Integration by Parts

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Calculus

October 25, 2023

Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts*.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

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$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

Formula 1 is called the **formula for integration by parts**. It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

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$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by Parts

EXAMPLE 1 Find $\int x \sin x \, dx$.

SOLUTION USING FORMULA 1 Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\begin{aligned}
\int x \sin x \, dx &= f(x)g(x) - \int g(x)f'(x) \, dx \\
&= x(-\cos x) - \int (-\cos x) \, dx \\
&= -x \cos x + \int \cos x \, dx \\
&= -x \cos x + \sin x + C
\end{aligned}$$

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

SOLUTION USING FORMULA 2 Let

$$u = x \qquad dv = \sin x \, dx$$

Then $$du = dx \qquad v = -\cos x$$

and so

$$\int x \sin x \, dx = \int \overbrace{x}^{u} \, \overbrace{\sin x \, dx}^{dv} = \overbrace{x}^{u} \, \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \, \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

Integration by Parts

NOTE Our aim in using integration by parts is to obtain a simpler integral than the one we started with. Thus in Example 1 we started with $\int x \sin x \, dx$ and expressed it in terms of the simpler integral $\int \cos x \, dx$. If we had instead chosen $u = \sin x$ and $dv = x \, dx$, then $du = \cos x \, dx$ and $v = x^2/2$, so integration by parts gives

$$\int x \sin x \, dx = (\sin x) \frac{x^2}{2} - \frac{1}{2} \int x^2 \cos x \, dx$$

Although this is true, $\int x^2 \cos x \, dx$ is a more difficult integral than the one we started with. In general, when deciding on a choice for u and dv, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x) \, dx$ can be readily integrated to give v.

Find the derivatives

EXAMPLE 2 Evaluate $\int \ln x \, dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x}\,dx \qquad v = x$$

Integrating by parts, we get

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \frac{dx}{x} \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f. ■

EXAMPLE 3 Find $\int t^2 e^t \, dt$.

SOLUTION Notice that t^2 becomes simpler when differentiated (whereas e^t is unchanged when differentiated or integrated), so we choose

$$u = t^2 \qquad dv = e^t \, dt$$

Then

$$du = 2t \, dt \qquad v = e^t$$

Integration by parts gives

$$\int t^2 e^t \, dt = t^2 e^t - 2 \int t e^t \, dt \tag{3}$$

The integral that we obtained, $\int te^t\,dt$, is simpler than the original integral but is still not obvious. Therefore, we use integration by parts a second time, this time with $u = t$ and $dv = e^t\,dt$. Then $du = dt$, $v = e^t$, and

$$\int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C$$

Putting this in Equation 3, we get

$$\begin{aligned}\int t^2e^t\,dt &= t^2e^t - 2\int te^t\,dt \\ &= t^2e^t - 2(te^t - e^t + C) \\ &= t^2e^t - 2te^t + 2e^t + C_1 \qquad \text{where } C_1 = -2C\end{aligned}$$

■

EXAMPLE 4 Evaluate $\int e^x \sin x \, dx$.

SOLUTION Neither e^x nor $\sin x$ becomes simpler when differentiated, but we try choosing $u = e^x$ and $dv = \sin x \, dx$ anyway. Then $du = e^x \, dx$ and $v = -\cos x$, so integration by parts gives

$$\boxed{4} \qquad \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

The integral that we have obtained, $\int e^x \cos x\,dx$, is no simpler than the original one, but at least it's no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again. This time we use $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

5 $$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

At first glance, it appears as if we have accomplished nothing because we have arrived at $\int e^x \sin x \, dx$, which is where we started. However, if we put the expression for $\int e^x \cos x \, dx$ from Equation 5 into Equation 4 we get

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

Integration by Parts

This can be regarded as an equation to be solved for the unknown integral. Adding $\int e^x \sin x \, dx$ to both sides, we obtain

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x \, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C$$

Integration by Parts

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between a and b, assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

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$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx}$$

EXAMPLE 5 Calculate $\int_0^1 \tan^{-1}x\,dx$.

SOLUTION Let

$$u = \tan^{-1}x \qquad dv = dx$$

Then

$$du = \frac{dx}{1 + x^2} \qquad v = x$$

So Formula 6 gives

$$\begin{aligned}
\int_0^1 \tan^{-1}x\,dx &= x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1 + x^2}\,dx \\
&= 1 \cdot \tan^{-1}1 - 0 \cdot \tan^{-1}0 - \int_0^1 \frac{x}{1 + x^2}\,dx \\
&= \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2}\,dx
\end{aligned}$$

To evaluate this integral we use the substitution $t = 1 + x^2$ (since u has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dt$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\int_0^1 \frac{x}{1 + x^2}\,dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\ln|t|\Big]_1^2$$

$$= \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Therefore $$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$ ■

EXAMPLE 6 Prove the reduction formula

$$\boxed{7} \qquad \int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\, dx$$

where $n \geqslant 2$ is an integer.

SOLUTION Let $\quad u = \sin^{n-1}x \qquad dv = \sin x\, dx$

Then $\quad du = (n-1)\sin^{n-2}x \cos x\, dx \qquad v = -\cos x$

so integration by parts gives

$$\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \cos^2 x\, dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx - (n-1)\int \sin^n x\, dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n \int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, dx$$

or
$$\int \sin^n x\, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$$

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\, dx$ in terms of $\int \sin x\, dx$ (if n is odd) or $\int (\sin x)^0\, dx = \int dx$ (if n is even).