# Problem Set 3

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Draw and illustrate in $\mathbb{R}^2$.

   (a) $\mathbf{e}_1 + \{n\mathbf{e}_2 | n \in \mathbb{N}\}$.

   (b) $\mathbf{e}_1 + \{\alpha\mathbf{e}_2 | \alpha \in \mathbb{R}\}$.

2. In $\mathbb{R}^2$, Is $\{\alpha\mathbf{e}_1|\alpha \in \mathbb{R}\} + \{\alpha\mathbf{e}_2|\alpha \in \mathbb{R}\} = \mathbb{R}^2$? What about $\{\alpha\mathbf{e}_1|\alpha \in \mathbb{R}\} + \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\} = \mathbb{R}^2$?

3. In $\mathbb{R}^3$ prove that $\left\{\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} |\alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\right\} = \mathbb{R}^3$. Do you use Gauss-Jordan Elimination (GJE) method somewhere?

   **Solution:** Put $A = \{\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\}$, $B = \{\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} |\alpha \in \mathbb{R}\}$, $C = \{\alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\}$. Then $A+B+C = \{a+b+c | a \in A, b \in B, c \in C\} \subset \mathbb{R}^3$.

   Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$. We want to find $\alpha, \beta, \gamma$ s.t. $\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. That is, need to solve $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. We may use GJE to find the values of $\alpha = \frac{x_1 - x_2 + x_3}{2}, \beta = x_2 - x_3, \gamma = \frac{-x_1 + x_2 + x_3}{2}$. But without doing so, we may find the determinant and conclude that the system has a unique solution. But, we will need GJE for higher order vectors.

4. Let $L_1$ and $L_2$ be two nonparallel lines passing through origin in $\mathbb{R}^3$. What is $L_1 + L_2$?

5. **(T)** Let $L_1$ and $L_2$ be two skewed (non parallel, nonintersecting) lines in $\mathbb{R}^3$? What is $L_1 + L_2$?

   **Solution:**

   A plane. Take $\mathbf{a} \in L_1$, $\mathbf{b} \in L_2$. Then $L_{1h} = L_1 - \mathbf{a}$ and $L_{2h} = L_2 - \mathbf{b}$ both pass through $\mathbf{0}$. Thus $L_1 + L_2 = \mathbf{a} + \mathbf{b} + L_{1h} + L_{2h}$. As $L_{1h} + L_{2h}$ is a plane, we are done.

   Alternately: Put $L_1' = L_1 + (\mathbf{b} - \mathbf{a})$. This is the line parallel to $L_1$ passing through $\mathbf{b}$. Then $L_1' + L_2$ is a plane parallel to both $L_1'$ and $L_2$ passing through $2\mathbf{b}$ (be clear, not $\mathbf{b}$, for example $L_1 := (1, y, 0)$ and $L_2 = (1, 0, z)$). So adding $\mathbf{a} - \mathbf{b}$ to it (that is, making the plane trace back $(\mathbf{b} - \mathbf{a})$) will give us the plane through $\mathbf{a} + \mathbf{b}$. So our answer is $L_1' + L_2 + \mathbf{a} - \mathbf{b}$ which is a plane.

6. **(T)** Fix a non-negative integer $n$ and let $\mathbb{R}[x; n]$ be the set of polynomials with real coefficients and degree less than or equal to $n$. That is, $\mathbb{R}[x; n] = \left\{\sum_{i=0}^{n} c_i x^i : c_0, c_1, \cdots, c_n \in \mathbb{R}\right\}$. Show that $\mathbb{R}[x; n]$ is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

   **Solution:** For $p(x) = \sum_{i=0}^{n} a_i x^i, q(x) = \sum_{i=0}^{n} b_i x^i, r(x) = \sum_{i=0}^{n} c_i x^i$, we define the following:

