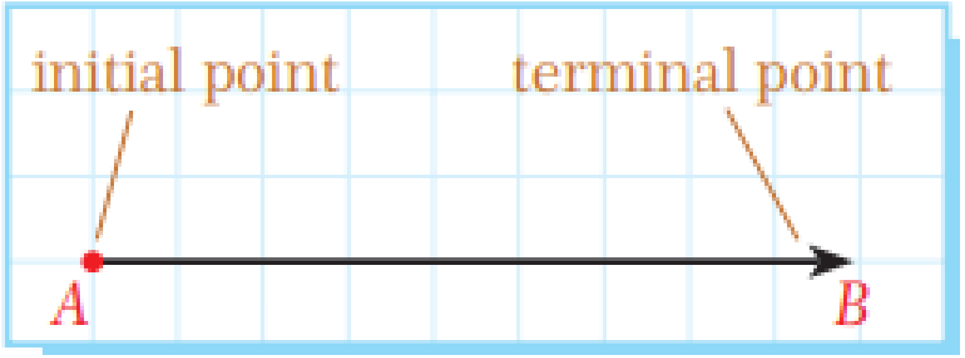
Vectors

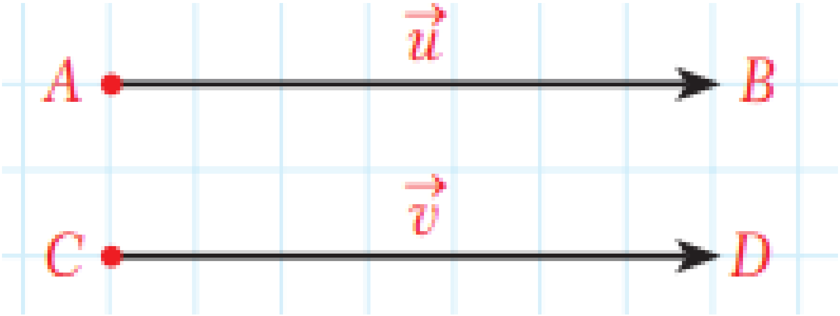
- Some of the quantities we measure in our daily lives are completely determined by their magnitudes, for example, length, mass, area, temperature, and energy. We call such quantities **scalar quantities**.
- Quantities such as displacement, velocity, acceleration, and other forces that have magnitude as well as direction are called **vector quantities.**

Directed line segment

A directed line segment in the plane is called a vector



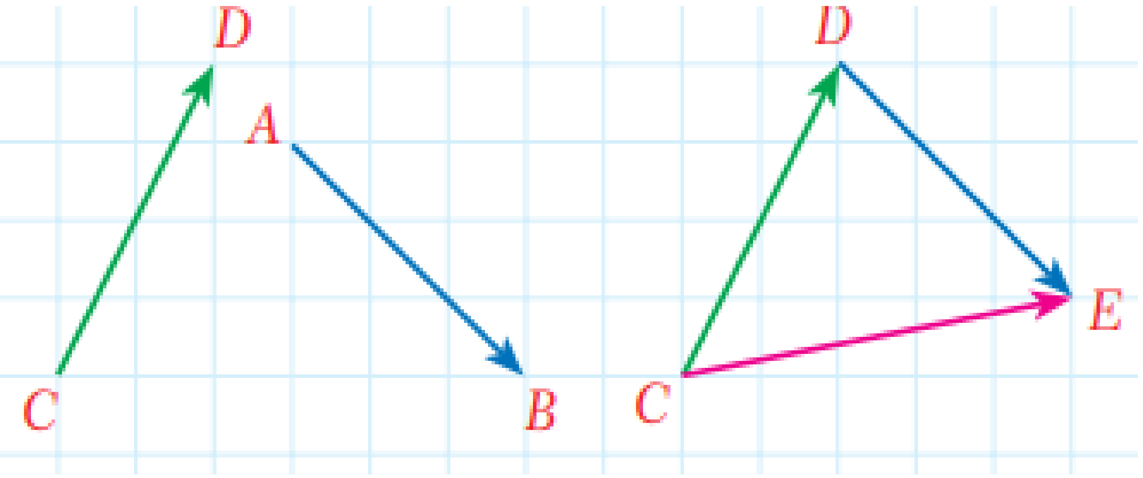
Two vectors that have the same direction and length are called equal vectors.

