Fall 2021 - MATH 1101 Discrete Structures
Lecture 10

PART 1. Relations and Their Properties.

PART 2. Combining Relations. Equivalence Relation

Exercises. Set 1 (Solved Problems)

Exercises. Set 2 (Supplementary Problems)

PART 1. RELATION AND THEIR PROPERTIES.

Introduction

Relationships between elements of sets occur in many contexts. Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, a person and a relative, and so on. In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number and one that is larger than it, a real number x and the value f(x) where f is a function, and so on. Relationships such as that between a program and a variable it uses, and that between a computer language and a valid statement in this language often arise in computer science.
Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets. Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.

Definitions and Examples.

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations. Recall that we have already defined *relation* in Lecture 1-2 as a subset of Cartesian Product of sets.

Again, consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A\times B$, which is read "A cross B". By definition,

$$A\times B=\{(a, b) \mid a\in A \text{ and } b\in B\}$$

One frequently writes A^2 instead of $A\times A$.

EXAMPLE 1. $\mathbf{R}$ denotes the set of real numbers and so $\mathbf{R}^2=\mathbf{R}\times\mathbf{R}$ is the set of ordered pairs of real numbers. The geometrical representation of $\mathbf{R}^2$ is the set of all points in the plane. Here each point P represents an ordered pair (a, b) of real numbers and vice versa; the vertical line through P meets the x-axis at a, and the horizontal line through P meets the y-axis at b. $\mathbf{R}^2$ is frequently called the *Cartesian plane*. ■

EXAMPLE 2. Let A={1, 2} and B={a, b, c}. Then

$$A\times B=\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

$B\times A=\{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

Also, $A\times A=\{(1, 1), (1, 2), (2, 1), (2, 2)\}$ ■

Definitions 1: Let A and B be sets. A *binary relation from* A *to* B is a subset of A × B.

Suppose *R* is a relation from *A* to *B*. Then *R* is a set of ordered pairs where each first element comes from *A* and each second element comes from *B*. That is, for each pair a ∈ A and b ∈ B, exactly one of the following is true:

(i) (a, b)∈R; we then say "*a* is R-related to *b*", written aRb
(ii) (a, b)∉R; we then say "*a* is not R-related to *b*".

The *domain* of a relation R is the set of all first elements of the ordered pairs which belong to *R*, and the *range* is the set of second elements.
If R is a relation from a set A to itself, that is, if R is a subset of A^2=A×A, then we say that R is a *relation on* A. ■

EXAMPLE 3.

. (a) A={1, 2, 3} and B ={x, y, z}, and let R={(1, y), (1, z), (3, y)}. Then R is a relation from A to B since R is a subset of A×B. With respect to this relation, 1Ry, 1Rz, 3Ry. The domain of R is {1, 3} and the range is {y, z}.

. (b) Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a, b), where a is a student enrolled in course b. For instance, if Jason Goodfriend and Deborah Sherman are enrolled in CS518, the pairs (Jason Goodfriend, CS518) and (Deborah Sherman, CS518) belong to R. If Jason Goodfriend is also enrolled in CS510, then the pair (Jason Goodfriend, CS510) is also in R. However, if Deborah Sherman is not enrolled in CS510, then the pair (Deborah Sherman, CS510) is not in R.

Note that if a student is not currently enrolled in any courses there will be no pairs in R that have this student as the first element. Similarly, if a course is not currently being offered there will be no pairs in R that have this course as their second element.

. (c) Let A be the set of cities in the U.S.A., and let B be the set of the 50 states in the U.S.A. Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the state b. For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R.

. (d) Set inclusion ⊆ is a relation on any collection of sets. For, given any pair of set A and B, either A⊆B or A⊄B.

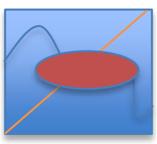
. (e) A familiar relation on the set **Z, Z**×**Z,** of integers is "*m* divides *n*." A common notation for this relation is to write *m*|*n* when *m* divides *n*. Thus 6|30 but 7 does not divide 25? Hence (7, 25) doesn't belong to the relation.

. (f) Consider the set *L* of lines in the plane. Perpendicularity, written "⊥," is a relation on *L*. That is, given any pair of lines *a* and *b*, either a⊥b or a is not perpendicular to b.

. (g) Similarly, "is parallel to," written "||," is a relation on *L* since either a||b or a is not parallel to b.

. (h) Let A be any set. An important relation on A (means a subset of A×A) is that of *equality*, {(a, a) | a∈A} which is usually denoted by "=". This relation is also called the *identity* or *diagonal* relation on A and it will also be denoted by Δ_A or simply Δ.

Square A×A – blue region. A – is an interval on OX axis, A – the same interval on OY axis



A×A – square; Diagonal relation, Δ_A on A is the diagonal of Square

. (k) Let A be any set. Then A×A is the subset of A×A and hence is a relation on *A* called the *universal relation*.

. (L) Let *A* be any set. Then ∅ is the subsets of A×A and hence is a relations on A called the *empty relation*. ■

EXAMPLE 4. How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of A×A. Because A×A has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of A×A. Thus, there are 2^{n^2} relations on a set with n elements.

For example, there are $2^{3^2}=2^9=512$ relations on the set {a, b, c}. ■

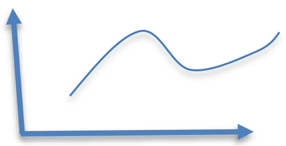
EXAMPLE 5. Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R={(a, b) | a divides b}?

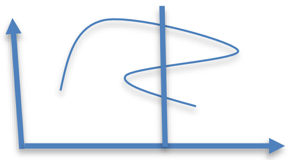
Solution: Because (a, b) is in R if and only if a and *b* are positive integers not exceeding 4 such that a divides b, we see that

R={(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}. ■

Functions as Relations

Recall from Lecture 1 that for arbitrary function f from a set A to a set B *grf* is the set of ordered pairs (a, b) such that f(a)=b. Because the *grf* is a subset of A×B, it is a relation from A to B. Moreover, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph. (a, b) is the point on a graph of function f(x) , it means that b=f(a)





Graph of a function. Not graph of a function

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.

$$R= \{(a, b)\in A\times B \mid a \text{ covers totally set } A, (a, b_1)\in R \text{ and } (a, b_2)\in R \text{ implies } b_1=b_2\}$$

This can be done by assigning to an element a of A the unique element b∈B such that (a, b)∈R.

A relation can be used to express a one-to-many relationship between the elements of the sets A and B, where an element of A may be related to more than one element of B. A function represents a relation where exactly one element of B is related to each element of A.

Relations are a generalization of graphs of functions; they can be used to express a much wider class of relationships between sets. (Recall that the graph of the function f from A to B is the set of ordered pairs (a, f(a)) for a∈A. ■

Pictorial Representatives of Relations

There are various ways of picturing relations.

Relations as graphs of the equations.

Let *S* be a relation on the set **R** of real numbers; that is, *S* is a subset of $\mathbf{R}^2=\mathbf{R}\times\mathbf{R}$. Frequently, *S* consists of all ordered pairs of real numbers which satisfy some given equation E(x, y)=0 (such as $x^2+y^2=25$).

Since $\mathbf{R}^2$ can be represented by the set of points in the plane, we can picture *S* by emphasizing those points in the plane which belong to *S*. The pictorial representation of the relation is sometimes called the *graph* of the relation.

EXAMPLE 6. The graph of the relation $S=\{(x, y)\in\mathbf{R}\times\mathbf{R} \mid x^2+y^2=25)$ is a circle having its center at the origin and radius 5. ■

Directed Graphs of Relations on Sets

There is an important way of picturing a relation *R* on a finite set. First we write down the elements of the set (elements are called *vertices*), and then we draw an arrow from each element *x* to each **element** *y* whenever *x* is related to *y* (arrows are called *directed edges*). This diagram is called the *directed graph of the relation*.

