Sequences and Series Tutorial

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BHOS

Calculus

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Find the general term for the sequences.

11. $\{2, 7, 12, 17, \ldots\}$

12. $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\right\}$

$\boxed{\textbf{13.}}$ $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$

14. $\{5, 1, 5, 1, 5, 1, \ldots\}$

11. $\{2, 7, 12, 17, \ldots\}$. Each term is larger than the preceding one by 5, so $a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

12. $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\right\}$. The numerator of the nth term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

13. $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

17–46 Determine whether the sequence converges or diverges. If it converges, find the limit.

17. $a_n = 1 - (0.2)^n$

18. $a_n = \dfrac{n^3}{n^3 + 1}$

19. $a_n = \dfrac{3 + 5n^2}{n + n^2}$

20. $a_n = \dfrac{n^3}{n + 1}$

21. $a_n = e^{1/n}$

22. $a_n = \dfrac{3^{n+2}}{5^n}$

23. $a_n = \tan\left(\dfrac{2n\pi}{1 + 8n}\right)$

24. $a_n = \sqrt{\dfrac{n + 1}{9n + 1}}$

17. $a_n = 1 - (0.2)^n$, so $\lim_{n\to\infty} a_n = 1 - 0 = 1$ by (9). Converges

18. $a_n = \dfrac{n^3}{n^3+1} = \dfrac{n^3/n^3}{(n^3+1)/n^3} = \dfrac{1}{1+1/n^3}$, so $a_n \to \dfrac{1}{1+0} = 1$ as $n \to \infty$. Converges

19. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{(3+5n^2)/n^2}{(n+n^2)/n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

20. $a_n = \dfrac{n^3}{n+1} = \dfrac{n^3/n}{(n+1)/n} = \dfrac{n^2}{1+1/n^2}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim_{n\to\infty} n^2 = \infty$ and $\lim_{n\to\infty} (1+1/n^2) = 1$. Diverges

21. Because the natural exponential function is continuous at 0, Theorem 7 enables us to write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n} = e^{\lim_{n\to\infty}(1/n)} = e^0 = 1.$$ Converges

21. Because the natural exponential function is continuous at 0, Theorem 7 enables us to write

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22. $a_n = \dfrac{3^{n+2}}{5^n} = \dfrac{3^2 3^n}{5^n} = 9\left(\frac{3}{5}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 9 \lim\limits_{n\to\infty} \left(\frac{3}{5}\right)^n = 9 \cdot 0 = 0$ by (9) with $r = \frac{3}{5}$. Converges

23. If $b_n = \dfrac{2n\pi}{1+8n}$, then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{(2n\pi)/n}{(1+8n)/n} = \lim\limits_{n\to\infty} \dfrac{2\pi}{1/n+8} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$. Since tan is continuous at $\frac{\pi}{4}$, by

Theorem 7, $\lim\limits_{n\to\infty} \tan\left(\dfrac{2n\pi}{1+8n}\right) = \tan\left(\lim\limits_{n\to\infty} \dfrac{2n\pi}{1+8n}\right) = \tan\dfrac{\pi}{4} = 1.$ Converges

24. Using the last limit law for sequences and the continuity of the square root function,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty} \frac{1+1/n}{9+1/n}} = \sqrt{\frac{1}{9}} = \frac{1}{3}. \quad \text{Converges}$$

57. For what values of r is the sequence $\{nr^n\}$ convergent?

58. (a) If $\{a_n\}$ is convergent, show that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geqslant 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

59. Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

57. If $|r| \geq 1$, then $\{r^n\}$ diverges by (9), so $\{nr^n\}$ diverges also, since $|nr^n| = n|r^n| \geq |r^n|$. If $|r| < 1$ then

$$\lim_{x\to\infty} xr^x = \lim_{x\to\infty} \frac{x}{r^{-x}} \stackrel{H}{=} \lim_{x\to\infty} \frac{1}{(-\ln r)r^{-x}} = \lim_{x\to\infty} \frac{r^x}{-\ln r} = 0,$$ so $\lim_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges

whenever $|r| < 1$.

58. (a) Let $\lim_{n\to\infty} a_n = L$. By Definition 2, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n + 1 > N \Leftrightarrow n > N - 1$. It follows that $\lim_{n\to\infty} a_{n+1} = L$ and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

(b) If $L = \lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1+\sqrt{5}}{2}$ (since L has to be nonnegative if it exists).

59. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

60–66 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

60. $a_n = (-2)^{n+1}$

$\boxed{\textbf{61.}}$ $a_n = \dfrac{1}{2n + 3}$

62. $a_n = \dfrac{2n - 3}{3n + 4}$

63. $a_n = n(-1)^n$

64. $a_n = ne^{-n}$

65. $a_n = \dfrac{n}{n^2 + 1}$

66. $a_n = n + \dfrac{1}{n}$

60. The terms of $a_n = (-2)^{n+1}$ alternate in sign, so the sequence is not monotonic. The first five terms are $4, -8, 16, -32$, and 64. Since $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} 2^{n+1} = \infty$, the sequence is not bounded.

61. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

62. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

$f'(x) = \dfrac{(3x+4)(2)-(2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$,

and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \geq 1$.

63. The terms of $a_n = n(-1)^n$ alternate in sign, so the sequence is not monotonic. The first five terms are $-1, 2, -3, 4$, and -5. Since $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} n = \infty$, the sequence is not bounded.

64. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$.

$[f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x) < 0$ for $x > 1$.] The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

65. $a_n = \dfrac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2+1}$, $f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$

for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

66. $a_n = n + \dfrac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \dfrac{1}{x}$ is increasing for $x > 1$. $[g'(x) = 1 - 1/x^2 > 0$

for $x > 1$.] The sequence is unbounded since $a_n \to \infty$ as $n \to \infty$. (It is, however, bounded below by $a_1 = 2$.)

69. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

69. $a_1 = 1$, $a_{n+1} = 3 - \dfrac{1}{a_n}$. We show by induction that $\{a_n\}$ is increasing and bounded above by 3. Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1} > a_n \Rightarrow \dfrac{1}{a_{n+1}} < \dfrac{1}{a_n} \Rightarrow -\dfrac{1}{a_{n+1}} > -\dfrac{1}{a_n}$. Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \Leftrightarrow P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem. If $L = \lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3+\sqrt{5}}{2}$.

77. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.

78. Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$.

(a) Show that if $0 \leqslant a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

(b) Deduce that $b^n[(n + 1)a - nb] < a^{n+1}$.

(c) Use $a = 1 + 1/(n + 1)$ and $b = 1 + 1/n$ in part (b) to show that $\{a_n\}$ is increasing.

(d) Use $a = 1$ and $b = 1 + 1/(2n)$ in part (b) to show that $a_{2n} < 4$.

77. **To Prove:** If $\lim\limits_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim\limits_{n\to\infty}(a_n b_n) = 0$.

Proof: Since $\{b_n\}$ is bounded, there is a positive number M such that $|b_n| \le M$ and hence, $|a_n|\,|b_n| \le |a_n|\,M$ for all $n \ge 1$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = 0$, there is an integer N such that $|a_n - 0| < \dfrac{\varepsilon}{M}$ if $n > N$. Then $|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n| \le |a_n|\,M = |a_n - 0|\,M < \dfrac{\varepsilon}{M}\cdot M = \varepsilon$ for all $n > N$. Since ε was arbitrary, $\lim\limits_{n\to\infty}(a_n b_n) = 0$.

78. (a) $$\frac{b^{n+1} - a^{n+1}}{b - a} = b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n$$
$$< b^n + b^{n-1}b + b^{n-2}b^2 + b^{n-3}b^3 + \cdots + bb^{n-1} + b^n = (n+1)b^n$$

(b) Since $b - a > 0$, we have $b^{n+1} - a^{n+1} < (n+1)b^n(b-a) \;\Rightarrow\; b^{n+1} - (n+1)b^n(b-a) < a^{n+1} \;\Rightarrow\; b^n[(n+1)a - nb] < a^{n+1}$.

(c) With this substitution, $(n+1)a - nb = 1$, and so $b^n = \left(1 + \dfrac{1}{n}\right)^n < a^{n+1} = \left(1 + \dfrac{1}{n+1}\right)^{n+1}$.

$\boxed{9.}$ Let $a_n = \dfrac{2n}{3n + 1}$.

(a) Determine whether $\{a_n\}$ is convergent.

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

9. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (12.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

47–51 Find the values of x for which the series converges. Find the sum of the series for those values of x.

$\boxed{\textbf{47.}}$ $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

48. $\sum_{n=1}^{\infty} (x-4)^n$

49. $\sum_{n=0}^{\infty} 4^n x^n$

50. $\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$

51. $\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$

47. $\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$ is a geometric series with $r = \frac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\frac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$; that is, $-3 < x < 3$. In that case, the sum of the series is $\frac{a}{1-r} = \frac{x/3}{1-x/3} = \frac{x/3}{1-x/3} \cdot \frac{3}{3} = \frac{x}{3-x}$.

48. $\sum_{n=1}^{\infty} (x-4)^n$ is a geometric series with $r = x - 4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x-4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\frac{x-4}{1-(x-4)} = \frac{x-4}{5-x}$.

49. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\frac{1}{1-4x}$.

50. $\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ is a geometric series with $r = \frac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\frac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x+3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is $\frac{1}{1-(x+3)/2} = \frac{2}{2-(x+3)} = -\frac{2}{x+1}$.

51. $\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r = \frac{\cos x}{2}$, so it converges $\Leftrightarrow$ $|r| < 1$. But $|r| = \frac{|\cos x|}{2} \leq \frac{1}{2}$ for all x. Thus, the series converges for all real values of x and the sum of the series is $\frac{1}{1-(\cos x)/2} = \frac{2}{2-\cos x}$.