Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment III

1. $\star$ Applying Euler's method with step size $h = 0.1$, find the approximate values for the solution of the differential equation

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

$x = 0.1, 0.2, 0.3$ and compare these values with the values of the solution $y = e^{4x} + e^{-3x}$ at these points.

Do the same problem with improved Euler's method.

Let us recall Euler's method. Let $y' = f(x, y)$ with $y(x_0) = y_0$ be the initial value problem. If the step size is h and $x_{i+1} = x_i + h$ for $i \geq 0$ in the domain of definition of the solution, then the approximate value of the solution at y_{i+1} is $y_{i+1} = y_i + hf(x_i, y_i)$ for $i \geq 0$.
The equation $y' = -3y + 7e^{4x}$ with $y(0) = 1$, i s of the form $y' = f(x, y)$ with $f(x, y) = -3y + 7e^{4x}$, $x_0 = 0$ and $y_0 = 2$.
Now we apply Euler's method to find the first approximate value

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 2 + (0.1)f(0, 2) = 2 + (0.1)(-3 \times 2 + 7) = 2.1 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) \\ &= 2.1 + (0.1)f(0.1, 2.1) = 2.514277288 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) \\ &= 3.317872752 \end{aligned}$$

With standard notation, let us recall the improved Euler's formula
$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right)$.

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right) \\ &= 2.257138644 \end{aligned}$$

By similar computation, we find that $y_2 = 2.826004666$ and $y_3 = 3.812671926$.
The actual value of the solution $y(x) = e^{4x} + e^{-3x}$ at the points 0.1, 0.2 and 0.3 are $y(0.1) = 2.232642918$, $y(0.2) = 2.774352665$ and $y(0.3) = 3.726686582$ respectively.

2. $\star$ Apply Euler's method with step size $h = 0.1$, $h = 0.05$ and $h = 0.025$ to the differential equation

$$y' - 2y = \frac{x}{1 + y^2}, \qquad y(1) = 7$$

to find approximate values $x = 1, 1.1, 1.2, 1.3$.

Leaving the computations, I just give the values below. $y_0 = 7$, $y_1 = 8.402$, $y_2 = 10.083936450$ and $y_3 = 12.101892354$.

3. $\star$ Use improved Euler's method to find the approximate values for the solution of the initial value problem

$$y' = -2y^2 + xy + x^2 \quad y(0) = 1$$

with step size $h = 0.1$ and $h = 0.05$.

Let us recall the improved Euler's formula

$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right)$.

Then the approximate value

$$\begin{aligned} y_1 &= y_0 + \frac{0.1}{2}\left(f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right) \\ &= 1 + \frac{0.1}{2}\left(f(0,1) + f(1.1, -0.2)\right) = 0.8405 \end{aligned}$$

Similar computations yield $y_2 = 0.733430846$ *and* $y_3 = 0.66160806$.

4. $\star$ Verify that $y_1 = x^2$ and $y_2 = \frac{1}{x^2}$ are a set of fundamental solutions of the differential equation $x^2y'' + xy' - 4y = 0$ in $(-\infty, 0) \cup (0, \infty)$.

   Find the solution y if (i) $(y(1), y'(1)) = (2, 0)$ and (ii) $(y(-1), y'(-1)) = (2, 0)$.

   Observe that $y_1(x) = x^2$ is a solution on the whole of $\mathbb{R}$.

   It is easy to see that $W(y_1, y_2) = (y_1y_2' - y_1'y_2) = -\frac{4}{x} \neq 0$ *if* $x \neq 0$. *Thus* $\{y_1, y_2\}$ *is a set of fundamental solutions of* $x^2y'' + xy' - 4y = 0$ *in* $(-\infty, 0) \cup (0, \infty)$.
   Let y *be the solutuion of* $x^2y'' + xy' - 4y = 0$ *with initial conditions* $y(1) = 2$ *and* $y'(1) = 0$. *We write this solution as* $y = c_1x^2 + c_2\frac{1}{x^2}$. *Then the initial condition shows that* $c_1 = c_2 = 1$. *Hence* $y(x) = x^2 + \frac{1}{x^2}$.

5. $\star$ Let $p, q : (a, b) \to \mathbb{R}$ be two continuous functions. Let $\{y_1, y_2\}$ be a fundamental set of solutions of the differential equations $y'' + py'1 + qy = 0$ in (a, b). Let y be the solution of the differential equation $y'' + py' + qy = 0$ with initial condition $y(x_0) = k_0$ and $y'(x_0) = k_1$. Show that $y = c_1y_1 + c_2y_2$ where $c_1 = \frac{y_2'(x_0)k_0 - y_2(x_0)k_1}{W(y_1,y_2)(x_0)}$ and $c_2 = \frac{y_1(x_0)k_1 - y_1'(x_0)k_0}{W(y_1,y_2)(x_0)}$.