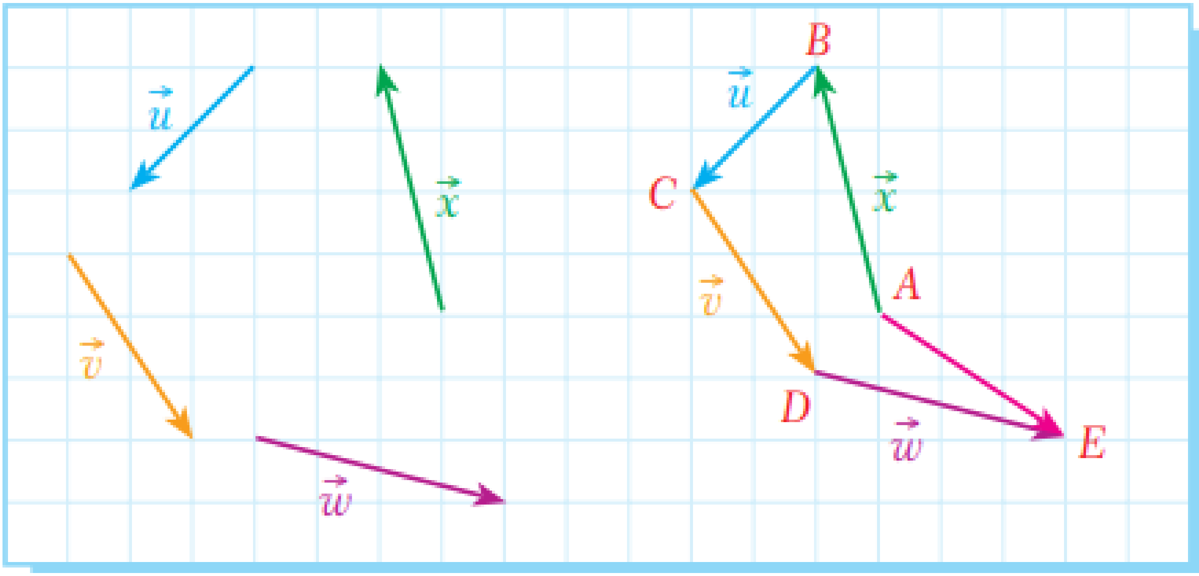


Addition of Vectors

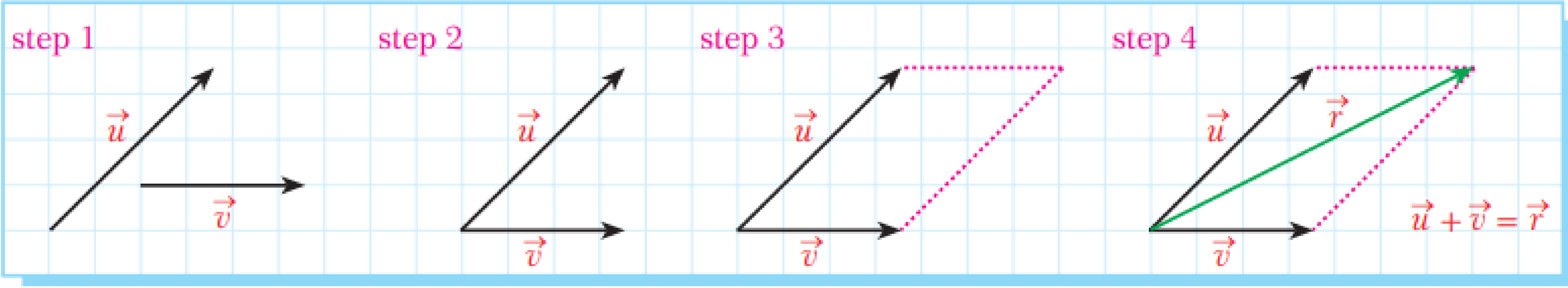
The Polygon Method

$$\vec{u} + \vec{v} + \vec{w} + \vec{x} = \overrightarrow{AE}.$$



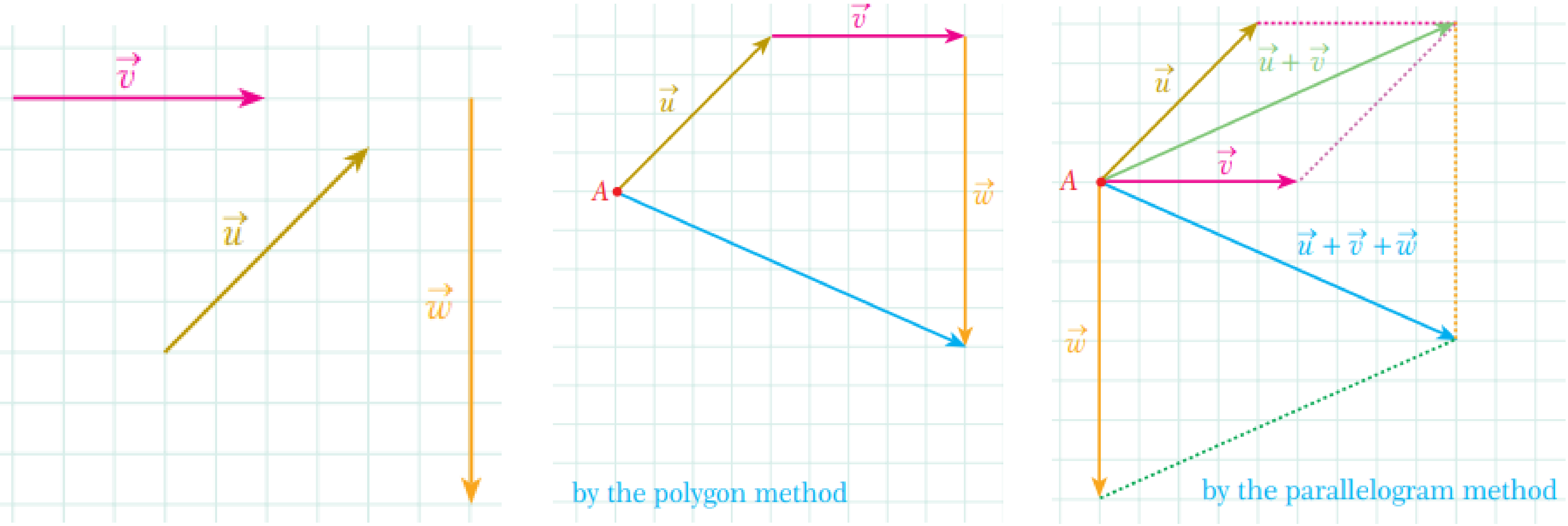


The Parallelogram Method



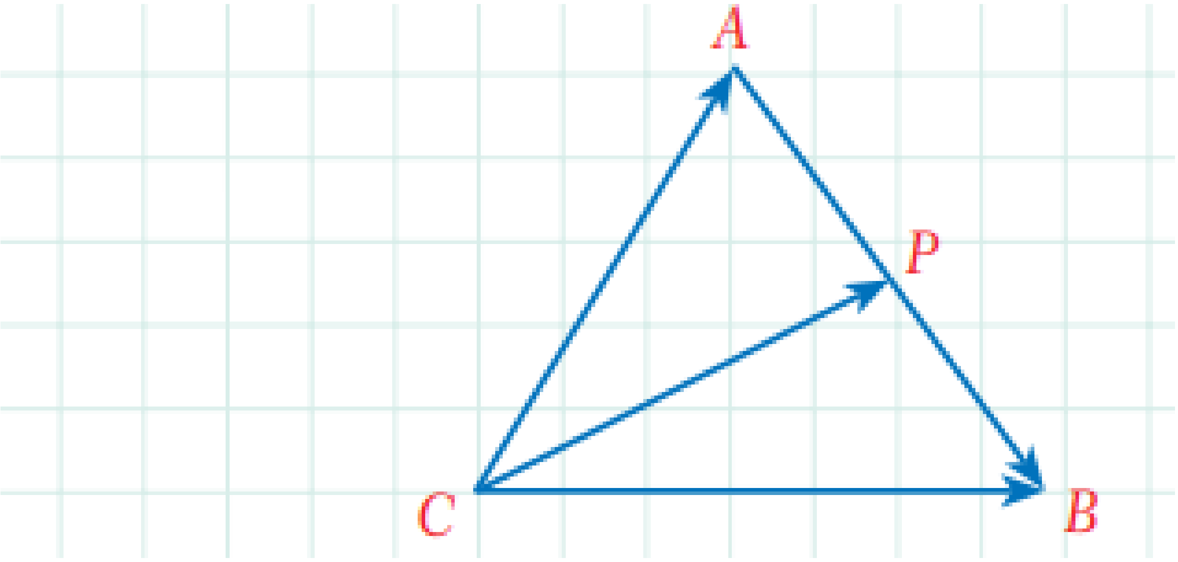
Example

Find $\vec{u} + \vec{v} + \vec{w}$ in the figure



Example 2

In a triangle ABC, P is the midpoint of $\overrightarrow{AB}$. Express $\overrightarrow{CP}$ in terms of $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

