BHOS

Computer Engineering

Fall 2023-2024

Calculus

Limit and Continuity

Exercises and Solutions

Instructor: Dr. Nijat Aliyev

October 3, 2023

Contents

1 Limits of Functions

Exercise 1.1. *Find the limits:*

1. $\lim_{x \to -5} (2x+7)$,
2. $\lim_{x \to 4} (-x^2 + 2x + 3)$,
3. $\lim_{x \to \frac{1}{2}} 2x^2(x+8)$,
4. $\lim_{x \to 3} \dfrac{3x+4}{3x^2+2x+1}$,
5. $\lim_{x \to 2} (3x^2+4)^{\frac{3}{4}}$.

Solution.

1. *Using Sum and Constant multiple Rules gives*

$$\lim_{x \to -5} (2x+7) = 2 \lim_{x \to -5} x + \lim_{x \to -5} 7 = 2(-5) + 7 = -3.$$

2. *Using Sum, Constant multiple and Power Rules gives*

$$\lim_{x \to 4} (-x^2 + 2x + 3) = -(\lim_{x \to 4} x)^2 + 2 \lim_{x \to 4} x + \lim_{x \to 4} 3 = -4^2 + 2 \cdot 4 + 3 = -5.$$

3. $\lim_{x \to \frac{1}{2}} 2x^2(x+8) = 2 \cdot \left(\frac{1}{2}\right)^2 \cdot (\frac{1}{2} + 8) = 2 \cdot \frac{1}{4} \cdot \frac{17}{2} = \frac{17}{4}$.
4. $\lim_{x \to 3} \frac{3x+4}{3x^2+2x+1} = \frac{3 \cdot 3+4}{3 \cdot 3^2 + 2 \cdot 3 + 1} = \frac{13}{34}$.
5. $\lim_{x \to 2} (3x^2+4)^{\frac{3}{4}} = (3 \cdot 2^2 + 4)^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 8$.

Exercise 1.2. *Suppose* $\lim_{x \to c} f(x) = 7$ *and* $\lim_{x \to c} g(x) = 3$. *Find:*

1. $\lim_{x \to c} f(x) \cdot g(x)$,
2. $\lim_{x \to c} (3f(x) \cdot g(x) + 1)$,
3. $\lim_{x \to c} (f(x) + 3g(x))$,
4. $\lim_{x \to c} (2f(x) \cdot g^2(x))$,

5. $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$.

Solution.

1. *Using Product Rule gives*

$$\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 7 \cdot 3 = 21.$$

2. *Using Constant multiple, Product and Sum Rules gives*

$$\lim_{x\to c}(3f(x)\cdot g(x)+1) = 3\lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) + \lim_{x\to c} 1 = 3\cdot 7\cdot 3 + 1 = 64.$$

3. *Using Sum and Constant multiple Rules gives*

$$\lim_{x\to c}(f(x)+3g(x)) = \lim_{x\to c} f(x) + 3\lim_{x\to c} g(x) = 7 + 3\cdot 3 = 16.$$

4. *Using Constant multiple, Product and Power Rules gives*

$$\lim_{x\to c}(2f(x)\cdot g^2(x)) = 2\lim_{x\to c} f(x)\cdot (\lim_{x\to c} g(x))^2 = 2\cdot 7\cdot 3^2 = 126.$$

5. *Using Quotient and Difference Rules gives*

$$\lim_{x\to c} \frac{f(x)}{f(x)-g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} f(x) - \lim_{x\to c} g(x)} = \frac{7}{7-3} = \frac{7}{4}.$$

Exercise 1.3. *Find the limits:*

1. $\lim_{x\to 4} \dfrac{x^2-16}{x-4}$,

2. $\lim_{x\to 0} \dfrac{\sqrt{5x+4}-2}{x}$,

3. $\lim_{x\to 0} \dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$,

4. $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-1}$,

5. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$,

6. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$,

7. $\lim_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$.

Solution.

1. *We cannot substitute $x = 4$ because it makes the denominator zero. We test the numerator to see if it is zero at $x = 4$ too. It is, so it has a factor of $(x-4)$ in common with the denominator. Canceling the $(x-4)$'s gives a simpler fraction with the same values as the original for $x \neq 4$:*

$$\frac{x^2-16}{x-4} = \frac{(x+4)(x-4)}{x-4} = (x+4) \quad if \quad x \neq 4.$$

Using the simpler fraction, we find the limit of these values as $x \to 4$ by substitution:

$$\lim_{x\to 4}\frac{x^2-16}{x-4} = \lim_{x\to 4}\frac{(x+4)(x-4)}{x-4} = \lim_{x\to 4}(x+4) = 4+4 = 8.$$

Note: *We will use this (Eliminating Zero Denominator method) for Exercise 3 and Exercise 4.*

2. *First we multiply the numerator and the denominator by the Conjugate fo the numerator. Then,*

$$\lim_{x\to 0}\frac{\sqrt{5x+4}-2}{x} = \lim_{x\to 0}\frac{(\sqrt{5x+4}-2)(\sqrt{5x+4}+2)}{x(\sqrt{5x+4}+2)} = \lim_{x\to 0}\frac{5x}{x(\sqrt{5x+4}+2)}$$
$$= \lim_{x\to 0}\frac{5}{(\sqrt{5x+4}+2)} = \frac{5}{2+2} = \frac{5}{4}.$$

3. $\displaystyle\lim_{x\to 0}\frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim_{x\to 0}\frac{\frac{x+1+x-1}{x^2-1}}{x} = \lim_{x\to 0}\frac{2x}{x(x^2-1)} = \lim_{x\to 0}\frac{2}{(x^2-1)} = \frac{2}{0^2-1} = -2.$

4. $\displaystyle\lim_{x\to 1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\frac{(x-1)(x+2)}{(x+1)(x-1)} = \lim_{x\to 1}\frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}.$

5. $\displaystyle\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$

6.

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}$$
$$= \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{x-1} = \lim_{x\to 1}(\sqrt{x+3}+2) = \sqrt{4}+2 = 4.$$

7.

$$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}=\lim_{x\to -1}\frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(\sqrt{x^2+8}+3)(x+1)}=\lim_{x\to -1}\frac{x^2-1}{(\sqrt{x^2+8}+3)(x+1)}$$
$$=\lim_{x\to -1}\frac{x-1}{\sqrt{x^2+8}+3}=\frac{-1-1}{(-1)^2+3}=-\frac{2}{4}=-\frac{1}{2}.$$

Exercise 1.4. *Calculate the following limits:*

1. $\lim\limits_{x\to 0}\dfrac{x^2-1}{2x^2-x-1}$,
2. $\lim\limits_{x\to 1}\dfrac{x^2-1}{2x^2-x-1}$.

Solution.

1. $\lim\limits_{x\to 0}\dfrac{x^2-1}{2x^2-x-1}=\dfrac{0^2-1}{2\cdot 0^2-0-1}=1.$
2. $\lim\limits_{x\to 1}\dfrac{x^2-1}{2x^2-x-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{2(x-1)(x+\frac{1}{2})}=\lim\limits_{x\to 1}\dfrac{x+1}{2x+1}=\dfrac{1+1}{2+1}=\dfrac{2}{3}.$

Exercise 1.5. *Prove the limit statements:*

1. $\lim\limits_{x\to 3}(x+4)=7$,
2. $\lim\limits_{x\to 1}\dfrac{2}{x}=2$,
3. $\lim\limits_{x\to 14}\sqrt{x-5}=3$.

Solution.

1. *Let $p=3$, $f(x)=x+4$ and $L=7$ in the definition of limit. For any given $\epsilon>0$ we have to find a suitable $\delta>0$ such that for all real $x\in R$, $0<|x-3|<\delta$ implies $|f(x)-7|<\epsilon$.*

   We find δ by working backward from the ϵ-inequality

$$|f(x)-L|<\epsilon,$$
$$|x+4-7|<\epsilon,$$
$$|x-3|<\epsilon.$$

   We can take $\delta=\epsilon$. If $0<|x-3|<\delta=\epsilon$ then

$$|x+4-7|=|x-3|<\epsilon,$$

   which proves that $\lim\limits_{x\to 3}(x+4)=7$.

2. *Let $p = 1$, $f(x) = \frac{2}{x}$ and $L = 2$. Let us take $\forall \epsilon > 0$. We must find $\delta > 0$ such that for all real x, for which $0 < |x - 1| < \delta$ ($x \neq 1$, $-\delta < x - 1 < \delta$), $|f(x) - 2| < \epsilon$ is true.*

   We solve the inequality
$$|f(x) - 2| < \epsilon.$$
   Then
$$\Big|\frac{2}{x} - 2\Big| < \epsilon,$$
$$-\epsilon < \frac{2}{x} - 2 < \epsilon,$$
$$2 - \epsilon < \frac{2}{x} < 2 + \epsilon,$$
   We can assume $\epsilon < 2$. Then
$$\frac{1}{2+\epsilon} < \frac{x}{2} < \frac{1}{2-\epsilon},$$
$$\frac{2}{2+\epsilon} < x < \frac{2}{2-\epsilon},$$
   We take δ to be the distance from $p = 1$ to the nearer endpoint of $\left(\frac{2}{2+\epsilon}, \frac{2}{2-\epsilon}\right)$. In other words, we take
$$\delta = \min\left\{1 - \frac{2}{2+\epsilon}, \frac{2}{2-\epsilon} - 1\right\} = \min\left\{\frac{\epsilon}{2+\epsilon}, \frac{\epsilon}{2-\epsilon}\right\} = \frac{\epsilon}{2+\epsilon}.$$
   Then for all x, $0 < |x - 1| < \delta = \dfrac{\epsilon}{2+\epsilon}$ implies $|f(x) - 2| < \epsilon$, ($\forall \epsilon < 2$), which proves that $\lim\limits_{x\to 1} \frac{2}{x} = 2$.

3. *Let us take $f(x) = \sqrt{x-5}$, $p = 14$ and $L = 3$ be given. Let $\epsilon > 0$. We want to find a positive number δ such that for all x $0 < |x - 4| < \delta$ ($x \neq 14$, $-\delta < x - 14 < \delta$) implies $|f(x) - 3| < \epsilon$.*

   We find δ by working backward from the ϵ-inequality
$$|f(x) - 3| = |\sqrt{x-5} - 3| = \Big|\frac{(\sqrt{x-5} - 3)(\sqrt{x-5} + 3)}{\sqrt{x-5} + 3}\Big|$$
$$= \Big|\frac{x - 14}{\sqrt{x-5} + 3}\Big| < \Big|\frac{x-14}{3}\Big| < \epsilon,$$
   Then,
$$|x - 14| < 3\epsilon.$$
   Thus, we take $\delta = 3\epsilon$.

   Then, whenever $|x - 14| < \delta = 3\epsilon$, it is true that $|f(x) - 3| < \Big|\dfrac{x-14}{3}\Big| < \dfrac{\delta}{3} = \dfrac{3\epsilon}{3} = \epsilon$, Which proves that $\lim\limits_{x\to 14} \sqrt{x-5} = 3$.

Exercise 1.6. *If* $3x^2+3 \le f(x) \le x^3+7$ *for* $0 \le x \le 5$, *find* $\lim_{x\to 2} f(x)$.

