Indian Institute of Technology Kanpur
Department of Physics

PHY103A **End Sem Examination** **Date: April 21, 2016**

Time: 3 hours Max Marks: 120

Name : Model Soln Roll No. :

Tutorial Section :

I pledge my honour as a gentleman/lady that during the examination I have neither given assistance nor received assistance.

(Signature)

- Your answer for each question must be in the **space provided.** **Your rough work elsewhere will not be graded.**
- There are total **SIX** questions in this booklet. (24 numbered pages +2 for rough works)
- No other paper must be in your possession.
- No calculators or cell phones are allowed.
- You are free to use the formulae provided in page number 2.

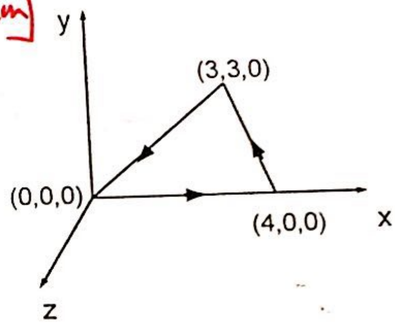
	MARKS
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
TOTAL	

1. Write only the answers in the space provided. *If any question has more than one answer, all the correct answers are required to get any credit.* Rough calculations can be done on the last page of this booklet(extra pages will be supplied if required).

(a) For $\vec{F} = 5y^2\,\hat{x} + 10(xy + xz)\,\hat{y} + 10(zx + zy)\,\hat{z}$, find out the line integral $\oint \vec{F}\cdot\vec{dl}$ over the path in the $x-y$ plane as shown in the figure: [3]

$\oint \vec{F}\cdot\vec{dl} =$ 0

[Hint : use Stoke's theorem]



(b) Consider an electric dipole $\vec{p} = p_0\hat{z}$ at the origin, find the force on another dipole $\vec{p_1} = p_1\hat{x}$ placed at $(0, 0, R)$. [3]

$\vec{F} = \dfrac{3 p_0 p_1}{4\pi\epsilon_0 R^4}\hat{x}$.

(c) Consider a large piece of magnetic material that has a constant magnetic field $\vec{B}_0$ and magnetization $\vec{M}$ so that $\vec{H}_0 = \frac{1}{\mu_0}\vec{B}_0 - \vec{M}$. A thin disk-shaped cavity perpendicular to $\vec{M}$ is hollowed out of the material. Find $\vec{B}$, $\vec{H}$ at the center of the cavity in terms of $\vec{B}_0, \vec{H}_0$ and $\vec{M}$. [2]

$\vec{B} =$ $\vec{B_0}$ $\qquad$ $\vec{H} =$ $\vec{H_0} + \vec{M}$

(d) An EM wave in free space is given by $\vec{E} = 10\sin(2x+3y+4z-\omega t)\hat{e}$. Which of the following is/are valid expression(s) for the unit vector $\hat{e}$? [2]

(Tick the right answer(s))

✓ (i) $(2\hat{x} - 4\hat{y} + 2\hat{z})/(2\sqrt{6})$ $\qquad$ (ii) $(2\hat{x} + 3\hat{y} + 4\hat{z})/(\sqrt{29})$

✓ (iii) $(3\hat{x} - 2\hat{y})/(\sqrt{13})$ $\qquad$ (iv) $(2\hat{x} - 3\hat{y} + 4\hat{z})/(\sqrt{29})$

(e) Find the divergence of $\vec{J} + \epsilon_0(\partial \vec{E}/\partial t)$. [2]

Ans: 0

(f) The magnetic field of an electromagnetic wave propagating in vacuum is given by $\vec{B}(z,t) = 6 \times 10^{-8} \hat{y} \cos(6 \times 10^5 z + \omega t)$ Tesla. Write down the corresponding electric field. [3]

$\vec{E}(z,t) =$ $-18\,\hat{x}\,\cos(6\times10^5 z + \omega t)$ V/m

(g) What is the state of polarization of the light represented by [2]

$$\vec{E} = E_0\ \hat{x}\ \cos(kz - \omega t) + E_0\ \hat{y}\ \sin(kz - \omega t + \pi/2).$$

Polarization state : Linear (linear/circular/elliptical)

Polarization direction : $\frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ ($\hat{x}, \hat{y}, \hat{z}$, left-handed, right-handed)

(h) Skin depth of a good conductor in the visible regime ($\lambda = 500$ nm) is 0.6 nm. What is the skin depth at $\lambda = 700$nm? [3]

Skin depth (d) = $\sqrt{\frac{700}{500}} \times 0.6$ nm

2. (a) Consider a point charge q covered by a surface in the form a spherical dome of radius R as shown in Fig. 1(a). Calculate the total electric flux through this surface. [2]

(b) A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from, the cylinder axis ($d+b < a$) as shown in Fig. 1(b). The current density $\vec{J} = J\hat{k}$ is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Find the magnitude and the direction of the magnetic field in the hole. [6]

(c) Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemi-spherical shell of dielectric (of dielectric constant ϵ_r, as shown in the Fig. 1(c).

