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BHOS

Calculus

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**DEFINITION** The **instantaneous rate of change** of $f$ with respect to $x$ at $x_0$ is the derivative

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When $D = 10$ m, the area is changing with respect to the diameter at the rate of $(\pi/2)10 = 5\pi \text{ m}^2/\text{m} \approx 15.71 \text{ m}^2/\text{m}$. ■

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Velocity (instantaneous velocity) is the derivative of position w.r.t time:

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

## Derivative of Trigonometric Functions

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**Proof:** We use the following identity

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If $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.
\end{aligned}$$

**Example:** Find the following derivatives.

**(a)** $y = x^2 - \sin x$:
$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) = 2x - \cos x$$

**(b)** $y = x^2 \sin x$:
$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x = x^2 \cos x + 2x \sin x.$$

**(c)** $y = \dfrac{\sin x}{x}$:
$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

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**Proof:** Using the identity $\cos(x+h) = \cos x \cos h - \sin x \sin h$

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

**Example:** Find the following derivatives

**(a)** $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x)$$
$$= 5 - \sin x.$$

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**(a)** $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x)$$
$$= 5 - \sin x.$$

**(b)** $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$
$$= \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos^2 x - \sin^2 x$$

**(c)** $y = \dfrac{\cos x}{1 - \sin x}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1 - \sin x)\dfrac{d}{dx}(\cos x) - \cos x\dfrac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \\
&= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\
&= \frac{1 - \sin x}{(1 - \sin x)^2} \\
&= \frac{1}{1 - \sin x}
\end{aligned}$$

Using the identities:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

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we can derive the derivatives of the other trigonometric functions

**The derivatives of the other trigonometric functions:**

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

**Example:** Find $\frac{d}{dx}(\tan x)$.

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**Solution :** We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

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**Example:** Find $y''$ if $y = \sec x$.

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**Solution :** $y = \sec x \Longrightarrow y' = \sec x \tan x \Longrightarrow y'' = \frac{d}{dx}(\sec x \tan x)$.

$$\begin{aligned} y'' &= \frac{d}{dx}(\sec x \tan x) \\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \\ &= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\ &= \sec^3 x + \sec x \tan^2 x \end{aligned}$$

**Example:** Find $y''$ if $y = \sec x$.

**Solution :** $y = \sec x \implies y' = \sec x \tan x \implies y'' = \frac{d}{dx}(\sec x \tan x)$.

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