Solution. *Since*

$$\lim_{x\to 2}(3x^2+3) = 3\cdot 2^2+3 = 15 \quad \text{and} \quad \lim_{x\to 2}(x^3+7) = 2^3+7 = 15.$$

The Sandwich (Squeeze) Theorem implies $\lim_{x\to 2} f(x) = 15$.

Exercise 1.7. *It can be shown that the inequality*

$$1-\frac{x^2}{6} \le \frac{x\sin x}{2-2\cos x} \le 1$$

holds for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}$$

Give reasons for your answer.

Solution. *Since*

$$\lim_{x\to 0}\left(1-\frac{x^2}{6}\right) = 1 \quad \text{and} \quad \lim_{x\to 0} 1 = 1$$

According to The Sandwich (Squeeze) Theorem

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x} = 1.$$

Exercise 1.8. *If* $\lim_{x\to 5}\frac{f(x)-6}{x-3} = 4$, *find* $\lim_{x\to 5} f(x)$.

Solution.

$$f(x)-6 = \frac{f(x)-6}{x-3}\cdot(x-3)$$

Because of this

$$\lim_{x\to 5}(f(x)-6) = \lim_{x\to 5}\frac{f(x)-6}{x-3}\cdot(x-3) = 4\cdot(5-3) = 8$$

Then,

$$\lim_{x\to 5} f(x) = 8+6 = 14.$$

Exercise 1.9. *If* $\lim_{x\to -1}\frac{f(x)}{x^2} = 5$, *find:*

1. $\lim_{x\to -1} \frac{f(x)}{x}$,
2. $\lim_{x\to -1} f(x)$.

Solution.

1. $\lim_{x\to -1} \dfrac{f(x)}{x} = \lim_{x\to -1} \left(\dfrac{f(x)}{x^2}\cdot x\right) = \lim_{x\to -1} \dfrac{f(x)}{x^2}\cdot \lim_{x\to -1} x = 5\cdot(-1) = -5.$
2. $\lim_{x\to -1} f(x) = \lim_{x\to -1} \left(\dfrac{f(x)}{x^2}\cdot x^2\right) = \lim_{x\to -1} \dfrac{f(x)}{x^2}\cdot \lim_{x\to -1} x^2 = 5\cdot(-1)^2 = 5.$

Exercise 1.10. *Find the one sided limits:*

1. $\lim_{x\to 3^+} \dfrac{\sqrt{x-2}}{x+3}$,
2. $\lim_{x\to 1^-} \dfrac{x^2+2x}{x-7}$,
3. $\lim_{x\to 1^-} \left(\dfrac{x+1}{x}\right)\left(\dfrac{2-x^2}{3x}\right)$,
4. $\lim_{x\to 3^+} \dfrac{x^3-27}{x-3}$,
5. $\lim_{x\to 0^-} \dfrac{\sqrt{6}-\sqrt{5x^2+11x+6}}{x}$.

Solution.

1. $\lim_{x\to 3^+} \dfrac{\sqrt{x-2}}{x+3} = \dfrac{\sqrt{3-2}}{3+3} = \dfrac{1}{6}.$
2. $\lim_{x\to 1^-} \dfrac{x^2+2x}{x-7} = \dfrac{1^2+2\cdot 1}{1-7} = \dfrac{3}{-6} = -\dfrac{1}{2}.$
3. $\lim_{x\to 1^-} \left(\dfrac{x+1}{x}\right)\left(\dfrac{2-x^2}{3x}\right) = \left(\dfrac{1+1}{1}\right)\left(\dfrac{2-1^2}{3\cdot 1}\right) = 2\cdot\dfrac{1}{3} = \dfrac{2}{3}.$
4. $\lim_{x\to 3^+} \dfrac{x^3-27}{x-3} = \lim_{x\to 3^+} \dfrac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x\to 3^+}(x^2+3x+9) = 3^2+3\cdot 3+9 = 27.$
5. $$\begin{aligned}\lim_{x\to 0^-} \frac{\sqrt{6}-\sqrt{5x^2+11x+6}}{x} &= \lim_{x\to 0^-} \frac{6-(5x^2+11x+6)}{x(\sqrt{6}+\sqrt{5x^2+11x+6})}\\ &= \lim_{x\to 0^-} \frac{x(5x+11)}{x(\sqrt{6}+\sqrt{5x^2+11x+6})} = \frac{11}{2\sqrt{6}}.\end{aligned}$$

Exercise 1.11. *Evaluate the limits:*

1. $\lim\limits_{x\to -2^+}(x+3)\dfrac{|x+2|}{x+2}$,
2. $\lim\limits_{x\to -2^-}(x+3)\dfrac{|x+2|}{x+2}$.

Solution.

1. *If* $x \to -2^+$, *then* $x > 2$,

   and $|x+2| = x+2$ *if* $x > 2$. *Therefore,*

$$\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+}(x+3)\frac{(x+2)}{x+2} = \lim_{x\to -2^+}(x+3) = -2+3 = 1.$$

2. *If* $x \to -2^-$, *then* $x < 2$,

   and $|x+2| = -(x+2)$ *if* $x < 2$. *Therefore,*

$$\lim_{x\to -2^-}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^-}(x+3)\frac{-(x+2)}{x+2} = \lim_{x\to -2^-}(-x-3) = -(-2)-3 = -1.$$

Exercise 1.12. *Prove that:*

1. $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$,
2. $\lim\limits_{x\to 0}(x^2\sin\dfrac{1}{x} + 3x^2 + 2) = 2$.

Solution.

1. *From Trigonometry, you know* $-1 < \sin\dfrac{1}{x} < 1$.

$$\text{If } x > 0 \quad \text{then} \quad -x < \sin\frac{1}{x} < x.$$

$$\text{If } x < 0 \quad \text{then} \quad -x > \sin\frac{1}{x} > x.$$

$$\lim_{x\to 0^+} x = \lim_{x\to 0^+}(-x) = 0.$$

$$\lim_{x\to 0^-} x = \lim_{x\to 0^-}(-x) = 0.$$

Because of this, in both case, using the Sandwich theorem

$$\lim_{x\to 0^+} x\sin\frac{1}{x} = 0.$$

$$\lim_{x\to 0^-} x\sin\frac{1}{x} = 0.$$

Right sided and left sided limits are equal at $x = 0$. *Hence,* $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$.

2. $-1 < \sin\frac{1}{x} < 1$, *therefore* $-x^2 < x^2\sin\frac{1}{x} < x^2$.

$$\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$$

According the Sandwich theorem

$$\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right) = 0.$$

Then,

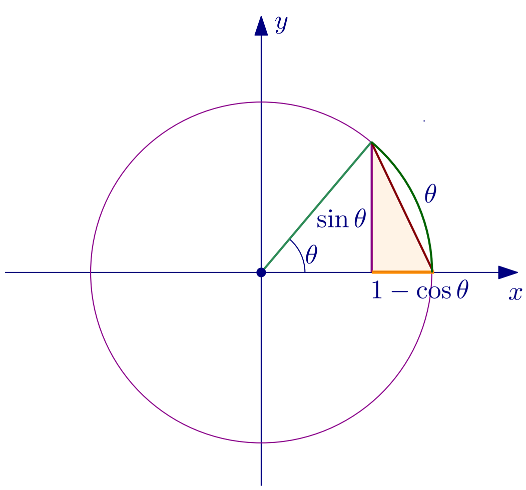
$$\lim_{x\to 0}\left(x^2\sin\frac{1}{x} + 3x^2 + 2\right) = \lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right) + \lim_{x\to 0}(3x^2+2) = 0 + 3\cdot 0^2 + 2 = 2.$$

Exercise 1.13. *Prove that:*

1. $\lim_{x\to 0}\sin x = 0$,
2. $\lim_{x\to 0}\cos x = 1$.

Solution.

First, let us prove that $-|x| \le \sin x \le |x|$ *and* $-|x| \le 1-\cos x \le |x|$ *are true.*
Take a circle with radius of 1 *and* θ *in the first quadrant.*



Using the Pythagorean Theorem gives,

$$\sin^2\theta + (1-\cos\theta)^2 \le \theta^2.$$

The terms on the left-hand side of inequality are both positive, so each is smaller than their sum and hence is less than or equal to θ^2*:*

$$\sin^2\theta \le \theta^2 \quad \textit{and} \quad (1-\cos\theta)^2 \le \theta^2$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \le |\theta| \quad \textit{and} \quad |1-\cos\theta| \le |\theta|$$

so

$$-|\theta| \le \sin\theta \le |\theta| \quad \textit{and} \quad |\theta| \le 1-\cos\theta \le |\theta|$$

To prove 1. and 2. we use these inequalities:

1. $-|x| \le \sin x \le |x|$, *and* $\lim\limits_{x\to 0}(-|x|) = \lim\limits_{x\to 0}|x| = 0.$

   Then the Sandwich theorem expresses that $\lim\limits_{x\to 0}\sin x = 0.$

2. $-|x| \le 1-\cos x \le |x|$, *and* $\lim\limits_{x\to 0}(-|x|) = \lim\limits_{x\to 0}|x| = 0.$

   Then the Sandwich theorem expresses that $\lim\limits_{x\to 0}(1-\cos x) = 0$. *Then,*

$$\lim_{x\to 0}\cos x = 1.$$

Exercise 1.14. *Prove that,* $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$

Solution.

We begin with positive values of θ less than $\dfrac{\pi}{2}$. According to accompanying figure,

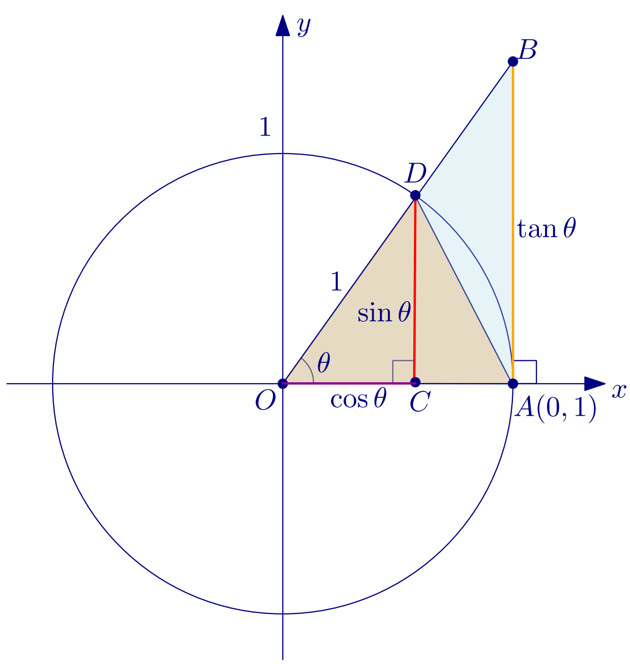
$$A_{\triangle OAD} < A_{\widehat{OAD}} < A_{\triangle OAB}$$

We can express these areas in terms of θ as follows:

$$A_{\triangle OAD} = \frac{1}{2}\cdot 1\cdot\sin\theta = \frac{1}{2}\sin\theta,$$

$$A_{\widehat{OAD}} = \frac{1}{2}r^2\theta = \frac{1}{2}\cdot 1^2\theta = \frac{\theta}{2},$$

$$A_{\triangle OAB} = \frac{1}{2} = \frac{1}{2}\cdot 1\cdot\tan\theta = \frac{1}{2}\tan\theta.$$



Thus,

$$\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta.$$

If we divide all three terms by the number $\frac{1}{2}\sin\theta$ *which is positive since* $0<\theta<\frac{\pi}{2}$*:*

$$1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta}.$$

Taking reciprocal reverses,

$$1>\frac{\sin\theta}{\theta}>\cos\theta,$$

Since $\lim\limits_{\theta\to 0^+}\cos\theta=1$*, The Sandwich theorem gives*

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1.$$

Note: If f ***is even function and*** $\lim\limits_{x\to 0^+}f(x)=L$ ***then***

$$\lim_{x\to 0}f(x)=L.$$

Using the previous note gives,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

Exercise 1.15. *Find the following limits:*

1. $\lim\limits_{x\to 0}\dfrac{\sin\sqrt{3}x}{\sqrt{3}x}$,
2. $\lim\limits_{x\to 0}\dfrac{\tan 5x}{x}$,
3. $\lim\limits_{x\to 0}\dfrac{x+x\cos x}{\sin x\cos x}$,
4. $\lim\limits_{x\to 0}\dfrac{x^3+x+\sin x}{2x}$,
5. $\lim\limits_{x\to 0}\dfrac{\sin 2x\cot 4x}{x\cot 3x}$.