(i) Find the electric field everywhere between the spheres. [3]

(ii) Calculate the surface charge density on the inner sphere. [3]

(iii) Calculate the bound surface charge density induced on the surface of the dielectric at $r = a$. [2]

(d) What is the electric charge density of a uniformly charged spherical surface of radius b. [2]

(e) Find the gradient of the scalar function $\phi = 1/|\vec{r}|$. [2]

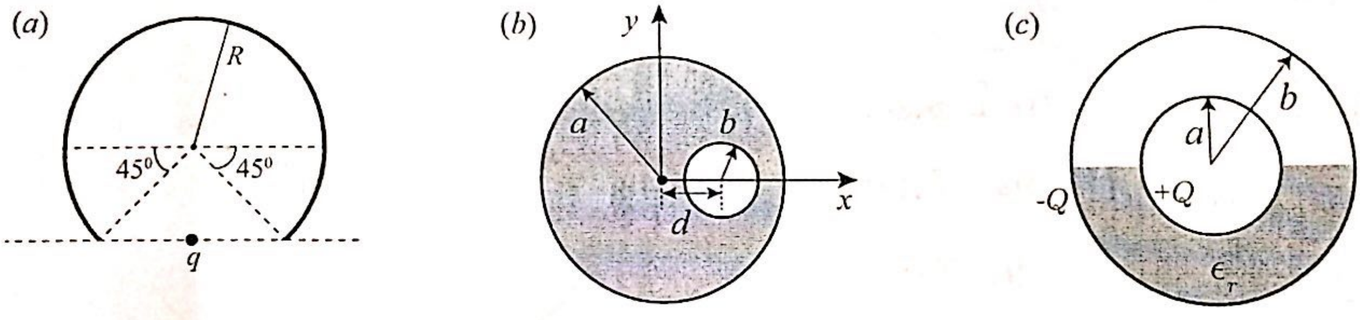


FIG. 1:

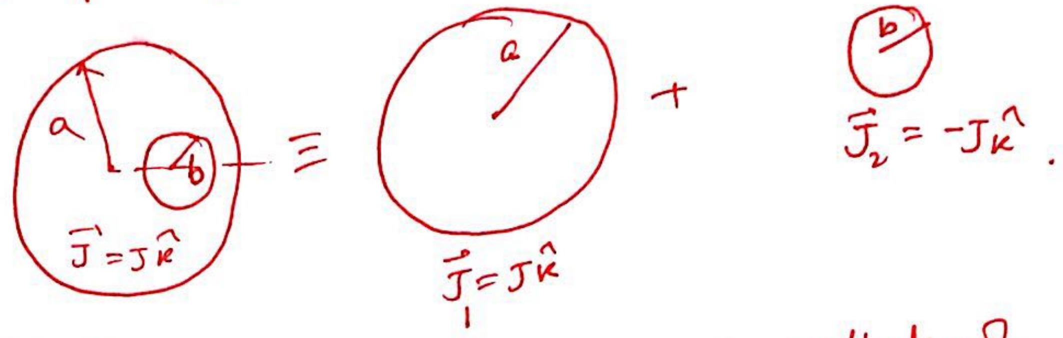
a) 

Total flux $= q/\epsilon_0$

half of the flux goes above the line and enters into the dome

$\Rightarrow$ flux through the given surface $= \left(\dfrac{q}{2\epsilon_0}\right)$

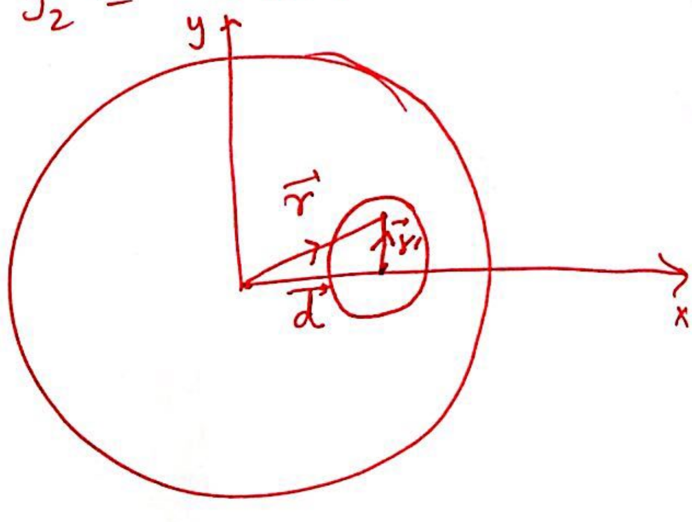
b) Apply principle of Superposition



Total $\vec{B} = \vec{B_1} + \vec{B_2}$

$\vec{B_1}$ = magnetic field due to the solid cylinder of radius a and carrying current density $\vec{J_1} = J\hat{k}$.

$\vec{B_2}$ = field due to the cylinder of radius b & carrying $\vec{J_2} = -J\hat{k}$.



Field due to $\vec{J_1}$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$.

$\Rightarrow B_1 \cdot 2\pi r = \mu_0 J \pi r^2$

$\Rightarrow B_1 = \frac{\mu_0 J r}{2}$

$\Rightarrow \vec{B_1} = \frac{\mu_0 J}{2}(\hat{k} \times \vec{r})$

Field due to $\vec{J_2}$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$. $\Rightarrow B_2 2\pi r' = -\mu_0 J \pi r'^2$

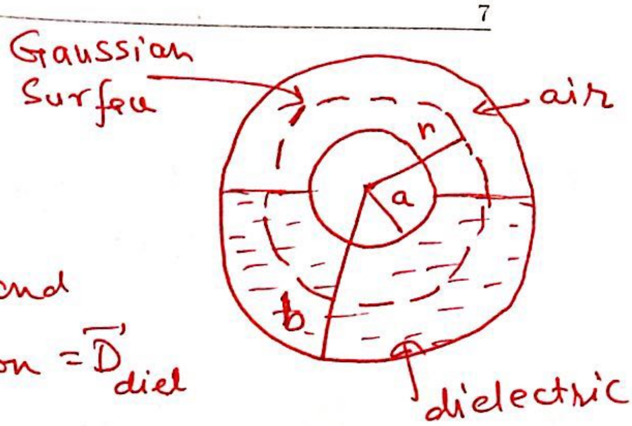
$\Rightarrow \vec{B_2} = -\frac{\mu_0 J}{2}(\hat{k} \times \vec{r}')$

$\Rightarrow \vec{B} = \vec{B_1} + \vec{B_2} = \frac{\mu_0 J}{2} \hat{k} \times (\vec{r} - \vec{r}') = \frac{\mu_0 J}{2}(\hat{k} \times \vec{d}) = \frac{\mu_0 J d}{2} \hat{j}$

$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 J d}{2} \hat{j}}$

c)

(i) $\oint \vec{D} \cdot d\vec{a} = Q_{f\,encl.}$



let,
Displacement vector in the 'air' region $= \vec{D}_{air}$, and that in the dielectric region $= \vec{D}_{diel}$

Because of symmetry in the problem, $\vec{D}$ only depends on r and has no ϕ-dependence.

