# Fall 2021 - MATH 1101 Discrete Structures
# Lecture 8-9

## FINITE SETS, COUNTING PRINCIPLES, ELEMENTS OF COMBINATORICS.

- **PART 1. Counting Elements in Finite Sets.**
- **PART 2. Inclusion–Exclusion Principle**
- **PART 3. Pigeonhole Principle**
- **PART 4. Factorial Function and Binomial Coefficients**
- **PART 5. Permutations and Combinations**

**Exercises. Set 1 (Solved Problems)**

**Exercises. Set 2 (Supplementary Problems)**

## PART 1. Counting Elements in Finite Sets

In this Part we introduce the counting principles in finite sets and the essentials of combinatorial techniques to find number of elements in finite sets. The following lemmas applies.

**Lemma 1.** Suppose $A$ and $B$ are finite disjoint sets. Then $A\cup B$ is finite and

$$n(A\cup B)=n(A)+n(B)$$

***Proof.*** In counting the elements of $A\cup B$, first count those that are in $A$. There are $n(A)$ of these. The only other elements of $A\cup B$ are those that are in $B$ but not in $A$. But since $A$ and $B$ are disjoint, no element of $B$ is in $A$, so there are $n(B)$ elements that are in $B$ but not in $A$. Therefore, $n(A\cup B)=n(A)+n(B)$. ■

For any sets $A$ and $B$, the set $A$ is the disjoint union of $A\backslash B$ and $A\cap B$, that is, $A=(A\backslash B)\cup(A\cap B)$ and $(A\backslash B)\cap(A\cap B)=\varnothing$. Thus Lemma 1 gives us the following useful result.

**Corollary 1.** Let $A$ and $B$ be finite sets. Then $n(A\backslash B)=n(A)-n(A\cap B)$ ■

**Corollary 2.** Let $A$ be a subset of a finite universal set $\mathbf{U}$. Then $n(A')=n(\mathbf{U})-n(A)$ ■

For example, suppose a class **U** with 45 students has 22 full-time students. Then there are $45-22=23$ part-time students in the class **U**.

**Lemma 2.** Suppose $A$ and $B$ are finite sets. Then $A\times B$ is finite and

$$n(A\times B)=n(A)\cdot n(B)$$

***Proof.*** Clear, by definition of $A\times B$. ■

**Example 1.** Let $A=\{2, 3\}$, $B=\{2, 3, 7\}$. Then

$$A\times B=\{(2, 2), (2, 3), (2, 7), (3, 2), (3, 3), (3, 7)\}$$

That is, $n(A\times B)=6=2\cdot 3=n(A)\cdot n(B)$ ■

Lemmas 1 and 2 form a base for the following two Principles

**(1) Sum Rule Principle:** Suppose A and B are disjoint sets. Then

$$n(A\cup B)=n(A)+n(B)$$

**(2) Product Rule Principle:** Let $A\times B$ be the Cartesian product of sets A and B. Then

$$n(A\times B)=n(A)\cdot n(B)$$

Both principles can be extended for the case of any finite number of finite sets. Namely:

If $A_1, \ldots, A_n$ are finite disjoint sets then

$$n(A_1 \cup \ldots \cup A_n) = n(A_1) + \ldots + n(A_n)$$

If $A_1, \ldots, A_n$ are finite sets then

$$n(A_1 \times \ldots \times A_n) = n(A_1) \cdot \ldots \cdot n(A_n)$$

Both Principles can be reformulated in other terminology as following:

**(1′) Sum Rule Principle:**

**IF** some event *E* can occur in *m* ways and a second event *F* can occur in *n* ways **and** suppose **both events cannot occur simultaneously THEN** *E* or *F* can occur in *m*+*n* ways.

**(2′) Product Rule Principle:**

**IF** there is an event E which can occur in *m* ways **and**, **independent of this event**, there is a second event F which can occur in *n* ways **THEN** combinations of E and F can occur in *mn* ways.

Another type of equivalent forms of Sum Rule and Product Rule Principles are:

**(1″) Sum Rule Principle:**

**IF** a task can be done **either** in one of *m* ways **or** in one of *n* ways, **where** none of the set of *m* ways is the same as any of the set of *n* ways, **THEN** there are *m*+*n* ways to do the task.

**(2″) Product Rule Principle:**

Suppose that a procedure can be broken down into ***a sequence*** of two tasks. **IF** there are *m* ways to do the first task **and** for each of these ways of doing the first task, there are *n* ways to do the second task, **THEN** there are *mn* ways to do the procedure.

**EXAMPLE 2.** Suppose SITE has 5 different "Calculus 1" sections, 3 different "Discrete Structures" sections, and 4 different "Linear Algebra" sections. Find the number of ways a student can choose: (a) one of each kind of sections; (b) just one of the sections.

***Solution***.

(a) We apply **(2″)** form of **Product Rule Principle.** The number *m* of ways a student can choose one of each kind of sections is: $m=5\cdot3\cdot4=60$

(b) We apply **(1″)** form of **Sum Rule Principle.** The number *n* of ways a student can choose just one of the sections is: $n=5+3+4=12$ ■

**EXAMPLE 3.** A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

***Solution***.

The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are 23+15+19 =57 ways to choose a project. ■

## PART 2. Inclusion–Exclusion Principle

There is in some sense an extension of the Sum Rule Principle even when A and B are not disjoint. This extension is called the **Inclusion–Exclusion Principle**, **IEP**. Namely:

**Theorem 1. (Inclusion–Exclusion Principle, IEP).** Suppose *A* and *B* are finite sets. Then $A\cup B$ and $A\cap B$ are finite and $n(A\cup B)=n(A)+n(B)-n(A\cap B)$.

That is, we find the number of elements in *A* or *B* (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A\cap B)$ (exclusion) since its elements were counted twice.

***Proof.*** If $A$ and $B$ are finite then, clearly, $A\cup B$ and $A\cap B$ are finite. Suppose we count the elements in $A$ and then count the elements in $B$. Then every element in $A\cap B$ would be counted twice, once in $A$ and once in $B$. Thus, $n(A\cup B)=n(A)+n(B)-n(A\cap B)$. ■

**EXAMPLE 4.** Suppose a list $A$ contains the 30 students in a mathematics class, and a list $B$ contains the 35 students in an English class and suppose there are 20 names on both lists. Find the number of students:
(a) only on list $A$,
(b) only on list $B$,
(c) on list $A$ or $B$ (inclusive "or" = at least in A or in B),
(d) on exactly one list.

***Solution.***
(a) List $A$ has 30 names and 20 are on list $B$; hence 30−20=10 names are only on list $A$.
(b) Similarly, 35−20=15 are only on list $B$.
(c) We seek $n(A\cup B)$. By inclusion–exclusion,
$n(A\cup B)=n(A)+n(B)-n(A\cap B)=30+35-20=45$.
In other words, we combine the two lists and then cross out the 20 names which appear twice.
(d) By (a) and (b), 10+15=25 names are only on one list; that is, $n(A\oplus B)=25$. ■

Mathematical induction may be used to further generalize this result to any finite number of finite sets. More precisely, the following theorem is valid.

**Theorem 2. (Inclusion–Exclusion Principle – general form).**
Let $A_1, A_2, \ldots, A_r$ be subsets of a universal set $U$. Suppose we let $\boldsymbol{s_k}$ denote the sum of the cardinalities of all possible $k$-tuple intersections of the sets, that is, the sum of all of the cardinalities $n(A_{i1}\cap A_{i2}\cap\cdot\cdot\cdot\cap A_{ik})$:

$$\boldsymbol{s_1}=\sum_i n(A_i),\qquad \boldsymbol{s_2}=\sum_{i<j} n(A_i\cap A_j),\qquad \boldsymbol{s_3}=\sum_{i1<i2<i3} n(A_{i1}\cap A_{i2}\cap A_{i3}),\ \ldots,$$
$$\boldsymbol{s_k}=\sum_{i1<i2<\cdots<ik} n(A_{i1}\cap A_{i2}\cap\ldots\cap A_{ik}).$$

Then

$$\boldsymbol{n(A_1\cup A_2\cup\cdot\cdot\cdot\cup A_r)=s_1-s_2+s_3-\cdot\cdot\cdot+(-1)^{r-1}s_r}$$

**Proof**. Exercise. (Apply Mathematical Induction). ■

For the particular case of three finite sets we get the formula:

$$\boldsymbol{n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C)}\quad ■$$

There exists an alternate form of the Theorem 2:

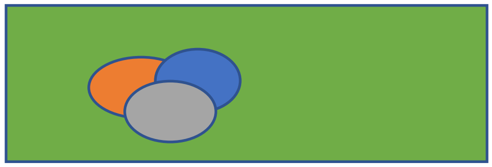
**Theorem 3. (Alternate form of Inclusion–Exclusion Principle).**
Let $A_1, A_2, \ldots, A_r$ be subsets of a universal set $\mathbf{U}$. Then the number $m$ of elements which do not appear in any of the subsets $A_1, A_2, \ldots, A_r$ of $\mathbf{U}$ is:

$$\boldsymbol{m=n(A_1'\cap A_2'\cap\ldots\cap A_r')=|\mathrm{U}|-s_1+s_2-s_3+\cdot\cdot\cdot+(-1)^r s_r=|\mathrm{U}|-n(A_1\cup A_2\cup\cdot\cdot\cdot\cup A_r)}\qquad \textbf{(*)}$$

**Proof**. Immediately follows from DeMorgan's Law

$n(A_1'\cap A_2'\cap\ldots\cap A_r')=n([A_1\cup A_2\cup\ldots\cup A_r]')=|\mathbf{U}|-n(A_1\cup A_2\cup\ldots\cup A_r)$ ■

Let us consider the case of three subsets $A_1, A_2, A_3$ of the universal set $\mathbf{U}$ to make the statement of the Theorem 3 more clear. Below $\mathbf{U}$ is the green rectangle, $A_1, A_2, A_3$ are orange, grey, and blue disks, respectively.



Then

$A_1'$ - complement of $A_1$ in **U** – elements which do not appear in $A_1$.

$A_2'$ - complement of $A_2$ in **U** - elements which do not appear in $A_2$.