Solution.

1. *We use the substution* $\sqrt{3}x=t$*,*

   then $x\to 0\iff t\to 0$*.*

   Because of this,

$$\lim_{x\to 0}\frac{\sin\sqrt{3}x}{\sqrt{3}x}=\lim_{t\to 0}\frac{\sin t}{t}=1.$$

Note: *We will apply this substitution method for following limits.*

2. $\lim\limits_{x\to 0}\dfrac{\tan 5x}{x}=\lim\limits_{x\to 0}\dfrac{5\sin 5x}{5x\cos 5x}=5\cdot\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos 5x}=5\cdot 1\cdot 1=5.$

3. $\lim\limits_{x\to 0}\dfrac{x+x\cos x}{\sin x\cos x}=\lim\limits_{x\to 0}\dfrac{x(1+\cos x)}{\sin x\cos x}=\lim\limits_{x\to 0}\dfrac{1+\cos x}{\frac{\sin x}{x}\cdot\cos x}=\dfrac{1+1}{1\cdot 1}=2.$

4. $\lim\limits_{x\to 0}\dfrac{x^3+x+\sin x}{2x}=\lim\limits_{x\to 0}\left(\dfrac{x^2}{2}+\dfrac{1}{2}+\dfrac{\sin x}{2x}\right)=1.$

5. $\lim\limits_{x\to 0}\dfrac{\sin 2x\cot 4x}{x\cot 3x}=\lim\limits_{x\to 0}\dfrac{2\sin 2x}{2x}\cdot\dfrac{4x\cdot\cos 4x}{4\sin 4x}\cdot\dfrac{3\sin 3x}{3x\cdot\cos 3x}=2\cdot\dfrac{1}{4}\cdot 3=\dfrac{3}{2}.$

Exercise 1.16. *Find the limits:*

1. $\lim\limits_{x\to 0}(3\sin x-2)$

2. $\lim\limits_{x\to 0}(x^2-2)(\cos^2 x-2)$

3. $\lim\limits_{x\to -\pi}\sqrt{x+4}\cos(x+\pi)$

Solution. *Find the limits:*

1. $\lim\limits_{x\to 0}(3\sin x-2)=3\lim\limits_{x\to 0}(\sin x)-\lim\limits_{x\to 0}2=-2$

2. $\lim\limits_{x\to 0}(x^2-2)(\cos^2 x-2)=(0^2-2)(1^2-2)=2.$

3. *Let us make substitution $x+\pi=t$, then $x\to -\pi\iff t\to 0$.*
$\lim\limits_{x\to -\pi}(\sqrt{x+4}\cos(x+\pi))=\lim\limits_{t\to 0}(\sqrt{t-\pi+4}\cos t)=\sqrt{0-\pi+4}=\sqrt{4-\pi}.$

Exercise 1.17. *Let f be a function defined by*

$$f(x)=\begin{cases}\dfrac{1}{x}, & x<-2\\ x^2-5 & -2<x\le 3\\ \sqrt{x+13}, & x>3\end{cases}$$

Find:

1. $\lim\limits_{x\to -2}f(x),$

2. $\lim\limits_{x\to 0}f(x).$

3. $\lim\limits_{x\to 3}f(x)$.

Solution. *We will determine the stated two-sided limit by first considering the corresponding one-sided limits.*

1. $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} \dfrac{1}{x} = -\dfrac{1}{2},$

   $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+} (x^2-5) = (-2)^2 - 5 = -1.$

   From which it follows that $\lim\limits_{x\to -2} f(x)$ *does not exist.*

2. $f(x) = x^2 - 5$ *on both sides of* 0, *therefore,*

   $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}(x^2-5) = 0^2 - 5 = -5$

3. $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (x^2-5) = 3^2 - 5 = 4,$

   $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \sqrt{x+13} = \sqrt{3+13} = 4.$

   Since left-sided and right sided limits are equal, 4, *we have* $\lim\limits_{x\to 3} f(x) = 4.$

Exercise 1.18. *Let* f *be a function defined by the expression*

$$f(x) = \begin{cases} x-1, & x \le 3 \\ 3x-7, & x > 3 \end{cases}$$

Find:

1. $\lim\limits_{x\to 3^-} f(x),$
2. $\lim\limits_{x\to 3^+} f(x),$
3. $\lim\limits_{x\to 3} f(x).$

Solution.

1. $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (x-1) = 3 - 1 = 2,$
2. $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (3x-7) = 3\cdot 3 - 7 = 2.$
3. *Since one-sided limits are equal, we have* $\lim\limits_{x\to 3} f(x) = 2.$

Exercise 1.19. *Let* f *be a function defined by*

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq 3 \\ k, & x = -3 \end{cases}$$

Find

1. *Find k so that* $f(-3) = \lim_{x \to -3} f(x)$

2. *With k assigned the value* $\lim_{x \to -3} f(x)$, *show that f can be expressed as a polynomial.*

Solution.

1. $k = f(-3) = \lim_{x \to -3} f(x) = \lim_{x \to -3} (x-3) = -3-3 = -6$

2. $$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq 3 \\ -6, & x = -3 \end{cases}$$

   That is, $f(x) = \dfrac{x^2-9}{x+3} = x-3$ *if* $x \neq 3$ *and* $f(x) = -6$ *if* $x = 3$. *So,* f *is equivalent to the polynomial* $g(x) = x - 3$.

Exercise 1.20. *Prove that*

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

Solution. *First we calculate one sided limits,*

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0)} 1 = 1 \qquad (x > 0)$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{(-x)}{x} = \lim_{x \to 0^-} (-1) = -1 \qquad (x < 0)$$

Since the one-sided limits are different, $\lim_{x \to 0} \frac{|x|}{x}$ *does not exist.*

Exercise 1.21. *At what points are the functions defined by following expressions continuous?*

1. $f(x) = \dfrac{x+3}{x^2-3x+2}$

2. $g(x) = \dfrac{3x}{(x+7)^2} + 5$

3. $h(x) = |x-2| + \sin x$

4. $l(x) = \dfrac{3x}{|x|-6} + x^2 + 4$

Solution.

1. *Since any rational function is continuous on its domain, The function f defined by $f(x) = \dfrac{x+3}{x^2-3x+2}$ is continuous any points at which $x^2-3x+2=0$. Let us solve $x^2-3x+2=0$*

$$x^2-3x+2=0 \Longleftrightarrow (x-2)(x-1)=0 \Longrightarrow x=2,\ x=1.$$

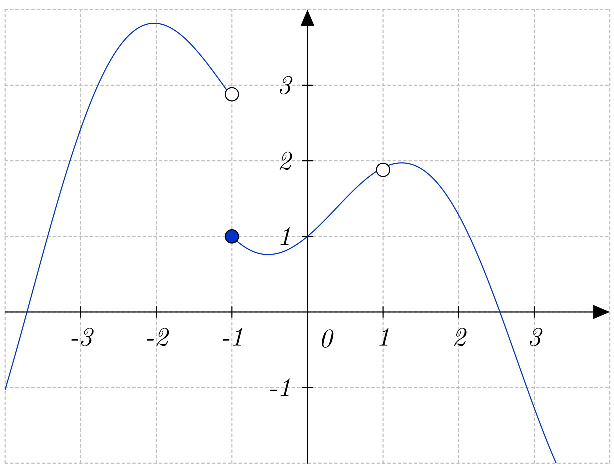
   Therefore, f is continuous on $\mathbb{R}\setminus\{1,2\}$.

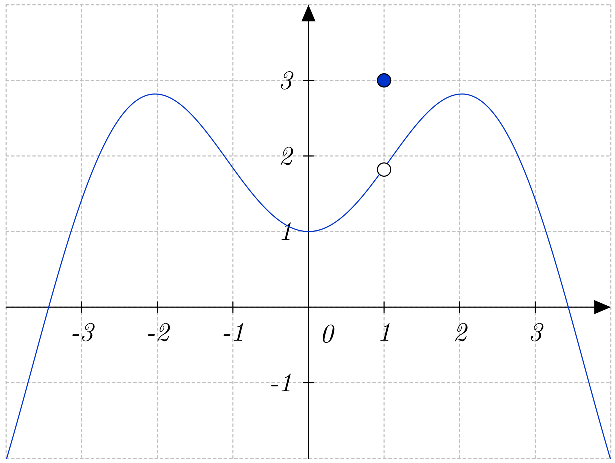
2. *A function defined by $m(x) = \dfrac{3x}{(x+7)^2}$ is continuous on the set of real numbers and a function defined by $n(x)=5$ (constant function) is also continuous on $\mathbb{R}$, therefore g will be continuous on $\mathbb{R}/\{-7\}$.*

3. *Absolute value function and trigonometric function are continuous on their domains, because of this h is also continuous on $\mathbb{R}$*

4. *The function l is continuous on $\mathbb{R}\setminus\{-6,6\}$.*

Exercise 1.22. *Say whether the functions represented by a graph are continuous on $[-2,3]$ If not, where do them fail to be continuous and why?*

(a) 



(b)

Solution.

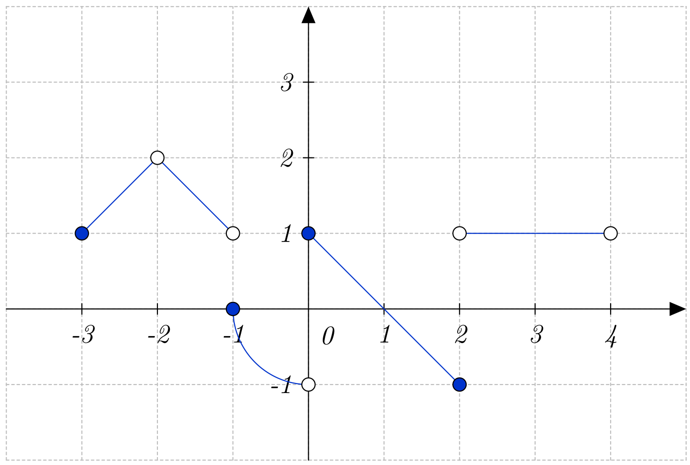
Geometric explanation: *Any function whose graph can be sketched over its domain in one continuous motion without lifting the pencil is a continuous function. Then the function represented in (a) is discontinuous. It is discontinuous at* $x = -1$ *and* $x = 1$.

