ADA University
BBA/BSE

Fall 2020-2021
Calculus I

Limit and Continuity

Exercises and Solutions

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May 12, 2019

Contents

1 Limits of Functions

Exercise 1.1. *Find the limits:*

1. $\lim_{x\to 4} \frac{x^2-16}{x-4}$,
2. $\lim_{x\to 0} \frac{\sqrt{5x+4}-2}{x}$,
3. $\lim_{x\to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$,
4. $\lim_{x\to 1} \frac{x^2+x-2}{x^2-1}$,
5. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$,
6. $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$,
7. $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$.

Solution.

1. *We cannot substitute $x = 4$ because it makes the denominator zero. We test the numerator to see if it is zero at $x = 4$ too. It is, so it has a factor of $(x-4)$ in common with the denominator. Canceling the $(x-4)$'s gives a simpler fraction with the same values as the original for $x \neq 4$:*

$$\frac{x^2-16}{x-4} = \frac{(x+4)(x-4)}{x-4} = (x+4) \quad if \quad x \neq 4.$$

Using the simpler fraction, we find the limit of these values as $x \to 4$ by substitution:

$$\lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = \lim_{x\to 4}(x+4) = 4+4 = 8.$$

Note: *We will use this (Eliminating Zero Denominator method) for Exercise 3 and Exercise 4.*

2. *First we multiply the numerator and the denominator by the Conjugate fo the numerator. Then,*

$$\lim_{x\to 0} \frac{\sqrt{5x+4}-2}{x} = \lim_{x\to 0} \frac{(\sqrt{5x+4}-2)(\sqrt{5x+4}+2)}{x(\sqrt{5x+4}+2)} = \lim_{x\to 0} \frac{5x}{x(\sqrt{5x+4}+2)}$$
$$= \lim_{x\to 0} \frac{5}{(\sqrt{5x+4}+2)} = \frac{5}{2+2} = \frac{5}{4}.$$

3. $\lim_{x\to 0}\frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}=\lim_{x\to 0}\frac{\frac{x+1+x-1}{x^2-1}}{x}=\lim_{x\to 0}\frac{2x}{x(x^2-1)}=\lim_{x\to 0}\frac{2}{(x^2-1)}=\frac{2}{0^2-1}=-2.$
4. $\lim_{x\to 1}\frac{x^2+x-2}{x^2-1}=\lim_{x\to 1}\frac{(x-1)(x+2)}{(x+1)(x-1)}=\lim_{x\to 1}\frac{x+2}{x+1}=\frac{1+2}{1+1}=\frac{3}{2}.$
5. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}=\lim_{x\to 9}\frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)}=\lim_{x\to 9}\frac{1}{\sqrt{x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}.$
6.
$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}$$
$$=\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{x-1}=\lim_{x\to 1}(\sqrt{x+3}+2)=\sqrt{4}+2=4.$$
7.
$$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}=\lim_{x\to -1}\frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(\sqrt{x^2+8}+3)(x+1)}=\lim_{x\to -1}\frac{x^2-1}{(\sqrt{x^2+8}+3)(x+1)}$$
$$=\lim_{x\to -1}\frac{x-1}{\sqrt{x^2+8}+3}=\frac{-1-1}{(-1)^2+3}=-\frac{2}{4}=-\frac{1}{2}.$$

Exercise 1.2. *Prove the limit statements:*

1. $\lim_{x\to 3}(x+4)=7$,
2. $\lim_{x\to 1}\frac{2}{x}=2$,
3. $\lim_{x\to 14}\sqrt{x-5}=3$.

Solution.

1. *Let $p=3$, $f(x)=x+4$ and $L=7$ in the definition of limit. For any given $\epsilon>0$ we have to find a suitable $\delta>0$ such that for all real $x\in R$, $0<|x-3|<\delta$ implies $|f(x)-7|<\epsilon$.*

   We find δ by working backward from the ϵ-inequality
$$|f(x)-L|<\epsilon,$$
$$|x+4-7|<\epsilon,$$
$$|x-3|<\epsilon.$$
   We can take $\delta=\epsilon$. If $0<|x-3|<\delta=\epsilon$ then
$$|x+4-7|=|x-3|<\epsilon,$$
   which proves that $\lim_{x\to 3}(x+4)=7$.

2. *Let* $p = 1$, $f(x) = \frac{2}{x}$ *and* $L = 2$. *Let us take* $\forall \epsilon > 0$. *We must find* $\delta > 0$ *such that for all real* x, *for which* $0 < |x - 1| < \delta$ $(x \neq 1, -\delta < x - 1 < \delta)$, $|f(x) - 2| < \epsilon$ *is true.*

