Problem Set 1

Problems marked **(T)** are for discussions in Tutorial sessions.

1. **(T)** If A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix, then show that $A(BD) = (AB)D$ (Associativity holds).

   Solution: Entry by entry for $1 \le i \le m$ and $1 \le j \le s$, we have

$$\begin{aligned}
[A(BD)]_{ij} &= \sum_{k=1}^{n}[A]_{ik}[BD]_{kj} = \sum_{k=1}^{n}[A]_{ik}\left(\sum_{l=1}^{p}[B]_{kl}[D]_{lj}\right) = \sum_{k=1}^{n}\sum_{l=1}^{p}[A]_{ik}[B]_{kl}[D]_{lj} \\
&= \sum_{l=1}^{p}\sum_{k=1}^{n}[A]_{ik}[B]_{kl}[D]_{lj} = \sum_{l=1}^{p}[D]_{lj}\left(\sum_{k=1}^{n}[A]_{ik}[B]_{kl}\right) \\
&= \sum_{l=1}^{p}[D]_{lj}[AB]_{il} = \sum_{l=1}^{p}[AB]_{il}[D]_{lj} = [(AB)D]_{ij}.
\end{aligned}$$

   Hence the result.

2. If A is an $m \times n$ matrix, B and C are $n \times p$ matrices and D is a $p \times s$ matrix, then show that

   (a) $A(B + C) = AB + AC$ (Distributive law holds).

   Solution: Entry by entry for $1 \le i \le m$ and $1 \le j \le p$, we have

$$\begin{aligned}
[A(B+C)]_{ij} &= \sum_{k=1}^{n}[A]_{ik}[B+C]_{kj} = \sum_{k=1}^{n}[A]_{ik}\left([B]_{kj} + [C]_{kj}\right) \\
&= \sum_{k=1}^{n}\left([A]_{ik}[B]_{kj} + [A]_{ik}[C]_{kj}\right) = \sum_{k=1}^{n}[A]_{ik}[B]_{kj} + \sum_{k=1}^{n}[A]_{ik}[C]_{kj} \\
&= [AB]_{ij} + [AC]_{ij} = [AB + AC]_{ij}.
\end{aligned}$$

   Hence the result.

   (b) $(B + C)D = BD + CD$ (Distributive law holds).

   Solution: Similar to part (a) with appropriate modifications.

3. **(T)** Let A and B be 2×2 real matrices such that $A\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x \\ y \end{bmatrix}$ for all $(x, y) \in \mathbb{R}^2$. Prove that $A = B$.

   Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. The given equation imply

$$x\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + y\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = x\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + y\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \tag{1}$$

Now, by substituting $x = 1$ and $y = 0$ in (1), we get

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \tag{2}$$

Similarly, by substituting $x = 0$ and $y = 1$ in (1), we get

$$\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \tag{3}$$

Equations (2) and (3) together imply the result.

4. Let A and B be $m \times n$ real matrices such that $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then, $A = B$

5. For two matrices A and B show that

   (a) $(A + B)^t = A^t + B^t$ if $A + B$ is defined.

   Solution: Let A and B be $m \times n$ matrices. Then, entry by entry for $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

   $$[(A + B)^t]_{ij} = [A + B]_{ji} = [A]_{ji} + [B]_{ji} = [A^t]_{ij} + [B^t]_{ij} = [A^t + B^t]_{ij}.$$

   Hence the result.

   (b) $(AB)^t = B^t A^t$ if AB is defined.

   Solution: Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then, entry by entry for $1 \leq i \leq p$ and $1 \leq j \leq m$, we have

   $$[(AB)^t]_{ij} = [AB]_{ji} = \sum_{k=1}^{n} [A]_{jk} [B]_{ki} = \sum_{k=1}^{n} [A^t]_{kj} [B^t]_{ik} = \sum_{k=1}^{n} [B^t]_{ik} [A^t]_{kj} = [B^t A^t]_{ij}.$$

   Hence the result.

6. If A and B are symmetric matrices, which of these matrices are necessarily symmetric?

   (a) $A^2 - B^2$

   Solution: Sum of two symmetric matrices is again symmetric. To see this, let C and D be two $n \times n$ symmetric matrices. Then, entry by entry for $1 \leq i \leq n$ and $1 \leq j \leq n$, we have

   $$[C + D]_{ij} = [C]_{ij} + [D]_{ij} = [C]_{ji} + [D]_{ji} = [C + D]_{ji}.$$

   Now, as A^2 and $-B^2$ are symmetric, so is $A^2 - B^2$.

