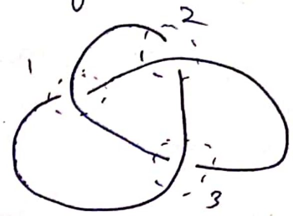
Knot Theory

Def: A Knot K is an embedding of the circle S^1 in $\mathbb{R}^3$; i.e. $f: S^1 \to \mathbb{R}^3$

$c(K)$ = crossing number = min number of crossing points over all diagrams of K



$u(K)$ = unknotting number = min. Number of ~~ext~~ exchanged needed to unknot K

<u>Knot Polynomials</u>: A very good method of distinguishing b/w "distinct" Knots.

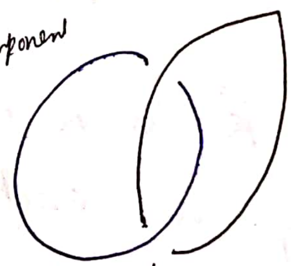
Defn A link is a finite ordered collection of knots that do not intersect each other

component

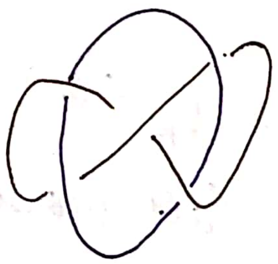
2_1^2

link

Knots



Hopf link



5_1^2

Borromean Ring 6_1^3

<u>Homework</u> Calculate Δ_K for all knots up to 6 crossing

Lecture

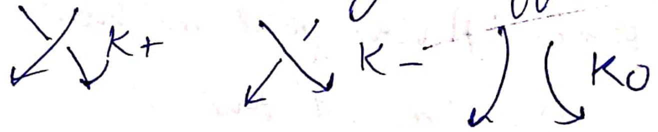
Knot Polynomial

① Alexander Conway Polynomial $\nabla_K(z)$

Given an Oriented knot (or link) $\nabla_K(z)$ is uniquely obtained from 2 axioms.

Axiom 1: $\nabla_O(z) = 1$

Axiom 2: Suppose K_+, K_-, K_0 are knots which are identical everywhere except at one crossing pt; where they differ



$$\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\,\nabla_{K_0}(z)$$

$$z = \sqrt{t} - \frac{1}{\sqrt{t}} \qquad \Delta_K(t) = \nabla_K\left(z = \sqrt{t} - \frac{1}{\sqrt{t}}\right)$$

Ex

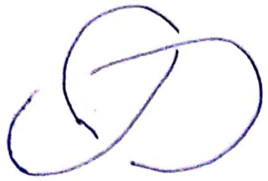
$V_{K^*}(t) = V_K(1/t)$ if K^* is the mirror image of K (all crossings are exchanged)

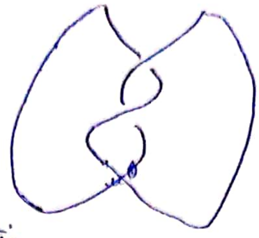
If $K \simeq K^* \Rightarrow V_K(\frac{1}{t}) = V_K(t)$ (Palindromic)

If $V_K(\frac{1}{t}) \neq V_K(t) \Rightarrow K \not\simeq K^* \to 3_1 \not\simeq 3_1^*$

Tait compiled first 800 knots by hand. A special type of knot is the "alternating knot".

Def^n A Alternating knot K is a knot which has at most one diagram in which the over & under crossing alternate.





→ K an alternating knot; $V_K(t)$ its Jones Polyn. Let max deg. $V_K(t) = m$, min degree $V_K(t) = n$

span $V_K(t) = m - n$. Crossing Number $c(K) = m - n$

$V_{3_1}(t) = t + t^3 - t^4$

$m = 4$

$n = 1$

$m - n = 3$

$c(3_1) = 3$

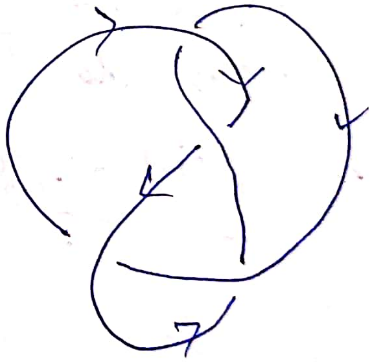
$V_{4_1} = \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$

$m = 2$

$n = -2$

Span $V_{4_1} = 2 - (-2) = 4$

$c(4_1) = 4$



Tait Conjecture Given an alternating knot K, all

Non Alternating ~~Notes~~ Knots

8_{19}

8_{20}

8_{21}

2 Variable Knot polynomial

<u>HOMFLY polynomial $P_K(v,z)$</u>

Let K be an oriented knot (or link) and D a diagram of K

<u>Axiom 1</u> $P_{O}(v,z) = 1$

<u>Axiom 2</u> $\frac{1}{v} P_{D_+}(v,z) - v P_{D_-}(v,z) = z P_{D_0}(v,z)$

<u>Sp case</u>

(1) $v = 1$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$ $P_K(1, \sqrt{t} - \frac{1}{\sqrt{t}}) = \Delta_K(t)$

<u>Home work</u> Calculate $V_K(t)$ and $P_K(v,z)$ for $3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3$ check span $= C(K)$

<u>Classical Knot Theory</u>

~~A classical~~

A given knot K is represented by many different knots diagram D. We would like to know what happens to D if

<u>3 - Reidmeister Moves (1928)</u>

1) Ω_1

2) Ω_2

3) Ω_3

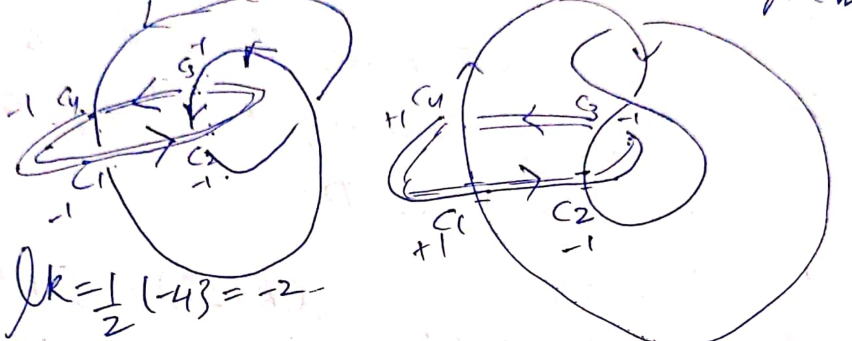
<u>Defn</u> If we can change a diagram D to another diagram D' by performing 3 R-Moves finitely many times. Then we'll say D is equv. to D'

$D \simeq D'$

Thm K, K' knots (or links), D, D' are their diagrams. Then $K \cong K' \iff D \simeq D'$

Linking Number (2 comp links)

$$lk(K_1, K_2) = \frac{1}{2}\{\text{sign}\, c_1 + \text{sign}\, c_2 + \dots + \text{sign}\, c_m\}$$



$lk = \frac{1}{2}\{-4\} = -2$

$lk = 0$

Thm $lk(K_1, K_2)$ is an invariant

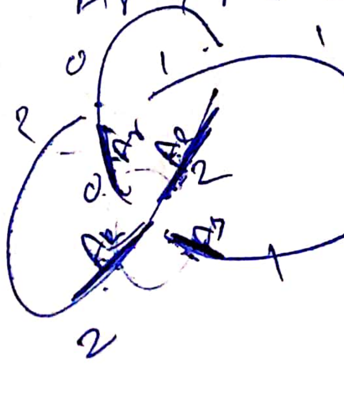
Strategy – We will show $lk(K_1, K_2)$ is unchanged under the 3 R moves $\Omega_1\ \Omega_2\ \Omega_3$

Defn Writhe $w(D)$ sum of all crossing points – Twist number. Not an invariant.

Defn A knot K is said to be tricolorable if we can colour it with 3 colours s.t. at each crossing point

① A_k & A_l same colour

② A_r, A_s distinct colours or all 3 have same colour.



We use $0, 1, 2$; a_1, a_2, a_3, a_4

$$a_1 + a_2 \equiv a_3 + a_4 \pmod 3$$

Lecture Recall

Reidemeister Moves

Def^n If D a regular diagram can be changed to another diagram D' by applying 3 R-moves finitely many times, we say $D \simeq D'$

Thm K, K' two knots (or link); and supp. D, D' are diagrams of K, K' resp. Then

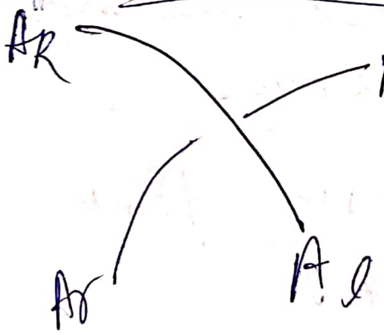
$K \simeq K' \Rightarrow D \simeq D'$

$L = \{K_1, K_2\}$

$L^* \Rightarrow$ mirror image of L $\qquad lk(L^*) = -lk(L)$

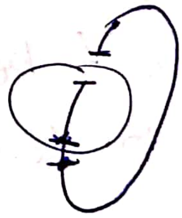
If $L \simeq L^* \Rightarrow lk(L) = 0$

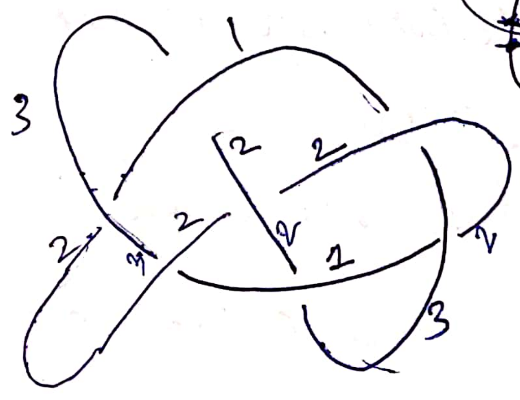
Tri colorability



(1) A_R + A_l have the same colour

(2) A_R, A_r, A_s have same colour or all have different colours.





