Solution of Assignment #6

#6·1

This problem is an extension of the problem in which a point charge is placed above an infinite grounded conducting plane. So, in order to obtain the correct potential, we need to put an infinite line charge $-\lambda$ running parallel to the x axis at a distance $-d$ from the x-y plane.

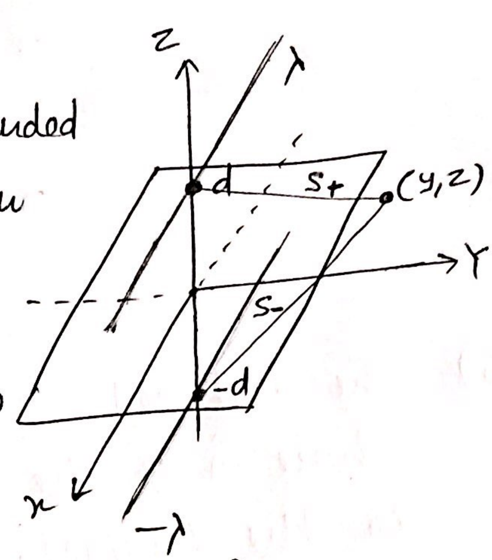


Fig:

(a) Let's first find the potential at a point (y, z) on the y-z plane. Assume that the distance of the point (y, z) from the top infinite line charge is given by the magnitude of the vector $\hat{s}_+$; similarly the distance of the point (y, z) from the bottom infinite line charge is given by the magnitude of the vector $\hat{s}_-$. Taking the potential $V = 0$ at x-y plane to be the reference, the potential due to top infinite line charge can be shown to be:

$$V_+ = -\frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{s_+}{d}\right)$$

Similarly the potential due to the bottom infinite line charge can be shown to be

$$V_- = +\frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{s_-}{d}\right)$$

Therefore the total potential at (y, z) is given by

$$V(y, z) = -\frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{s_+}{d}\right) + \frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{s_-}{d}\right)$$

$$= \frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{S_-}{S_+}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{S_-^2}{S_+^2}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]$$

(b) The induced charge density is given by $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$, where $\hat{n}$ is the vector in the direction perpendicular to the surface. In the present problem, $\hat{n} = \hat{z}$. Therefore,

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}$$

$$= -\epsilon_0 \frac{\partial}{\partial z}\left[\frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]\right]\bigg|_{z=0}$$

$$= -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2}\right]\bigg|_{z=0}$$

$$= -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2d}{y^2+z^2} + \frac{2d}{y^2+d^2}\right]$$

$$= -\frac{\lambda d}{\pi(y^2+d^2)^2}$$

\# 6.2 In this problem, a point r on the sphere is represented in the (r, θ, ϕ) coordinates. An observation point r_0 is represented in the (r_0, θ_0, ϕ_0) coordinates. The angle between the vectors r and r_0 is represented by α. So, the formula for multipole expansion of potential V can be written as

$$V(r_0) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r_0^{n+1}} \int r^n P_n(\cos\alpha)\, \rho(r)\, d\tau$$

Now, since we are interested in observation points along z-axis only, the angle α between the two vectors is essentially the angle θ. Therefore we can write the above expression as

$$V(r_0) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int r^n P_n(\cos\theta)\, \rho(r)\, d\tau$$

Let's calculate the above potential term by term in the limit $z \gg R$.

1. Monopole Term:

We have

$$V_{mono}(r_0) = \frac{1}{4\pi\epsilon_0 z} \int \rho(r)\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0 z} \int \left[k \frac{R}{r^2} (R-2r) \sin\theta \right] r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{1}{4\pi\epsilon_0 z} kR \int (R-2r) \sin^2\theta\, dr\, d\theta\, d\phi$$

The r-Integral yields

$$\int_0^R (R-2r)\, dr = (Rr - r^2)\Big|_0^R = R^2 - R^2 = 0$$

Therefore, the monopole potential is zero:

$$V_{mono}(r_0) = \frac{1}{4\pi\epsilon_0 z} kR \int (R-2r) \sin^2\theta\, dr\, d\theta\, d\phi = 0$$