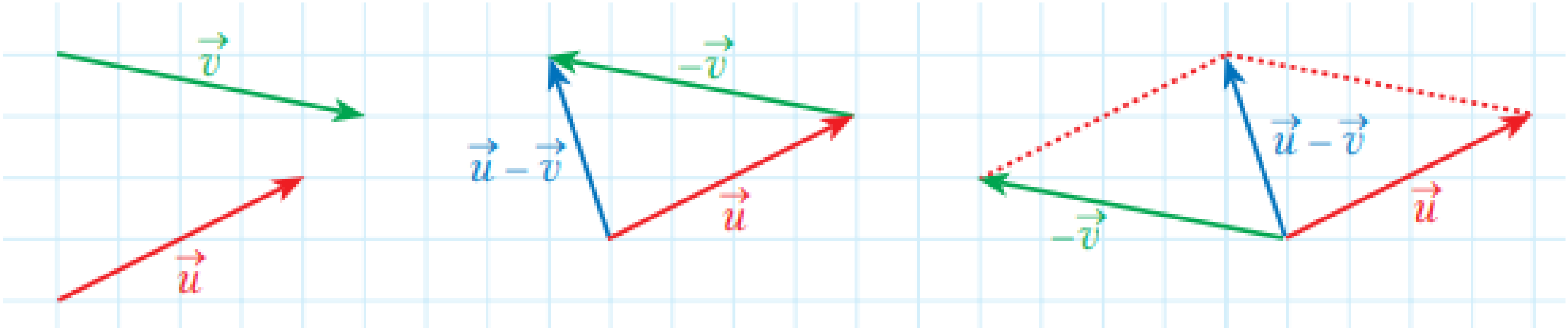


Solution

$$
\begin{array}{rrcl}
 & \overrightarrow{CP} & = & \overrightarrow{CA} + \overrightarrow{AP} \\
+ & \overrightarrow{CP} & = & \overrightarrow{CB} + \overrightarrow{BP} \\
\hline
 & 2\cdot\overrightarrow{CP} & = & \overrightarrow{CA} + \overrightarrow{CB} + \underbrace{\overrightarrow{AP} + \overrightarrow{BP}}_{\vec{0}} \\
 & \overrightarrow{CP} & = & \frac{1}{2}\cdot(\overrightarrow{CA} + \overrightarrow{CB})
\end{array}
$$

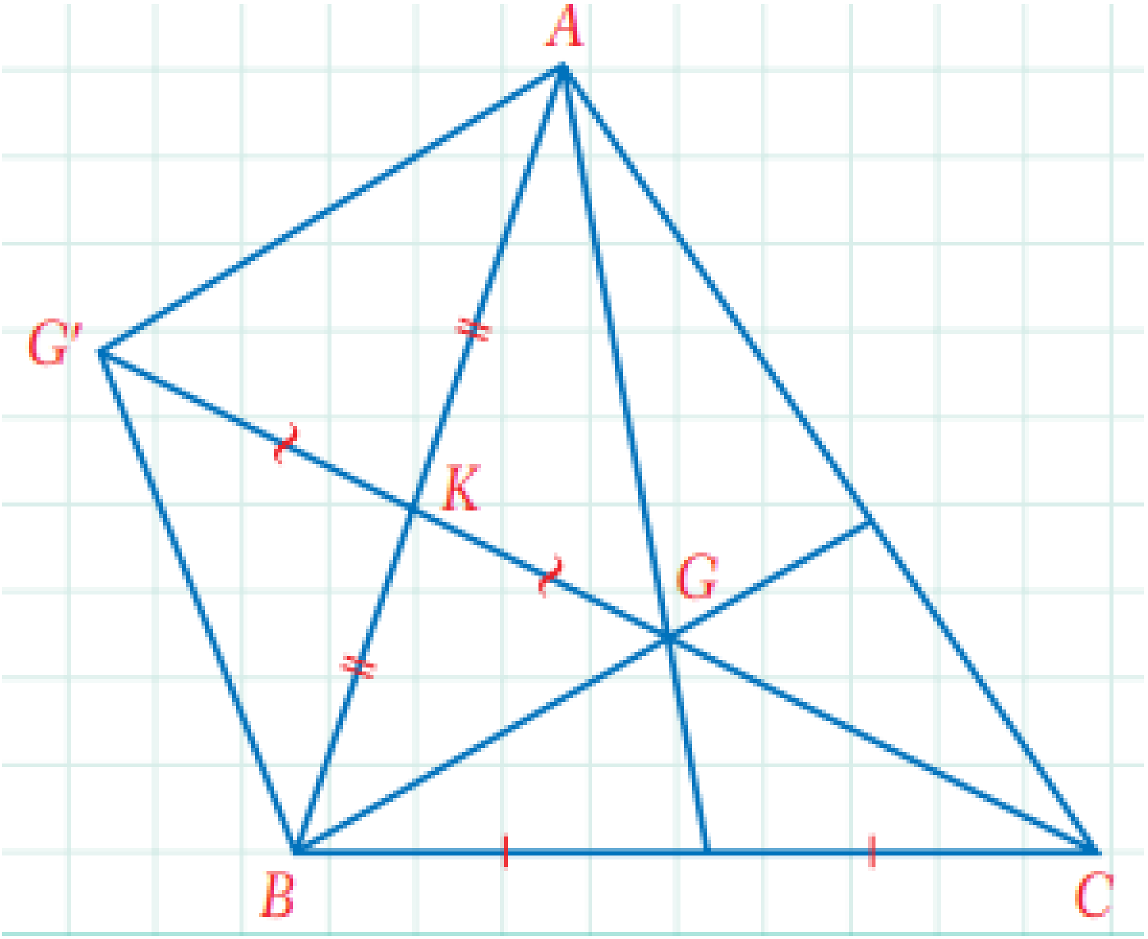
Subtraction of Vectors

$(\vec{u} - \vec{v} = \vec{u} + (-\vec{v}))$



Example 3

In a triangle ABC, G is the centroid. Find $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$.



Multiplication of a Vector by a Scalar

For a real number a and a vector $\vec{u}$,

1. if $a > 0$ then vector $a \cdot \vec{u}$ has the same direction as $\vec{u}$ and the length $|a \cdot \vec{u}| = a \cdot |\vec{u}|$.
2. if $a < 0$ then vector $a \cdot \vec{u}$ has the opposite direction to $\vec{u}$ and the length $|a \cdot \vec{u}| = |a| \cdot |\vec{u}|$.
3. if $a = 0$ then $a \cdot \vec{u} = \vec{0}$.