   We solve the inequality
$$|f(x) - 2| < \epsilon.$$
   Then
$$\left|\frac{2}{x} - 2\right| < \epsilon,$$
$$-\epsilon < \frac{2}{x} - 2 < \epsilon,$$
$$2 - \epsilon < \frac{2}{x} < 2 + \epsilon,$$
   We can assume $\epsilon < 2$. *Then*
$$\frac{1}{2+\epsilon} < \frac{x}{2} < \frac{1}{2-\epsilon},$$
$$\frac{2}{2+\epsilon} < x < \frac{2}{2-\epsilon},$$
   We take δ *to be the distance from* $p = 1$ *to the nearer endpoint of* $\left(\frac{2}{2+\epsilon}, \frac{2}{2-\epsilon}\right)$. *In other words, we take*
$$\delta = \min\left\{1 - \frac{2}{2+\epsilon}, \frac{2}{2-\epsilon} - 1\right\} = \min\left\{\frac{\epsilon}{2+\epsilon}, \frac{\epsilon}{2-\epsilon}\right\} = \frac{\epsilon}{2+\epsilon}.$$
   For $\epsilon \geq 2$,

   we take δ *to be the distance from* $p = 1$ *to the nearer endpoint of* $\left(0, \frac{2}{2+\epsilon}\right)$.

   In other words, we take
$$\delta = \min\left\{1 - \frac{2}{2+\epsilon}, 1\right\} = \min\left\{\frac{\epsilon}{2+\epsilon}, 1\right\} = \frac{\epsilon}{2+\epsilon}.$$
   Then for all x, $0 < |x - 1| < \delta = \frac{\epsilon}{2+\epsilon}$ *implies* $|f(x) - 2| < \epsilon$, *which proves that* $\lim_{x\to 1} \frac{2}{x} = 2$.

3. *Let us take* $f(x) = \sqrt{x-5}$, $p = 14$ *and* $L = 3$ *be given. Let* $\epsilon > 0$. *We want to find a positive number* δ *such that for all* x $0 < |x - 4| < \delta$ $(x \neq 14, -\delta < x - 14 < \delta)$ *implies* $|f(x) - 3| < \epsilon$.

   We find δ *by working backward from the* ϵ*-inequality*

$$
\begin{aligned}
|f(x)-3| = |\sqrt{x-5}-3| &= \left|\frac{(\sqrt{x-5}-3)(\sqrt{x-5}+3)}{\sqrt{x-5}+3}\right| \\
&= \left|\frac{x-14}{\sqrt{x-5}+3}\right| < \left|\frac{x-14}{3}\right| < \epsilon,
\end{aligned}
$$

Then,

$$|x-14| < 3\epsilon.$$

Thus, we take $\delta = 3\epsilon$.

Then, whenever $|x-14| < \delta = 3\epsilon$, *it is true that* $|f(x)-3| < \left|\frac{x-14}{3}\right| < \frac{\delta}{3} = \frac{3\epsilon}{3} = \epsilon$,
Which proves that $\lim\limits_{x\to 14}\sqrt{x-5} = 3$.

Exercise 1.3. *If* $3x^2+3 \le f(x) \le x^3+7$ *for* $0 \le x \le 5$, *find* $\lim\limits_{x\to 2} f(x)$.

Solution. *Since*

$$\lim_{x\to 2}(3x^2+3) = 3\cdot 2^2+3 = 15 \quad \textit{and} \quad \lim_{x\to 2}(x^3+7) = 2^3+7 = 15.$$

The Sandwich (Squeeze) Theorem implies $\lim\limits_{x\to 2} f(x) = 15$.

Exercise 1.4. *It can be shown that the inequality*

$$1-\frac{x^2}{6} \le \frac{x\sin x}{2-2\cos x} \le 1$$

holds for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}$$

Give reasons for your answer.

Solution. *Since*

$$\lim_{x\to 0}\left(1-\frac{x^2}{6}\right) = 1 \quad \textit{and} \quad \lim_{x\to 0} 1 = 1$$

According to The Sandwich (Squeeze) Theorem

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x} = 1.$$

Exercise 1.5. *Evaluate the limits:*

1. $\lim\limits_{x\to -2^+}(x+3)\dfrac{|x+2|}{x+2}$,

2. $\lim\limits_{x\to -2^-}(x+3)\dfrac{|x+2|}{x+2}$.

Solution.

1. *If $x\to -2^+$, then $x>2$,*

   and $|x+2| = x+2$ if $x>2$. Therefore,

$$\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+}(x+3)\frac{(x+2)}{x+2} = \lim_{x\to -2^+}(x+3) = -2+3 = 1.$$

2. *If $x\to -2^-$, then $x<2$,*

   and $|x+2| = -(x+2)$ if $x<2$. Therefore,

$$\lim_{x\to -2^-}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^-}(x+3)\frac{-(x+2)}{x+2} = \lim_{x\to -2^-}(-x-3) = -(-2)-3 = -1.$$

Exercise 1.6. *Find the following limits:*

1. $\lim\limits_{x\to 0}\dfrac{\sin\sqrt{3}x}{\sqrt{3}x}$,

2. $\lim\limits_{x\to 0}\dfrac{\tan 5x}{x}$,

3. $\lim\limits_{x\to 0}\dfrac{x+x\cos x}{\sin x\cos x}$,

4. $\lim\limits_{x\to 0}\dfrac{x^3+x+\sin x}{2x}$,

5. $\lim\limits_{x\to 0}\dfrac{\sin 2x\cot 4x}{x\cot 3x}$.

Solution.

1. *We use the substution $\sqrt{3}x = t$,*

   then $x\to 0 \iff t\to 0$.

