Review of Functions

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BHOS

Calculus

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Function

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The set of all actual output values of $f(x)$ as x varies throughout the domain of f is called the range of . and is denoted by R_f.

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Note:

1. The domain and the range of a function can be any sets of objects but usually they are real numbers (in calculus course).

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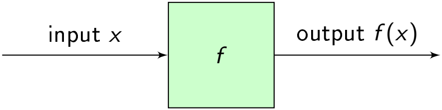

Figure 1: A diagram showing a function as a kind of machine.

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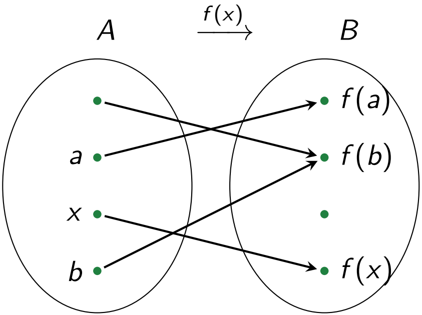

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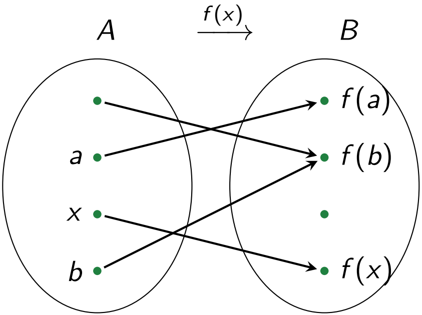

Figure 2: A function f from a set A to a set B.

Domain of f is A, codomain is B, the range is $\{f(x), f(a), f(b)\}$

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Notice that a function can have the same value at two different input elements in the domain (as occurs with $f(b)$ in Figure 2), but each input element x is assigned a single output value $f(x)$.

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Domain and range of some functions

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Function	Domain	Range
$f(x) = x^2$	$\mathbb{R} = (-\infty, \infty)$	$[0, \infty)$
$f(x) = \frac{1}{x}$	$\mathbb{R} \setminus \{0\}$	$\mathbb{R} \setminus \{0\}$
$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$f(x) = \sqrt{1-x}$	$[1, \infty)$	$[0, \infty)$
$f(x) = \sqrt{1-x^2}$	$[0, \infty)$	$[0, 1]$

Graphs of function

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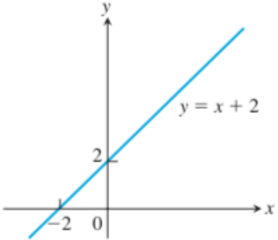

Figure 3: The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

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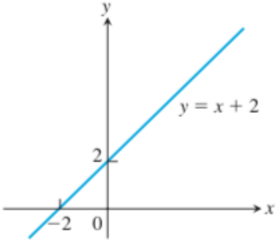

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If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

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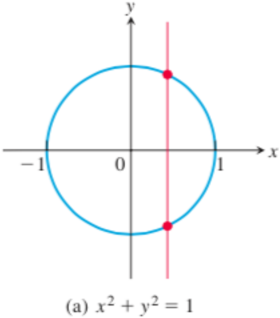

Figure 4: The circle $x^2 + y^2 = 1$ is not the graph of a function, since it fails vertical line test.

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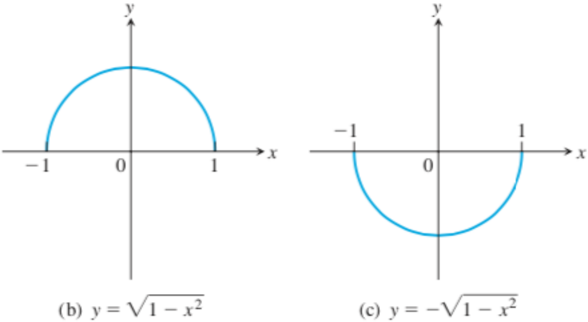

Figure 5: The upper semi-circle is the graph of the function $y = \sqrt{1 - x^2}$, while the lower semi-circle is the graph of the function $y = -\sqrt{1 - x^2}$.

Piecewise-Defined Functions

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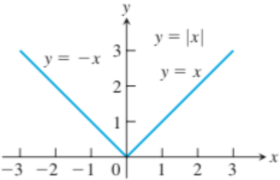

Figure 6: The absolute value function has domain (∞, ∞) and range $[0, \infty)$.

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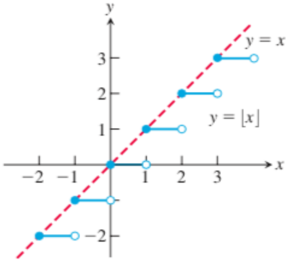

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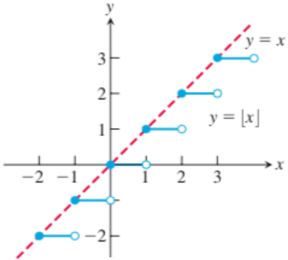

Figure 7: The graph of $y = \lfloor x \rfloor$ lies on or below the line $y = x$.