Example 4

Points A, B, C, and M are on the same line. M is between A and C. $\overrightarrow{AB} = 2{\cdot}\overrightarrow{AC}$. Express the vector $\overrightarrow{MC}$ in terms of the vectors $\overrightarrow{MA}$ and $\overrightarrow{MB}$.

Answers

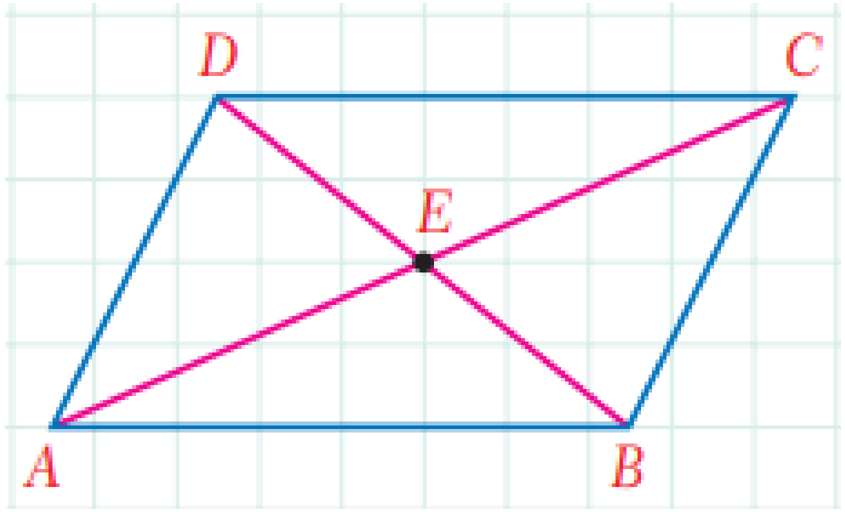
$$\overrightarrow{MC} = \frac{1}{2}\cdot(\overrightarrow{MA} + \overrightarrow{MB})$$

Parallel Vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors. $\vec{a}$ and $\vec{b}$ are called **parallel vectors** if and only if $\vec{a} = k \cdot \vec{b}$ where $k \neq 0$ and $k \in \mathbb{R}$. We write $\vec{\boldsymbol{a}} || \vec{\boldsymbol{b}}$ to show that two vectors are parallel.

Example 5

Prove that the diagonals of a parallelogram intersect at their midpoints by using vectors.



Practice

- Point O is in the plane of a triangle ABC. Point G is the centroid of triangle ABC. Show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3 \cdot \overrightarrow{OG}$.

In a triangle ABC, $|BD| = |DE| = |EC|$, and $E, D \in [BC]$. If $|\overrightarrow{AD} + \overrightarrow{AE}| = 9$ cm, find $|\overrightarrow{AB} + \overrightarrow{AC}|$.

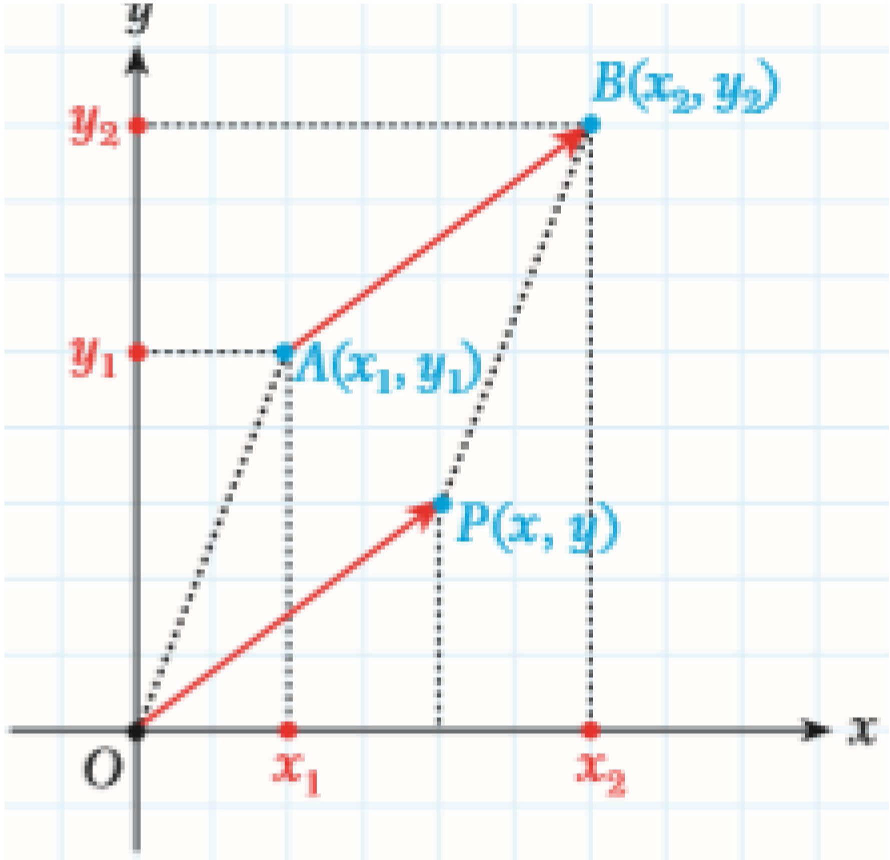
In a quadrilateral $ABCD$, E and F are the midpoints of the diagonals AC and BD respectively.
Show that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4 \cdot \overrightarrow{EF}$.

Position Vector

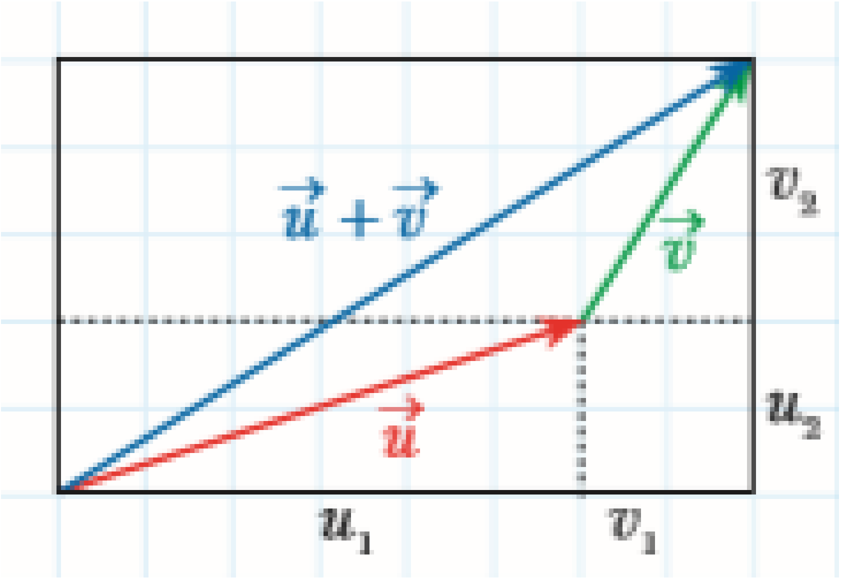
A vector $\overrightarrow{OP}$ whose initial point is at the origin of the rectangular coordinate plane and which is parallel to a vector $\overrightarrow{AB}$ is called the **position vector** of $\overrightarrow{AB}$ in the plane.