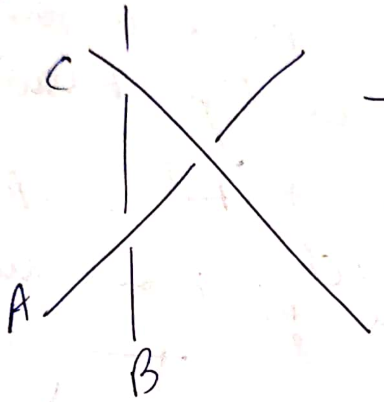
Ref If diag. D is of K which is 3 colourable, then every diag. D' is 3 colourable. Such a knot (or link) K is 3 colourable.

Proof Tricolourability is preserved under the 3 R-moves

Ω_1 ✓

Ω_2 ✓

Ω_3 ✓



→ Case 1: A, B, C same colour

Case 2 A, B diff

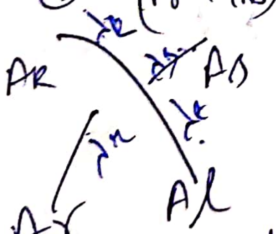
Case 3 A, B, C diff

Case 4 (A, B) same colour
C diff

p-colourability ; p - a prime,

① $\alpha_k = \alpha_l$

② $(\alpha_r + \alpha_s) \equiv (\alpha_k + \alpha_l) \pmod{p}$



Assign numbers 0, 1, 2 . . . p-1 to A_k, A_l, A_r, A_s s.t.
Call them $\alpha_k, \alpha_l, \alpha_r, \alpha_s$

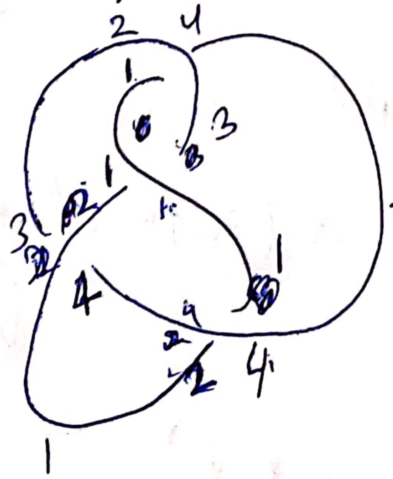
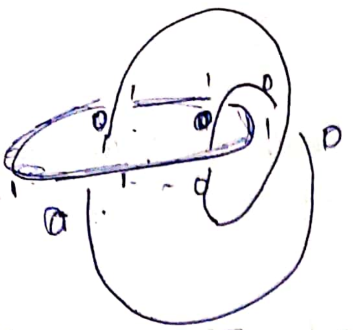
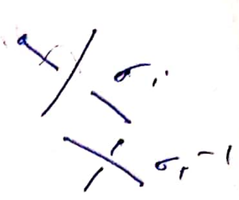


figure 8 → 5 colourable.

$2 \equiv 4$



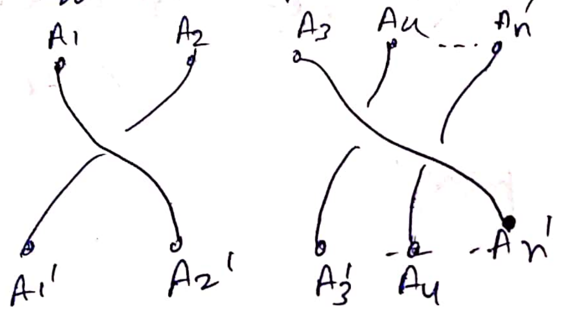
$\to$ 2 colourable

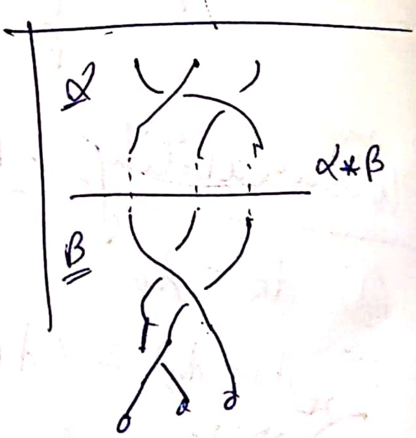


Braid Group $(B_n, *)$

Def^n Take Points $A_1, A_2, \ldots A_n$ in a row and points $A_1', A_2', \ldots, A_n'$ directly below.

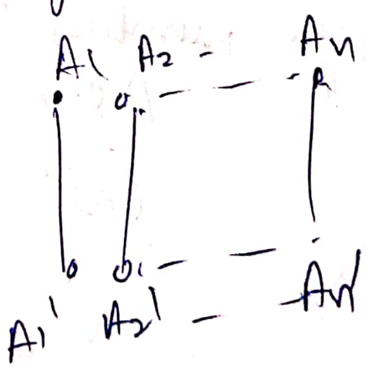
Connect $A_1, \ldots A_n$ to $A_1' \ldots A_n'$ by strings which do not intersect, move monotone down





Claim: B_n is a group

① **Id.**



$\alpha * \beta = \beta * \alpha$ (Not always True)

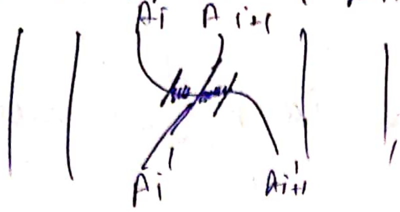
② **Associativity**

$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$

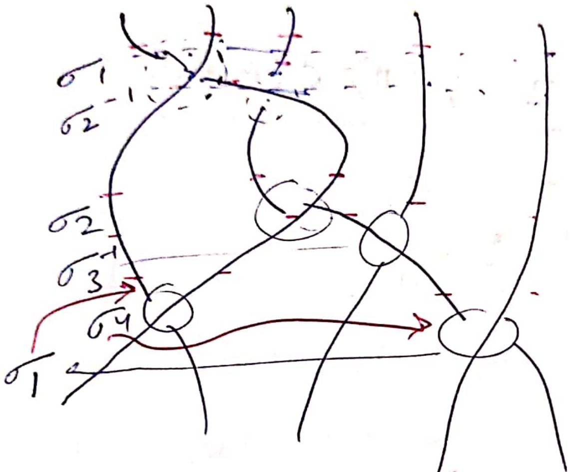
① α^{-1} mirror img of α.

Consider σ_1

⇒ $A_1, A_2 \ldots A_n$ all go to $A_1', \ldots A_n'$
except $A_i \to A_{i+1}'$ & $A_{i+1} \to A_i'$



⇒ elementary generator



$\sigma_1 \circ \sigma_2^{-1} \circ \sigma_2 \circ \sigma_3^{-1} \circ \sigma_4 \circ \sigma_1$

Lecture

Braid Group B_n and Braid Closures

→ Any braid α in B_n can be written as a product of σ_i & σ_j^{-1}

Braid Word

B_n : Braid Group on n-strings

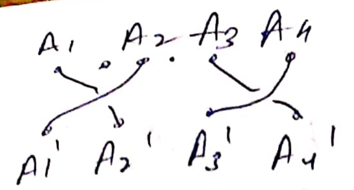
Note: Braid words are not unique

B_2

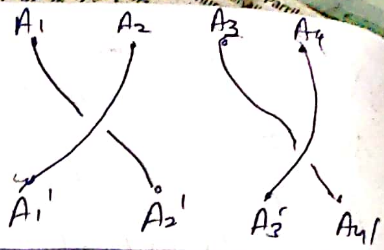
σ_1^3 σ_1^n σ_1^{-m}

B_3 $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$

$\sigma_1\sigma_3$



$\sigma_3\sigma_1$



$\alpha, \beta \in B_n$ then $\alpha \sim \beta$ if we can go from α to β by finitely many elementary knot moves.

$\sigma_1\sigma_3 = \sigma_3\sigma_1$

i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$; ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

$\sigma_1\sigma_2\sigma_1$ $\Rightarrow$ $\sigma_2\sigma_1\sigma_2$

By moving this string (sliding) we can chang $\sigma_1\sigma_2\sigma_1$ to $\sigma_2\sigma_1\sigma_2$

$\therefore$ They are same.