For the Gaussian surface as shown in the figure

$$D_{air} \cdot 2\pi r^2 + D_{diel} \cdot 2\pi r^2 = Q$$

$$\Rightarrow \epsilon_0 E \, 2\pi r^2 + \epsilon_0 \epsilon_r E \, 2\pi r^2 = Q$$

$$\Rightarrow E = \frac{Q}{2\pi \epsilon_0 (1+\epsilon_r) r^2} \Rightarrow \boxed{\vec{E} = \frac{Q}{2\pi\epsilon_0(1+\epsilon_r) r^2}\hat{r}}$$

ii) Boundary condition $D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$

For the inner surface at $r=a$

$$D^{\perp}_{below} = 0 \Rightarrow D^{\perp}_{above} = \sigma_f$$

$\Rightarrow$ In 'air' region: $\sigma_{air} = D_{air} = \epsilon_0 E = \dfrac{Q}{2\pi(1+\epsilon_r) a^2}$

In "dielectric" region: $\sigma_{diel} = D_{diel} = \epsilon_0 \epsilon_r E = \dfrac{\epsilon_r Q}{2\pi(1+\epsilon_r) a^2}$.

(iii) In the dielectric region

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \epsilon_0 \epsilon_r \vec{E} - \epsilon_0 \vec{E} = \epsilon_0 (\epsilon_r - 1)\vec{E}$$

Bound surface charge at $r=a$

$$\boxed{\sigma_b|_{r=a} = \vec{P}\cdot\hat{n}|_{r=a} = \vec{P}\cdot(-\hat{r})|_{r=a} = \frac{1-\epsilon_r}{1+\epsilon_r}\frac{Q}{2\pi a^2}}$$

d) $\rho(r) = \frac{q}{4\pi b^2}\,\delta(r-b)$ if q = total charge.

e) $\vec{\nabla}\frac{1}{|\vec{r}|} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\frac{1}{\sqrt{x^2+y^2+z^2}}$

$$= \frac{-x\hat{x} - y\hat{y} - z\hat{z}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\vec{r}}{r^3}.$$

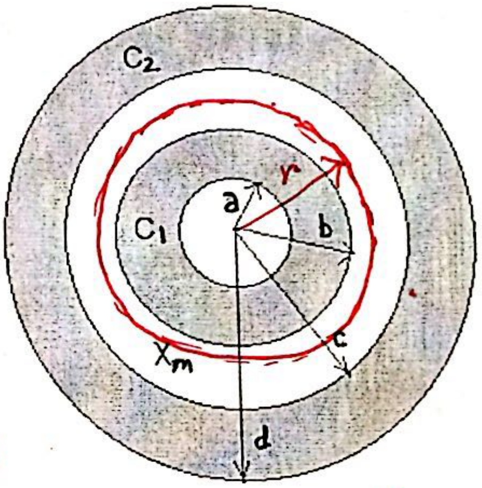
3. (a) Consider a long conducting thick cylinder (C_1) of inner radius a and outer radius b. It is surrounded by another thick cylinder (C_2) of inner radius c and outer radius d, as shown in the figure. The cylinders C_1 and C_2 carry equal currents in opposite directions flowing along the axis of the cylinders. The currents are distributed uniformly in the conductors. The conductors are separated by an insulator with susceptibility χ_m.

(i) Find $\vec{H}$, $\vec{B}$, and $\vec{M}$ everywhere.

(ii) Find all the bound currents.

(iii) Plot $|H|$ as a function of radial distance r. [4 + 3 + 3]

1) $\oint \vec{H} \cdot \vec{dl} = I_{f\,encl.}$



From symmetry

$\vec{H} = H(r)\,\hat{\varphi}$

Considering Amperian loops as shown:

$$\vec{H} = \begin{cases} 0 & 0 \le r \le a \\ \dfrac{I(r^2-a^2)}{2\pi r(b^2-a^2)}\hat{\varphi} & a \le r \le b \\ \dfrac{I}{2\pi r}\hat{\varphi} & b \le r \le c \\ \dfrac{I(d^2-r^2)}{2\pi r(d^2-c^2)} & c \le r \le d \\ 0 & r \ge d \end{cases}$$

$I\hat{k}$ = current in C_1

current in C_2 $= -I\hat{k}$

$$\left.\begin{array}{l} \vec{B} = \mu_0 \vec{H} \\ M = 0 \end{array}\right\} \text{except for } b \le r \le c$$

for $b \le r \le c$ $\begin{cases} \vec{B} = \mu_0(1+\chi_m)\vec{H} = \dfrac{\mu_0(1+\chi_m)}{2\pi r} \\ \& \ \vec{M} = \chi_m H = \dfrac{\chi_m I}{2\pi r}\hat{\varphi} \end{cases}$

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ii) $\vec{J}_b = \bar{\nabla} \times \bar{M} = \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\, \frac{\chi_m I}{2\pi \cancel{r}}\right)\hat{z}$

