Fall 2021 - MATH 1101 Discrete Structures
Lecture 6

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Introduction.

Mathematical induction is one of the powerful proof methods in Mathematics and Computer Science. A major goal of Part 1 is to give a thorough understanding of **Principle of Mathematical Induction I** (or, Regular Induction, RI). In Part 2 we introduce and discuss the **Principle of Mathematical Induction II** (or Strong Induction, SI) which is equivalent to RI.

There is the third equivalent form of Mathematical Induction called as **Well-Ordering Property** (WOP) which we will discuss in the next **Lecture 7** together with Structural Induction: a technique for proving results about recursively defined sets.

PART 1. PRINCIPLE OF MATHEMATICAL INDUCTION I (RI).

Mathematical induction being one of the powerful proof techniques in Mathematics and Computer Science can be used to prove a tremendous variety of results. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities in Mathematics.

Understanding how to construct proof by mathematical induction is one of the key goals of learning discrete mathematics.

We describe **how** mathematical induction can be used and **why** it is a valid proof technique. It is extremely important to note that *mathematical induction can be used only to prove results obtained in some other way. It is not a tool for discovering formulae or theorems.*

Principle of Mathematical Induction I

(=Regular Induction, RI; = Mathematical Induction, MI).

Let P(n), n∈**N,** be a propositional function with domain **N** (set of positive integers), that is, **for each n∈N, P(n) is a proposition depending on n, which is either true or false.**

In general, mathematical induction can be used to prove statements that assert that

$$P(n) \textit{ is true for all positive integers } n.$$

A proof by mathematical induction has two parts, a **basis step**, where we show that P(1) is true, and an **inductive step**, where we show that for all positive integers k, **if P(k) is true, then P(k+1) is true**. Therefore, we can accumulate steps of Principle of Mathematical Induction I in the following table.

Principle of Mathematical Induction I (or MI, or RI).

To prove that P(n) is true for all positive integers n, where P (n) is a propositional function, we complete two steps:

Basis Step. Verify that P(1) is true.

Inductive Step. Show that if P(k) is true, then P(k+1) is true for all positive integers k.

Or, equivalently, we show that the conditional statement $P(k)\rightarrow P(k+1)$ is true for all positive integers k.

In other words, in inductive step we need to show that P(k+1) cannot be false when P(k) is true. This can be accomplished by assuming that P(k) is true and showing that *under this hypothesis* P(k+1) must also be true.

Using the Propositional Algebra language, the Regular Induction technique can be stated as the following compound proposition

$$[P(1)\wedge\forall k(P(k)\rightarrow P(k+1))]\rightarrow\forall nP(n),$$

when the domain is the set of positive integers.

Often, we will need to show that P(n) is true for n=b, b+1, b+2, ... , where b is an integer other than 1.We can use regular induction to accomplish this, as long as we change the basis step by replacing P(1) with P(b). In other words, to use mathematical induction to show that P(n) is true for n=b, b+1, b+2, ... , where b is an integer other than 1, we show that P(b) is true in the basis step. In the inductive step, we show that if P(k) is true, then P(k+1) is true for k=b, b+1, b+2, ... **Note that b can be negative, zero, or positive.**

We summarize discussions around Regular Induction as the following template.

Template for Proofs by Regular Induction

1. Express the statement that is to be proved in the form "for all n≥b, P(n)" for a fixed integer b.
2. Write out the words "Basis Step." Then show that P(b) is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
   - 3.1. State, and clearly identify, the inductive hypothesis, in the form "assume that P(k) is true for an arbitrary fixed integer k≥b."
   - 3.2. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what P(k+1) says.
   - 3.3. Prove the statement P(k+1) making use the assumption P(k). Be sure that your proof is valid for all integers k with k≥b, taking care that the proof works for small values of k, including k=b.
   - 3.4. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
4. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, P(n) is true for all integers n with n≥b

Remark: **In a proof by regular induction it is *not* assumed that P(k) is true for all positive integers k≥b! It is only shown that *if it is assumed* that P(k) is true, then P(k+1) is also true.**

An important point needs to be made about requar induction before we commence a study of its use. The good thing about regular induction is that it can be used to prove a conjecture once it is having been made (and is true). The bad thing about it is that it cannot be used to find new theorems. Mathematicians sometimes find proofs by regular induction unsatisfying because they do not provide insights as to why theorems are true. Many theorems can be proved in many ways, including by regular induction. Proofs of these theorems by methods other than regular induction are often preferred because of the insights they bring.

EXAMPLE 1. Show that if n is a positive integer, then

$1+2+\cdots+n=\frac{n(n+1)}{2}$.

Solution: Let P(n) be the propositional function that the sum of the first n positive integers, 1+2+⋯+n is n(n+1)/2, that is, P(n)="1+2+⋯+n=n(n+1)/2". We must do two things to prove that P(n) is true for n=1, 2, 3, ... , . Namely,

- We must show that P(1) is true and
- If P(k) is true, then P(k+1) is true for an arbitrary positive integer k.

Basis Step. P(1) is true, because $1=\frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in n(n+1)/2.)