The function represented in (b) is discontinuous. It is discontinuous at $x = 1$.

Exercise 1.23. *The function* f *given by*

$$f(x) = \begin{cases} x+4, & -3 \le x < -2 \\ -x, & -2 < x < -1 \\ -\sqrt{1-x^2}, & -1 \le x < 0 \\ -x+1, & 0 \le x \le 2 \\ 1, & 2 < x < 4 \end{cases}$$

is graphed in the accompanying figure.



1. *Does* $f(-3)$ *exist?*

2. *Does* $\lim_{x \to -2} f(x)$ *exist?*

3. *Does* $\lim_{x \to 2} f(x)$ *exist?*

4. *Does* $\lim_{x \to 2^-} f(x) = f(-1)$*?*

5. *Does* $\lim_{x \to -1^-} f(x)$ *exist?*

6. *Does* $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$ *?*

7. *Does* $\lim_{x \to -1^-} f(x) = f(-1)$*?*

8. *Define the points at which* f *is discontinuous.*

Solution.

Note: Geometric explanation.

1. $f(-3)$ *exists.*

2. *The limit of a function does not depend on how the function is defined at the point being approached. If we approach* -2 *from both sides, results will be the same. Because of this* $\lim_{x \to -2} f(x)$ *exists*

3. *If we approach* 2 *from both sides, results will be the different. Because of this* $\lim_{x \to 2} f(x)$ *doesn't exist.*

4. *According to the graph* $\lim_{x \to 2^-} f(x) = 1$ *but* $f(-1) = 0$*. So* $\lim_{x \to 2^-} f(x) \neq f(-1)$

5. $\lim_{x \to -1^-} f(x)$ *exist*

6. $\lim_{x \to -1^-} f(x) = 1$ *and* $\lim_{x \to -1^+} f(x) = 0$*. Therefore* $\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

7. *Does* $\lim_{x \to -1^-} f(x) = 1$ *and* $f(-1) = 0$*. So* $\lim_{x \to -1^-} f(x) \neq f(-1)$

8. $f(x)$ *is discontinuous at* $x = -2, -1, 0, 2, 4$*.*

Exercise 1.24. *At what points is the function* f *defined by the following expression continuous?*

$$f(x) = \begin{cases} \dfrac{x^2 - 6x + 8}{x - 4}, & x \neq 4 \\ 2, & x = 4 \end{cases}$$

Solution. $f(x) = \dfrac{x^2-6x+8}{x-4} = \dfrac{(x-2)(x-4)}{x-4} = x-2$ *if* $x \neq 4$. *Since every polynomial is continuous everywhere,* $f(x)$ *is continuous on* $\mathbb{R}\setminus\{4\}$. *Lets consider* $x=4$. *Since*

$$\lim_{x\to 4}\frac{x^2-6x+8}{x-4} = \lim_{x\to 4}(x-2) = 2,$$

$$f(4) = 2$$

$f(x)$ *is continuous at* $x=4$. *Therefore,* $f(x)$ *is continuous everywhere.*

Exercise 1.25. *At what points is the function* f *defined by*

$$f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9}, & x \neq -3 \text{ and } x \neq 3 \\ 9, & x = 3 \\ 1, & x = -3 \end{cases}$$

continuous?

Solution.

$$f(x) = \frac{x^3-27}{x^2-9} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \frac{x^2+3x+9}{x+3}$$

if $x \neq -3$ *and* $x \neq 3$. *So* f *is continuous for* $x \neq -3$ *and* $x \neq 3$.

Consider $x=-3$ *and* $x=3$.

If $x=-3$,

$$\lim_{x\to -3} f(x) = \lim_{x\to -3}\frac{x^3-27}{x^2-9} = \lim_{x\to -3}\frac{x^2+3x+9}{x+3}$$

does not exist. Therefore, f *is not continuous at* $x=-3$.

If $x=3$,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x^2+3x+9}{x+3} = \frac{3^2+3\cdot 3+9}{3+3} = \frac{27}{6} = \frac{9}{2} = 4.5 \neq 1 = f(-3)$$

Because of this f *is not continuous at* $x=3$. *Finally,* f *is continuous on* $\mathbb{R}\setminus\{-3,3\}$.

Exercise 1.26. *Let* g *be a function defined by* $g(x) = \dfrac{x^2-25}{x-5}$. *Define* $g(5)$ *in a way that extends* g *to be continuous at* $x=5$.

Solution. *If* $\lim_{x\to c} f(x) = f(c)$ *then it is continuous at* c*. Hence, since*

$$\lim_{x\to 5} g(x) = \lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5}(x+5) = 10$$

we define $g(5) = 10$*, that extends* g *to be continuous at* $x = 5$*.*

Exercise 1.27. *Let* g *be a function defined by* $h(x) = \dfrac{x^2+3x-2}{x-2}$ *Define* h *in a way that extends* h *to be continuous at* $x = 2$*.*

Solution. *Since*

$$\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2+3x-2}{x-2} = \lim_{x\to 2}(x+1) = 3$$

we define $h(2) = 3$*, that extends* h *to be continuous at* $x = 2$*.*

Exercise 1.28. *Let* f *be a function defined by*

$$f(x) = \begin{cases} x^2-3, & x < 3 \\ 4ax, & x \geq 3 \end{cases}$$

For what value of a *is the function* f *continuous at every* x*?*

Solution. f *is polynomial if* $x \neq 3$*, so it is continuous for* $x \neq 3$*.*
Consider $x = 3$*.*

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-3) = 6,$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 4ax = 12a,$$
$$f(3) = 4a\cdot 3 = 12a.$$

Since f *is continuous if and only if it is right continuous and left continuous,*

$$\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x).$$

Then, $6 = 12a = 12a$*, and* $a = 1/2$*.*

Exercise 1.29. *Let* g *be a function defined by*

$$g(x) = \begin{cases} \dfrac{x-b}{b+1}, & x \leq 0 \\ x^2+b, & x > 0 \end{cases}$$

For what value of b *is the function* g
continuous at every x*?*

Solution. *$g$is polynomial if $x \neq 0$, because of this it is continuous at $x \neq 0$.*

Consider $x = 0$.

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{x-b}{b+1} = \frac{-b}{b+1},$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x^2+b) = b,$$

$$g(0) = \frac{-b}{b+1}.$$

$$\lim_{x\to 0^-} g(x) = g(0) = \lim_{x\to 0^+} g(x).$$

Then, $\dfrac{-b}{b+1} = b = \dfrac{-b}{b+1}$, hence, $b = 0$ or $b = 2$.

Exercise 1.30. *Let f be a function defined by*

$$f(x) = \begin{cases} ax^3 + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

For what values of a and b is f continuous at every x?

Solution. *$f$is polynomial if $x \neq 0$ and $x \neq 2$, because of this it is continuous at $x \neq 0$ and $x \neq 2$.*

Let us consider $x = 0$ and $x = 2$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (ax^3+2b) = 2b,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2+3a-b) = 3a-b,$$

$$f(0) = 2b.$$

If $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$. then, f is continuous at $x = 0$. Hence, $2b = 2b = 3a - b$, finally, $a = b$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2+3a-b) = 4+3a-b,$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-5) = 1,$$

$$f(2) = 4+3a-b.$$

$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$. *Then, $4+3a-b = 4+3a-b = 1$, so $b - 3a = 3$.*

We get the system of equations

$$\begin{cases} a = b \\ b - 3a = 3 \end{cases}$$

Therefore $f(x)$ is continuous if $a = -3/2$ and $b = -3/2$.

Exercise 1.31. *Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function defined by*

$$f(x) = \begin{cases} 1, & if \;\; x \;\; is\ rational \\ 0, & if \;\; x \;\; is\ irrational \end{cases}$$

is discontinuous at every point.

Solution.

Suppose p is rational, then $f(p) = 1$. Let us choose $\epsilon = \frac{1}{3}$. For any δ there is an irrational number x in the interval $(p-\delta, p+\delta)$ for which $f(x) = 0$. Then for this x, $0 < |x-p| < \delta$ but $|f(x) - f(p)| = 1 > \frac{1}{3}$ so $\lim\limits_{x\to p} f(x) \neq f(p)$. Hence f is discontinuous at p rational.

If p irrational, it is proved in the similar method.

Exercise 1.32. *Explain why the equation $\cos x = x$ has at least one solution.*

Solution. *Let us take the function f defined by $f(x) = \cos x - x$. A zero of f is the root of $\cos x = x$.*

If $x = -\frac{\pi}{2}$, $f(-\frac{\pi}{2}) = \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} > 0$.

If $x = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = -\frac{\pi}{2} < 0$.

Since f is continuous over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the Intermediate Value Theorem implies that there is some c on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that $f(c) = 0$. Therefore c is the solution of $\cos x = x$.

Exercise 1.33. *Show that there is a root of the equation $x^3 + 3x^2 - x = 1$.*

Solution. *Let f be a function defined by $f(x) = x^3 + 3x^2 - x - 1$*

If $x = 0$, $f(0) = -1 < 0$.

If $x = 1$, $f(1) = 1^3 + 3\cdot 1^2 - 1 - 1 = 2 > 0$.

Since f is continuous over the interval $[0,1]$, the Intermediate Value Theorem implies that $f(c) = 0$ for some $c \in [0,1]$. Therefore c is the solution of $x^3 + 3x^2 - x = 1$.

Exercise 1.34. ***A fixed point theorem*** *Suppose that a function f is continuous on the closed interval $[0,1]$ and that $0 \le f(x) \le 1$ for every x in $[0,1]$. Show that there must exist a number c in $[0,1]$ such that $f(c) = c$ (c is called a fixed point of f).*

Solution. *Let us define $F(x) = f(x) - x$, that is continuous on $[0,1]$.*

Since $F(0) = f(0) - 0 = f(0) \ge 0$ and $F(1) = f(1) - 1 \le 0$ according the IVT $F(c) = 0$ for some $c \in [0,1]$. Hence, $f(c) = c$.

Exercise 1.35. *Prove that the equation $8\sqrt{x} - \sqrt{1-x} = 5$ has at least one solution.*

Solution. *Let f be a function defined by $f(x) = 8\sqrt{x} - \sqrt{1-x} - 5$*

If $x = 0$, $f(0) = -6 < 0$.

If $x = 1$, $f(1) = 3 > 0$.
Since f is continuous over the interval $[0,1]$, the Intermediate Value Theorem implies that $f(c) = 0$ for some $c \in [0,1]$. Therefore c is the solution of $8\sqrt{x} - \sqrt{1-x} = 5$.

Exercise 1.36. *Show that:*

1. $\lim\limits_{x\to\infty} \frac{1}{x} = 0$,
2. $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$.

Solution.