$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\chi_m I}{2\pi}\right)\hat{z} = 0$

At $r = b$ and c, $\vec{K}_b = \vec{M} \times \hat{n}$

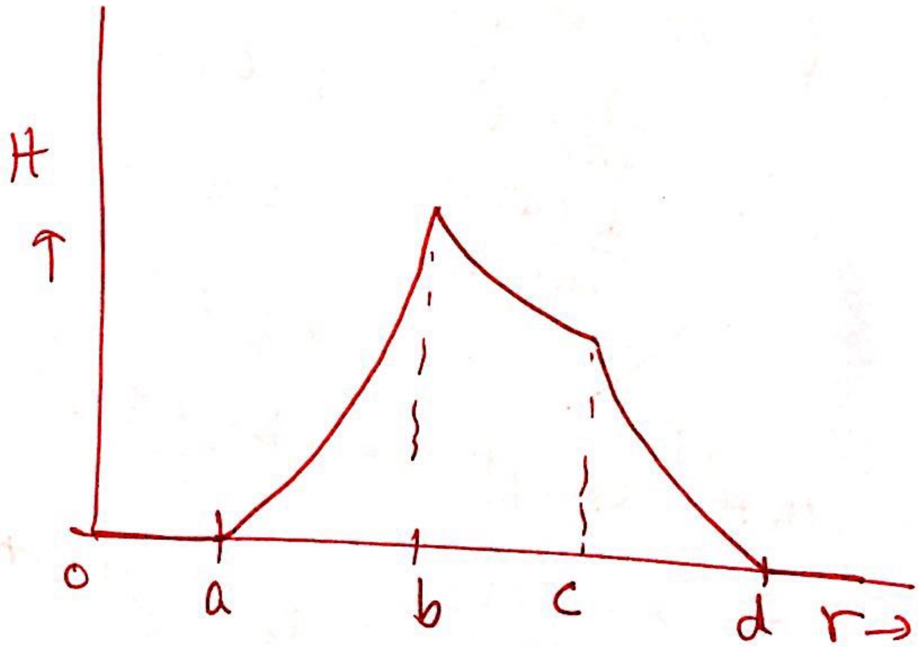
At $r = b$, $\hat{n} = -\hat{r}$ $\quad \vec{K}_b\Big|_{r=b} = \frac{\chi_m I}{2\pi b}(\hat{\varphi} \times -\hat{r})$

$= \frac{\chi_m I}{2\pi b}\hat{z}$

At $r = c$, $\hat{n} = \hat{r}$,

$\vec{K}_b\Big|_{r=c} = \frac{\chi_m I}{2\pi c}(\hat{\varphi} \times \hat{r}) = -\frac{\chi_m I}{2\pi c}\hat{z}$

iii)



(b) The electric and magnetic fields in a frame S are given by $\vec{E} = E_0\hat{y}$ and $\vec{B} = B_0\hat{z}$. Find the fields in the frame S' moving with velocity $\frac{E_0}{B_0}\hat{x}$ with respect to S. If an electron is released at rest in frame S, describe the motion of the electron as observed from the moving frame S'. [6]

$$E_x' = E_x = 0 \,, \quad E_y' = \gamma\left(E_y - vB_z\right) \qquad \left\{ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} , \quad \vec{v} = \frac{E_0}{B_0}\hat{x} \right.$$

$$E_y' = \gamma\left(E_0 - \frac{E_0}{B_0}B_0\right) = 0$$

$$E_z' = \gamma\left(E_z + vB_y\right) = 0$$

$$B_x' = B_x = 0$$

$$B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right) = 0$$

$$B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right) = \gamma\left(B_0 - \frac{v}{c^2}E_0\right)$$

$$= \gamma\left(B_0 - \frac{v^2}{c^2}B_0\right) \quad \left[\because \frac{E_0}{B_0} = v\right]$$

$$= \gamma B_0\left(1 - \frac{v^2}{c^2}\right) = \cancel{\gamma}\frac{B_0}{\gamma^{\cancel{2}}}$$

$$= \frac{B_0}{\gamma} = \frac{B_0}{\sqrt{1 - \frac{E_0^2}{B_0^2c^2}}}$$

In s'-frame $\boxed{\vec{E}' = 0 \;\&\; \vec{B}' = \frac{B_0}{\gamma}\hat{z}}$

The electron is released at rest in S-frame

$\Rightarrow$ In s'-frame, velocity of the electron $= -\vec{v}$

$= -\frac{E_0}{B_0}\hat{x}$.

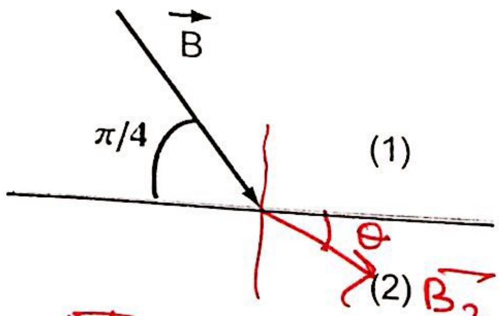
The electron will experience only the magnetic field in $\hat{z}$-direction.

$\Rightarrow$ Move in a circle with radius $\frac{mv^{\cancel{2}}}{R} = e\left(\frac{B_0}{\gamma}\right)\cancel{v}$

$\Rightarrow$ radius $\boxed{R = \frac{mv\gamma}{eB_0}}$

(c) Consider a charge free and current free interface between two homogeneous media. The dielectric permitivity and magnetic permeability of medium (1) are (ϵ_1, μ_1) and for medium(2) are (ϵ_2, μ_2). If the magnetic field in medium (1) makes an angle of $\pi/4$ with the interface, what is the angle with the interface that magnetic field makes on the other side inside medium (2).