Figure 1 below shows the directed graph of the relation *R* on the set A={1,2,3,4}:

R={(1,2), (2,2), (2,4), (3,2), (3,4), (4,1), (4,3)}

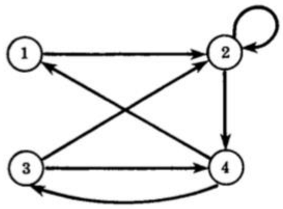


Figure 1

Observe that there is an arrow from 2 to itself, since 2 is related to 2 under *R*. We will intensively use the directed graph representation of relations, especially for the transitive relations. ■

Pictures of Relations on Finite Sets

Suppose A and B are finite sets. There are two ways of picturing a relation R from A to B.

(i) Form a rectangular array (matrix) whose rows are labeled by the elements of A and whose columns are labeled by the elements of *B*. Put a 1 or 0 in each position of the array according as $a\in$A is or is not related to $b\in$B. This array is called the *matrix of the relation*.

(ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a\in$A to $b\in$B whenever *a* is related to *b*. This picture will be called the *arrow diagram* of the relation.

Figure 2 below pictures the relation R={(1, y), (1, z), (3, y)}) by the above two ways.

	x	**y**	**z**
1	0	1	1
2	0	0	0
3	0	1	0

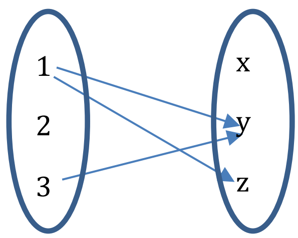
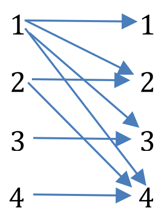


Figure 2. (i) – matrix form, (ii) – arrow diagram form

Figure 3 below demonstrates directed graph (or arrow diagram) and table (matrix) form for the relation from Example 5.

Figure 3



R	**1**	**2**	**3**	**4**
1	x	x	x	x
2		x		x
3			x	
4				x

Properties of Relations

There are several properties that are used to classify relations on a set. We will introduce the most important of these here.

Reflexive Relation.

In some relations an element is always related to itself. For instance, let R be the relation on the set of all people consisting of pairs (x, y) where x and y have the same mother and the same father. Then xRx for every person x.

Definition 2: A relation R on a set A is called *reflexive* if $(a, a)\in R$ for every element $a\in A$. ■

Remark: We see that a relation on A is reflexive if every element of A is related to itself. Therefore, a relation R on the set A is reflexive iff $\Delta_A \subseteq R$ ■

EXAMPLE 7. Consider the following relations on A={1, 2, 3, 4}:

R_1={(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)},

R_2={(1,1), (1,2), (2,1)},

R_3={(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)},

R_4={(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)},

R_5={(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)},

R_6={(3, 4)}.

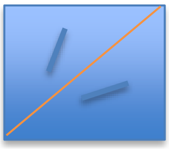
Which of these relations are reflexive? ■

Solution: Only R_3 and R_5 contain the diagonal Δ_A therefore they are reflexive. ■

EXAMPLE 8.

Which of relations from Example 3 are reflexive?

Solution: 3(d), 3(g), 3(h), and 3(k) contain the diagonal Δ_A therefore they are reflexive. ■



Symmetric and Antisymmetric Relations.

In some relations an element is related to a second element if and only if the second element is also related to the first element. The relation consisting of pairs (x, y), where x and y are students at your school with at least one common class has this property.

Definition 3: A relation R on a set A is called *symmetric* if:

$$(b, a)\in R \text{ whenever } (a, b)\in R, \text{ for all } a, b\in A.$$

A relation R on a set A is called *antisymmetric* if:

$$(a, b) \in R \text{ and } (b, a) \in R, \text{ then } a = b$$

■

That is, a relation is symmetric if and only if a is related to b implies that b is related to a.

A relation is antisymmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a. That is, R is antisymmetric if $a\neq b$ and $(a, b)\in R$ then $(b, a)\notin R$. That is, the only way to have a related to b and b related to a is for a and b to be the same element. ■

The terms *symmetric* and *antisymmetric* are not opposites, because a relation can have both of these properties or may lack both of them. A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b), where $a\neq b$.

For example, the relation R={(1, 3), (3, 1), (2, 3)} is neither symmetric nor antisymmetric. On the other hand, the relation G={(1, 1), (2, 2)} is both symmetric and antisymmetric. ■

EXAMPLE 9.

Which of the relations from Example 7 are symmetric and which are antisymmetric?

Solution: The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2, the only thing to check is that both (2, 1) and (1, 2) are in the relation. For R_3, it is necessary to check that both (1, 2) and (2, 1) belong to the relation, and (1, 4) and (4, 1) belong to the relation. The reader should verify that none of the other relations is symmetric. **This is done by finding a pair (a, b) such that it is in the relation but (*b, a*) is not.**

R_4, R_5, and R_6 are all antisymmetric. **For each of these relations there is no pair of elements *a* and *b* with *a*≠*b* such that both (*a, b*) and (*b, a*) belong to the relation.** The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b) with a≠b such that (a, b) and (b, a) are both in the relation. Thus, *R* is not antisymmetric if there exist distinct elements *a* and *b* in *A* such that *aRb* and *bRa*. ■

EXAMPLE 10.

Consider the relations on the set of integers:

$R_1=\{(a, b)|a\leq b\}$, $R_2=\{(a, b)|a>b\}$, $R_3=\{(a, b)|a=b \text{ or } a=-b\}$,

$R_4=\{(a, b)|a=b\}$, $R_5=\{(a, b)|a=b+1\}$, $R_6=\{(a, b)|a+b\leq 3\}$.

Which of the relations from Example 10 are symmetric and which are antisymmetric?

Solution: The relations R_3, R_4, and R_6 are symmetric:

R_3 is symmetric, for if a=b or a=−b, then b=a or b=−a.

R_4 is symmetric because a=b implies that b=a.

R_6 is symmetric because a+b≤3 implies that b+a≤3.

The reader should verify that none of the other relations is symmetric.

The relations R_1, R_2, R_4, and R_5 are antisymmetric:

R_1 is antisymmetric because the inequalities a≤b and b≤a imply that a=b.

R_2 is antisymmetric because it is impossible that a>b and b>a.

R_4 is antisymmetric, because two elements are related with respect to R_4 if and only if they are equal.

R_5 is antisymmetric because it is impossible that a=b+1 and b=a+1.

The reader should verify that none of the other relations is antisymmetric. ■

Transitive Relation.

Let R be the relation consisting of all pairs (x, y) of students at your school, where x has taken more credits than y. Suppose that x is related to y and y is related to z. This means that x has taken more credits than y and y has taken more credits than z. We can conclude that x has taken more credits than z, so that x is related to z. What we have shown is that R has the transitive property, which is defined as follows.

Definition 4: A relation R on a set A is called *transitive* if whenever (a, b)∈R and (b, c)∈R, then (a,c)∈R, for all a, b, c∈A. ■

EXAMPLE 11. Which of the relations in Example 7 are transitive?

R_1={(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)},

R_2={(1,1), (1,2), (2,1)},

R_3={(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)},

R_4={(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)},
R_5={(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)},
R_6={(3, 4)}.

Solution. R_4, R_5, and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does. For instance, R_4 is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R_4. The reader should verify that R_5 and R_6 are transitive. R_1 is not transitive because (3,4) and (4,1) belong to R_1, but (3,1) does not. R_2 is not transitive because (2, 1) and (1, 2) belong to R_2, but (2, 2) does not. R_3 is not transitive because (4, 1) and (1, 2) belong to R_3, but (4, 2) does not. ■

EXAMPLE 12. Which of the relations in Example 10 are transitive?