Inductive Step. For the inductive hypothesis we assume that P (k) holds for an arbitrary positive integer k. That is, we assume that

$$1+2+\cdots+k=\frac{k(k+1)}{2}$$

Under this assumption, it must be shown that P(k+1) is true, namely, that

$$1+2+\cdots+k+(k+1)=\frac{(k+1)\big((k+1)+1\big)}{2}=\frac{(k+1)(k+2)}{2}$$

is also true. When we add k+1 to both sides of the equation in P(k), we obtain

$$\begin{aligned}1+2+\cdots+k+(k+1)&=\frac{k(k+1)}{2}+(k+1)\\&=\frac{k(k+1)+2(k+1)}{2}\\&=\frac{(k+1)(k+2)}{2}\end{aligned}$$

This last equation shows that P(k+1) is true under the assumption that P(k) is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that P(n) is true for all positive integers n. That is, we have proven that 1+2+⋯+n=n(n+1)/2 for all positive integers n. ■

As we noted, *mathematical induction is not a tool for finding theorems about all positive integers. Rather, it is a proof method for proving such results once they are conjectured.* In Example 2, using mathematical induction to prove a summation formula, we will both formulate and then prove a conjecture.

EXAMPLE 2. Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$1=1$; $1+3=4$; $1+3+5=9$; $1+3+5+7=16$; $1+3+5+7+9=25$.

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is, $1+3+5+\cdots+(2n-1)=n^2$. We need a method to *prove* that this *conjecture* is correct, if in fact it is.

Let P(n) denote the propositional function that the sum of the first n odd positive integers is n^2, that is $P(n)=$"$1+3+5+\cdots+(2n-1)=n^2$". Our conjecture: P(n) is true for all positive integers. To use mathematical induction to prove this conjecture, we must first complete the basis step; that is, we must show that P(1) is true. Then we must carry out the inductive step; that is, we must show that P(k+1) is true when P(k) is assumed to be true. We now attempt to complete these two steps.

Basis Step. P(1) states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1. The basis step is complete.

Inductive Step. To complete the inductive step we must show that P(k+1) is true when P(k) is assumed to be true for every positive integer k. To do this, we first assume the inductive hypothesis. The inductive hypothesis is the statement that P(k) is true for an arbitrary positive integer k, that is,

$1+3+5+\cdots+(2k-1)=k^2$

(Note that the kth odd positive integer is $(2k-1)$, because this integer is obtained by adding 2 a total of $k-1$ times to 1). Note that P(k+1) is the statement that

$1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2$.

Assuming that P(k) is true, it follows that

$1+3+5+\ldots+(2k-1)+(2k+1)=[1+3+\ldots+(2k-1)]+(2k+1)=k^2+(2k+1)=k^2+2k+1=(k+1)^2$.

This shows that P(k+1) follows from P(k). Note that we used the inductive hypothesis P(k) in the second equality to replace the sum of the first k odd positive integers by k^2.

We have now completed both the basis step and the inductive step. That is, we have shown that P(1) is true and the conditional statement $P(k)\rightarrow P(k+1)$ is true for all positive integers k. Consequently, by the principle of mathematical induction we can conclude that P(n) is true for all positive integers n. That is, we know that $1+3+5+\cdots+(2n-1)=n^2$ for all positive integers n. ■

Proving Inequalities.

Mathematical induction can be used to prove a variety of inequalities that hold for all positive integers greater than a given particular positive integer.

EXAMPLE 3. Use mathematical induction to prove that $n<2^n$ for all positive integers n.

Solution: Let P(n) be the propositional function defined as $P(n)=$"$n<2^n$".

Basis Step. P(1) is true, because $1<2^1=2$. This completes the basis step.

Inductive Step. We first assume the inductive hypothesis that P(k) is true for an arbitrary positive integer k. That is, the inductive hypothesis P(k) is the statement that $k<2^k$. To complete the inductive step, we need to show that if P (k) is true, then P(k+1), which is the statement that $k+1<2^{k+1}$, is true. That is, we need to show that if $k<2^k$, then $k+1<2^{k+1}$ for the positive integer k. To show it we first add 1 to both sides of $k<2^k$, and then note that $1\le 2^k$. This tells us that $k+1<2^k+1\le 2^k+2^k=2\cdot 2^k=2^{k+1}$

This shows that P(k+1) is true, namely, that $k+1<2^{k+1}$, based on the assumption that P(k) is true. The induction step is complete.

Therefore, because we have completed both the basis step and the inductive step, we have shown that $n<2^n$ is true for all positive integers n. ■

EXAMPLE 4. Use mathematical induction to prove that $2^n<n!$ for every integer n with $n\geq4$. (Note that this inequality is false for n=1, 2, and 3.). In other words, factorial function growth faster than exponential one.

Solution: Let P(n) be the propositional function that $2^n<n!$, or P(n)="$2^n<n!$"

Basis Step. To prove the inequality for $n \geq 4$ requires that the basis step be P(4). Note that P(4) is true, because $2^4=16<24=4!$

Inductive Step. For the inductive step, we assume that P(k) is true for an arbitrary integer k with $k\geq4$. That is, we assume that $2^k<k!$ for the positive integer k with $k\geq4$. We must show that under this hypothesis, P(k+1) is also true. That is, we must show that if $2^k<k!$ for an arbitrary positive integer k where $k\geq4$, then $2^{k+1}<(k+1)!$. We have

$2^{k+1}=2 \cdot 2^k$	(by definition of exponent)
$<2\cdot k!$	(by the inductive hypothesis)
$<(k+1)k!$	(because $2<k+1$)
$=(k+1)!$	(by definition of factorial function).

This shows that P(k+1) is true when P(k) is true. This completes the inductive step of the proof. We have completed both steps. Hence, by mathematical induction P(n) is true for all integers n with $n\geq4$. That is, we have proved that $2^n<n!$ is true for all integers n with $n\geq4$. ■

Application to Sets.

Mathematical induction can be used to prove many results about sets.

EXAMPLE 5. Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Solution: Let P(n) be the propositional function defined as P(n)="a set with n elements has 2^n subsets".

