given positive integer $\boldsymbol{m}$ define a binary function F: $\mathbf{Z}\times\mathbf{Z}\rightarrow\{0, 1, 2, \ldots, \boldsymbol{m}\mathbf{-1}\}$ as follows. We know that for any pair (s, t)$\in\mathbf{Z}\times\mathbf{Z}$ there exist **uniquely defined** integers q and r, $0\leq r<\boldsymbol{m}$ such that,

$$s-t=q\boldsymbol{m}+r \qquad (\#_2)$$

(here (s-t) plays the role of the integer *a* from formula ($\#_1$)).

Set F(s, t)=*r*.

Definition 4. *Integers s and t are called equal by* **mod *m***, denotation s≡t **mod *m***, if r=0 in ($\#_2$) or equivalently, (s-t)=$q\boldsymbol{m}$ for some $q\in\mathbf{Z}$.

In some sources (**mod m**) equality is also called as (**mod m)** congruence.

Clear that

- if $\boldsymbol{m}$=1 then all integers are equal each other by **mod 1**.
- If $\boldsymbol{m}$=2 then two even numbers are equal each other by **mod 2** as well as two odd numbers also are equal each other by **mod 2**. However no even number is equal to odd number by **mod 2.**

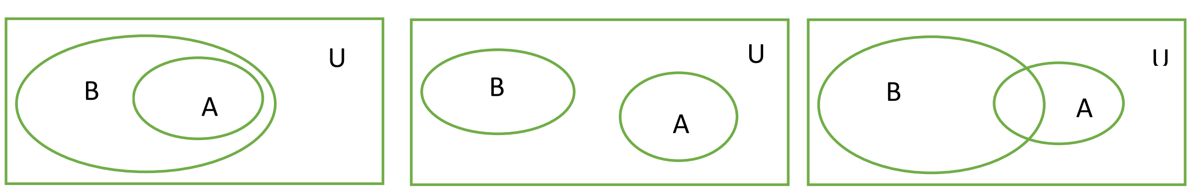
We will use (**mod *m***) relation in forthcoming lectures. ■

Venn Diagrams

A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane. The universal set **U** is represented by the interior of a rectangle, and the other sets are represented by ellipses (circles) lying within the rectangle.

The interior of the circle symbolically represents the elements of the set, while the exterior represents elements that are not members of the set. For instance, in a two-set Venn diagram, one circle may represent the group of all wooden objects, while the other circle may represent the set of all tables. The overlapping region, or intersection, would then represent the set of all wooden tables. Shapes other than circles can be employed.

If $A\subseteq B$, then the ellipse representing *A* will be entirely within the ellipse representing *B* as in Figure 3(a). If *A* and *B* are disjoint, then the ellipse representing *A* will be separated from the ellipse representing *B* as in Figure 3(b).



(a) $A\subseteq B$ (b) A and B are disjoint (c)

Figure 3

However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Figure 3(c).

Arguments and Venn Diagrams

Many verbal statements are essentially statements about sets and can therefore be described by Venn diagrams. Hence Venn diagrams can sometimes be used to determine whether an argument is valid.

EXAMPLE 9. Determine the validity of the following argument:

$S1$: All my friends are musicians.
$S2$: Akif is my friend.
$S3$: None of my neighbors are musicians.
S : Akif is not my neighbor.

The premises $S1$ and $S3$ lead to the Venn diagram in Figure 4. By $S2$, Akif belongs to the set of friends which is disjoint from the set of neighbors. Thus, S is a valid conclusion and so the argument is valid.

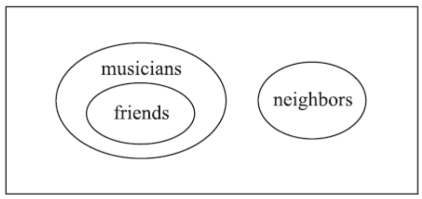


Figure 4

PART 3. Set Operations. Algebra of Sets. Duality

Set Operations.

This section introduces the basic operations of union, intersection, and complement.

1. Union and Intersection

Both **union** and **intersection** operations are so-called binary operations on the collection of subsets of **U**. These operations assign new sets, denoted by $A\cup B$ and $A\cap B$ respectively to each pair of sets, say (A, B) according to the definitions:

$$(A, B) \mapsto A\cup B=\{x\in \mathbf{U} \mid x\in A \text{ or } x\in B\} \text{ - } \textbf{union}$$
$$(A, B) \mapsto A\cap B=\{x\in \mathbf{U} \mid x\in A \text{ and } x\in B\} \text{ - } \textbf{intersection}$$

Here "or" in $A\cup B$ is used in the sense of and/or (at least one of sets A or B).
Using the definition of intersection, sets A and B are said to be ***disjoint*** or *nonintersecting* if $A\cap B=\emptyset$.

Suppose $S=A\cup B$ and $A\cap B=\emptyset$. Then S is called the ***disjoint union*** of A and B.
Both operations can be visualized by Venn diagrams as below

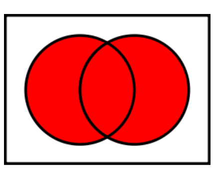
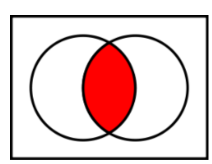


Figure 5.



(a) Union of two sets A∪B (b) Intersection of two sets A∩B

Theorem 3. For any sets A and B, we have:
(i) $A\cap B\subseteq A\subseteq A\cup B$

(ii) $A\cap B\subseteq B\subseteq A\cup B$.

Proof. Every element x in $A\cap B$ belongs to both A and B; hence x belongs to A and x belongs to B. Thus $A\cap B$ is a subset of A and of B; namely $A\cap B\subseteq A$ and $A\cap B\subseteq B$ therefore $A\cap B\subseteq A\cup B$.
The operation of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem. ■

Theorem 4. The following are equivalent:

(1) $A\subseteq B$, (2) $A\cap B=A$, (3) $A\cup B=B$.

Proof.

(1)⇒(2). Suppose $A\subseteq B$ and let $x\in A$. Then $x\in B$, hence $x\in A\cap B$ and $A\subseteq A\cap B$. By Theorem 1.3, $(A\cap B)\subseteq A$. Therefore $A\cap B=A$.
(2)⇒(1). Suppose $A\cap B=A$ and let $x\in A$. Then $x\in(A\cap B)$; hence $x\in A$ and $x\in B$. Therefore, $A\subseteq B$.
(1)⇒(3). Suppose again that $A\subseteq B$. Let $x\in(A\cup B)$. Then $x\in A$ or $x\in B$. If $x\in A$, then $x\in B$ because $A\subseteq B$. In either case, $x\in B$. Therefore $A\cup B\subseteq B$. By Theorem 1.3, $B\subseteq A\cup B$. Therefore $A\cup B = B$.
(3)⇒(1). Now suppose $A\cup B=B$ and let $x\in A$. Then $x\in A\cup B$ by definition of the union of sets. Hence $x\in B=A\cup B$. Therefore $A\subseteq B$.
Finally, from (3)⇒(1) and (1)⇒(2) it follows (3)⇒(2). And from (2)⇒(1) and (1)⇒(3) it follows (2)⇒(1).
Thus $A\subseteq B$, $A\cup B=A$ and $A\cup B=B$ are equivalent. ■

2. Complement

The **absolute complement** operation or, simply, **complement** operation, is so-called unary operation on the collection of subsets of **U**. This operation assigns a new set, denoted by A′, to each set A, according to the following definition:
$A'=\{x \mid x\in\mathbf{U}, x\notin A\}=\mathbf{U}-A$.
Some texts denote the complement of A by A^c or $\bar{A}$.

Next two binary operations can be expressed in terms of operations above.

The ***relative complement*** of a set A with respect to a set B or, simply, the *difference* of B and A, denoted by $B\backslash A$ (in that order), is the set of elements which belong to B but which do not belong to A; that is

$$B\backslash A=\{x \mid x\in B, x\notin A\}$$

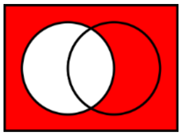
The set $B\backslash A$ is read "B minus A." Many texts denote $B\backslash A$ by $B–A$

The ***symmetric difference*** of sets A and B, denoted by $A\oplus B$, (or $A\Delta B$ in some texts) consists of those elements which belong to A **or** B **but not both (**it means exclusive "or"**)**.