2. Dipole Term:

We have

$$V_{dip}(r_0) = \frac{1}{4\pi\epsilon_0 z^2} \int r P_1(\cos\theta)\, \rho(r)\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0 z^2} \int r \cos\theta\, \rho(r)\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0 z^2} \int r\cos\theta \left[k \frac{R}{r^2}(R-2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi$$

The θ integral yields

$$\int_0^{\pi} \sin^2\theta \cos\theta\, d\theta = \left.\frac{\sin^3\theta}{3}\right|_0^{\pi}$$

$$= \frac{1}{3}(0-0)$$

$$= 0$$

Therefore, the dipole potential is also zero:

$$V_{dip}(r_0) = \frac{1}{4\pi\epsilon_0 z^2} \int r\cos\theta \left[k \frac{R}{r^2}(R-2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 0$$

3. Quadrupole Term:

We have

$$V_{quad}(r_0) = \frac{1}{4\pi\epsilon_0 z^3} \int r^2 P_2(\cos\theta)\, \rho(r)\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0 z^3} \int r^2 (3\cos^2\theta - 1)/2\, \rho(r)\, d\tau$$

$$= \frac{1}{8\pi\epsilon_0 z^3}\int r^2(3\cos^2\theta-1)\left[k\frac{R}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta\, dr\, d\theta\, d\phi$$

The three integrals can be calculated as follows:

r - integral : $\int_0^R r^2(R-2r)\,dr$

$$= \left(\frac{r^3}{3}R-\frac{r^4}{2}\right)\Big|_0^R$$

$$= -\frac{R^4}{6}$$

θ - integral : $\int_0^\pi (3\cos^2\theta-1)\sin^2\theta\, d\theta$

$$= 2\int_0^\pi \sin^2\theta\, d\theta - 3\int_0^\pi \sin^4\theta\, d\theta$$

$$= 2\left(\frac{\pi}{2}\right) - 3\left(\frac{3\pi}{8}\right)$$

$$= \pi\left(1-\frac{9}{8}\right)$$

$$= -\frac{\pi}{8}$$

ϕ - integral $\int_0^\pi d\phi$

$$= 2\pi$$

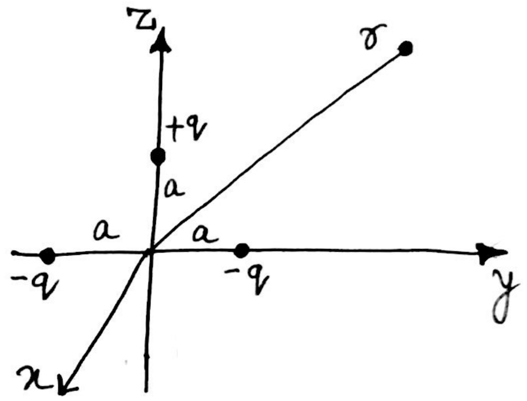
Therefore,

$$V_{quad}(r_0) = \frac{kR}{8\pi\epsilon_0 z^3}\left(-\frac{R^4}{6}\right)\times\left(-\frac{\pi}{8}\right)\times(2\pi)$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{k\pi^2 R^5}{48z^3}$$

So, the approximate potential is the potential of the quadrupole :

$$V(r_0) \approx V_{quad}(r_0) = \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48 z^3}$$



6.3 The total charge of the three-charge system is $-q$.

Therefore the monopole contribution is not zero and is given by

$$V_{mono} = \frac{1}{4\pi\epsilon_0} \frac{-q}{r}$$
$$= -\frac{q}{4\pi\epsilon_0 r}$$

The dipole moment p of the charge system is

$$p = qa\hat{z} + [-qa - q(-a)]\hat{y}$$
$$= qa\hat{z}$$

So, the dipole contribution to the potential is

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{p \cdot \hat{r}}{r^2}$$
$$= \frac{1}{4\pi\epsilon_0} \frac{qa\cos\theta}{r^2}$$