   Because of this,

$$\lim_{x\to 0}\frac{\sin\sqrt{3}x}{\sqrt{3}x} = \lim_{t\to 0}\frac{\sin t}{t} = 1.$$

   Note: *We will apply this substitution method for following limits.*

2. $\lim\limits_{x\to 0}\dfrac{\tan 5x}{x}=\lim\limits_{x\to 0}\dfrac{5\sin 5x}{5x\cos 5x}=5\cdot\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos 5x}=5\cdot 1\cdot 1=5.$

3. $\lim\limits_{x\to 0}\dfrac{x+x\cos x}{\sin x\cos x}=\lim\limits_{x\to 0}\dfrac{x(1+\cos x)}{\sin x\cos x}=\lim\limits_{x\to 0}\dfrac{1+\cos x}{\frac{\sin x}{x}\cdot\cos x}=\dfrac{1+1}{1\cdot 1}=2.$

4. $\lim\limits_{x\to 0}\dfrac{x^3+x+\sin x}{2x}=\lim\limits_{x\to 0}\left(\dfrac{x^2}{2}+\dfrac{1}{2}+\dfrac{\sin x}{2x}\right)=1.$

5. $\lim\limits_{x\to 0}\dfrac{\sin 2x\cot 4x}{x\cot 3x}=\lim\limits_{x\to 0}\dfrac{2\sin 2x}{2x}\cdot\dfrac{4x\cdot\cos 4x}{4\sin 4x}\cdot\dfrac{3\sin 3x}{3x\cdot\cos 3x}=2\cdot\dfrac{1}{4}\cdot 3=\dfrac{3}{2}.$

Exercise 1.7. *Let f be a function defined by*

$$f(x)=\begin{cases}\dfrac{1}{x}, & x<-2\\ x^2-5 & -2<x\le 3\\ \sqrt{x+13}, & x>3\end{cases}$$

Find:

1. $\lim\limits_{x\to -2} f(x)$,
2. $\lim\limits_{x\to 0} f(x)$.
3. $\lim\limits_{x\to 3} f(x)$.

Solution. *We will determine the stated two-sided limit by first considering the corresponding one-sided limits.*

1. $\lim\limits_{x\to -2^-} f(x)=\lim\limits_{x\to -2^-}\dfrac{1}{x}=-\dfrac{1}{2},$

   $\lim\limits_{x\to -2^+} f(x)=\lim\limits_{x\to -2^+}(x^2-5)=(-2)^2-5=-1.$

   From which it follows that $\lim\limits_{x\to -2} f(x)$ *does not exist.*

2. $f(x)=x^2-5$ *on both sides of* 0, *therefore,*

   $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}(x^2-5)=0^2-5=-5$

3. $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(x^2-5)=3^2-5=4,$

   $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}\sqrt{x+13}=\sqrt{3+13}=4.$

   Since left-sided and right sided limits are equal, 4, *we have* $\lim\limits_{x\to 3} f(x)=4$.

Exercise 1.8. *Let f be a function defined by*

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq 3 \\ k, & x = -3 \end{cases}$$

Find

1. *Find k so that $f(-3) = \lim\limits_{x \to -3} f(x)$*
2. *With k assigned the value $\lim\limits_{x \to -3} f(x)$, show that f can be expressed as a polynomial.*

Solution.

1. $k = f(-3) = \lim\limits_{x \to -3} f(x) = \lim\limits_{x \to -3} (x-3) = -3-3 = -6$
2. $$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq 3 \\ -6, & x = -3 \end{cases}$$

   That is, $f(x) = \dfrac{x^2-9}{x+3} = x-3$ if $x \neq 3$ and $f(x) = -6$ if $x = 3$. So, f is equivalent to the polynomial $g(x) = x-3$.

Exercise 1.9. *Prove that*

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

Solution. *First we calculate one sided limits,*

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0)} 1 = 1 \qquad (x > 0)$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{(-x)}{x} = \lim_{x \to 0^-} (-1) = -1 \qquad (x < 0)$$

Since the one-sided limits are different, $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

Exercise 1.10. *At what points are the functions defined by following expressions continuous?*

1. $f(x) = \dfrac{x+3}{x^2-3x+2}$

2. $g(x) = \dfrac{3x}{(x+7)^2} + 5$

3. $h(x) = |x-2| + \sin x$

4. $l(x) = \dfrac{3x}{|x|-6} + x^2 + 4$

Solution.

1. *Since any rational function is continuous on its domain, The function f defined by $f(x) = \dfrac{x+3}{x^2-3x+2}$ is continuous any points at which $x^2 - 3x + 2 = 0$. Let us solve $x^2 - 3x + 2 = 0$*

$$x^2 - 3x + 2 = 0 \iff (x-2)(x-1) = 0 \implies x = 2,\ x = 1.$$

Therefore, f is continuous on $\mathbb{R} \setminus \{1, 2\}$.