$A_3'$ - complement of $A_3$ in **U** - elements which do not appear in $A_3$.

$A_i' \cap A_j'$ is the set elements of which do not appear in $A_i$ and in $A_j$ simultaneously.

$A_1' \cap A_2' \cap A_3'$ - is the set elements which do not appear in $A_1$, $A_2$ and $A_3$ simultaneously, it means that elements do not appear in the union $A_1 \cup A_2 \cup A_3$.

It easy to understand that in this case we have:

$n(A_1' \cap A_2' \cap A_3') = n([A_1 \cup A_2 \cup A_3]') = |\mathbf{U}| - n(A_1 \cup A_2 \cup A_3)$

**EXAMPLE 5.** Let **U** be the set of positive integers not exceeding 1000. Then $|U|=1000$. Let $S$ is the set of such integers which are not divisible by 3, 5, 7. For instance 8, 11, 13, 16, 17, 19, 22, … all are elements of S.

Find $|S|$.

***Solution.*** Let

$A$ be the subset of integers which are divisible by 3, $A=\{3, 6, 9, \ldots, 996, 999\}$, $a \in A \Leftrightarrow a|3$

B be the subset of integers which are divisible by 5, $B=\{5, 10, 15, \ldots, 995, 1000\}$ $b \in A \Leftrightarrow b|5$

$C$ be the subset of integers which are divisible by 7. $C=\{7, 14, 21, \ldots, 987, 994\}$, $c \in C \Leftrightarrow c|7$

Then:

$A'$: $a_1 \in A' \Leftrightarrow a_1 \notin A \Leftrightarrow$ $a_1$ is not divisible by 3

$B'$: $b_1 \in B' \Leftrightarrow b_1 \notin B \Leftrightarrow$ $b_1$ is not divisible by 5

$C'$: $c_1 \in C' \Leftrightarrow c_1 \notin C \Leftrightarrow$ $c_1$ is not divisible by 7

Thus:

$A' \cap B' \cap C'$ = positive integers from U which are not divisible by 3, 5, 7

Therefore S= $A' \cap B' \cap C'$.

Hence to find n(S) we need to find $n(A' \cap B' \cap C')$. By the identity (*) (Theorem 3) we have

$n(A' \cap B' \cap C') = n([A \cup B \cup C]') = |\mathbf{U}| - n(A \cup B \cup C)$

Our problem now is reduced now to the finding of $n(A \cup B \cup C)$.

From the Theorem 2 we have

$$\boldsymbol{n(A \cup B \cup C) = [n(A) + n(B) + n(C)] - [n(A \cap B) + n(A \cap C) + n(B \cap C)] + [n(A \cap B \cap C)]}$$

$|A| = \mathrm{int}(1000/3) = 333$, $|B| = \mathrm{int}(1000/5) = 200$, $|C| = \mathrm{int}(1000/7) = 142$,

Next three numbers:

$|A \cap B| = \mathrm{int}(1000/15) = 66$, $|A \cap C| = \mathrm{int}(1000/21) = 47$, $|B \cap C| = \mathrm{int}\ (1000/35) = 28$,

$|A \cap B \cap C| = \mathrm{int}(1000/105) = 9$

Thus, by the Inclusion–Exclusion Principle, (first, Theorem 3 then Theorem 2), we have:

$|S| = 1000 - n(A \cup B \cup C) = 1000 - [n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)] =$

$1000 - (333 + 200 + 142) - (66 + 47 + 28) + 9] = 1000 - 675 + 141 - 9 = 457.$ ■

# PART 3. Pigeonhole Principle

Many results in combinational theory come from the following almost obvious statement.

**Pigeonhole Principle (PP):** *If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.*
***Proof.*** We prove the pigeonhole principle using a proof by contraposition. Assume the contrary. Suppose that none of the *k* boxes contains more than one object. Then the total number of objects would be at most *k*. This is a contradiction, because there are at least *k*+1 objects. ■

**Corollary 3.** A function $f:X\rightarrow Y$, $|X|\geq(k+1)$, $|Y|=k$, is not one-to-one.

***Proof.*** Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that f(x)=y. Because the domain contains k+1 or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one. ■

This principle can be applied to many problems where we want to show that a given situation can occur.

**EXAMPLE 6.**

(a) Assume that there are 13 professors at the department. Then two of the professors (pigeons) were born in the same month (pigeonholes).

(b) In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

(c) How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
***Solution***. There are 101 possible scores on the final (from 0 to 100). The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score. ■

The pigeonhole principle is a useful tool in many proofs, including proofs of surprising results, such as that given in the next Example 7.

**EXAMPLE 7.** Show that for every integer *n* there is a multiple of *n* that has only 0s and 1s in its decimal expansion.

***Solution***. Let *n* be a positive integer. Consider the list A of (*n*+1) integers 1, 11, 111, ..., 11 ... 1, where the last integer in this list is the integer with (*n*+1) 1s in its decimal expansion. Note that there are *n* possible remainders when an integer is divided by *n*: for arbitrary integer *m* there exist integers *p* and *q* such that $m=pn+q$, here $0\leq q<n$. Since the list A contains (*n*+1) integers then by the pigeonhole principle there must be two integers with the *same remainder q* when divided by *n*. Let they will be a and b:

| | | | | | | |
|---|---|---|---|---|---|---|
| a= | 1 | 1 | 1 | 1 | 1 | 1 |
| b= | | | 1 | 1 | 1 | 1 |
| a-b= | 1 | 1 | 0 | 0 | 0 | 0 |

| |
|---|
| a=mn+q |
| b=kn+q |
| a-b=(m-k)n |

The result of subtraction of smaller of these integers (for instance, b) from larger one (for instance, a) is a multiple of n, in one side, and, from other side, it has a decimal expansion consisting entirely of 0s and 1s. ■

**Generalized Pigeonhole Principle (GPP):**

The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. **However, even more can be said when the number of objects exceeds a multiple of the number of boxes.** For instance, among any set of 21 decimal digits there must be 3 that are the same. This follows because when 21 objects are distributed into 10 boxes, one box must have more than 2 objects.

**GPP** *If N objects are placed into k boxes, then there is at least one box containing at least* $\lceil N/k \rceil$ *objects.*

Or equivalently,

**GPP.** *If N pigeonholes are occupied by* $kN+1$ *or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by* $k+1$ *or more pigeons.*

> **Note**. $\lceil x \rceil$ is the *ceiling* (round up) *function* which assigns to the real number x the smallest integer that is greater than or equal to x. For example, $\lceil 3{,}2 \rceil$=4, $\lceil 0{,}3 \rceil$=1, $\lceil -3{,}2 \rceil$=-3, $\lceil 3 \rceil$=3.

***Proof of* GPP.** Assume the contrary. Suppose that none of the boxes contains more than $(\lceil N/k \rceil - 1)$ objects. Then, the total number of objects is at most $k(\lceil N/k \rceil - 1) < k((\frac{N}{k}+1)-1) = N$, where the inequality $\lceil N/k \rceil < (N/k)+1$ has been used. This is a contradiction because there are N objects in total. ■

**EXAMPLE 8.** Among 75 people there are at least $\lceil 75/12 \rceil$=7 who were born in the same month.■

**EXAMPLE 9.** What is the minimum number of students required in a Discrete Structures class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

***Solution.*** The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer *N* such that $\lceil N/5 \rceil$=6. The smallest such integer is N=5·5+1=26. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade. ■

**A common type of problem solved by GPP**. *Find a minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes.*

***Solution.*** When we have N objects, the **GPP** tells us there must be at least *r* objects in one of the boxes as long as $\lceil N/k \rceil \geq r$. The smallest integer *N* with $(N/k) > r-1$, namely, $N=k(r-1)+1$, is the smallest integer satisfying the inequality $\lceil N/k \rceil \geq r$. Could a smaller value of N suffice? The answer is no, because if we had k(r−1) objects, we could put (r–1) of them in each of the k boxes and no box would have at least r objects.

When thinking about problems of this type, it is useful to consider how you can avoid having at least r objects in one of the boxes as you add successive objects. To avoid adding a r-th object to any box, you eventually end up with (r−1) objects in each box. There is no way to add the next object without putting an r-th object in that box. ■

There are a variety of applications of GPP in different areas.

**EXAMPLE 10.** What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form *NXX-NXX-XXXX*, where the first three digits form the area code, *N* represents a digit from 2 to 9 inclusive, and *X* represents any digit.)

***Solution.*** Clear that there are eight million different phone numbers of the form *NXX-XXXX*. Hence, **by the GPP**, among 25 million telephones, at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil=4$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different.

**EXAMPLE 11.** During a month with 30 days, a volleyball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

***Solution:*** Let $a_j$ be the total number of games played on first j days of the month. Then $a_1, a_2, ..., a_{30}$ is a strongly increasing sequence of distinct positive integers, with $1 \le a_j \le 45$. Moreover, $(a_1+14), (a_2+14), ..., (a_{30}+14)$ is also a strongly increasing sequence of distinct positive integers, with $15 \le (a_j+14) \le 59$.

The 60 positive integers $a_1, a_2, ..., a_{30}, (a_1+14), (a_2+14), ..., (a_{30}+14)$ are all less than or equal to 59. Hence, **by the pigeonhole principle** two of these integers are equal. Because

- the integers $a_j$ are all distinct (j=1, 2, ..., 30) and
- the integers $(a_j+14)$ are all distinct, (j=1, 2, ..., 30),

there must be indices i and j with $a_i=a_j+14$. This means that exactly 14 games were played from day (j+1) to the day i. ■

Example 12 below demonstrates non-trivial application of GGP in number theory.

**EXAMPLE 12.** Show that among any (n+1) positive integers not exceeding 2n there must be an integer that divides one of the other integers.