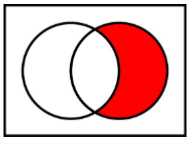
$$A\oplus B=(A\cup B)\backslash(A\cap B) \text{ or } A\oplus B=(A\backslash B)\cup(B\backslash A)$$

Note. Because of exclusive or property sometimes *symmetric difference $A\oplus B$ is called also union of sets A and B by modulo 2* (we take union of A and B and delete from the union elements belonging both A and B.

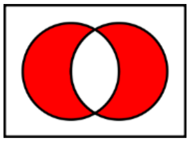
We use Venn diagrams for the visualization of the last three operations (results of operations are presented by red color).



Absolute complement of A (white) in U, $A'=\mathbf{U}-A$



Relative Complement of A (left) in B (right) $B-A=A'\cap B$



Symmetric difference of two sets $A\oplus B=(A\backslash B)\cup(B\backslash A)$

Generalized Set Operations

The set operations of union and intersection can be extended to any number of sets as follows. Consider first a finite number of sets, say, $A_1, A_2, ..., A_m$. The union and intersection of these sets are denoted and defined, respectively, by

$A_1 \cup A_2 \cup \dots \cup A_m = \bigcup_{i=1}^{m} A_i = \{x \mid x \in A_i$ for **some** $A_i\}$

$A_1 \cap A_2 \cap \dots \cap A_m = \bigcap_{i=1}^{m} A_i = \{x \mid x \in A_i$ for **every** $A_i\}$

EXAMPLE.

- Find $B = \bigcap_{i=1}^{10} A_i$ where A_i is a circle on the plane $\mathbf{R}^2$ with radius i (i=1, 2, …, 10) that is, $A_i=\{(x, y)\in\mathbf{R}^2 | x^2+y^2=i\}$.

  Solution. Since circles A_i and A_j are mutually disjoint if $i\neq j$ ($A_i \cap A_j = \emptyset$) hence B= $\emptyset$.

- Find $C = \bigcap_{i=1}^{5} D_i$ where D_i is an interval [4, i+2].

  Solution. Under i=1 D_i=[4, 3]=$\emptyset$ (right endpoint must be not less than left endpoint to avoid empty set). Since one of sets in intersection is empty so result is also empty.

Now let Ω be any collection of sets. The union and the intersection of the sets in the collection Ω is denoted and defined, respectively, by

$\cup(A|A\in\Omega)=\{x \mid x\in A$ for **some** $A\in\Omega\}$

$\cap(A|A\in\Omega)=\{x \mid x\in A$ for **every** $A\in\Omega\}$

That is, the union consists of those elements which belong to at least one of the sets in the collection Ω and the intersection consists of those elements which belong to every set in the collection Ω. ■

Fundamental Products

Consider n distinct sets $A_1, A_2, ..., A_n$. A *fundamental product* of the sets is a set of the form

$$A_1^* \cap A_2^* \cap \dots \cap A_n^* \quad \text{where } A_i^*=A \text{ or } A_i^*=A'$$

Theorem 5.

(i) There are $m=2^n$ such fundamental products.

(ii) Any two such fundamental products are disjoint.

(iii) The universal set **U** is the disjoint union of all fundamental products.

Proof. Exercise. (Check the result for n=3). ■

Partitions

Let S be a nonempty set. A ***partition*** of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}_{i\in I}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i (or equivalently, $S=\bigcup_i A_i$)

(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$A_j \neq A_k$ then $A_j \cap A_k = \emptyset$

A partition concept is intensively used in different branches of mathematics.

There is an intimate connection between mappings and partitions into classes, as shown by the following examples:

EXAMPLE 10. Let f be a mapping of a set A into a set B and partition A into sets, each consisting of all elements with the same image b=f(a)∈B. This gives a partition of A into classes. For example, suppose f projects the xy-plane onto the x-axis, by mapping the point (x, y) into the point (x, 0). Then the preimages of the points of the x-axis are vertical lines, and the representation of the plane as the union of these lines is the decomposition into classes corresponding to f. ■

EXAMPLE 11. Given any partition of a set A into classes, let B be the set of these classes and associate each element a∈A with the class (i.e., element of B) to which it belongs. This gives a mapping of A into B. For example, suppose we partition three-dimensional space into classes by assigning to the same class all points which are equidistant from the origin of coordinates.

Then every class is a sphere of a certain radius. The set of all these classes can be identified with the set of points on the half-line $[0, \infty)$, each point corresponding to a possible value of the radius. In this sense, the decomposition of space into concentric spheres corresponds to the mapping of space into the half-line $[0, \infty)$. Note that, this mapping is not a bijection, because all points on a sphere are mapped to a single point on $[0, \infty)$. ■

Let *x* be any real number. Then *x* lies between two integers called the floor and ceiling of *x*. Specifically,

$\lfloor x \rfloor$, called *floor* of *x* (or *integral part* of *x*), denotes the greatest integer that does not exceed *x*.

$\lceil x \rceil$, called *ceiling* of *x*, denotes the least integer that is not less than *x*.

x-$\lfloor x \rfloor$ called the *fractional part* of *x*.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$;

EXAMPLE 12.

$\lfloor 3.14 \rfloor=3$, $\lfloor \sqrt{5} \rfloor=2$, $\lfloor -3.14 \rfloor=-4$, $\lfloor -4 \rfloor=-4$, $\lfloor 4 \rfloor=4$

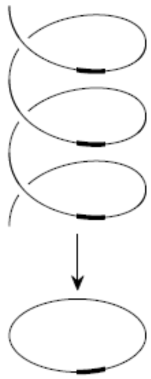
$\lceil 3.14 \rceil=4$, $\lceil \sqrt{5} \rceil=3$, $\lceil -3.14 \rceil=-3$, $\lceil -4 \rceil=-4$, $\lceil 4 \rceil=4$

Fractional part of:

3.14 is: $3.14-\lfloor 3.14 \rfloor=3.14-3=0.14$

(-3.14) is: $(-3.14)-\lfloor -3.14 \rfloor=(-3.14)-(-4)=0.86$ ■

EXAMPLE 13. Suppose we assign all real numbers with the **same fractional part** to the same class. Then the mapping corresponding to this partition has the effect of "winding" the real line onto a circle of unit circumference.



■

Algebra of Sets. Duality

Definition 5. Algebra of sets is an algebraic structure which contains:

a) A family, say $\mathcal{F}$, of subsets of the universal set U, including the empty set, ∅, and universal set U itself;

and three operations

b) Two binary operations $\mathcal{F}\times\mathcal{F}\to\mathcal{F}$
   a. Union. (A, B) $\longmapsto$ (A∪B) , and
   b. Intersection (A, B) $\longmapsto$ (A∩B)

c) Unary operation: Complement A $\longmapsto$A′=**U**-A
   Denotation for Algebra of Sets $\mathcal{A}$ =<$\mathcal{F}$, ∅, **U**, ∪, ∩, ′ > ■

Algebra $\mathcal{A}$ satisfies various laws (identities) which are listed in Table 1 below. In fact, we formally state this as:

Theorem 6. Algebra $\mathcal{A}$ satisfies the laws in Table 1.

Table 1. Laws of the Algebra of Sets

Idempotent laws:	(1a) A∪A=A	(1b) A∩A=A
Associative laws:	(2a) (A∪B)∪C=A∪(B∪C)	(2b) (A∩B)∩C=A∩(B∩C)
Commutative laws:	(3a) A∪B=B∪A	(3b) A∩B=B∩A
Distributive laws:	(4a) A∪(B∩C)=(A∪B)∩(A∪C)	(4b) A∩(B∪C)=(A∩B)∪(A∩C)
Identity laws:	(5a) A∪∅=A	(5b) A∩**U**=A
	(6a) A∪**U**=**U**	(6b) A∩∅=∅
Involution laws:	(7) (A′)′=A	
Complement laws:	(8a) A∪A′=**U**	(8b) A∩A′=∅
	(9a) **U**′=∅	(9b) ∅′=**U**
DeMorgan's laws:	(10a) (A∪B)′=A′∩B′	(10b) (A∩B)′=A′∪B′

Proof. We prove here, as example, two identities (Distributive law 4a and DeMorgan law 10a) and left all others to readers as exercises.