[Vector Addition:]

$$(p+q)(x) = \sum_{i=0}^{n} (a_i + b_i)x^i \in \mathbb{R}[x;n]. \tag{1}$$

[Scalar Multiplication:] for $\alpha \in \mathbb{R}$,

$$(\alpha p)(x) = \sum_{i=0}^{n} (\alpha a_i)x^i \in \mathbb{R}[x;n]. \tag{2}$$

Verify all vector space requirements:

i. Clearly, $p+q = q+p$ as

$$(p+q)(x) = \sum_{i=0}^{n} (a_i + b_i)x^i = \sum_{i=0}^{n} (b_i + a_i)x^i = (q+p)(x).$$

ii. $(p+q)+r = p+(q+r)$ as

$$(p+q)(x) + r(x) = \sum_{i=0}^{n} (a_i + b_i)x^i + \sum_{i=0}^{n} c_i x^i = \sum_{i=0}^{n} ((a_i + b_i) + c_i)x^i =$$
$$\sum_{i=0}^{n} (a_i + (b_i + c_i))x^i = \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} (b_i + c_i)x^i = p(x) + (q+r)(x).$$

iii. The *zero* polynomial, $z(x) = 0$, satisfies $p + z = p$ as

$$(p+z)(x) = \sum_{i=0}^{n} (a_i + 0)x^i = \sum_{i=0}^{n} a_i x^i.$$

iv. For all $p(x) \in \mathbb{R}[x;n]$, there is $(-p)(x) := \sum_{i=0}^{n} (-a_i)x^i$ such that

$$(p + (-p))(x) = \sum_{i=0}^{n} (a_i + (-a_i))x^i = \sum_{i=0}^{n} 0x^i = 0 = z(x)$$

v. For all $\alpha, \beta \in \mathbb{R}$ and $p(x) \in \mathbb{R}[x;n]$, $\alpha(\beta p) = (\alpha\beta)p$ as

$$(\alpha(\beta p))(x) = \sum_{i=0}^{n} \alpha(\beta a_i)x^i = \sum_{i=0}^{n} (\alpha\beta)a_i x^i = ((\alpha\beta)p)(x).$$

vi. For all $\alpha \in \mathbb{R}$, $\alpha(p+q) = \alpha p + \alpha q$ as

$$\begin{aligned}(\alpha(p+q))(x) &= \sum_{i=0}^{n} \alpha(a_i + b_i)x^i = \sum_{i=0}^{n} (\alpha a_i + \alpha b_i)x^i = \sum_{i=0}^{n} \alpha a_i x^i + \sum_{i=0}^{n} \alpha b_i x^i \\ &= (\alpha p)(x) + (\alpha q)(x) = ((\alpha p) + (\alpha q))(x)\end{aligned}$$

vii. For all $\alpha, \beta \in \mathbb{R}$ and $p(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x; n]$, $(\alpha + \beta)p = \alpha p + \beta p$ as

$$((\alpha + \beta)p)(x) = \sum_{i=0}^{n} (\alpha + \beta) a_i x^i = \sum_{i=0}^{n} (\alpha a_i + \beta a_i) x^i = (\alpha p)(x) + (\beta p)(x) = ((\alpha p) + (\beta p))(x).$$

viii. For all $p(x) \in \mathbb{R}[x; n]$, $1(p) = p$ as

$$(1p)(x) = \sum_{i=0}^{n} (1 a_i) x^i = \sum_{i=0}^{n} a_i x^i = p(x).$$

7. Show that the space of all real $m \times n$ matrices is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

   **Solution:** Similar to Problem 4; a straightforward verification of all vector space requirements.

8. Let $\mathbb{M}_n(\mathbb{R})$ be the set of all $n \times n$ real matrices. Then, from above we see that $\mathbb{M}_n(\mathbb{R})$ is a real vector space. Now, prove the following:

   (a) $\mathbb{S} = \{A \in \mathbb{M}_n(\mathbb{R}) : A^t = A$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (b) Fix $A \in \mathbb{M}_n(\mathbb{R})$. Define $\mathbb{U} = \{B \in \mathbb{M}_n(\mathbb{R}) : AB = BA\}$. Then, $\mathbb{U}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (c) Let $\mathbb{W} = \{a_0 I + a_1 A + \cdots + a_m A^m : m \text{ is a non-negative integer}, a_i \in \mathbb{R}\}$. Then, $\mathbb{W}$ is a subspace of $\mathbb{U}$.

9. In $\mathbb{R}$, consider the addition $x \oplus y = x + y - 1$ and $a.x = a(x - 1) + 1$. Show that $\mathbb{R}$ is a real vector space with respect to these operations with additive identity 1 (note that 0 is NOT the additive identity).

   **Solution:** Again, an easy verification of all vector space requirements.

10. **(T)** Which of the following are subspaces of $\mathbb{R}^3$:

$$(a)\ \{(x, y, z) \mid x \geq 0\},\ (b)\ \{(x, y, z) \mid x + y = z\},\ (c)\ \{(x, y, z) \mid x = y^2\}.$$

   **Solution:**

   (a) Not a subspace : $-1(1, 0, 0)$ does not belong to the set.

