MSO202A: Assignment-I Solutions

1. For any $z \in \mathbb{C}$, show that

   (a) Re(iz)=$-$Im z

   Soln: $z = x + iy \implies iz = -y + ix \implies \text{Re}(iz) = -y = -\text{Im } z$

   (b) z is a real number iff $z = \bar{z}$

   Soln: $z = x + iy \implies \bar{z} = x - iy$. Hence $z = \bar{z} \implies y = 0$ and hence z is real

   (c) $|\text{Re } z| \leq |z|$ and $|\text{Im } z| \leq |z|$

   Soln: $|z|^2 = x^2 + y^2 \geq x^2 \implies |z| \geq |x|$ etc.

   (d) $|\text{Im }(1 - \bar{z} + z^2)| < 3, \quad \forall z < 1$

   Soln: $|\text{Im }(1 - \bar{z} + z^2)| \leq |1 - \bar{z} + z^2| \leq 1 + |\bar{z}| + |z|^2 < 1 + 1 + 1 = 3$

2. Prove the following:

   (a) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

   Soln: $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

   (b) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

   Soln: Proceeding similar to (a), $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$ and hence the result follows

   (c) $|z_1 + z_2| \leq |z_1| + |z_2|$ and equality holds iff one is a nonnegative scalar multiple of other.

   Soln: $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2) \leq |z_1|^2 + |z_2|^2 + 2|\text{Re}(z_1\bar{z}_2)| \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$. Hence $|z_1 + z_2| \leq |z_1| + |z_2|$.

   Equality holds if $\text{Re}(z_1\bar{z}_2) = |z_1z_2| = t \geq 0$ and $\text{Im}(z_1\bar{z}_2) = 0$. Hence, $z_1\bar{z}_2 = t \implies z_1 = t/\bar{z}_2 = \alpha z_2$, where $\alpha = t/|z_2|^2$.

3. Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C} : |z| < 1\}$.

   Soln: Suppose that it has a solution z_0. Then $|z_0| < 1$. Now $5 = |z_0^4 + z_0| \leq |z_0|^4 + |z_0| \leq 2$, which is a contradiction.

4. Let $\lambda \in \mathbb{C}$ be such that $0 < |\lambda| < 1$. Then show that

   (a) $|z - \lambda| < |1 - \bar{\lambda}z|$ if $|z| < 1$.

   Soln: $|z - \lambda|^2 - |1 - \bar{\lambda}z|^2 = (z - \lambda)(\bar{z} - \bar{\lambda}) = (1 - \bar{\lambda}z)(1 - \lambda\bar{z}) = |z|^2 + |\lambda|^2 - 1 - |\lambda|^2|z|^2 = (|z|^2 - 1)(1 - |\lambda|^2)$.

   Hence if $|z| < 1$, then $|z - \lambda|^2 - |1 - \bar{\lambda}z|^2 < 0 \implies |z - \lambda| < |1 - \bar{\lambda}z|$.

   (b) $|z - \lambda| = |1 - \bar{\lambda}z|$ if $|z| = 1$.

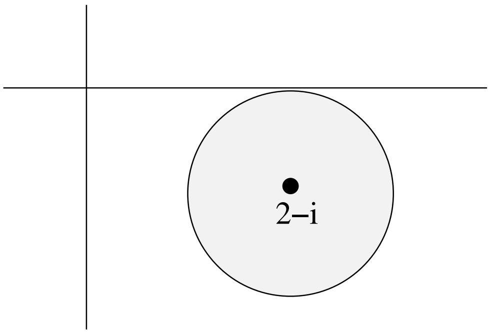
   Soln: Similar to (a)

   (c) $|z - \lambda| > |1 - \bar{\lambda}z|$ if $|z| > 1$.

   Soln: Similar to (a)

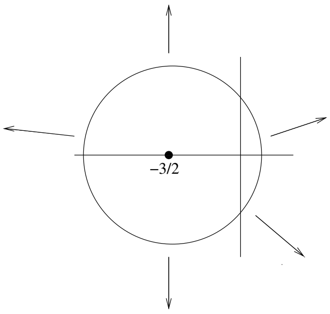
5. Sketch each of the following set of complex numbers and determine which ones of these are domains:

(a) $S = \{z : |z - 2 + i| \leq 1\}$.



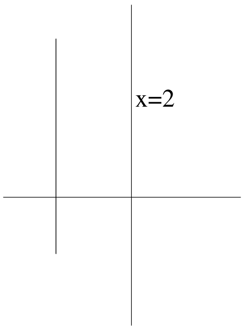
Region includes the interior and boundary of the unit disk centred at $(2, -1)$. It is closed and connected. Being closed, it is not a domain.