Eg $w_1 = \sigma_1 \sigma_2 \sigma_4^{-1} \sigma_1 \sigma_2 \sigma_4$; $w_2 = \sigma_2\sigma_1\sigma_2^2$

$\sigma_1 \sigma_1 \sigma_2 \sigma_2 \cancel{\sigma_4^{-1} \sigma_4}$

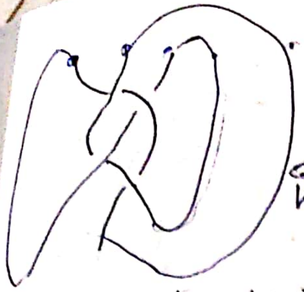
$\sim \sigma_1\sigma_2\sigma_1\sigma_2$

$\sim \sigma_2\sigma_1\sigma_2\sigma_2$

$\sim \sigma_2\sigma_1\sigma_2^2$

→ Given a braid α in B_n. we ~~from~~ form its closure as follows. Join A_1 to A_1' ; A_2 to A_2' ; ... A_n to A_n' with large non intersecting arcs outside the braid diagram

4. Diff b/w Links & Knots!



σ_1^2 closure → Hopf Link

Alexander's Thm (1920's) Any knot (or link) can be obtained in the closure of some braid α

Proof Suppose D is a diagram of the Knot K

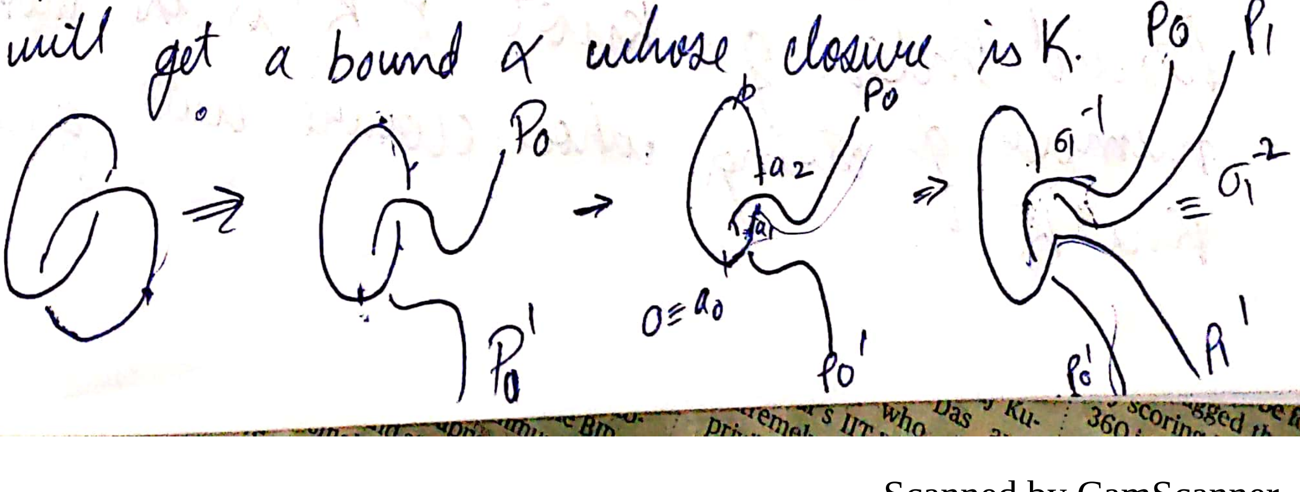
Step1 We will cut D at a point P_0 (Not a crossing point) & pull the loose ends apart at $P_0 P_0'$

Step2 Let us assume the remaining diagram will have at least one maxima b & one minima a

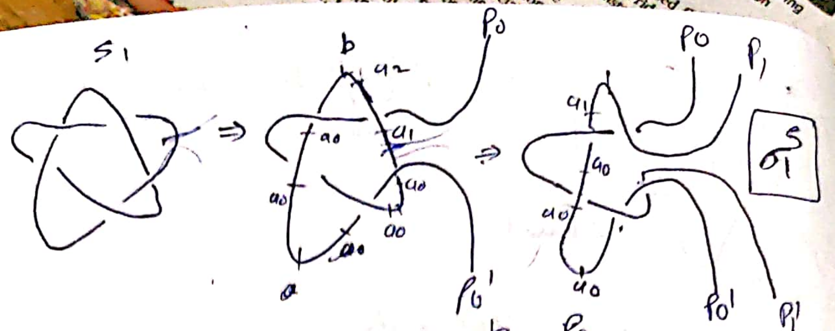
→ We ~~assume~~ divide the strand $\overline{ab}$ intersects with the crossing $a_1 -- a_n$ such that $a_i a_{i+1}$. Intersects at only 1 crossing point

Step 3 Replace $\overline{a_0 a_1}$ by the larger arc $a_0 P_1 P_1 a_1$ s.t. all crossings remain as they were. Using the same method on $\overline{a_1 a_2}$; $\overline{a_2 a_3}$; ...; $\overline{a_{n-1} a_n}$, we will get a braid α whose closure is K.

Eg

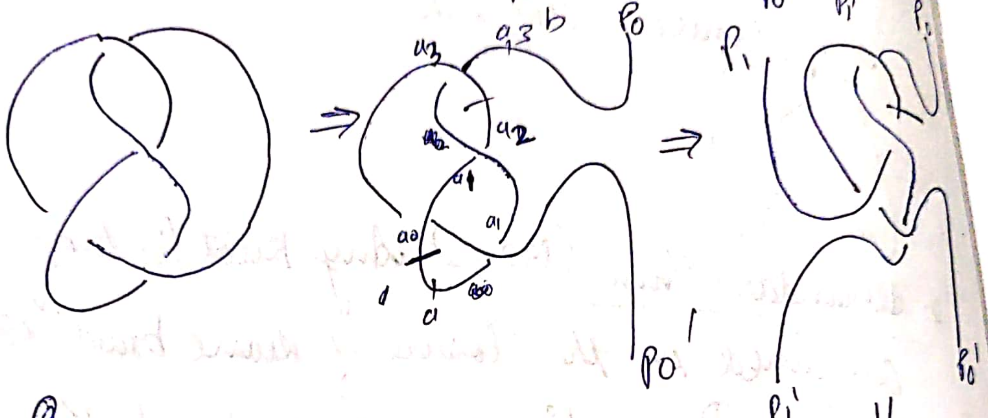


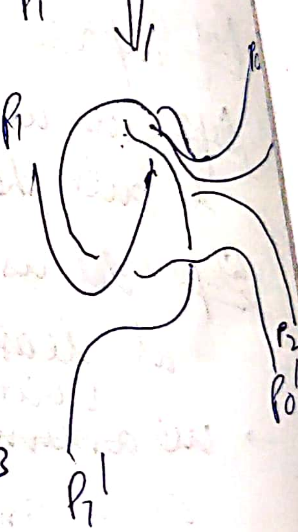
Ex



$\boxed{\sigma_1^5}$

Ex

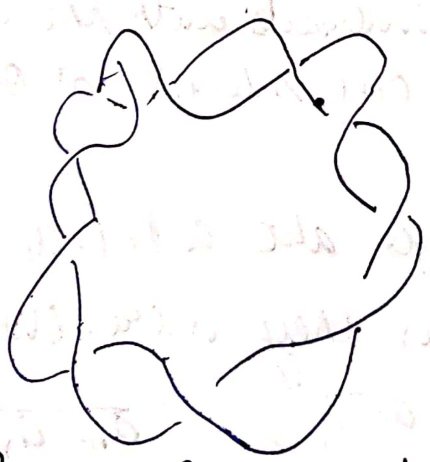




$\sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1}$

HW 6_3, 7_1 find the braid α by Alex. Thm.

Ex



HW
For $5_2, 6_1, 6_2, 6_3$
find the braid α by Alex. Thm

Braid Index of a knot $b(K)$ is the minimum number of strings whose closure will give the knot K.

Lecture

Braid closures

B_n : Braid group on n-string

$B_n = \{\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid 1)\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|\geq 2,\ 2)\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\}$

$B_2 = \{\sigma_1\}$ σ_1^{n} or σ_1^{-n}

$B_3 = \{\sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$

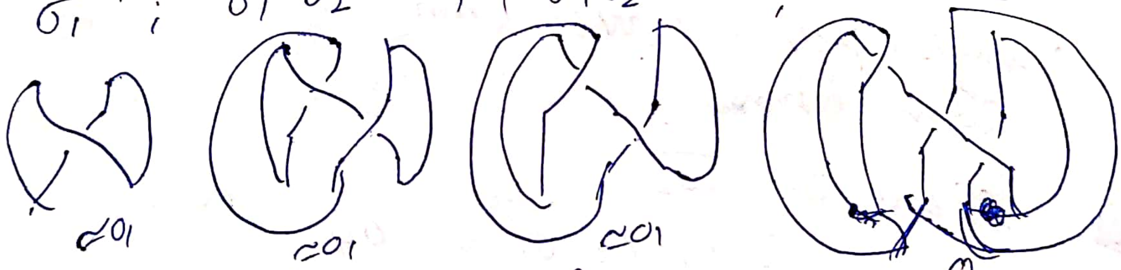
$B_4 = \{\sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1,\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3\}$

Given an n-braid α, we form its closure $\bar{\alpha}$ which is a knot (link) K

Note If $\alpha \sim \beta \Rightarrow \bar{\alpha} = K \simeq \bar{\beta}$

If $\bar{\alpha} = \bar{\beta}$?