Therefore, the potential up to two terms in the multipole expansion is given by

$$V \approx V_{mono} + V_{dip} = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{qa\cos\theta}{r^2}\right)$$

The electric field E is given by

$$E = -\nabla V \approx \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\hat{r} + \frac{a}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})\right]$$

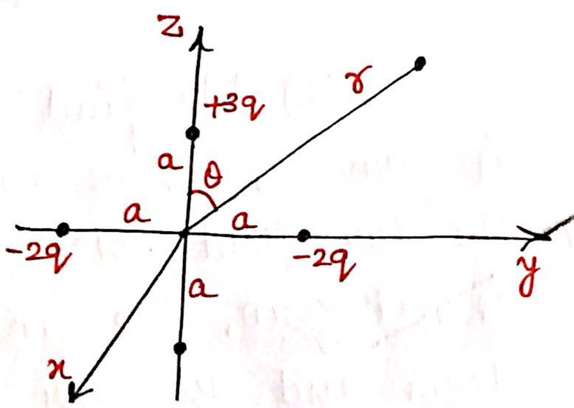
6.4 As far as force is concerned, this problem is the same if we remove the grounded conducting plate and simply put an image charge $+2q$ at $z=-d$ and an image charge $-q$ at $z=-3d$. Therefore, the force on the charge $+q$ is given by

$$F = \frac{1}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} + \frac{-q}{(6d)^2}\right]\hat{z}$$

$$= \frac{q^2}{4\pi\epsilon_0 d^2}\left[\frac{-1}{2} + \frac{1}{8} + \frac{-1}{36}\right]\hat{z}$$

$$= -\frac{q^2}{4\pi\epsilon_0 d^2}\,\frac{29}{72}\,\hat{z}$$

6.5 The total charge of the system is zero.



So, the monopole term in the potential would be zero. Now, let's calculate the dipole term, We have

$$p = (3qa - qa)\hat{z} + (-2qa - 2q(-a))\hat{y}$$

$$= 2qa\hat{z}$$

Therefore,

$$V \simeq V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{p.\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2qa\hat{z}.\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}$$

6.6 (a) From the symmetry in the problem it is clear that the net dipole moment will be in the z direction. Therefore :

$$p \equiv \int r \rho(r)\, d\tau = \int z\hat{z}\rho(r)\, d\tau = \int z\rho(r)\, d\tau\, \hat{z}$$

$$= \int z\sigma\, da\, \hat{z}$$

$$= \int (R\cos\theta)(k\cos\theta) R^2 \sin\theta\, d\theta\, d\phi\, \hat{z}$$

$$= 2\pi R^3 k \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta\, \hat{z}$$

$$= 2\pi R^3 k \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} \hat{z}$$

$$= \frac{4\pi R^3 k}{3} \hat{z}$$

(b) We find that the total charge of the system is zero. So, there will be no monopole contribution to the far-field potential. However since the dipole moment is not zero, the first contribution will be from the dipole term and the far-field potential can be written as

$$V \approx V_{dipole} = \frac{1}{4\pi\epsilon_0} \frac{p\cdot\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{4\pi R^3 k}{3} \frac{\hat{z}\cdot\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{4\pi R^3 k}{3} \frac{\cos\theta}{r^2}$$

$$= \frac{kR^3}{3\epsilon_0} \frac{\cos\theta}{r^2}$$

6.7 The electric field of a pure dipole is given by

$$E_{dip}(r) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left(3p\cos\theta\,\hat{r} - p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}\right)$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left[3(p\cdot\hat{r})\hat{r} - (p\cdot\hat{r})\hat{r} - (p\cdot\hat{\theta})\hat{\theta}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(p\cdot\hat{r})\hat{r} - p\right]$$

6.8 (a) The field due to the dipole p_1 is given by

$$E_1(r,\theta) = \frac{p_1}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