[4]



Boundary conditions

$$\begin{cases} B^{\perp}_{above} = B^{\perp}_{below} \Rightarrow B \sin\frac{\pi}{4} = B_2 \sin\theta & -(1) \\ \dfrac{B^{\parallel}_{above}}{\mu_1} = \dfrac{B^{\parallel}_{below}}{\mu_2} \Rightarrow \dfrac{B\cos\frac{\pi}{4}}{\mu_1} = \dfrac{B_2 \cos\theta}{\mu_2} & -(\end{cases}$$

$$\Rightarrow \tan\theta = \frac{\mu_1}{\mu_2}$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{\mu_1}{\mu_2}\right)}$$

4. (a) The current I flowing along the edges of the cube as shown in Fig. a, produces a magnetic field in the centre of the cube of magnitude B_0. Consider another cube where the current I flows along a path shown in Fig. b.

(i) What is the direction of magnetic field in Fig. a? [1]

(ii) What magnetic field will now exist at the centre of the cube in Fig. b?

[Hint : Look for the symmetry in the problem] [3]

(iii) What is the magnitude of the magnetic field in Fig. b in terms of B_0? [1]

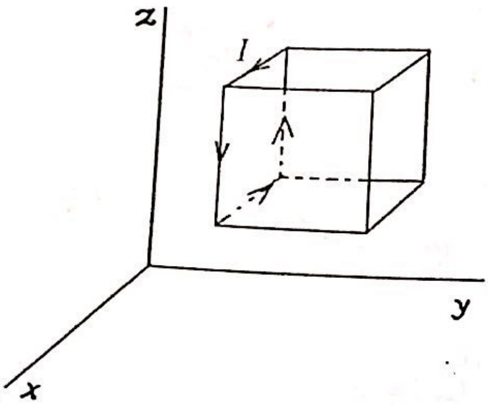


Fig. a

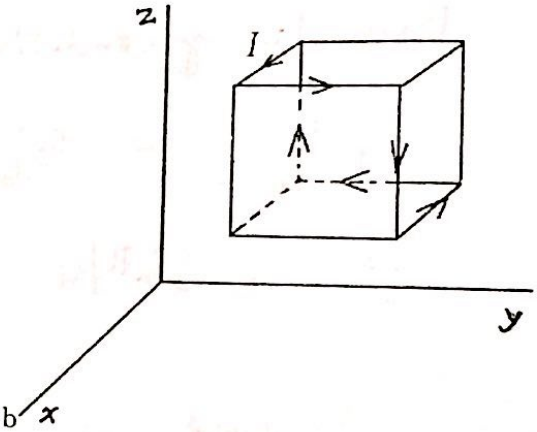
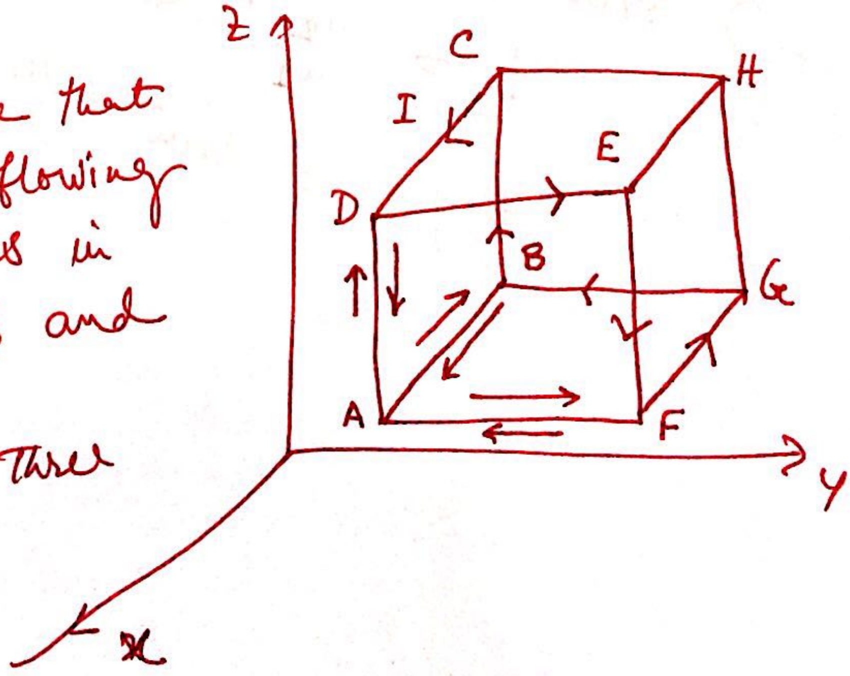


Fig. b

(i) By Bio-Savart law $\vec{B}$ will be in $\hat{y}$ direction $\rightarrow \vec{B} = B_0 \hat{y}$.

ii) We can assume that current I is flowing in both directions in the arms AD, AB and AF.

This will give three complete loops ABCD, AFGB & ADEF



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For ABCD : $\vec{B}_1 = B_0 \hat{y}$ [$\equiv$ as in fig. a]

for AFGB : $\vec{B}_2 = B_0 \hat{z}$

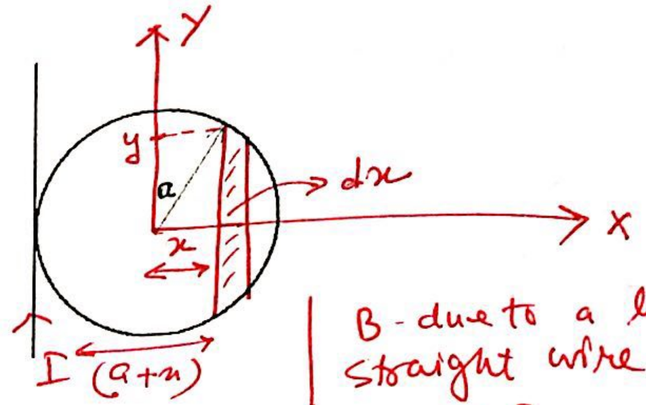
For ADEF : $\vec{B}_3 = -B_0 \hat{x}$

$\Rightarrow$ Total $\vec{B} = B_0(-\hat{x} + \hat{y} + \hat{z})$

iii) $|\vec{B}| = \sqrt{3}\, B_0 \Rightarrow \frac{|\vec{B}|}{B_0} = \sqrt{3}$.

