Fourth Quiz, November 10, 2018

CS345: Algorithms II

Time minutes 45 Max Marks 35

Instructions: Please write legibly and to the point. Be precise in your answer.

Question 1. [Marks 10].

Prove or disprove.

(a) A system of linear equations can have no solution,

(b) A system of linear equations can have exactly one solution.

(c) A system of linear equations can have exactly two solutions.

(d) A system of linear equations can have countably infinite solutions but not uncountable infinite solutions.

Solution

Let us assume that the system has n variables and m equations. If a system of linear equations $A.x = b$ has two equations, explicitly or implicitly (i.e., two equations can be created by linearly combining the given equations, corresponding to two parallel hyperplanes), then their intersection is empty. Such a system is called *inconsistent.*

Otherwise if $rank(A) = r$, then any linearly independent r hyperplanes intersect in to an $n-r$ dimensional plane, called *solution plane.* A point in $\mathbb{R}^n$ is a solution of this system if and only if that point is in the solution plane.

(a) If the system is inconsistent, then it has no solution. .....True.

(b) If the system is consistent and the solution plane is $n-r=0$ dimensional, then there will be exactly one solution because a zero-dimensional plane has one point. ......True.

(c) No plane in $\mathbb{R}^n$ can have finite number of points except the 0-dimensional plane, which has exactly one point. ....False.

(d) No plane in $\mathbb{R}^n$ can have countable number of points, except zero-dimensional plane, because all other planes contain at least one (infinite) line which is isomorphic to the real line. ......False.

Question 2. [Marks 10].

Let

$$A(x_1, \ldots, x_n = \begin{bmatrix} x_1 & x_1^2 & x_1^3 & \ldots & x_1^n \\ x_2 & x_2^2 & x_2^3 & \ldots & x_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_n & x_n^2 & x_n^3 & \ldots & x_n^n \end{bmatrix}$$

Let $f_n(x_1, x_2, \ldots, x_n) = det(A(x_1, \ldots, x_n))$.

Determine a recurrence relation for f_n.

Solution Using explicit formula for the determinant we get.

$$|A| = x_1.\begin{bmatrix} x_2^2 & x_2^3 & \ldots & x_2^n \\ x_3^2 & x_3^3 & \ldots & x_3^n \\ \ldots & \ldots & \ldots & \ldots \\ x_n^2 & x_n^3 & \ldots & x_n^n \end{bmatrix} - x_2.\begin{bmatrix} x_1^2 & x_1^3 & \ldots & x_1^n \\ x_3^2 & x_3^3 & \ldots & x_3^n \\ \ldots & \ldots & \ldots & \ldots \\ x_n^2 & x_n^3 & \ldots & x_n^n \end{bmatrix} + \cdots + (-1)^{n-1}x_n.\begin{bmatrix} x_1^2 & x_1^3 & \ldots & x_1^n \\ x_2^2 & x_2^3 & \ldots & x_2^n \\ \ldots & \ldots & \ldots & \ldots \\ x_{n-1}^2 & x_{n-1}^3 & \ldots & x_{n-1}^n \end{bmatrix}$$

Next use the fact that if

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ \lambda.a_{j1} & \lambda.a_{j2} & \dots & \lambda.a_{jn} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then $|A| = \lambda.|B|$.
So we have

$$|A| = \prod_i x_i. \begin{bmatrix} x_2 & x_2^2 & \dots & x_2^{n-1} \\ x_3 & x_3^2 & \dots & x_3^{n-1} \\ \dots & \dots & \dots & \dots \\ x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} - \prod_i x_i. \begin{bmatrix} x_1 & x_1^2 & \dots & x_1^{n-1} \\ x_3 & x_3^2 & \dots & x_3^{n-1} \\ \dots & \dots & \dots & \dots \\ x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} + \dots + (-1)^{n-1} \prod_i x_i. \begin{bmatrix} x_1 & x_1^2 & \dots & x_1^{n-1} \\ x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots \\ x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{bmatrix}$$

So we get $f_n(x_1, \dots, x_n) = \prod_i x_i.(f_{n-1}(x_2, x_3, \dots, x_{n-1}) - f_{n-1}(x_1, x_3, \dots, x_{n-1}) + \dots + (-1)^{n-1} f_{n-1}(x_1, x_2, \dots, x_{n-2}))$.

Question 3. [Marks 15].

