Limit and Continuity

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BHOS

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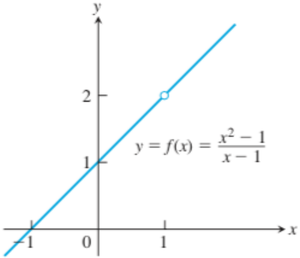

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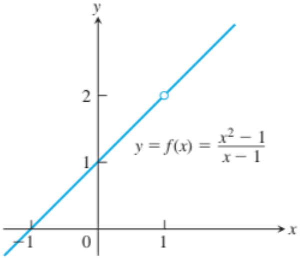

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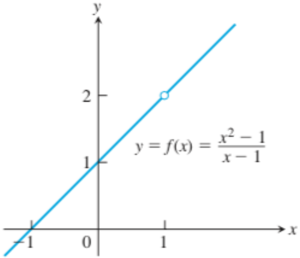

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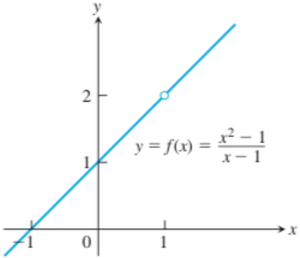

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Even though f is not defined at $x = 1$, it is defined at each point near $x = 1$.

So, we can make the value of $f(x)$ as close as we want to $y = 2$ by making x close enough to $x = 1$.

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Definition: If as x is sufficiently close to x_0, $f(x)$ is arbitrarily close to a number $L \in \mathbb{R}$, we say that, $f(x)$ tends to L as x goes to x_0.

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Denote: $\lim_{x \to x_0} f(x) = L$ or $f(x) \to L$ as $x \to x_0$.

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We say,

"the limit of $f(x)$ as x approaches x_0, equals L" or

"$f(x)$ approaches L as x approaches x_0."

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Investigate the behavior of $f(x) = x^2 - x + 2$ for values of f near $x = 2$.

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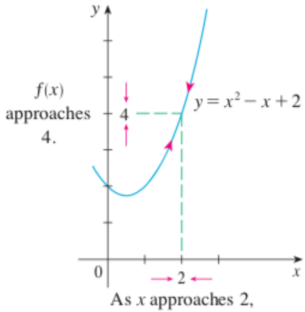

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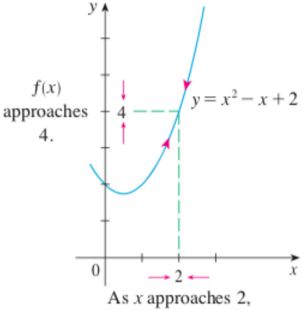

We can make $f(x)$ values as close as we want to 4 by taking x sufficiently close to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
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Figure 1: Table showing the values of f for values of x close to 2 but not equal to 2.

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When x is close to 2 (on either side of 2), $f(x)$ is close to 4.

In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2.

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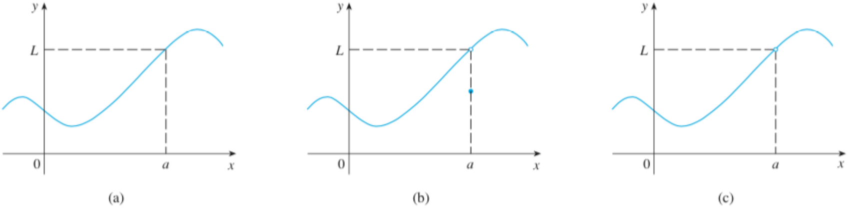
It only mattes how f is defined near x_0.


y
L
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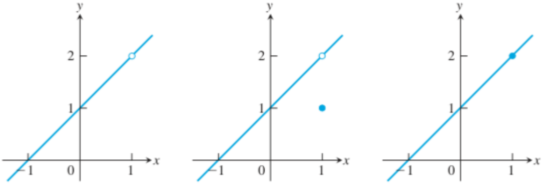
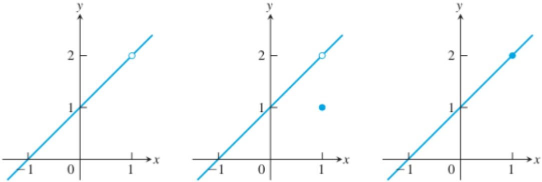


(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$



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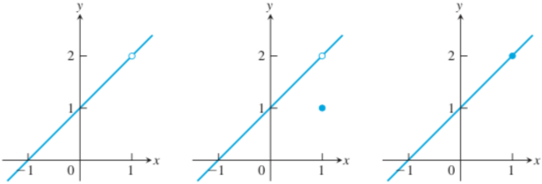
The function f has limit value 2 as x goes to 1, even though f is not defined at $x = 1$.



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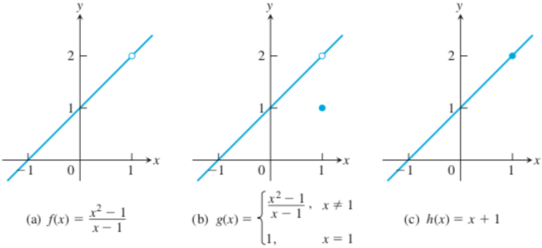

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The function h has limit value as x goes to 1 and it equals to its value at $x = 1$.

Example:

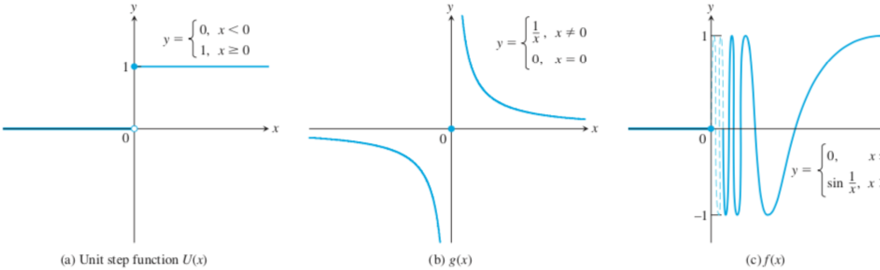
Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

Solution:


y
x
0
1
$y = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$
(a) Unit step function $U(x)$
$y = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(b) $g(x)$
-1
(c) $f(x)$

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(a) If f is the **identity function** $f(x) = x$, then for any value of c

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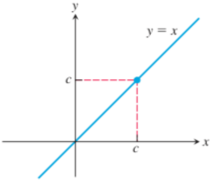

(a) Identity function

(b) If f is the constant function $f(x) = k$ then for any value of c

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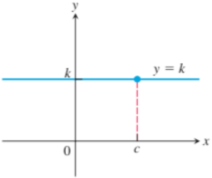

(b) Constant function

Limit Laws

Let $L, M, c, k \in \mathbb{R}$ and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

Limit Laws

Example: Evaluate the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

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(a) $\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ — Sum and Difference Rules

$= c^3 + 4c^2 - 3$ — Power and Multiple Rules

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5} = \dfrac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$ — Quotient Rule

$= \dfrac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ — Sum and Difference Rules

$= \dfrac{c^4 + c^2 - 1}{c^2 + 5}$ — Power or Product Rule

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)}$ — Root Rule with $n = 2$

$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$ — Difference Rule

$= \sqrt{4(-2)^2 - 3}$ — Product and Multiple Rules

$= \sqrt{16 - 3}$

$= \sqrt{13}$

Limit of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is any polynomial, then

$$\lim_{x \to x_0} P(x) = P(x_0).$$

Example: Find the following limit $\lim_{x\to 1}(x^5 - 3x^4 - x + 7)$.

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Solution:
Let $P(x) = x^5 - 3x^4 - x + 7$.

$$\lim_{x\to 1}(x^5 - 3x^4 - x + 7) = P(1) = 4.$$

Limit of Rational Functions

If $P(x), Q(x)$ are any two polynomials with $Q(x_0) \neq 0$, then

$$\lim_{x \to x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)}.$$

Example: Find the limit $\lim_{x \to -1} \frac{x^2+x+1}{x-1}$.

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Solution:
Let $P(x) = x^2 + x + 1$ and $Q(x) = x - 1$. Then, $Q(-1) = 2$ and $P(-1) = 1$.

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Example: Find the limit $\lim_{x \to -1} \frac{x^2+x+1}{x-1}$.

Solution:
Let $P(x) = x^2 + x + 1$ and $Q(x) = x - 1$. Then, $Q(-1) = 2$ and $P(-1) = 1$.

Note that, $Q(-1) \neq 0$. So,

$$\lim_{x \to -1} \frac{x^2 + x + 1}{x - 1} = \frac{P(-1)}{Q(-1)} \frac{1}{2}.$$

Eliminating Common Factors

If $Q(x_0) = 0$. Then we check if $P(x_0) = 0$, too. If so, then we $P(x)$ and $Q(x)$ have common factors $x - x_0$.

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Example: Find the limit $\lim_{x\to 2} \frac{(x^2+x-6)}{x^2-2x}$.

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Then, we can rearrange the function as:

$$f(x) = \frac{(x^2 + x - 6)}{x^2 - 2x} = \frac{(x-2)(x+3)}{x(x-2)} = \frac{(x+3)}{x}.$$

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$$\lim_{x \to 2} \frac{(x^2 + x - 6}{x^2 - 2x} = \lim_{x \to 2} \frac{(x+3)}{x} = 5/2.$$

Sandwich Theorem

Theorem: Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in an open interval I that containing x_0, except possibly at $x = x_0$ itself. Suppose further that,

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L.$$

Then, $\lim_{x \to x_0} f(x) = L$.

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Then, $\lim_{x \to x_0} f(x) = L$.

Example: Suppose $1 - x^2/2 \leq f(x) \leq\leq 1 + x^2/2$ for all $x \neq 0$. Find $\lim_{x \to 0} f(x)$.

Solution:
Since $\lim_{x \to 0}(1 - x^2/2) = 1$ and $\lim_{x \to 0}(1 + x^2/2) = 1$, by Sandwich Theorem $\lim_{x \to 0} f(x) = 1$.