Derivation of Logistic Regression

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Our hypothesis function can be defined as;

$$h_\theta(x) = g(\theta^T x)$$

Where;

$$g(z) = \frac{1}{1 + e^{-z}}$$

Cost function will be;

$$J(\theta) = -\frac{1}{m} \sum_{i=1}^{m} \left[y^{(i)} \log\left(h_\theta(x^{(i)})\right) + (1 - y^{(i)}) \log\left(1 - h_\theta(x^{(i)})\right) \right]$$

Cost function's vectorized implementation is;

$$h = g(X\theta)$$

$$J(\theta) = \frac{1}{m} \left(-y^T \log(h) - (1-y)^T \log(1-h)\right)$$

Now, coming to Gradient Descent

Remember the general form of the gradient descent is:

Repeat {

$$\theta_j = \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

}

So, gradient descent for the logistic regression is;

Repeat {

$$\theta_j = \theta_j - \frac{\alpha}{m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

}

And its vectorized form is;

$$\theta = \theta - \frac{\alpha}{m} X^T (g(X\theta) - y)$$

So, lets come to derivation;

Derivative of sigmoid function;

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$\sigma\ (x)' = \left(\frac{1}{1+e^{-x}}\right)'$$

$$= -\frac{(1+e^{-x})'}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2} = \sigma(x)\sigma(1-x)$$

Now take partial derivative of cost function;

$$\frac{\partial}{\partial\theta_j}J(\theta) = \frac{\partial}{\partial\theta_j} - \frac{1}{m}\sum_{i=1}^{m}[y^{(i)}\log\left(h_\theta(x^{(i)})\right) + (1-y^{(i)})\log\left(1-h_\theta(x^{(i)})\right)]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}\left[y^{(i)}\frac{\partial}{\partial\theta_j}\log\left(h_\theta(x^{(i)})\right) + (1-y^{(i)})\frac{\partial}{\partial\theta_j}\log\left(1-h_\theta(x^{(i)})\right)\right]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}\left[\frac{y^{(i)}\frac{\partial}{\partial\theta_j}h_\theta(x^{(i)})}{h_\theta(x^{(i)})} + \frac{(1-y^{(i)})\frac{\partial}{\partial\theta_j}\left(1-h_\theta(x^{(i)})\right)}{1-h_\theta(x^{(i)})}\right]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}\left[\frac{y^{(i)}\frac{\partial}{\partial\theta_j}\sigma(\theta^T x^{(i)})}{h_\theta(x^{(i)})} + \frac{(1-y^{(i)})\frac{\partial}{\partial\theta_j}\left(1-\sigma(\theta^T x^{(i)})\right)}{1-h_\theta(\theta^T x^{(i)})}\right]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}\left[\frac{y^{(i)}\sigma(\theta^T x^{(i)})\left(1-\sigma(\theta^T x^{(i)})\right)\frac{\partial}{\partial\theta_j}\theta^T x^{(i)}}{h_\theta(x^{(i)})} + \frac{-(1-y^{(i)})\sigma(\theta^T x^{(i)})\left(1-\sigma(\theta^T x^{(i)})\right)\frac{\partial}{\partial\theta_j}\theta^T x^{(i)}}{1-h_\theta(x^{(i)})}\right]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}\left[y^{(i)}\left(1-h_\theta(x^{(i)})\right)x_j^{(i)} - (1-y^{(i)})h_\theta(x^{(i)})x_j^{(i)}\right]$$

$$= -\frac{1}{m}\sum_{i=1}^{m}[y^{(i)} - y^{(i)}h_\theta(x^{(i)}) - h_\theta(x^{(i)}) + y^{(i)}h_\theta(x^{(i)})]x_j^{(i)}$$

$$= -\frac{1}{m}\sum_{i=1}^{m}[y^{(i)} - h_\theta(x^{(i)})]x_j^{(i)}$$

$$= \frac{1}{m}\sum_{i=1}^{m}[h_\theta(x^{(i)}) - y^{(i)}]x_j^{(i)}$$