$$\overrightarrow{OP} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x, y) = (x_2 - x_1, y_2 - y_1).$$

$$\vec{u} = (u_1, u_2) \qquad |\vec{u}| = \sqrt{u_1^2 + u_2^2}.$$



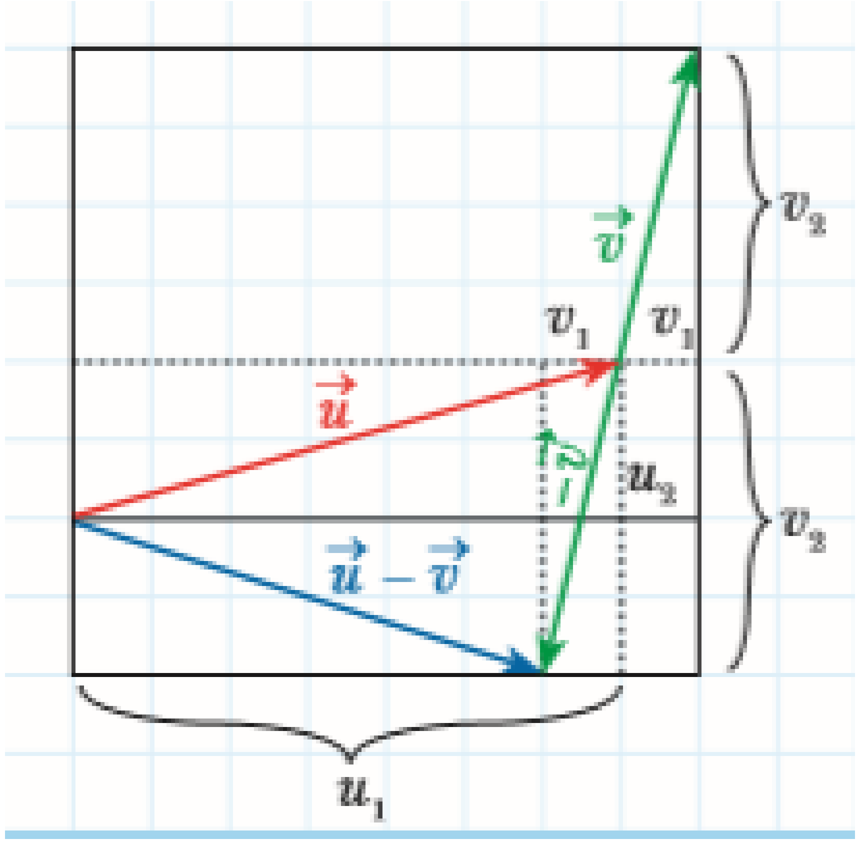
Addition of Vectors



If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, then

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$$

Subtraction of Vectors



If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ then

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2).$$

Multiplication of a Vector by a Scalar

Let $\vec{v} = (v_1, v_2)$ and $c \in \mathbb{R}$, then $c \cdot \vec{v} = (c \cdot v_1, c \cdot v_2)$.

A vector of length 1 is called a **unit vector**.

For example, the vector $\vec{w} = (\frac{3}{5}, \frac{4}{5})$ is a unit vector.

Parallel Vectors

for any $c \neq 0$, $\vec{u} \neq 0$, and $\vec{v} \neq 0$, $\vec{u} \parallel \vec{v}$ if and only if $\vec{u} = c \cdot \vec{v}$.

It follows that if $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, then $(u_1, u_2) = (c \cdot v_1, c \cdot v_2)$.

So $\vec{u} \parallel \vec{v}$ if and only if $\dfrac{u_1}{v_2} = \dfrac{u_2}{v_2} = c$.

Linear Combination of Vectors (optional)

Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ be vectors in the plane and let $c_1, c_2, \ldots, c_k$ be scalars.

The expression $c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2 + \ldots + c_k \cdot \vec{u}_k$ is called a **linear combination** of the vectors.

Express $\vec{v} = (12, 5)$ as a linear combination of the vectors $\vec{u}_1 = (2, 1)$ and $\vec{u}_2 = (3, 2)$.

Let $c_1, c_2 \in \mathbb{R}$. Then $\vec{v} = c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2$.

$(12, 5) = c_1 \cdot (2, 1) + c_2 \cdot (3, 2)$

$(12, 5) = (2 \cdot c_1, c_1) + (3 \cdot c_2, 2 \cdot c_2)$

$(12, 5) = (2 \cdot c_1 + 3 \cdot c_2, c_1 + 2 \cdot c_2)$

$c_2 = -2$

$c_1 = 5 + 4 = 9$

$\vec{v} = 9 \cdot \vec{u}_1 - 2 \cdot \vec{u}_2.$

DOT PRODUCT (Scalar Product)

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be two vectors in the plane.

The **dot product** of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u}\cdot\vec{v}$, is defined by

$$\vec{u}\cdot\vec{v} = u_1\cdot v_1 + u_2\cdot v_2.$$

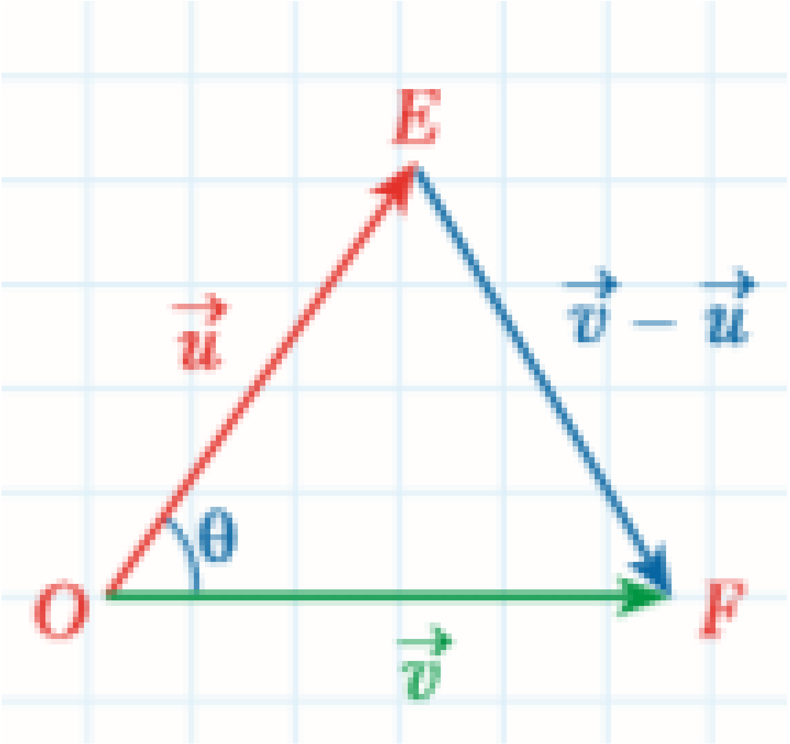
The definition of the dot product gives us the following properties.