(4a). Let x∈A∪(B∩C) ⇔ (x∈A or x∈(B∩C)) ⇔ (x∈A or (x∈B and x∈C)) ⇔ (x∈(A∪B) and x∈(A∪C)) ⇔ x∈(A∪B)∩(A∪C).

(10a). Let x∈(A∪B)′ ⇔ x∈ **U**-(A∪B) ⇔ x∉(A∪B) ⇔ x∉A and x∉B ⇔ x∈A′ and x∈B′ ⇔ x∈(A′∩B′) ■

Duality

The identities in Table 1 are arranged in pairs, as, for example, (2a) and (2b). We now consider the principle behind this arrangement. Suppose E is an equation of set algebra. The dual E^* of E is the equation obtained by replacing each occurrence of ∪, ∩, **U** and ∅ in E by ∩, ∪, ∅, and **U**, respectively. For example,

the dual of (**U**∩A)∪(B∩A)=A is (∅∪A)∩(B∪A)=A

Observe that the pairs of laws in Table 1 are duals of each other. It is a property of set algebra, called the *principle of duality*: if an equation E is an identity, then its dual E^* is also an identity.

Relationship between Set Operations and Functions

In this section we present fundamental relationship between union and intersection operations in one side and functions in other side. Specifically, we propose the following three theorems. We prove only one of them (Theorem 7). Proofs of remaining two theorems are exercises.

Theorem 7. The preimage of the union of two sets is the union of the preimages of the sets:

$$f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B).$$

Theorem 8. The preimage of the intersection of two sets is the intersection of the preimages of the sets:

$$f^{-1}(A\cap B)=f^{-1}(A)\cap f^{-1}(B).$$

Theorem 9. The image of the union of two sets equals the union of the images of the sets :

$$f(A\cup B)=f(A)\cup f(B).$$

Note. The common way how to prove all three theorems is

Step 1 Show that left-hand side (LHS) is the subset of right-hand side (RHS)

Step 2. Conversely, RHS is the subset of LHS.

Remark 1. Surprisingly enough, the image of the intersection of two sets does not necessarily equal the intersection of the images of the sets. For example, suppose the mapping f projects the xy-plane onto the x-axis, carrying the point (x, y) into the (x, 0). Then the segments $0\le x\le 1$, $y=0$ and $0\le x\le 1$, $y=1$ do not intersect, although their images coincide.

Remark 2. Theorems 7-9 continue to hold for unions and intersections of an arbitrary number (finite or infinite) of sets.

Proof (Theorem 7).

Check that $f^{-1}(A\cup B)\subset f^{-1}(A)\cup f^{-1}(B)$. Really, let $x\in f^{-1}(A\cup B) \Rightarrow f(x)\in A\cup B \Rightarrow$ f(x) belongs to A or B or both. Without losing generality, let $f(x)\in A \Rightarrow x\in f^{-1}(A) \Rightarrow x\in f^{-1}(A)\cup f^{-1}(B)$. Hence the relation $f^{-1}(A\cup B)\subset f^{-1}(A)\cup f^{-1}(B)$ is proved.

Now we check the relation $f^{-1}(A)\cup f^{-1}(B)\subset f^{-1}(A\cup B)$. Really, let $x\in f^{-1}(A)\cup f^{-1}(B) \Rightarrow$ x belongs to $f^{-1}(A)$ or $f^{-1}(B)$ or both B. Without losing generality, let $x\in f^{-1}(A) \Rightarrow f(x)\in A \Rightarrow f(x)\in A\cup B \Rightarrow x\in f^{-1}(A\cup B)$. Theorem is proved. ∎

Part 4. Equivalence of Sets. The Power (Cardinality) of a Set

(Additional Reading – mandatory for math students)

From Part 2 we know that two sets are called equivalent if there exist a bijection between them. The next question seems very natural: given two sets X and Y how to understand (or to predict) whether a bijection exist?

To compare two *finite sets* A and B we can apply two methods.

Method 1. We can count the number of elements in each set and then compare the two numbers. Alternatively,

Method 2. We can try to establish a bijection between A and B. *It is clear that a bijection between two finite sets can be set up if and only if the two sets have the same number of elements*. For example, to ascertain whether or not the number of students in an assembly is the same as the number of seats in the auditorium, there is no need to count the number of students and the number of seats. We need merely observe whether or not there are empty seats or students with no place to sit down (see also Example 3 in Part 2 above). If the students can all be seated with no empty seats left, i.e., if there is a bijection between the set of students and the set of seats, then these two sets obviously have the same number of elements.

The important point here is that the Method 1 (counting elements) works only for finite sets, while the Method 2 (setting up a bijection) works for infinite sets as well as for finite sets.

Infinite sets: Countable and Uncountable sets.

The simplest infinite set is the set $\mathbf{Z}_+$ of all positive integers.

Definition 6. An infinite set is called **countable** if its elements can be put in bijection with those of $\mathbf{Z}_+$. In other words, a countable set is a set whose elements can be numbered $a_1, a_2, ..., a_n$. By an **uncountable** set we mean, of course, an infinite set which is not countable. ■

From transitivity property of a bijection we immediately conclude that all countable sets are equivalent (because each of them are equivalent to $\mathbf{Z}_+$).

We now give examples of countable sets and discover an unexpected property (Examples 14-16) of them: **countable set is equivalent to its proper subset**. Later (Theorem 13) it will be proved in general case: *any infinite set is equivalent to its proper subset*. Clear that such a property is impossible for a finite set and therefore this property is a characteristic property of all infinite sets.

EXAMPLE 14. The set $\mathbf{Z}$ of all integers (positive, negative and zero) is countable. In fact, we can set up the following bijection between $\mathbf{Z}$ and the set $\mathbf{Z}_+$ of all positive integers:

0, -1, 1, -2, 2, ...	$\mathbf{Z}$
1, 2, 3, 4, 5, ...	$\mathbf{Z}_+$

More explicitly, we associate the nonnegative integer $n\geq 0$ with the odd number $2n+1$, and the negative integer $n<0$ with the even number $2|n|$, that is,

$$n \leftrightarrow 2n+1 \qquad \text{if } n>0,$$

$$n \leftrightarrow 2|n| \qquad \text{if } n<0$$

(the symbol $\leftrightarrow$ denotes a bijection). ■

EXAMPLE 15. The set of all positive even numbers is countable, as shown by the obvious correspondence $n \leftrightarrow 2n$. ■

EXAMPLE 16. The set 2, 4, 8, ..., 2^n, ... of powers of 2 is countable, as shown by the obvious correspondence $n \leftrightarrow 2^n$. ■

EXAMPLE 17. The set $\mathbf{Q}$ of all rational numbers is countable.

Proof. We first note that every rational number $\boldsymbol{r}$ can be written uniquely as an irreducible fraction p/q, $q>0$. Call the sum $|p|+q$ the "height" of the rational number $\boldsymbol{r}$. Clear that number of irreducible fractions with given height $\boldsymbol{n}>0$ is fintie. For example,

if $\boldsymbol{n}=1$ we have only one fraction, namely 0/1

if $\boldsymbol{n}=2$ we have two fractions, -1/1 and 1/1

if $\boldsymbol{n}=3$ we have four fractions, -2/1, -1/2, 1/2, 2/1, We can now arrange all rational numbers in order of increasing height (with the numerators increasing in each set of rational numbers of the same height). In other words, we first count the rational numbers of height 1, then those of height 2 (suitably arranged), those of height 3, and so on. In this way, we assign every rational number a unique positive integer, i.e., we set up a bijection between the set **Q** of all rational numbers and the set $\mathbf{Z}_+$ of all positive integers. ■

Elementary properties of countable sets.

Below we prove two important properties of countable sets.

Theorem 10. Every subset of a countable set is finite or countable.

Proof. Let A be countable, with elements $a_1, a_2, \ldots$, and let B be a subset of A. Among the elements $a_1, a_2, \ldots$, let $a_{n1}, a_{n2}, \ldots$ be those in the set B. If the set of numbers $n_1, n_2, \ldots$ has a largest number, then B is finite. Otherwise B is countable (consider the correspondence $i \leftrightarrow a_{ni}$). ■

Next statement asserts that a family of countable sets is closed with respect to union operation.

