Limit and Continuity

Dr. Nijat Aliyev

BHOS

Calculus

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Avarage Rate of Change

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Solution:

The average speed of the rock during a given time interval is the change in distance, call Δy, divided by the length of the time interval, say Δt.

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(a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\,\frac{\text{ft}}{\text{sec}}$$

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EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Avarage Rate of Change

Solution

The average speed of the rock over a time interval $[t_0, t_0 + h]$, with the length of $\Delta t = h$, is

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Let us see the following pattern

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

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$$= \frac{32h + 16h^2}{h} = 32 + 16h.$$

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Given any function $y = f(x)$, the average rate of change of y with respect to (input) x over the interval $[x_1, x_2]$ is given by the ratio of the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, and the length $\Delta x = x_2 - x_1 = h$ of the interval.

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DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

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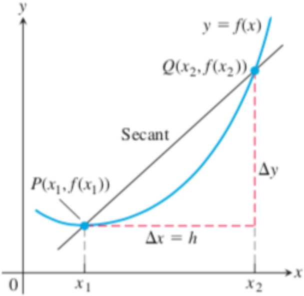

FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

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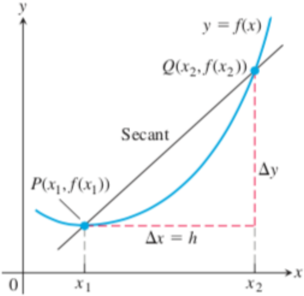

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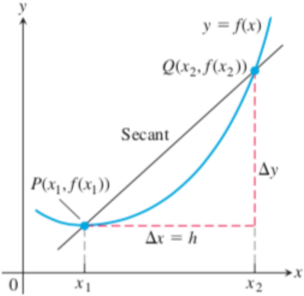

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We will see that this procedure leads to defining the slope of a curve at a point.

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What is meant by the slope of a curve at a point P on the curve?

If there is a tangent line to the curve at P-a line that just touches the curve like the tangent to a circle-it would be reasonable to identify the slope of the tangent as the slope of the curve at P.

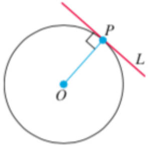

FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

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3. If the **limit** exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

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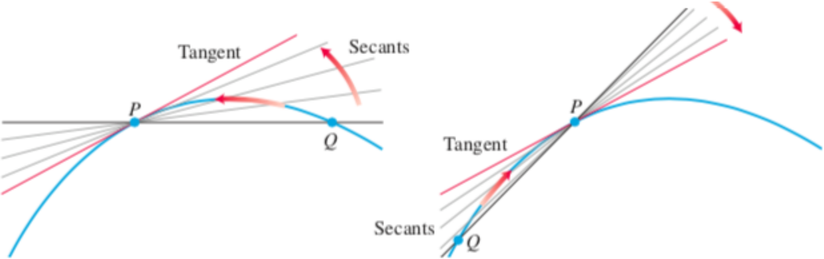

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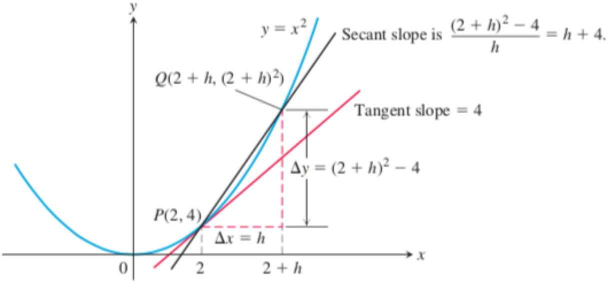

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The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$

$$y = 4x - 4.$$

Instantaneous Rates of Change and Tangent Lines

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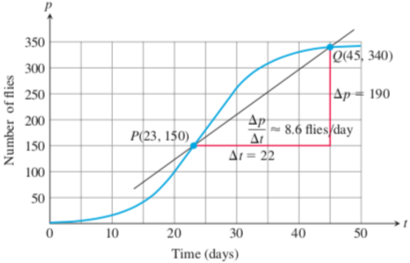
Example 4: Figure below shows how a population p of fruit flies (Drosophila) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time t, and the points joined by a smooth curve (colored blue in Figure). Find the average growth rate from day 23 to day 45.



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This average is the slope of the secant through the points P and Q on the graph in Figure.

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Example 5: How fast was the number of flies in the population of Example 4 growing on day 23?

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We first examine the average rates of change over increasingly short time intervals starting at day 23.

Geometrically, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve.

Instantaneous Rates of Change and Tangent Lines

Q	Slope of $PQ = \Delta p/\Delta t$ (flies / day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$

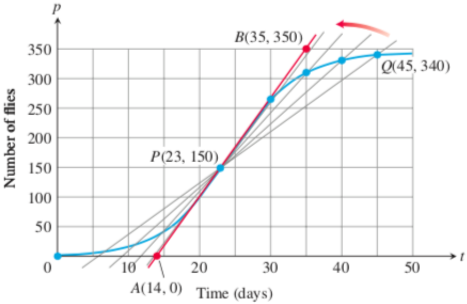

Figure 1: The positions and slopes of four secants through the point P on the fruit fly graph (Example 5).

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$$\frac{350-0}{35-14} \approx 16.7$$

So, on day 23 the population was increasing at a rate of about 16.7 flies/day

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Fort his, we need the concept of a limit.