Derivative

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BHOS

Calculus

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We considered the derivative of f at a fixed point a as

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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If we let the number a vary, in other words if we replace a by a variable x, we get

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

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If we want to indicate the value of a derivative $\frac{dy}{dx}$ at specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a}$$

Example: Find a formula for the derivative of $f(x) = x^3 - x$.

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Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

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$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

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f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
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$f'(x)$ exists if $x > 0$. So, the domain of f' is $(0, \infty)$.

Example: Let $f(x) = \frac{1-x}{2+x}$. Find f'.

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$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h}$$

$$= \lim_{h\to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{-3h}{h(2+x+h)(2+x)} = \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2}$$

Definition

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval $(a, b), (a, \infty), (-\infty, b), (-\infty, \infty)$ if it is differentiable at each point in the interval.

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$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

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Example: Where is the function $f(x) = |x|$ differentiable?

Solution: If $x > 0$, then $|x| = x$ and we can choose h small enough so that $x + h > 0$ and hence $|x + h| = x + h$. So for $x > 0$ we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

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For $x = 0$ we have to investigate

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|0+h| - |0|}{h} \qquad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

and $$\lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

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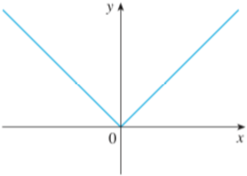
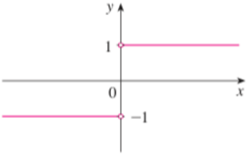


(a) $y = f(x) = |x|$



(b) $y = f'(x)$

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Taking the limit of both sides as $x \to a$,

$$\begin{aligned}
\lim_{x \to a}(f(x) - f(a)) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) \\
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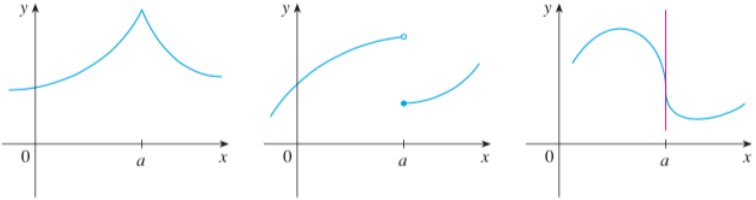
$$\lim_{x \to a} f(x) = f(a)$$

Note: The converse of the Theorem is false; that is, there are functions that are continuous but not differentiable.

A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has:



(a) A corner (b) A discontinuity (c) A vertical tangent

Higher order derivatives

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Example:

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative:	$y' = 3x^2 - 6x$
Second derivative:	$y'' = 6x - 6$
Third derivative:	$y''' = 6$
Fourth derivative:	$y^{(4)} = 0.$