(b) A circular ring of radius a is insulated from an infinitely long straight wire in tangential direction. Find the mutual inductance. [5]

Let us choose the co-ordinates as shown in the fig.



B - due to a long straight wire $= \frac{\mu_0 I}{2\pi r}$

Area element

$$dS = 2y\,dx$$

where $y^2 + x^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$

$$\Rightarrow dS = 2\sqrt{a^2-x^2}\,dx$$

Φ = flux through the ring $= \frac{\mu_0 I}{2\pi} \int_{-a}^{+a} \frac{2\sqrt{a^2-x^2}}{(a+x)}\,dx$

$$= \frac{\mu_0 I a}{\pi} \int_{-1}^{+1} \sqrt{\frac{1-u}{1+u}}\,du \quad (\text{where } x = au)$$

let $u = \cos 2\theta \Rightarrow du = -2\sin 2\theta\, d\theta$

$$\Rightarrow \Phi = -\frac{\mu_0 I a}{\pi} \int_{\pi/2}^{0} \frac{\sin\theta}{\cos\theta} 4 \sin\theta \cos\theta\, d\theta$$

$$= \frac{4\mu_0 I a}{\pi} \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \mu_0 I a$$

$\Rightarrow$ Mutual inductance $\boxed{M = \Phi/I = \mu_0 a}$

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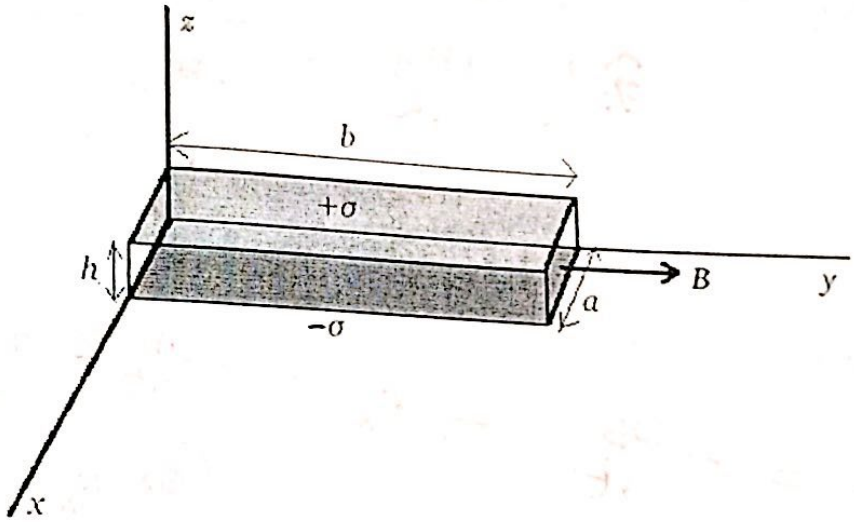
(c) The top and bottom surfaces of a rigid non-conducting box are uniformly charged with surface charge densities $+\sigma$ and $-\sigma$, respectively. The positively charged surface occupies the region $0 \leq x \leq a$ and $0 \leq y \leq b$ of the plane $z = h$, while the negatively charged surface occupies the region $0 \leq x \leq a$ and $0 \leq y \leq b$ of the $z = 0$ plane(see figure). Inside the box, there is a uniform magnetic field $\mathbf{B} = B_0\hat{y}$. Assume that $h << a, b$.

(i) Find the Poynting vector inside the box. [3]

(ii) Calculate the momentum contained in the EM fields within the box. [2]

(iii) Estimate the momentum imparted to the box when the magnetic field is switched off quasi-statically in a time interval τ. [4]

(iv) Compare the magnitudes of part (ii) and (iii) above. [1]



i) $\vec{E} = -\frac{\sigma}{\epsilon_0}\hat{z}$, $\vec{B} = B_0\hat{y}$

Poynting vector $\boxed{\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\sigma B_0}{\mu_0\epsilon_0}\hat{x}}$

ii) Momentum density in the box

$$\vec{P}_{em} = \frac{\vec{S}}{c^2} = \frac{\sigma B_0 \hat{x}}{\mu_0\epsilon_0 c^2} = \sigma B_0\hat{x} \quad [\because c^2 = \frac{1}{\mu_0\epsilon_0}]$$

Total momentum in the box

$$\boxed{\vec{P}_{tot} = \sigma B_0\hat{x}\,(hab) = (\sigma B_0\, abh)\hat{x}}$$

iii) When $\vec{B}$ is switched off in time τ

$\Rightarrow$ induced electric field given by

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}.$$

Since $h << a$, E_x mainly contributes

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t} \Rightarrow \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = 0$$

$$\hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = 0$$

$$-\hat{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = -\frac{\partial B}{\partial t}\hat{y}$$

Taking

$E_y \simeq E_z = 0$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B}{\partial t} \rightarrow E_x = -\frac{\partial B}{\partial t} z + \cancel{C}$$