$\overline{\sigma_1^{-1}}$; $\overline{\sigma_1^{-1}\sigma_2}$; $\overline{\sigma_1\sigma_2^{-1}}$; $\overline{\sigma_1\sigma_2^{-1}\sigma_3^{-1}}$



$\simeq O_1$ $\quad$ $\simeq O_1$ $\quad$ $\simeq O_1$ $\quad$ O_1

B_2 $\quad$ B_3 $\quad$ B_3 $\quad$ B_4

$\overline{\sigma_1^{-1}}$; $\overline{\sigma_1^{-1}\sigma_2}$; $\overline{\sigma_1\sigma_2^{-1}}$; $\overline{\sigma_1\sigma_2^{-1}\sigma_3^{-1}}$ all give $O_1 \Rightarrow$ Different braids α, β giving $\bar{\alpha} = \bar{\beta}$

$\Rightarrow$ We want '$\sim$' such that if $\alpha \sim \beta \Rightarrow \bar{\alpha} \simeq \bar{\beta}$

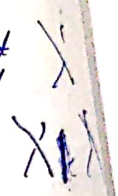
Answered by Markov in 1935

Def^n Suppose B_∞ is the union of the groups $B_1, B_2, \ldots$

i.e. $B_\infty = \bigcup_{k\geq 1} B_k$. we perform 2 operators (Called Markov Moves)

1) If $\beta \in B_n$ then M_1 - first Markov move.
Conjugation $\beta \longrightarrow \gamma\beta\gamma^{-1}$ where $\gamma \in B_n$

2) M_2 - The Second Markov moves β an ~~n-bound~~ n-braid



Stabilization $\left.\begin{array}{l}\beta \to \beta\sigma_n \\ \text{or } \beta \to \beta\sigma_n^{-1}\end{array}\right\} n \to n+1$

Def^n Suppose $\alpha, \beta \in B_\infty$. If we can transform α into β by performing the Markov moves M_1, M_2 and their inverses finitely many times, we ~~call~~ will say α is Markov equivalent to β ~~or~~ & we write
$\alpha \sim_M \beta$

Markov's Thm (1935) Suppose K_1 & K_2 are oriented knots (or links) formed as closures of braids β_1 and β_2 resp.
Then $K_1 \simeq K_2 \iff \beta_1 \sim_M \beta_2$

| a complete proof of Markov Thm
from Birman

Ex Show $\overline{\omega_1} = \overline{\omega_2}$

$\omega_1 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_4$

$\omega_2 = \sigma_2^2 \sigma_3 \sigma_1 \sigma_3^{-1} \sigma_4$

~~$\sigma_3^{-1} \sigma_2^2 \sigma_3 \sigma_1 \sigma_3^{-1} \sigma_4$~~

$(\sigma_3^{-1} \sigma_4 \sigma_2^2 (\sigma_3) \sigma_1$ ~~σ_3^{-1}~~ ~~σ_3~~

σ_1

Scanned by CamScanner

~~$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3$~~

Ex a) $\sigma_1^{-2}\sigma_2^{2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{2}$

b) $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3 \simeq_M \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

c) $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1} \simeq_M \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

d) $\sigma_2^{2}\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_2\sigma_1^{2}$

(a, b, c) are conjugate to each other.

Braid Index $b(K)$

A knot (link) can be formed from a infinite number of braid $\exists$ a braid which has the least no. of string. The no. of string of $\alpha = b(K)$ is the braid index of K. Recall Homfly Pol.

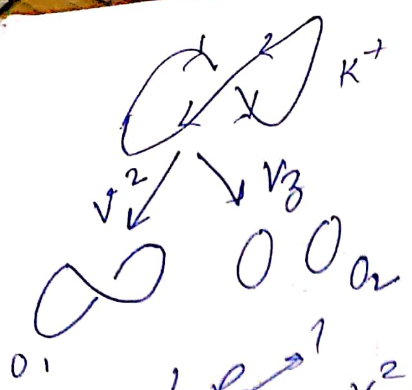
$$b(K) \geq \frac{1}{2}\left(v\text{-span}\,P_K(v,z)\right) + 1 \quad (*)$$

(*) is an equality for all knots up to 10 crossings except 9_{42}, 9_{41}, 10_{132}, 10_{156}, 10_{156}

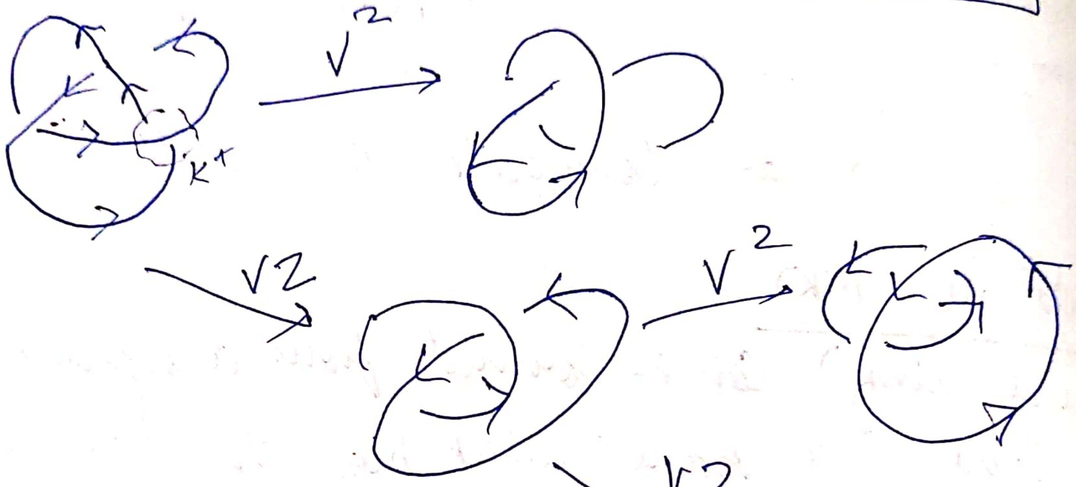
$$\boxed{b(K) = \frac{1}{2}(v\text{-span}\,P) + 1} \; *$$

$\Rightarrow$ Calculate Homfly for 3_1

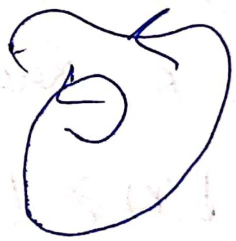
$$\frac{1}{v}P_{K^+} - vP_{K^-} = zP_{K_0}$$



$$1\,\cancel{P_{O_1}} = v^2 \cancel{P_{O_1}} + vz\,P_{O_2} \Rightarrow \boxed{\frac{1-v^2}{vz} = P_{O_2}}$$



$$\left(\frac{1-v^2}{vz}\right) v^2 vz + (vz)^2$$



$$+v^2 = 2v^2 - v^4 + v^2z^2$$

$$\frac{1}{2}(\text{v-span}+1) = \frac{1}{2}(2)+1$$

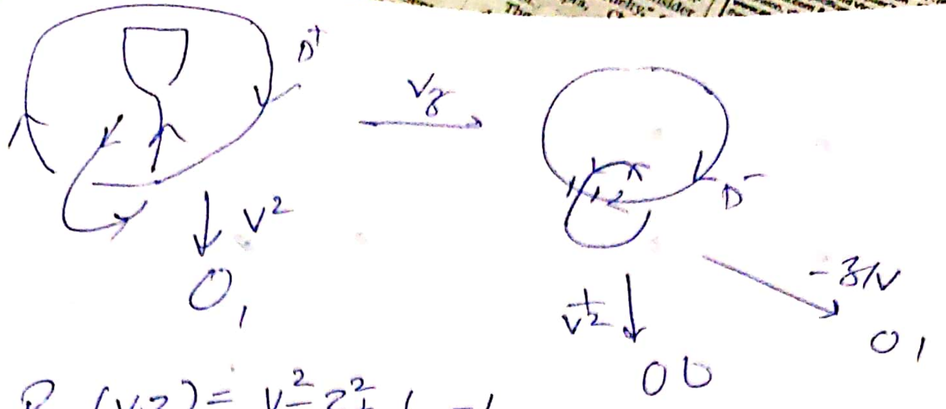
Lecture

$$b(K) = \frac{1}{2}\left(\text{v-span } P_K(v,z)\right) + 1$$

axiom 2 $\quad \frac{1}{v} P_{D+} - v P_{D-} = z P_{D_0}$

$$P_{D+} = v^2 P_{D-} + vz\,P_{D_0}$$

$$P_{D-} = \frac{1}{v^2} P_{D+} - \frac{z}{v} P_{D_0}$$



$P_{4_1}(v,z) = v^2 - z^2 + \frac{1}{v^2} - 1$

max v-deg $= 2$

min v-deg $= -2$

v-span $= 2-(-2) = 4$

$\frac{1}{2}$ v-span $= 2$

$b(4_1) = 3$

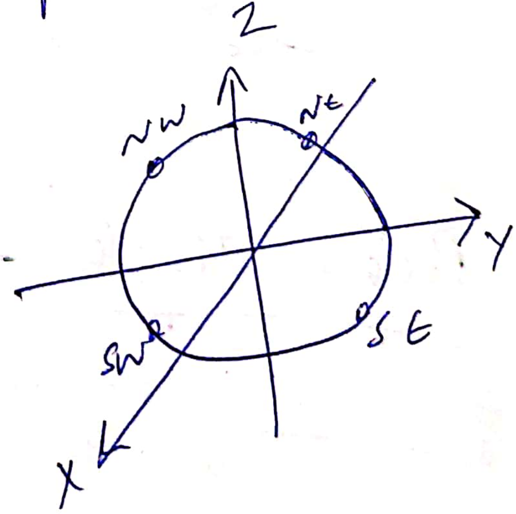
$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

$b(K) = \frac{1}{2}(\text{v-span } P_{4_1}) + 1$

$= 2 + 1 = 3 = b(4_1)$

$(*)$ is true for 4_1

Rational knots



$S^2 = \{x^2+y^2+z^2=1\}$

we fix 4 point on S^2

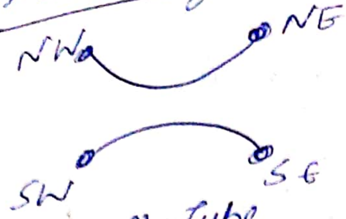
$NE = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$NW = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

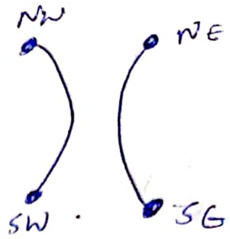
$SE = (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$

$SW = (0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$

Trivial tangles



0-type trivial tangle

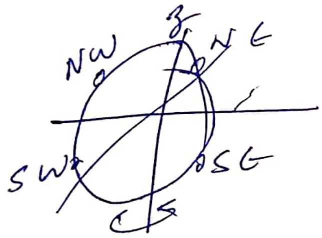


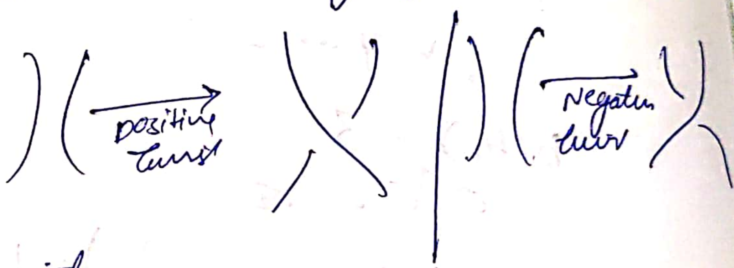
0-0 type trivial tangle.