***Solution:*** Write each of the $n + 1$ integers $a_1, a_2, ..., a_{n+1}$ as a power of 2 times an odd integer (any integer can be written as a product of powers of prime numbers). In other words, let $a_j=2^{k_j} \cdot q_j$ for $j = 1, 2, ..., n+1$, where $k_j$ is a nonnegative integer and $q_j$ is odd. The integers $q_1, q_2, ..., q_{n+1}$ are all odd positive integers less than $2n$. Because there are only $n$ odd positive integers less than $2n$, it follows from **the pigeonhole principle** that two of the integers $q_1, q_2, ..., q_{n+1}$ must be equal. Therefore, there are distinct integers $i$ and $j$ such that $q_i = q_j$. Let $q$ be the common value of $q_i$ and $q_j$. Then, $a_i=2^{k_i} \cdot q_i$, $a_j=2^{k_j} \cdot q_j$. It follows that if $k_i < k_j$, then $a_i$ divides $a_j$; while if $k_i > k_j$, then $a_j$ divides $a_i$. ■

## PART 4. Factorial Function and Binomial Coefficients

We discuss two important mathematical functions frequently used in combinatorics.

### Factorial Function

**Definition 1**. The product of the positive integers from 1 to $n$ inclusive is denoted by $n!$, read "*n* factorial." Namely:

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (n-2)(n-1)n = n(n-1)(n-2) \cdot ... \cdot 3 \cdot 2 \cdot 1 \qquad (1)$$

Accordingly,

$$1!=1 \text{ and } n!=n(n-1)!$$

It is also convenient to define $0!=1$. ■

**EXAMPLE 13.**

(a) 3! =3 • 2 • 1=6, 4! =4 • 3 • 2 • 1=24, 5!=5 • 4!=5(24)=120.

(b) $\frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}$ = [multiply numerator and denominator by (12-3)!=9!] = $\frac{12\cdot 11\cdot 10\cdot 9!}{3\cdot 2\cdot 1\cdot 9!}$ = (by formula (1)) = $\frac{12!}{3!\cdot 9!}$

Now we generalize (b) (in (b) n=12, r=3).

(c) $\frac{n\cdot(n-1)\cdot\ldots\cdot(n-r+1)}{r\cdot(r-1)\cdot(r-2)\cdot\ldots\cdot 1}$= [multiply numerator and denominator by (n-r)!] =

= $\frac{n(n-1)\ldots(n-r+1)\cdot(n-r)!}{r(r-1)(r-2)\ldots 1\cdot(n-r)!}$ = (by definition of n!) = $\frac{n!}{r!\cdot n-r!}$

## Binomial Coefficients

**Definition 2** Symbol $\binom{n}{r}$, read "*nCr*" or "*n* Choose *r*," where *r* and *n* are positive integers with *r*≤*n* is defined as follows:

$$\binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 1} \tag{2}$$

or, equivalently, according to arguments above as:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{3}$$ ■

**Corollary 4.** $\binom{n}{r}$ has exactly *r* factors in the numerator and in the denominator by (2). ■

**Lemma 3.** $\binom{n}{r} = \binom{n}{n-r}$

***Proof.*** Trivially follows from the definition of $\binom{n}{r}$ and relation n-(n-r)=r. ■

**Corollary 5**. 0!=1 and definition of $\binom{n}{r}$ implies

$$\binom{n}{0} = \binom{n}{n} = \binom{0}{0} = 1$$ ■

**EXAMPLE 14.** Compute $\binom{10}{7}$.

By Corollary 4 there will be 7 factors in both the numerator and the denominator. However, 10−7=3. Thus, we use Lemma 3 to compute:

$\binom{10}{7} = \binom{10}{3} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}$ =120 ■

**Theorem 4. (Pascal's identity).**

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r} \tag{4}$$

***Proof.***

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{(r)!(n-r)!} = \frac{n!}{(r-1)!(n-r+1))!} + \frac{n!}{(r)!(n-r)!} =$$

$$= \frac{n!}{(r-1)!(n-r)!(n-r+1)} + \frac{n!}{(r-1)!r(n-r)!} = \frac{rn!+(n-r+1)n!}{(r-1)!r(n-r)!(n-r+1)} = \frac{n!(r+(n-r+1))}{r!(n-r+1)!}$$

$$= \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}$$ ■

The numbers $\binom{n}{r}$ are called ***binomial coefficients***, since they appear as the coefficients in the expansion of $(a+b)^n$. Specifically:

**Theorem 5 (Binomial Theorem).**

$$(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r. \qquad (5)$$

***Proof.*** We aim to prove that $(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r=$

$=a^nb^0+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{r}a^{n-r}b^r+\cdots+\binom{n}{n-1}a^1b^{n-1}+\binom{n}{n}a^0b^n=$

$=$ [ taking into account that $\binom{n}{0}=\binom{n}{n}=1$ ] $=$

$=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{r}a^{n-r}b^r+\cdots+\binom{n}{n-1}a^1b^{n-1}+b^n$

Now we apply Mathematical Induction.

***Basis Step.*** Formula is clearly true for n=1 (and moreover for n=2)

***Inductive Step.*** We first assume the inductive hypothesis that the formula is true for an arbitrary positive integer n. That is,

$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{r}a^{n-r}b^r+\cdots+\binom{n}{n-1}a^1b^{n-1}+b^n$

Now we consider expansion

$(a+b)^{n+1}=(a+b)(a+b)^n=(a+b)[a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{r}a^{n-r}b^r+\cdots+\binom{n}{n-1}a^1b^{n-1}+b^n]=$

$=a^{n+1}+[1+\binom{n}{1}]a^nb+[\binom{n}{1}+\binom{n}{2}]a^{n-1}b^2+\ldots+[\binom{n}{r-1}+\binom{n}{r}]a^{n-(r-1)}b^r+\ldots+[\binom{n}{n-1}+1]ab^n+b^{n+1}.$

From Pascal's identity (Theorem 10), it follows that $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$

thus,

$(a+b)^{n+1}=a^{n+1}+\left[\binom{n+1}{1}\right]a^nb+\left[\binom{n+1}{2}\right]a^{n-1}b^2+\ldots+\left[\binom{n+1}{r}\right]a^{n-(r-1)}b^r+\ldots+\left[\binom{n+1}{n}\right]ab^n+b^{n+1}.$

Hence the result is true for n+1. Therefore, the result is true for all positive integers n. ■

**EXAMPLE 15.**

$(a+b)^1=\binom{1}{0}a+\binom{1}{1}b=1a+1b=a+b$

$(a+b)^2=\binom{2}{0}a^2b^0+\binom{2}{1}a^1b^1+\binom{2}{2}a^0b^2=a^2+2ab+b^2.$

**Corollary 6**. $2^n=\sum_{r=0}^{n}\binom{n}{r}$. That is, the sum of all Binomial Coefficients is $2^n$.

***Proof.*** Take a=b=1 in Binomial Theorem

Left part of the identity (1) yields $2^n$. On the right hand side we obtain:

$$1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{r}+\cdots+\binom{n}{n-1}+1=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{r}+\cdots+\binom{n}{n-1}+\binom{n}{n}$$

Therefore,

$$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{r}+\cdots+\binom{n}{n-1}+\binom{n}{n} \qquad (6)$$

Q.E.D ■

**Corollary 7**. Last expression for $2^n$ has another known form. Applying definition of $\binom{n}{r}$ to the last identity yields:

$$2^n=1+n+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)}{3!}+\ldots+\frac{n(n-1)(n-2)\ldots(n-(r-1))}{r!}+\ldots+\frac{n(n-1)(n-2)\ldots(n-(n-2))}{(n-1)!}+\frac{n(n-1)\ldots(n-(n-1))}{n!}.$$ ■

Corollaries 5 or 6 provides very known decompositions for $2^n$ for n=1, 2, 3… below

n=0: $1=2^0$

n=1: $1+1=2^1$

n=2: $1+2+1=2^2$

n=3: $1+3+3+1=2^3$

n=4: $1+4+6+4+1=2^4$

n=5: $1+5+10+10+5+1=2^5$

n=6: $1+6+15+20+15+6+1=2^6$

……………………………………

These decompositions determine so-called *Pascal's triangle* below (Table 2). Here are the top 6 rows of the *Pascal's triangle*.

**Table 2 *Pascal's triangle (*n=6*)***

Numbers in column **Σ** of Table 1 show the sum of elements of a relative row.

| Row # | | | | | | | | | | | | | | | **Σ** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | **1** | | | | | | | | $2^0=1$ |
| 1 | | | | | | **1** | | **1** | | | | | | | $2^1=2$ |
| 2 | | | | | **1** | | **2** | | **1** | | | | | | $2^2=4$ |
| 3 | | | | **1** | | **3** | | **3** | | **1** | | | | | $2^3=8$ |
| 4 | | | **1** | | **4** | | **6** | | **4** | | **1** | | | | $2^4=16$ |
| 5 | | **1** | | **5** | | **10** | | **10** | | **5** | | **1** | | | $2^5=32$ |
| 6 | **1** | | **6** | | **15** | | **20** | | **15** | | **6** | | **1** | | $2^6=64$ |

The numbers along the sides are all 1s, and each entry in the middle is the sum of the two entries above it. These are just the binomial coefficients $\binom{n}{r}$ arranged in a table. By making it a triangle rather than a rectangle, you can see the two relationships $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$ and $\binom{n}{r}=\binom{n}{n-r}$ more clearly. Table 3 (below) is the same as Table 2, but the entries are labelled with binomial coefficients.

**Historical reference about Pascal's triangle**. Blaise Pascal (1623-1662) and Pierre de Fermat (1601-1665) studied these binomial coefficients in the context of probability in the 1600s. Their correspondence resulted in some of the first significant theory of probability and a systematic study of binomial coefficients. Because of Pascal's publication of their results, a particular arrangement of the binomial coefficients in a triangle is called Pascal's triangle. It was known in Europe for a couple of centuries before Pascal, and it was known much longer in Islamic mathematics, in India, and in China.

**Table 3 Pascal's triangle with binomial coefficients *(n=6)***

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

# PART 5. Permutations and Combinations

In this Part we study two classes of problems related with counting in finite sets:

Class 1. Counting problems which can be solved by finding the *number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.*

Class 2. Counting problems which can be solved by finding *the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.*

For example,

- in how many ways can we select ***m*** students from a group of ***n*** students to stand in line for a picture? (Related with Class 1).
- how many different committees of ***m*** students can be formed from a group of ***n*** students? (Related with Class 2).

We develop methods to answer questions such as these. To do it we introduce first the notion of *permutation*.

