Series Tutorial

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Calculus

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Strategy for Series Tests

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 12.4 have this form. (The value of p should be chosen as in Section 12.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Test the series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 3^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

1. $\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ for all $n \geq 1$. $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series $\left[|r| = \frac{1}{3} < 1\right]$, so $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ converges by the Comparison Test.

2. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2n+1)^n}{n^{2n}}\right|} = \lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty} \left(\frac{2}{n} + \frac{1}{n^2}\right) = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ converges by the Root Test.

3. $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n+2} = 1$, so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n}{n+2}$ does not exist. Thus, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ diverges by the Test for Divergence.

4. $b_n = \frac{n}{n^2+2} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since $\left(\frac{x}{x^2+2}\right)' = \frac{(x^2+2)(1) - x(2x)}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2} < 0$ for $x \geq \sqrt{2}$. Also, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+2} = \lim_{n\to\infty} \frac{1/n}{1+2/n^2} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ converges by the Alternating Series Test.

5. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2 2^n}{(-5)^{n+1}} \cdot \frac{(-5)^n}{n^2 2^{n-1}}\right| = \lim_{n\to\infty} \frac{2(n+1)^2}{5n^2} = \frac{2}{5} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = \frac{2}{5}(1) = \frac{2}{5} < 1$, so the series $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$ converges by the Ratio Test.

6. Use the Limit Comparison Test with $a_n = \frac{1}{2n+1}$ and $b_n = \frac{1}{n}$: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+(1/n)} = \frac{1}{2} > 0$.

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{2n+1}$. [*Or:* Use the Integral Test.]

7. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\int \frac{1}{x\sqrt{\ln x}}\,dx \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{2^{k+1}}{(k+2)(k+3)}\cdot\frac{(k+1)(k+2)}{2^k}\right| = \lim_{k\to\infty}\left(2\cdot\frac{k+1}{k+3}\right) = 2 > 1,$$ so the series diverges.

Or: Use the Test for Divergence.

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^2}{e^{k+1}}\cdot\frac{e^k}{k^2}\right| = \lim_{k\to\infty}\left[\left(\frac{k+1}{k}\right)^2\cdot\frac{1}{e}\right] = 1^2\cdot\frac{1}{e} = \frac{1}{e} < 1,$$ so the series converges.

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1,\infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1,\infty)$ as well, and we can apply the Integral Test. $\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the integral converges, and hence, the series converges.

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

2. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

3. $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$

$\boxed{\textbf{4.}}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^4}$

7. $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

2. The series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has positive terms and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right] = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

3. $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$. Using the Ratio Test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n}\right| = \lim_{n\to\infty} \left|\frac{-10}{n+1}\right| = 0 < 1$, so the series is absolutely convergent.

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty} \frac{2^n}{n^4} = \infty$, so $\lim_{n\to\infty} (-1)^{n-1} \frac{2^n}{n^4}$ does not exist.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ is a divergent p-series $(p = \frac{1}{4} \le 1)$, so the given series is conditionally convergent.

6. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series $(p = 4 > 1)$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is absolutely convergent.

7. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left[\frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k}\right] = \lim_{k\to\infty}\frac{k+1}{k}\left(\frac{2}{3}\right)^1 = \frac{2}{3}\lim_{k\to\infty}\left(1+\frac{1}{k}\right) = \frac{2}{3}(1) = \frac{2}{3} < 1$, so the series $\sum_{n=1}^{\infty} k\left(\frac{2}{3}\right)^k$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

8. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)!}{100^{n+1}}\cdot\frac{100^n}{n!}\right] = \lim_{n\to\infty}\frac{n+1}{100} = \infty$, so the series $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges by the Ratio Test.

9. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(1.1)^{n+1}}{(n+1)^4}\cdot\frac{n^4}{(1.1)^n}\right] = \lim_{n\to\infty}\frac{(1.1)n^4}{(n+1)^4} = (1.1)\lim_{n\to\infty}\frac{1}{\frac{(n+1)^4}{n^4}} = (1.1)\lim_{n\to\infty}\frac{1}{(1+1/n)^4}$

$= (1.1)(1) = 1.1 > 1,$

so the series $\sum_{n=1}^{\infty}(-1)^n\frac{(1.1)^n}{n^4}$ diverges by the Ratio Test.

