Area Between Two Curves

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BHOS

Calculus

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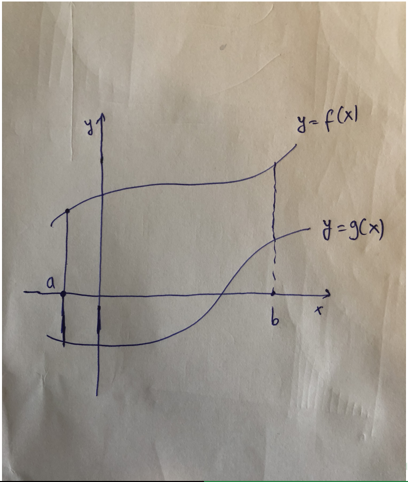

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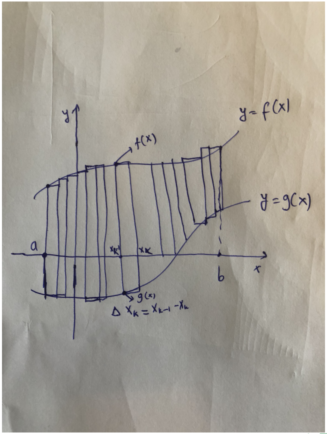

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$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n}[f(c_k) - g(c_k)]\Delta x_k = \int_a^b [f(x) - g(x)]dx$$

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Then you can integrate the function $f - g$ for the area between the intersections.

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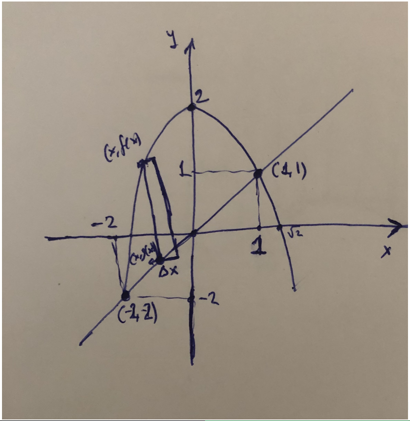

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$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx = \int_{-2}^{1} [2 - x^2 - x]dx \\ &= 2x - \frac{x^3}{3} - \frac{x^2}{2}\Big|_{-2}^{1} \\ &= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - \frac{4}{2}\right) = 5 - \frac{1}{2} = 9/2. \end{aligned}$$

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Then we integrate to find the area of each subregion and sum them up.

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The upper boundary is the graph of $f(x) = \sqrt{x}$ while the lowr boundary changes from $g(x) = 0$ for $x \in [0, 2]$ to $g(x) = x - 2$ for $x \in [2, 4]$.

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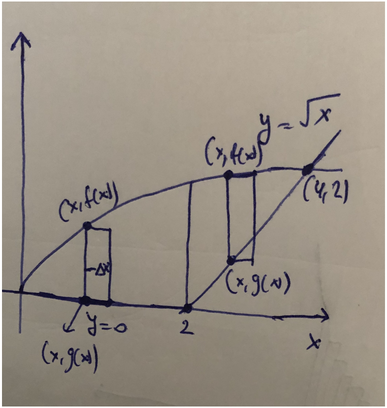

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To find the right-hand limit we solve $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x :

$$\begin{aligned} \sqrt{x} &= x - 2 \\ x &= x^2 - 4x + 4 \\ x^2 - 5x + 4 &= 0 \\ (x-4)(x-1) &= 0 \\ x = 1, \quad x &= 4 \end{aligned}$$

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Only $x = 4$ satisfies the equation. So, the right-hand limit is $b = 4$.

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Summing the area of the regions A_1 and A_2 we find the total area:

$$
\begin{aligned}
A = A_1 + A_2 &= \int_0^2 \sqrt{x}dx + \int_2^4 (\sqrt{x} - x + 2)dx \\
&= \frac{2}{3}x^{3/2}\Big|_0^2 + (\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x)\Big|_2^4 = 10/3.
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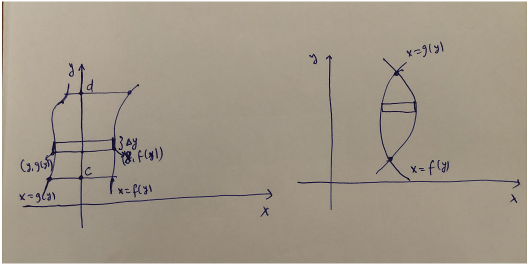