The field due to the dipole p_1 at p_2 is given by substituting $\theta = \pi/2$ in the above equation which yields

$$E_1(r,\theta) = \frac{p_1}{4\pi\epsilon_0 r^3}\hat{\theta}$$

$$= \frac{p_1}{4\pi\epsilon_0 r^3}(-\hat{z})$$

So, the torque on p_2 is

$$N_2 = p_2 \times E_1 = p_2(\hat{y}) \times \frac{p_1}{4\pi\epsilon_0 r^3}(-\hat{z})$$

$$= \frac{p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{x})$$

(b) The force on dipole p due to the point charge is given by

$$F = (p\cdot\nabla)E = (p\cdot\nabla)E_x\hat{x} + (p\cdot\nabla)E_y\hat{y} + (p\cdot\nabla)E_z\hat{z}$$

The electric field due to the point charge q is

$$E = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

So we have

$$F_x = (p\cdot\nabla)E_x = \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{qp_x}{4\pi\epsilon_0}\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{(-3/2)x\times 2x}{(x^2+y^2+z^2)^{5/2}}\right] +$$

$$\frac{qp_y}{4\pi\epsilon_0}\left[\frac{(-3/2)x * 2y}{(x^2+y^2+z^2)^{5/2}}\right] + \frac{qp_z}{4\pi\epsilon_0}\left[\frac{(-3/2)x\times 2z}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{(x^2+y^2+z^2)^{3/2}} - \frac{3x(p_x x + p_y y + p_z z)}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x(p\cdot r)}{r^5}\right]$$

Therefore the force F is

$$F = (p\cdot\nabla)E_x\hat{x} + (p\cdot\nabla)E_y\hat{y} + (p\cdot\nabla)E_z\hat{z}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x(p\cdot r)}{r^5}\right]\hat{x} + \frac{q}{4\pi\epsilon_0}\left[\frac{p_y}{r^3} - \frac{3y(p\cdot r)}{r^5}\right]\hat{y}$$

$$+ \frac{q}{4\pi\epsilon_0}\left[\frac{p_z}{r^3} - \frac{3z(p\cdot r)}{r^5}\right]\hat{z}$$

$$F = \frac{q}{4\pi\epsilon_0}\left[p - 3\hat{r}(p\cdot\hat{r})\right]$$

6.9

(a) Bound charges are calculated as follows:

$$\rho_b = -\nabla\cdot P = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2};$$

$$\sigma_b = P\cdot\hat{n} = \frac{k}{r}\hat{r}\cdot(\hat{r}) = \frac{k}{b} \quad (\text{at } r=b)$$

$$= \frac{k}{r}\hat{r}\cdot(-\hat{r}) = -\frac{k}{a} \quad (\text{at } r=a)$$

(b) Because of the spherical symmetry, the Gauss's law implies $E = \frac{Q_{enc}}{4\pi\epsilon_0 r^2}\hat{r}$. Therefore we have

for $r<a$; $Q_{enc}=0$; so $E=0$

for $r>b$; $Q_{enc}=0$; so $E=0$

for $a<r<b$; $Q_{enc} = \frac{-k}{a}4\pi a^2 + \int_a^r \frac{-k}{r'^2}4\pi r'^2 dr'$

$$= -4\pi ka - 4\pi k(r-a)$$

$$= -4\pi kr;$$

So $\quad E = -\frac{k}{\epsilon_0 r}\hat{r}$

(c) Since there are no free charges, the Gauss's law in the presence of dielectric gives

$$\oint D\cdot da = 0 \Rightarrow D = 0 \text{ at all } r$$

But we have $D = \epsilon_0 E + P$ and so $E = -\frac{P}{\epsilon_0}$,

Therefore we have

for $r<a$; $E = -\frac{P}{\epsilon_0} = 0$

for $r>b$; $E = -\frac{P}{\epsilon_0} = 0$

for $a<r<b$; $E = -\frac{P}{\epsilon_0} = -\frac{k}{\epsilon_0 r}\hat{r}$