1. *Let $\epsilon > 0$ be given. We must find a number M such that for all x*

$$x > M \quad \Longrightarrow \quad \left|\frac{1}{x} - 0\right| < \epsilon$$

We find M by working backward from the ϵ-inequality:

$$\left|\frac{1}{x} - 0\right| < \epsilon \quad \Longleftrightarrow \quad \left|\frac{1}{x}\right| < \epsilon \quad \Longleftrightarrow \quad -\epsilon < \frac{1}{x} < \epsilon \quad \Longrightarrow \quad x > \frac{1}{\epsilon}$$

We take all x which satisfy $\epsilon > \frac{1}{x} > 0$ for which ϵ-inequality is true. Hence $x > \frac{1}{\epsilon}$
Then, if we take $M = \frac{1}{\epsilon}$, for all x

$$x > M = \frac{1}{\epsilon} \quad \Longrightarrow \quad \left|\frac{1}{x} - 0\right| < \epsilon$$

which yields that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.

2. *Let $\epsilon > 0$ be given. We must find a number N such that for all x*

$$x < N \quad \Longrightarrow \quad \left|\frac{1}{x} - 0\right| < \epsilon$$

We find N by working backward from the ϵ-inequality:

$$\left|\frac{1}{x}-0\right|<\epsilon \iff \left|\frac{1}{x}\right|<\epsilon \iff -\epsilon<\frac{1}{x}<\epsilon$$

We take all x which satisfies $-\epsilon<\frac{1}{x}<0$ for which ϵ-inequality is true. Hence $x<-\frac{1}{\epsilon}$

Then, if we take $N=-\frac{1}{\epsilon}$, for all x

$$x<N=-\frac{1}{\epsilon} \implies \left|\frac{1}{x}-0\right|<\epsilon$$

which yields that $\lim\limits_{x\to-\infty}\frac{1}{x}=0$.

Exercise 1.37. *Find the following limits:*

1. $\lim\limits_{x\to\infty}\left(3+\frac{1}{x}\right)$,
2. $\lim\limits_{x\to\infty}\left(\frac{1}{x^3}+\frac{2}{x^2}\right)$.

Solution.

1. $\lim\limits_{x\to\infty}\left(3+\frac{1}{x}\right)=\lim\limits_{x\to\infty}3+\lim\limits_{x\to\infty}\frac{1}{x}=3+0=3.$
2. $\lim\limits_{x\to\infty}\left(\frac{1}{x^3}+\frac{2}{x^2}\right)=0.$

Exercise 1.38. *Find the following limits:*

1. $\lim\limits_{x\to\infty}\frac{3x^2+2x-1}{x^2+7}$,
2. $\lim\limits_{x\to\infty}\frac{5x^3+4x-8}{2x^4+3x^3-5x}$,
3. $\lim\limits_{x\to-\infty}\frac{-x^2-6x+1}{2x^4-3x^2-6}$,
4. $\lim\limits_{x\to-\infty}\frac{10x^5}{-2x^5+x^4}$,
5. $\lim\limits_{x\to\infty}\sqrt{x-5}-\sqrt{x-7}$,
6. $\lim\limits_{x\to\infty}\sqrt{2x^2-2x}-\sqrt{2x^2+3x}$,

7. $\lim\limits_{x\to-\infty} \sqrt{x^2+8}+x.$

Solution.

First we divide the numerator and numerator by the highest power of the denominator:

1.

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2+7} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{7}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{2}{x}-\frac{1}{x^2}}{1+\frac{7}{x^2}} = \frac{3+0}{1+0} = 3.$$

2.

$$\lim_{x\to\infty} \frac{5x^3+4x-8}{2x^4+3x^3-5x} = \lim_{x\to\infty} \frac{\frac{5x^3}{x^4}+\frac{4x}{x^4}-\frac{8}{x^4}}{\frac{2x^4}{x^4}+\frac{3x^3}{x^4}-\frac{5x}{x^4}} = \lim_{x\to\infty} \frac{\frac{5}{x}+\frac{4}{x^3}-\frac{8}{x^4}}{2+\frac{3}{x}-\frac{5}{x^3}} = \frac{0-0+0}{2-0-0} = 0.$$

3.

$$\lim_{x\to-\infty} \frac{-x^2-6x+1}{2x^4-3x^2-6} = \lim_{x\to-\infty} \frac{-\frac{x^2}{x^4}-\frac{6x}{x^4}+\frac{1}{x^4}}{\frac{2x^4}{x^4}-\frac{3x^2}{x^4}-\frac{6}{x^4}} = \lim_{x\to-\infty} \frac{-\frac{1}{x^2}-\frac{6}{x^3}+\frac{1}{x^4}}{2-\frac{3}{x^2}-\frac{6}{x^4}} = \frac{0}{2} = 0.$$

4.

$$\lim_{x\to-\infty} \frac{10x^5}{-2x^5+x^4} = \lim_{x\to-\infty} \frac{\frac{10x^5}{x^5}}{\frac{-2x^5}{x^5}+\frac{x^4}{x^5}} = \lim_{x\to-\infty} \frac{10}{-2+\frac{1}{x}} = \frac{10}{-2} = -5.$$

5.

$$\lim_{x\to\infty} \sqrt{x-5}-\sqrt{x-7} = \lim_{x\to\infty} \frac{(\sqrt{x-5}-\sqrt{x-7})(\sqrt{x-5}+\sqrt{x-7})}{\sqrt{x-5}+\sqrt{x-7}}$$

$$= \lim_{x\to\infty} \frac{x-5-x+7}{\sqrt{x-5}+\sqrt{x-7}} = \lim_{x\to\infty} \frac{2}{\sqrt{x-5}+\sqrt{x-7}} = 0.$$

6.

$$\lim_{x\to\infty} \sqrt{2x^2-2x}-\sqrt{2x^2+3x} = \lim_{x\to\infty} \frac{(\sqrt{2x^2-2x}-\sqrt{2x^2+3x})(\sqrt{2x^2-2x}+\sqrt{2x^2+3x})}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}$$

$$= \lim_{x\to\infty} \frac{2x^2-2x-2x^2-3x}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}} = \lim_{x\to\infty} \frac{-5x}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}$$

$$= \lim_{x\to\infty} \frac{-5}{\sqrt{2-\frac{2}{x}}+\sqrt{2+\frac{3}{x}}} = \frac{-5}{2+2} = -\frac{5}{4}.$$

7.

$$\lim_{x\to-\infty}\sqrt{x^2+8}+x=\lim_{x\to-\infty}\frac{(\sqrt{x^2+8}+x)(\sqrt{x^2+8}-x)}{\sqrt{x^2+8}-x}=\lim_{x\to-\infty}\frac{x^2+8-x^2}{\sqrt{x^2+8}-x}$$
$$=\lim_{x\to-\infty}\frac{8}{|x|\sqrt{1+\frac{8}{x^2}}-x}=\lim_{x\to-\infty}\frac{8}{-x\sqrt{1+\frac{8}{x^2}}-x}=\lim_{x\to-\infty}\frac{8}{-x\left(\sqrt{1+\frac{8}{x^2}}+1\right)}=0.$$

Exercise 1.39. *Find the limits:*

1. $\lim\limits_{x\to 0^+}\dfrac{2}{5x}$,
2. $\lim\limits_{x\to 0^-}\dfrac{2}{5x}$,
3. $\lim\limits_{x\to 6^+}\dfrac{1}{x-6}$,
4. $\lim\limits_{x\to 0^-}\dfrac{1}{x-6}$.

Solution.

1. $\lim\limits_{x\to 0^+}\dfrac{2}{5x}=+\infty$. *(since the numerator and denominator are positive)*
2. $\lim\limits_{x\to 0^-}\dfrac{2}{5x}=-\infty$. *(since the numerator is positive and the denominator is negative)*
3. $\lim\limits_{x\to 6^+}\dfrac{1}{x-6}=+\infty$. *(since the numerator and denominator are positive)*
4. $\lim\limits_{x\to 0^-}\dfrac{1}{x-6}=-\infty$. *(since the numerator is positive and the denominator is negative)*

Exercise 1.40. *Find the limits:*

1. $\lim\limits_{x\to -3^+}\dfrac{2x}{5x+15}$,
2. $\lim\limits_{x\to -3^-}\dfrac{2x}{5x+15}$,
3. $\lim\limits_{x\to -2}\dfrac{1}{(x+2)^2}$,
4. $\lim\limits_{x\to 7}\dfrac{2x}{(x-7)^2}$.

Solution.

1. $\lim\limits_{x\to -3^+} \dfrac{2x}{5x+15} = \lim\limits_{x\to -3^+} \dfrac{2x}{5(x+3)} = -\infty.$ *(since the numerator is negative and the denominator is positive)*

2. $\lim\limits_{x\to -3^-} \dfrac{2x}{5x+15} = \lim\limits_{x\to -3^-} \dfrac{2x}{5(x+3)} = \infty.$ *(since the numerator and denominator are negative)*

3. $\lim\limits_{x\to -2} \dfrac{1}{(x+2)^2} = \infty.$ *(since the left and right hand limits are the same, ∞)*

4. $\lim\limits_{x\to 7} \dfrac{2x}{(x-7)^2} = \infty.$ *(since the left and right hand limits are the same, ∞)*

Exercise 1.41. *Find the limit of f defined by $f(x) = \dfrac{x^2}{4} - \dfrac{3}{x}$,*

1. *as $x \to 0^+$,*

2. *as $x \to 0^-$,*

3. *as $x \to 5$,*

4. *as $x \to -2$,*

Solution.

1. $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \left(\dfrac{x^2}{4} - \dfrac{3}{x}\right) = \lim\limits_{x\to 0^+} \dfrac{x^2}{4} - \lim\limits_{x\to 0^+} \dfrac{3}{x} = -\infty.$ *(since $0 - \infty = -\infty$)*

2. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \left(\dfrac{x^2}{4} - \dfrac{3}{x}\right) = \lim\limits_{x\to 0^-} \dfrac{x^2}{4} - \lim\limits_{x\to 0^-} \dfrac{3}{x} = \infty.$ *(since $0 - (-\infty) = \infty$)*

3. $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} \left(\dfrac{x^2}{4} - \dfrac{3}{x}\right) = \dfrac{5^2}{4} - \dfrac{3}{5} = 5.65.$

4. $\lim\limits_{x\to -2} f(x) = \lim\limits_{x\to -2} \left(\dfrac{x^2}{4} - \dfrac{3}{x}\right) = \dfrac{(-2)^2}{4} - \dfrac{3}{-2} = 2.5.$

Exercise 1.42. *Find the limit of f given by $f(x) = \dfrac{x^2-5x+6}{x^3-9x}$,*

1. *as $x \to -3^+$,*

2. *as $x \to -3^-$,*

3. *as $x \to -3$,*

4. *as* $x \to 0^-$,

Solution.

1. $\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{x^2-5x+6}{x^3-9x} = \lim_{x\to -3^+} \frac{(x-2)(x-3)}{x(x-3)(x+3)} = \lim_{x\to -3^+} \frac{(x-2)}{x(x+3)} = \infty.$

2. $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{x^2-5x+6}{x^3-9x} = \lim_{x\to -3^-} \frac{(x-2)}{x(x+3)} = -\infty.$

3. $\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x^2-5x+6}{x^3-9x}$ *does not exist, because one sided limits are different.*

4. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2-5x+6}{x^3-9x} = \lim_{x\to 0^-} \frac{(x-2)}{x(x+3)} = \infty.$

Exercise 1.43. *Find the limit of f which is defined by* $f(x) = \frac{1}{x^{2/3}} + \frac{1}{x-1}$,

1. *as* $x \to 0^+$,
2. *as* $x \to 0^-$,
3. *as* $x \to 1^+$,
4. *as* $x \to 1^-$.