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (commutative property)
2. $\vec{u}\cdot(\vec{v} + \vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ (associative property)
3. $c\cdot(\vec{u}\cdot\vec{v}) = (c\cdot\vec{u})\cdot\vec{v}$
4. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$
5. $\vec{u}\cdot\vec{v} \geq 0$, and $\vec{u}\cdot\vec{u} = 0$ if and only if $\vec{u} = \vec{0}$.

Angle Between Two Vectors

Let θ be the angle measure between two non-zero vectors $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta.$$



Prove by using the law of cosines.

Perpendicular and Parallel Vectors

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$.

Let θ be the angle measure between nonzero vectors $\vec{u}$ and $\vec{v}$.

Then $\vec{u} \parallel \vec{v}$ if and only if $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}|$ or $\vec{u} \cdot \vec{v} = -|\vec{u}| \cdot |\vec{v}|$.

Example 6

Find the area of the triangle with vertices $A(2, 3)$, $B(0, 1)$, $C(3, 2)$.

Solution

$$A(\widehat{ABC}) = \frac{|\overrightarrow{AH}| \cdot |\overrightarrow{BC}|}{2}$$

$$\overrightarrow{AH} = (x_0 - 2,\ y_0 - 3)$$

$$\overrightarrow{BC} = (3,\ 1)$$

$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 3 \cdot (x_0 - 2) + (y_0 - 3) = 0$$

$$3x_0 + y_0 - 6 - 3 = 0$$

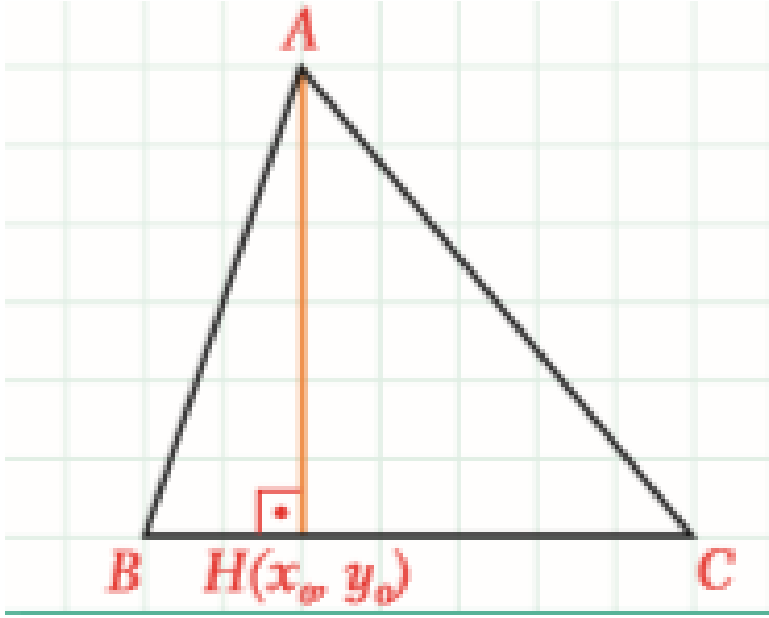
$$3x_0 + y_0 = 9$$

$$\overrightarrow{BH} = k \cdot \overrightarrow{HC}$$

$$\overrightarrow{BH} = (x_0,\ y_0 - 1)$$

$$\overrightarrow{HC} = (3 - x_0,\ 2 - y_0)$$

$$\frac{x_0}{3 - x_0} = \frac{y_0 - 1}{2 - y_0}$$



$x_0 = 2.4$ and $y_0 = 1.8$.

$$A(\widehat{ABC}) = \frac{1}{2} \cdot \frac{4}{\sqrt{10}} \cdot \sqrt{10} = 2$$

Practice 2

Write the equation of the line passing through $A(-1, -1)$ which is perpendicular to $\vec{u} = (3, 4)$.

Find the area of a rhombus with vertices $A(2, 0)$, $B(-3, 3)$, $C(-8, 0)$, and $D(-3, -3)$.

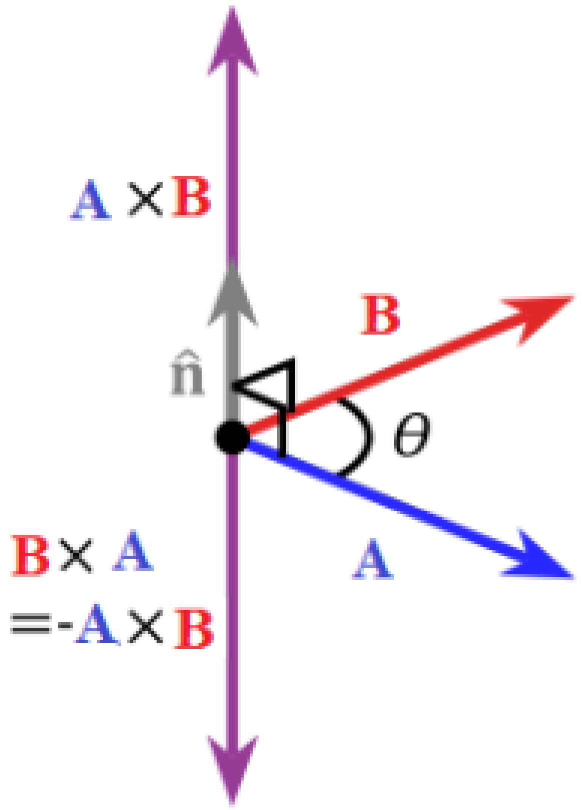
In a rectangle $ABCD$, $\overrightarrow{DC} = 3 \cdot \overrightarrow{AD}$ and point E is on DC. Find the quantity $\overrightarrow{AE} \cdot \overrightarrow{BE}$ given $\overrightarrow{DE} = 2 \cdot \overrightarrow{EC}$ and $|\overrightarrow{AD}| = 3$ cm.

Show that the altitudes of an acute-angled triangle are concurrent using vectors.