   (b) Is a subspace.

   (c) Not a subspace : $(1, 1, 0) + (4, 2, 0)$ is not in the set. Since the relation is non-linear, closure is a problem.

11. Find the condition on real numbers $a, b, c, d$ so that the set $\{(x, y, z) \mid ax + by + cz = d\}$ is a subspace of $\mathbb{R}^3$.

   **Solution:** If $d = 0$, then this is a subspace. For it to be a subspace, $(0, 0, 0)$ had to be in the space and hence $d = 0$.

12. **(T)** Let $W_1$ and $W_2$ be subspaces of a vector space $V$ such that $W_1 \cup W_2$ is also a subspace. Prove that one of the spaces $W_i,\ i = 1, 2$ is contained in the other.

    **Solution:** Suppose $W_1$ is not a subset of $W_2$. Then to prove the result, we have to show that $W_2$ is a subset of $W_1$.

    Let $\mathbf{w}_2 \in W_2$. To show that $W_2$ is contained in $W_1$, we need to show that $\mathbf{w}_2 \in W_1$. Since $W_1 \not\subset W_2$, we can choose $\mathbf{w}_1 \in W_1$ such that $\mathbf{w}_1 \notin W_2$. Then $\mathbf{w}_2 - \mathbf{w}_1 \in W_1 \cup W_2$ as it is a subspace but $\mathbf{w}_2 - \mathbf{w}_1 \notin W_2$ because then $\mathbf{w}_1 = \mathbf{w}_2 - (\mathbf{w}_2 - \mathbf{w}_1) \in W_2$. So, $\mathbf{w}_2 - \mathbf{w}_1 \in W_1 \Rightarrow \mathbf{w}_2 = (\mathbf{w}_2 - \mathbf{w}_1) + \mathbf{w}_1 \in W_1$.

13. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be $n$ vectors from a vector space $V$ over $\mathbb{R}$. Define **linear span** of this set of vectors as

    $$\mathrm{LS}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} : c_1, c_2, \ldots, c_n \in \mathbb{R}\},$$

    that is, the set of all linear combinations of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Show that $\mathrm{LS}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$ is a subspace of $V$.

    **Solution:** If $\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n}$ and $\mathbf{w} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots d_n\mathbf{v_n}$, then

    $$\mathbf{u} + \mathbf{w} = (c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots (c_n + d_n)\mathbf{v_n} \in \mathrm{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$$

    and

    $$\alpha\mathbf{u} = (\alpha c_1)\mathbf{v_1} + (\alpha c_2)\mathbf{v_2} + \cdots (\alpha c_n)\mathbf{v_n} \in \mathrm{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$$

    for $\alpha \in \mathbb{R}$. Rest is straightforward.

14. **(T)** Show that $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} = \mathrm{LS}(\{(1, 0, 0, -1),\ (0, 1, 0, 1),\ (0, 0, 1, 1)\}$ and hence is a subspace of $\mathbb{R}^4$.

    **Solution:**

    $$\begin{aligned} (x_1, x_2, x_3, x_4) &\in \{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} \\ \Leftrightarrow (x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3, -x_1 + x_2 + x_3) \text{ as } x_4 = -x_1 + x_2 + x_3 \\ &= x_1(1, 0, 0, -1) + x_2(0, 1, 0, 1) + x_3(0, 0, 1, 1) \end{aligned}$$

    Moreover, $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\}$ is a subspace of $\mathbb{R}^4$ because it is a linear span of vectors in $\mathbb{R}^4$.

15. Suppose $S$ and $T$ are two subspaces of a vector space $V$. *Define* the **sum**

    $$S + T = \{\mathbf{s} + \mathbf{t} : \mathbf{s} \in S, \mathbf{t} \in T\}.$$

    Show that $S + T$ satisfies the requirements for a vector space. Moreover, $LS(S \cup T) = S + T$.

    **Solution:** Straightforward to check all vector space requirements.

16. **(T)** Find all the subspaces of $\mathbb{R}^2$.