2. *A function defined by $m(x) = \dfrac{3x}{(x+7)^2}$ is continuous on the set of real numbers and a function defined by $n(x) = 5$ (constant function) is also continuous on $\mathbb{R}$, therefore g will be continuous on $\mathbb{R}/\{-7\}$.*

3. *Absolute value function and trigonometric function are continuous on their domains, because of this h is also continuous on $\mathbb{R}$*

4. *The function l is continuous on $\mathbb{R} \setminus \{-6, 6\}$.*

Exercise 1.11. *At what points is the function f defined by the following expression continuous?*

$$f(x) = \begin{cases} \dfrac{x^2-6x+8}{x-4}, & x \neq 4 \\ 2, & x = 4 \end{cases}$$

Solution. $f(x) = \dfrac{x^2-6x+8}{x-4} = \dfrac{(x-2)(x-4)}{x-4} = x - 2$ *if $x \neq 4$. Since every polynomial is continuous everywhere, $f(x)$ is continuous on $\mathbb{R} \setminus \{4\}$. Lets consider $x = 4$. Since*

$$\lim_{x \to 4} \frac{x^2-6x+8}{x-4} = \lim_{x \to 4}(x-2) = 2,$$

$$f(4) = 2$$

$f(x)$ is continuous at $x = 4$. Therefore, $f(x)$ is continuous everywhere.

Exercise 1.12. *At what points is the function f defined by*

$$f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9}, & x \neq -3 \ \text{ and } \ x \neq 3 \\ 9, & x = 3 \\ 1, & x = -3 \end{cases}$$

continuous?

Solution.

$$f(x) = \frac{x^3-27}{x^2-9} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \frac{x^2+3x+9}{x+3}$$

if $x \neq -3$ and $x \neq 3$. So f is continuous for $x \neq -3$ and $x \neq 3$.

Consider $x = -3$ and $x = 3$.

If $x = -3$,

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x^3-27}{x^2-9} = \lim_{x\to -3} \frac{x^2+3x+9}{x+3}$$

does not exist. Therefore, f is not continuous at $x = -3$.

If $x = 3$,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2+3x+9}{x+3} = \frac{3^2+3\cdot 3+9}{3+3} = \frac{27}{6} = \frac{9}{2} = 4.5 \neq 1 = f(-3)$$

Because of this f is not continuous at $x = 3$. Finally, f is continuous on $\mathbb{R} \setminus \{-3, 3\}$.

Exercise 1.13. *Let g be a function defined by $g(x) = \dfrac{x^2-25}{x-5}$. Define $g(5)$ in a way that extends g to be continuous at $x = 5$.*

Solution. *If $\lim\limits_{x\to c} f(x) = f(c)$ then it is continuous at c. Hence, since*

$$\lim_{x\to 5} g(x) = \lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5}(x+5) = 10$$

we define $g(5) = 10$, that extends g to be continuous at $x = 5$.

Exercise 1.14. *Let g be a function defined by $h(x) = \dfrac{x^2+3x-2}{x-2}$ Define h in a way that extends h to be continuous at $x = 2$.*

Solution. *Since*
$$\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2+3x-2}{x-2} = \lim_{x\to 2}(x+1) = 3$$
we define $h(2) = 3$, that extends h to be continuous at $x = 2$.

Exercise 1.15. *Let f be a function defined by*
$$f(x) = \begin{cases} x^2-3, & x < 3 \\ 4ax, & x \geq 3 \end{cases}$$
For what value of a is the function f continuous at every x?

Solution. *$f$is polynomial if $x \neq 3$, so it is continuous for $x \neq 3$.*
Consider $x = 3$.
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(x^2-3) = 6,$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 4ax = 12a,$$
$$f(3) = 4a \cdot 3 = 12a.$$
Since f is continuous if and only if it is right continuous and left continuous,
$$\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x).$$
Then, $6 = 12a = 12a$, and $a = 1/2$.

Exercise 1.16. *Let g be a function defined by*
$$g(x) = \begin{cases} \dfrac{x-b}{b+1}, & x \leq 0 \\ x^2+b, & x > 0 \end{cases}$$
For what value of b is the function g
continuous at every x?

Solution. *$g$is polynomial if $x \neq 0$, because of this it is continuous at $x \neq 0$.*
Consider $x = 0$.
$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{x-b}{b+1} = \frac{-b}{b+1},$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(x^2+b) = b,$$
$$g(0) = \frac{-b}{b+1}.$$
$$\lim_{x\to 0^-} g(x) = g(0) = \lim_{x\to 0^+} g(x).$$
Then, $\dfrac{-b}{b+1} = b = \dfrac{-b}{b+1}$, hence, $b = 0$ or $b = 2$.

Exercise 1.17. *Let f be a function defined by*

$$f(x) = \begin{cases} ax^3 + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

For what values of a and b is f
continuous at every x?

Solution. *$f$is polynomial if $x \neq 0$ and $x \neq 2$, because of this it is continuous at $x \neq 0$ and $x \neq 2$.*

Let us consider $x = 0$ and $x = 2$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (ax^3 + 2b) = 2b,$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 3a - b) = 3a - b,$$

$$f(0) = 2b.$$

If $\lim\limits_{x \to 0^-} f(x) = f(0) = \lim\limits_{x \to 0^+} f(x)$. then, f is continuous at $x = 0$. Hence, $2b = 2b = 3a - b$, finally, $a = b$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 3a - b) = 4 + 3a - b,$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 5) = 1,$$

$$f(2) = 4 + 3a - b.$$

$\lim\limits_{x \to 2^-} f(x) = f(2) = \lim\limits_{x \to 2^+} f(x)$. Then, $4 + 3a - b = 4 + 3a - b = 1$, so $b - 3a = 3$.