Solution.

1. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\frac{1}{x^{2/3}} + \frac{1}{x-1}\right) = \lim_{x\to 0^+} \frac{1}{x^{2/3}} + \lim_{x\to 0^+} \frac{1}{x-1} = \infty.$

2. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(\frac{1}{x^{2/3}} + \frac{1}{x-1}\right) = \lim_{x\to 0^-} \frac{1}{x^{2/3}} + \lim_{x\to 0^-} \frac{1}{x-1} = \infty.$

3. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(\frac{1}{x^{2/3}} + \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{1}{x^{2/3}} + \lim_{x\to 1^+} \frac{1}{x-1} = \infty.$

4. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\frac{1}{x^{2/3}} + \frac{1}{x-1}\right) = \lim_{x\to 1^-} \frac{1}{x^{2/3}} + \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$

Exercise 1.44. *Let f be defined as follows. Find the horizontal asymptotes of the graph of f.*

1. $f(x) = \frac{4x^3+2x+1}{x^3+3x^2}$,

2. $f(x) = \frac{x^2}{-2x^2+6x+10}$,

3. $f(x) = \dfrac{|x|^3 - 3x}{2x^3 + 3x^2 - 1}$,

4. $f(x) = \dfrac{x^3 + x^2 - 4x - 6}{x + 3}$.

Solution.

1.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x^3 + 2x + 1}{x^3 + 3x^2} = \lim_{x\to-\infty} \frac{4 + \dfrac{2}{x^2} + \dfrac{1}{x^3}}{1 + \dfrac{3}{x}} = 4,$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^3 + 2x + 1}{x^3 + 3x^2} = \lim_{x\to\infty} \frac{4 + \dfrac{2}{x^2} + \dfrac{1}{x^3}}{1 + \dfrac{3}{x}} = 4.$$

These limits imply that the line of $y = 4$ is the horizontal asymptote of the graph of f on both the right and the left.

2.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{-2x^2 + 6x + 10} = -\frac{1}{2},$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{-2x^2 + 6x + 10} = -\frac{1}{2}.$$

Then the line of $y = -\frac{1}{2}$ is the horizontal asymptote of the graph of f on both the right and the left (or at $-\infty$ and $-\infty$).

3.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{|x|^3 - 3x}{2x^3 + 3x^2 - 1} = \lim_{x\to-\infty} \frac{-x^3 - 3x}{2x^3 + 3x^2 - 1} \quad (since \quad x < 0)$$
$$= \lim_{x\to-\infty} \frac{-1 - \dfrac{3}{x^2}}{2 + \dfrac{3}{x} - \dfrac{1}{x^3}} = -\frac{1}{2}.$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{|x|^3 - 3x}{2x^3 + 3x^2 - 1} = \lim_{x\to\infty} \frac{x^3 - 3x}{2x^3 + 3x^2 - 1} \quad (since \quad x > 0)$$
$$= \lim_{x\to\infty} \frac{1 - \dfrac{3}{x^2}}{2 + \dfrac{3}{x} - \dfrac{1}{x^3}} = \frac{1}{2}.$$

Theare are the horizontal asymptotes of $y = -\frac{1}{2}$ and $y = \frac{1}{2}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3+x^2-4x-6}{x+3} = \lim_{x\to\infty} \frac{x^2+x-4-\dfrac{6}{x}}{1+\dfrac{3}{x}} = \infty.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3+x^2-4x-6}{x+3} = \lim_{x\to-\infty} \frac{x^2+x-4-\dfrac{6}{x}}{1+\dfrac{3}{x}} = \infty.$$

The graph of f has no horizontal asymptote.

Exercise 1.45. *Find the horizontal asymptotes of the graph of f given by:*

1. $f(x) = 3\cdot e^{2x}$,
2. $f(x) = x - \sqrt{x^2+9}$.

Solution.

1. *Since*

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (3\cdot e^{2x}) = 0,$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (3\cdot e^{2x}) = \infty.$$

the graph of f has a horizontal asymptote of $y=0$ only at $-\infty$.

2.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x-\sqrt{x^2+9}) = \lim_{x\to-\infty} \left(x-|x|\sqrt{1+\frac{9}{x^2}}\right)$$
$$= \lim_{x\to-\infty} \left(x+x\sqrt{1+\frac{9}{x^2}}\right) = \lim_{x\to-\infty} x\left(1+\sqrt{1+\frac{9}{x^2}}\right) = -\infty,$$

This limit implies that the curve of f has no horizontal asymptote at $-\infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x-\sqrt{x^2+9}) = \lim_{x\to\infty} \frac{(x-\sqrt{x^2+9})(x+\sqrt{x^2+9})}{(x+\sqrt{x^2+9})}$$
$$= \lim_{x\to\infty} \frac{x^2-x^2-9}{x+\sqrt{x^2+9}} = \lim_{x\to\infty} \frac{-9}{x+|x|\sqrt{1+\dfrac{9}{x^2}}}$$
$$= \lim_{x\to\infty} \frac{-9}{x+x\sqrt{1+\dfrac{9}{x^2}}} = \lim_{x\to-\infty} \frac{-9}{x\left(1+\sqrt{1+\dfrac{9}{x^2}}\right)} = 0.$$

Then the line of $y=0$ will be the horizontal asymptote of f at ∞.

Exercise 1.46. *Find the horizontal asymptotes of the graph of f defined by*

1. $f(x) = \cos\dfrac{1}{x}$,
2. $f(x) = \dfrac{1}{x^2}\cos x$,
3. $f(x) = x\sin\dfrac{1}{x}$.

Solution.

1.
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\cos\frac{1}{x} = 0,$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\cos\frac{1}{x} = 0,$$
Therefore the line of $y = 0$ is the horizontal asymptote of the curve f at both $-\infty$ and $+\infty$.

2.
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{1}{x^2}\cos x\right) = 0, \quad (from\ the\ Sandwich\ theorem)$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{1}{x^2}\cos x\right) = 0.$$
Hence, the line of $y = 0$ is the horizontal asymptote of the curve f at both $-\infty$ and $+\infty$.

3.
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(x\sin\frac{1}{x}\right) = 0,$$
Let us make the substitution $t = \dfrac{1}{x}$, then $t \to -\infty \Longleftrightarrow x \to 0^-$, hence,
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(x\sin\frac{1}{x}\right) = \lim_{t\to 0^-}\left(\frac{1}{t}\sin t\right) = 1,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x\sin\frac{1}{x}\right) = 0,$$
We substitute $t = \dfrac{1}{x}$, then $t \to +\infty \Longleftrightarrow x \to 0^+$, hence,
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x\sin\frac{1}{x}\right) = \lim_{t\to 0^+}\left(\frac{1}{t}\sin t\right) = 1.$$
Therefore, the line of $y = 1$ is the horizontal asymptote of the curve f on both the right and the left (or at $-\infty$ and $-\infty$).

Exercise 1.47. *Find the vertical asymptotes of the graph of f defined by*

1. $f(x) = \dfrac{1}{x-2}$,
2. $f(x) = \dfrac{x^2-3x+2}{x^2-2x}$,
3. $f(x) = \sec x, \quad x \in \Big[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\Big]$,
4. $f(x) = \tan x$.

Solution.

1. *We consider the point* 2, *such that* $\lim_{x\to 2}\left|\dfrac{1}{x-2}\right| = +\infty$. *Since*

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\frac{1}{x-2} = -\infty,$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{1}{x-2} = +\infty,$$

the line of the equation $x = 2$ is a vertical asymptote of f both from the right and from the left.

2. $f(x) = \dfrac{x^2-3x+2}{x^2-2x} = \dfrac{(x-2)(x-1)}{x(x-2)}$.

Since,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0^-}\frac{(x-2)(x-1)}{x(x-2)} = \lim_{x\to 0^-}\frac{x-1}{x} = -\infty,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{1}{x-2}\lim_{x\to 0^+}\frac{(x-2)(x-1)}{x(x-2)} = \lim_{x\to 0^+}\frac{x-1}{x} = +\infty,$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2^-}\frac{x-1}{x} = \frac{1}{2},$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2^+}\frac{x-1}{x} = \frac{1}{2}.$$

The line of $x = 0$ is a vertical asymptote of f both from the right and from the left.

3. $f(x) = \sec x = \dfrac{1}{\cos x}$, *and* $\cos x = 0$ *if* $x = -\dfrac{\pi}{2}$ *and* $x = \frac{\pi}{2}$, *so we find,*

$$\lim_{x\to -\frac{\pi}{2}^+} f(x) = \lim_{x\to -\frac{\pi}{2}^+}\sec = \lim_{x\to -\frac{\pi}{2}^+}\frac{1}{\cos x} = +\infty,$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} \sec = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x} = +\infty,$$

Because of these the lines of the equations $x = -\dfrac{\pi}{2}$ *is vertical asymptote of the graph of* f *from the left,* $x = -\dfrac{\pi}{2}$ *is vertical asymptotes of the graph of* f *from the right.*

4. $f(x) = \tan x = \dfrac{\sin x}{\cos x}$*, and* $\cos x = 0$ *if* $x = \dfrac{\pi}{2} + \pi k$ *where* $k \in \mathbb{Z}$*, so we find,*

$$\lim_{x\to(\frac{\pi}{2}+\pi k)^+} f(x) = \lim_{x\to(\frac{\pi}{2}+\pi k)^+} \sec = \lim_{x\to(\frac{\pi}{2}+\pi k)^+} \frac{1}{cosx} = -\infty$$

$$\lim_{x\to(\frac{\pi}{2}+\pi k)^-} f(x) = \lim_{x\to(\frac{\pi}{2}+\pi k)^-} \sec = \lim_{x\to(\frac{\pi}{2}+\pi k)^-} \frac{1}{cosx} = +\infty$$

Because of this the lines of the equations $x = \dfrac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ *are vertical asymptotes of the graph of* f *both from the right and from the left.*

Exercise 1.48. *Find the oblique asymptote of the graph of* f *defined by*

1. $f(x) = \dfrac{2x^2 + 3x - 1}{x - 7}$,
2. $f(x) = \dfrac{x^3 + 3x^2 - 3}{x^2 - 2}$,
3. $f(x) = \sqrt{x^2 + 3x - 1} - x$,
4. $f(x) = \dfrac{x^{3/2} + 2x - 4}{\sqrt{x} - 1}$.

Solution.

1. *If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asymptote. We find an equation for the asymptote by dividing numerator by denominator to express it as a linear function plus a remainder that goes to zero as* $x \to \pm\infty$ *(this method is only for rational functions)*

$$f(x) = \frac{2x^2 + 3x - 1}{x - 7} = (2x + 17) + \frac{118}{x - 7},$$

where $\dfrac{118}{x - 7} \to 0$ *as* $x \to \pm\infty$.

Becaise of this the slant line defined by $y = 2x + 17$ *is the oblique asymptote of the graph of* f.

2. *(General method) for the oblique asymptote which is defined by the expression* $y = ax+b$ *we define the slope as* $a = \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x}$ *and* b *as* $b = \lim\limits_{x\to\pm\infty}(f(x) - ax)$

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\dfrac{x^3+3x^2-3}{x^2-2}}{x} = \lim_{x\to-\infty} \frac{x^3+3x^2-3}{x^3-2x} = 1,$$

$$\lim_{x\to-\infty} \left(\frac{x^3+3x^2-3}{x^2-2} - x\right) = \lim_{x\to-\infty} \frac{3x^2+2x-3}{x^2-2} = 3,$$

so the slant line of the equation $y = x - 3$ *is the oblique asymptote to the curve at* $-\infty$.

