1) Find the inverses of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

2) Suppose A is invertible. Show that if $AB = AC$, then $B = C$. Give an example of a nonzero matrix A such that $AB = AC$ but $B \neq C$.

3) Find 2×2 invertible matrices A and B such that $A + B \neq 0$ and $A + B$ is not invertible.

4) Find x, y, z such that A is symmetric, where

(a) $A = \begin{bmatrix} 2 & x & 3 \\ 4 & 5 & y \\ z & 1 & 7 \end{bmatrix}$, (b) $A = \begin{bmatrix} 7 & -6 & 2x \\ y & z & -2 \\ x & -2 & 5 \end{bmatrix}$.

5) Suppose A is a square matrix. Show (a) $A + A^T$ is symmetric, (b) $A - A^T$ is skew-symmetric, (c) $A = B + C$, where B is symmetric and C is skew-symmetric.

6) Write $A = \begin{bmatrix} 4 & 5 \\ 1 & 3 \end{bmatrix}$ as the sum of a symmetric matrix B and a skew-symmetric matrix C.

7) Solve

(a) $\begin{aligned} 2x - y - 4z &= 2 \\ 4x - 2y - 6z &= 5 \\ 6x - 3y - 8z &= 8 \end{aligned}$ (b) $\begin{aligned} x + 2y - z + 3t &= 3 \\ 2x + 4y + 4z + 3t &= 9 \\ 3x + 6y - z + 8t &= 10 \end{aligned}$

8) Write v as a linear combination of u_1, u_2, u_3, where

(a) $v = (4, -9, 2)$, $u_1 = (1, 2, -1)$, $u_2 = (1, 4, 2)$, $u_3 = (1, -3, 2)$;

(b) $v = (1, 3, 2)$, $u_1 = (1, 2, 1)$, $u_2 = (2, 6, 5)$, $u_3 = (1, 7, 8)$;

9) Reduce each of the following matrices to echelon form and then to row canonical form:

(a) $\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 & 5 & 7 \\ 3 & 6 & 4 & 9 & 10 & 11 \\ 1 & 2 & 4 & 3 & 6 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 12 \\ 0 & 0 & 4 & 6 \\ 0 & 2 & 7 & 10 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 8 & 5 & 10 \\ 1 & 7 & 7 & 11 \\ 3 & 11 & 7 & 15 \end{bmatrix}$

10) Find the inverse of each of the following matrices (if it exists):

$$A=\begin{bmatrix}1 & -2 & -1\\ 2 & -3 & 1\\ 3 & -4 & 4\end{bmatrix},\qquad B=\begin{bmatrix}1 & 2 & 3\\ 2 & 6 & 1\\ 3 & 10 & -1\end{bmatrix},\qquad C=\begin{bmatrix}1 & 3 & -2\\ 2 & 8 & -3\\ 1 & 7 & 1\end{bmatrix},$$

11) Find the LU factorization of each of the following matrices:

$$\text{(a)}\ \begin{bmatrix}1 & -1 & -1\\ 3 & -4 & -2\\ 2 & -3 & -2\end{bmatrix},\ \text{(b)}\ \begin{bmatrix}1 & 3 & -1\\ 2 & 5 & 1\\ 3 & 4 & 2\end{bmatrix},\ \text{(c)}\ \begin{bmatrix}2 & 3 & 6\\ 4 & 7 & 9\\ 3 & 5 & 4\end{bmatrix}.$$

12) Determine whether or not W is a subspace of $\mathbf{R}^3$ where W consists of all vectors (a,b,c) in $\mathbf{R}^3$ such that (a) $a=3b$, (b) $a \le b \le c$, (c) $ab=0$, (d) $a+b+c=0$, (e) $b=a^2$, (f) $a=2b=3c$.

13) Let V be the vector space of n-square matrices over a field K. Show that W is a subspace of V if W consists of all matrices $A=[a_{ij}]$ that are

(a) symmetric ($A^T=A$ or $a_{ij}=a_{ji}$), (b) (upper) triangular, (c) diagonal, (d) scalar.

14) Let $AX=B$ be a nonhomogeneous system of linear equations in n unknowns; that is, $B \neq 0$. Show that the solution set is not a subspace of K^n.