Following is an incomplete algorithm to compute the reciprocal of an n-bit number P, $\lfloor 2^{2n-1}/P \rfloor$, modified to handle arbitrary n.

(i) Complete the algorithm

(ii) Complete the error analysis

Input: A positive integer $P = \langle p_1 p_2 \dots p_n \rangle$ where n is positive integer.
Output: $A = \lfloor 2^{2n-1}/P \rfloor$
if $n = 1$ **then**
  return $\langle 10 \rangle$;
else
  $n_1 := \lceil n/2 \rceil$;
  $n_2 := \lfloor n/2 \rfloor$;
  $P_1 := \langle p_1 p_2 \dots p_{n_1} \rangle$;
  $C := Reciprocal\_General(n_1, P_1)$;
  $D := 2^{+n_1}.C - P.C^2$;
  `/* What should be □?  */`
  $A := \lfloor D/2^{2n_1 - 1} \rfloor$;
  `/* fix the error in` A `*/`
  ————————-;
  return A;
end

Algorithm 1: *Reciprocal_General*(n, P)

Error Analysis
Complete the following analysis by determining α, β, and k. Show all steps of the deduction.

1. Fact 1: $2^{2n_1} - 1 < P_1.C \leq 2^{2n_1}$.
2. Fact 2: $D/2^{2n_1-1} - 1 < A \leq D/2^{2n_1-1}$.
   So $PD/2^{2n_1-1} - P < PA \leq PD/2^{2n_1-1}$.
3. Using $D = 2^{\square}.C - PC^2$, determine bounds for $PD/2^{2n_1-1}$:
   $2^{2n-1} - \alpha \leq PD/2^{2n_1-1} \leq 2^{2n-1}$.
4. Combine (2) and (3) to find bounds for $P.A$:
   $2^{2n-1} - \beta \leq P.A \leq 2^{2n-1}$
5. Find an upperbound in terms of P for β: $\beta \leq k.P$.

Therefore error in A can be at most k.

Solution

(3) $PD = 2^{n+n_1}PC - P^2C^2 = 2^{n+n_1}PC - P^2C^2 - 2^{2n+2n_1-2} + 2^{2n+2n_1-2} = 2^{2n+2n_1-2} - (2^{n+n_1-1} - PC)^2$. So $PD/2^{2n_1} = 2^{2n-1} - (1/2^{2n_1-1}(2^{n+n_1-1} - PC)^2$

So

$$2^{2n-1} - \alpha \leq PD/2^{2n_1-1} \leq 2^{2n-1}$$

where $\alpha = (2^{n+n_1-1} - PC)^2/2^{2n_1-1}$.

(4) Next let us use fact 2 to get bounds for $P.A$. The fact is that $PD/2^{2n_1-1} - P < PA \leq PD/2^{2n_1-1}$ so combining it with bounds for $PD/2^{2n_1-1}$ we get $2^{2n-1} - \alpha - P \leq PD/2^{2n_1-1} - P \leq PA \leq PD/2^{2n_1-1} \leq 2^{2n-1}$ so

$$2^{2n-1} - (\alpha + P) \leq PA \leq 2^{2n-1}.$$

(5) Now we will find an upperbound for $\beta = \alpha + P$ in the form $k.P$ for some integer k.

$\alpha + P = (2^{n+n_1-1} - PC)^2/2^{2n_1-1} + P = P + (2^{n+n_1-1} - 2^{n_2}P_1C - P_2C)^2 = P + (2^{n_2}(2^{2n_1-1} - P_1C) - P_2C)^2/2^{2n_1-1}$ because $P = 2^{n_2}.P_1 + P_2$.

Simplify it to get $\alpha + P \leq P + \max\{2^{2n_2}(2^{2n_1-1} - P_1C) - P_2C)^2, P^2C^2\}.(1/2^{2n_1-1}) \leq P + \max\{2^{2n_2}P_1^2, P^2C^2\}.(1/2^{2n_1-1})$ from the Fact 1.

Since $2^{n-1} \leq P < 2^n$, $P_1 < 2^{n_1}$, $C < 2^n$, and $P_2 < 2^{n_2}$, $\max\{2^{2n_2}P_1^2, P^2C^2\} \leq 2^{2n_1+2n_2} = 2^{2n}$. This gives

$$\alpha + P \leq P + 2^{2n}/2^{2n_1-1} = P + 2^{2n_2+1} \leq P + 2^{n+1} \leq 4.2^{n-1} \leq P + 4P = 5P$$

So $k = 5$.