Theorem 11. The union of finite or countable number of countable sets is itself countable.

Proof. We can assume that no two of the sets $A_1, A_2, \ldots$ have elements in common, since otherwise we could consider the sets

$$A_1,\ A_2\text{-}A_1,\ A_3\text{-}(A_1 \cup A_2),\ \ldots$$

instead, which are countable by Theorem 10 and have the same union as the original sets. Suppose we write the elements of $A_1, A_2, \ldots$ in the form of an infinite table

$$\begin{array}{l} a_{11}, a_{12}, a_{13}, a_{14}, \ldots \\ a_{21}, a_{22}, a_{23}, a_{24}, \ldots \\ a_{31}, a_{32}, a_{33}, a_{34}, \ldots \\ a_{41}, a_{42}, a_{43}, a_{44}, \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \qquad (4.1)$$

where the elements of the set A_1, appear in the first row, the elements of the set A_2, appear in the second row, and so on.

We now count all the elements in (4.1) "diagonally," i.e., first we choose a_{11}, then a_{12}, then a_{21}, and so on, moving in the way shown in the following table

$$\begin{array}{ccccccc} a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & \cdots \\ & \swarrow & & \nearrow & & \swarrow & & \cdots \\ a_{21} & & a_{22} & & a_{23} & & a_{24} & \cdots \\ \downarrow & \nearrow & & \swarrow & & & & \cdots \\ a_{31} & & a_{32} & & a_{33} & & a_{34} & \cdots \\ & \swarrow & & & & & & \\ a_{41} & & a_{42} & & a_{43} & & a_{44} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad (4.2)$$

It is clear that this procedure associates a unique positive integer to each element in each of the sets $A_1, A_2, \ldots$ thereby establishing a bijection between the union of the sets $A_1, A_2, \ldots$ and the set $\mathbf{Z}_+$ of all positive integers. ■

Two important properties of infinite sets.

First result says that among the infinite sets countable set is the smallest one.

Theorem 12. Every infinite set has a countable subset.

Proof. Let M be an infinite set and a_1, any element of M. Being infinite, M contains an element a_2, distinct from a_1, an element a_3, distinct from both a_1 and a_2, and so on. Continuing this process (which can never terminate due to a "shortage" of elements, since M is infinite), we get a countable subset $A=\{a_1, a_2, ... a_n, ... \}$ of the set M. ■

Remark 1. Theorem 12 shows that countable sets are the "smallest" infinite sets. The question of whether there exist **uncountable (infinite) sets** will be considered later in this Part.

In Part 2 Definition 3 we noticed that two sets X and Y are equivalent if there is a bijection $f:X\to Y$ between them. We also demonstrated there very surprising Example 4 which establishes equivalences (the same "number" of elements) between infinite set Y=**R** and its proper subset $X=(-\pi/2, \pi/2)$ and mentioned that this unexpectable result is a special property of "infinite" sets.

We demonstrate two interesting examples of equivalent sets more.

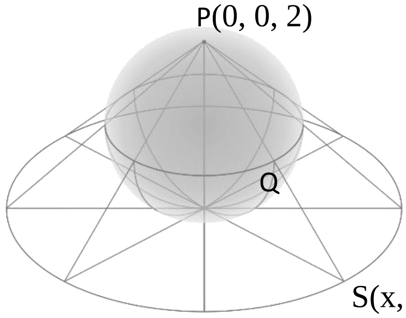
EXAMPLE 18. Define the unit sphere S^2 in the space R^3 as following

$S^2=\{(x, y, z)\in \mathbf{R^3} | x^2+y^2+(z-1)^2=1\}$ - sphere in R^3 with radius 1 centerred at (0, 0, 1).

Then the (north) pole point of S^2 is the point P(0, 0, 2).

Statement. The plane $\mathbf{R^2}$ is equivalent to the set S^2-{P}.

In fact, a bijection between the points of the two sets can be established by using stereographic projection π: S^2-{P}$\to$ $\mathbf{R^2}$

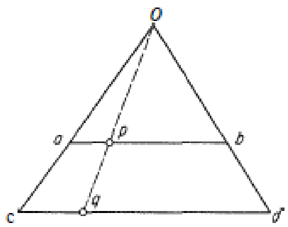


A stereographic projection from the north pole P onto the plane z=0.

We take any point $Q\in S^2$-{P} and take the uniquely defined line L through P and Q. Then the intersection of L and the plane z=0 (say point S) is the inuquely defined image of Q under the mapping π, $\pi(Q)=S$. Geometrically, it is clear that π is the bijection. ■

EXAMPLE 19. Any closed intervals [a, b], a<b, and [c, d], c<d are equivalent (they contain the "same" number of points).

Proof. Figure below shows how to set up a one-to-one correspondence between them. Here two points p and q correspond to each other if and only if they lie on the same ray emanating from the point 0 in which the extensions of the line segments ac and bd intersect.



■

Note. In Particular, if $c<a<b<d$, then we find that closed interval $[c, d]$ is equivalent to its proper subinterval $[a, b]$. This result is similar to the Examples 4, 14-17 which establish an equivalence between infinite set and its proper subset. This fact is characteristic of all infinite sets as shown by the next theorem 13 and can be used to define such sets.

The second important result about infinite sets which can be considered as characteristic property (definition) of them is the Theorem 13 below.

Theorem 13. Every infinite set is equivalent to one of its proper subsets.

Proof. According to Theorem 12, any infinite set M contains a countable subset, say:

$$A=\{a_1, a_2, \ldots, a_n, \ldots\}.$$

Partition A into two countable subsets

$$A_1=\{a_1, a_3, a_5, \ldots\}, \qquad A_2=\{a_2, a_4, a_6, \ldots\}.$$

Obviously, we can establish a bijection between the countable sets A and A_1, say, $a_n \leftrightarrow a_{2n-1}$. This bijection can be extended to a one-to-one correspondence between the sets $A\cup(M-A)=M$ and $A_1\cup(M-A)=M-A_2$, by simply assigning x itself to each element $x \in M-A$. But $M-A_2$ is a proper subset of M. ■

Uncountable sets.

In considerations above we demonstarted examples of countable sets. Now the question arises: **do there exist infinite uncountable sets?** The existence of such sets is shown by the following theorem 14.

Theorem 14. The set of real numbers in the closed unit interval [0, 1] is uncountable.

Proof. Assume the contrary. Suppose we have somehow managed to count some or all of the real numbers in [0, 1], arranging them in a list

$\alpha_1=0, a_{11}a_{12}\ldots a_{1n}\ldots,$

$\alpha_2=0, a_{21}a_{22}\ldots a_{2n}\ldots,$

$\alpha_3=0, a_{31}a_{32}\ldots a_{3n}\ldots,$ (4.3)

....

$\alpha_n=0, a_{n1}a_{n2}\ldots a_{nn}\ldots,$

where a_{ik}, is the k-th digit in the decimal expansion of the number α_i.

Consider the decimal

$\beta=0, b_1b_2\ldots b_n\ldots$ (4.4)

constructed by Cantor diagonalization process as follows:

for b_1, choose any digit (from 0 to 9) different from a_{11},

for b_2, any digit different from a_{22}, and so on, and in general

for b_n, any digit different from a_{nn}.

Then the decimal (4.4) cannot coincide with any decimal in the list (4.3). In fact, β differs from α_1, in at least the first digit, from α_2, in at least the second digit, and so on, since in general, $b_n \neq a_{nn}$ for all n. Thus no list of countable real numbers in the interval [0, 1] can include all the real numbers in [0, 1].