Solution. The relations R_1, R_2, R_3, R_4 are transitive. R_1 is transitive because $a \leq b$ and $b \leq c$ imply that $a \leq c$. R_2, is transitive because $a>b$ and $b>c$ imply that $a>c$. R_3 is transitive because $a=\pm b$ and $b=\pm c$ imply that $a = \pm c$. R_4 is clearly transitive, as the reader should verify. R_5 is not transitive because (2, 1) and (1, 0) belong to R_5, but (2, 0) does not. R_6 is not transitive because (2, 1) and (1, 2) belong to R_6, but (2, 2) does not. ■

At the next part we face an interesting characterization of transitivity property in terms of composite operation over relations.

PART 2. COMBINING RELATIONS. EQUIVALENCE RELATION.

Combining Relations (Operation over Relations)

Because relations from A to B are subsets of A×B, two relations from A to B can be combined in any way two sets can be combined.

EXAMPLE 13. Let A={1,2,3}, B={1,2,3,4} and R_1, R_2 are the following subsets of A×B:
R_1={(1,1), (2,2), (3,3)}, R_2={(1, 1), (1, 2), (1, 3), (1, 4)}.
Then
$R_1 \cup R_2$={(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)}, $R_1 \cap R_2$={(1, 1)}
$R_1 - R_2$={(2, 2), (3, 3)}, $R_2 - R_1$={(1,2),(1,3),(1,4)}. ■

EXAMPLE 14. Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose that R_1 consists of all ordered pairs (*a*, *b*), where *a* is a student who has taken course *b*, and R_2 consists of all ordered pairs (*a*, *b*), where *a* is a student who requires course *b* to graduate. What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$?

Solution. The relation $R_1 \cup R_2$ consists of all ordered pairs (*a*, *b*), where *a* is a student who either has taken course *b* or needs course *b* to graduate;
$R_1 \cap R_2$ is the set of all ordered pairs (*a*, *b*), where *a* is a student who has taken course *b* and needs this course to graduate;
$R_1 - R_2$ is the set of ordered pairs (*a*, *b*), where *a* has taken course *b* but does not need it to graduate; that is, *b* is an elective course that *a* has taken;
$R_2 - R_1$ is the set of all ordered pairs (*a*, *b*), where *b* is a course that *a* needs to graduate but has not taken.

Definition 5: Let *R* be any relation from a set *A* to a set *B*. The *inverse* of *R*, denoted by R^{-1}, is the relation from *B* to *A* which consists of those ordered pairs which, when reversed, belong to *R*; that is, $R^{-1}=\{(b, a) | (a, b) \in R\}$ ■

EXAMPLE 15. Let A={1, 2, 3} and B={x, y, z}. If R={(1, y), (1, z), (3, y)} then the inverse of R is R^{-1}={(y, 1), (z, 1), (y, 3)}. Clearly, if R is any relation, then $(R^{-1})^{-1}=R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then

R^{-1} is also a relation on A. ■

There is another way that relations are combined that is analogous to the composition of functions.

Definition 6: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite* of R and S, denoted as (S∘R), is a relation from A to C, subset of A×C, consisting of ordered pairs (a, c)∈A×C such that there exists an element b∈B such that (a, b)∈R and (b, c)∈S. ■

The important property of composition of relations is associative property.

Theorem 1: Let A, B, C and D be sets. Suppose R is a relation from A to B, and S is a relation from B to C, and T is a relation from C to D. Then

$$T\circ(S\circ R)=(T\circ S)\circ R$$

That is, the composition of relations is associative.

Proof. We need to show that each ordered pair in T∘(S∘R) belongs to (T∘S)∘R, and vice versa. Suppose (a, d) belongs to T∘(S∘R). Then there exists c∈C such that (a, c)∈S∘R and (c, d)∈T. Since (a, c)∈S∘R, there exists b∈B such that (a, b)∈R and (b, c)∈S. Since (b, c)∈S and (c, d)∈T , we have (b, d)∈T∘S; and since (a, b)∈R and (b, d)∈T∘S, we have (a, d)∈(T∘S)∘R Therefore, T∘(S∘R)⊆(T∘S)∘R. Similarly (T∘S)∘R⊆T∘(S∘R). Both inclusion relations prove T∘(S∘R)=(T∘S)∘R. ■

QUESTION. How to find the composite of the relations? We suggest 3 equivalent methods to it.

Method 1. Finding of Intermediate elements.

Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation, as Examples 16 and 17 illustrate.

EXAMPLE 16. What is the composite of the relations R and S, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with R={(1, 1),(1, 4),(2, 3),(3, 1),(3, 4)} and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with S={(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?

Solution: S∘R is constructed using all ordered pairs in R and ordered pairs in S, where **the second element of the ordered pair in R agrees with the first element of the ordered pair in S.** For example, the ordered pairs (2, 3) in R and (3, 1) in S produce the ordered pair (2, 1) in S∘R. Computing all the ordered pairs in the composite, we find:

S∘R={(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)}. ■

EXAMPLE 17. *Composing the Parent Relation with Itself.* Let R be the relation on the set of all people such that (a, b)∈R if person a is a parent of person b. Then (a, c)∈R∘R if and only if there is a person b such that (a, b)∈R and (b, c)∈R, that is, if and only if there is a person b such that a is a parent of b and b is a parent of c. In other words, (a, c)∈R∘R if and only if a is a grandparent of c. ■

Method 2. We apply arrow diagrams for R and S to find the composition.

EXAMPLE 18. Let A={1, 2, 3, 4}, B={a, b, c, d}, C={x, y, z} and let R={(1, a), (2, d), (3, a), (3, b), (3, d)} and S={(b, x), (b, z), (c, y), (d, z)}

Consider the arrow diagrams of R and S as in Figure. 4.

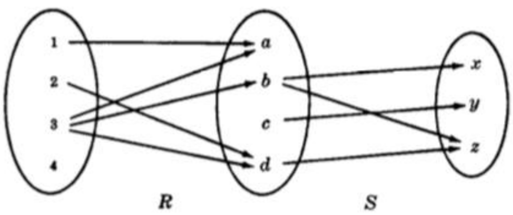


Figure. 4

Observe that there is an arrow from 2 to *d* which is followed by an arrow from *d* to *z*. We can view these two arrows as a "path" which "connects" the element 2∈A to the element z∈C. Thus: 2(S∘R)z since 2Rd and dSz. Similarly there is a path from 3 to *x* and a path from 3 to *z*. Hence 3(S∘R)x and 3(S∘R)z.

No other element of *A* is connected to an element of *C*. Accordingly, S∘R={(2, z), (3, x), (3, z)} ■

Method 3. Composition of Relations and Matrices

There is another way of finding S∘R. Let M_R and M_S denote respectively the matrix representations of the relations R and S. Then multiplying M_R and M_S we obtain the matrix M_RM_S

M_R

	a	b	c	d
1	1	0	0	0
2	0	0	0	1
3	1	1	0	1
4	0	0	0	0

M_S

	x	y	z
a	0	0	0
b	1	0	1
c	0	1	0
d	0	0	1

M_RM_S

	x	y	z
1	0	0	0
2	0	0	1
3	1	0	2
4	0	0	0

The nonzero entries in this matrix tell us which elements are related by S∘R. Thus, M = M_RM_S and $M_{S\circ R}$ have the same nonzero entries.