Basis Step. P(0) is true, because a set with zero elements, the empty set, has exactly $2^0=1$ subset, namely, itself.

Inductive Step. For the inductive hypothesis we assume that P(k) is true for an arbitrary nonnegative integer k, that is, we assume that every set with k elements has 2^k subsets. It must be shown that under this assumption, P(k+1), which is the statement that every set with k+1 elements has 2^{k+1} subsets, must also be true. To show this, let T be a set with k+1 elements. Then, it is possible to write $T=S\cup\{a\}$, where a is one of the elements of T and $S=T-\{a\}$ (and hence $|S|=k$). The subsets of T can be obtained in the following way. For each subset X of S there are exactly two subsets of T, namely, X and $X\cup\{a\}$. These constitute all the subsets of T and are all distinct. We now use the inductive hypothesis to conclude that S has 2^k subsets, because it has k elements. We also know that there are two subsets of T for each subset of S. Therefore, there are $2\cdot2^k=2^{k+1}$ subsets of T. This finishes the inductive argument.

Because we have completed the basis step and the inductive step, by mathematical induction it follows that P (n) is true for all nonnegative integers n. That is, we have proved that a set with n elements has 2^n subsets whenever n is a nonnegative integer. ■

PART 2. PRINCIPLE OF MATHEMATICAL INDUCTION II
(STRONG INDUCTION, SI)

In this section we introduce another equivalent form of mathematical induction, called **Strong Induction**, which can often be used when we cannot easily prove a result using Mathematical (Regular) Induction. The basis step of a proof by Strong Induction is the same as a proof of the same result using Mathematical Induction. That is, in a Strong Induction, a proof that P(n) is true for all positive integers n, the basis step shows that P(1) is true. However, the inductive steps in these two proof methods are different. In a proof by Mathematical Induction, the inductive step shows that *if the inductive hypothesis* P(k) *is true, then* P(k+1) *is also true*. In a proof by Strong Induction, the inductive step shows that *if* P(j) *is true for all positive integers not exceeding k, then* P(k+1) *is true*. That is, for the inductive hypothesis we assume that P(j) is true for j=1, 2, ..., k.

We prove (see Theorem 1 below) that both forms of induction are equivalent. This means that a proof using one of these two principles can be rewritten as a proof using another one. Just as it is sometimes the case that it is much easier to see how to prove a result using Strong Induction rather than Mathematical Induction.

Principle of Mathematical Induction II (=Strong Induction).

To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete two steps:

Basis Step. Verify that the proposition P(1) is true.

Inductive Step. Show that **if** all P(1), P(2), …, P(k) are true **then** P(k+1) is true for all positive integers k.

Or, equivalently, we show that the conditional statement $[P(1)\wedge P(2)\wedge\cdots\wedge P(k)]\rightarrow P(k+1)$ is true for all positive integers k.

Note that when we use Strong Induction to prove that P(n) is true for all positive integers n, our inductive hypothesis is the assumption that P(j) is true for j=1, 2, ... , k. That is, the inductive hypothesis includes all k statements P(1), P(2), ... , P(k). Because we can use all k statements P(1), P(2), ... , P(k) to prove P(k+1), rather than just the statement P(k) as in a proof by Mathematical Induction, Strong Induction is a more flexible proof technique.

Theorem 1. Strong Induction **is equivalent** to Regular (Mathematical) Induction.

Proof.

1. **Strong Induction ⇒ Regular Induction.**

Hypothesis: Assume that two following statements:

- P(1) is true and $(*_1)$
- $[P(1)\wedge P(2)\wedge\cdots\wedge P(k)]\rightarrow P(k+1)$ is true for all positive integers k $(*_2)$

imply that P(n) is true for all positive n.

Conclusion: We **must prove** that two statements

- P(1) is true and $(*_3)$

- $P(k) \rightarrow P(k+1)$ is true for all positive integer k $(*_4)$

imply that P(n) is true for all positive n.

In other words, **having $(*_3)$ and $(*_4)$ we must show that P(n) is true for all positive n.**

Really, under k=1, from:

- ✓ $(*_3)$ which is: P(1) is true,
- ✓ $(*_4)$ which is: $P(1) \rightarrow P(2)$ is true, and
- ✓ definition of conditional statement

we obtain that P(2) must be true.

Now, since P(2) is true and $P(2) \rightarrow P(3)$ is true, therefore P(3) must be true by definition of conditional statement. By similar arguments we find out that all P(1), P(2), …, P(k) are true for all positive k. Hence, the pair $(*_3)$ and $(*_4)$ implies the validity of the pair $(*_1)$ and $(*_2)$. Hence, both steps of Strong Induction are satisfied. So P(n) is true for all positive n as claimed.

Conversely,

2. Regular Induction ⇒ Strong Induction

Hypothesis: Assume that two following statements:

- P(1) is true and $(*_3)$
- $P(k) \rightarrow P(k+1)$ is true for all positive integer k $(*_4)$

imply that P(n) is true for all positive n.

Conclusion: We **must prove** that two statements

- P(1) is true and $(*_1)$
- $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k $(*_2)$

imply that P(n) is true for all positive n.

In other words, **having $(*_1)$ and $(*_2)$ we must show that P(n) is true for all positive n.**

Really, condition $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ is true means P(k) is true. Hence, the pair $[(*_1), (*_2)]$ implies the validity of the pair $[(*_3), (*_4)]$. Hence, both steps of Regular Induction are satisfied. So P(n) is true for all positive n as claimed. ■

Below we provide several examples how to apply Strong Induction.