The above argument must be refined slightly since certain numbers, namely those of the form $p/10^q$, can be written as decimals in two ways,

$$\frac{1}{2} = \frac{5}{10} = 0.5000 \ldots. = 0.4999 \ldots$$

Really, 0.499.....=0.4+$(\frac{9}{100} + \frac{9}{1000} + \cdots)$. The term in paranthesis is the sum (S) of geometric series $\frac{9}{100}, \frac{9}{1000}$, ... with ratio r=(1/10). From secondary school program it is known that

S=$\frac{a_1}{1-r} = \frac{\frac{9}{100}}{1-\frac{1}{10}} = \frac{10}{100} = \frac{1}{10}$. Hence, 0.499...=0.4+0.1=0.500...=$\frac{1}{2}$

Thus, the fact that two decimals are distinct does not necessarily mean that they represent distinct real numbers. However, this difficulty disappears if in constructing β, we require that β contain neither zeros nor nines, for example by setting b_{nn}=2 if a_{nn}=1 and b_n=1 if $a_{nn} \neq 1$. ■

Thus the set [0, 1] is uncountable. Other examples of uncountable sets equivalent to [0,1] are

1) The set of points in any closed interval [a, b];

2) The set of points on the real line;

3) The set of points in any open interval (a, b);

4) The set of all points in the plane or in space;

5) The set of all points on a sphere or inside a sphere;

6) The set of all lines in the plane;

7) The set of all continuous real functions of one or several variables.

Power (=Cardinality) of a Set.

Definition 7. Given any two sets M and N, suppose M and N are equivalent, M. Then M and N are said to have the **same power (cardinality)**. ■

Thus, a cardinality is something shared by equivalent sets. We use symbol |·| to denote cardinality. Hence if M∼N then we write |M|=|N|.

If M and N are equivalent finite sets, then M and N have the same number of elements, and the concept of the cardinality of a set reduces to the usual notion of **the number of elements** in a set, that is, |M|=|N|=k, here k=0, 1, 2, 3, …

Definition 8. The power of the set $\mathbf{Z}_+$ of all positive integers, and hence the **cardinality of any countable set**, is denoted by the symbol $\aleph_0$, read "aleph null." A set equivalent to the set of real numbers in the interval [0, 1], and hence to the set of all real numbers, is said to have the **cardinality of the continuum**, denoted by ***c*** (first letter of the word “continuum”). ■

For the cardinalities of finite sets (that is, for the positive integers), we have the notions of "greater than" and "less than," as well as the notion of equality (ordering property). We now show how these concepts are extended to the case of infinite sets.

Let A and B be any two sets, with cardinalities |A| and |B|, respectively. If A is equivalent to B, then |A|=|B| by the Definition 7.

Below we a bit extend Definition 7 in the form of Definition 9.

Definition 9. Given two sets A and B we say

- |A|=|B| if there is a bijection f from A to B
- |A|≤|B| if there is an injection f from A to B
- |A|<|B| if there is an injection f from A to B but no bijective function from A to B (that is, no surjective function from A to B). ■

Logically, we have one more case:

-) A and B are not equivalent, and neither has a subset equivalent to the other.

Last case would obviously show the existence of powers that cannot be compared, but it can be proved (based on so-called well-ordering theorem) that this case is impossible. We use this fact without proof.

If |A|≤|B| and |B|≤|A|, then A and B are equivalent and hence have the same cardinality, as shown by the Cantor-Bernstein theorem (Theorem 16, next section).

Therefore, we see that **any two sets A and B**: either have the same cardinality: **|A|=|B|** or else satisfy one of the relations: **|A|**<**|B|** or **|A|>|B|.**

For example, it is clear that, $\aleph_0 < c$ (why?).

Remark 3. A very deep problem of the existence of powers between $\aleph_0$ and c is called **continuum hypothesis** (abbreviated as CH). It states, there is no set whose cardinality is strictly between that of the integers and the real numbers, that is, there is no D, such that $\aleph_0 < |D| < c$. This problem has been solved in 1963 by Paul Cohen.

As a rule, however, the infinite sets encountered in Calculus are either countable or else have the cardinality of the continuum. The infinite sets encountered in Discrete Structure are mainly countable.

So far, we established that

- countable sets (with cardinality $\boldsymbol{\aleph_0}$) are the "smallest" infinite sets;
- there are infinite sets of cardinalities greater than that of a countable set, namely sets with the cardinality of the continuum.

It is natural to ask whether there are infinite sets of cardinalities greater than that of the continuum or, more generally, whether there is a "largest" cardinality. This question is answered by the following Theorem 15.

Theorem 15. Given any set M, let $\mathcal{M}$ be the set whose elements are all possible subsets of M. Then the power of $\mathcal{M}$ is greater than the power of the original set M.

Proof. Clearly, the power μ of the set $\mathcal{M}$ cannot be less than the power m of the original set M, since the "single-element subsets" (or "singletons") of M form a subset of $\mathcal{M}$ equivalent to M. Thus, we need only show that m and μ do not coincide, that is, $m\neq\mu$.

Assume the contrary. Suppose a bijection $f:M\to\mathcal{M}$

$$a\leftrightarrow A, \qquad b\leftrightarrow B, \ldots$$

has been established between the elements $a, b, \ldots$ of M and certain elements A, B, … of $\mathcal{M}$ (i.e., certain subsets of M). We assert that A, B, … do not exhaust all the elements of $\mathcal{M}$, i.e., all the subsets of M. To see this, *let* X *be the set of elements of* M *which do not belong to their "associated subsets"*. More exactly, if $a\leftrightarrow A$ we assign a to X if $a\notin A$, but not if $a\in A$.

Clearly, X is a subset of M and hence an element of $\mathcal{M}$. Since f is a bijection then there exists $x\in M$ such that $f(x)=X$, that is, $x\leftrightarrow X$. Question: whether x belongs to X. Suppose $x\notin X$. Then $x\in X$, since, by definition, X contains every element not contained in its associated subset. On the other hand, suppose $x\in X$. Then $x\notin X$, since X consists precisely of those elements which do not belong to their associated subsets. In any event, the element x corresponding to the subset X *must simultaneously belong to* X *and not belong to* X. But this is impossible!

It follows that there is no such element x. Therefore, no bijection can be established between the sets M and $\mathcal{M}$, i.e., $m\neq\mu$ and we obtain that $\mu<m$. Theorem is proved. ■

Thus, given any set M, there is a set $\mathcal{M}$ of larger power, a set $\mathcal{M}^*$ (set of all possible subsets of $\mathcal{M}$) of still larger power, and so on indefinitely. In particular, there is no set of "largest" power.

Cantor-Bernstein Theorem (optional reading).

The Cantor-Bernstein (sometimes in literature Cantor–Schröder–Bernstein or Schröder–Bernstein) Theorem is one of the important facts in the set theory and asserts that two sets are equivalent if each of them contains a subset which is equivalent to another set.

Below we provide two different proofs of this Theorem.

Theorem 16. (Cantor-Bernstein). Given any two sets A and B, suppose A contains a subset A_1, equivalent to B, while B contains a subset B_1, equivalent to A. Then A and B are equivalent.

***Proof* 1.** (Cantor-Bernstein Theorem**).** The following proof is attributed to Julius König (J. König (1906). "Sur la théorie des ensembles". Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. 143: 110–112.)

Since A is equivalent to $B_1\subseteq B$ then there exist an injective function $f:A\to B$, such that $f(A)=B_1$. (f is the bijection between A and B_1). By the similar arguments there exist an injective function $g:B\to A$, such that $g(B)=A_1$ (g is the bijection between B and A_1).

Assume without loss of generality that A and B are disjoint and imagine (for this part of proof) we work with set $A\cup B$. For any $a\in A$ and $b\in B$ we can form a unique two-sided sequences ($\#_1$) and ($\#_2$) of elements that are **alternately** in A and B, by repeatedly applying

f and g^{-1} to go from A to B (**where defined)** and

g and f^{-1} to go from B to A (**where defined**);

$$S_a: \quad \ldots \rightarrow f^{-1}(g^{-1}(a)) \rightarrow g^{-1}(a) \rightarrow a \rightarrow f(a) \rightarrow g(f(a)) \rightarrow \ldots \ (\#_1)$$

$$S_b: \quad \ldots \rightarrow g^{-1}(f^{-1}(b)) \rightarrow f^{-1}(b) \rightarrow b \rightarrow g(b) \rightarrow f(g(b)) \rightarrow \ldots \ (\#_2)$$

The inverses f^{-1} and g^{-1} are understood as partial functions at this stage of the proof, that is, domain of f^{-1} is a subset of B and domain of g^{-1} is a subset of A.

For any particular $a \in A$ and $b \in B$, this sequence may **terminate to the left or not**, at a point where f^{-1} or g^{-1} is not defined.

