Continuity

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BHOS

Calculus

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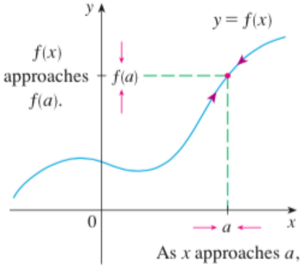
Geometrically, if the graph of a function $y = f(x)$ can be sketched at $x = a$ in one unbroken motion, then f is a continuous function.



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If a function f is not continuous at a point a in its domain then we say that f is discontinuous at $x = a$.

To be continuous, f must satisfy the following conditions:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Recall: $\lim_{x \to a} f(x)$ exist if and only if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and are equal.

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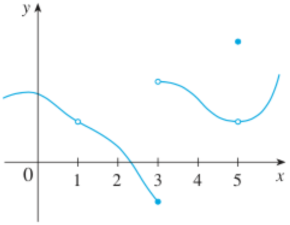

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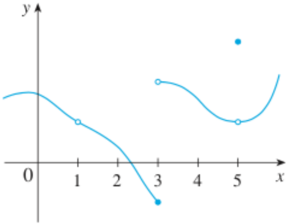

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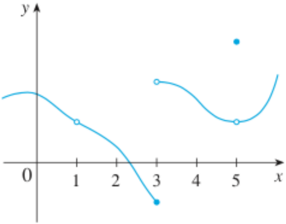

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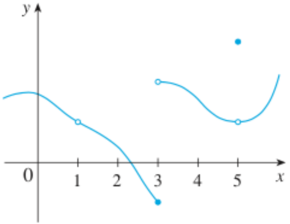

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1. There is a break at $x = 1$. The reason is that $f(1)$ is not defined.
2. There is a break at $x = 3$. $f(3)$ is defined. But $\lim_{x\to 3}$ does not exist. Why?
3. f is not continuous at $x = 5$. $f(5)$ is defined and limit exists. But $\lim_{x\to 5} \neq f(5)$.

Example: At which points are the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

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(a) $f(2)$ is not defined. So f is discontinuous at $x = 2$.

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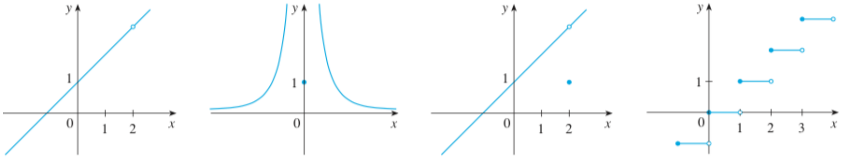
(d) $\lim_{x\to n}[[x]]$ does not exist for $n \in \mathbb{Z}$



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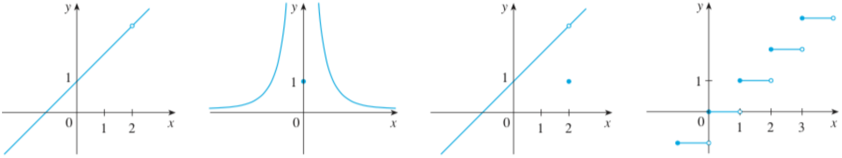

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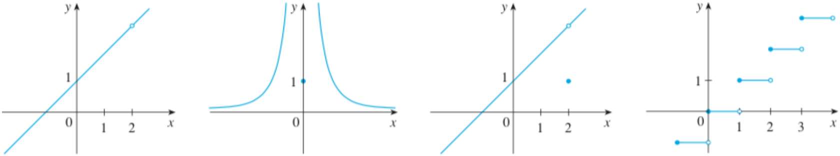

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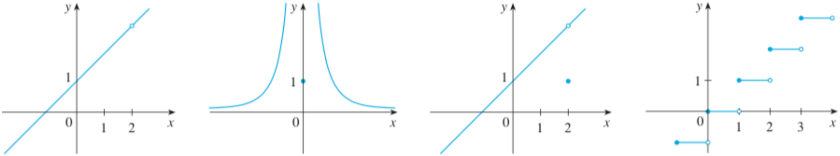

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The discontinuities in part (d) are called jump discontinuities because the function “jumps” from one value to another.

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A function f is called right continuous (or continuous from the right) at a point $x = a$ if

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and f is continuous from the left at a if

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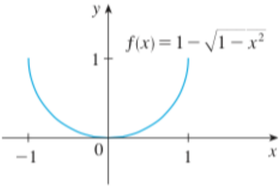

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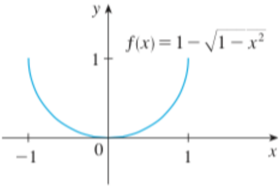

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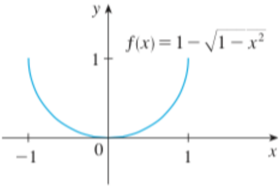

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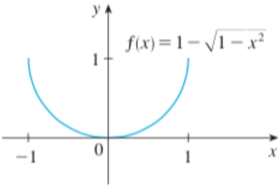

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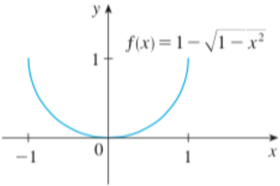

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Theorem

If f, g are continuous at $x = a$ and c is any constant then the following functions are also continuous at $x = a$:

1. $f \pm g$
2. cf
3. fg
4. $\frac{f}{g}$ *provided that* $g(a) \neq 0$.

Proof: Let us prove only first one. f, g continuous at a means that $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. Then,

$$\lim_{x \to a}(f+g)(x) = \lim_{x \to a}[f(x)+g(x)] = \lim_{x \to a} f(x)+\lim_{x \to a} g(x) = f(a)+g(a) = (f+g)(a)$$

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$$\lim_{x \to 2} \frac{x^2+1}{x^3-1} = f(2) = 5/7.$$

Theorem:
The following types of functions are continuous at every number in their domains:

polynomials

rational functions

root functions

trigonometric functions

inverse trigonometric functions

exponential functions

logarithmic functions

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Therefore, $f(x)$ is continuous everywhere except where denominator is zero.

So, $f(x)$ is continuous on $(0, \infty) \setminus \{1\} = (0, 1) \cup (1, \infty)$

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If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

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$$\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)$$

$$= \arcsin\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right) = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

Theorem

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Proof.

Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$ we can apply previous theorem to get

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(g(a))$$

Thus, we get

$$\lim_{x \to a} f(g(x)) = f(g(a)).$$

Example:

(a) $f(x) = \cos(x^3 + 2x)$

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(c) $f(x) = e^{x^2}$