15) Show that the following functions f, g, h are linearly independent:

(a) $f(t) = e^t,\ g(t) = \sin t,\ h(t) = t^2$; (b) $f(t) = e^t,\ g(t) = e^{2t},\ h(t) = t$.

16) Show that $u = (a, b)$ and $v = (c, d)$ in K^2 are linearly dependent if and only if $ad - bc = 0$.

17) Suppose u, v, w are linearly independent vectors. Prove that S is linearly independent where

$S = \{u + v - 2w,\ u - v - w,\ u + w\}$;

18) Find a subset of u_1, u_2, u_3, u_4 that gives a basis for $W = \text{span}(u_i)$ of $\mathbf{R}^5$, where

(a) $u_1 = (1, 1, 1, 2, 3),\quad u_2 = (1, 2, -1, -2, 1),\quad u_3 = (3, 5, -1, -2, 5),\quad u_4 = (1, 2, 1, -1, 4)$

(b) $u_1 = (1, -2, 1, 3, -1),\quad u_2 = (-2, 4, -2, -6, 2),\quad u_3 = (1, -3, 1, 2, 1),\quad u_4 = (3, -7, 3, 8, -1)$

19) Find a basis and the dimension of the solution space W of each of the following homogeneous systems:

$$\begin{aligned} x + 2y - 2z + 2s - t &= 0 \\ x + 2y - z + 3s - 2t &= 0 \\ 2x + 4y - 7z + s + t &= 0 \end{aligned}$$

20) Find the rank of each of the following matrices:

(a) $\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & -3 & -2 \\ 1 & 3 & -2 & 0 \\ 3 & 8 & -7 & -2 \\ 2 & 1 & -9 & -10 \end{bmatrix}$

21) Determine which of the following subspaces of $\mathbf{R}^3$ are identical:

$U_1 = \text{span}[(1,1,-1),\ (2,3,-1),\ (3,1,-5)]$, $U_2 = \text{span}[(1,-1,-3),\ (3,-2,-8),\ (2,1,-3)]$

$U_3 = \text{span}[(1,1,1),\ (1,-1,3),\ (3,-1,7)]$

22) Find a basis for (i) the row space and (ii) the column space of each matrix M:

(a) $M = \begin{bmatrix} 0 & 0 & 3 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \end{bmatrix}$, (b) $M = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 3 \\ 3 & 6 & 5 & 2 & 7 \\ 2 & 4 & 1 & -1 & 0 \end{bmatrix}$.

23) Let A and B be arbitrary $m \times n$ matrices. Show that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

24)

The vectors $u_1 = (1,2,0)$, $u_2 = (1,3,2)$, $u_3 = (0,1,3)$ form a basis S of $\mathbf{R}^3$. Find the coordinate vector $[v]$ of v relative to S where (a) $v = (2,7,-4)$, (b) $v = (a,b,c)$.

25)

Let $V = \mathbf{M}_{2,2}$. Find the coordinate vector $[A]$ of A relative to S where

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \quad \text{and} \quad \text{(a)} \quad A = \begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}, \quad \text{(b)} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

26)

Find the dimension and a basis of the subspace W of $\mathbf{P}_3(t)$ spanned by

$$u = t^3 + 2t^2 - 3t + 4, \qquad v = 2t^3 + 5t^2 - 4t + 7, \qquad w = t^3 + 4t^2 + t + 2$$

27) Show that the following mappings are linear:

(a) $F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (x + 2y - 3z,\ 4x - 5y + 6z)$.

(b) $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (ax + by,\ cx + dy)$, where a, b, c, d belong to $\mathbf{R}$.

28) Show that the following mappings are not linear:

(a) $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (x^2, y^2)$.

(b) $F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (x + 1,\ y + z)$.

29) Find a linear mapping $F: \mathbf{R}^3 \to \mathbf{R}^3$ whose image is spanned by $(1, 2, 3)$ and $(4, 5, 6)$.

30) Let $F: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $F(x, y) = (4x + 5y,\ 2x - y)$.

(a) Find the matrix A representing F in the usual basis E.