Proposition 1. For distinct points c and d from $A \cup B$ the sequences S_c and S_d are disjoint or completely the same, that is, $S_c \cap S_d = \emptyset$ or $S_c = S_d$.

Proof. Exercise. (Immediately follows from the fact that f and g are injective). ■

Thus, each point $c \in A \cup B$ belongs to only sequence S_c and such sequences form the partition of $A \cup B$:

- $S_c \neq \emptyset$
- $\bigcup_{c \in A \cup B} S_c = A \cup B$
- If $S_c \cap S_d \neq \emptyset$ then $S_c \cap S_d = \emptyset$

Regarding structure of a sequence S_c, no matter, case ($\#_1$) or ($\#_1$). We introduce the definitions.

Call a sequence S_c, an **A-stopper** if it stops (under moving to the left) at an element of A, or a **B-stopper** if it stops at an element of B. Otherwise, call it **doubly infinite** if all the elements are distinct or cyclic if it repeats.

Finally, we can define a bijection $h: A \rightarrow B$ at least by two different ways:

Method 1:

$$h(a) = \begin{cases} f(a), & \text{if } S_a \text{ is A} - \text{stopper} \\ g^{-1}(a), & \text{if } S_a \text{ is B} - \text{stopper} \\ g^{-1}(a), & \text{if } S_a \text{ is double infinite or cyclic} \end{cases}$$

Method 2

$$h(a) = \begin{cases} f(a), & \text{if } S_a \text{ is A} - \text{stopper} \\ g^{-1}(a), & \text{if } S_a \text{ is B} - \text{stopper} \\ f(a), & \text{if } S_a \text{ is double infinite or cyclic} \end{cases}$$

Proposition. 2. $h: A \rightarrow B$ is a surjective.

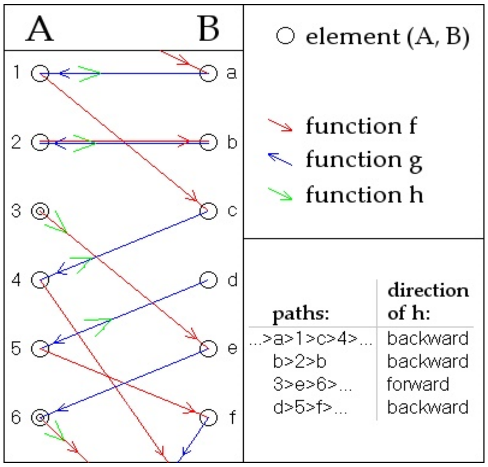
Clear, since f and g both injective. ■

Proposition 3. $h: A \rightarrow B$ is a surjective.

Proof. Exercise. ■

Thus, h is a bijection. Theorem is proved. ■

EXAMPLE 20. Below we provide an example to demonstrate König's definition of a bijection $h: A \rightarrow B$ (see picture and explanation below).



König's definition of a bijection h:A→B from given example injections f:A→B and g:B→A. An element in A and B is denoted by a number and a letter, respectively.

We use Method 1 to define h.

The sequence 3→e→6→... is an A-stopper, leading to the definitions h(3)=f(3)=e, h(6)=f(6), ...

The sequence d→5→f→... is a B-stopper, leading to $h(5)=g^{-1}(5)=d$, ...

The sequence ... →a→1→c→4→... is doubly infinite, leading to $h(1)=g^{-1}(1)=a$, $h(4)=g^{-1}(4)=c$, ... The sequence b→2→b is cyclic, leading to $h(2)=g^{-1}(2)=b$.

***Proof* 2**. (Cantor-Bernstein Theorem). The proof presented here belongs to Leo Goldmakher (https://web.williams.edu/Mathematics/lg5/CanBer.pdf) and it is modeled on the argument given in A. N. Kolmogorov and S. V. Fomin, Introductory Real Analysis, translated by R. A. Silverman, Dover Publications, New York, 1975.

We use several lemmas within the ***Proof 2***.

For the simplicity denote $A_0=A$ and $B_0=B$. By hypothesis, there is a bijections $f:A_0\to B_l$ and $g: B_0\to A_1$, so that

$$f(A_0)=B_l \subseteq B_0 \qquad (4.5) \qquad\qquad g(B_0)=A_1\subseteq A_0 \qquad (4.6)$$

Consider the following Table 1.

	Left Column	**Right Column**
0	$A_0 = A$	$B_0 = B$
1	$A_1 = g(B_0)$	$B_1 = f(A_0)$
2	$A_2 = g(B_1)$	$B_2 = f(A_1)$
...	....	....
n	$A_n = g(B_{n-1})$	$B_n = f(A_{n-1})$
	....	....

Rule. To obtain the n-th set B_n, n=0, 1, 2, ..., in the right column we apply the bijection ***f*** to the (n-1)-th set A_{n-1} in left column. Similarly, to get the n-th set A_n in left column we apply the bijection ***g*** to the (n-1)-th set B_{n-1} in right column.

Since f and g are both bijections so we obtain the following chains of equivalences

$$A=A_0\sim B_1\sim A_2\sim B_3\sim A_4\sim ... \qquad (4.7)$$

$$B=B_0\sim A_1\sim B_2\sim A_3\sim B_4\sim ... \qquad (4.8)$$

Since $B_1 \subseteq B_0$ so $A_2=g(B_1)\subseteq g(B_0)\subseteq A_1$. Similarly, since $A_1 \subseteq A_0$ so $B_2=f(B_1)\subseteq f(B_0)\subseteq B_1$

In general, by the similar aruments it can be shown that

$$A=A_0\supseteq A_1\supseteq A_2\supseteq\ldots \qquad (4.9)$$

$$B=B_0\supseteq B_1\supseteq B_2\supseteq\ldots \qquad (4.10)$$

Remark 2. If $A_n=A_{n+1}$ for some n, then (4.7) and (4.8) immediately imply that A~B and theorem is proved. Therefore, we assume that the inclusions in (4.9) and (4.10) are all **strict**. That is,

$$A=A_0\supset A_1\supset A_2\supset\ldots \qquad (4.9')$$

$$B=B_0\supset B_1\supset B_2\supset\ldots \qquad (4.10')$$

The following Lemma 1 is almost clear and we use it to continue the proof of the theorem.

Lemma 1. Suppose we have sets $\{X_i\}$ and $\{Y_i\}$ satisfying $X_i \sim Y_i$ for all i. If all the X_i are pairwise disjoint, and all the Y_i are pairwise disjoint, then

$$\cup_i X_i \sim \cup_i Y_i \qquad (4.11)$$

Proof. Exercise. ■

Thus, to continue our line of argument, we require analogues of the sets A_i which are pairwise disjoint. For each n, set

$$A_n^* = A_n - A_{n+1} \qquad (4.12)$$

If $A_n^* = \varnothing$ for some n, then $A_n=A_{n+1}$ and, by Remark above, we obtain that A~B and the theorem is proved. Therefore, we assume that $A_n^* \neq \varnothing$ for all n.

Clear that $A_n^* \neq \varnothing$ for all n because of (4.9′)

Lemma 2. All A_n^* are are pairwise disjoint.