Note. Number 2 in matrix M (intersection of 3rd row and z-column) shows that there exist 2 ways to connect element 3 and element z ((3, b), (b, z) and (3, d), (d, z)). ■

Transitive Relation (continuation).

The transitive property has an interesting description in terms of the composition of relations.

Definition 7: Let R be a relation on the set A. The powers R^n, n=1, 2, 3,..., are defined recursively by R^1=R and $R^{n+1}=R^n\circ R$. ■

Theorem 2: A relation R on a set A is transitive if and only if, for every n ≥ 1, we have $R^n\subseteq R$.

Proof. We suppose that $R^n\subseteq R$ for n = 1, 2, 3, . . ., in particular, $R^2\subseteq R$. To see that this implies R is transitive, note that if (a, b)∈R and (b, c)∈R, then by the definition of composition, (a, c)∈ R^2. Because $R^2\subseteq R$, this means that (a, c)∈R. Hence, R is transitive.
Conversely, let a relation *R* is transitive. We will use mathematical induction to prove $R^n\subseteq R$. Note that the theorem is trivially true for n=1. Assume that $R^n\subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step, we must show that this implies that R^{n+1} is also a subset of R. To show this, assume that (a, b)∈R^{n+1}. Then, because $R^{n+1}=R^n\circ R$, there is an element x with x∈A such that (a, x)∈R and (x, b)∈R^n. The inductive hypothesis, namely, that $R^n\subseteq R$, implies that (x, b)∈R. Furthermore, because R is transitive, and (a, x)∈R, (x, b)∈R, it follows that (a, b)∈R. This shows that $R^{n+1}\subseteq R$, completing the proof. ■

Definition 8: Let A be a set, *a*∈A, *b*∈B and let R be a relation on A**.** A **path (*of length n*) in R from *a* to *b*** is defined as a sequence of elements (*a*, x_1, x_2, ..., x_{n-1}, *b*) in A with property that (*a*, x_1)∈R, (x_1, x_2)∈R,..., and (x_{n-1}, *b*)∈R. Sometimes we denote path $x_0x_1x_2...x_{n-1}x_n$.

A path of length n≥1 that begins and ends at the same vertex is called a *circuit* or *cycle*. Any empty relation can be vied as a path of length ***zero*** from *a* to *a* in **R** ■

Theorem 3: Let R be a relation on A. There is a path of length n≥1 in R from *a* to *b* iff (*a, b*)∈R^n.

Proof. From the definition 9 we immediately see that existence of the path of the length n from a to b means that the pair (*a, b*)∈R^n. We need to proof the theorem only in opposite direction.

Assume that $(a, b) \in R^n$. We need to show that there exists a path of length n from a to b. We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, hence, the theorem is true when $n=1$. Assume that the theorem is true for an arbitrarily chosen positive integer n. Let $(a, b) \in R^{n+1} = R^n \circ R$. By definition of composition of relations it means that there exist an element $c \in A$ such that $(a, c) \in R$, and $(c, b) \in R^n$. Therefore, according to our inductive hypothesis there exists a path of length one from a to c, and there exist a path of the length n from c to b. Last two paths together determine the path of length (n+1) from a to b in R. This completes the proof. ■

Closure Properties

Consider a given set A and the collection of all relations on A. Let P be a property of such relations, such as being symmetric or being transitive. A relation with property P will be called a P-relation.

Definition 9. The P-closure of an arbitrary relation R on A, written $P(R)$, is a P-relation such that

$$R \subseteq P(R) \subseteq S \text{ for every } P\text{-relation } S \text{ containing } R.$$

In other words, $P(R)$ is the smallest P-relation containing the relation R. ■

We write

reflexive(R), symmetric(R), and transitive(R)

for the reflexive, symmetric, and transitive closures of R. In general, P(R) may not exist. However, there is a general situation where P(R) will always exist. Suppose P is a property such that there is at least one P-relation containing R and that the intersection of any P-relations is again a P-relation. Then one can prove (Exercise) that

$$P(R) = \cap(S \mid S \text{ is a } P\text{-relation and } R \subseteq S)$$

Thus, one can obtain P(R) from the "top-down," that is, as the intersection of relations. However, one usually wants to find P(R) from the "bottom-up," that is, by adjoining elements to R to obtain P(R). We do it below.

Reflexive and Symmetric Closures

The next theorem tells us how to obtain easily the reflexive and symmetric closures of a relation. Here $\Delta_A = \{(a, a) \mid a \in A\}$ is the diagonal or equality relation on A.

Theorem 4: Let R be a relation on a set A. Then:

(i) $R \cup \Delta_A$ is the reflexive closure of R.

(ii) $R \cup R^{-1}$ is the symmetric closure of R. In other words, reflexive(R) is obtained by simply adding to R those elements (a, a) in the diagonal which do not already belong to R, and symmetric(R) is obtained by adding to R all pairs (b, a) whenever (a, b) belongs to R. ■

EXAMPLE 19. Consider the relation R={(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3)} on the set A={1, 2, 3, 4}. Then reflexive(R)=R∪{(2, 2), (4, 4)} and symmetric(R)=R∪{(4, 2), (3, 4)} ■

Transitive Closure.

We now show that finding the transitive closure of a relation is equivalent to determining which pairs of vertices in the associated directed graph are connected by a path. With this in mind, we define a new relation.

Definition 10: Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b, it follows that R^* is the union of all the sets R^n. In other words, $R^* = \bigcup_{i=1}^{\infty} R^i$ ■

The connectivity relation is useful in many models.

EXAMPLE 20. Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^* ?

Solution: The relation R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ and $(c, b) \in R$, that is, if there is a person c such that a has met c and c has met b. Similarly, R^n consists of those pairs (a, b) such that there are people $x_1, x_2, ..., x_{n-1}$ such that a has met x_1, x_1 has met x_2, ..., and x_{n-1} has met b.

The relation R^* contains (a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met the next person in the sequence. ■

There are many interesting conjectures about R^*. We will use graphs to model this application later.

EXAMPLE 21. Let R be the relation on the set of all subway stops in Baku that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n when n is a positive integer? What is R^* ?

Solution: The relation R^n contains (a, b) if it is possible to travel from stop a to stop b by making at most (n–1) changes of trains. The relation R^* consists of the ordered pairs (a, b) where it is possible to travel from stop a to stop b making as many changes of trains as necessary. (The reader should verify these statements.) ■

EXAMPLE 22. Let R be the relation on the set of all regions in Azerbaijan that contains (a, b) if regions a and b have a common border. What is R^n, where $n \geq 1$? What is R^* ?

Solution: The relation R^n consists of the pairs (a, b), where it is possible to go from region a to region b by crossing exactly n region borders. R^* consists of the ordered pairs (a, b), where it is possible to go from region a to region b crossing as many borders as necessary. (The reader should verify these statements.) The only ordered pairs not in R^* are those containing regions that are not connected to the continental Azerbaijan (i.e., those pairs containing Nakhchivan). ■

Theorem 5 shows that the transitive closure of a relation and the associated connectivity relation are the same.

Theorem 5: R^* is the transitive closure of R, that is, transitive(R)=R^*

Proof. Exercise. **Hint.** We must show three facts:

1. $R \subseteq R^*$ (it is clear by definition of R);
2. R^* is the transitive relation on A
3. If S any transitive relation on A, such that $R \subseteq S$ then $R^* \subseteq S$. ■

Now that we know that the transitive closure equals the connectivity relation, we turn our attention to the problem of computing this relation. Let A be a finite set, $a, b \in A$ and R be a relation on A. We do not need to examine arbitrarily long paths to determine whether there is a path from a to b in R**.** As Lemma 1 shows, it is sufficient to examine paths with length no more than n, where n is the number of elements in the set.

Lemma 1: Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n-1$.