(b) Find the matrix B representing F in the basis $S = \{u_1, u_2\} = \{(1, 4),\ (2, 9)\}$.

(c) Find P such that $B = P^{-1}AP$.

31) Let **D** denote the differential operator; that is, $\mathbf{D}(f(t)) = df/dt$. Each of the following sets is a basis of a vector space V of functions. Find the matrix representing **D** in each basis:

(a) $\{e^t, e^{2t}, te^{2t}\}$. (b) $\{1, t, \sin 3t, \cos 3t\}$.

32) Verify that the following is an inner product on $\mathbf{R}^2$, where $u = (x_1, x_2)$ and $v = (y_1, y_2)$:

$$f(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$$

33) Show that each of the following is not an inner product on $\mathbf{R}^3$, where $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$:

(a) $\langle u, v\rangle = x_1y_1 + x_2y_2$, (b) $\langle u, v\rangle = x_1y_2x_3 + y_1x_2y_3$.

34) Find a basis for the subspace W of $\mathbf{R}^5$ orthogonal to the vectors $u_1 = (1, 1, 3, 4, 1)$ and $u_2 = (1, 2, 1, 2, 1)$.

35) Let U be the subspace of $\mathbf{R}^4$ spanned by

$$v_1 = (1,1,1,1), \qquad v_2 = (1,-1,2,2), \qquad v_3 = (1,2,-3,-4)$$

(a) Apply the Gram–Schmidt algorithm to find an orthogonal and an orthonormal basis for U.

(b) Find the projection of $v = (1,2,-3,4)$ onto U.

36) Suppose $v = (1,2,3,4,6)$. Find the projection of v onto W, or, in other words, find $w \in W$ that minimizes $\|v - w\|$, where W is the subspace of $\mathbf{R}^5$ spanned by

(a) $u_1 = (1,2,1,2,1)$ and $u_2 = (1,-1,2,-1,1)$, (b) $v_1 = (1,2,1,2,1)$ and $v_2 = (1,0,1,5,-1)$.

37) Evaluate each of the following determinants:

$$\text{(a)} \begin{vmatrix} 1 & 2 & -1 & 3 & 1 \\ 2 & -1 & 1 & -2 & 3 \\ 3 & 1 & 0 & 2 & -1 \\ 5 & 1 & 2 & -3 & 4 \\ -2 & 3 & -1 & 1 & -2 \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 6 & 2 \\ 0 & 0 & 2 & 3 & 1 \end{vmatrix}, \quad \text{(c)} \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 6 & 5 & 1 \\ 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 2 & 3 \end{vmatrix}$$

38) Find the determinant of each of the following linear transformations:

(a) $T:\mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x,y) = (2x - 9y,\ \ 3x - 5y)$,

(b) $T:\mathbf{R}^3 \to \mathbf{R}^3$ defined by $T(x,y,z) = (3x - 2z,\ \ 5y + 7z,\ \ x + y + z)$,

39) Find the volume $V(S)$ of the parallelopiped S in $\mathbf{R}^3$ determined by the following vectors:

$u_1 = (1,2,-3)$, $u_2 = (3,4,-1)$, $u_3 = (2,-1,5)$,

40) Solve the following systems by determinants:

(a) $\begin{cases} 2x - 5y + 2z = 2 \\ x + 2y - 4z = 5 \\ 3x - 4y - 6z = 1 \end{cases}$, (b) $\begin{cases} 2z + 3 = y + 3x \\ x - 3z = 2y + 1 \\ 3y + z = 2 - 2x \end{cases}$

41) Let $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$.

(a) Find eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal.

(c) Find A^8 and $f(A)$ where $f(t) = t^4 - 5t^3 + 7t^2 - 2t + 5$.

(d) Find a matrix B such that $B^2 = A$.

42) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real matrix. Find necessary and sufficient conditions on a, b, c, d so that A is diagonalizable—that is, so that A has two (real) linearly independent eigenvectors.

43) For each of the following symmetric matrices B, find its eigenvalues, a maximal orthogonal set S of eigenvectors, and an orthogonal matrix P such that $D = P^{-1}BP$ is diagonal:

(a) $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, (b) $B = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & 17 \end{bmatrix}$