Vertical twist

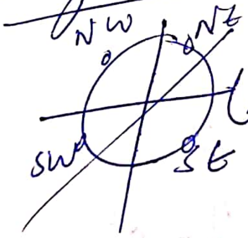
Keep North hemisphere and southpole fixed & rotate 180° about z axis

SW & SE will exchange position

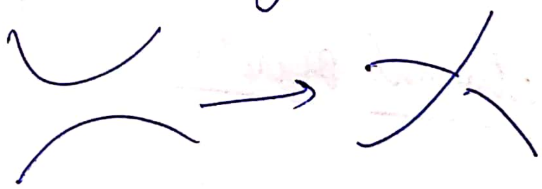




Horizontal ~~do~~ twist



↳ NE & SE exchange position



Rational tangles $T(a_1, a_2, \ldots, a_n)$

If n is odd start with $T(0)$

Perform a_1 horizontal twist

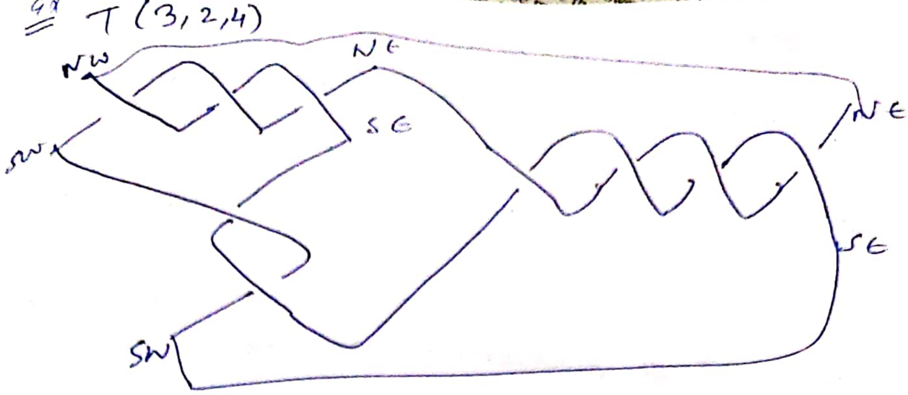
then " a_2 Vertical "

" " a_3 Horiz. "

⋮ ⋮ ⋮

" " a_n Horiz. "

Eg $T(3,2,4)$



Connect NW to NE, SW to SE by non-intersecting arcs this will give a knot or link $T(a_1, \dots, a_n)$

Case 2 n even

Start with $T(0,0)$

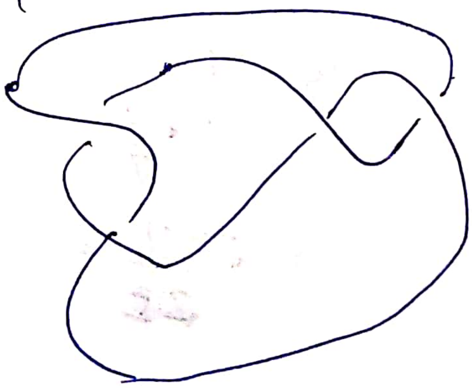
Perform a_1 vertical twist

then a_2 hor. "

⋮

a_n hor. "

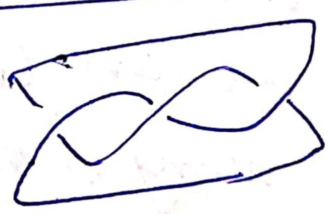
Eg. $T(2,2)$



⇐ figure 8

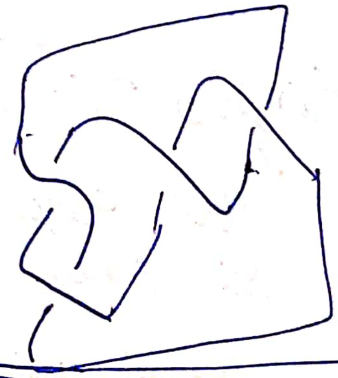
Conway Notation

3



3_1

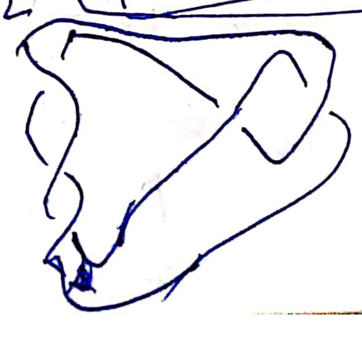
22



5

5_1

32



Conway's Notation

Only knots not obtained by this way

8_{21} 8_{20} 8_{19} 8_{18} 8_{17} 8_{16} 8_{15} 8_{10} 8_5

Let $T(a_1, a_2, \ldots, a_n)$ be an n-tangle. Consider the associated rational number

$$[a_n, a_{n-1}, \ldots, a_2, a_1] = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cdots}}$$

Eg $[2, 2] = 2 + \frac{1}{2} = \frac{5}{2}$

Eg $T(7, 2, 1, 2)$

$$[2, 1, 2, 7] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{7}}} \qquad 2 + \cfrac{1}{1 + \cfrac{7}{15}}$$

$$\frac{15}{22} + 2 = \frac{59}{22}$$

Eg $T(-7, -3, 3)$

$$[3, -3, -7] = 3 + \cfrac{1}{-3 + \cfrac{1}{7}}$$

$$3 + \cfrac{1}{-3 + \cfrac{1}{-7}} \qquad 3 + \frac{7}{22} =$$

$$\frac{59}{22}$$

Tangles $T(a_1, \ldots, a_n)$ and $T(b_1, \ldots, b_m)$ are equivalent if we can convert one to the other by finitely many elementary knot moves.

$T(a_1, a_2, \ldots, a_n)$ corresponds to the fraction

$$\frac{p}{q} = [a_n, a_{n-1}, \ldots, a_1]$$

Thm If T

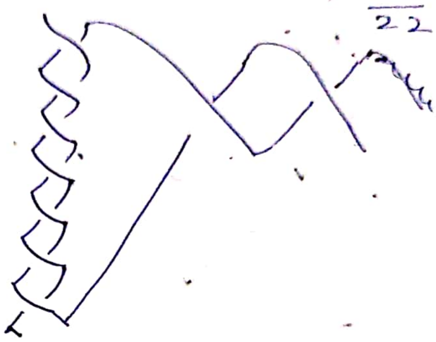
$[a_n,$

$T(-7$

Thm

Thm If $T(a_1, \ldots, a_n) \simeq T(b_1, b_2, \ldots, b_m)$ then
$[a_n, a_{n-1}, \ldots, a_1] = [b_m, b_{m-1}, \ldots, b_1]$
The converse also holds.

$T(7, -3, 3)$ complete $T(7, 2, 1, 2) \Rightarrow \frac{59}{22}$



Thm Suppose K, K' are rational knots having rational number $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ respectively.
K and K' are equal if & only if (iff)

(1) $\alpha = \alpha'$; $\beta \equiv \beta' \pmod{\alpha}$

or

(2) $\alpha = \alpha'$, $\beta\beta' \equiv 1 \pmod{\alpha}$

$K = \frac{\alpha}{\beta}$, $K^* = \frac{-\alpha}{\beta} = \frac{\alpha}{-\beta}$

$K \simeq K^*$; $-\beta^2 \equiv 1 \pmod{\alpha}$

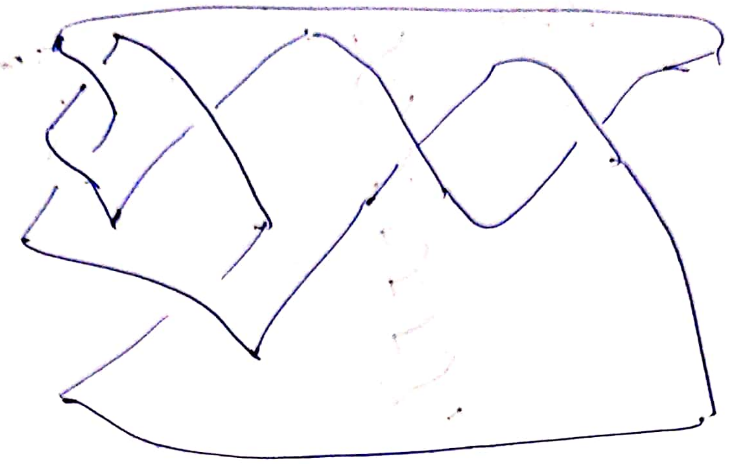
Ex $[2, 2] = 2 + \frac{1}{2} = \frac{5}{2} = \frac{\alpha}{\beta}$

$[-2, -2] = \frac{-5}{2} = \frac{5}{-2} = \frac{\alpha'}{\beta'}$ $\quad \alpha = \alpha'$ $\quad -4 \equiv 1 \pmod{5}$