$C = 0$ (as $E_x = 0$ if $\frac{\partial B}{\partial t} = 0$)

$$\Rightarrow E_x = -\frac{\partial B}{\partial t} z$$

$\Rightarrow$ Force on the box due to switching off $\bar{B}$

$$\bar{F} = \frac{d\vec{p}}{dt} = q\vec{E} = \left(-\sigma ab\, E_x\Big|_{z=0} + \sigma ab\, E_x\Big|_{z=h}\right)\hat{x}$$

$$= -\sigma ab \frac{\partial B}{\partial t} h\, \hat{x}$$

$\Rightarrow$ Momentum imparted to the box

$$\bar{p} = \int_0^{\tau} \bar{F}\, dt = -\sigma abh \int_{B_0}^{0} \frac{\partial B}{\partial t} dt\, \hat{x}$$

$$\Rightarrow \boxed{\bar{p} = \sigma abh B_0 \hat{x}}$$

= momentum in the EM fields inside the box obtained in (ii)

5. An infinitely long solenoid of radius a, with N turns per unit length, carries a current I_s. Coaxial with the solenoid, at radius $b >> a$, is a circular ring of wire of resistance R. When the current in the solenoid is gradually decreased, a current I_r is induced in the ring;

(i) Calculate the induced current I_r in the ring in terms of dI_s/dt. [3]

(ii) Find induced electric field inside and outside the solenoid. [5]

(iii) Find induced magnetic field at the surface ($r = a$) of the solenoid ($b >> a$). [5]

(iv) Calculate the Poynting vector just outside the solenoid and integrate it over the whole surface of the solenoid. How is this power connected to the power dissipated in the ring?

[3 + 4]

i) Induced emf in the ring

$$\mathcal{E} = -\frac{d\phi}{dt}$$

$\Phi = \pi a^2 B$, where $B = \mu_0 n I_s$

$\mathcal{E} = I_r R \Rightarrow \boxed{I_r = -\frac{\mu_0 \pi a^2 n}{R} \frac{dI_s}{dt}}$.

ii) $\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi a^2 \mu_0 n I_s\right)$

$$\Rightarrow \vec{E} = \begin{cases} -\frac{\mu_0}{2} n r \frac{dI_s}{dt} \hat{\phi} & (r<a) \\ -\frac{\mu_0}{2} \frac{n a^2}{r} \frac{dI_s}{dt} \hat{\phi} & (r>a) \end{cases}$$

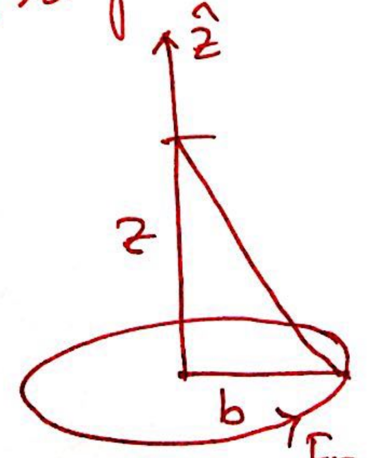
(iii) for $b >> a$

Magnetic field at the surface of the solenoid

Magnetic field at the axis of the ring

$\equiv$ field on the axis of the ring carrying current I_r

$$\boxed{= \frac{\mu_0 I_r}{2} \frac{b^2}{(b^2+z^2)^{3/2}} \hat{z}}$$



iv) Poynting vector

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

$$= -\frac{1}{4}\mu_0 I_r \frac{dI_s}{dt} \frac{ab^2 n}{(b^2+z^2)^{3/2}} \hat{r}$$

Power radiated by the solenoid

$$P = \int \vec{S} \cdot d\vec{a}$$

$$= \int_{-\infty}^{+\infty} S \cdot 2\pi a \, dz$$

$$= -\pi a^2 \mu_0 n \frac{dI_s}{dt} I_r$$

$$= I_r^2 R$$

= Power dissipated in the ring

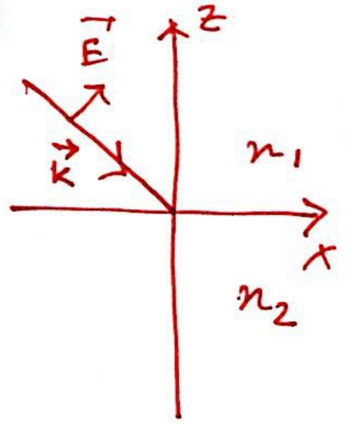
6. (a) A linearly polarized EM wave of frequency $\nu = 10^{14}$ Hz and intensity $I = 1\ KW/m^2$ traveling in air($n = 1$) is incident on the surface (at $z = 0$) of another material with refractive index $n = \sqrt{3}$.

(i) It is found that at a particular angle of incidence, there is no reflected wave. Write down the expression for the electric and magnetic fields for the incident wave at that angle. Evaluate $\vec{E}, \vec{B}$ and $\vec{k}$. [10]

(ii) Plot the reflection coefficient R and transmission coefficient T as a function of the incident angle (θ_I) for the above EM wave. Mark the axes in the plot. [4]

i) At a particular angle of incidence there is no reflection

$\Rightarrow$ ~~the~~ wave is TM polarized and the particular angle is Brewster's angle (θ_B)