(b) $S = \{z : |2z + 3| > 4\}$.



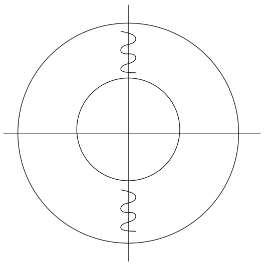
Region includes the exterior of the disk of radius 2 centred at $(-3/2, 0)$. It is open and connected and hence it is a domain.

(c) $S = \{z : |z - 1| = |z - 3|\}$.



Region is the points on the line $x = 2$. It is not an open set and hence it is not a domain.

(d) $S = \{z : 1 < |z| < 2,\ \operatorname{Re} z \neq 0\}$.



Region includes the annular region of the concentric circles of radii 1 and 2 with center at the origin and the y-axis excluded. It is open but not connected and hence it is not a domain.

6. If z and w are such that Im $z > 0$ and Im $w > 0$, then show that

$$\left|\frac{z-w}{z-\bar{w}}\right| < 1.$$

Soln: $|z-w|^2 - |z-\bar{w}|^2 = 2\,\mathrm{Re}(zw - z\bar{w}) = 4\,\mathrm{Im}(w)\mathrm{Re}(iz) = -8\,\mathrm{Im}(z)\,\mathrm{Im}(w) < 0$. Hence the result.

7. Let $z = i/(-2-2i)$.

   (a) Express z in polar form

   Soln: $z = -\frac{1}{4} - \frac{i}{4}$. Hence $|z| = \frac{\sqrt{2}}{4}$ and $Arg(z) = -\frac{3\pi}{4}$. Thus, $z = \frac{1}{2\sqrt{2}}e^{-i3\pi/4}$.

   (b) Express z^5 in polar and Cartesian form

   Soln: $z^5 = \frac{1}{128\sqrt{2}}e^{-i15\pi/4} = \frac{1}{128\sqrt{2}}e^{i\pi/4}$. Also, $z^5 = \frac{1}{128\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \frac{1+i}{256}$.

   (c) Express $z^{1/5}$ in Cartesian form

   Soln: $z = \frac{1}{2^{3/2}}e^{-i3\pi/4+i2k\pi}$. Hence, $z_k = \frac{1}{2^{3/10}}e^{-i3\pi/20+i2k\pi/5}$, $k = 0, 1, 2, \cdots, 5$. Thus, $z_0 = \frac{1}{2^{3/10}}\left(\cos -3\pi/20 + i\sin -3\pi/20\right)$, $z_1 = \frac{1}{2^{3/10}}\left(\cos 5\pi/20 + i\sin 5\pi/20\right)$, $z_2 = \frac{1}{2^{3/10}}\left(\cos 13\pi/20 + i\sin 13\pi/20\right)$, $z_3 = \frac{1}{2^{3/10}}\left(\cos 21\pi/20 + i\sin 21\pi/20\right)$, $z_4 = \frac{1}{2^{3/10}}\left(\cos 29\pi/20 + i\sin 29\pi/20\right)$.

8. Prove that for $z, w \in \mathbb{C}$

$$|1 - z\bar{w}|^2 - |z-w|^2 = (1-|z|^2)(1-|w|^2).$$

Soln: Same as Q.4(a).

Using this result show that if $|w| < 1$, then the function

$$f_w(z) = \frac{z-w}{1-z\bar{w}}$$

maps the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ onto itself and the unit circle $S = \{z \in \mathbb{C} : |z| = 1\}$ onto itself.

Soln: If $|w| < 1$, Then $|1 - z\bar{w}|^2 - |z-w|^2 > 0$ and hence $|f_w(z)| < 1$. Let z_0 be such that $|z_0| < 1$. Now define $z_1 = (z_0 + w)/(1 + z_0\bar{w})$. Using 4(a) again, we can show that $|z_1| < 1$ and clearly $f_w(z_1) = z_0$.

Applying 4(b), we can prove the part concerning the circle.

9. Prove de Moivre's theorem: Given $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Use this result to find

$$(a)(1+i\sqrt{3})^{99} \quad (b)\left(\frac{1+i}{\sqrt{2}}\right)^{10}.$$

Soln: Let $z = (\cos\theta + i\sin\theta)$. Then, by direct multiplication, we find $z^2 = (\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta)$. Let it be true for $n = m$. Then $z^{m+1} = z^m\,z = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta) = (\cos(m+1)\theta + i\sin(m+1)\theta)$. Hence, proved.