Proof: Suppose there is a path from a to b in R. Let m be the length of the shortest such path. Suppose that $x_0, x_1, x_2, ..., x_{m-1}, x_m$, where $x_0=a$ and $x_m=b$, is such a path.

Suppose that $a=b$ and that $m>n$, so that $m \geq n+1$. By the pigeonhole principle, because there are n vertices in A, among the m vertices $x_0, x_1, x_2, ..., x_{m-1}$, at least two are equal (Figure 5).

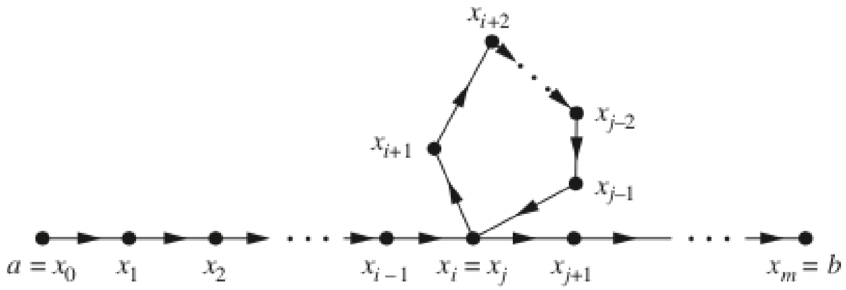


Figure 5. Producing a Path with Length Not Exceeding *n*.

Suppose that $x_i=x_j$ with $0\le i<j\le m-1$. Then the path contains a circuit from x_i to itself. This circuit can be deleted from the path from *a* to b, leaving a path, namely, $x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_{m-1}, x_m$ from *a* to *b* of shorter length. Hence, the path of shortest length must have length less than or equal to n.
The case where $a\neq b$ is left as an exercise. ■

From the Theorem 5 and Lemma 1 we get the following theorem.

Theorem 6. Suppose A is a finite set with *n* elements. Then transitive(R)=$R^*=R\cup R^2\cup \ldots \cup R^n$

Proof. From Lemma 1, we see that the transitive closure of R is the union of R, R^2, R^3,..., and R^n. This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i, for some positive integer *i* with $i \le n$. ■

EXAMPLE 23. Consider the relation R={(1, 2), (2, 3), (3, 3)} on A={1, 2, 3}. Then:

$$R^2= R\circ R = \{(1, 3), (2, 3), (3, 3)\} \text{ as } R=\{(1, 2), (2, 3), (3, 3)\}, R=\{(1, 2), (2, 3), (3, 3)\}$$

$$\text{and } R^3=R^2\circ R=\{(1, 3), (2, 3), (3, 3)\}$$

Accordingly, transitive (R) = {(1, 2), (2, 3), (3, 3), (1, 3)} ■

Equivalence Relation

Definition 11: Consider a nonempty set A. A relation *R* on A, $R\subset A\times A$ is an *equivalence relation* if *R* is reflexive, symmetric, and transitive. That is, *R* is an equivalence relation on A if it has the following three properties:

(1) For every a∈A, *aRa*. (2) If *aRb*, then *bRa*. (3) If *aRb* and *bRc*, then *aRc*. ■

The general idea behind an equivalence relation is that it is a classification of objects which are in some way “alike.” In fact, the relation “=” of equality on any set A is an equivalence relation, that is, (1) a=a for every a∈A. (2) If a=b, then b=a. (3) If a=b, b=c, then a=c.

Other equivalence relations below.

EXAMPLE 24

(a) Let *L* be the set of lines and let *T* be the set of triangles in the Euclidean plane.

(i) The relation “is parallel to or identical to” is an equivalence relation on *L*.

(ii) The relations of congruence and similarity are equivalence relations on *T*.

(b) The relation ⊆ of set inclusion **is not** an equivalence relation. It is reflexive and transitive, but it is not symmetric since A⊆B does not imply B⊆A.

(c) Let *m* be a fixed positive integer. Two integers *a* and *b* are said to be *congruent modulo m*, written

$$a\equiv b \pmod m$$

if *m* divides (a−b). For example, for the modulus m = 4, we have

$11 \equiv 3 \pmod 4)$ $\{3, 7, 11, 15, 19, 23, \ldots\}$
and $22 \equiv 6 \pmod 4)$, $\{2, 6, 10, 14, 18, 22, \ldots\}$

since 4 divides $11-3=8$ and 4 divides $22 - 6 = 16$. This relation of congruence modulo *m* is an important equivalence relation. ■

EXAMPLE 25

Suppose that R is the relation on the set of strings of English letters such that aRb if and only if L(a)=L(b), where L(x) is the length of the string x. Show that R is an equivalence relation.

Solution: Because L(a)=L(a), it follows that aRa whenever a is a string, so that R is reflexive. Next, suppose that aRb, so that L(a)=L(b). Then bRa, because L(b)=L(a). Hence, R is symmetric. Finally, suppose that aRb and bRc. Then L(a)=L(b) and L(b)=L(c). Hence, L(a)=L(c), so aRc. Consequently, R is transitive. Therefore, R is an equivalence relation. ■

EXAMPLE 26

Let *n* be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that:

$$\mathbf{s}R_n\mathbf{t}$$

if and only if

s=**t,** or both **s** and **t** have at least *n* characters and the first *n* characters of **s** and **t** are the same.

That is,

- a string of fewer than *n* characters is related only to itself;
- a string **s** with at least *n* characters is related to a string **t** if and only if **t** has at least *n* characters and **t** begins with the *n* characters at the start of **s**.

For example, let n = 3 and let S be the set of all bit strings. Then $\mathbf{s}R_3\mathbf{t}$ either when **s = t** or both **s** and **t** are bit strings of length 3 or more that begin with the same three bits. For instance, $01R_301$ and $00111R_300101$, but $01\not{R}_3010$ and $01011\not{R}_301110$.

Task. Show that for every set S of strings and every integer $n>0$, R_n is an equivalence relation on S.

Solution: The relation R_n is reflexive because **s=s**, so that $\mathbf{s}R_n\mathbf{s}$ whenever s is a string in S. If $\mathbf{s}R_n\mathbf{t}$, then either **s=t** or **s** and **t** are both at least *n* characters long that begin with the same *n* characters. This means that $\mathbf{t}R_n\mathbf{s}$.We conclude that R_n is symmetric.

Now suppose that $\mathbf{s}R_n\mathbf{t}$ and $\mathbf{t}R_n\mathbf{u}$. Then either **s=t** or **s** and **t** are at least *n* characters long and **s** and **t** begin with the same *n* characters, and either **t=u** or **t** and **u** are at least n characters long and **t** and **u** begin with the same n characters. From this, we can deduce that either **s=u** or both **s** and **u** are n characters long and **s** and **u** begin with the same n characters (because in this case we know that **s**, **t**, and **u** are all at least n characters long and both **s** and **u** begin with the same *n* characters as **t** does). Consequently, R_n is transitive. It follows that R_n is an equivalence relation. ■

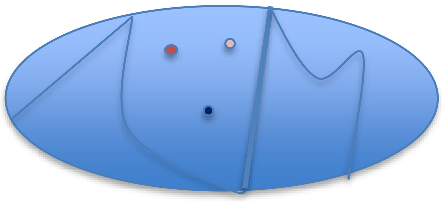
Equivalence Relations and Partitions

This subsection explores the relationship between equivalence relations and partitions on a non-empty set *S*. Recall first that a *partition P* of *S* is a collection $\{A_i\}$ of nonempty subsets of *S*, $A_i \subseteq S$ for all *i* with the following two properties:

(1) Each $a \in S$ belongs to some A_i, that is, $S = \cup A_i$

(2) If $A_i \neq A_j$ then $A_i \cap A_j = \emptyset$.