EXAMPLE 6. Use **Strong Induction** to show that any integer n>1 can be written as the product of primes.

Solution: Let P(n) be the propositional function defined as P(n)="the integer n can be written as the product of primes".

<u>Basis Step.</u> P(2) is true, because 2 can be written as the product of one prime, itself. (Note that P(2) is the first case we need to establish.)

<u>Inductive Step.</u> The inductive hypothesis is the assumption that P(j) is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k. To complete the inductive step, it must be shown

that P(k+1) is true under this assumption, that is, that (k+1) is the product of primes.
There are two cases to consider, namely, when (k+1) is prime and when (k+1) is composite.

Case 1. If (k+1) is prime, we immediately see that P(k+1) is true.

Case 2. If (k+1) is composite then it can be written as the product of two positive integers a and b, k+1=ab, with $2 \leq a \leq b < k+1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if (k+1) is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.

Remark: Because 1 can be thought of as the *empty* product of no primes, we could have started the proof in Example 10 with P(1) as the basis step. We chose not to do so because many people find this confusing. ■

Note. It can be shown that an integer has at most one factorization into primes. Example 6 shows there is at least one such factorization. These two results, considered together, prove the fundamental theorem of arithmetic, which asserts that every nonnegative integer can be written uniquely as the product of primes in nondecreasing order.

EXAMPLE 7. It is known that Emin from the class Discrete Structures can run *one km or two km*. Use **Strong Induction** to show that if Emin can always run *two more km* once he has run a specified number of km, then Emin can run any number of kilometers.

Solution. Let P(n) be the statement: P(n)=”Emin can run *n* km”.

Basis step: We are told Emin can run one km, so P(1) is true.

Inductive step: Assume the inductive hypothesis, that Emin can run *any number of km from 1 to k.* We must prove that Emin can run (k+1) km. Two cases are possible:

Case 1. k=1. Then k+1=2 and since we are already told that Emin can run two km, so the problem solved.

Case 2. k>1. By the inductive hypothesis Emin can run *any number of km from 1 to k* , therefore, he can run (k–1) km. On the other hand, by the hypothesis of the Example, Emin can always run two km more, so he can run $(k-1)+2=k+1$ kilometers as claimed ■

We can slightly modify Strong Induction to handle a wider variety of situations. We can adapt Strong Induction to handle cases where the inductive step is valid only for integers greater than a particular integer.

Alternative form in the Strong Induction. Let b be a fixed integer and j a fixed positive integer. The form of Strong Induction we need tells us that *P(n) is true for all integers n with n≥b if we can complete these two steps*:

Basis Step. Verify that the propositions P(b), P(b+1), ..., P(b+j) are true.

Inductive Step.

Show that if all P(b), P(b+1), ..., P(k) are true for every integer $k \geq b+j$ then P(k+1) is also true.

Or, equivalently: we show that $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq b+j$.

It can be proved that the alternative form is equivalent to strong induction.

EXAMPLE 8. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using SI.

Solution. First, we must create a conjecture then use strong induction to prove it. It is clear that \$1 and \$3 cannot be formed and that \$2 and \$4 can be formed. Consider amounts starting from 4 dollars.

\$4=2(2-dollars);   \$5=1(5-dollars);   \$6=3(2-dollars);   \$7=1(2-dollars)+1(5-dollars);

\$8=4(2-dollars);   \$9=2(2-dollars)+1(5-dollars);   \$10=2(5-dollars) an so on.

Conjecture. Starting from amount of \$4 any amount can be formed using 2 and 5-dollars bills.

Below we prove this conjecture using strong induction.

Let P(n) be the statement that we can form n dollars using just 2-dollar and 5-dollar bills. We want to prove that P(n) is true for all $n \geq 5$.

Basis step: For the basis step, note that 4=2+2, 5=5, and 6=2+2+2.

Inductive step: Assume the inductive hypothesis, that P(j) is true for all j with $4 \leq j \leq k$, where $k \geq 6$. We want to show that P(k+1) is true, that is, any amount of (k+1) dollars can be formed using 2 and 5-dollars bills. Since $k \geq 6$ so $(k-1) \geq 5$. On the other hand, by inductive hypothesis, we know that P(k−1) is true, that is, we can form (k−1) dollars using two-dollar bills and five-dollar bills. Add another 2-dollar bill, and we have formed (k+1) dollars. ■

EXAMPLE 9. Let P(n) be the propositional function (statement) as following: P(n)="postage of n cents can be formed using just 3-cent stamps and 5-cent stamps". Below are parts of a strong induction proof that P(n) is true for $n \geq 8$.

a) Show that the statements P(8), P(9), and P(10) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 10$.

e) Explain why these steps show that this statement is true whenever $n \geq 8$.

Solution.

a) P(8) is true, because we can form 8 cents of postage with one 3-cent stamp and one 5-cent stamp. P(9) is true, because we can form 9 cents of postage with three 3-cent stamps. P(10) is true, because we can form 10 cents of postage with two 5-cent stamps.

b) The statement that using just 3-cent and 5-cent stamps we can form j cents postage for all j with $8 \leq j \leq k$, where we assume that $k \geq 10$.

c) Assuming the inductive hypothesis, we can form k+1 cents postage using just 3-cent and 5-cent stamps.

d) Because $k \geq 10$, we know that P(k−2) is true, that is, that we can form k−2 cents of postage. Put one more 3-cent stamp on the envelope, and we have formed k+1 cents of postage.

e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 8. ■

Some results can be readily proved using either MI or SI (see section EXERCISES. SET 1 (Solved Problems), exercises 1.5, 1.6).