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\dfrac{x^3+3x^2-3}{x^2-2}}{x} = \lim_{x\to+\infty} \frac{x^3+3x^2-3}{x^3-2x} = 1,$$

$$\lim_{x\to+\infty} \left(\frac{x^3+3x^2-3}{x^2-2} - x\right) = \lim_{x\to+\infty} \frac{3x^2+2x-3}{x^2-2} = 3,$$

then the line defined by $y = x - 3$ *is also the oblique asymptote of the curve at* $+\infty$.

3.

$$\begin{aligned} a = \lim_{x\to-\infty} \frac{f(x)}{x} &= \lim_{x\to-\infty} \frac{\sqrt{x^2+3x-1} - x}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x}{x} \\ &= \lim_{x\to-\infty} \frac{-x\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x}{x} = \lim_{x\to-\infty} \frac{-x\left(\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} + 1\right)}{x} \\ &= \lim_{x\to-\infty} -\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}} + 1\right) = -2. \end{aligned}$$

$$\begin{aligned} b = \lim_{x\to-\infty}(f(x) - ax) &= \lim_{x\to-\infty}(\sqrt{x^2+3x-1} - x - (-2x)) = \lim_{x\to-\infty}(\sqrt{x^2+3x-1} + x) \\ &= \lim_{x\to-\infty} \frac{(\sqrt{x^2+3x-1}+x)(\sqrt{x^2+3x-1}-x)}{\sqrt{x^2+3x-1}-x} = \lim_{x\to-\infty} \frac{x^2+3x-1-x^2}{\sqrt{x^2+3x-1}-x} \\ = \lim_{x\to-\infty} \frac{3x-1}{\sqrt{x^2+3x-1}-x} &= \lim_{x\to-\infty} \frac{3x-1}{|x|\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x} = \lim_{x\to-\infty} \frac{3x-1}{-x\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} - x} \\ &= \lim_{x\to-\infty} \frac{3x-1}{-x\left(\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} + 1\right)} = \lim_{x\to-\infty} \frac{-3+\frac{1}{x}}{\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}} + 1} = -\frac{3}{2}. \end{aligned}$$

Then, the slant line of $y = -2x - \frac{3}{2}$ will be the oblique asymptote to f at $-\infty$.

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\sqrt{x^2+3x-1}-x}{x} = \lim_{x\to+\infty} \frac{|x|\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-x}{x}$$
$$= \lim_{x\to+\infty} \frac{x\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-x}{x} = \lim_{x\to+\infty} \frac{x\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-1\right)}{x}$$
$$= \lim_{x\to+\infty} \left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-1\right) = 0.$$

Since $a = 0$, the curve has no oblique asymptote.

$$b = \lim_{x\to+\infty}(f(x)-ax) = \lim_{x\to+\infty}(\sqrt{x^2+3x-1}-x-0) = \lim_{x\to+\infty}(\sqrt{x^2+3x-1}-x)$$
$$= \lim_{x\to+\infty} \frac{(\sqrt{x^2+3x-1}-x)(\sqrt{x^2+3x-1}+x)}{\sqrt{x^2+3x-1}+x} = \lim_{x\to+\infty} \frac{x^2+3x-1-x^2}{\sqrt{x^2+3x-1}+x}$$
$$= \lim_{x\to+\infty} \frac{3x-1}{\sqrt{x^2+3x-1}+x} = \lim_{x\to+\infty} \frac{3x-1}{|x|\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+x} = \lim_{x\to+\infty} \frac{3x-1}{x\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+x}$$
$$= \lim_{x\to+\infty} \frac{3x-1}{x\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+1\right)} = \lim_{x\to+\infty} \frac{3-\frac{1}{x}}{\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+1} = \frac{3}{2}.$$

Therefore, it has a horizontal asymptote defined by $y = \dfrac{3}{2}$ at $+\infty$.

4. The function is defined on for all $x \geq 0$, because of this we investigate a limit at $+\infty$

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\frac{x^{3/2}+2x-4}{\sqrt{x}-1}}{x} = \lim_{x\to+\infty} \frac{x^{3/2}+2x-4}{x^{3/2}-x} = 1.$$

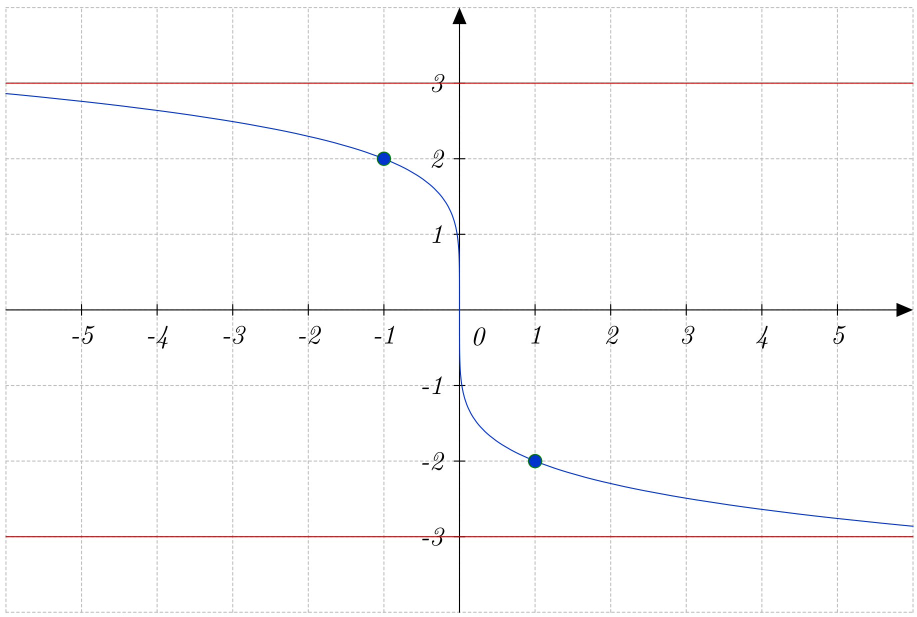
$$b = \lim_{x\to+\infty}(f(x)-ax) = \lim_{x\to+\infty}\left(\frac{x^{3/2}+2x-4}{\sqrt{x}-1}-x\right)$$
$$= \lim_{x\to+\infty} \frac{3x-4}{\sqrt{x}-1} = \lim_{x\to+\infty} \frac{3\sqrt{x}-\frac{4}{\sqrt{x}}}{1-\frac{1}{\sqrt{x}}} = +\infty.$$

Since $b = +\infty$, the curve does not an oblique asymptote.

Exercise 1.49. *Sketch the graph of a function defined by the expression $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)*

$$f(0) = 1, \quad f(1) = -2, \ f(-1) = 2, \quad \lim_{x\to-\infty} f(x) = 3, \quad \lim_{x\to+\infty} f(x) = -3.$$

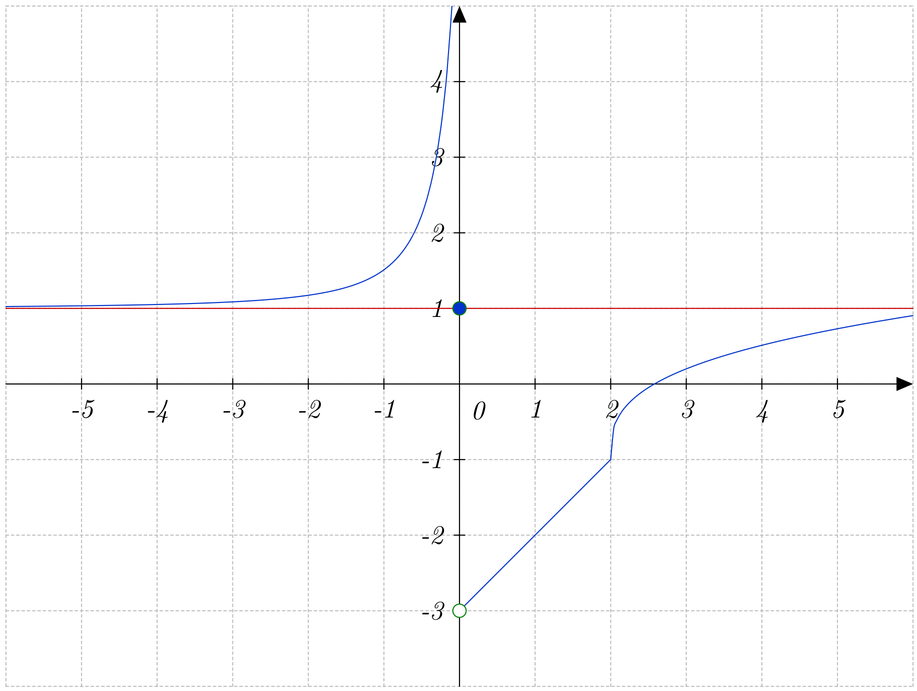
Solution. *One of the choices for the graph of the function can be as follows*



Exercise 1.50. *Sketch the graph of a function defined by $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)*

$$f(0) = 1, \quad \lim_{x\to-\infty} f(x) = 1, \quad \lim_{x\to+\infty} f(x) = 1 \quad \lim_{x\to 0^+} f(x) = -3, \quad \lim_{x\to 0^-} f(x) = +\infty.$$

Solution. *One of the choices is as follows*



Exercise 1.51. *Let f is defined by $f(x) = \sqrt{x^2 - 7x}$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f =]-\infty, 0[\cup]7, +\infty[$.

(a) *H.A.*

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{x^2 - 7x} = \lim_{x\to-\infty} |x|\sqrt{1 - \frac{7}{x}} = \lim_{x\to-\infty} -x\sqrt{1 - \frac{7}{x}} = -\infty,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty} \sqrt{x^2 - 7x} = \lim_{x\to+\infty} |x|\sqrt{1 - \frac{7}{x}} = \lim_{x\to+\infty} x\sqrt{1 - \frac{7}{x}} = +\infty.$$

Therefore, the curve of f has no horizontal asymptote at $\pm\infty$.

(b) *V.A.*

Since there is no point a such that $\lim_{x\to a} |f(x)| = +\infty$ the graph of f has no vertical asymptote.

(c) *O.A.*

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2 - 7x}}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{1 - \frac{7}{x}}}{x}$$
$$= \lim_{x\to-\infty} \frac{-x\sqrt{1 - \frac{7}{x}}}{x} = \lim_{x\to-\infty} \left(-\sqrt{1 - \frac{7}{x}}\right) = -1,$$

$$\begin{aligned}
b = \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(f(x)-ax) = \lim_{x\to-\infty}(\sqrt{x^2-7x}+x) \\
= \lim_{x\to-\infty}\frac{(\sqrt{x^2-7x}+x)(\sqrt{x^2-7x}-x)}{\sqrt{x^2-7x}-x} = \lim_{x\to-\infty}\frac{x^2-7x-x^2}{\sqrt{x^2-7x}-x} \\
= \lim_{x\to-\infty}\frac{-7x}{|x|\sqrt{1-\frac{7}{x}}-x} = \lim_{x\to-\infty}\frac{-7x}{-x\sqrt{1-\frac{7}{x}}-x} \\
= \lim_{x\to-\infty}\frac{-7x}{-x\Big(\sqrt{1-\frac{7}{x}}+1\Big)} = \lim_{x\to-\infty}\frac{7}{\sqrt{1-\frac{7}{x}}+1} = \frac{7}{2}.
\end{aligned}$$

Hence, the graph of f has an oblique asymptote of $y=-x+\frac{7}{2}$ at $-\infty$.

