MSO202A: Assignment-I

1. For any $z \in \mathbb{C}$, show that

   (a) $\text{Re}(iz) = -\text{Im } z$

   (b) z is a real number iff $z = \bar{z}$

   (c) $|\text{Re } z| \leq |z|$ and $|\text{Im } z| \leq |z|$

   (d) $|\text{Im } (1 - \bar{z} + z^2)| < 3, \quad \forall z < 1$

2. Prove the following:

   (a) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

   (b) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

   (c) $|z_1 + z_2| \leq |z_1| + |z_2|$ and equality holds iff one is a nonnegative scalar multiple of other.

3. Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C} : |z| < 1\}$.

4. Let $\lambda \in \mathbb{C}$ be such that $0 < |\lambda| < 1$. Then show that

   (a) $|z - \lambda| < |1 - \bar{\lambda}z|$ if $|z| < 1$.

   (b) $|z - \lambda| = |1 - \bar{\lambda}z|$ if $|z| = 1$.

   (c) $|z - \lambda| > |1 - \bar{\lambda}z|$ if $|z| > 1$.

5. Sketch each of the following set of complex numbers and determine which ones of these are domains:

   (a) $S = \{z : |z - 2 + i| \leq 1\}$.

   (b) $S = \{z : |2z + 3| > 4\}$.

   (c) $S = \{z : |z - 1| = |z - 3|\}$.

   (d) $S = \{z : 1 < |z| < 2, \ \text{Re}\, z \neq 0\}$.

6. If z and w are such that $\text{Im } z > 0$ and $\text{Im } w > 0$, then show that

$$\left|\frac{z - w}{z - \bar{w}}\right| < 1.$$

7. Let $z = i/(-2 - 2i)$.

   (a) Express z in polar form

   (b) Express z^5 in polar and Cartesian form

   (c) Express $z^{1/5}$ in Cartesian form

8. Prove that for $z, w \in \mathbb{C}$

$$|1 - z\bar{w}|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2).$$

Using this result show that if $|w| < 1$, then the function

$$f_w(z) = \frac{z - w}{1 - z\bar{w}}$$

maps the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ onto itself and the unit circle $S = \{z \in \mathbb{C} : |z| = 1\}$ onto itself.

9. Prove de Moivre's theorem: Given $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Use this result to find

$$(a)(1 + i\sqrt{3})^{99} \quad (b)\left(\frac{1+i}{\sqrt{2}}\right)^{10}.$$

10. Show that

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \qquad z \neq 1.$$

Use this result to deduce that

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

11. Discuss the convergence of the following sequences:

$$(a)\, \{\cos\left(\frac{n\pi}{2}\right) + i^n\}, \quad (b)\, \{i^n \sin\left(\frac{n\pi}{4}\right)\}, \quad (c)\, \{\frac{1}{n} + i^n\}$$

12. Let $z = re^{i\theta}$, $w = Re^{i\phi}$, $0 \leq r < R$. For a fixed w, find

$$\lim_{r \to R} \operatorname{Re}\left(\frac{w + z}{w - z}\right).$$

13. If $1 = z_0, z_1, z_2, \cdots, z_{n-1}$ are distinct n-th roots of unity, then prove that

$$\Pi_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j$$

14. Check whether the following functions can be defined a at $z = 0$ so that they become continuous at $z = 0$:

$$(a)\, f(z) = \frac{|z|^2}{z}, \quad (b)\, f(z) = \frac{z+1}{|z| - 1}, \quad (c)\, f(z) = \frac{\bar{z}}{z}.$$