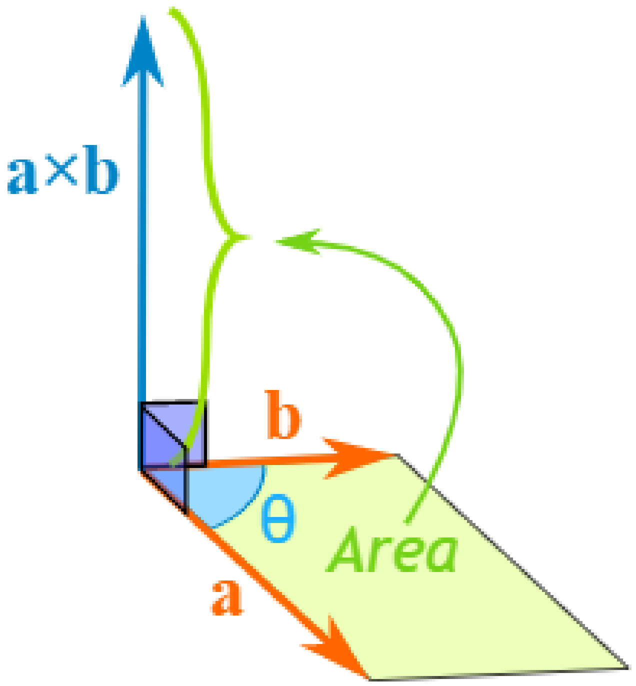
Vector (cross) product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

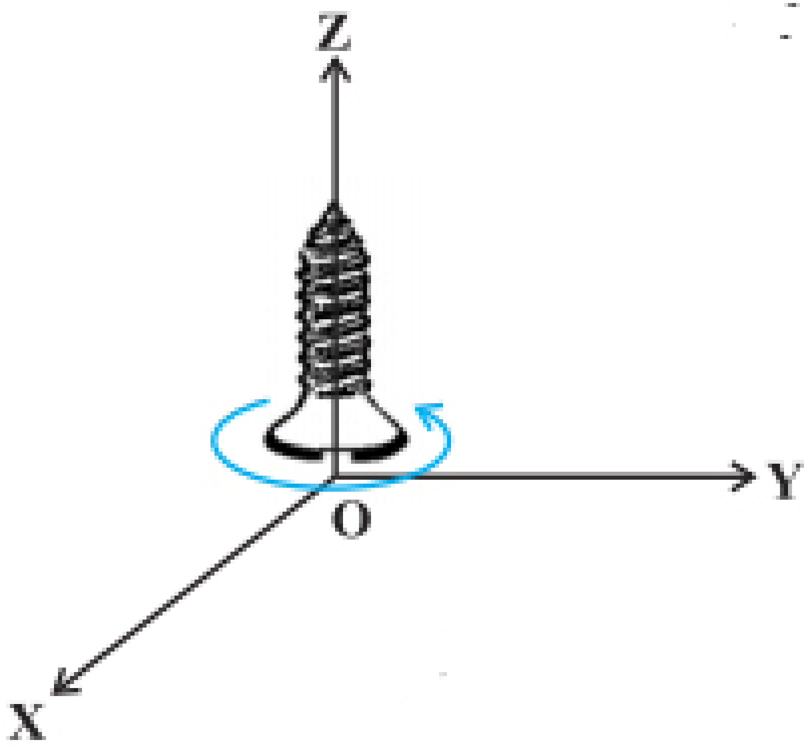
n is the unit vector at right angles to both **a** and **b**



The magnitude of the cross product equals the area of a parallelogram with vectors **a** and **b** for sides:



Direction of the cross product is determined By " Right Hand Rule".



PROPERTIES of Cross Product

Anticommutative Property	$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
Distributive Property	$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
Multiplication by a Constant	$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
Cross Product of the Zero Vector	$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
Cross Product of a Vector with Itself	$\mathbf{v} \times \mathbf{v} = \mathbf{0}$

Component Form of Vector Product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 7

Find the cross product of $\vec{v} = (3, 6, 8)$ and $\vec{w} = (2, -4, 7)$.

$$\begin{vmatrix} i & j & k \\ 3 & 6 & 8 \\ 2 & -4 & 7 \end{vmatrix} = (6(7) - 8(-4))i - (3(7) - 2(8))j + (3(-4) - (6)(2))k$$

$$= 74i - 5j - 24k$$

$$= (74, -5, -24).$$

Practice 3

If $\vec{w} = \langle 1, 6, -8 \rangle$ and $\vec{v} = \langle 4, -2, -1 \rangle$ compute $\vec{w} \times \vec{v}$.

$A = (3, 4, 0)$, $B = (3, 6, 3)$ and $C = (1, 2, 1)$ are the three vertices of a triangle.

Calculate the area of the triangle.

Find a vector that is orthogonal to the plane containing the points $P = (3, 0, 1)$, $Q = (4, -2, 1)$ and $R = (5, 3, -1)$.

Are the vectors $\vec{u} = \langle 1, 2, -4 \rangle$, $\vec{v} = \langle -5, 3, -7 \rangle$ and $\vec{w} = \langle -1, 4, 2 \rangle$ are in the same plane?

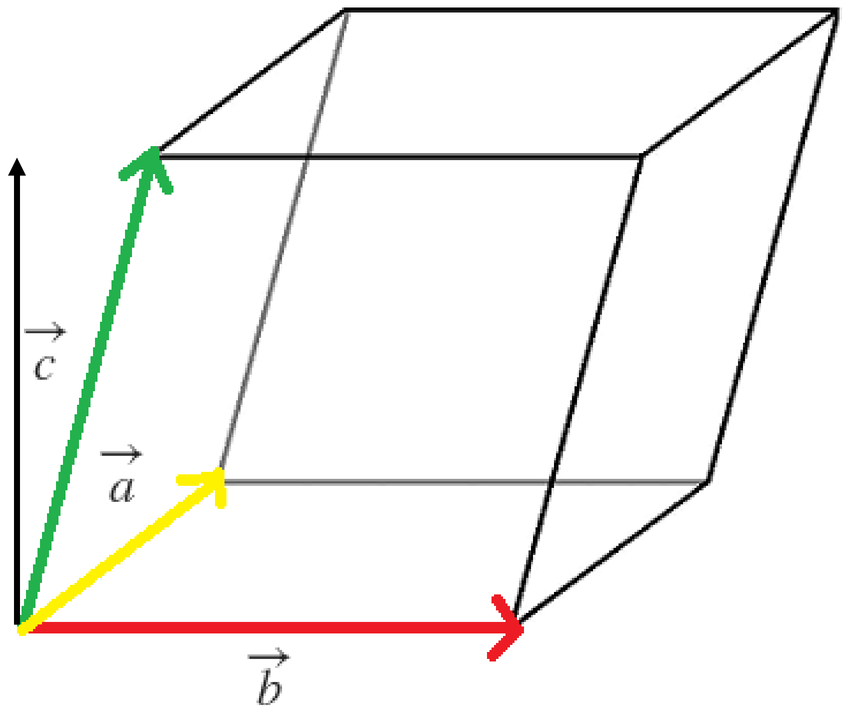
Example 8

Show that the cross product $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$ is equal to the volume of of the parallelepiped.