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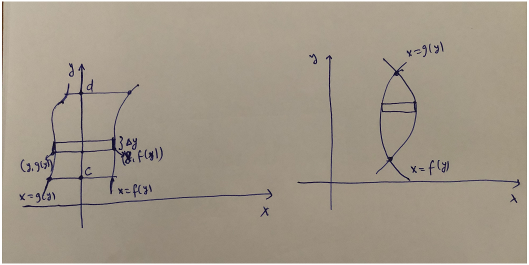

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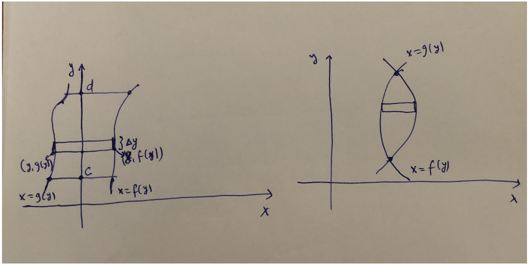

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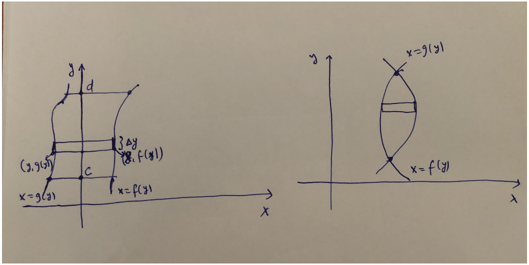

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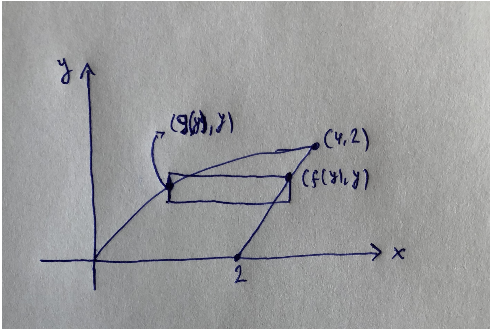
Here f always denotes the right-hand curve and g denotes the left-hand curve, so $f(y) - g(y)$ is always non-negative.

Area between two curves

Example:
Evaluate the region in the previous example by integrating with respect to y.
Solution
Let us first sketch the region and a typical **horizontal** rectangle based on a partition of y values



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The upper limit of integration is $d = 2$. ($y = -1$ is the intersection point below the x axis)

Area between two curves

The area of the region is

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\begin{aligned}
A &= \int_c^d [f(y) - g(y)]dy = \int_0^2 [y + 2 - y^2)]dy \\
&= \left(\frac{y^2}{2} + 2y - \frac{y^3}{3} \right) \Bigg|_0^2 \\
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If integrating with respect to x needs to do more work, then we can switch to the integrating with respect to y.

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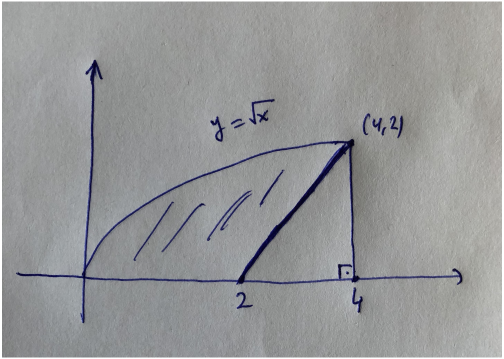

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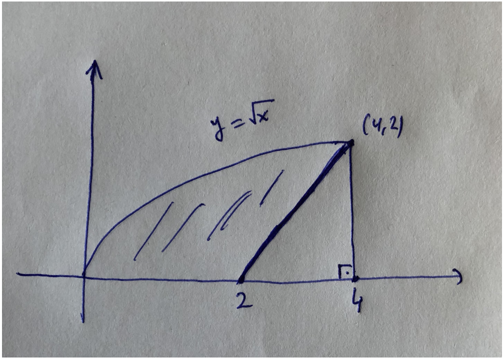

Looking at the figure we see that the desired region is the area between the curve $y = \sqrt{x}$ for $x \in [0, 4]$, minus the area of the right triangle with height and base of 2.

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Therefore the area is

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\begin{aligned}
A &= \int_0^4 \sqrt{x}dx - \frac{1}{2}(2)(2) \\
&= \frac{2}{3}x^{3/2}\Big|_0^4 - 2 \\
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Again we get the same result by doing even less work.