$\beta\beta' \equiv 1$

$6_3 = T(2,1,1,2)$ $\quad [2,1,1,2]$

$$2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}} = 2+\frac{3}{\cancel{14}-\cancel{3}\,8} = \frac{13}{5}$$



6_3

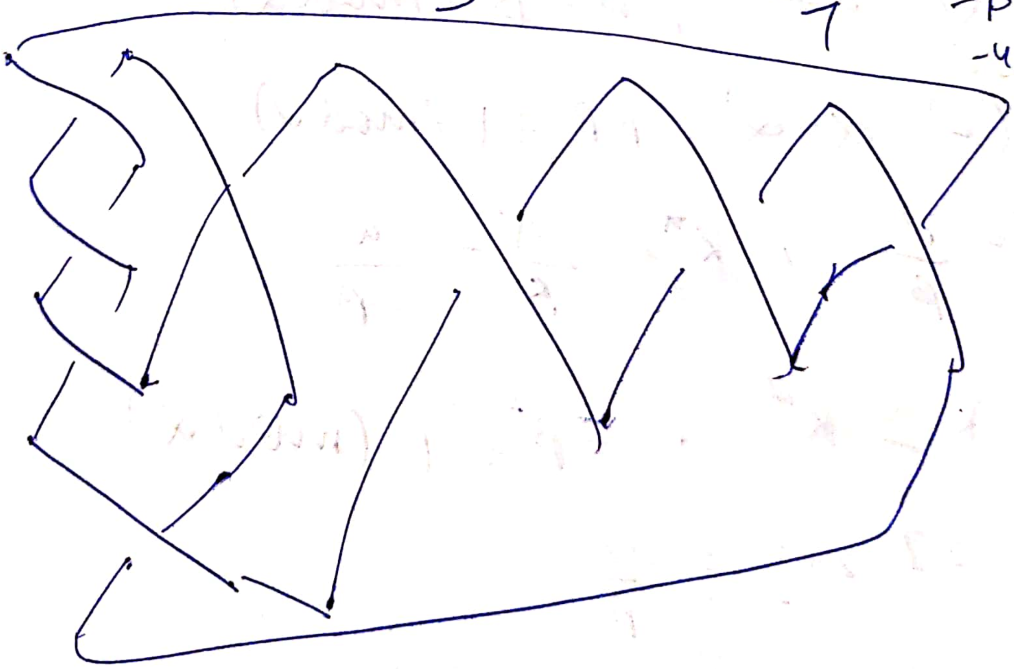
$8_9 = T(3,1,1,3)$

$$3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}} = 3+\frac{4}{\cancel{14}-\cancel{7}\,4}$$

$$= \frac{25}{7}$$

$-\beta^2 \equiv 1 \bmod (25)$.

$-49 \equiv 1 \bmod (25)$



Lecture

$y \; H(x,t) \quad H(x,0) = f, \; H(x,1) = g$

Fundamental Group (Topological Invariant)

Munkres - Topology 2nd Ed

Def^n (Homotopy)

Let f, f' are cont. maps of the space X (subset of $\mathbb{R}^n$) into the space Y (subset of $\mathbb{R}^m$); we say that f is <u>homotopic to</u> f' if there is a cont. map

$H: X \times I \to Y$ s.t. $H(x,0) = f(x)$

$H(x,t)$ and $H(x,1) = f'(x)$

$t \in I$, "time",

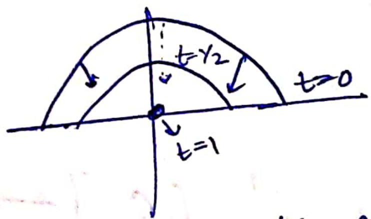
f is homotopic to f', $f \sim f'$

Ex $f: [0,1] \to \mathbb{R}^2$

$f(x) = (\cos \pi x, \sin \pi x)$

$f': [0,1] \to \mathbb{R}^2$; $f(x) = (0,0)$

$H(x,t) = (1-t) f(x)$



<u>Homotopy is an equiv. reln on the set of cont. fns $f: X \to Y$</u>

<u>Reflexive</u> $f \sim f$. $H: X \times I \to Y$

$H(x,t) = f(x)$

<u>Symmetry</u> If $f \sim f'$ then $f' \sim f$

If $f \sim f' \Rightarrow$

Transitive $f \sim f'$ & $f' \sim f''$
to show $f \sim f''$

Let $\left.\begin{array}{l} F(x,0) = f(x) \\ F(x,1) = f'(x) \end{array}\right\} f \sim f''$

$\left.\begin{array}{l} G(x,0) = f'(x) \\ G(x,1) = f''(x) \end{array}\right\} f' \sim f''$

Define $H : X \times I \to Y$

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

H is cont^n & $H(x,0) = f(x)$, $H(x,1) = f''(x)$

Defn (Path homotopy)

Path $f : [0,1] \to X$; f is cont.

$f(0) = x_0$ $[0,1] \to x_0$

$f(1) = x_0$

Let $f, f' : [0,1] \to X$ be cont. paths in X

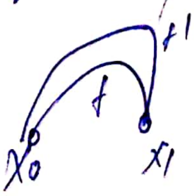
s.t. $f(0) = f'(0) = x_0$ and $f(1) = f'(1) = x_1$.

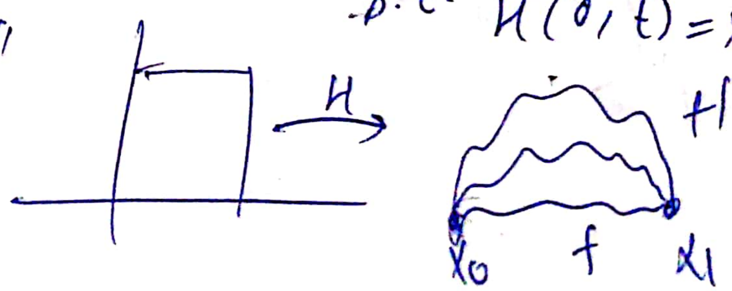
f is said to be path homotopy to f' if

$\exists H : I \times I \to X$ s.t. $H(x,0) = f(x)$

$H(x,1) = f'(x)$

s.t. $H(0,t) = x_0$, $H(1,t) = x_1$





Notation : $\sim_p$

$f \sim_p f'$, f is path homotopic to f'

$\sim_p$ is an equiv. reln

Reflexive $f \sim_p f$ $\quad H(x,t) = f(x)$

Symm $f \sim_p f' \Rightarrow f' \sim_p f$

$H'(x,t) = H(x, 1-t)$

Transitivity $f \sim_p f', f' \sim_p f'' \Rightarrow f \sim_p f'$

Proof: Same as before

Ex $f(x) = (\cos \pi x, \sin \pi x)$

$g(x) = (\cos \pi x, 2 \sin \pi x)$

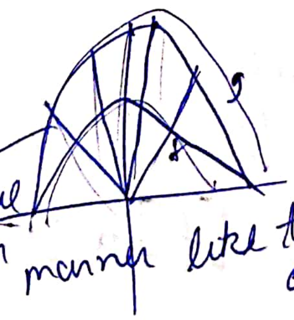
$f(0) = (1,0) = g(0)$

$f(1) = (-1,0) = g(1)$

$H(x,t) = \cancel{f(x)} (\cos \pi x, (2-t) \sin \pi x)$

$H(x,0) = (\cos \pi x, 2 \sin \pi x)$

$H(x,1) = (\cos \pi x, \sin \pi x)$

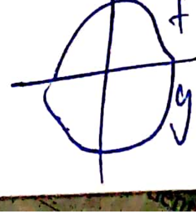


$\Rightarrow H(x,t) = (1-t) f(x) + t g(x)$

when we have a cont^n manner like this example

Ex Same works for $f(x) = (\cos \pi x, \sin \pi x)$

$g(x) = (\cos \pi x, -\sin \pi x)$

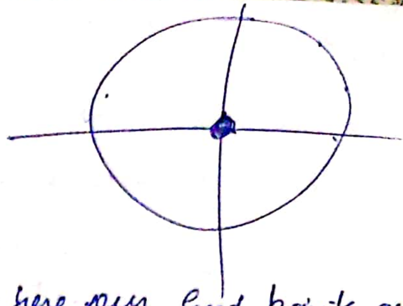


$\Rightarrow (1-t) f(x) + t g(x)$

Eg $f : I \to \mathbb{R}^2 \setminus \{0,0\}$

$f(x) = (\cos \pi x, \sin \pi x)$

$g(x) = (\cos \pi x, -\sin \pi x)$

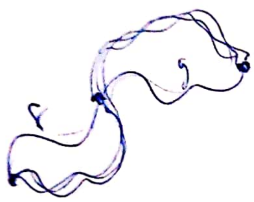


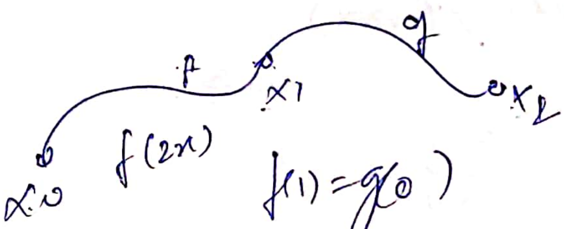
$f \not\simeq_p g \Rightarrow$ here our End points are fixed

but $f \sim g$ ← here no such Restriction is there

$H(x,t) = f[(1-t)x]$

Defn If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2, we define the composition $f * g = h(x) = \begin{cases} f(2x) & 0 \le x \le \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \le x \le 1 \end{cases}$





$f(1) = g(0)$

$h(x)$

$[f] * [g] = [f * g]$

Def $f * g$ is well-defined on path equivalent classes & (i) associativity $[f] * ([g] * [h]) = ([f] * [g]) * [h]$

(2) Right & Left Identity.