EXAMPLE 10. Let P(n) be a propositional function. Determine for which positive integers n the statement P(n) must be true, and justify your answer, if

a) P(1) is true; for all positive integers n, if P(n) is true, then P(n+2) is true.

b) P(1) and P(2) are true; for all positive integers n, if P(n) and P(n+1) are true, then P(n+2) is true.

c) P(1) is true; for all positive integers n, if P(n) is true, then P(2n) is true.

d) P(1) is true; for all positive integers n, if P(n) is true, then P(n+1) is true.

Solution.

a) The inductive step here allows us to conclude that P(3), P(5), ... are all true, but we can conclude nothing about P(2), P(4), ...

b) Using strong induction, we establish that P(n) is true for all positive integers n.

c) The inductive step here enables us to conclude that P(2), P(4), P(8), P(16), …are all true, but we can conclude nothing about P(n) when n is not a power of 2.

d) This is mathematical induction; we can conclude that P(n) is true for all $n\in$**N.** ■

We must be very careful when applying RI or SI as a method of proof. An incorrect declaration of a propositional function or an erroneously specified basic step, or, especially, in most cases, an inductive step may give incorrect results a priori.

EXAMPLE 11. What is wrong with this "proof" that "all horses are the same color"?

Let P(n) be the propositional function: "all the horses in a set of n horses are the same color".

"Proof".

Basis Step: Clearly, P(1) is true.

Inductive Step: Assume that P(k) is true, so that all the horses in any set of k horses are the same color. Consider any k+1 horses; number these as horses 1, 2, 3, . . . , k, k+1. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Because the set of the first k horses and the set of the last k horses overlap, all k+1 must be the same color. This shows that P(k+1) is true and finishes the proof by induction.

Solution. The two sets do not overlap if n+1=2: conditional statement $P(1)\rightarrow P(2)$ is false. ■

EXAMPLE 12. What is wrong with this ***"proof"*** by **Strong Induction**?

"Theorem". For every nonnegative integer n, 5n=0.

"Proof".

Basis Step: $5\cdot 0=0$.

Inductive Step: Suppose that 5j=0 for all nonnegative integers j with $0\le j\le k$. Write k+1=i+j, where i and j are natural numbers less than k+1. By the inductive hypothesis, 5(k+1)=5(i+j)=5i+5j=0+0=0.

Solution. The error is in going from the base case n=0 to the next case, n=1; we cannot write 1 as the sum of two smaller natural numbers. ■

Note. The reader interested in **using Strong Induction in computational geometry** is strongly encouraged to study the example from the textbook [2] (pages 338-340). To make our Lecture Notes as convenient as possible, we copy here the corresponding part from [2].

EXAMPLE (Additional reading – optional!)

Computational geometry is a part of discrete mathematics that studies computational problems involving geometric objects. Computational geometry is used extensively in computer graphics, computer games, robotics, scientific calculations, and a vast array of other areas.

First, we introduce some terminology. A **polygon** is a closed geometric figure consisting of a sequence of line segments s_1, s_2, ..., s_n, called **sides**. Each pair of consecutive sides, s_i and s_{i+1}, i=1, 2, ... , n−1, as well as the last side s_n and the first side s_1, of the polygon meet at a common endpoint, called a **vertex**. A polygon is called **simple** if no two nonconsecutive sides intersect.

Every simple polygon divides the plane into two regions: its **interior**, consisting of the points inside the curve, and its **exterior**, consisting of the points outside the curve. This last fact is complicated to prove. It is a special case of the famous Jordan curve theorem, which tells us that every simple curve divides the plane into two regions.

A polygon is called **convex** if every line segment connecting two points in the interior of the polygon lies entirely inside the polygon. (A polygon that is not convex is said to be **nonconvex**.) Figure 1 displays some polygons; polygons (a) and (b) are convex, but polygons (c) and (d) are not. A **diagonal** of a simple polygon is a line segment connecting two nonconsecutive vertices of the polygon, and a diagonal is called an **interior diagonal** if it lies entirely inside the polygon, except for its endpoints. For example, in polygon (d), the line segment connecting a and f is an interior diagonal, but the line segment connecting a and d is a diagonal that is not an interior diagonal.

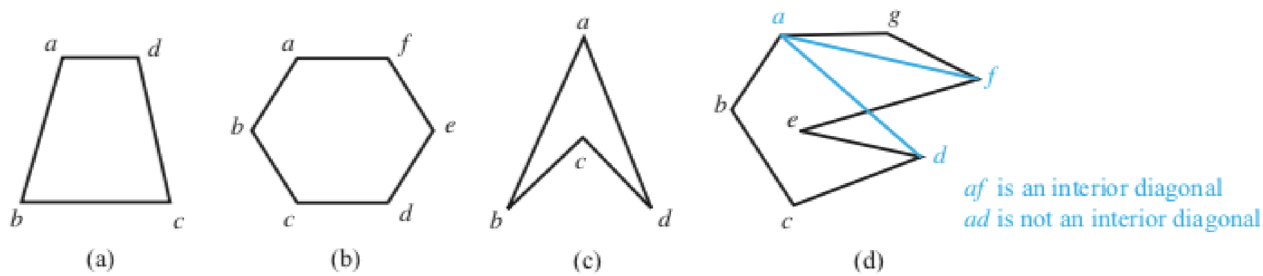

FIGURE 1. Convex and Nonconvex Polygons.

One of the most basic operations of computational geometry involves dividing a simple polygon into triangles by adding nonintersecting diagonals. This process is called **triangulation**. Note that a simple polygon can have many different triangulations, as shown in Figure 2.