## Permutations

We start to discuss bijections of a finite set onto itself (so-called permutations).

**Definition 3**. Any bijection f :X→X is called a *permutation of n symbols*, or *permutation* on X. In other words, a permutation of a set of distinct objects is an ordered arrangement of these objects. ■

**EXAMPLE 16.** Let X={1, 2, …, 6}. Let f= $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 5 & 1 & 2 \end{pmatrix}$ . Here f is the permutation of 6 elements. 1st row is X, 2nd row is f(X) – permutation of X – namely, f(1)=4, f(2)=3, …, f(6)=2. ■

Two permutations f and g on X are different if there exist at least one number m, 1≤m≤n, such that f(m)≠g(m). Let S(n) be the set of all permutations on a set of n elements. We usually are interested in the number of such permutations without listing them. We denote the number of permutations of n elements by P(n). Hence, |S(n)|=P(n).

**Theorem 6**. There are n! different permutations on the set X={1, 2, 3, …, n}, that is, P(n)=n!

***Proof.*** Apply Mathematical Induction.

***Basis Step.*** Let n=1. Then there is only one bijection (identity mapping) {1}→{1}. So P(1)=1=1!

***Inductive Step.*** Assume that P(k)=k! for an arbitrary integer k>1. We have to show that P(k+1)=(k+1)!. Let X={1, 2, 3, …, k, k+1}. Define the following partition of the set S(k+1): $S^i(k+1)$ is the subset of S(k+1) consisting of permutations taking (k+1) to *i*=1, 2, … k+1, that is, $S^i(k+1)=\{f\in S(k+1) \mid f(k+1)=i\}$, *i*=1, 2, … k+1. Clear that

- each $S^i(k+1)\neq\varnothing$
- $S^i(k+1)\cap S^j(k+1)=\varnothing$ if $i\neq j$
- $\bigcup_{i=1}^{k+1} S_{k+1}^{i}$= S(k+1).

So the family ($S^i(k+1)$, *i*=1, 2, … k+1) of subsets S(k+1) form the partition of S(k+1). This partition contains (k+1) subsets. By definition, $S^i(k+1)$ consists of all bijections f: X→ X, such that f(k+1)=*i*. Restriction of any such f on the set X-{k+1} yields the bijection $\bar{f}$ : X-{k+1} → X-{$i$} and domain and codomain of $\bar{f}$ consists of k elements. By inductive hypothesis there exists exactly k! such bijections $\bar{f}$. Therefore, for each *i*=1, 2, … k+1, $|S^i(k+1)|$=k! Thus, by Sum Principle, P(k+1)=|S(k+1)|=$\sum_{i=1}^{k+1} |S^i(k+1)|$= k!+k!+ … +k!=(k+1)k!=(k+1)! ■

We also are interested in ***ordered*** arrangements of **some of the elements** of a set. An ordered arrangement of r elements of a set is called an **r-permutation** or "*a permutation of the n objects taken r at a time*".

**Definition 4**. Any arrangement of **any r of n** objects (r≤n) in a given order is called an "r-permutation" or "a permutation of the n objects taken r at a time.

**Examples.**

Let Y={1, 2, 3, 4}. Then:

(i) (4, 1, 2, 3); (3, 4, 1, 2); (2, 4, 1, 3) are permutations of Y or 4-permutations taken all at a time. Total number of such permutations is 4!=24.

(ii) (3, 1, 4); (1, 4, 3); (3, 2, 4); (4, 3, 2); (3, 1, 2), etc are 3-permutations of Y, or permutations of the four elements taken three at a time.

(iii) (1, 4); (3, 2); (1, 3); (1, 3); (2, 4), (4, 1), etc are 2-permutations of Y, or permutations of the four elements taken two at a time.

Let S(n, r) is the set of all r-permutations. We are interested in the number |S(n, r)| of such permutations which is denoted as P(n, r)=|S(n, r)|.

Our goal is to find a formula to calculate P(n, r). Next example provides the argumentation which we will use to get a formula for P(n, r).

**EXAMPLE 17.** Find the number m of 4-permutations of the set {A, B, C, D, E, F}. In other words, find the number of "four-letter words" using only the given six letters **without repetition** of a letter in a 4-letter word.

***Solution***. Let us represent the general four-letter word by the following four positions:

$$\_\_\_, \quad \_\_\_, \quad \_\_\_, \quad \_\_\_$$

The first letter can be chosen in 6 ways; following this the second letter can be chosen in 5 ways; the third letter can be chosen in 4 ways and, finally, the fourth letter can be chosen in 3 ways. Write each number of ways in its appropriate position as follows:

$$\underline{\ 6\ }, \ \underline{\ 5\ }, \ \underline{\ 4\ }, \ \underline{\ 3\ },$$

By the Product Rule there are $m=6\cdot5\cdot4\cdot3=\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1}=\frac{6!}{2!}=\frac{6!}{(6-4)!}$ =360 possible 4-letter words which use the set of six letters **without repetition** of used letters. Namely, there are 360 4-permutations on a set of 6 distinct elements. ■

Generalize the result of Example 17 as the theorem below.

**Theorem 7.**

$$P(n, r)=n(n-1)(n-2)\ \dots\ (n-r+1) = \frac{n!}{(n-r)!} \qquad (7)$$

***Proof.*** Arguments here are the same as in Example 17. We try to find all possible ways to arrange "r-letter word" using only the given n objects **without repetition**. The first "letter" can be chosen in n ways; following this the second letter can be chosen in (n-1) ways; the third letter can be chosen in (n-2) ways and, finally, the r-th letter can be chosen in (n-r+1) ways.
By the **Product Rule** there are $m=n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r+1)$ possible "r-letter words" **without repetition** which can be built up by using n given elements. ■

**Corollary 8.** If $r=n$ we find the Theorem 6 as a particular case of the Theorem 7. ■

**Permutations with Repetitions**

Frequently we want to know the number of permutations of a multiset, that is, a set of objects some of which are alike. We will let $P(n; n_1, n_2, \dots, n_r)$ denote the number of permutations of n objects of which $n_1$ are alike, $n_2$ are alike, . . ., $n_r$ are alike. The general formula follows:

**Theorem 8.**

$$P(n; n_1, n_2, \dots, n_r) = \frac{n!}{n_1!\, n_2! \dots n_r!} \qquad (8)$$

We demonstrate the way how to proof the above theorem by a particular example. Suppose we want to form all possible five-letter "words" using the letters from the word "BABBY." Now there are 5! = 120 permutations of the objects $B_1$, A, $B_2$, $B_3$, Y, where the three B's are distinguished. Observe that the following six permutations

$$B_1B_2B_3AY,\ B_1B_3B_2AY,\ B_2B_1B_3AY,\ B_2B_3B_1AY,\ B_3B_1B_2AY,\ B_3B_2B_1AY$$

produce the same word when the subscripts are removed, that is, if $B_1=B_2=B_3=B$. The 6 comes from the fact that there are $3!=3\cdot2\cdot1=6$ different ways of placing the three B's in the first three positions in the permutation. This is true for each set of three positions in which the B's can appear. Accordingly, the number of different five-letter words that can be formed using the letters from the word "BABBY" is: $P(5; 3)=\frac{5!}{3!}=4\cdot5=20$. ■

**EXAMPLE 18.** Find the number m of seven-letter words that can be formed using the letters of the word "BENZENE."

***Solution***. We seek the number of permutations of 7 objects of which 3 are alike (the three E's), and 2 are alike (the two N's). By Theorem 13-2,

$m=P(7; 3, 2)=\frac{7!}{3!2!}=7\cdot6\cdot5\cdot2=420$. ■

### Ordered Samples

Many problems are concerned with choosing an element from a set *S*, say, with *n* elements. When we choose one element after another, say, *r* times, we call the choice an *ordered sample* of size *r*. We consider two cases.

**(1) Sampling with replacement**

Here the element is replaced in the set S before the next element is chosen. Thus, each time there are n ways to choose an element (repetitions are allowed). The Product Rule tells us that the number of such samples is:

$$n\cdot n\cdot n\cdots n\cdot n \text{ (r factors)}=n^r$$

**(2) Sampling without replacement**

Here the element is not replaced in the set S before the next element is chosen. Thus, there is no repetition in the ordered sample. Such a sample is simply an r-permutation. Thus, the number of such samples is:

$$P(n, r)=n(n-1)(n-2)\cdot\cdot\cdot(n-r+1)=\frac{n!}{(n-r)!} \qquad (9)$$

**EXAMPLE 19.** A class contains 8 students. Find the number n of samples of size 3:

(a) With replacement; (b) Without replacement.

***Solution.***

(a) Each student in the ordered sample can be chosen in 8 ways; hence, there are $n=8\cdot8\cdot8=8^3=512$ samples of size 3 **with replacement**.

(b) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $n = 8\cdot7\cdot6 = 336$ samples of size 3 **without replacement**. ■

## Combinations

Let *X* be a finite set with *n* elements. Any subset of X, consisting of *r*, $0\le r\le n$, distinct elements from X is called a *r-combination* of these *n* elements. In other words, an *r-combination* is any selection of *r* of the distinct elements where order does not count.

For instance, if X={1, 2, 3, 4} then {1, 3, 4}={3, 1, 4}={4, 3, 1}=…

The number of such combinations is denoted by C(n, r) (other texts may use nCr, $C_{n,r}$, or $C^n{}_r$).

Before we give the general formula for C(n, r) consider an example.

**EXAMPLE 20.** Find the number of combinations of 4 objects, *A*, *B*, *C*, *D*, taken 3 at a time. In other words, find the number of all 3-combinations of 4 objects.

***Solution***. Let X={A, B, C, D}. Each 3-combination is a selection (subset) consisting of three distinct elements from a 4-element set X. We have only four different 3-combinations in a set of four elements:

{A, B, C}, {A, B, D}, {A, C, D}, {B, C, D}

Recall that order of elements does not count.