$e_{x_0} : I \to x_0$

$e_{x_1} : I \to x_1$

$[f] * [e_{x_1}] = [f]$

and

$[e_{x_0}] * [f] = [f]$.

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(3) Given $f : I \to X$

$f(0) = x_0$, $f(1) = x$

Inverse of f: $\bar{f} = f(1-x)$

$[f] * [\bar{f}] = [e_{x_0}]$

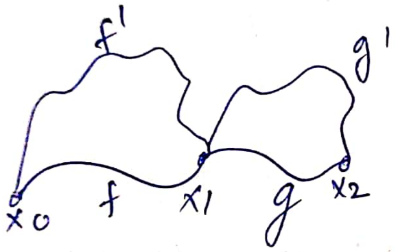
$[\bar{f}] * [f] = [e_{x_1}]$

<u>Thm</u> $*$ is well-defined.

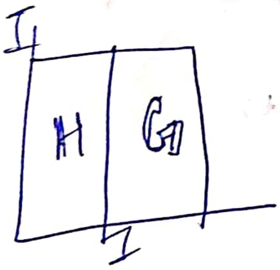
Proof $[f]*[g] = [f*g]$

To show if $f' \sim_p f$ & $g' \sim_p g$

then $f*g \sim_p f'*g'$



$H : I \times I \to X$	$G : I \times I \to X$
$H(x,0) = f(x)$	$G(x,0) = g(x)$
$H(x,1) = f'(x)$	$G(x,1) = g'(x)$



$F : I \times I \to X$

$$F(x,t) = \begin{cases} H(2x,t) & 0 \le x \le \frac{1}{2} \\ G(2x-1,t) & \frac{1}{2} \le x \le 1 \end{cases}$$

$F(x,0) = f(2x)$ $\quad 0 \le x \le \frac{1}{2}$

$g(2x-1)$ $\quad \frac{1}{2} \le x \le 1$

<u>Munkres Section</u> 51

Q 1, 2, 3

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Lecture

#51 # 1, 2 & 3

#2 (a) Given spaces X & Y, let $[X, Y]$ denote the set of homotopy classes of cont. maps of X into Y. Let $I = [0,1]$ (a) Show that for any X, the set $[X, I]$ has a single element.

Sol $f: X \to I$ s.t. $f(x) = 0 \; \forall x \in V$

claim: For any $g: X \to I$ s.t. it is continuous, it will be homotopic ~~to~~ to $f(x)$

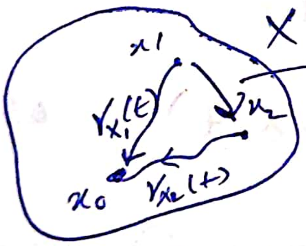
Define $H(x,t) = (1-t) f(x) + t g(x)$

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element

3

(b) X: contractible $\Rightarrow$

$id: X \to X$ is homotopic to a const map



$\gamma_{x_1, x_2} = \gamma_{x_1}(t) * \gamma_{x_2}^{-1}(t)$

$\gamma_{x_1}(t) = H(x_1, t)$

(c) Y: contractible

$H(y,0) = y$

$H(y,1) = y_0$

$X \to Y \to y_0$

$[X, Y]$

$G: X \to Y$

$G(x, 0) = x$ $\quad$ $G = (1-t)X + Y$

$G(x, 1) = y$

$Z \to Y$

d) If ... the

Le...

i

p

d) If X is contractible and Y is path connected then $[X, Y]$ has a single element. (7)

~~$f: X \to X$~~

$\exists$ s.t. $f(x) = x_0 \; \forall x$

$H = X \times I \to X$

$H = (id, x)$

$G: X \to Y$

Lecture

Let f be a path in X from x_0 to x_1; and g a path from x_1 to x_2; we define a composition $f * g$ given by the path h, as
$$h(s) = \begin{cases} f(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ g(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

$[f]$ denotes the path homotopy equiv. class of f.

$$[f] * [g] = [f * g]$$

Claim $(*)$ satisfies groupoid properties

$[f] * [g]$ is defined only for paths f, g s.t. $f(1) = g(0)$

Thm $*$ is well defined on path homotopy classes and satisfies

1) associativity: $[f] * ([g] * [h]) \simeq ([f] * [g]) * [h]$

2) Identity: Let e_x denote const. path $e: I \to X$
$e(s) = x \; \forall s \in I$
point

Then $[f] * [e_{x_1}] = [f]$; $[e_{x_0}] * [f] = [f]$

3) We define $\bar{f}(s) = f(1-s)$; Then $[f] * [\bar{f}] = [e_{x_0}]$
Reverse of f
$[\bar{f}] * [f] = [e_{x_1}]$

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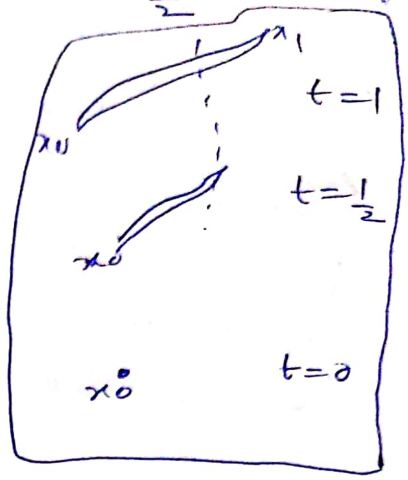
Proof of 3: We will prove $f * \bar{f} \simeq_p e_{x_0}$

Let $H(s,t) : I \times I \to X$ defined by

$$H(s,t) = \begin{cases} f(2ts) & ; 0 \le s \le \frac{1}{2} \\ H(2t(1-s)) & ; \frac{1}{2} \le s \le 1 \end{cases}$$

$$f * \bar{f} = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ f(2(1-s)) & \frac{1}{2} \le s \le 1 \end{cases}$$



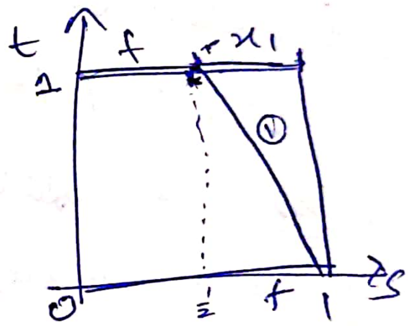
$$H(s,1) = \begin{cases} f(2s) \\ H(2(1-s)) \end{cases}$$

$$H(s,0) = e_{x_0}$$

Proof of 2: We will prove $f \simeq_p f * e_{x_1}$

$$f * e_{x_1} = f(2s) \quad 0 \le s \le \frac{1}{2}$$

$$= x_1 \quad \frac{1}{2} \le s \le 1$$



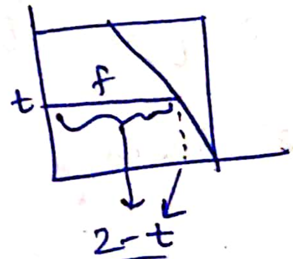
Slope of line 1 $= \dfrac{1}{-\frac{1}{2}} = -2$

Equation of line 1: $t = -2s + c$

$t = 0$, $s = 1 \Rightarrow c = 2$

$\to t = -2s + 2$

$s = \dfrac{2-t}{2}$



$H : I \times I \to X$

$$H(s,t) = \begin{cases} f\left(\dfrac{2s}{2-t}\right) & 0 \le s \le \dfrac{2-t}{2} \\ x_1 & \dfrac{2-t}{2} \le s \le 1 \end{cases}$$

Cont.

h_1 (2 2) → can also be written as (1 1 1 1)

6_3 (2, 1, 1, 2)

8_9 T(3, 1, 1, 3)

T(n, 1, 1, n) → verify

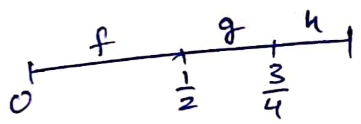
T(1, 1, 1, 1)

$(1,1,1,1) = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}} = 1 + \cfrac{2}{3} = \frac{5}{3}$

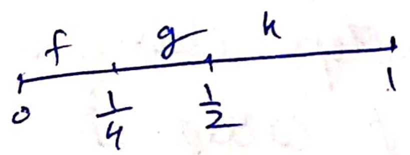
Lecture Continue

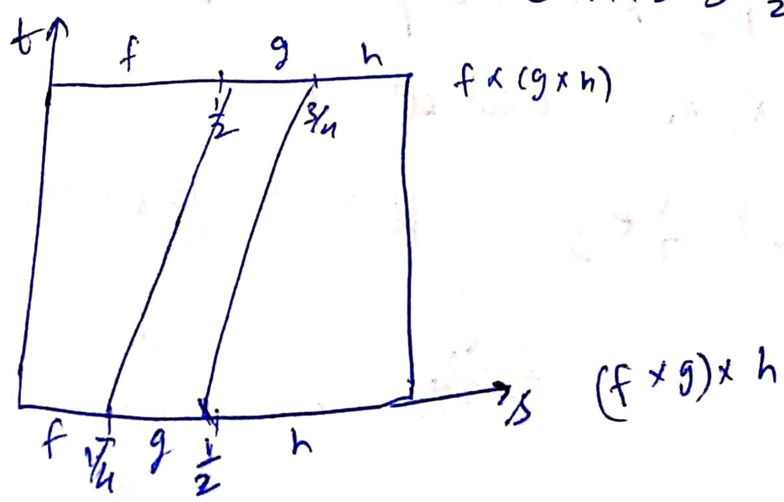
Proof of 1 $f * (g * h) \simeq_p (f * g) * h$

$$f * (g * h) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(4(s - \frac{1}{2})) & \frac{1}{2} \le s \le \frac{3}{4} \\ h(4(s - \frac{3}{4})) & \frac{3}{4} \le s \le 1 \end{cases}$$



$$(f * g) * h = \begin{cases} f(4s) & 0 \le s \le \frac{1}{4} \\ g(4(s - \frac{1}{4})) & \frac{1}{4} \le s \le \frac{1}{2} \\ h(2(s - \frac{1}{2})) & \frac{1}{2} \le s \le 1 \end{cases}$$