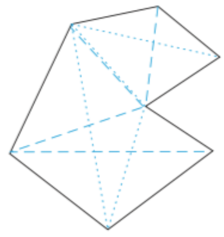

FIGURE 2 **Triangulations of a Polygon.**

Two different triangulations of a simple polygon with seven sides into five triangles, shown with dotted lines and with dashed lines, respectively.

Perhaps the most basic fact in computational geometry is that it is possible to triangulate every simple polygon, as we state in Theorem 2. Furthermore, this theorem tells us that every triangulation of a simple polygon with n sides includes $n-2$ triangles.

Theorem 2. *A simple polygon with n sides, where n is an integer with $n \geq 3$, can be triangulated into $n-2$ triangles.*

It seems obvious that we should be able to triangulate a simple polygon by successively adding interior diagonals. Consequently, a proof by strong induction seems promising. However, such a proof requires this crucial lemma.

Lemma. *Every simple polygon with at least four sides has an interior diagonal.*

Although Lemma seems particularly simple, it is surprisingly tricky to prove. There is a variety of incorrect proofs thought to be correct were commonly seen in books and articles.

Proof of the Theorem 2. We will prove this result using strong induction. Let T (n) be the statement that every simple polygon with n sides can be triangulated into $n-2$ triangles.

Basis Step: T(3) is true because a simple polygon with three sides is a triangle. We do not need to add any diagonals to triangulate a triangle; it is already triangulated into one triangle, itself. Consequently, every simple polygon with $n=3$ has can be triangulated into $n-2=3-2=1$ triangle.

Inductive Step: For the inductive hypothesis, we assume that T(j) is true for all integers j with $3 \leq j \leq k$. That is, we assume that we can triangulate a simple polygon with j sides into $(j-2)$ triangles whenever $3 \leq j \leq k$. To complete the inductive step, we must show that when we assume the inductive hypothesis, P(k+1) is true, that is, that every simple polygon with (k+1) sides can be triangulated into $(k+1)-2=k-1$ triangles.

So, suppose that we have a simple polygon P with (k+1) sides. Because $k+1 \geq 4$, Lemma tells us that P has an interior diagonal *ab*. Now, *ab* splits P into two simple polygons Q, with s sides, and R, with t sides. The sides of Q and R are the sides of P, together with the side *ab*, which is a side of both Q and R. Note that $3 \leq s \leq k$ and $3 \leq t \leq k$ because both Q and R have at least one fewer side than P does (after all, each of these is formed from P by deleting at least two sides and replacing these sides by the diagonal *ab*). Furthermore, the number of sides of P is two less than the sum of the numbers of sides of Q and the number of sides of R, because each side of P is a side of either Q or of R, but not both, and the diagonal *ab* is a side of both Q and R, but not P. That is, $k+1=s+t-2$.

We now use the inductive hypothesis. Because both $3 \leq s \leq k$ and $3 \leq t \leq k$, by the inductive hypothesis we can triangulate Q and R into $s-2$ and $t-2$ triangles, respectively. Next, note that these triangulations together produce a triangulation of P. (Each diagonal added to triangulate one of these smaller polygons is also a diagonal of P.) Consequently, we can triangulate P into a total of $(s-2)+(t-2)=s+t-4=(k+1)-2$ triangles. This completes the proof by strong induction: we have shown that every simple polygon with n sides, $n \geq 3$, can be triangulated into $n-2$ triangles. ■

Proof of Lemma. Suppose that P is a simple polygon drawn in the plane (Figure 3).

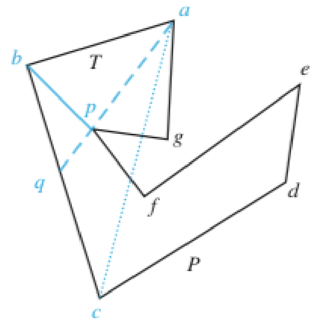

FIGURE 3. Constructing an Interior Diagonal of a Simple Polygon.

T is the triangle *abc* *p* is the vertex of *P* inside *T* such that the $\angle bap$ is smallest *bp* must be an interior diagonal of *P*.

Furthermore, suppose that *b* is the point of P or in the interior of P with the least *y*-coordinate among the vertices with the smallest *x*-coordinate. Then b must be a vertex of P, for if it is an interior point, there would have to be a vertex of P with a smaller *x*-coordinate. Two other vertices each share an edge with b, say *a* and c. It follows that the angle in the interior of P formed by *ab* and *bc* must be less than 180 degrees (otherwise, there would be points of P with smaller *x*-coordinates than *b*).

Now let T be the triangle $\triangle abc$. If there are no vertices of P on or inside T, we can connect *a* and *c* to obtain an interior diagonal. On the other hand, if there are vertices of P inside T, we will find a vertex *p* of P on or inside T such that *bp* is an interior diagonal. (This is the tricky part. *Ho* noted that in many published proofs of this lemma a vertex *p* was found such that *bp* was not necessarily an interior diagonal of P.) The key is to select a vertex *p* such that the angle $\angle bap$ is smallest. To see this, note that the ray starting at *a* and passing through *p* hits the line segment *bc* at a point, say *q*. It then follows that the triangle $\triangle baq$ cannot contain any vertices of P in its interior. Hence, we can connect *b* and *p* to produce an interior diagonal of P. Locating this vertex *p* is illustrated in Figure 3. ■

EXERCISES. SET 1 (Solved Problems).

