MSO202A: Assignment-V

1. Find

   (a) Taylor series of the function $f(z) = 1/z^2$ in powers of $z - 1$.

   (b) Laurent series of the function $f(z) = 1/z^2$ for $\{z : |z - 1| > 1\}$.

2. (a) Find Laurent series of the function $f(z) = \frac{6z+8}{(2z+3)(4z+5)}$ in the region

   (i) $\{z \in \mathbb{C} : |z| < \frac{5}{4}\}$ (ii) $\{z \in \mathbb{C} : \frac{5}{4} < |z| < \frac{3}{2}\}$ (iii) $\{z \in \mathbb{C} : |z| > \frac{3}{2}\}$

   (b) Find Laurent series of the function $f(z) = \frac{1}{z^3 - z^4}$ in the region

   (i) $\{z \in \mathbb{C} : 0 < |z| < 1\}$ (ii) $\{z \in \mathbb{C} : |z| > 1\}$

3. Find the Laurent series of the function $f(z) = \exp(z + \frac{1}{z})$ around $z = 0$. Hence, show that (for $n \geq 0$)

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta} \cos n\theta \, d\theta = \sum_{j=0}^{\infty} \frac{1}{(n+j)!j!}.$$

4. Is there a polynomial $P(z)$ such that $P(z)e^{1/z}$ is an entire function? Justify your answer.

5. Which of the following singularities are removable/pole:

   (i) $\frac{\sin z}{z^2 - \pi^2}$ at $z = \pi$

   (ii) $\frac{\sin \pi z}{(z-\pi)^2}$ at $z = \pi$ pi must not be present

   (iii) $\frac{z \cos z}{1 - \sin z}$ at $z = \pi/2$

6. Suppose f and g are two analytic functions in a neighbourhood of a point $z_0 \in \mathbb{C}$ such that $g(z_0) \neq 0$ and f has a simple zero at z_0. Prove that

$$\text{Res}\left(\frac{g}{f} : z_0\right) = \frac{g(z_0)}{f'(z_0)}$$

7. Let f be analytic in a domain Ω and γ be a simple closed curve in Ω in the counterclockwise sense. Suppose z_0 is the only zero of f in the region enclosed by Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i m,$$

   where m is the order of zero of f at z_0.

8. Find the isolated singularities and compute the residue of the functions

$$(i)\ \frac{e^z}{z^2 - 1} \qquad (ii)\ \frac{3z}{z^2 + iz + 2} \qquad (iii)\ \cot \pi z \qquad (iv)\ \frac{\pi \cot \pi z}{(z + 1/2)^2}$$

9. Evaluate

$$(i)\ \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{2n}}, \quad n \geq 1 \qquad (ii)\ \int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2 + a^2} dx \qquad (iii)\ \int_0^{\pi} \sin^{2n}\theta \, d\theta$$

n

10. Compute the following integrals

$$(i) \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \qquad (ii) \int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \qquad (iii) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx, \quad 0 < a < 1.$$