Each 3-combination above, as a set with 3 elements, determines 3!=6 permutations of the elements as follows:

{A, B, C}: ABC, ACB, BAC, BCA, CAB, CBA
{A, B, D}: ABD, ADB, BAD, BDA, DAB, DBA
{A, C, D}: ACD, ADC, CAD, CDA, DAC, DCA
{B, C, D}: BDC, BDC, CBD, CDB, DBC, DCB

Thus, **the number of 3-combinations** (which is equal to 4) multiplied by 3! gives us **the number of 3-permutations of 4 objects**; that is,

$$C(4, 3)\cdot 3! = P(4, 3) \text{ or } C(4, 3)=P(4, 3)/3!$$

But P(4, 3)=4·3·2=24 and 3!=6; hence C(4, 3)=4 as noted above. ■

**Theorem 9.**

$$C(n, r)=\frac{P(n,r)}{r!}=\frac{n!}{r!(n-r)!} \tag{10}$$

***Proof.*** As indicated above in Example 20, any *r*-combination of n objects determines *r*! number of *r*-permutations of the objects in the same combination; that is, P(*n, r)* = *r*! C(*n*, *r*). Accordingly, we obtain the formula (10). ■

Theorem 9 demonstrates the remarkable relation between combinations and binomial coefficients. Namely, we have the following Corollary

**Corollary 9**.

$$C(n, r)=\binom{n}{r} \tag{11}$$

**Note**. Due to Corollary 9 **we shall use C(*n*, *r*) and $\boldsymbol{\binom{n}{r}}$ interchangeably.**

**EXAMPLE 21.** How many bit strings of length n contain exactly *r* 1s?

***Solution***. The **positions** of *r* 1s in a bit string of length n form an *r*-combination of the set {1, 2, 3, ..., n}. Hence, there are C(n, r) bit strings of length n that contain exactly *r* 1s. ■

**EXAMPLE 22.** Suppose that there are 5 faculty members in the mathematics department and 9 in the computer science department. How many ways are there to select a committee to develop a

Discrete Structures course content at a school if the committee is to consist of 2 faculty members from the mathematics department and 3 from the computer science department?

***Solution***. By the product rule, the answer is the product of the number of 2-combinations of a set with 5 elements and the number of 3-combinations of a set with 9 elements. By Theorem 9, the number of ways to select the committee is

$$C(5, 2)\cdot C(9, 3)= \frac{5!}{2!3!}\cdot\frac{9!}{3!6!}=10\cdot 84=840$$ ■

Below we provide another proof of the **Example 5** from the Lecture Notes 10-11. Remind you that in Lecture Notes 10-11 we used Mathematical Induction to prove the following result:

**Theorem 10.** Let $S$ be a finite set with n elements, $|S|=n$, then $|P(S)|=2^n$, here P(S) is the collection of all subsets of S.

***Proof.*** Let $C_k$ be the collection of all subsets of S each of which consist of k distinct elements from S, $0\le k\le n$. Using counting terminology, each element of $C_k$ is k-combination of elements of S. Therefore, $C_k$ is simply the set of all k-combination of elements of S.

Clear that $P(S)=\bigcup_{k=0}^{k=n} C_k$. By the definition of $C_k$'s they are disjoint subsets of P(S). Therefore, according to the Sum Rule Principle,

$$|P(S)|=\sum_{k=0}^{k=n}|C_k| \qquad (12)$$

here **$|C_k|$ - number of elements in collection $C_k$, in other words, number of all distinct k-element subsets of S** or, in counting terminology, number of all k-combinations of the elements of S. Hence

$$|C_k|=C(n, k)=\binom{n}{k} \qquad (13)$$

Substitution of the values from (13) into 12 yields

$$|P(S)|=\sum_{k=0}^{k=n}\binom{n}{k} \qquad (14)$$

Or, in detailed form

$$|P(S)|= \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{r}+\cdots+\binom{n}{n-1}+\binom{n}{n} \qquad (15)$$

But RHS (right-hand side) of (15) is $2^n$ (we use formula (6)). Theorem is proved. ■

**EXAMPLE 23. Number of Onto Functions**

Let $A$ and $B$ be sets such that $|A|=6$ and $|B|=4$. We want to find the number of surjective (onto) functions from $A$ onto $B$.

***Solution.*** Let $b_1$, $b_2$, $b_3$, $b_4$ be the four elements in $B$. Let **U** be the set of all functions from $A$ into $B$. Furthermore, let $F_1$ be the set of functions which do not send any element of $A$ into $b_1$ that is, $b_1$ is not in the range of any function in $F_1$. Similarly, let $F_2$, $F_3$, and $F_4$ be the sets of functions which do not send any element of $A$ into $b_2$, $b_3$, and $b_4$, respectively.

We are looking for the number of functions in $S=F_1'\cap F_2'\cap F_3'\cap F_4'$, that is, those functions which do send at least one element of $A$ into $b_1$, at least one element of $A$ into $b_2$, and so on. We will use the alternate form Inclusion–Exclusion Principle (Theorem 3) as follows.

(i) For each function in **U**, there are 4 choices for each of the 6 elements in $A$; hence $|U|=4^6=4096$.

(ii) There are $C(4, 1)=4$ sets $F_i$. For each $F_i$, there are 3 choices for each of the 6 elements in $A$, hence $|F_i|=3^6=729$.

(iii) There are $C(4, 2)=6$ pairs $F_i \cap F_j$. In each case, there are 2 choices for each of the 6 elements in $A$, hence $|F_i \cap F_j|=2^6=64$.

(iv) There are $C(4, 3)=4$ triplets $F_i \cap F_j \cap F_k$. In each case, there is only one choice for each of the 6 elements in $A$. Hence $|F_i \cap F_j \cap F_k|=1^6=1$.

(v) $F_1 \cap F_2 \cap F_3 \cap F_4$ has no element, that is, is empty. Hence $|F_1 \cap F_2 \cap F_3 \cap F_4|=0$. By the alternate form of Inclusion–Exclusion Principle (Theorem 9),

$|S|=|F_1' \cap F_2' \cap F_3' \cap F_4'|=4^6-C(4, 1)3^6+C(4, 2)2^6-C(4, 3)1^6=4096-2916+384-4=1560$ ■

The above result is true in general case. Namely:

**Theorem 11.** Suppose $|A|=m$ and $|B|=n$ where $m\geq n$. Then the number $N$ of surjective (onto) functions from $A$ onto $B$ is:

$$N=n^m-C(n, 1)(n-1)^m+C(n, 2)(n-2)^m-\cdot\cdot\cdot+(-1)^{n-1}C(n, n-1)1^m$$

***Proof.*** Exercise. ■

## EXERCISES. SET 1 (Solved Problems).

### Sum and Product Rules

**1.1.** Suppose a bookcase shelf has 6 Programming Languages texts, 4 Computer Network texts, 5 Machine Learning texts, and 3 Databases texts.

Find the number n of ways a student can choose:

(a) one of the texts; (b) one of each type of text.

***Solution.*** (a) Here the **Sum Rule** applies; hence, n=6+4+5+3=18.

(b) Here the **Product Rule** applies; hence, n=6·4·5·3=360.

**1.2.** A Theory of Computations class contains 12 male students and 7 female students. Find the number n of ways that the class can elect: (a) 1 class representative; (b) 2 class representatives, 1 male and 1 female; (c) 1 president and 1vice president.

***Solution.***

(a) Here the **Sum Rule** is used; hence, n=12+7=19.

(b) Here the **Product Rule** is used; hence, n=12·7=84.

(c) There are 19=12+7 ways to elect the president, and then 18=19-1 ways to elect the vice president. Thus n=19·18=342.

**1.3.** There are four bus lines between A and B, and three bus lines between B and C. Find the number m of ways that a man can travel by bus:

(a) from A to C by way of B;

(b) roundtrip from A to C by way of B;

(c) roundtrip from A to C by way of B but without using a bus line more than once.

***Solution.***

(a) There are 4 ways to go from A to B and 3 ways from B to C; hence n = 4·3 = 12.

(b) There are 12 ways to go from A to C by way of B, and 12 ways to return. Thus n=12·12=144.

(c) The man will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

$$A \rightarrow B \rightarrow C \rightarrow B \rightarrow A$$

The man can travel 4 ways from A to B and 3 ways from B to C, but he can only travel 2 ways from C to B and 3 ways from B to A since he does not want to use a bus line more than once. Thus, by the **Product Rule**, $n=4\cdot3\cdot2\cdot3=72$.

**1.4.** What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

```
k:=0
for i1:=1 to n1
    for i2:=1 to n2



        for im:=1 to nm
            k:=k+1
```

***Solution:*** The initial value of k is zero. Each time the nested loop is traversed, 1 is added to k. Let $T_i$ be the task of traversing the *i*-th loop. Then the number of times the loop is traversed is the number of ways to do the tasks $T_1, T_2, \ldots, T_m$. The number of ways to carry out the task $T_j$ , $j = 1, 2, \ldots, m$, is $n_j$ , because the j-th loop is traversed once for each integer $i_j$ with $1\leq i_j \leq n_j$. By the **Product Rule**, it follows that the nested loop is traversed $n_1\cdot n_2\cdot\ldots\cdot n_m$ times. Hence, the final value of k is $n_1\cdot n_2\cdot\ldots\cdot n_m$. ■

**1.5.** What is the value of k after the following code has been executed? Here $n_1, n_2, \ldots, n_m>0$.

```
k:=0
for i1:= 1 to n1
    k:=k+1
for i2:= 1 to n2
    k:=k+1
.
.
.
for im:=1 to nm
    k:=k+1
```

***Solution.*** The initial value of k is zero. We have m different loops. Each time a loop is traversed, 1 is added to k. To determine the value of k after this code has been executed, we need to determine how many times we traverse a loop. There are $n_i$ ways to traverse the i-th loop. Because we only traverse one loop at a time, the **Sum Rule** shows that the final value of k, which is the number of ways to traverse one of the m loops is $n_1+n_2+\cdots+n_m$. ■

**1.6.** Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

***Solution:*** Let P be the total number of possible passwords, and let P6, P7, and P8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the **Sum Rule**, P=P6+P7+P8. We will now find P6, P7, and P8. Finding P6 directly is difficult. To find P6 it is easier to find the number of strings of uppercase letters and digits that are six characters long and subtract from this the number of strings with no digits. By the **Product Rule**, the number of strings of six characters is $36^6$ (36=26 (uppercase letters)+10 digits), and the number of strings with no digits is $26^6$. Hence,

$$P6=36^6-26^6=2{,}176{,}782{,}336-308{,}915{,}776=1{,}867{,}866{,}560.$$

Similarly, we have

$$P7=36^7-26^7=78{,}364{,}164{,}096-8{,}031{,}810{,}176=70{,}332{,}353{,}920$$

and

$$P8=36^8-26^8=2{,}821{,}109{,}907{,}456-208{,}827{,}064{,}576=2{,}612{,}282{,}842{,}880.$$

Consequently,

$P=P6+P7+P8=2{,}684{,}483{,}063{,}360.$ ■

**1.7. Counting Internet Addresses.** In the Internet, which is made up of interconnected physical networks of computers, each computer (or more precisely, each network connection of a computer) is assigned an Internet address. In IPv4 Internet Protocol, now in use, an address is a string of 32 bits. It begins with a network number (netid). The netid is followed by a host number (hostid), which identifies a computer as a member of a particular network (Table 3).