In other words, a partition *P* of *S* is a subdivision of *S* into disjoint nonempty sets.



Definition 12: Suppose *R* is an equivalence relation on a set *S, R*⊆*S*×*S*. For each a∈S, let [a] denote the set of elements of *S* to which *a* is related under *R*; that is:

$$[a]=\{x\in S \mid (a, x)\in R\}$$

We call [*a*] the *equivalence class* of *a* in *S*; any b∈[a] is called a *representative* of the class [*a*]. The collection of all equivalence classes of elements of *S* under an equivalence relation *R* is denoted by *S/R*, that is,

$$S/R = \{[a] \mid a \in S\}$$

It is called the *quotient set* of *S* by *R*. ■

The fundamental property of a quotient set is contained in the following theorem.

Theorem 7: Let R be an equivalence relation on a set *S*. Then *S/*R is a partition of *S*. Specifically:

(i) For each *a* in *S*, we have $a\in[a]$, that is, $S=\bigcup_{a\in S}[a]$.
(ii) [*a*]=[*b*] if and only if (*a, b*)∈R.
(iii) If [*a*]≠[*b*], then [*a*] and [*b*] are disjoint.

Conversely, given a partition $\{A_i \mid i \in I\}$ of the set *S*, there is an equivalence relation R on *S* such that the sets A_i are the equivalence classes.

Proof.

⇒. (Let R be an equivalence relation on a set *S*. Then *S/*R is a partition of *S*.**)**

(i): Since R is reflexive, (*a, a*)∈R for every *a*∈A and therefore a∈[a].

(ii): Suppose (*a, b*)∈R. We want to show that [*a*]=[*b*]. Let *x*∈[*b*]; then (*b, x*)∈R. But by hypothesis, (*a, b*)∈R and so, by transitivity, (*a, x*)∈R. Accordingly *x*∈[a]. Thus [*b*]⊆[*a*]. To prove that [*a*]⊆[*b*] we observe that (*a, b*)∈R implies, by symmetry, that (*b, a*)∈R. Then, by a similar argument, we obtain [a]⊆[b]. Consequently, [a]=[b].
On the other hand, if [*a*]=[*b*], then, by (i), b∈[*b*] = [*a*]; hence (a, b)∈R.

(iii): We prove the equivalent contrapositive statement:

$$\text{If } [a]\cap[b]\neq\emptyset \text{ then } [a]=[b]$$

If $[a]\cap[b]\neq\emptyset$, then there exists an element $x\in A$ with $x\in[a]\cap[b]$. Hence (*a, x*)∈R and (*b, x*) ∈R. By symmetry, (*x, b*)∈R and by transitivity, (*a, b*)∈R. Consequently by (ii), [*a*]=[*b*].

⇐ Conversely, (Any partition of S generates an equivalence relation R on S such that elements of the partition are the equivalence classes)

To see this, assume that $\{A_i \mid i\in I\}$ is a partition on S. Let R be the relation on S consisting of the pairs (*a, b*), where *a* and *b* belong to the same subset A_i in the partition. To show that R is an equivalence relation we must show that R is reflexive, symmetric, and transitive.

We see that *(a, a)*∈R for every *a*∈S, because *a* is in the same subset as itself. Hence, R is reflexive. If (*a, b*)∈R, then *b* and *a* are in the same subset of the partition, so that (*b, a*)∈R as well. Hence, R is symmetric. If (*a, b*)∈R and (*b, c*)∈R, then *a and b* are in the same subset X in the partition, and

b and c are in the same subset Y of the partition. Because the subsets of the partition are disjoint and b belongs to X and Y, it follows that X=Y. Consequently, a and c belong to the same subset of the partition, so $(a, c)\in$R. Thus, R is transitive.

It follows that R is an equivalence relation. The equivalence classes of R consist of subsets of S containing related elements, and by the definition of R, these are the subsets of the partition. ■

EXAMPLE 27.

(a) Consider the relation R={(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)} on S={1, 2, 3}. One can show that R is reflexive, symmetric, and transitive, that is, that R is an equivalence relation. Also:

$$[1]=\{1,2\},\ [2]=\{1,2\},\ [3]=\{3\}$$

Observe that [1]=[2] and that S/R={[1], [3]} is a partition of S. One can choose either {1, 3} or {2, 3} as a set of representatives of the equivalence classes.

(b) Let R_5 be the relation of congruence modulo 5 on the set **Z** of integers denoted by

$$x\equiv y \pmod 5$$

This means that the difference $x - y$ is divisible by 5. Then R_5 is an equivalence relation on **Z**. The quotient set $\mathbf{Z}/R_5$ contains the following five equivalence classes:

$A_0 = \{\ldots,-10, -5, 0, 5, 10, \ldots\}$ $\quad$ $A_1 = \{\ldots,-9, -4, 1, 6, 11, \ldots\}$

$A_2 = \{\ldots,-8, -3, 2, 7, 12, \ldots\}$ $\quad$ $A_3 = \{\ldots,-7, -2, 3, 8, 13, \ldots\}$

$A_4 = \{\ldots,-6, -1, 4, 9, 14, \ldots\}$

Any integer x, uniquely expressed in the form $x=5q+r$ where $0\le r<5$, is a member of the equivalence class A_r, where r is the remainder. As expected, **Z** is the disjoint union of equivalence classes A_0 A_1, A_2, A_3, A_4. Usually one chooses {0, 1, 2, 3, 4} as a set of representatives of the equivalence classes.■

EXERCISES. SET 1 (Solved Problems)

Relations and their graphs

1. Find the number of relations from A={a, b, c} to B ={1, 2}.

Solution.

There are 3·2=6 elements in A×B, and hence there are $m = 2^6 = 64$ subsets of A×B. Thus, there are 64 relations from A to B. ■

2. Given A={1, 2, 3, 4} and B={x, y, z}. Let R be the following relation from A to B:

$$R=\{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(a) Determine the matrix of the relation and draw the arrow diagram of R.

(c) Find the inverse relation R^{-1} of R.

(d) Determine the domain and range of R.

Solution.

(a). Observe that the rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that the entry in the matrix corresponding to a∈A and b∈B is 1 if a is related to b and 0 otherwise. Observe also that there is an arrow from a∈A to b∈B iff a is related to b, i.e., iff (a, b)∈R (see Figure 6.)

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0
4	1	0	1

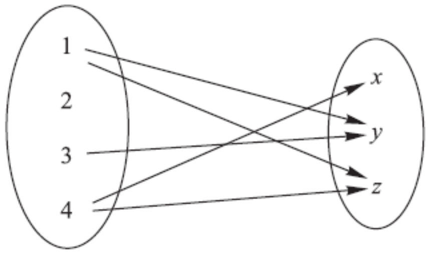


Figure 6.

(c) Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1}=\{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

Observe that by reversing the arrows in Figure 6, we obtain the arrow diagram of R^{-1}.

(d) The domain of R, Dom(R), consists of the 1st elements of the ordered pairs of R, and the range of R, Ran(R), consists of the 2nd elements. Thus, Dom(R)={1,3,4} and Ran(R)={x,y,z} ■

3. Let A {1, 2, 3}, B={a, b, c}, and C={x, y, z}. Consider the following relations R and S from A to B and from B to C, respectively.

$$R=\{(1, b), (2, a), (2, c)\} \text{ and } S=\{(a, y), (b, x), (c, y), (c, z)\}$$

(a) Find the composition relation $S \circ R$.

(b) Find the matrices M_R, M_S, and $M_{S \circ R}$ of the respective relations R, S, and $S \circ R$, and compare $M_{R \circ S}$ to the product $M_R M_S$.