We get the system of equations

$$\begin{cases} a = b \\ b - 3a = 3 \end{cases}$$

Therefore $f(x)$ is continuous if $a = -3/2$ and $b = -3/2$.

Exercise 1.18. *Explain why the equation $\cos x = x$ has at least one solution.*

Solution. *Let us take the function f defined by $f(x) = \cos x - x$. A zero of f is the root of $\cos x = x$.*

If $x = -\frac{\pi}{2}$, $f(-\frac{\pi}{2}) = \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} > 0$.

If $x = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = -\frac{\pi}{2} < 0$.

Since f is continuous over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the Intermediate Value Theorem implies that there is some c on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that $f(c) = 0$. Therefore c is the solution of $\cos x = x$.

Exercise 1.19. *Show that there is a root of the equation* $x^3 + 3x^2 - x = 1$.

Solution. *Let* f *be a function defined by* $f(x) = x^3 + 3x^2 - x - 1$

If $x = 0$, $f(0) = -1 < 0$.

If $x = 1$, $f(1) = 1^3 + 3 \cdot 1^2 - 1 - 1 = 2 > 0$.

Since f *is continuous over the interval* $[0, 1]$, *the Intermediate Value Theorem implies that* $f(c) = 0$ *for some* $c \in [0, 1]$. *Therefore* c *is the solution of* $x^3 + 3x^2 - x = 1$.

Exercise 1.20. *Find the following limits:*

1. $\lim\limits_{x\to\infty} \left(3 + \dfrac{1}{x}\right)$,
2. $\lim\limits_{x\to\infty} \left(\dfrac{1}{x^3} + \dfrac{2}{x^2}\right)$.

Solution.

1. $\lim\limits_{x\to\infty} \left(3 + \dfrac{1}{x}\right) = \lim\limits_{x\to\infty} 3 + \lim\limits_{x\to\infty} \dfrac{1}{x} = 3 + 0 = 3$.
2. $\lim\limits_{x\to\infty} \left(\dfrac{1}{x^3} + \dfrac{2}{x^2}\right) = 0$.

Exercise 1.21. *Find the following limits:*

1. $\lim\limits_{x\to\infty} \dfrac{3x^2 + 2x - 1}{x^2 + 7}$,
2. $\lim\limits_{x\to\infty} \dfrac{5x^3 + 4x - 8}{2x^4 + 3x^3 - 5x}$,
3. $\lim\limits_{x\to-\infty} \dfrac{-x^2 - 6x + 1}{2x^4 - 3x^2 - 6}$,
4. $\lim\limits_{x\to-\infty} \dfrac{10x^5}{-2x^5 + x^4}$,
5. $\lim\limits_{x\to\infty} \sqrt{x - 5} - \sqrt{x - 7}$,
6. $\lim\limits_{x\to\infty} \sqrt{2x^2 - 2x} - \sqrt{2x^2 + 3x}$,
7. $\lim\limits_{x\to-\infty} \sqrt{x^2 + 8} + x$.

Solution.

First we divide the numerator and numerator by the highest power of the denominator:

1.

$$\lim_{x\to\infty}\frac{3x^2+2x-1}{x^2+7}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{7}{x^2}}=\lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{1}{x^2}}{1+\frac{7}{x^2}}=\frac{3+0}{1+0}=3.$$

2.

$$\lim_{x\to\infty}\frac{5x^3+4x-8}{2x^4+3x^3-5x}=\lim_{x\to\infty}\frac{\frac{5x^3}{x^4}+\frac{4x}{x^4}-\frac{8}{x^4}}{\frac{2x^4}{x^4}+\frac{3x^3}{x^4}-\frac{5x}{x^4}}=\lim_{x\to\infty}\frac{\frac{5}{x}+\frac{4}{x^3}-\frac{8}{x^4}}{2+\frac{3}{x}-\frac{5}{x^3}}=\frac{0-0+0}{2-0-0}=0.$$

3.

$$\lim_{x\to-\infty}\frac{-x^2-6x+1}{2x^4-3x^2-6}=\lim_{x\to-\infty}\frac{-\frac{x^2}{x^4}-\frac{6x}{x^4}+\frac{1}{x^4}}{\frac{2x^4}{x^4}-\frac{3x^2}{x^4}-\frac{6}{x^4}}=\lim_{x\to-\infty}\frac{-\frac{1}{x^2}-\frac{6}{x^3}+\frac{1}{x^4}}{2-\frac{3}{x^2}-\frac{6}{x^4}}=\frac{0}{2}=0.$$

4.

$$\lim_{x\to-\infty}\frac{10x^5}{-2x^5+x^4}=\lim_{x\to-\infty}\frac{\frac{10x^5}{x^5}}{\frac{-2x^5}{x^5}+\frac{x^4}{x^5}}=\lim_{x\to-\infty}\frac{10}{-2+\frac{1}{x}}=\frac{10}{-2}=-5.$$

5.