**Solution:** Let $W$ be a subspace of $\mathbb{R}^2$. Assume that $W \neq \{0\}$, then there exists $0 \neq (w_1, w_2) \in W$. If the span, $L(\{(w_1, w_2)\}) = W$, then $W$ is a line through origin. If $L(\{(w_1, w_2)\})$ is a proper subset of $W$ then we show that $W = \mathbb{R}^2$. Let $(u_1, u_2) \in W \setminus L(\{(w_1, w_2)\})$. So, $(u_1, u_2) \neq \alpha(w_1, w_2)$ for all $\alpha \in \mathbb{R}$. So, $A = \begin{bmatrix} w_1 & u_1 \\ w_2 & u_2 \end{bmatrix}$ is invertible. Therefore, we see that for any $(x, y) \in \mathbb{R}^2$, we need to find $\alpha, \beta \in \mathbb{R}$ such that the system $(x, y) = \alpha(w_1, w_2) + \beta(u_1, u_2)$ has a solution. Note that the above system reduces to $A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Such $\alpha, \beta$ exist as $A$ is invertible.

17. **(T)** Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, with $a_{ij} \in \mathbb{C}$. Then, we define the following 4 fundamental subspaces:

(a) The column space of $A$ is defined as

$$\operatorname{col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{C}^n\} = \operatorname{LS}(A(:,1), \ldots, A(:,n)) = \operatorname{LS}\left(\left\{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}\right\}\right)$$

(b) The column space of $A^*$ is defined as

$$\operatorname{col}(A^*) = \operatorname{LS}(A^*(1,:), \ldots, A^*(m,:)) = \{A^*\mathbf{x} : \mathbf{x} \in \mathbb{C}^m\}.$$

(c) The null space of $A$ is defined as

$$\text{Null Space}(A) = \mathcal{N}(A) = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\}.$$

(d) The null space of $A^*$ is defined as

$$\text{Null Space}(A^*) = \mathcal{N}(A^*) = \{\mathbf{x} \in \mathbb{C}^m : A^*\mathbf{x} = \mathbf{0}\}.$$

**Important: In case the matrix $A$ has real entries, the spaces col$(A^*)$ and Null Space$(A^*)$ are called the row-space of $A$ and the left-null space of $A$, respectively**

Now, determine the above 4 mentioned fundamental spaces for the following matrices.

(i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$ (iii) $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

(iv) Suppose $B$ and $C$ are two $m \times n$ matrices and $S = \text{col}(B)$ and $T = \text{col}(C)$, then $S + T$ is a column space of what matrix $M$?

**Solution:** (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ Reduced Row Echelon Form of A.

So, the solutions are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = -z \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Thus, $\mathcal{N}(A) = \text{LS}\left(\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}\right)$.

(iv) Let $M = \begin{bmatrix} B & C \end{bmatrix}$. In other words, $M$ is an $m \times (2n)$ matrix whose first $n$ columns are same as columns of $B$ and next $n$ columns are same as columns of $C$. It is easy to see that if $\mathbf{u} \in \text{col}(M)$ then $\mathbf{u} \in S + T$. Similarly, if $\mathbf{u} \in S + T$ then $\mathbf{u} = \mathbf{s} + \mathbf{t}$ where $\mathbf{s}$ is a linear combination of columns of $B$ and $\mathbf{t}$ is a linear combination of columns of $C$ which implies that $\mathbf{u} \in \text{col}(M)$.

18. Construct a matrix whose column space contains $[\ 1\ \ 1\ \ 1\ ]^T$ and whose null space is the line of multiples of $[\ 1\ \ 1\ \ 1\ \ 1\ ]^T$.

**Solution:** Clearly, the matrix we are looking for is a $3 \times 4$ matrix with rank 3. Two such matrices are

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & -3 \\ 1 & 2 & 3 & -6 \\ 1 & 4 & 9 & -14 \end{bmatrix}.$$

19. **(T)** Suppose $A$ is an $m$ by $n$ matrix of rank $r$.

(a) If $A\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$, what is the column space of $A$?

**Solution:** There must be a pivot in every row, so $r = m$ and the column space of $A$ is all of $\mathbb{R}^m$.

(b) In part (a), what are all equations or inequalities that must hold between the numbers $m$, $n$ and $r$?

**Solution:** We always have $r \leq n$. From (a), we know that $r = m$. From these, we deduce that $m \leq n$.

(c) Give a specific example of a 3 by 2 matrix $A$ of rank 1 with first row $[\ 2\ \ 5\ ]$. Describe the column space, $\text{col}(A)$, and the null space $N(A)$ completely.