Solution.

(a) Draw the arrow diagram of the relations R and S as in Figure 7. Observe that 1 in A is "connected" to x in C by the path 1→b→x; hence (1, x) belongs to $S \circ R$. Similarly, (2, y) and (2, z) belong to $S \circ R$. We have $S \circ R=\{(1, x), (2, y), (2, z)\}$

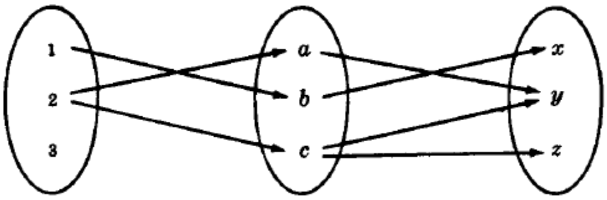


Figure 7.

(b) The matrices of M_R, M_S, and $M_{R \circ S}$ follow:

M_R

	a	b	c
1	0	1	0
2	1	0	1
3	0	0	0

M_S,

	x	y	z
a	0	1	1
b	1	0	0
c	0	1	1

$M_{S \circ R}$

	x	y	z
1	1	0	0
2	0	1	1
3	0	0	0

Multiplying M_R and M_S we obtain $M_R M_S$ as following:

$M_R M_S$

	x	y	z
1	1	0	0
2	0	2	1
3	0	0	0

Observe that $M_{S \circ R}$ and $M_R M_S$ have the same zero entries. Entry (**2, y**)=2 shows that there are 2 possible ways from element 2 to element y: one – through **a**, another - through **c**.

Important Note. Nonzero elements of $M_R M_S$ shows number of paths from source to destination while nonzero elements of $M_{S \circ R}$ shows only existence of such paths. ■

4. Consider the relation R={(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)} on A={1, 2, 3, 4}.

(*a*) Draw its directed graph. (*b*) Find $R^2 = R \circ R$.

Solution.

(*a*) For each (*a*, *b*)∈R, draw an arrow from *a* to *b* as in Figure 8.

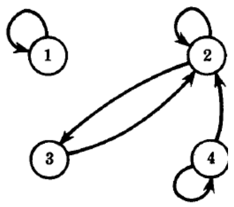


Figure 8

(*b*) For each pair (*a*, *b*)∈R, find all (*b*, *c*)∈R. Then (*a*, *c*)∈R^2. Thus,

$$R^2=\{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$$

5. Let R and S be the following relations on A={1, 2, 3}:

R={(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)}, S= {(1, 2), (1, 3), (2, 1), (3, 3)}

Find (a) R∪S, R∩S, R′; (b) S∘R; (c) $S^2 = S \circ S$.

Solution.

(a) Treat R and S simply as sets, and take the usual intersection and union. For R′, use the fact that A×A is the universal relation on A.

R∩S={(1, 2), (3, 3)}

R∪S={(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)}

R′={(1, 3), (2, 1), (2, 2), (3, 2)}

(b) For each pair (a, b)∈R, find all pairs (b, c)∈S. Then (a, c)∈S∘R.

For example, (1, 1)∈R and (1, 2), (1, 3)∈S; hence (1, 2) and (1, 3) belong to S∘R. Thus,

S∘R= {(1, 2), (1, 3), (1, 1), (2, 3), (3, 2), (3, 3)}

(c) Following the algorithm in (b), we get

S^2=S∘S={(1, 1), (1, 3), (2, 2), (2, 3), (3, 3)} ■

Types of Relations and Closure Properties

6. Consider the following five relations on the set $A=\{1, 2, 3\}$:

$R=\{(1, 1), (1, 2), (1, 3), (3, 3)\}$, $\emptyset$ = empty relation

$S=\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$, $A\times A$ = universal relation

$T=\{(1, 1), (1, 2), (2, 2), (2, 3)\}$

Determine whether or not each of the above relations on A is:

(a) reflexive; (b) symmetric; (c) transitive; (d) antisymmetric.

Solution.

(a) R is not reflexive since $2\in A$ but $(2, 2)\notin R$. T is not reflexive since $(3, 3)\notin T$ and, similarly, $\emptyset$ is not reflexive. S and $A\times A$ are reflexive.

(b) R is not symmetric since $(1, 2)\in R$ but $(2, 1)\notin R$, and similarly T is not symmetric. S, $\emptyset$, and $A\times A$ are symmetric.

(c) T is not transitive since (1, 2) and (2, 3) belong to T, but (1, 3) does not belong to T. The other four relations are transitive.

(d) S is not antisymmetric since $1\neq 2$, and (1, 2) and (2, 1) both belong to S. Similarly, $A\times A$ is not antisymmetric.

The other three relations are antisymmetric. ■

7. Give an example of a relation R on $A=\{1, 2, 3\}$ such that:

(a) R is both symmetric and antisymmetric.

(b) R is neither symmetric nor antisymmetric.

(c) R is transitive but $R\cup R^{-1}$ is not transitive.

Solution.

There are several such examples. One possible set of examples follows:

(a) $R=\{(1, 1), (2, 2)\}$; (b) $R=\{(1, 2), (2, 3)\}$; (c) $R=\{(1, 2)\}$. ■

8. Consider the relation $R=\{(a, a), (a, b), (b, c), (c, c)\}$ on the set $A=\{a, b, c\}$. Find:

(a) reflexive(R); (b) symmetric(R); (c) transitive(R).

Solution.

(a) The reflexive closure on R is obtained by adding all diagonal pairs of $A\times A$ to R which are not currently in R. Hence,

$\text{reflexive}(R)=R\cup\{(b, b)\}=\{(a, a), (a, b), (b, b), (b, c), (c, c)\}$

(b) The symmetric closure on R is obtained by adding all the pairs in R^{-1} to R which are not currently in R. Hence,

$\text{symmetric}(R)=R\cup\{(b, a), (c, b)\}=\{(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)\}$

(c) The transitive closure on R, since A has three elements, is obtained by taking the union of R with $R^2=R\circ R$ and $R^3=R\circ R\circ R$. Note that

$R^2=R\circ R=\{(a, a), (a, b), (a, c), (b, c), (c, c)\}$

$R^3=R\circ R\circ R=\{(a, a), (a, b), (a, c), (b, c), (c, c)\}$

Hence,

$\text{transitive}(R)=R\cup R^2\cup R^3=\{(a, a), (a, b), (a, c), (b, c), (c, c)\}$ ■

Equivalence Relations and Partitions

9. Let A be a set of nonzero integers and let ≈ be the relation on A×A defined by

$$(a, b)\approx(c, d) \text{ whenever } ad=bc$$

Prove that ≈ is an equivalence relation.

Solution.

We must show that ≈ is reflexive, symmetric, and transitive.

(i) Reflexivity: We have $(a, b)\approx(a, b)$ since $ab=ba$. Hence ≈ is reflexive.

(ii) Symmetry: Suppose $(a, b)\approx(c, d)$. Then $ad=bc$. Accordingly, $cb=da$ and hence $(c, d)=(a, b)$. Thus, ≈ is symmetric.

(iii) Transitivity: Suppose $(a, b)\approx(c, d)$ and $(c, d)\approx(e, f)$. Then $ad=bc$ and $cf=de$. Multiplying corresponding terms of the equations gives $(ad)(cf)=(bc)(de)$. Canceling $c\neq0$ and $d\neq0$ from both sides of the equation yields $af=be$, and hence $(a, b)\approx(e, f)$. Thus, ≈ is transitive. Accordingly, ≈ is an equivalence relation.

10. Let R be the following equivalence relation on the set A={1, 2, 3, 4, 5, 6}:

R={(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)}

Find the partition of A induced by R, i.e., find the equivalence classes of R.