**Table 3 Internet Addresses (IPv4).**

| **Bit Number** | **0** | **1** | **2** | **3** | **4** | **8** | **16** | **24** | **31** |
|---|---|---|---|---|---|---|---|---|---|
| Class A | 0 | netid | | | | hosted | | | |
| Class B | 1 | 0 | netid | | | | hosted | | |
| Class C | 1 | 1 | 0 | netid | | | | hosted | |
| Class D | 1 | 1 | 1 | 0 | Multicast Address | | | | |
| Class E | 1 | 1 | 1 | 1 | 0 | Address | | | |

Three forms of addresses are used, with different numbers of bits used for netids and hostids. Class A addresses, used for the largest networks, consist of 0, followed by a 7-bit netid and a 24-bit hostid. Class B addresses, used for medium-sized networks, consist of 10, followed by a 14-bit netid and a 16-bit hostid. Class C addresses, used for the smallest networks, consist of 110, followed by a 21-bit netid and an 8-bit hostid. There are several restrictions on addresses because of special uses: 1111111 is not available as the netid of a Class A network, and the hostids consisting of all 0s and all 1s are not available for use in any network. A computer on the Internet has either a Class A, a Class B, or a Class C address. (Besides Class A, B, and C addresses, there are also Class D addresses, reserved for use in multicasting when multiple computers are addressed at a single time, consisting of 1110 followed by 28 bits, and Class E addresses, reserved for future use, consisting of 11110 followed by 27 bits. Neither Class D nor Class E addresses are assigned as the IPv4 address of a computer on the Internet).

Table 3 illustrates IPv4 addressing. (Limitations on the number of Class A and Class B netids have made IPv4 addressing inadequate; IPv6, a new version of IP, uses 128-bit addresses to solve this problem.)

**Problem.** How many different IPv4 addresses are available for computers on the Internet?

***Solution.*** Let $x$ be the number of available addresses for computers on the Internet, and let $x_A$, $x_B$, and $x_C$ denote the number of Class A, Class B, and Class C addresses available, respectively. By the **Sum Rule**, $x=x_A+x_B+x_C$.

To find $x_A$, note that there are $2^7-1=127$ Class A netids, recalling that the netid 1111111 is unavailable. For each netid, there are $2^{24}-2=16{,}777{,}214$ hostids, recalling that the hostids consisting of all 0s and all 1s are unavailable. Consequently, $x_A=127\cdot 16{,}777{,}214$ $=2{,}130{,}706{,}178$.

To find $x_B$ and $x_C$, note that there are $2^{14}$=16,384 Class B netids and $2^{21}$=2,097,152 Class C netids. For each Class B netid, there are $2^{16}-2$=65,534 hostids, and for each Class C netid, there are $2^8-2$=254 hostids, recalling that in each network the hostids consisting of all 0s and all 1s are unavailable. Consequently, $x_B$=1,073,709,056 and $x_C$=532,676,608.

We conclude that the total number of IPv4 addresses available is $x=x_A+x_B+x_C$=2,130,706,178+1,073,709,056+532,676,608=3,737,091,842. ■

**1.8.** How many one-to-one functions are there from a set with ***m*** elements to one with ***n*** elements?

**Solution:** First note that when m>n there are no one-to-one functions from a set with m elements to a set with n elements.

Now let m≤n. Suppose the elements in the domain are $a_1$, $a_2$, ..., $a_m$. There are n ways to choose the value of the function at $a_1$. Because the function is one-to-one, the value of the function at $a_2$ can be picked in (n–1) ways (because the value used for $a_1$ cannot be used again).

In general, the value of the function at $a_k$ can be chosen in (n−k+1) ways. By the **Product Rule**, there are n(n−1)(n−2)···(n−m+1) one-to-one functions from a set with m elements to one with n elements.

For example, there are 5·4·3=60 one-to-one functions from a set with three elements to a set with five elements. ■

**1.9.** How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

***Solution.*** We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00.

We can construct a bit string of length eight that begins with a 1 in $2^7$=128 ways. This follows by the **Product Rule,** because the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways. Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6$=64 ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.

Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5$=32 ways to construct such a string. This follows by the **Product Rule**, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way. Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to construct a bit string of length eight that begins with a 1 or that ends with 00, equals 128+64−32=160. ■

## Permutations and Combinations

**1.10.** State the essential difference between permutations and combinations, with examples.

***Solution.*** Order counts with permutations, such as words, sitting in a row, and electing a president, vice president, and treasurer.

Order does not count with combinations, such as committees and teams (without counting positions). The product rule is usually used with permutations, since the choice for each of the ordered positions may be viewed as a sequence of events. ■

**1.11.** Find: (a) P(7, 3); (b) P(14, 2).

***Solution.*** Recall P(n, r) = n(n-1) … (n-r+1) has r factors beginning with n, therefore:

(a) $P(7, 3)=7\cdot6\cdot5=210$; (b) $P(14, 2)=14\cdot13=182$. ■

**1.12.** Find the number m of ways that 5 people can arrange themselves:

(a) In a row of chairs; (b) Around a circular table.

***Solution.*** (a) Here $m=P(5)=P(5, 5)=5!$ ways.

(b) One person can sit at any place at the table. The other 4 people can arrange themselves in 4! ways around the table; that is $m=4!$

Exercise (b) is an example of a ***circular*** permutation. In general, n objects can be arranged in a circle in $(n-1)!$ ways. ■

**1.13.** Find the number n of distinct permutations that can be formed from all the letters of each word: (a) THOSE; (b) UNUSUAL; (c) SOCIOLOGICAL.

***Solution.*** This problem concerns permutations with repetitions.

(a) $n=5!=120$, since there are 5 letters and no repetitions.

(b) $n=\frac{7!}{3!}=7\cdot6\cdot5\cdot4=840$ since there are 7 letters of which 3 are U and no other letter is repeated.

(c) $n=\frac{12!}{3!2!2!2!}$ since there are 12 letters of which 3 are O, 2 are C, 2 are I, and 2 are L. ■

**1.14.** A class contains 8 students. Find the number n of samples of size 3:

(a)With replacement; (b)Without replacement.

***Solution.***

(a) Each student in the ordered sample can be chosen in 8 ways; hence, there are $n=8\cdot8\cdot8=8^3=512$ samples of size 3 with replacement.

(b) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $n=8\cdot7\cdot6=336$ samples of size 3 without replacement. ■

**1.15.** Find n if $P(n, 2) = 20$.

***Solution.*** $P(n, 2)=n(n-1)=n^2-n$. Thus, we get $n^2-n=20$ or $n^2-n-20=0$ or $(n-5)(n+4)=0$

Since n must be positive, the only answer is $n = 5$. ■

**1.16.** A class contains 13 students with 8 men and 5 women. Find the number *n* of ways to:

(a) Select a 6-member committee from the students.

(b) Select a 6-member committee with 3 men and 3 women.

(c) Elect a president, vice president, and treasurer.

***Solution.***

(a) This concerns combinations, not permutations, since order does not count in a committee.

There are "13 choose 6" such committees, i.e., $n=C(13, 6)=\binom{13}{6}=\frac{13\cdot12\cdot11\cdot10\cdot9\cdot8}{6\cdot5\cdot4\cdot3\cdot2\cdot1}$ $=1716$.

(b) The 3 men can be chosen from the 8 men in $C(8, 3)$ ways, and the 3 women can be chosen from the 5 women in $C(5, 3)$ ways. Thus, by the **Product Rule**:

$$n=\binom{8}{3}\binom{5}{3}=\frac{8\cdot7\cdot6}{3\cdot2\cdot1}\cdot\frac{5\cdot4\cdot3}{3\cdot2\cdot1}=56\cdot10=510$$

(c) This concerns permutations, not combinations since order does count.

Thus, $n=P(13, 3)=13\cdot12\cdot11$. ■

**1.17.** Find the number m of committees of 3 with a given chairperson that can be selected from 9 people.

***Solution.*** The chairperson can be chosen in 9 ways and, following this, the other 2 on the committee can be chosen from the 8 remaining in C(8, 2) ways.
Thus $m=9\cdot C(8, 2)=9\cdot 28=252$. ■

**1.18.** A box contains 9 blue pens and 7 red pens. Find the number of ways two pens can be drawn from the box if: (a) They can be any color. (b) They must be the same color.

***Solution.***

(a) There are "16 choose 2" ways to select 2 of the 16 pens. Thus,

$$n=C(16, 2)=\binom{16}{2}=\frac{16!}{2!14!}=\frac{16\cdot 15}{2\cdot 1}=120$$

(b) There are C(9, 2)=36 ways to choose 2 of the 9 blue pens, and C(7, 2)=21 ways to choose 2 of the 7 red pens. By the Sum Rule, n = 36 + 21 = 43. ■

## Inclusion–Exclusion Principle

**1.19.** Each student in Liberal Arts at some college has

(A) a mathematics requirement and

(B) a science requirement.

A poll of 140 sophomore students shows that:

60 completed (A); 45 completed (B); 20 completed both (A) and (B).

Use a Venn diagram (and/or) **IEP** to find the number of students who have completed:

(a) At least one of (A) and (B); (b) exactly one of (A) or (B); (c) neither (A) nor (B).