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x-5}-\sqrt{x-7}&=\lim_{x\to\infty}\frac{(\sqrt{x-5}-\sqrt{x-7})(\sqrt{x-5}+\sqrt{x-7})}{\sqrt{x-5}+\sqrt{x-7}}\\&=\lim_{x\to\infty}\frac{x-5-x+7}{\sqrt{x-5}+\sqrt{x-7}}=\lim_{x\to\infty}\frac{2}{\sqrt{x-5}+\sqrt{x-7}}=0.\end{aligned}$$

6.

$$\lim_{x\to\infty}\sqrt{2x^2-2x}-\sqrt{2x^2+3x}=\lim_{x\to\infty}\frac{(\sqrt{2x^2-2x}-\sqrt{2x^2+3x})(\sqrt{2x^2-2x}+\sqrt{2x^2+3x})}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}$$

$$=\lim_{x\to\infty}\frac{2x^2-2x-2x^2-3x}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}=\lim_{x\to\infty}\frac{-5x}{\sqrt{2x^2-2x}+\sqrt{2x^2+3x}}$$

$$=\lim_{x\to\infty}\frac{-5}{\sqrt{2-\frac{2}{x}}+\sqrt{2+\frac{3}{x}}}=\frac{-5}{2+2}=-\frac{5}{4}.$$

7.

$$\lim_{x\to-\infty}\sqrt{x^2+8}+x=\lim_{x\to-\infty}\frac{(\sqrt{x^2+8}+x)(\sqrt{x^2+8}-x)}{\sqrt{x^2+8}-x}=\lim_{x\to-\infty}\frac{x^2+8-x^2}{\sqrt{x^2+8}-x}$$
$$=\lim_{x\to-\infty}\frac{8}{|x|\sqrt{1+\frac{8}{x^2}}-x}=\lim_{x\to-\infty}\frac{8}{-x\sqrt{1+\frac{8}{x^2}}-x}=\lim_{x\to-\infty}\frac{8}{-x\left(\sqrt{1+\frac{8}{x^2}}+1\right)}=0.$$

Exercise 1.22. *Find the limits:*

1. $\lim\limits_{x\to0^+}\frac{2}{5x}$,
2. $\lim\limits_{x\to0^-}\frac{2}{5x}$,
3. $\lim\limits_{x\to6^+}\frac{1}{x-6}$,
4. $\lim\limits_{x\to0^-}\frac{1}{x-6}$.

Solution.

1. $\lim\limits_{x\to0^+}\frac{2}{5x}=+\infty.$ *(since the numerator and denominator are positive)*
2. $\lim\limits_{x\to0^-}\frac{2}{5x}=-\infty.$ *(since the numerator is positive and the denominator is negative)*
3. $\lim\limits_{x\to6^+}\frac{1}{x-6}=+\infty.$ *(since the numerator and denominator are positive)*
4. $\lim\limits_{x\to0^-}\frac{1}{x-6}=-\infty.$ *(since the numerator is positive and the denominator is negative)*

Exercise 1.23. *Let f be defined as follows. Find the horizontal asymptotes of the graph of f.*

1. $f(x)=\dfrac{4x^3+2x+1}{x^3+3x^2}$,
2. $f(x)=\dfrac{x^2}{-2x^2+6x+10}$,
3. $f(x)=\dfrac{|x|^3-3x}{2x^3+3x^2-1}$,

4. $f(x) = \dfrac{x^3 + x^2 - 4x - 6}{x + 3}$.

Solution.

1.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x^3 + 2x + 1}{x^3 + 3x^2} = \lim_{x\to-\infty} \frac{4 + \dfrac{2}{x^2} + \dfrac{1}{x^3}}{1 + \dfrac{3}{x}} = 4,$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^3 + 2x + 1}{x^3 + 3x^2} = \lim_{x\to\infty} \frac{4 + \dfrac{2}{x^2} + \dfrac{1}{x^3}}{1 + \dfrac{3}{x}} = 4.$$

These limits imply that the line of $y = 4$ is the horizontal asymptote of the graph of f on both the right and the left.

2.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{-2x^2 + 6x + 10} = -\frac{1}{2},$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{-2x^2 + 6x + 10} = -\frac{1}{2}.$$

Then the line of $y = -\frac{1}{2}$ is the horizontal asymptote of the graph of f on both the right and the left (or at $-\infty$ and $-\infty$).

3.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{|x|^3 - 3x}{2x^3 + 3x^2 - 1} = \lim_{x\to-\infty} \frac{-x^3 - 3x}{2x^3 + 3x^2 - 1} \quad (\textit{since} \quad x < 0)$$
$$= \lim_{x\to-\infty} \frac{-1 - \dfrac{3}{x^2}}{2 + \dfrac{3}{x} - \dfrac{1}{x^3}} = -\frac{1}{2}.$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{|x|^3 - 3x}{2x^3 + 3x^2 - 1} = \lim_{x\to\infty} \frac{x^3 - 3x}{2x^3 + 3x^2 - 1} \quad (\textit{since} \quad x > 0)$$
$$= \lim_{x\to\infty} \frac{1 - \dfrac{3}{x^2}}{2 + \dfrac{3}{x} - \dfrac{1}{x^3}} = \frac{1}{2}.$$

Theare are the horizontal asymptotes of $y = -\frac{1}{2}$ and $y = \frac{1}{2}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3+x^2-4x-6}{x+3} = \lim_{x\to\infty} \frac{x^2+x-4-\dfrac{6}{x}}{1+\dfrac{3}{x}} = \infty.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3+x^2-4x-6}{x+3} = \lim_{x\to-\infty} \frac{x^2+x-4-\dfrac{6}{x}}{1+\dfrac{3}{x}} = \infty.$$

The graph of f has no horizontal asymptote.