Mathematical Induction

1.1. Prove that for all nonnegative integers n, $n\geq 0$ the identity $1+2+2^2+2^3+\ldots+2^n=2^{n+1}-1$ is true.

Solution. Let P(n)="$1+2+2^2+2^3+\ldots+2^n=2^{n+1}-1$".

Basis step: P(0) is true since $1=2^1-1$.

Inductive step: Assuming P(k) is true, we add 2^{k+1} to both sides of P(k), obtaining

$$1+2+2^2+2^3+\ldots+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$$
$$1+2+2^2+2^3+\ldots+2^k+2^{k+1}=2(2^{k+1})-1$$
$$1+2+2^2+2^3+\ldots+2^{k+1}=2^{k+2}-1$$

Last identity which is P(k+1). That is, P(k+1) is true whenever P(k) is true. By the principle of induction, P(n) is true for all n. ■

1.2. Prove that 5 divides n^5-n whenever n is a nonnegative integer.

Solution. Let P(n) is defined as "n^5-n is divisible by 5."

Basis step: P(0) is true because $0^5-0=0$ is divisible by 5.

Inductive step: Assume that P(k) is true, that is, k^5-5 is divisible by 5. Then $(k+1)^5-(k+1)=(k^5+5k^4+10k^3+10k^2+5k+1)-(k+1)=(k^5-k)+5(k^4+2k^3+2k^2+k)$ is also divisible by 5, because both terms in this sum are divisible by 5. ■

1.3. Prove using **mathematical induction** that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Note. We have already proved this result at the end of Part 3 (see discussions immediately after identity (2′)). Below we provide a new proof based on mathematical induction.

Solution. Let P(n) be the statement that a set with n elements has $n(n-1)/2$ two-element subsets.

Basis step: P(2), the basis case, is true, because a set with two elements has one subset with two elements - namely, itself - and $2(2-1)/2=1$.

Inductive step: Now assume that P(k) is true. Let S be a set with (k+1) elements. Choose an element a in S and let $T=S-\{a\}$. A two-element subset of S either contains a or does not. Those subsets not containing a are the subsets of T with two elements; by the inductive hypothesis there are $k(k-1)/2$ of these. There are k subsets of S with two elements that contain a, because such a subset contains a and one of the k elements in T. Hence, there are $k(k-1)/2+k=(k+1)k/2$ two-element subsets of S. This completes the inductive proof. ■

1.4. Let P(n) be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline **a strong induction proof** that P(n) is true for $n\geq 8$.

a) Show that the statements P(8), P(9), and P(10) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k\geq 10$.

e) Explain why these steps show that this statement is true whenever $n\geq 8$.

Solution.

Basis step:

a) P(8) is true, because we can form 8 cents of postage with one 3-cent stamp and one 5-cent stamp. P(9) is true, because we can form 9 cents of postage with three 3-cent stamps. P(10) is true, because we can form 10 cents of postage with two 5-cent stamps.

Inductive step:

b) The statement that using just 3-cent and 5-cent stamps we can form j cents postage for all j with $8\leq j\leq k$, where we assume that $k\geq 10$

c) Assuming the inductive hypothesis, we can form (k+1) cents postage using just 3-cent and 5-cent stamps

d) Because $k\geq 10$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed k+1 cents of postage.

e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 8. ■

1.5. Prove using the **mathematical induction** that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution. Let P(**n**) be the statement: P(**n**) = "**postage of n cents can be formed using 4-cent and 5-cent stamps".**

Basis Step. Postage of 12 cents can be formed using three 4-cent stamps.

Inductive Step. The inductive hypothesis is the statement that P(k) is true, where $k\geq 12$. That is, under this hypothesis, postage of k cents can be formed using 4-cent and 5-cent stamps. To complete the inductive step, we need to show that when we assume P(k) is true, then

P(k+1) is also true where $k \geq 12$. That is, we need to show that if we can form postage of k cents using 4-cent and 5-cent stamps, then we can form postage of k+1 cents using 4-cent and 5-cent stamps. So, assume the inductive hypothesis is true; that is, assume that we can form postage of k cents using 4-cent and 5-cent stamps.

We consider two cases,

- when at least one 4-cent stamp has been used and
- when no 4-cent stamps have been used.

Case 1. Suppose that **at least one 4-cent stamp** was used to form postage of k cents. Then we can replace this stamp with a 5-cent stamp to form postage of k+1 cents.

Case 2. **No 4-cent stamps** were used to form postage of k cents. We can form postage of k cents using only 5-cent stamps. Moreover, because $k \geq 12$, we needed at least **three** 5-cent stamps to form postage of k cents. So, we can replace three 5-cent stamps with four 4-cent stamps to form postage of k+1 cents. This completes the inductive step.

Because we have completed the basis step and the inductive step, we know that P(n) is true for all $n \geq 12$. That is, we can form postage of n cents, where $n \geq 12$ using just 4-cent and 5-cent stamps. This completes the proof by mathematical induction. ∎

1.6. Prove using the **strong mathematical induction** that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution. Let P(**n**) be the statement: P(**n**) = "**postage of n cents can be formed using 4-cent and 5-cent stamps".**

Basis Step. We can form postage of 12, 13, 14, and 15 cents using the following combinations:

12 cent – 3×4 cent+0×5 cent

13 cent – 2×4 cent+1×5 cent

14 cent – 1×4 cent+2×5 cent

15 cent – 0×4 cent+3×5 cent

This shows that P(12), P(13), P(14), and P(15) are true. This completes the basis step.

