Section 52: Problem 6 Solution

> Working problems is a crucial part of learning mathematics. No one can learn topology merely by poring over the definitions, theorems, and examples that are worked out in the text. One must work part of it out for oneself. To provide that opportunity is the purpose of the exercises.
>
> — *James R. Munkres*

Show that if X is path connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphisms of the groups involved. More precisely, let $h: X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1 , and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes."

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \downarrow \hat{\alpha} & & \downarrow \hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

$$\begin{aligned} \hat{\beta} \circ (h_{x_0})_*([f]) = \; & [\overline{\beta}] \star (h_{x_0})_*([f]) \star [\beta] = [h \circ \overline{\alpha}] \star [h \circ f] \star [h \circ \alpha] = \\ = \; & [h \circ (\overline{\alpha} \star f \star \alpha)] = (h_{x_1})_*([\overline{\alpha} \star f \star \alpha]) = (h_{x_1})_* \circ \hat{\alpha}([f]). \end{aligned}$$