Exercise 1.24. *Find the vertical asymptotes of the graph of f defined by*

1. $f(x) = \dfrac{1}{x-2}$,
2. $f(x) = \dfrac{x^2-3x+2}{x^2-2x}$,
3. $f(x) = \sec x, \quad x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$,
4. $f(x) = \tan x$.

Solution.

1. *We consider the point 2, such that $\lim_{x\to 2}\left|\dfrac{1}{x-2}\right| = +\infty$. Since*

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{x-2} = -\infty,$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{x-2} = +\infty,$$

the line of the equation $x = 2$ is a vertical asymptote of f both from the right and from the left.

2. $f(x) = \dfrac{x^2-3x+2}{x^2-2x} = \dfrac{(x-2)(x-1)}{x(x-2)}$.

Since,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0^-} \frac{(x-2)(x-1)}{x(x-2)} = \lim_{x\to 0^-} \frac{x-1}{x} = -\infty,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x-2} \lim_{x\to 0^+} \frac{(x-2)(x-1)}{x(x-2)} = \lim_{x\to 0^+} \frac{x-1}{x} = +\infty,$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2^-} \frac{x-1}{x} = \frac{1}{2},$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 2^+} \frac{x-1}{x} = \frac{1}{2}.$$

The line of $x=0$ is a vertical asymptote of f both from the right and from the left.

3. $f(x) = \sec x = \dfrac{1}{\cos x}$, *and* $\cos x = 0$ *if* $x = -\dfrac{\pi}{2}$ *and* $x = \frac{\pi}{2}$, *so we find,*

$$\lim_{x\to -\frac{\pi}{2}^+} f(x) = \lim_{x\to -\frac{\pi}{2}^+} \sec = \lim_{x\to -\frac{\pi}{2}^+} \frac{1}{\cos x} = +\infty,$$
$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^-} \sec = \lim_{x\to \frac{\pi}{2}^-} \frac{1}{\cos x} = +\infty,$$

Because of these the lines of the equations $x = -\dfrac{\pi}{2}$ *is vertical asymptote of the graph of f from the left,* $x = -\dfrac{\pi}{2}$ *is vertical asymptotes of the graph of f from the right.*

4. $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, *and* $\cos x = 0$ *if* $x = \dfrac{\pi}{2} + \pi k$ *where* $k \in \mathbb{Z}$, *so we find,*

$$\lim_{x\to (\frac{\pi}{2}+\pi k)^+} f(x) = \lim_{x\to (\frac{\pi}{2}+\pi k)^+} \sec = \lim_{x\to (\frac{\pi}{2}+\pi k)^+} \frac{1}{cosx} = -\infty$$

$$\lim_{x\to (\frac{\pi}{2}+\pi k)^-} f(x) = \lim_{x\to (\frac{\pi}{2}+\pi k)^-} \sec = \lim_{x\to (\frac{\pi}{2}+\pi k)^-} \frac{1}{cosx} = +\infty$$

Because of this the lines of the equations $x = \dfrac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ *are vertical asymptotes of the graph of f both from the right and from the left.*

Exercise 1.25. *Find the oblique asymptote of the graph of f defined by*

1. $f(x) = \dfrac{2x^2+3x-1}{x-7}$,
2. $f(x) = \dfrac{x^3+3x^2-3}{x^2-2}$,
3. $f(x) = \sqrt{x^2+3x-1} - x$,
4. $f(x) = \dfrac{x^{3/2}+2x-4}{\sqrt{x}-1}$.

Solution.

1. *If the degree of the numerator of a rational function is* 1 *greater than the degree of the denominator, the graph has an oblique or slant line asymptote. We find an equation for the asymptote by dividing numerator by denominator to express it as a linear function plus a remainder that goes to zero as* $x \to \pm\infty$ *(this method is only for rational functions)*

$$f(x) = \frac{2x^2+3x-1}{x-7} = (2x+17) + \frac{118}{x-7},$$

where $\dfrac{118}{x-7} \to 0$ *as* $\quad x \to \pm\infty$.

Becaise of this the slant line defined by $y = 2x+17$ *is the oblique asymptote of the graph of* f.

2. *(General method) for the oblique asymptote which is defined by the expression* $y = ax+b$ *we define the slope as* $a = \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x}$ *and* b *as* $b = \lim\limits_{x\to\pm\infty}(f(x) - ax)$

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\dfrac{x^3+3x^2-3}{x^2-2}}{x} = \lim_{x\to-\infty} \frac{x^3+3x^2-3}{x^3-2x} = 1,$$

$$\lim_{x\to-\infty}\left(\frac{x^3+3x^2-3}{x^2-2} - x\right) = \lim_{x\to-\infty} \frac{3x^2+2x-3}{x^2-2} = 3,$$

so the slant line of the equation $y = x-3$ *is the oblique asymptote to the curve at* $-\infty$.