Find the radius of convergence and interval of convergence of the series.

$\boxed{3.}$ $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

4. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3}$

6. $\sum_{n=1}^{\infty} \sqrt{n}\, x^n$

$\boxed{7.}$ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

8. $\sum_{n=1}^{\infty} n^n x^n$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$

10. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$

11. $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$

12. $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$

13. $\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$

14. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\boxed{15.}$ $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

16. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$

3. If $a_n = \frac{x^n}{\sqrt{n}}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{\sqrt{n+1}/\sqrt{n}}\right| = \lim_{n\to\infty}\frac{|x|}{\sqrt{1+1/n}} = |x|$.

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges because it is a p-series with $p = \frac{1}{2} \leq 1$. When $x = -1$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-1, 1)$.

4. If $a_n = \frac{(-1)^n x^n}{n+1}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+2}\cdot\frac{n+1}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/(n+1)} = |x|$.

By the Ratio Test, the series $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

5. If $a_n = \dfrac{(-1)^{n-1}x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)xn^3}{(n+1)^3}\right| = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^3|x|\right] = 1^3\cdot|x| = |x|.$$ By the Ratio Test, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n^3}$ converges when $|x|<1$, so the radius of convergence $R=1$. Now we'll check the endpoints, that is, $x=\pm1$. When $x=1$, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test. When $x=-1$, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}(-1)^n}{n^3} = -\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ converges because it is a constant multiple of a convergent p-series $[p=3>1]$. Thus, the interval of convergence is $I=[-1,1]$.

6. $a_n = \sqrt{n}\,x^n$, so we need $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n} = \lim_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x| = |x| < 1$ for convergence (by the Ratio Test), so $R=1$. When $x=\pm1$, $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\sqrt{n} = \infty$, so the series diverges by the Test for Divergence. Thus, $I=(-1,1)$.

7. If $a_n = \dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = |x|\cdot 0 = 0 < 1$ for *all* real

So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

8. Here the Root Test is easier. If $a_n = n^n x^n$ then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} n|x| = \infty$ if $x \neq 0$, so $R = 0$ and $I = \{0\}$.

9. If $a_n = (-1)^n \dfrac{n^2 x^n}{2^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n^2 x^n}\right| = \lim_{n\to\infty}\left|\frac{x(n+1)^2}{2n^2}\right| = \lim_{n\to\infty}\left[\frac{|x|}{2}\left(1+\frac{1}{n}\right)^2\right] = \frac{|x|}{2}(1)^2 = \tfrac{1}{2}|x|.$$ By th

Ratio Test, the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2 x^n}{2^n}$ converges when $\frac{1}{2}|x| < 1 \quad \Leftrightarrow \quad |x| < 2$, so the radius of convergence is $R = 2$.

When $x = \pm 2$, both series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2(\pm 2)^n}{2^n} = \sum_{n=1}^{\infty}(\mp 1)^n n^2$ diverge by the Test for Divergence since

$\lim_{n\to\infty}\left|(\mp 1)^n n^2\right| = \infty$. Thus, the interval of convergence is $I = (-2, 2)$.

10. If $a_n = \frac{10^n x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1} x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n x^n}\right| = \lim_{n\to\infty}\left|\frac{10x\, n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\frac{10|x|}{(1+1/n)^3} = \frac{10|x|}{1^3} = 10|x|$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{10^n x^n}{n^3}$ converges when $10|x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{10}$, so the radius of convergence is $R = \frac{1}{10}$.

When $x = -\frac{1}{10}$, the series converges by the Alternating Series Test; when $x = \frac{1}{10}$, the series converges because it is a p-series with $p = 3 > 1$. Thus, the interval of convergence is $I = \left[-\frac{1}{10}, \frac{1}{10}\right]$.