**Solution:** Just use multiples of [2 5] for the other rows. For example, $\begin{bmatrix} 2 & 5 \\ 4 & 10 \\ 0 & 0 \end{bmatrix}$. Column space will be the line in $\mathbb{R}^3$ consisting of all multiples of your first column. The null space will be the line in $\mathbb{R}^2$ consisting of all multiples of the null space solution $\begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$.

(d) Suppose the right side $\mathbf{b}$ is same as the first column in your example (part c). Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

**Solution:** Adding the particular solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the null space solution from (c), we get the complete solution $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$.

20. Suppose the matrix $A$ has row reduced echelon form $R$ :

$$A = \begin{bmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ & (row & 3) & \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) What can you say immediately about row 3 of $A$?

**Solution:** Because row 3 of $R$ is all zeros, row 3 of $A$ must be a linear combination of row 1 and row 2 of $A$.

(b) What are the numbers $a$ and $b$?

**Solution:** After one step of elimination, we have

$$\begin{bmatrix} 1 & 2 & 1 & b \\ 0 & a-4 & -1 & 8-2b \\ & (row & 3) & \end{bmatrix}.$$

From $R$, we see that the second column of $A$ is not a pivot column, so $a = 4$. Continuing with elimination, we get to

$$\begin{bmatrix} 1 & 2 & 0 & 8-b \\ 0 & 0 & 1 & 2b-8 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Comparing this to $R$, we see that $b = 5$.

(c) Describe all solutions of $R\mathbf{x} = \mathbf{0}$. Which among row spaces, column spaces and null spaces are the same for $A$ and for $R$.

**Solution:** Setting the free variables $x_2$ and $x_4$ to 1 and 0, and vice versa, and solving $R\mathbf{x} = \mathbf{0}$, we get the null space solution

$$\mathbf{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

The row space and the null space are always the same for $A$ and $R$ whereas column space is different (row operations preserve row space but change column space).

21. **(T)** Suppose that $A$ is a $3 \times 3$ matrix. What relation is there between the null space of $A$ and the null space of $A^2$? How about the null space of $A^3$?

**Solution:** The null space of $A$ is contained in the null space of $A^2$. The reason is that if $A\mathbf{x} = \mathbf{0}$, *i.e.*, if $\mathbf{x}$ is in the null space of $A$, then $A^2\mathbf{x} = A(A\mathbf{x}) = \mathbf{0}$. Thus, $\mathbf{x}$ is also in the null space of $A^2$. Similarly, we have

$$N(A) \subseteq N(A^2) \subseteq N(A^3) \subseteq \ldots.$$

Note that one can prove that if $A$ is an $n \times n$ matrix, then one has $N(A^n) = N(A^{n+1}) = \ldots.$

22. Suppose $R$ (an $m \times n$ matrix) is in row reduced echelon form $\begin{pmatrix} I_r & F \\ 0 & 0 \end{pmatrix}$, with $r$ non-zero rows and first $r$ pivot columns. Describe the column space and null space of $R$.

**Solution:** The column space is the space of all vectors whose last $m - r$ coordinates are zero. This is clear since rank of the matrix $R$ is $r$ and the first $r$ columns of $R$ are independent.

Denote by $f_{ij}$ the entry in the the $(i, j)$ position in $F$. The null space of $R$ is the space of all linear combinations of the $n - r$ vectors

$$\begin{bmatrix} -f_{11} \\ -f_{21} \\ \vdots \\ -f_{r1} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -f_{12} \\ -f_{22} \\ \vdots \\ -f_{r2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} -f_{1(n-r)} \\ -f_{2(n-r)} \\ \vdots \\ -f_{r(n-r)} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Clearly, these vectors are linearly independent and therefore the dimension of the null space is $n - r$.

23. **(T)** Let $W_1 = \text{span}\left\{\left[1\ 1\ 0\right]^T, \left[-1\ 1\ 0\right]^T\right\}$ and $W_2 = \text{span}\left\{\left[1\ 0\ 2\right]^T, \left[-1\ 0\ 4\right]^T\right\}$. Show that $W_1 + W_2 = \mathbb{R}^3$. Give an example of a vector $v \in \mathbb{R}^3$ such that $v$ can be written in two different ways in the form $v = v_1 + v_2$, where $v_1 \in W_1, v_2 \in W_2$.

**Solution:** $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right\} \subseteq W_1 + W_2$ and is linearly independent which means $W_1 + W_2 = \mathbb{R}^3$. Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \in W_2$, we have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W_1 + W_2$. Note that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \in W_1$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{5}{6}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \in W_2$, so

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in W_1 + W_2.$$