   (b) $(A + B)(A - B)$

   Solution: It is not symmetric in general. For example, take $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$.
   Then, $(A + B)(A - B) = \begin{bmatrix} -4 & 4 \\ -4 & 4 \end{bmatrix}$, not symmetric.

(c) ABA

Solution: Always symmetric.

$$(ABA)^t = ((AB)(A))^t = A^t(AB)^t = A^tB^tA^t = ABA.$$

(d) $ABAB$

Solution: It is not symmetric in general. For example, take $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $ABAB = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$, which is not symmetric.

7. Prove that every square matrix can be uniquely written as a sum of a Hermitian matrix ($A^* = A$) and a skew-Hermitian matrix ($A^* = -A$).

8. Give examples of 3×3 nonzero matrices A and B such that

(a) $A^n = 0$, for some $n > 1$.

Solution:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^2 = 0. \qquad A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^3 = 0.$$

(b) $B^3 = B$.

Solution:

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow B^3 = B.$$

9. Show by an example that if $AB \neq BA$ then $(A+B)^2 = A^2 + 2AB + B^2$ need not hold.

Solution: Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Clearly, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = BA$. A straightforward calculation shows that

$$(A+B)^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} = A^2 + 2AB + B^2.$$

10. If $AB = BA$ then show that $(A+B)^m = \sum_{i=0}^{m} \binom{m}{i} A^{m-i}B^i$.

Solution: Proof by induction on m.
Base step: Clearly, the result is true for $m = 0$ and $m = 1$.

Induction step: Assume that the result is true for some m. Consider,

$$\begin{aligned}
(A+B)^{m+1} &= (A+B)(A+B)^m = (A+B)\sum_{i=0}^{m}\binom{m}{i}A^{m-i}B^i \\
&= \sum_{i=0}^{m}\binom{m}{i}A^{m-i+1}B^i + \sum_{i=0}^{m}\binom{m}{i}BA^{m-i}B^i \\
&= \sum_{i=0}^{m}\binom{m}{i}A^{m-i+1}B^i + \sum_{i=0}^{m}\binom{m}{i}A^{m-i}B^{i+1} \text{ (as } AB = BA) \\
&= \sum_{i=0}^{m}\binom{m}{i}A^{m-i+1}B^i + \sum_{i=1}^{m+1}\binom{m}{i-1}A^{m-i+1}B^i \\
&= A^{m+1} + \sum_{i=1}^{m}\left\{\binom{m}{i} + \binom{m}{i-1}\right\}A^{m-i+1}B^i + B^{m+1} \\
&= A^{m+1} + \sum_{i=1}^{m}\binom{m+1}{i}A^{m+1-i}B^i + B^{m+1} \text{ (using Pascal's rule)} \\
&= \sum_{i=0}^{m+1}\binom{m+1}{i}A^{m+1-i}B^i
\end{aligned}$$

Thus, whenever result holds for m, it also holds for $m+1$. Hence proved.

11. If an $n \times n$ real matrix A satisfies the relation $AA^t = 0$ then show that $A = 0$. Is the same true if A is a complex matrix? Show that if A is a $n \times n$ complex matrix and $A\bar{A}^t = 0$ then $A = 0$.

    Solution:

$$AA^t = 0 \Rightarrow \mathrm{Tr}(AA^t) = 0 \Rightarrow \sum_{i=1}^{n}[AA^t]_{ii} = 0 \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}[A]_{ij}[A^t]_{ji} = 0 \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}[A]_{ij}[A]_{ij} = 0$$

We, therefore, have $[A]_{ij} = 0$ for all $1 \le i \le n$, $1 \le j \le n$ and thus $A = 0$.

To see that this result is not true for matrices with complex entries, one can consider the non-zero matrix $A = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ for which $AA^t = 0$.

For complex matrices,

$$A\bar{A}^t = 0 \Rightarrow \mathrm{Tr}(A\bar{A}^t) = 0 \Rightarrow \sum_{i=1}^{n}[A\bar{A}^t]_{ii} = 0 \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}[A]_{ij}[\bar{A}^t]_{ji} = 0 \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}[A]_{ij}[\bar{A}]_{ij} = 0$$

We, therefore, have $[A]_{ij} = 0$ for all $1 \le i \le n$, $1 \le j \le n$ and thus $A = 0$.

12. Find two 2×2 invertible matrices A and B such that $A \neq cB$, for any scalar c and $A + B$ is not invertible.

Solution: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We have, $A - cB = \begin{bmatrix} 1 & -c \\ -c & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all c. Clearly,

$$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

has a non-trivial null-space, for example,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

and hence is not invertible.

13. Let A and B be two $n \times n$ invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: Let $D = B^{-1}A^{-1}$. Then

$$(AB)D = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$D(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

imply that D is the inverse of AB.