Inductive Step. The inductive hypothesis is the statement that **P(j) is true for $12 \leq j \leq k$, where k is an integer with $k \geq 15$**. To complete the inductive step, **we assume that we can form postage of j cents, where $12 \leq j \leq k$, using 4-cent and 5-cent stamps.** We need to show that under the assumption P(k+1) is true. That is, we can also form postage of (k+1) cents **using 4-cent and 5-cent stamps.**

Using the inductive hypothesis, we can assume that P(k−3) is true because $k-3 \geq 12$, that is, we can form postage of k−3 cents using just 4-cent and 5-cent stamps. To form postage of (k+1) cents, we need only add another 4-cent stamp to the stamps we used to form postage of k−3 cents. That is, we have shown that if the inductive hypothesis is true, then P(k+1) is also true. This completes the inductive step.

Because we have completed the basis step and the inductive step of a strong induction proof, we know by strong induction that P(n) is true for all integers n with $n \geq 12$. That is, we know that every postage of n cents, where n is at least 12, can be formed using 4-cent and 5-cent stamps. This finishes the proof by strong induction. ∎

EXERCISES. SET 2. (Supplementary Problems)

2.1. Prove by Mathematical Induction that if $A_1, A_2, ..., A_n$ and B are sets, then $(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$.

2.2. Prove by Mathematical Induction that if $A_1, A_2, ..., A_n$ and B are sets, then $(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap An) - B$.

2.3. Prove: $2+4+6+\cdots+2n=n(n+1)$

2.4. Prove: $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$

2.5. Prove: $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

2.6. Prove: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}$

2.7. Prove: $\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\cdots+\frac{1}{(4n-3)\cdot(4n+1)}=\frac{n}{4n+1}$

2.8. Prove 7^n-2^n is divisible by 5 for all $n \in \mathbf{N}$

2.9. Prove n^3-4n+6 is divisible by 3 for all $n \in \mathbf{N}$

2.10. Use the identity $1+2+3+\cdots+n=n(n+1)/2$ to prove that $1^3+2^3+3^3+\cdots+n^3=(1+2+3+\cdots+n)^2$

2.11. Sums of Geometric Progressions. Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term (a) and common ratio (r):

$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1}-a}{r-1}$ when $r\neq 1$, where n is nonnegative integer.

2.12. Prove that $1\cdot 1!+2\cdot 2!+\ldots+n\cdot n! = (n+1)!-1$ whenever n is a positive integer.

2.13. Prove that if $h>-1$, then $1+nh\leq(1+h)n$ for all nonnegative integers n.

2.14. Suppose that m and n are positive integers with $m>n$ and $f:\{1, 2, ..., m\}\rightarrow\{1, 2, ..., n\}$. Use mathematical induction on the variable n to show that f is not one-to-one.

2.15. Let P(n) be the statement that $n!<n^n$, where n is an integer greater than 1. Use mathematical induction to show that statement is true for all $n\geq 2$.

2.16. Prove that for every positive integer n,

$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)=n(n+1)(n+2)(n+3)/4$.

2.17. Prove that $n^2-7n+12$ is nonnegative whenever n is an integer with $n\geq 3$.

2.18. Prove that $3^n<n!$ if n is an integer greater than 6.

2.19. Prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever n is a positive integer.

2.20. Suppose that a and b are real numbers with $0<b<a$. Prove that if n is a positive integer, then $a^n-b^n\leq na^{n-1}(a-b)$.

2.21. Prove that 6 divides n^3-n whenever n is a nonnegative integer.

2.22. Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

2.23. Use mathematical induction to prove that the derivative of $f(x)=x^n$ equals nx^{n-1} whenever n is a positive integer.

Hint. For the inductive step, use the product rule for derivatives.

2.24. Suppose that we want to prove that for all positive integers n.

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-1}{2n}<\frac{1}{\sqrt{3n}}$$

a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

b) Show that mathematical induction can be used to prove the stronger inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

for all n>1, which, together with a verification for the case where n=1, establishes the weaker inequality we originally tried to prove using mathematical induction.

2.24. Let P(n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for $n \geq 18$.

a) Show statements P(18), P(19), P(20), and P(21) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

2.25. Suppose that P(n) is a propositional function. Determine for which nonnegative integers n the statement P(n) must be true if

a) P(0) is true; for all nonnegative integers n, if P(n) is true, then P(n+2) is true.

b) P(0) is true; for all nonnegative integers n, if P(n) is true, then P(n+3) is true.

c) P(0) and P(1) are true; for all nonnegative integers n, if P(n) and P(n+1) are true, then P(n+2) is true.

d) P(0) is true; for all nonnegative integers n, if P(n) is true, then P(n+2) and P(n+3) are true.

2.26. What is wrong with this "proof"?

"Theorem". For every positive integer n, if x and y are positive integers with max(x, y)=n, then x=y.

Basis Step: Suppose that n=1. If max(x, y)=1 and x and y are positive integers, we have x=1 and y=1.

Inductive Step: Let k be a positive integer. Assume that whenever max(x, y)=k and x and y are positive integers, then x=y. Now let max(x, y)=k+1, where x and y are positive integers. Then max(x−1, y−1)=k, so by the inductive hypothesis, x−1=y−1. It follows that x=y, completing the inductive step.

2.27. Find the **flaw** with the following "proof" that $a^n=1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0=1$ is true by the definition of a^0.

Inductive Step: Assume that $a^j=1$ for all nonnegative integers j with $j \leq k$. Then note that

$a^{k+1}=\frac{a^k \cdot a^k}{a^{k-1}}=\frac{1 \cdot 1}{1}=1.$

2.28. Find the **flaw** with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of (k+1) cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.