***Solution.*** Translating the above data into set notation yields:

$n(A)=60$, $n(B)=45$, $n(A\cap B)=20$, $n(\mathbf{U})=140$

Draw a Venn diagram of sets $A$ and $B$ and assign numbers to the four regions as in Figure 3:

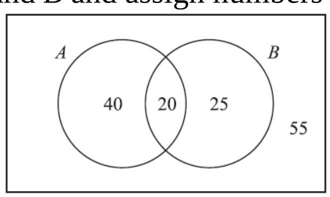



Figure 3

20 completed both $A$ and $B$, so $n(A\cap B)=20$.

60−20=40 completed $A$ but not $B$, so $n(A\backslash B)=40$.

45−20=25 completed $B$ but not $A$, so $n(B\backslash A)=25$.

140−20−40−25=55 completed neither A nor B.

*(a)* By the Venn diagram: 20+40+25=85 completed $A$ or $B$.
The same result we get alternately, by the **IEP**: $n(A\cup B)=n(A)+n(B)-n(A\cap B)=60+45-20=85$.

(*b*) 40+25=65 completed exactly one requirement. That is, $n(A\oplus B)=65$.

(*c*) 55 completed neither requirement, i.e. $n(A'\cap B')=n[(A\cup B)']=140-85=55$. ■

**1.20.** In a survey of 120 people, it was found that:

65 read *Newsweek* magazine, 20 read both *Newsweek* and *Time*,
45 read *Time*, 25 read both *Newsweek* and *Fortune*,
42 read *Fortune*, 15 read both *Time* and *Fortune*,
8 read all three magazines.

($a$) Find the number of people who read at least one of the three magazines.
($b$) Fill in the correct number of people in each of the eight regions of the Venn diagram in Figure 4a (answer is presented in Figure 4b).
($c$) Find the number of people who read exactly one magazine.

***Solution.*** Let $N$, $T$, and $F$ denote the set of people who read *Newsweek*, *Time*, and *Fortune*, respectively.

($a$) We want to find $n(N\cup T\cup F)$. By **IEP** we have

$$n(N\cup T\cup F)=n(N)+n(T)+n(F)-n(N\cap T)-n(N\cap F)-n(T\cap F)+n(N\cap T\cap F)=$$
$$=65+45+42-20-25-15+8=100$$

($b$) The required Venn diagram is obtained as follows (Figure 4):

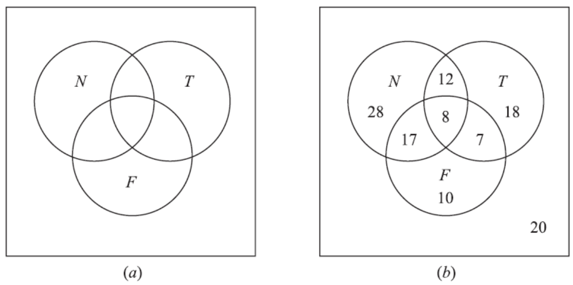



Figure 4

- $n(N\cap T\cap F)=8$ read all three magazines;
- $n(N\cap T)-n(N\cap T\cap F)=20-8=12$ read *Newsweek* and *Time* but not all 3 magazines;
- $n(N\cap F)-n(N\cap T\cap F)=25-8=17$ read *Newsweek* and *Fortune* but not all 3 magazines;
- $n(T\cap F)-n(N\cap T\cap F)=15-8=7$ read *Time* and *Fortune* but not all 3 magazines,
- $n(N)-[n(N\cap T)-n(N\cap T\cap F)]-[(n(N\cap F)-n(N\cap T\cap F)]-[n(N\cap T\cap F)]=65-12-17-8=$**28 read only *Newsweek***,
- $n(T)-[n(N\cap T)-n(N\cap T\cap F)]-[(n(T\cap F)-n(N\cap T\cap F)]-[n(N\cap T\cap F)]=45-12-7-8=$**18 read only *Time***,
- $n(F)-[n(N\cap F)-n(N\cap T\cap F)]-[(n(T\cap F)-n(N\cap T\cap F)]-[n(N\cap T\cap F)]=42-17-7-8=$**10 read only *Fortune*,**
- $n(\mathbf{U})-n(N\cup T\cup F)=120-100=20$ **read no magazine at all**.

($c$) **28+18+10=56** read exactly one of the magazines. ■

**1.21.** Let A, B, C, D denote, respectively, the following courses: Algorithms and Complexity, Bioinformatics, Computer Organization, and Databases. Find the number N of 3$^{rd}$ year students in the Department given the data:

| | | | |
|---|---|---|---|
| 15 take A, | 7 take A and B, | 6 take B and D, | 6 take B, C, D, |
| 20 take B, | 8 take A and C, | 5 take C and D, | 8 take A, C, D, |
| 25 take C, | 12 take A and D, | 8 take A, B, C, | 4 take all four, |
| 17 take D, | 10 take B and C, | 4 take A, B, D, | 11 take none. |

***Solution.*** Let T be the number of students who take at least one course. The number N of students of the Department is the union of those who take no courses and those who take at least one course, that is, N=11+T. **By the Inclusion–Exclusion Principle** (**IEP**, Theorem 8): T=s1−s2+s3−s4, where:

s1=15+20+25+17=77, s2=7+8+12+10+6+5=48,
s3=8+4+6+8=26, s4=4.

Thus T=77–48+26–4=51, and N=11+T=62. ■

## Pigeonhole Principle

**1.22.** Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

***Solution.*** Because there are four possible remainders (0, 1, 2, 3) when an integer is divided by 4, the pigeonhole principle implies that given five integers, at least two have the same remainder. ■

**1.23.** Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

***Solution.*** Let

$$a, a+1, ..., a+(n-1) \qquad (1)$$

be the integers in the sequence. Remind you that two integers c and d are equal by **mod** $n$ if there exists an integer p such that

$$c\text{-}d=pn, \text{ or, equivalently, } c\text{-}d=0, \pm n, \pm 2n, \pm 3n, \ldots \qquad (2)$$

Consider two arbitrarily chosen and distinct elements of (1), say $a+j$ and $a+k$, here $0 \le k < j \le n-1$. Since $0 < j\text{-}k < n$ so $0 < (a+j)-(a+k) < n$. Hence, the difference $(a+j)-(a+k)$ does not satisfy (2) and therefore, $(a+j) \neq (a+k)$ **mod** $n$. Thus, the integers $(a+i)$ **mod** $n$, $i=0, 1, 2, ..., n-1$, are distinct. Because there are $n$ possible values for $(a+i)$ **mod** $n$ and there are $n$ different integers in the set, each of these values is taken on exactly once. It follows that there is exactly one integer in the sequence that is divisible by $n$. ■

**1.24.** Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.

b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

***Solution.***

a) If there were fewer than 9 freshmen, fewer than 9 sophomores, and fewer than 9 juniors in the class, there would be no more than 8 with each of these three class standings, for a total of at most 24 students, contradicting the fact that there are 25 students in the class.

b) If there were fewer than 3 freshmen, fewer than 19 sophomores, and fewer than 5 juniors, then there would be at most 2 freshmen, at most 18 sophomores, and at most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class. ■

**1.25.** Consider a tournament with n players where each player plays against every other player. Suppose each player wins at least once. Show that at least 2 of the players have the same number of wins.

***Solution.*** Each player will win anywhere from 1 up to (n–1) games (pigeonholes). There are n players (pigeons). ■

**1.26.** Suppose 5 points are chosen at random in the interior of an equilateral triangle T where each side has length two inches. Show that the distance between two of the points must be less than one inch.

***Solution.*** Draw three lines between the midpoints of the sides of T. This partitions T into 4 equilateral triangles (pigeonholes) where each side has length 1. Two of the 5 points (pigeons) must lie in one of the triangles. Hence, the distance between these two points must be less than one inch. ■

**1.27.** Consider any set $X = \{x_1, x_2, ..., x_7\}$ of seven distinct integers. Show that there exists x, y∈X such that x+y or x−y is divisible by 10.

***Solution.*** Let $r_i$ be the remainder when $x_i$ is divisible by 10. Consider the six pigeonholes: $H_1=\{x_i|r_i=0\}$, $H_2=\{x_i|r_i=5\}$, $H_3=\{x_i|r_i=1 \text{ or } 9\}$, $H_4=\{x_i|r_i=2 \text{ or } 8\}$, $H_5=\{x_i|r_i=3 \text{ or } 7\}$, $H_6=\{x_i|r_i=4 \text{ or } 6\}$. Then all seven numbers from X will be distributed between $H_k$'s. and therefore at least one of $H_k$'s must contain at least pair of numbers from X, say x and y.

If x and y belong to $H_1$ or $H_2$ then both x+y and x-y are divisible by 10.

If x and y belong to $H_3$ then three cases are possible:

Case 1. Last digit in both x and y is 1. Then x-y is divisible by 10.

Case 2. Last digit in both x and y is 9. Then x-y is divisible by 10.

Case 3. Last digit of x is 9 and last digit of y is 1. Then x+y is divisible by 10.

If x and y belong to $H_4$ then three cases are possible: las

Case 1. Last digit in both x and y is 2. Then x-y is divisible by 10.

Case 2. Last digit in both x and y is 8. Then x-y is divisible by 10.

Case 3. Last digit of x is 8 and last digit of y is 2. Then x+y is divisible by 10.

The similar arguments are true for pigeonholes $H_5$ and $H_6$. ■

**1.28.** Find the minimum number of students needed to guarantee that five of them belong to the same class (Freshman, Sophomore, Junior, Senior).

***Solution.*** Apply **GPP** as follows. Here the *k*=4 classes are the pigeonholes (boxes). The minimum number of students needed to guarantee that five of them belong to the same class is the smallest integer N such that $\lceil N/4 \rceil=5$. The smallest such integer is $N=4\cdot4+1=17$. If you have only 16 students, it is possible that equal (by 4) numbers of students belong to each of 4 classes and no five students belong to the same class. Thus, 17 is the minimum number of students needed to guarantee that five of them belong to the same class. ■

**1.29.** Let L be a list (not necessarily in alphabetical order) of the 26 letters in the English alphabet (which consists of 5 vowels, A, E, I, O, U, and 21 consonants).