Position of the curve of f with respect to the asymptote of $y=-x+\frac{7}{2}$ at $-\infty$:

$$\begin{aligned}
\lim_{x\to-\infty}(f(x)-y) = \lim_{x\to-\infty}\Big(\sqrt{x^2-7x}-\Big(-x+\frac{7}{2}\Big)\Big) \\
= \lim_{x\to-\infty}\Big(\sqrt{x^2-7x}+x-\frac{7}{2}\Big) = \lim_{x\to-\infty}\Big(\sqrt{x^2-7x}+x\Big)-\frac{7}{2} \\
= \lim_{x\to-\infty}\frac{7}{\sqrt{1-\frac{7}{x}}+1}-\frac{7}{2} = 0^-.
\end{aligned}$$

Hence, The graph of f is below the oblique asymptote of $y=-x+\frac{7}{2}$ at $-\infty$.

$$\begin{aligned}
a = \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\sqrt{x^2-7x}}{x} = \lim_{x\to+\infty}\frac{|x|\sqrt{1-\frac{7}{x}}}{x} \\
= \lim_{x\to+\infty}\frac{x\sqrt{1-\frac{7}{x}}}{x} = \lim_{x\to+\infty}\sqrt{1-\frac{7}{x}} = 1.
\end{aligned}$$

$$b = \lim_{x\to+\infty} f(x) = \lim_{x\to-\infty}(f(x) - ax) = \lim_{x\to+\infty}(\sqrt{x^2-7x} - x)$$
$$= \lim_{x\to+\infty} \frac{(\sqrt{x^2-7x}-x)(\sqrt{x^2-7x}+x)}{\sqrt{x^2-7x}+x} = \lim_{x\to+\infty} \frac{x^2-7x-x^2}{\sqrt{x^2-7x}+x}$$
$$= \lim_{x\to+\infty} \frac{-7x}{|x|\sqrt{1-\frac{7}{x}}+x} = \lim_{x\to+\infty} \frac{-7x}{x\sqrt{1-\frac{7}{x}}+x}$$
$$= \lim_{x\to+\infty} \frac{-7x}{x\left(\sqrt{1-\frac{7}{x}}+1\right)} = \lim_{x\to+\infty} \frac{-7}{\sqrt{1-\frac{7}{x}}+1} = -\frac{7}{2}$$

Therefore, the graph of f has an oblique asymptote of $y = x - \frac{7}{2}$ at $+\infty$.

Position of the curve of f with respect to the asymptote of $y = x - \frac{7}{2}$ at $+\infty$:

$$\lim_{x\to+\infty}(f(x) - y) = \lim_{x\to+\infty}\left(\sqrt{x^2-7x} - \left(x - \frac{7}{2}\right)\right)$$
$$= \lim_{x\to+\infty}\left(\sqrt{x^2-7x} - x + \frac{7}{2}\right) = \lim_{x\to+\infty}\left(\sqrt{x^2-7x} - x\right) + \frac{7}{2}$$
$$= \lim_{x\to+\infty} \frac{-7}{\sqrt{1-\frac{7}{x}}+1} + \frac{7}{2} = 0^-$$

The graph of f is also below the oblique asymptote of $y = x - \frac{7}{2}$ at $+\infty$.

Exercise 1.52. *Let f is defined by $f(x) = \frac{x^2-3x+7}{x+3}$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f = \mathbb{R} \setminus \{-3\}$.

(a) H.A.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-3x+7}{x+3} = \lim_{x\to-\infty} \frac{x-3+\frac{7}{x}}{1+\frac{3}{x}} = -\infty,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2-3x+7}{x+3} = \lim_{x\to+\infty} \frac{x-3+\frac{7}{x}}{1+\frac{3}{x}} = +\infty.$$

Therefore, the curve of f has no horizontal asymptote.

(b) V.A.

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{x^2-3x+7}{x+3} = -\infty$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{x^2-3x+7}{x+3} = +\infty$$

Therefore the line of $x=-3$ is the vertical asymptote of the the graph of f. The graph of f is on the left of the horizontal asymptote of $x=-3$ as $y\to-\infty$, on the right of the horizontal asymptote of $x=-3$ as $y\to+\infty$.

(c) O.A.

Note: *At $f(x)=\dfrac{x^2-3x+7}{x+3}$ the degree of the numerator in only one greater then the degre of the denominator, becauce of this the curve of f has an oblique asymptote.*

$$f(x) = \frac{x^2-3x+7}{x+3} = x-6+\frac{25}{x+3},$$

where $\frac{25}{x+3}\to 0$ as $x\to\pm+\infty$.

Hence, the line of $y=x-6$ is the oblique asymptote of the graph of f at $\pm+\infty$.

Since

$$\lim_{x\to+\infty}(f(x)-y) = \lim_{x\to+\infty}\Big(\frac{x^2-3x+7}{x+3}-(x-6)\Big) = \lim_{x\to+\infty}\frac{25}{x+3} = 0^+$$

the curve of f is above the asymptote of $y=x-6$ at $+\infty$.

$$\lim_{x\to-\infty}(f(x)-y) = \lim_{x\to-\infty}\Big(\frac{x^2-3x+7}{x+3}-(x-6)\Big) = \lim_{x\to-\infty}\frac{25}{x+3} = 0^-$$

the curve of f is below the asymptote of $y=x-6$ at $-\infty$.

Exercise 1.53. *Let f is defined by $f(x)=\dfrac{3x^2+2}{x^2+7}$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f=\mathbb{R}$.

(a) H.A. Since

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{3x^2+2}{x^2+7} = 3,$$

and

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty}\frac{3x^2+2}{x^2+7} = 3,$$

the curve of f has the horizontal asymptote of $y = 3$ at both $-\infty$ and $+\infty$.

$$\lim_{x\to-\infty}(f(x)-y) = \lim_{x\to-\infty}\Big(\frac{3x^2+2}{x^2+7}-3\Big) = \lim_{x\to-\infty}\frac{3x^2+2-3x^2-21}{x^2+7} = \lim_{x\to-\infty}\frac{-19}{x^2+7} = 0^-$$

Therefore, the graph of f is below the line of $y = 3$ at $-\infty$.

$$\lim_{x\to+\infty}(f(x)-y) = \lim_{x\to+\infty}\Big(\frac{3x^2+2}{x^2+7}-3\Big) = \lim_{x\to+\infty}\frac{3x^2+2-3x^2-21}{x^2+7} = \lim_{x\to+\infty}\frac{-19}{x^2+7} = 0^-$$

Therefore, the graph of f is below the line of $y = 3$ at $+\infty$ as well.

(b) *V.A.*

There is no point p such that $\lim\limits_{x\to p}|f(x)| = +\infty$, so the curve of f has no vertical asymptote.

(c) *O.A. Since the degree of the numerator and denominator are equal in $f(x)$ which defines the function f, f the curve of f has no oblique asymptote.*

*(**Another explanation:** Since $a = \lim\limits_{x\to\pm+\infty}\frac{f(x)}{x} = 0$, (slope is 0), the curve of f has no oblique asymptote.)*

Exercise 1.54. *Let f is defined by $f(x) = \dfrac{x^2}{x-2} + |x-1|$. Find all the asymptotes to the curve of f and determine the position of the curve of f with respect to its asymptotes.*

Solution. $D_f = \mathbb{R}\setminus\{2\}$.

(a) *H.A. Since*

$$\begin{aligned}\lim_{x\to-\infty}f(x) &= \lim_{x\to-\infty}\frac{x^2}{x-2}+|x-1| = \lim_{x\to-\infty}\Big(\frac{x^2}{x-2}-x+1\Big)\\ &= \lim_{x\to-\infty}\frac{x^2-x^2+2x+x-2}{x-2} = \lim_{x\to-\infty}\frac{3x-2}{x-2} = \lim_{x\to-\infty}\frac{3-\frac{2}{x}}{1-\frac{2}{x}} = 3,\end{aligned}$$

the curve of f has the horizontal asymptote of $y = 3$ at $-\infty$.

$$\lim_{x\to-\infty}(f(x)-y) = \lim_{x\to-\infty}\Big(\frac{x^2}{x-2}+|x-1|-3\Big) = \lim_{x\to-\infty}\Big(\frac{3-\frac{2}{x}}{1-\frac{2}{x}}-3\Big) = 0^-.$$

Hence, the curve is below of the horizontal asymptote of $y = 3$ at $-\infty$.

Since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{x-2} + |x-1| = \lim_{x\to+\infty} \left(\frac{x^2}{x-2} + x - 1\right)$$
$$= \lim_{x\to+\infty} \frac{x^2 + x^2 - 2x - x + 2}{x-2} = \lim_{x\to+\infty} \frac{2x^2 - 3x + 2}{x-2} = +\infty,$$

the curve of f does not have a horizontal asymptote at $+\infty$.

(b) *V.A.*

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2}{x-2} + |x-1| = -\infty,$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2}{x-2} + |x-1| = +\infty.$$

Therefore the line of $x = 2$ is the vertical asymptote of the the graph of f. The graph of f is on the left of the vertical asymptote of $x = 2$ as $y \to -\infty$, on the right of the horizontal asymptote of $x = 2$ as $y \to +\infty$.

(c) *O.A. Since the curve of f has horizontal asymptote at $-\infty$, we will consider the limit of f at only $+\infty$.*

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\frac{x^2}{x-2} + |x-1|}{x} = \lim_{x\to+\infty} \frac{\frac{2x^2-3x+2}{x-2}}{x}$$
$$= \lim_{x\to+\infty} \frac{2x^2 - 3x + 2}{x^2 - 2x} = 2,$$

$$b = \lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} (f(x) - ax) = \lim_{x\to+\infty} \left(\frac{x^2}{x-2} + |x-1| - 2x\right)$$
$$= \lim_{x\to+\infty} \left(\frac{2x^2 - 3x + 2}{x-2} - 2x\right) = \lim_{x\to+\infty} \frac{2x^2 - 3x + 2 - 2x^2 + 4x}{x-2}$$
$$= \lim_{x\to+\infty} \frac{x+2}{x-2} = 1.$$

Then, the curve of f has the oblique asymptote of $y = 2x + 1$ at $+\infty$.

$$\lim_{x\to+\infty} (f(x) - y) = \lim_{x\to+\infty} \left(\frac{x^2}{x-2} + |x-1| - 2x - 1\right)$$
$$= \lim_{x\to+\infty} \left(\left(\frac{x^2}{x-2} + |x-1| - 2x\right) - 1\right) = \lim_{x\to+\infty} \left(\frac{x+2}{x-2} - 1\right)$$
$$= \lim_{x\to+\infty} \frac{x + 2 - x + 2}{x-2} = \lim_{x\to+\infty} \frac{4}{x-2} = 0^+.$$

Therefore, the curve of f is above the oblique asymptote of $y = 2x + 1$ at $+\infty$.