   Since the $\{y_1, y_2\}$ *is a set of fundmantal solutions, the solution* y *with* $y(x_0) = k_0$ *and* $y'(x_0) = k_1$ *can be written as* $y = c_1y_1 + c_2y_2$ *for some constants* c_1 *and* c_2. *Differentiating this equation and evaluating at the point* x_0, *we get two equations*

   $$c_1y_1(x) + c_2y_2(x) = k_0 \text{ and } c_1y_1'(x_0) + c_2y_2'(x_0) = k_1$$

   We re-write this equation as a matrix equation

   $$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} k_0 & k_1 \end{pmatrix}.$$

   Since the matrix $\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}$ *is invertible we can solve uniquely for* c_1 *and* c_2 *and they turn out to be the values given in the problem.*

6. $\star$ In the following problems, use the method of reduction of order to find a solution y_2 that is not a constant multiple of the solution y_1.

   (a) $y'' - 2ay' + a^2y = 0, \qquad y_1(x) = e^{ax}$.

   (b) $4x^2 \sin xy'' - 4x(x \cos x + \sin x)y' + (2x \cos x + 3 \sin x)y = 0, \qquad y_1(x) = x^{\frac{1}{2}}$ for $x > 0$.

If y_1 and y_2 are two solutions of $y'' + py' + qy = 0$ then we know that $y_1y_2' - y_1'y_2 = Ke^{P(x)}$ where $P(x) = -\int p(x)dx$ and K is a constant. If $y_2 = uy_1$, then substituting $y' = u'y_1 + uy_1'$ in the equation $y_1y_2' - y_1'y_2 = Ke^{P(x)}$, we get $u'y_1^2 = Ke^{P(x)}$.

In the first problem $p = -2a$. Applying this formula we get $u = ax + b$ for some constant sa and b. Hence $y_2(x) = xe^{ax}$ is a solution that is not a constant multiple of y_1.

In the second problem applying this formula $p = -4x(x\cos x + \sin x)$ and $P(x) = 4x^2 \sin x + 4x\cos x - 4\sin x$. Thus $u' = K\frac{e^{4x^2 \sin x + 4x\cos x - 4\sin x}}{x}$ and we need to integrate this to get y_2.

This exercise shows that though the method is neat and clear, it may not be easy to calculate the integrals in the formula !

7. Use Euler's method to find the approximate value of the following initial value problems.
   (a) $y' + \frac{2}{x}y = \frac{3}{x^3} + 1$, $\quad y(1) = 1;\ h = 0.1, h = 0.05$ at $x = 1.0, 1.1, 1.2, 1.3, 1.4$.
   Compare these approximate values with the values of the exact solution $y = \frac{1}{3x^2}(9\ln x + x^3 + 2)$.
   (b) $(3y^2 + 4y)y' + 2x + \cos x = 0$, $\quad y(0) = 1;\ h = 0.1h = 0.05$ at $x = 0, 0.1, 0.2, 0.3, 0.4$

8. In the exrcises given below, use improved Euler's method to find the approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$ where x_0 is the initial point and $i = 1, 2, 3$.
   (a) $y' = 2x^2 + 3y^2 - 2$, $\quad y(2) = 1; \quad h = 0.05$.
   (b) $y' = y + \sqrt{x^2 + y^2}$, $\quad y(0) = 1; \quad h = 0.1$.
   (c) $y' + x^2y = \sin(xy)$ $\quad y(1) = \pi; \quad h = 0.2$.

9. Let $p, q : (a, b) \to \mathbb{R}$ be two continuous functions and y_1 be a solution of the differential equation $y'' + py' + qy = 0$ in (a, b). Let $y_2(x) = ky_1(x)$ for all $x \in (a, b)$ and k is a constant. Show that $W(y_1, y_2) \equiv 0$ in (a, b).

10. Let $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ for all $x \in \mathbb{R}$. Show that
    (a) the two functions y_1 and y_2 are linearly independent in any interval (a, b) such that $a < 0 < b$.
    (b) the wronskian $W(y_1, y_2) \equiv 0$ in $\mathbb{R}$.
    (c) the functions y_1 and y_2 can't be solutions of an ordinary differential equation $y'' + py' + qy = 0$.

11. Let y_1 and y_2 be two solutions of $x^2y'' + xy' + (x^2 - n^2)y = 0$ in $(0, \infty)$ with $(y_1(0), y_1'(0)) = (1, 0)$ and $(y_2(0), y_2'(0)) = (0, 1)$. Compute $W(y_1, y_2)$.