14. Let A be a nilpotent matrix. Show that $I + A$ is invertible.

Solution: As A is nilpotent, there exists an $N > 0$ such that $A^N = 0$. Define

$$B = \sum_{n=0}^{N-1} (-1)^n A^n.$$

We have

$$\begin{aligned} (I + A)B &= (I + A)\left(\sum_{n=0}^{N-1}(-1)^n A^n\right) = \sum_{n=0}^{N-1}(-1)^n A^n + \sum_{n=0}^{N-1}(-1)^n A^{n+1} \\ &= \sum_{n=0}^{N-1}(-1)^n A^n - \sum_{n=1}^{N-1}(-1)^n A^n = I \end{aligned}$$

and

$$\begin{aligned} B(I + A) &= \left(\sum_{n=0}^{N-1}(-1)^n A^n\right)(I + A) = \sum_{n=0}^{N-1}(-1)^n A^n + \sum_{n=0}^{N-1}(-1)^n A^{n+1} \\ &= \sum_{n=0}^{N-1}(-1)^n A^n - \sum_{n=1}^{N-1}(-1)^n A^n = I \end{aligned}$$

and, therefore, B is the inverse of $I + A$.

15. Let A be a 5×5 invertible matrix with row sums 1. That is $\sum_{j=1}^{5} a_{ij} = 1$ for $1 \leq i \leq 5$. Then, prove that the sum of all the entries of A^{-1} is 5.

    Solution: Let $\mathbf{e}$ be 5×1 vector of all 1's. Then, we are given that $A\mathbf{e} = \mathbf{e}$. Hence, $\mathbf{e} = A^{-1}\mathbf{e}$. Therefore,
$$\sum_{i=1}^{5}\sum_{j=1}^{5}(A^{-1})_{ij} = \mathbf{e}^t A^{-1}\mathbf{e} = \mathbf{e}^t\left(A^{-1}\mathbf{e}\right) = \mathbf{e}^t\mathbf{e} = 5.$$

16. **(T)** Let A and B be two $n \times n$ matrices. Define
$$\text{Tr }(A) = \sum_{i=1}^{n} a_{ii}.$$

    Then show that $\text{Tr }(AB) = \text{Tr }(BA)$. Hence or otherwise, show that if A is invertible then $\text{Tr }(ABA^{-1}) = \text{Tr }(B)$. Furthermore, show that there do not exist matrices A and B such that $AB - BA = cI$, for any $c \neq 0$.

    Solution: $\text{Tr }(AB) = \text{Tr }(BA)$ follows from a straightforward calculation shown below:
$$\sum_{i=1}^{n}[AB]_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}[BA]_{jj}.$$

    Now let $D = BA^{-1}$. We have,
$$\text{Tr}(ABA^{-1}) = \text{Tr}(AD) = \text{Tr}(DA) = \text{Tr}(BA^{-1}A) = \text{Tr}(B).$$

17. **(T)** The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$, $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

    Solution: As the parabola passes through point $(1, 4)$, we have
$$(4) = a + b(1) + c(1)^2$$

    leading to the equation
$$a + b + c = 4.$$

    Similarly for points $(2, 8)$ and $(3, 14)$, we get
$$\begin{aligned} a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14. \end{aligned}$$

    We can obtain a, b and c as a solution to
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix}.$$

Carry out Gauss-elimination as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

We can thus obtain the solution to the given linear system by solving the equivalent system

$$\begin{aligned} a + b + c &= 4 \\ b + 3c &= 4 \\ 2c &= 2 \end{aligned}$$

The solution is $a = 2, b = 1$ and $c = 1$.

18. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & x & y \\ 1 & x^2 & y^2 \end{bmatrix}$ with x and y distinct numbers different from 1. Is A invertible?

19. **(T)** Find the numbers a and b such that

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

Solution: It is straight forward to see that a and b satisfy the 2×2 linear system

$$\begin{aligned} 4a - 3b &= 1 \\ a - 2b &= 0 \end{aligned}$$

to which the answer is $a = 2/5$ and $b = 1/5$.

20. Let J be an $n \times n$ matrix with every entry 1. Determine condition(s) on a and b such that the $n \times n$ matrix $bJ + (a - b)I$ is invertible. Find α and β in terms of a and b such that the inverse has the form $\alpha J + \beta I$.

Solution: Check that $J^2 = nJ$. The symmetry of the matrix $bJ + (a - b)I$ motivates us to try to assume that $\alpha J + \beta I$ may be the inverse for some choice of α and β. Now, multiply the two matrices to get the required conditions.