(a) Show that L has a sublist consisting of four or more consecutive consonants.

(b) Assuming L begins with a vowel, say A, show that L has a sublist consisting of five or more consecutive consonants.

***Solution.***

(a) The five vowels partition L into n=6 sublists (pigeonholes) of consecutive consonants. We apply GPP. Nere N=21 (consonants). Then $\lceil N/6 \rceil=4$. Hence some sublist has at least four consecutive consonants.

(b) Since L begins with a vowel, the remainder of the vowels partition L into n=5 sublists. Here k+1=5 and so k=4. Hence kn+1=21. Thus, some sublist has at least five consecutive consonants. ■

**1.30.** (a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least five cards of the same suit are chosen?

(b) How many must be selected to guarantee that at least four diamonds are selected?

***Solution.***

(a) This subtask can be solved by applying GPP as follows. Let each suit (diamond, club, hearts, spade) means a pigeonhole (box). Using the GPP, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards.

Consequently, we know that at least 5 cards of one suit are selected if $\lceil N/4 \rceil\geq5$. The smallest integer N such that $\lceil N/4 \rceil\geq5$ is $N=4\cdot4+1=17$, so seventeen cards suffice.

(b) It is not needed to apply PP for the this subtask. In each suit there are 13 cards: 2, 3, 4, 5, 6, 7, 8, 9, 10 Jack, Queen, King, Ace. Hence in the worst case all 39=3(13) cards are all clubs, hearts, and spades. Therefore, any next card starting from $40^{th}$ is diamond. Hence, we need 39+4=43 cards to guarantee that at least four diamonds are selected. ■

## EXERCISES. SET 2 (Supplementary Problems with Answers)

### Finite Sets and Counting Principles

### Sum and Product Rules

**2.1.** A store sells clothes for men. It has 3 kinds of jackets, 7 kinds of shirts, and 5 kinds of pants. Find the number of ways a person can buy: (a) one of the items; (b) one of each of the three kinds of clothes.

**2.2.** Suppose a code consists of five characters, two letters followed by three digits. Find the number of: (a) codes; (b) codes with distinct letter; (c) codes with the same letters.

### Permutations

**2.3.** Find the number of automobile license plates where: (a) Each plate contains 2 different letters followed by 3 different digits. (b) The first digit cannot be 0.

**2.4.** Find the number m of ways a judge can award first, second, and third places in a contest with 18 contestants.

**2.5.** Find the number of ways 5 large books, 4 medium-size books, and 3 small books can be placed on a shelf where: (a) there are no restrictions; (b) all books of the same size are together.

**2.6.** A debating team consists of 3 boys and 3 girls. Find the number of ways they can sit in a row where: (a) there are no restrictions; (b) the boys and girls are each to sit together; (c) just the girls are to sit together.

**2.7.** Find the number of ways 5 people can sit in a row where: (a) there are no restrictions; (b) two of the people insist on sitting next to each other.

**2.8.** Find the number of ways 5 people can sit around a circular table where: (a) there are no restrictions; (b) two of the people insist on sitting next to each other.

**2.9.** Suppose repetitions are not permitted. (a) Find the number of three-digit numbers that can be formed from the six digits 2, 3, 5, 6, 7, and 9. (b) How many of them are less than 400? (c) How many of them are even?

**2.10.** Find $n$ if: (a) $P(n, 4)=42\cdot P(n, 2)$; (b) $2P(n, 2)+50 = P(2n, 2)$.

### Permutations with Repetitions, Ordered Samples

**2.11.** Find the number of permutations that can be formed from all the letters of each word: (a) QUEUE; (b) COMMITTEE; (c) PROPOSITION; (d) BASEBALL.

**2.12.** Suppose we are given 4 identical red flags, 2 identical blue flags, and 3 identical green flags. Find the number $m$ of different signals that can be formed by hanging the 9 flags in a vertical line.

**2.13.** A box contains 12 lightbulbs. Find the number $n$ of ordered samples of size 3: (a) with replacement; (b) without replacement.

**2.14.** A class contains l0 students. Find the number $n$ of ordered samples of size 4: (a) with replacement; (b) without replacement.

## Combinations

**2.15.** A restaurant has 6 different desserts. Find the number of ways a customer can choose: (a) 1 dessert; (b) 2 of the desserts; (c) 3 of the desserts.

**2.16.** A class contains 9 men and 3 women. Find the number of ways a professor can select a committee of 4 from the class where there is:(a) no restrictions; (b) 2 men and 2 women; (c) exactly one woman; (d) at least one woman.

**2.17.** A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner where:

(a) There are no restrictions.
(b) Two of the friends are married to each other and will not attend separately.
(c) Two of the friends are not speaking with each other and will not attend together.

**2.18.** A class contains 8 men and 6 women and there is one married couple in the class. Find the number *m* of ways a teacher can select a committee of 4 from the class where the husband or wife but not both can be on the committee.

**2.19.** A box has 6 blue socks and 4 white socks. Find the number of ways two socks can be drawn from the box where:

(a) There are no restrictions. (b) They are different colors. (c) They are the same color.

**2.20.** A women student is to answer 10 out of 13 questions. Find the number of her choices where she must answer:

(a) the first two questions;
(b) the first or second question but not both;
(c) exactly 3 out of the first 5 questions;
(d) at least 3 of the first 5 questions.

## Inclusion–Exclusion Principle

**2.21.** Suppose 32 students are in a Programming class (P) and 24 students are in a Calculus class C, and suppose 10 students are in both classes. Find the number of students who are:

(a) in class P or in class C; (b) only in class P; (c) only in class C.

**2.22.** Consider all integers from 1 up to and including 100. Find the number of them that are:

(a) odd or the square of an integer; (b) even or the cube of an integer.

**2.23.** In a class of 30 students, 10 got A on the first test, 9 got A on a second test, and 15 did not get an A on either test. Find: the number of students who got:

(a) an A on both tests; (b) an A on the first test but not the second;
(c) an A on the second test but not the first.

**2.24.** Consider all integers from 1 up to and including 300. Find the number of them that are divisible by:

(a) at least one of 3, 5, 7;
(b) 3 and 5 but not by 7;
(c) by 5, but by neither 3 nor 7;
(d) by none of the numbers 3, 5, 7.

**2.25.** Find the number m of elements in the union of sets A, B, C, D where:

(i) A, B, C, D have 50, 60, 70, 80 elements, respectively.
(ii) Each pair of sets has 20 elements in common.
(iii) Each three of the sets has 10 elements in common.
(iv) All four of the sets have 5 elements in common.

## Pigeonhole Principle

**2.26.** How many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

**2.27.** A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle, location in the aisle, and shelf. There are 50 aisles, 85 horizontal locations in each aisle, and 5 shelves throughout the warehouse.
What is the least number of products the company can have so that at least two products must be stored in the same bin?

**2.28.** Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.

## Answers. SET 2.

**2.1.** (a) 15; (b) 105

**2.2.** (a) $26^2 \cdot 10^3$; (b) $26 \cdot 25 \cdot 10^3$; (c) $26 \cdot 10^3$;

**2.3.** (a) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468000$; (b) $26 \cdot 25 \cdot 9 \cdot 9 \cdot 8 = 421200$.

**2.4.** $m = 18 \cdot 17 \cdot 16 = 4896$.

**2.5.** (a) 12!; (b) 3!5!4!3!=103680.

**2.6.** (a) 6! =720; (b) $2 \cdot 3! \cdot 3! = 72$; (c) $4 \cdot 3! \cdot 3! = 144$.

**2.7.** (a) 120; (b) 48.

**2.8.** (a) 24; (b) 12.

**2.9.** (a) $P(6, 3) = 120$; (b) $2 \cdot 5 \cdot 4 = 40$; (c) $2 \cdot 5 \cdot 4 = 40$.

**2.10.** (a) 9; (b) 5.

**2.11.** (a) 30; (b) 9!/[2!2!2!]=45360; (c) 11!/[2!3!2!]=1663200; (d) 8!/[2!2!2!]=5040.

**2.12.** $m$=9!/[4!2!3!]=1260.

**2.13.** (a) $12^3 = 1728$; (b) $P(12, 3) = 1320$.

**2.14.** (a) $10^4 = 10000$; (b) $P(10, 4) = 5040$.

**2.15.** (a) $C(6, 1) = 6$; (b) $C(6, 2) = 15$; (c) $C(6, 3) = 20$.

**2.16.** (a) $C(12, 4)$; (b) $C(9, 2) \cdot C(3, 2) = 108$; (c) $C(9, 3) \cdot C(3, 1) = 252$; (d) 9 + 108 + 252 = 369 or $C(12,4) - C(9,4) = 369$.

**2.17.** (a) $C(11, 5) = 462$; (b) $C(9, 5) + C(9, 3) = 126 + 84 = 210$; (c) $C(9, 5) + 2C(9, 4) = 378$.

**2.18.** $C(12, 4) + 2C(12, 3) = 935$.

**2.19.** (a) $C(10, 2) = 45$; (b) $C(6, 1) \cdot C(4, 1) = 6 \cdot 4 = 24$; (c) $C(6, 2) + C(4, 2) = 21$ or $45 - 24 = 21$.

**2.20.** (a) $C(11, 8) = 165$; (b) $2 \cdot C(11, 9) = 110$; (c) $C(5, 3) \cdot C(8, 7) = 80$;
(d) $C(5, 3) \cdot C(8, 7) + C(5, 4) \cdot C(8, 6) + C(5, 5) \cdot C(8, 5) = 80 + 140 + 56 = 276$.

**2.21.** (a) 46; (b) 22; (c) 14.

**2.22.** (a) 55; (b) 52.

**2.23.** (a) 4; (b) 6; (c) 5.

**2.24.** (a) $100 + 60 + 42 - 20 - 14 - 8 + 2 = 162$; (b) $20 - 2 = 18$; (c) $60 - 20 - 8 + 2 = 34$; (d) $300 - 162 = 138$.

**2.25.** $m$=175

**2.26.** 4

**2.27.** 21, 251

**2.28.** 4, 3, 2, 1, 8, 7, 6, 5, 12, 11,10, 9, 16, 15, 14, 13.