At Brewster's angle $\theta_I + \theta_T = \pi/2$

$$\theta_T = \pi/2 - \theta_I$$

Snell's law $\Rightarrow$ $n_1 \sin\theta_I = n_2 \sin\theta_T$

$$\Rightarrow \theta_I = \tan^{-1}(\sqrt{3}) = \pi/3$$

$$I = 1\,kw/m^2 \Rightarrow \frac{1}{2} c\epsilon_0 E_0^2 = 10^3\ W/m^2$$

$$E_0 = \sqrt{\frac{2\times10^3}{3\times10^8 \times 8.85\times10^{-12}}}\ V/m = \sqrt{\frac{20}{3\times8.85}}\times10^3\ V/m$$

$$K = \frac{\omega}{c} = \frac{2\pi\nu}{c} = \frac{2\pi\times10^{14}}{3\times10^8}\ m^{-1}$$

$$\hat{k} = -\cos\theta_I\,\hat{z} + \sin\theta_I\,\hat{x} = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z}$$

direction of polarization

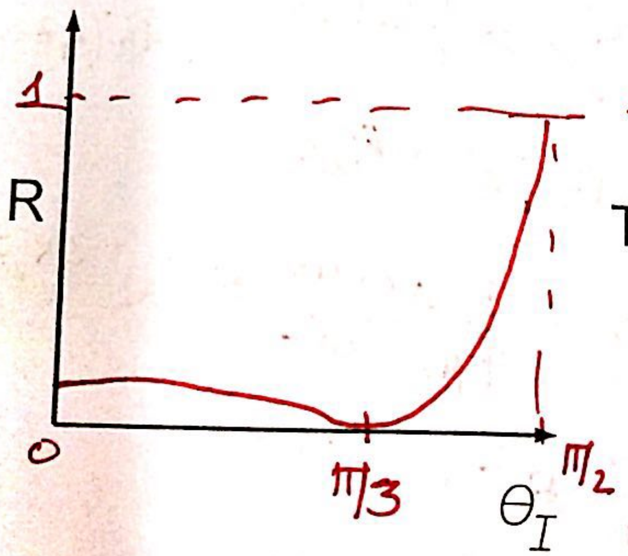
$$\hat{n} = \cos\left(\frac{\pi}{2} - \theta_I\right)\hat{z} + \sin\left(\frac{\pi}{2} - \theta_I\right)\hat{x}$$

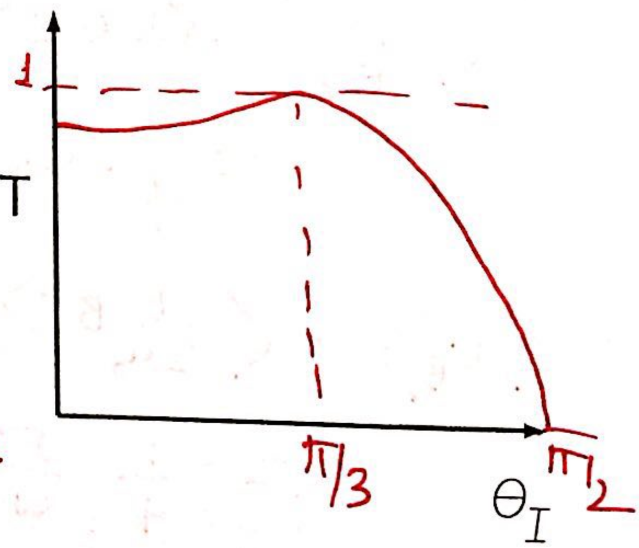
$$= \frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z}.$$

$$\vec{E} = E_0 \hat{n} \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$= \sqrt{\frac{20}{3\times 8.85}} \times 10^3 \left(\frac{\hat{x}}{2} + \frac{\sqrt{3}}{2}\hat{z}\right) \cos\left[\frac{2\pi}{3}\times 10^6\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}z\right) - 2\pi\times 10^{14} t\right] \text{V/m}$$

$$\vec{B} = \frac{\hat{k}\times\vec{E}}{c} = \frac{\sqrt{\frac{20}{3\times 8.85}}\times 10^3 \,\hat{y}}{3\times 10^8} \cos\left[\frac{2\pi}{3}\times 10^6\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}z\right) - 2\pi\times 10^{14} t\right] \text{T}$$





(b) Calculate the time averaged energy density of an EM wave in a good conductor. The E-field is given by $\vec{E} = E_0\ e^{-k_2 z} \sin(k_1 z - \omega t + \delta)\hat{x}$. Find the ratio of the energies contained in the electric field and the magnetic fields, if $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma = 6 \times 10^7$ S/m, $\omega = 10^6$ rad/s. [4 + 2]

For the conductor

$$k_{1,2} = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} \pm 1\right]^{1/2}$$

for good conductor $\frac{\sigma}{\omega\epsilon} >> 1$

$$k_1 = k_2 = \sqrt{\frac{\mu\omega\sigma}{2}}.$$

$$\vec{E} = E_0 e^{-k_2 z} \sin(k_1 z - \omega t + \delta)\,\hat{x}$$

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = \frac{|k|\, E\, \hat{n}}{\omega} \qquad \hat{n} = \hat{k} \times \hat{E}$$

Average energy density

$$\langle u_E \rangle = \left\langle \frac{1}{2}\epsilon E^2 \right\rangle = \frac{1}{2}\epsilon_0 E_0 e^{-2k_2 z} \langle \sin^2(k_1 z - \omega t + \delta)\rangle$$

$$= \frac{1}{4}\epsilon_0 E_0^2 e^{-2k_2 z}$$

$$\langle u_B \rangle = \left\langle \frac{1}{2\mu_0} B^2 \right\rangle = \frac{1}{4\mu_0}\frac{\sigma\omega\mu_0}{\omega^2} E_0^2 e^{-2k_2 z}$$

$$= \frac{1}{4}\frac{\sigma}{\omega} E_0^2 e^{-2k_2 z}.$$

Total $\langle u \rangle = \langle u_E \rangle + \langle u_B \rangle$

and $$\frac{\langle u_B \rangle}{\langle u_E \rangle} = \frac{\sigma}{\omega\epsilon_0} >> 1.$$