Proof. Let n<k. We prove that $A_n^* \cap A_k^* = \emptyset$. Assume the contrary. Let $x\in A_n^* \cap A_k^* = (A_n - A_{n+1}) \cap (A_k - A_{k+1})$, $n+1\leq k$. Hence $x\in A_n$, $x \notin A_{n+1}$ and $x\in A_k$, $x \notin A_{k+1}$. But two conditions $x \notin A_{n+1}$ and $x\in A_k$, contradict each other because of (4.9′) [since $n+1\leq k$, so $A_k \subseteq A_{n+1}$ and last implies that if $x\in A_k$ then $x\in A_{n+1}$ which is impossible because $x \notin A_{n+1}$]. Hence $A_n^* \cap A_k^* = \emptyset$. ■

Similarly, for each n we set

$$B_n^* = B_n - B_{n+1} \qquad (4.13)$$

Again $B_n^* \neq \varnothing$ for all n and all

Lemma 2′. All B_n^* are are pairwise disjoint. ■

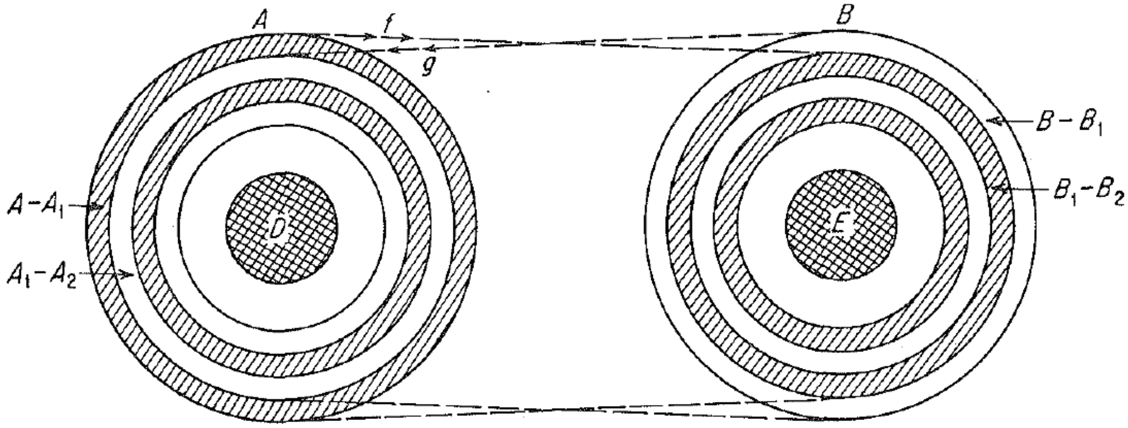
Lemma 3. FollowingThe following equavalences are true

$$A_0^* \sim B_1^* \sim A_2^* \sim B_3^* \sim A_4^* \sim \ldots \qquad (4.14)$$

$$B_0^* \sim A_1^* \sim B_2^* \sim A_3^* \sim B_4^* \sim \ldots \qquad (4.15)$$

Proof. Trivially follows from definitions A_n^* and B_k^*. Really, let's check, for example, $A_0^* \sim B_1^*$. We have: $A_0^* = A_0 - A_1$, $B_1^* = B_1 - B_2 = f(A_0) - f(A_1) = f(A_0 - A_1)$. Last equality follows from $A_1 \subset A_0$ and the fact that f is a bijection. Finally, $f(A_0 - A_1) = f(A_0^*)$. Hence $A_0^* \sim B_1^*$. ■

Equivalences (4.14), (4.15) are dual or analogous to (4.7), (4.8) and presented in figure below



We continue to proof the theorem. Since $A_0^* \sim B_1^*$ (4.14) and $A_1^* \sim B_0^*$ (4.15), and because of Lemmas (2) and (2′) we apply Lemma 1 and obtain

$(A_0^* \cup A_1^*) \sim (B_0^* \cup B_1^*)$.

More generally, we deduce that for all *n*

$$(A_{2n}^* \cup A_{2n+1}^*) \sim (B_{2n}^* \cup B_{2n+1}^*). \qquad (4.16)$$

Taking the union over all n and once again applying Lemma 1, we conclude that

$$(\bigcup_{n\geq 0} A_n^*) \sim (\bigcup_{n\geq 0} B_n^*) \qquad (4.17)$$

Left hand side (LHS) and Right hand side (RHS) of (4.17), denoted by $\tilde{A}$ and $\tilde{B}$ respectively, look a lot like A and B. But in general it is not true. Recall that

$$\tilde{A} = (A_0 - A_1) \cup (A_1 - A_2) \cup \ldots \cup (A_n - A_{n+1}) \cup \ldots \qquad (4.18)$$

$$\tilde{B} = (B_0 - B_1) \cup (B_1 - B_2) \cup \ldots \cup (B_n - B_{n+1}) \cup \ldots \qquad (4.19)$$

$$\tilde{A} \sim \tilde{B} \qquad (4.17')$$

Clear that $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$. For instance, if $x \in \tilde{A}$, then there exist n such that $x \in (A_n - A_{n+1}) \Rightarrow x \in A_n \Rightarrow x \in A$ because of (4.9′).

Now let us bring the light to the structure of A-$\tilde{A}$ and B-$\tilde{B}$. Let $x \in A - \tilde{A}$. Last means that

$$x \in A = A_0 \qquad (4.20)$$

and $x \notin \tilde{A}$. (18) implies that

$$x \notin A_0 - A_1 \qquad (4.20_1)$$

$$x \notin A_1 - A_2 \qquad (4.20_2)$$

....

$$x \notin A_n - A_{n+1} \qquad (4.20_{n+1})$$

...

System of relations (4.20)-(4.20_1)–(4.20_{n+1}) imples that $x \in A_n$, $n \geq 0$. Hence $x \in \bigcap_{n\geq 0} A_n$. Denote $D = \bigcap_{n\geq 0} A_n$. Therefore, $A - \tilde{A} \subseteq D$.

Conversly. Let $x \in D \Rightarrow x \in A_n$, $n \geq 0 \Rightarrow x \notin A_n - A_{n+1}$ for all $n \geq 0 \Rightarrow$ [because of (4.18)] $\Rightarrow x \notin \tilde{A}$ $\Rightarrow x \in A - \tilde{A}$. Hence, $D \subseteq A - \tilde{A}$. Therefore, $\bigcap_{n \geq 0} A_n = D = A - \tilde{A}$.

Thus, we proved that sets $\tilde{A}$ and D form a partition of the set A:

(i). $\tilde{A} \cap D = \varnothing$

(ii) $\tilde{A} \cup D = A$

By similar arguments it can be proved that $\tilde{B}$ and $E = \bigcap_{n \geq 0} B_n$ form a partition of B, that is:

(iii) $\tilde{B} \cap E = \varnothing$

(iv) $\tilde{B} \cup E = B$

Ealier we proved that $\tilde{A} \sim \tilde{B}$ (see 4.17′). We show below that $D \sim E$.

Lemma 4. f(D)=E and g(E)=D.

Proof. We show that f(D)=E. Let $x \in D \Rightarrow x \in A_n$, $n \geq 0 \Rightarrow f(x) \in B_n$, $n \geq 1 \Rightarrow f(x) \in B_0 \Rightarrow f(x) \in B_n$, $n \geq 0 \Rightarrow f(x) \in E = \bigcap_{n \geq 0} B_n$. Hence, $f(D) \subset E$.

Conversly, Let $y \in E \Rightarrow y \in B_n$, $n \geq 0 \Rightarrow \exists!\ x \in A_0$ such that $f(x) = y \in B_1$ [since f is the bijection]. As $B_n = f(A_{n-1})$ and $y \in B_n \Rightarrow x \in A_{n-1}$, $n \geq 1 \Rightarrow x \in \bigcap_{n \geq 0} A_n = D \Rightarrow y = f(x) \in f(D)$. Hence, $E \subset f(D)$.

Therefore, f(D)=E.

By similar arguments we can prove that g(E)=D. ■

Finally, we apply Lemma 1 to the pairs $(\tilde{A}, D)$ and $(\tilde{B}, E)$ and obtain that $A \sim B$. Theorem 15 is proved. ■

EXERCISES. SET 1 (Solved Problems)

Sets, Operations, Duality

1.1. Which of these sets are equal: $\{x, y, z\}$, $\{z, y, z, x\}$, $\{y, x, y, z\}$, $\{y, z, x, y\}$?

***Solution*.** They are all equal. Order and repetition do not change a set. ■

1.2. List the elements of each set where **N** = {1, 2, 3, …}.

(*a*) $A = \{x \in \mathbf{N} \mid 2 < x < 10\}$

(*b*) $B = \{x \in \mathbf{N} \mid x \text{ is odd}, x < 9\}$

(c) $C = \{x \in \mathbf{N} \mid (x-1.5)^2 = 0\}$

***Solution*.**

(*a*) *A* consists of the positive integers between 2 and 10; hence $A = \{3, 4, 5, 6, 7, 8, 9\}$.

(*b*) *B* consists of the odd positive integers less than 9; hence $B = \{1, 3, 5, 7\}$.