Solution.

Those elements related to 1 are 1 and 5 hence [1]={1, 5}

We pick an element which does not belong to [1], say 2. Those elements related to 2 are 2, 3, and 6, hence [2]={2, 3, 6}

The only element which does not belong to [1] or [2] is 4. The only element related to 4 is 4. Thus [4]={4}

Accordingly, the following is the partition of A induced by R: [{1, 5}, {2, 3, 6}, {4}]

EXERCISES. SET 2 (Supplementary Problems)

Relations

1. Consider the relation R={(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)} on A={1, 2, 3, 4}.

(a) Find the matrix M_R of R.

(b) Find the domain and range of R.

(c) Find R^{-1}.

(d) Draw the directed graph of R.

(e) Find the composition relation $R \circ R$.

(f) Find $R \circ R^{-1}$ and $R^{-1} \circ R$.

2. Let A={1, 2, 3, 4}, B={a, b, c}, C={x, y, z}. Consider the relations R from A to B and S from B to C as follows:

R={(1, b), (3, a), (3, b), (4, c)} and S={(a, y), (c, x), (a, z)}

(a) Draw the diagrams of R and S.

(b) Find the matrix of each relation R, S (composition) $S \circ R$.

(c) Write R^{-1} and the composition $S \circ R$ as sets of ordered pairs.

3. Let R and S be the following relations on B={a, b, c, d}:

R={(a, a), (a, c), (c, b), (c, d), (d, b)} and S={(b, a), (c, c), (c, d), (d, a)}

Find the following composition relations: (a) $R \circ S$; (b) $S \circ R$; (c) $R \circ R$; (d) $S \circ S$.

Properties of Relations

4. Suppose C is a collection of relations S on a set A, and let T be the intersection of the relations S in C, that is, $T = \cap(S \mid S \in C)$ Prove:

(a) If every S is symmetric, then T is symmetric.

(b) If every S is transitive, then T is transitive.

5. Let R and S be relations on a set A. Assuming A has at least three elements, state whether each of the following statements is true or false. If it is false, give a counterexample on the set A={1,2,3}:

(a) If R and S are symmetric then $R \cap S$ is symmetric.

(b) If R and S are symmetric then $R \cup S$ is symmetric.

(c) If R and S are reflexive then $R \cap S$ is reflexive.

(d) If R and S are reflexive then $R \cup S$ is reflexive.

(e) If R and S are transitive then $R \cup S$ is transitive.

(f) If R and S are antisymmetric then $R \cup S$ is antisymmetric.

(g) If R is antisymmetric, then R^{-1} is antisymmetric.

(h) If R is reflexive then $R \cap R^{-1}$ is not empty.

(i) If R is symmetric then $R \cap R^{-1}$ is not empty.

6. Suppose R and S are relations on a set A, and R is antisymmetric. Prove that $R \cap S$ is antisymmetric.

7. Let A={0, 1, 2, 3} and relation R is defined as R={(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)}. Find the a) reflexive closure of R. b) symmetric closure of R.

8. Let A={1, 2, 3, 4} and R={(1, 2), (1, 4), (3, 3), (4, 1)}. Find the smallest relation S containing R such that S is:

a) reflexive and transitive.

b) symmetric and transitive.

c) reflexive, symmetric, and transitive.

Equivalence Relations and Partitions

9. Which of these relations on the set of all functions from Z to Z are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1)=g(1)\}$

b) $\{(f, g) \mid f(0)=g(0) \text{ or } f(1)=g(1)\}$

c) $\{(f, g) \mid f(x)-g(x)=1 \text{ for all } x\in Z\}$

d) $\{(f, g) \mid \text{for some } C\in Z, \text{ for all } x\in Z, f(x)-g(x)=C\}$

e) $\{(f, g) \mid f(0)=g(1) \text{ and } f(1)=g(0)\}$

10. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that f(x)=f(y).

a) Show that R is an equivalence relation on A.

b) What are the equivalence classes of R?

11. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d))\in R$ if and only if a+d=b+c. Show that R is an equivalence relation.

12. (a) Show that the relation R on the set of all differentiable functions from **R** to **R** consisting of all pairs (f, g) such that $f'(x)=g'(x)$ for all real numbers x is an equivalence relation.

(b) Which functions are in the same equivalence class as the function $f(x)=x^2$?

13. a) What is the equivalence class of (1, 2) with respect to the equivalence relation in Exercise 11?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 11. [Hint: Look at the difference a−b corresponding to (a, b).]

14. Find the smallest equivalence relation on the set {a, b, c, d, e} containing the relation {(a, b), (a, c), (d, e)}.

15. Which of these collections of subsets are partitions of {1, 2, 3, 4, 5, 6}?

a) {1, 2}, {2, 3, 4}, {4, 5, 6} b) {1}, {2, 3, 6}, {4}, {5}

c) {2, 4, 6}, {1, 3, 5} d) {1, 4, 5}, {2, 6}

16. Which of these collections of subsets are partitions of the set of bit strings of length 8?

a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01

b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11

c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11

d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00

e) the set of bit strings that contain 3k ones for some nonnegative integer k; the set of bit strings that contain 3k+1 ones for some nonnegative integer k; and the set of bit strings that contain 3k+2 ones for some nonnegative integer k.

Answers to Supplementary Problems

5. All are true except: (e) R = {(1, 2)}, S = {(2, 3)}; (f) R = {(1, 2)}, S = {(2, 1)}.

7. a) {(0, 0), (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0), (3, 3)}
b) {(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0)}

8. a) {(1, 1), (1, 2), (1, 4), (2, 2), (3, 3), (4, 1), (4, 2), (4, 4)}
b) {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (4, 1), (4, 2), (4, 4)}
c) {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (4, 1), (4, 2), (4, 4)}

9. a) Equivalence relation; b) Not transitive; c) Not reflexive, not symmetric, not transitive;

d) Equivalence relation; e) Not reflexive, not transitive;

10. a) (x, x)∈R because f(x) =f(x). Hence, R is reflexive. (x, y)∈R if and only if f(x)=f(y), which holds if and only if f(y)=f(x) if and only if (y, x)∈R. Hence, R is symmetric. If (x, y)∈R and (y, z)∈R, then f(x) =f(y) and f(y)=f(z). Hence, f(x)=f(z). Thus, (x, z)∈R. It follows that R is transitive.

b) The sets $f^{-1}(b)$ for b in the range of f.

11. For reflexivity, ((a, b), (a, b))∈R because a+b=b+a. For symmetry, if ((a, b), (c, d))∈R, then a+d=b+c, so c+b=d+a, so ((c, d), (a, b))∈R. For transitivity, if ((a, b), (c, d))∈R and ((c, d), (e, f))∈R, then a+d=b+c and c+e=d+f, so a+d+c+e=b+c+d+f, so a+e=b+f, so ((a, b), (e, f))∈R. An easier solution is to note that by algebra, the given condition is the same as the condition that f((a, b))=f((c, d)), where f((x, y))=x−y; therefore by Exercise 10 this is an equivalence relation.

12. (a) This follows from Exercise 10, where the function f from the set of differentiable functions (from **R** to **R**) to the set of functions (from **R** to **R**) is the differentiation operator.

(b) The set of all functions of the form $g(x) = x^2+C$ for some constant C

13. (a) [(1, 2)] = {(a, b) | a-b=−1}={(1, 2), (3, 4), (4, 5), (5, 6), . . .}

(b) Each equivalence class can be interpreted as an integer (negative, positive, or zero); specifically, [(a, b)] can be interpreted as a−b.

14. {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (d, e), (e, d), (e, e)}

15. a) No b) Yes c) Yes d) No

16. (a), (c), (e)