(a) $(1+i\sqrt{3}) = 2(\cos\pi/3 + i\sin\pi/3)$ and hence $(1+i\sqrt{3})^{99} = 2^{99}(\cos 33\pi + i\sin 33\pi) = -2^{99}$.

(b) $\left(\frac{1+i}{\sqrt{2}}\right) = \cos\pi/4 + i\sin\pi/4$ and hence $\left(\frac{1+i}{\sqrt{2}}\right)^{10} = \cos 10\pi/4 + i\sin 10\pi/4 = i\sin\pi/2 = i$

10. Show that

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \qquad z \neq 1.$$

Use this result to deduce that

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}}.$$

Soln: $(1 - z)(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1}$ and hence $1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1 - z)$ where $z \neq 1$.

Let $z = \cos\theta + i \sin\theta = e^{i\theta}$. Then

$$\sum_{k=0}^{n} e^{ik\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Equating real part from both side we get

$$\sum_{k=0}^{n} \cos k\theta = \mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right).$$

Now

$$\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{(1 - e^{i(n+1)\theta})(1 - e^{-i\theta})}{|1 - e^{i\theta}|^2}$$

Hence

$$\mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) = \frac{1 - \cos\theta + \cos n\theta - \cos(n+1)\theta}{|1 - e^{i\theta}|^2}$$

Now $|1 - e^{i\theta}|^2 = 4\sin^2\theta/2$ and $1 - \cos\theta + \cos n\theta - \cos(n + 1) = 2\sin^2\theta/2 + 2\sin\theta/2 \sin(n + 1/2)\theta$ etc.

11. Discuss the convergence of the following sequences:

$$(a)\, \{\cos\left(\frac{n\pi}{2}\right) + i^n\}, \quad (b)\, \{i^n \sin\left(\frac{n\pi}{4}\right)\}, \quad (c)\, \{\frac{1}{n} + i^n\}$$

Soln: If a sequnce convergence, then every subsequnce must also converge. Let each of the sequence be denoted by $\{z_n\}$.

(a) $\{z_{2m}\} = \{2(-1)^m\}$ which does not converge.

(b) $\{z_{2m}\} = \{(-1)^m \sin m\pi/2\}$. This subsequence has every even term zero and every odd term nonzero. Hence it does not converge.

(c) $\{z_{2m}\} = \{1/2m + (-1)^m\}$ which does not converge.

12. Let $z = re^{i\theta}$, $w = Re^{i\phi}$, $0 \leq r < R$. For a fixed w, find

$$\lim_{r \to R} \mathrm{Re}\left(\frac{w + z}{w - z}\right).$$

Soln: Clearly, $w \neq 0$.

$$\mathrm{Re}\left(\frac{w + z}{w - z}\right) = \frac{|w|^2 - |z|^2}{|w - z|^2} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

If $\arg(w) = \arg(z)$, then

$$\lim_{r \to R} \mathrm{Re}\left(\frac{w + z}{w - z}\right) = \lim_{r \to R} \frac{R + r}{R - r} = \infty$$

If $\arg(w) \neq \arg(z)$, then

$$\lim_{r\to R} \operatorname{Re}\left(\frac{w+z}{w-z}\right) = \lim_{r\to R} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi-\theta) + r^2} = 0.$$

13. If $1 = z_0, z_1, z_2, \cdots, z_{n-1}$ are distinct n-th roots of unity, then prove that

$$\Pi_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j$$

Soln: We have

$$z^n - 1 = (z-1)\Pi_{j=1}^{n-1}(z-z_j) \implies \Pi_{j=1}^{n-1}(z-z_j) = \frac{z^n-1}{z-1} = \sum_{j=0}^{n-1} z^j \qquad (z \neq 1)$$

The result is also true for $z = 1$.

14. Check whether the following functions can be defined a at $z = 0$ so that they become continuous at $z = 0$:

$$(a)\, f(z) = \frac{|z|^2}{z}, \quad (b)\, f(z) = \frac{z+1}{|z|-1}, \quad (c)\, f(z) = \frac{\bar{z}}{z}.$$

Soln: (a) $|f(z) - 0| = |z| \to 0$ as $z \to 0$. Hence $f(0) = 0$ makes f continuous at $z = 0$.

(b) $|f(z) + 1| = \frac{|z+|z||}{||z|-1|} \to 0$ as $z \to 0$. Hence $f(0) = -1$ makes f continuous at $z = 0$.

(c) As $z \to 0$, $f(z) \to 1$ along $y = 0$ and $f(z) \to -1$ along $x = 0$. Since limit does not exist, it cannot be made continuous at $z = 0$.