(*c*) No positive integer satisfies $(x-1.5)^2 = 0$; hence $C = \emptyset$, the empty set. ■

1.3. Let **U**={1, 2, …, 9} be the universal set, and let

$A = \{1, 2, 3, 4, 5\}$, $C = \{5, 6, 7, 8, 9\}$, $E = \{2, 4, 6, 8\}$,

$B = \{4, 5, 6, 7\}$, $D = \{1, 3, 5, 7, 9\}$, $F = \{1, 5, 9\}$.

Find: (*a*) $A \cup B$ and $A \cap B$; (*b*) $A \cup C$ and $A \cap C$; (*c*) $D \cup F$ and $D \cap F$.

***Solution*.** Recall that the union $X \cup Y$ consists of those elements in either *X* or *Y* (or both), and that the intersection $X \cap Y$ consists of those elements in both *X* and *Y*

(*a*) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$ and $A \cap B = \{4, 5\}$

(*b*) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \mathbf{U}$ and $A \cap C = \{5\}$

(*c*) $D \cup F = \{1, 3, 5, 7, 9\} = D$ and $D \cap F = (1, 5, 9) = F$

Observe that $F \subseteq D$, so by Theorem 4 we must have $D \cup F = D$ and $D \cap F = F$. ■

1.4. Prove: $B\backslash A=B\cap A'$. Thus, the set operation of difference can be written in terms of the operations of intersection and complement.

Solution. $B\backslash A = \{x \mid x \in B, x\notin A\}=\{x \mid x\in B, x\in A'\}=B\cap A'$. ■

1.5. Prove: $(A\cup B)\backslash(A\cap B)=(A\backslash B)\cup(B\backslash A)$. (Thus either one may be used to define $A\oplus B$.)

Solution. Using $X\backslash Y=X\cap Y'$ and the laws in Table 1, including DeMorgan's Law, we obtain:

$(A\cup B)\backslash(A\cap B)=(A\cup B)\cap(A\cap B)'=(A\cup B)\cap(A'\cup B')$

$= (A\cap A')\cup(A\cap B')\cup(B\cap A')\cup(B\cap B')$

$= \emptyset\cup(A\cap B')\cup(B\cap A')\cup\emptyset$

$= (A\cap B')\cup(B\cap A')=(A\backslash B)\cup(B\backslash A)$ ■

1.6. Let $\mathbf{N}=\{1, 2, 3,\dots\}$ and, for each $n\in\mathbf{N}$, Let $A_n=\{n, 2n, 3n,\dots\}$. Find:

(a) $A_3\cap A_5$; (b) $A_4\cap A_6$;

(c) $\bigcup_{i\in P} A_i$ where $P=\{2, 3, 5, 7, 11, \dots\}$ is the set of prime numbers.

Solution.

(a) Those numbers which are multiples of both 3 and 5 are the multiples of 15; hence $A_3\cap A_5=A_{15}$.

(b) The multiples of 12 and no other numbers belong to both A_4 and A_6, hence $A_4\cap A_6=A_{12}$.

(c) Every positive integer except 1 is a multiple of at least one prime number; hence $\bigcup_{i\in P} A_i=\{2, 3, 4, \dots\}=N\backslash\{1\}$ ■

1.7. Write the dual of: (*a*) $(\mathbf{U}\cap A)\cup(B\cap A)=A$; (*b*) $(A\cap\mathbf{U})\cap(\emptyset\cup AC)=\emptyset$.

Solution. Interchange ∪ and ∩ and also **U** and ∅ in each set equation:

(*a*) $(\emptyset\cup A)\cap(B\cup A)=A$; (*b*) $(A\cup\emptyset)\cup(\mathbf{U}\cap AC)=\mathbf{U}$. ■

Partitions

1.8. Let $S=\{a, b, c, d, e, f, g\}$. Determine which of the following are partitions of S:

(*a*) $P1=[\{a, c, e\}, \{b\}, \{d, g\}]$, (*c*) $P3=[\{a, b, e, g\}, \{c\}, \{d, f\}]$,

(*b*) $P2=[\{a, e, g\}, \{c, d\}, \{b, e, f\}]$, (*d*) $P4=[\{a, b, c, d, e, f, g\}]$.

Solution.

(*a*) $P1$ is not a partition of S since $f\in S$ does not belong to any of the cells.

(*b*) $P2$ is not a partition of S since $e\in S$ belongs to two of the cells.

(*c*) $P3$ is a partition of S since each element in S belongs to exactly one cell.

(*d*) $P4$ is a partition of S into one cell, S itself. ■

EXERCISES. SET 2.

Sets, Operations

2.1. Which of the following sets are equal?

$A=\{x \mid x^2-4x+3=0\}$, $C=\{x \mid x\in N, x<3\}$, $E=\{1, 2\}$, $G=\{3, 1\}$,

$B=\{x \mid x^2-3x+2=0\}$, $D=\{x \mid x\in N, x \text{ is odd}, x<5\}$, $F=\{1, 2, 1\}$, $H=\{1, 1, 3\}$.

2.2. Prove that if $A\cup B=A$ and $A\cap B=A$, then $A=B$.

2.3. Let A and B be any sets. Prove:

(a) A is the disjoint union of $A\backslash B$ and $A\cap B$.

(b) $A\cup B$ is the disjoint union of $A\backslash B$, $A\cap B$, and $B\backslash A$.

2.4. Prove the Absorption Laws: (a) $A\cup(A\cap B)=A$; (b) $A\cap(A\cup B)=A$.

2.5. Show that in general $(A-B)\cup B\neq A$. if $A\cup B=A$ and $A\cap B=A$, then $A=B$.

2.6. Let $A=\{2, 4, \dots, 2n, \dots)$ and $B=\{3, 6, \dots, 3n, \dots \}$.

Find A∩B and A-B.

2.7. Prove theorems 7-9:

2.8. Let *f*: X→Y, and B⊆Y. Prove that f^{-1}(Y-B)=X-f^{-1}(B), that is, pre-image of the complement is equal to the complement of pre-image.

Whether the similar statement is true for the image of the complement? If yes, then prove. Otherwise provide a counterexample.

2.9. Let A_n be the set of all positive integers divisible by n. Find the sets

(a) $\bigcup_{n=2}^{\infty} A_n$ (b) $\bigcap_{n=2}^{\infty} A_n$

2.10*. (*Familiarity with sup and inf from Calculus 1 is required*). Let b≥a+2. Find

(a) $\bigcup_{n=1}^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}]$ (b) $\bigcap_{n=1}^{\infty}(a-\frac{1}{n}, b+\frac{1}{n})$

Use Theorem 11 from Part 4 for the exercises 2.11-2.14.

2.11*. Prove that the set A of all polynomials with rational coefficients is countable (use the Theorem 11).

2.12*. A number λ is called an algebraic one if it is a root of a polynomial P(t) with rational coefficients. Prove that the set of all algebraic numbers is countable.

2.13*. Prove that the set of all rational intervals on the straight-line **R** is countable. (An interval *I*⊂**R** is called rational if its end points are rational numbers).

2.14*. Prove that set of all points of the plane with rational coordinates is countable.

2.15*. Let M be any infinite set and A any countable set. Prove that M is equivalent to M∪A.

2.16*. Prove the existence of uncountably many **transcendental numbers**, i.e., numbers which are not algebraic. (Hint: Use Theorems 11 and 14).

2.17*. Prove that a set with an uncountable subset is itself uncountable.

2.18*. Prove that the set of all real functions (more generally, functions taking values in a set containing at least two elements) defined on a set M is of power greater than the power of M. In particular, prove that the power of the set of all real functions (continuous and discontinuous) defined in the interval [0, 1] is greater than *c*.

Hint. Use the fact that the set of all characteristic functions (i.e., functions taking only the values 0 and 1) on M is equivalent to the set of all subsets of M.

2.18*. Give an indirect proof of the equivalence of the closed interval [a, b], the open interval (a, b) and the half-open interval [a, b) or (a, b].

Hint. Use Theorem 15.

2.19*. Prove that the union of a finite or countable number of sets each of power c is itself of power *c*.

2.20*. Prove that each of the following sets has the power of the continuum:

a) The set of all infinite sequences of positive integers;

b) The set of all ordered n-tuples of real numbers;

c) The set of all infinite sequences of real numbers.

2.21*. Prove the Proposition 3 in Theorem 16: