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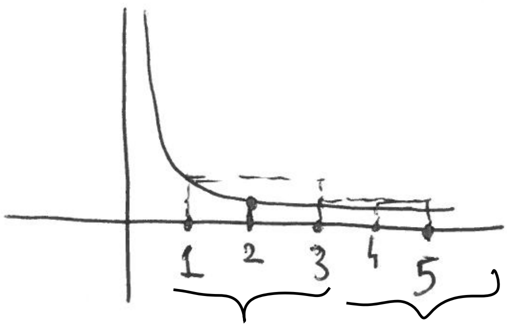


Use midpoint rule to estimate the area under the curve $y = \frac{1}{x}$ for $n=2$ and $n=4$ rectangles

$[1, 5]$

(a) $n = 2$; $\Delta x = \frac{5-1}{2} = 2$

$c_1 = 2$ $f(c_1) = \frac{1}{2}$

$c_2 = 4$ $f(c_2) = \frac{1}{4}$

$A \approx f(c_1) \cdot 2 + f(c_2) \cdot 2 = 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 3/2$

b) $n = 4$; $\Delta x = \frac{5-1}{4} = 1$: $[1, 2], [2, 3], [3, 4], [4, 5]$

$c_1 = \frac{1+2}{2} = 3/2$; $c_2 = \frac{2+3}{2} = 5/2$, $c_3 = \frac{3+4}{2} = 7/2$, $c_4 = \frac{4+5}{2} = 9/2$

$A \approx \sum_{i=1}^{4} f(c_i) \Delta x = \left(f(3/2) + f(5/2) + f(7/2) + f(9/2) \right) \cdot 1$

$= \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}.$

2. Determine the values of the following limits

$$\underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sin\left(\frac{i}{n}\right)}$$

Sol.

$$\frac{1}{n}\sum_{i=1}^{n} \sin\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1-0}{n} \sin\left(0 + i \cdot \frac{1-0}{n}\right)$$

$$= \sum_{i=1}^{n} \Delta x \, f(x_i)$$

where $\Delta x = \frac{1-0}{n}$, $x_i = 0 + i\,\Delta x$, $f(x) = \sin x$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sin\left(\frac{i}{n}\right) = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x}_{\text{Rieman Sum}}$$

$$= \int_0^1 f(x)\,dx = \int_0^1 \sin x\,dx = -\cos x \Big|_0^1 = 1 - \cos 1.$$

3. Express the limits as definite integrals

$\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \, \Delta x_k$, P is a partition of $[0,2]$

Sol: This is defn. of Riemann Sum with $f(x) = x^2$, So,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \, \Delta x_k = \int_0^2 f(x)\,dx = \int_0^2 x^2\,dx.$$

4 Show that $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

Sol :

By Max - Min Inequality

$$(1-0)\cdot \min f \le \int_0^1 f(x)\,dx \le \max f \,(1-0)$$

where $f(x) = \sqrt{x+8}$ Note that $\max f = \sqrt{1+8}$

and $\min f = \sqrt{0+8}$ on $[0,1]$. (f is increasing)

$\Rightarrow$ $\min f = \sqrt{8} = 2\sqrt{2}$ and $\max f = \sqrt{9} = 3$.

$\Rightarrow$ $2\sqrt{2} \le \int_0^1 f(x)\,dx \le 3$

5. Evaluate the integral

$$I = \int_0^{\pi/3} 2\sec^2 x \, dx.$$

Sol: $I = 2 \int_0^{\pi/3} \sec^2 x \, dx$ Note that $\frac{d}{dx}(\tan x) = 1 + \tan^2 x = \sec^2 x$

So, $I = 2 \int_0^{\pi/3} \frac{d}{dx}(\tan x)\, dx \underset{\text{FTC}}{=} 2 \tan x \Big|_0^{\pi/3} =$

$$= 2\left(\tan(\pi/3) - \cancel{\tan 0}\right) = 2\sqrt{3}.$$

6. Find dy/dx if $y = \int_{\sqrt{x}}^{0} \sin(t^2)\, dt.$

Sol: $y = \int_{\sqrt{x}}^{0} \sin(t^2)\, dt = -\int_0^{\sqrt{x}} \sin(t^2)\, dt$

Let $u = \sqrt{x}$. Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (Chain Rule)

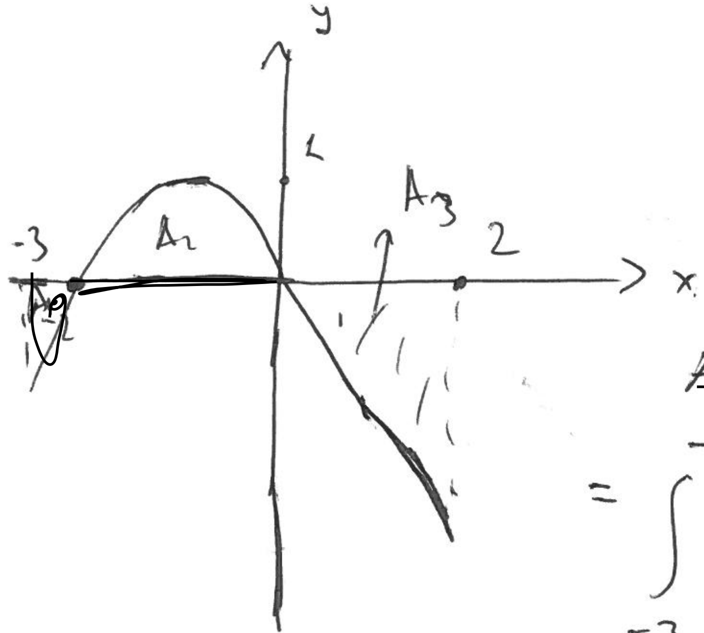
and $y = -\int_0^{u} \sin(t^2)\, dt \underset{\text{FTC}}{\Rightarrow} \frac{dy}{du} = -\sin(u^2)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u^2) \cdot \frac{1}{2\sqrt{x}} = -\sin\left((\sqrt{x})^2\right) \cdot \frac{1}{2\sqrt{x}}$$

$$= -\frac{\sin x}{2\sqrt{x}}.$$

7 Find the total area between the region and the x-axis

$$y = -x^2 - 2x, \quad -3 \le x \le 2$$



$$-x^2 - 2x = 0 \Rightarrow -x(x+2) = 0$$

$$\Rightarrow x = 0 \; ; \; x = -2$$

$$A = |A_1| + |A_2| + |A_3|$$

$$= \int_{-3}^{-2} (0 - (-x^2 - 2x))\,dx + \int_{-2}^{0} (-x^2 - 2x - 0)\,dx$$

$$+ \int_{0}^{2} [0 - (-x^2 - 2x)]\,dx$$

$$= \left(\frac{x^3}{3} + x^2\right)\Big|_{-3}^{-2} + \left(-\frac{x^3}{3} - x^2\right)\Big|_{-2}^{0} + \left(\frac{x^3}{3} + x^2\right)\Big|_{0}^{2} = 28/3$$

8. Using Substitution formula evaluate the integral

(a) $I = \int_0^1 r\sqrt{1-r^2}\, dr.$

Sol: $I = \int_0^1 r\sqrt{1-r^2}\, dr.$ Let $u = 1-r^2$

$\Rightarrow du = -2r\, dr \Rightarrow r\,dr = -\frac{1}{2} du$

$\left.\begin{array}{l} r=0 \Rightarrow u=1 \\ r=1 \Rightarrow u=0 \end{array}\right\} \Rightarrow I = \int_1^0 -\frac{1}{2}\sqrt{u}\, du$

$= -\frac{1}{2}\left(\int_1^0 u^{1/2} du\right) = -\frac{1}{2} \frac{u^{3/2}}{\frac{3}{2}}\Big|_1^0$

$= -\frac{1}{2}\cdot\frac{2}{3} u^{3/2}\Big|_1^0 = \frac{1}{3}.$

(b) $I = \int_0^{\pi/4} \tan x \sec^2 x \, dx$

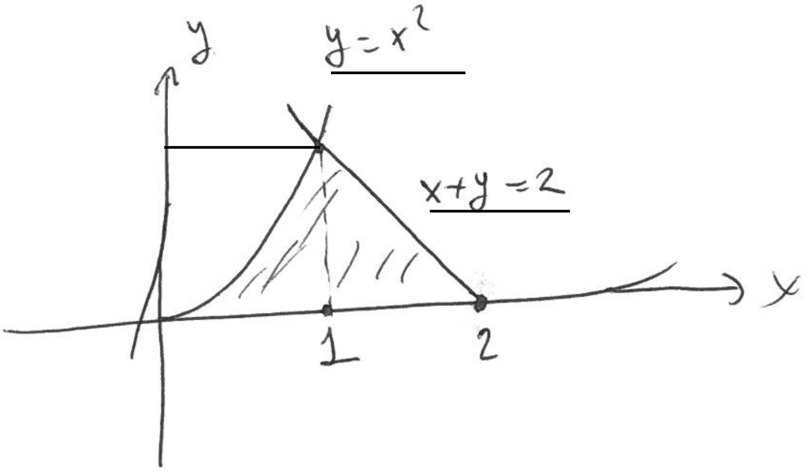
Sol: Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$

$x = 0 \Rightarrow u = \tan 0 = 0$

$x = \pi/4 \Rightarrow u = \tan(\pi/4) = 1$

$$I = \int_0^1 u \, du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}$$

9 Find the area of the shaded region



Two curves intersects at $x^2 = 2-x \Rightarrow x=1$

From 0 to 1, upper curve is $y=x^2$ and lower curve is $y=0$ (x- axis)

From 1 to 2 upper curve is $y=2-x$ lower curve is $y=0$.

$$\Rightarrow A = \int_0^1 x^2 dx + \int_1^2 (2-x)\,dx.$$

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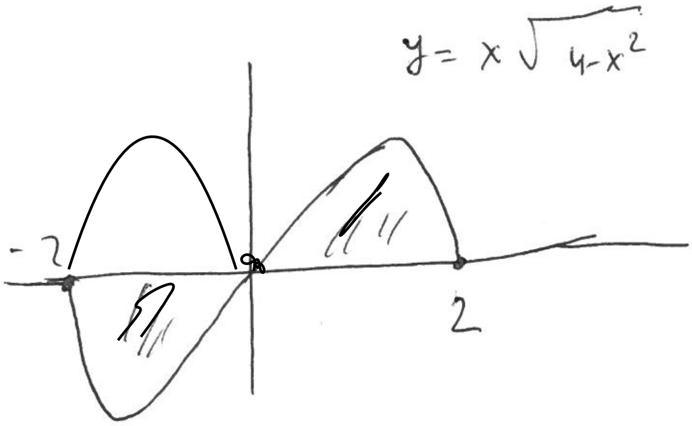

Sol. Note that $f(x) = x\sqrt{4-x^2}$ is an ~~even~~ odd function

So, $$A = \int_{-2}^{2} x\sqrt{4-x^2}\,dx = 2\underbrace{\int_{0}^{2} x\sqrt{4-x^2}\,dx}_{\text{can be calculated by subst. rule.}}$$

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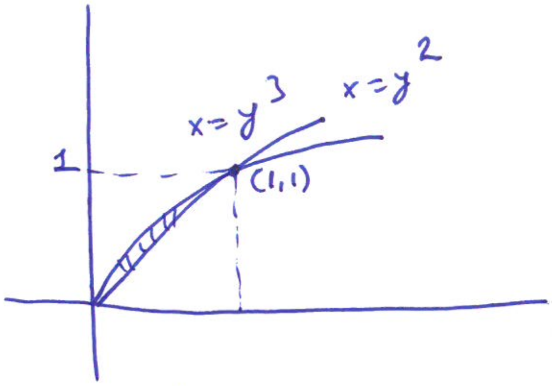

$$A = \int_0^1 (y^2 - y^3)\,dy = \left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

or

$$A = \int_0^1 \left(\sqrt[3]{x} - \sqrt{x}\right)dx$$