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\dfrac{x^3+3x^2-3}{x^2-2}}{x} = \lim_{x\to+\infty} \frac{x^3+3x^2-3}{x^3-2x} = 1,$$

$$\lim_{x\to+\infty}\left(\frac{x^3+3x^2-3}{x^2-2} - x\right) = \lim_{x\to+\infty} \frac{3x^2+2x-3}{x^2-2} = 3,$$

then the line defined by $y = x-3$ *is also the oblique asymptote of the curve at* $+\infty$.

3.

$$\begin{aligned} a = \lim_{x\to-\infty} \frac{f(x)}{x} &= \lim_{x\to-\infty} \frac{\sqrt{x^2+3x-1}-x}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}-x}{x} \\ &= \lim_{x\to-\infty} \frac{-x\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}-x}{x} = \lim_{x\to-\infty} \frac{-x\left(\sqrt{1+\dfrac{3}{x}-\dfrac{1}{x^2}}+1\right)}{x} \\ &= \lim_{x\to-\infty} -\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+1\right) = -2. \end{aligned}$$

$$
\begin{aligned}
b = \lim_{x\to-\infty}(f(x)-ax) &= \lim_{x\to-\infty}(\sqrt{x^2+3x-1}-x-(-2x)) = \lim_{x\to-\infty}(\sqrt{x^2+3x-1}+x) \\
&= \lim_{x\to-\infty}\frac{(\sqrt{x^2+3x-1}+x)(\sqrt{x^2+3x-1}-x)}{\sqrt{x^2+3x-1}-x} = \lim_{x\to-\infty}\frac{x^2+3x-1-x^2}{\sqrt{x^2+3x-1}-x} \\
= \lim_{x\to-\infty}\frac{3x-1}{\sqrt{x^2+3x-1}-x} &= \lim_{x\to-\infty}\frac{3x-1}{|x|\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-x} = \lim_{x\to-\infty}\frac{3x-1}{-x\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-x} \\
&= \lim_{x\to-\infty}\frac{3x-1}{-x\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+1\right)} = \lim_{x\to-\infty}\frac{-3+\frac{1}{x}}{\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+1} = -\frac{3}{2}.
\end{aligned}
$$

Then, the slant line of $y=-2x-\frac{3}{2}$ will be the oblique asymptote to f at $-\infty$.

$$
\begin{aligned}
a = \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\sqrt{x^2+3x-1}-x}{x} &= \lim_{x\to+\infty}\frac{|x|\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-x}{x} \\
= \lim_{x\to+\infty}\frac{x\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-x}{x} &= \lim_{x\to+\infty}\frac{x\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-1\right)}{x} \\
&= \lim_{x\to+\infty}\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}-1\right) = 0.
\end{aligned}
$$

Since $a=0$, the curve has no oblique asymptote.

$$
\begin{aligned}
b = \lim_{x\to+\infty}(f(x)-ax) &= \lim_{x\to+\infty}(\sqrt{x^2+3x-1}-x-0) = \lim_{x\to+\infty}(\sqrt{x^2+3x-1}-x) \\
&= \lim_{x\to+\infty}\frac{(\sqrt{x^2+3x-1}-x)(\sqrt{x^2+3x-1}+x)}{\sqrt{x^2+3x-1}+x} = \lim_{x\to+\infty}\frac{x^2+3x-1-x^2}{\sqrt{x^2+3x-1}+x} \\
= \lim_{x\to+\infty}\frac{3x-1}{\sqrt{x^2+3x-1}+x} &= \lim_{x\to+\infty}\frac{3x-1}{|x|\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+x} = \lim_{x\to+\infty}\frac{3x-1}{x\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+x} \\
&= \lim_{x\to+\infty}\frac{3x-1}{x\left(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+1\right)} = \lim_{x\to+\infty}\frac{3-\frac{1}{x}}{\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+1} = \frac{3}{2}.
\end{aligned}
$$

Therefore, it has a horizontal asymptote defined by $y=\dfrac{3}{2}$ at $+\infty$.

4. *The function is defined on for all $x \geq 0$, because of this we investigate a limit at $+\infty$*

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\dfrac{x^{3/2}+2x-4}{\sqrt{x}-1}}{x} = \lim_{x\to+\infty} \frac{x^{3/2}+2x-4}{x^{3/2}-x} = 1.$$

$$b = \lim_{x\to+\infty}(f(x)-ax) = \lim_{x\to+\infty}\left(\frac{x^{3/2}+2x-4}{\sqrt{x}-1} - x\right)$$

$$= \lim_{x\to+\infty} \frac{3x-4}{\sqrt{x}-1} = \lim_{x\to+\infty} \frac{3\sqrt{x}-\dfrac{4}{\sqrt{x}}}{1-\dfrac{1}{\sqrt{x}}} = +\infty.$$

Since $b = +\infty$, the curve does not an oblique asymptote.