$\lfloor 2.3 \rfloor = 2$, $\lfloor 1.7 \rfloor = 1$, $\lfloor 0 \rfloor = 0$, $\lfloor 5 \rfloor = 5$, $\lfloor -0.3 \rfloor = -1$, $\lfloor 0.15 \rfloor = 0$, $\lfloor -4 \rfloor = -4$,

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Definition

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.
1. If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is said to be increasing on I.
2. If $f(x_2) < f(x1)$ whenever $x_2 > x_1$, then f is said to be decreasing on I.

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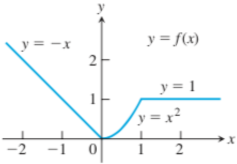

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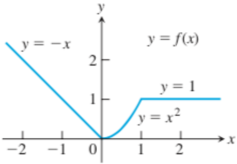

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Symmetry: Even and Odd Functions

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A function $y = f(x)$ is said to be an

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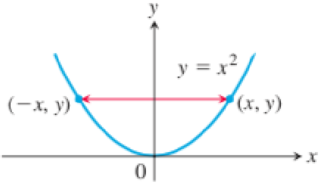

Figure 9: The graph of $y = x^2$ (an even function) is symmetric about the $y-$axis.

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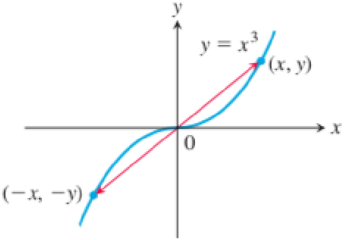

Figure 10: The graph of $y = x^3$ (an odd function) is symmetric about the origin.

Composite Functions

Definition

If f and g are functions, the composite function $f \circ g$ (" f composed with g ") is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f :

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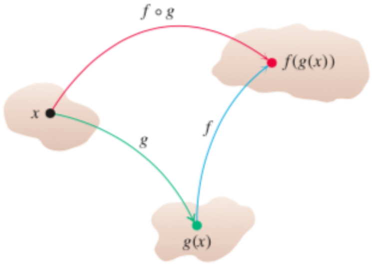

Figure 11: Arrow diagram for $f \circ g$

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Solution

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

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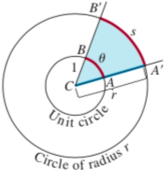

FIGURE [redacted] The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

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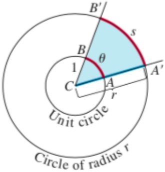

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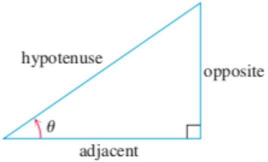

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Trigonometric Functions

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Trigonometric Functions

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Periods of Trigonometric Functions

Period π:

$$\tan(x+\pi) = \tan x$$
$$\cot(x+\pi) = \cot x$$

Period 2π:

$$\sin(x+2\pi) = \sin x$$
$$\cos(x+2\pi) = \cos x$$
$$\sec(x+2\pi) = \sec x$$
$$\csc(x+2\pi) = \csc x$$

Trigonometric Functions

Even and odd trigonometric functions:

Trigonometric Functions

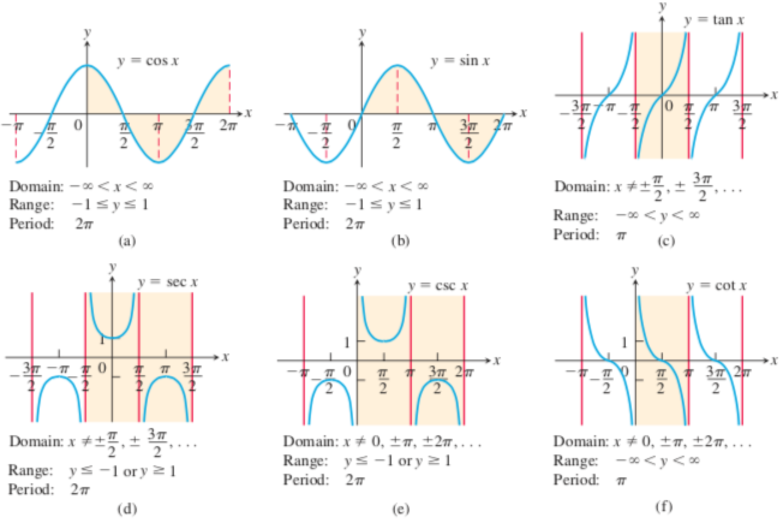
Even and odd trigonometric functions:

Even
$\cos(-x) = \cos x$
$\sec(-x) = \sec x$

Odd
$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$
$\cot(-x) = -\cot x$

Trigonometric Functions

Graphs of six basic trigonometric functions:



Trigonometric Identities

$$\cos^2\theta + \sin^2\theta = 1.$$

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Addition Formulas:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Double-Angle Formulas

$$\cos 2u = \cos^2 u - \sin^2 u$$
$$\sin 2u = 2 \sin u \cos u$$

Trigonometric Identities

Half-Angle Formulas

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$