Differentiation Rules

Nijat Aliyev

BHOS

Calculus

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Differentiation Rules

Calculate the following limit

$$\lim_{x \to 0} \frac{\sqrt{3x+1} - 1}{x}$$

Differentiation Rules

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$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(x) = 1$$

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

Differentiation Rules

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If $f(x) = x^n$, and n is any positive integer, then

$$\frac{df}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}.$$

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If $f(x) = x^n$, and n is any positive integer, then

$$\frac{df}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof: Note that,

$$z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

So,

$$\begin{aligned} f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} &= \lim_{z \to x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \to x} \frac{(z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{z - x} \\ &= \lim_{z \to x} (z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-}) = nx^{n-1}. \end{aligned}$$

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for all x where the powers x^n and x^{n-1} are defined.

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Example: Find the derivative of the following functions

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

Differentiation Rules

Solutions:

(a) $\dfrac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}x^{(2/3)-1} = \dfrac{2}{3}x^{-1/3}$

(c) $\dfrac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right) = \dfrac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\dfrac{4}{x^5}$

(e) $\dfrac{d}{dx}(x^{-4/3}) = -\dfrac{4}{3}x^{-(4/3)-1} = -\dfrac{4}{3}x^{-7/3}$

(f) $\dfrac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \dfrac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \dfrac{\pi}{2}\right)x^{1+(\pi/2)-1} = \dfrac{1}{2}(2 + \pi)\sqrt{x^\pi}$

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Proof:

Let $g(x) = cf(x)$. Then

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= cf'(x)$$

Differentiation Rules

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$$\text{(a)} \quad \frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$$

$$\text{(b)} \quad \frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x] = (-1)\frac{d}{dx}(x) = -1(1) = -1$$

Differentiation Rules

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If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof:

Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x)
\end{aligned}$$

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6. **The Difference Rule**

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

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6. **The Difference Rule**

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Example:

$$\begin{aligned}
&\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5) \\
&\qquad = \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5) \\
&\qquad = 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0 \\
&\qquad = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6
\end{aligned}$$

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Example: Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

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Solution:

Horizontal tangents occur where the derivative is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$$

$$= 4x^3 - 12x + 0 = 4x(x^2 - 3)$$

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$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$$
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So the curve has horizontal tangents when $x = 0, x = \sqrt{3}, -\sqrt{3}$

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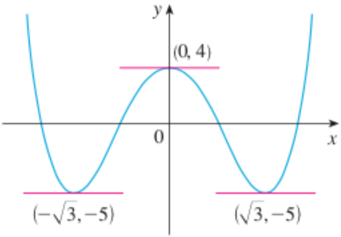

Differentiation Rules

7. **Product Rule** If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Differentiation Rules

7. Product Rule If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Proof:

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)[f(x+h) - f(x)] + f(x)[g(x+h) - g(x)]}{h} \\
&= \lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}$$

Example:

(a) If $f(x) = xe^x$, find $f'(x)$.
(b) Find the nth derivative, $f^{(n)}(x)$.

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Solution:

(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^x)$$

$$= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1 = (x + 1)e^x$$

(b) Using the Product Rule a second time, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx}[(x+1)e^x] \\ &= (x+1)\frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x+1) \\ &= (x+1)e^x + e^x \cdot 1 = (x+2)e^x \end{aligned}$$

Further applications of the Product Rule give

$$f'''(x) = (x+3)e^x \qquad f^{(4)}(x) = (x+4)e^x$$

In fact, each successive differentiation adds another term e^x, so

$$f^{(n)}(x) = (x+n)e^x$$

Differentiation Rules

8. Quotient Rule: If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g^2(x)}$$

provided that $g(x) \neq 0$.

Example:

$$\text{Let } y = \frac{x^2 + x - 2}{x^3 + 6}. \text{ Then}$$

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Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$\begin{aligned}
y' &= \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2} \\
&= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2} \\
&= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2} \\
&= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}
\end{aligned}$$

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According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1+x^2)\dfrac{d}{dx}(e^x) - e^x\dfrac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2}$$

$$= \frac{e^x(1-x)^2}{(1+x^2)^2}$$

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$$= \frac{e^x(1-x)^2}{(1+x^2)^2}$$

So the slope of the tangent line at $(1, \frac{1}{2}e)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 0$$

This means that the tangent line at $(1, \frac{1}{2}e)$ is horizontal and its equation is $y = \frac{1}{2}e$.

Table of Differentiation Rules

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$