$$\mathrm{A}_{\text{base}} = \left\| \vec{a} \times \vec{b} \right\|.$$

$$\text{Height} = \|\vec{c}\| \cos\theta.$$

$$\begin{aligned} \mathrm{A} &= \mathrm{A}_{\text{base}} \times \text{Height} \\ &= \left\| \vec{a} \times \vec{b} \right\| \|\vec{c}\| \cos\theta \\ &= (\vec{a} \times \vec{b}) \cdot \vec{c}. \ \square \end{aligned}$$



Mixed product of Vectors

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

PROPERTIES

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

PROPERTIES That Relate The Cross Product And The Dot Product

A) $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

B) $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

C) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

D) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

E) $\left\|\vec{a} \times \vec{b}\right\|^2 = \|\vec{a}\|^2 \left\|\vec{b}\right\|^2 - (\vec{a} \cdot \vec{b})^2$

The coplanarity condition of three vectors.

Coplanar vectors are defined as vectors that are lying on the same in a three-dimensional plane.

If there are three vectors in a 3d-space and their scalar triple product (mixed product) is zero, then these three vectors are coplanar.

Practice 4

Does the notation $\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{V}} \times \vec{\boldsymbol{W}}$ make sense?
Why? How about the notation $\overrightarrow{\boldsymbol{U} \times} \vec{\boldsymbol{V}} \times \vec{\boldsymbol{W}}$ **?**

Show that x = {1; 1; 1}, y = {1; 3; 1} and z = {2; 2; 2} are three coplanar vectors.

Find the volume of the parallelepiped formed by the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and $\mathbf{c} = 3\mathbf{j} + \mathbf{k}$.

Review Practice 5

In the figure,

$d_1: x + y = 0$,

$A(\sqrt{6}, 0)$,

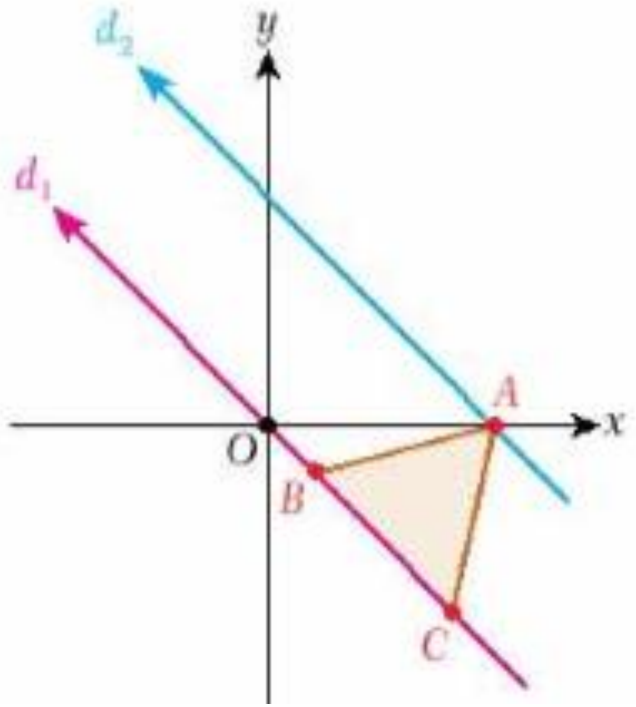
and $d_1 \parallel d_2$

are given.

If ΔABC is an

equilateral triangle,

find its area.

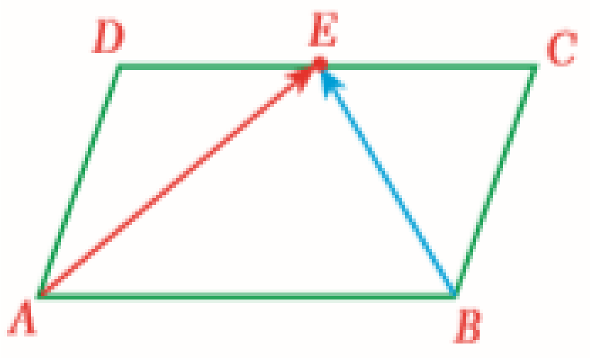


Find the lengths of the altitudes of the triangle bounded by the lines $x - y = 1$, $4x + 3y = 4$, and $x + y = 2$.

Review Practice 5

$A(2, 5)$, $B(-1, 3)$, $C(m, 6)$, and $\overrightarrow{AB} \perp \overrightarrow{BC}$ are given. Find m.

In the figure, $ABCD$ is a parallelogram and $|BC| = 1$, $|DC| = 2$, $|DE| = |EC|$. Find $(\overrightarrow{BC} + \overrightarrow{CE}) \cdot (\overrightarrow{AD} + \overrightarrow{DE})$.



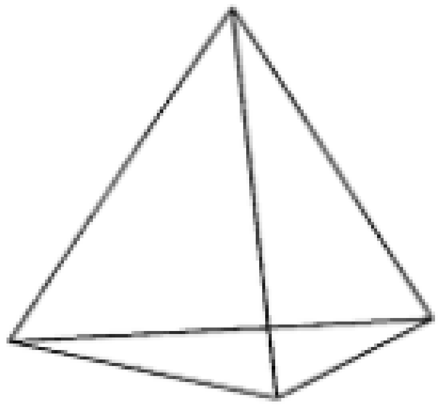
Show that the centroid of a triangle divides a median in the ratio 1:2 using vectors.

Review Practice 5

If $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, show that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$.

Find a unit vector perpendicular to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$.

Find the volume of tetrahedron whose vertices are A(1,1,0) B(-4,3,6) C(-1,0,3) and D(2,4,-5).



tetrahedron

$$\frac{1}{6} \times \left| \left\{ \left(\overrightarrow{AB} \times \overrightarrow{AC} \right) . \overrightarrow{AD} \right\} \right. \qquad \frac{1}{6} \times |-36| = \frac{36}{6} = 6$$

Review Practice 5

Exercise . Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AB = BA$, where $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Show that

$$\begin{vmatrix} x-2 & x-3 & x-4 \\ x+1 & x-1 & x-3 \\ x-4 & x-7 & x-10 \end{vmatrix} = 0.$$