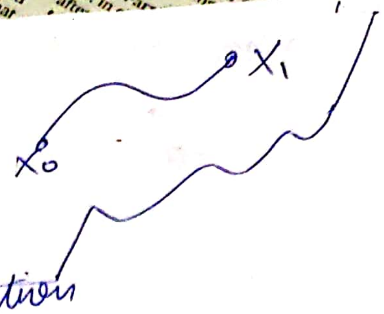
Scanned by CamScanner

Proof

$f : [0,1] \longrightarrow X$

$f' : [0,\alpha] \longrightarrow X$

$f'(s) = f\left(\frac{s}{\alpha}\right)$

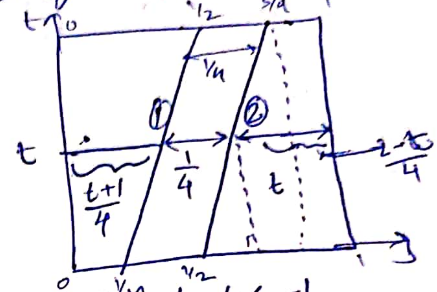


Reparamatrization

To show $(f*g)*h \simeq_p f*(g*h)$ the homotopy

$H : I \times I \longrightarrow X$

$$H(s,t) = \begin{cases} f\left(\frac{4s}{t+1}\right) & s \in \left[0, \frac{t+1}{4}\right] \\ g\left\{4\left(s - \frac{t+1}{4}\right)\right\} & s \in \left[\frac{t+1}{4}, \frac{t+2}{4}\right] \\ h\left[\frac{4}{2-t}\left(s - \frac{t+2}{4}\right)\right] & s \in \left[\frac{t+2}{4}, 1\right] \end{cases}$$

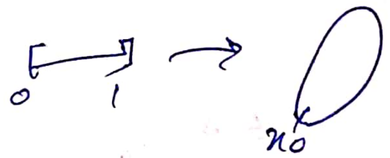


line① $t = 4s - 1$

$s = \frac{t+1}{4}$

Def^n Let $\alpha : [0,1] \to X$ be a cont. map s.t. $\alpha(0) = x_0 = \alpha(1)$

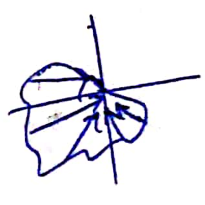
α is called a loop based at x_0.



Def^n X a space; $x_0 \in X$ let α be a loop in X based at x_0; & let $[\alpha]$ be its path homotopy class of α.

$[\alpha] * [\beta] = [\alpha * \beta]$ is a group called the fund. group $\pi_1(X, x_0)$ based at x_0

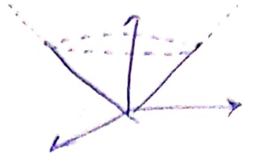
Ex $\pi_1(\mathbb{R}^2, 0) = \{e\}$ ($H(x,t) = (1-t)f(x)$



Ex Let C, any convex subset of $\mathbb{R}^n$, $x_0 \in C$

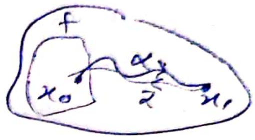
$\pi_1(C, x_0) = \{e\}$

Ex. $D: z^2 = x^2 + y^2$



$\pi_1(D, z_0) = \{e\}$
$(0,0)$

Thm Let X be path-connected $x_0, x_1 \in X$ Then
$\pi_1(X, x_0) \cong \pi_1(X, x_1)$



Pf Let α be a path from x_0 to x_1
we define a map

$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ as

Define $\hat{\alpha}[f] = [\bar{\alpha}] * [f] * [\alpha]$

$\hat{\alpha}$ is a group homomorphism

Porush

$= \alpha(1)$

To show $\hat{\alpha}([f]*[g]) = \hat{\alpha}[f] * \hat{\alpha}[g]$

$\hat{\alpha}([f]*[g]) = \hat{\alpha}[[f*g]) - -$

$$G_1 \underset{\phi_2}{\overset{\phi_1}{\rightleftarrows}} G_2 \qquad G_1 \cong G_2$$

If $\left.\begin{matrix}\phi_1 \circ \phi_2 = id \\ \phi_2 \circ \phi_1 = id\end{matrix}\right\} \Rightarrow$ G_1 & G_1 are Isomorphic

$\hat{\alpha} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ homo.

$(\hat{\bar{\alpha}}) = \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$

~~Can~~ Prove $\hat{\alpha} \cdot (\hat{\bar{\alpha}}) = id$ and $\hat{\bar{\alpha}} \circ \hat{\alpha} = id$.

Section 52 Q 1, 2, 3, 4, 5

Def^n

Fundamental Group and its Properties

Def^n Let X be a space $X \subset \mathbb{R}^n$; let x_0 be a point of X. a path $f: [0,1] \to X$ s.t. $f(0) = f(1) = x_0$ is called a loop based at x_0; let $[f]$ denote the path homotopy class of f

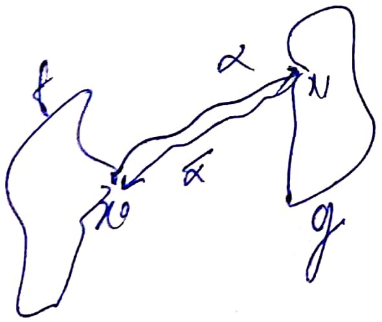
$$\pi_1(X, x_0) = \{ [f] \mid f \text{ as the group loops} \}$$ is called fundamental gp of X based at x_0

Thm X: path connected space, $x_0, x_1 \in X$

Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Claim: $\hat{\alpha}: \pi_1(X, x_0) \to \pi(X, x_1)$ is gp isomor

Let $\beta = \bar{\alpha}$ $\quad \hat{\beta}: \pi_1(X, x_1) \to \pi_1(X, x_0)$



$\hat{\beta}[g] = [\bar{\beta}] * [g] * [\beta] = [\alpha] * [g] * [\bar{\alpha}]$

$\Rightarrow \pi_1(X, x_0) \cong \pi_1(X, x_1)$

$\hat{\alpha}[f] = [\bar{\alpha}] * [f] * [\alpha]$

To show $\hat{\beta} \circ \hat{\alpha}[f] = \hat{\beta}([\bar{\alpha}] * [f] * [\alpha])$

$= [\alpha] * [\bar{\alpha}] * [f] * [\bar{\alpha}] * [\bar{\alpha}]$

$= [f]$

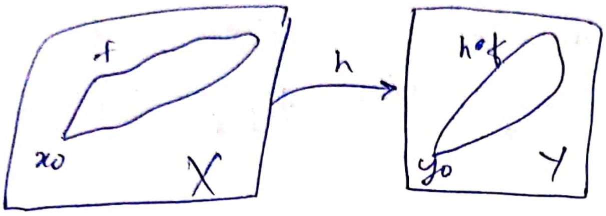
Similarly $\hat{\alpha} \circ \hat{\beta}[g] = [g]$ $\square$

Goal : If X_{x_0} is homeo to Y_{y_0} then $\pi_1(X, x_0) \underset{isom}{\cong} \pi_1(Y, y_0)$

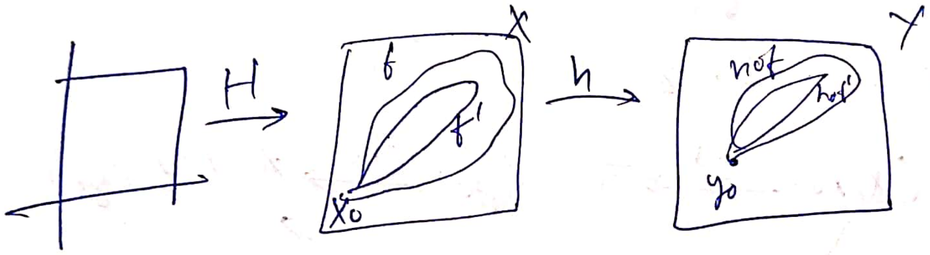
$\underline{\text{def}^n}$ Suppose $h : X \to Y$ a cont. map. we define a corresponding map (homomorphism)

$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

by $h_*[f] = [h \circ f]$



well define
$f \sim f'$
$h \circ f \sim h \circ f'$



Given $h : (X, x_0) \to (Y, h(x_0) = y_0)$.

$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

$h_*[f] = h \circ f$

$\underline{\text{To show}}$ $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a gp. homomorphism

$h_*([f] * [g]) = h_*[f] * h_*[g]$

$[f * g] = f(2s) \quad 0 \le s \le \frac{1}{2}$

$g(2s-1) \quad \frac{1}{2} \le s \le 1$

$\underline{\underline{\text{Lhs}}}$

$$h_*([f] * [g]) = \begin{cases} h(f(2s)) & 0 \le s \le \frac{1}{2} \\ h(g(2s-1)) & \frac{1}{2} \le s \le 1 \end{cases}$$

$h_*[f] * h_*[g] = h_*[f(2s)]$ $\quad 0 \le s \le \frac{1}{2} = h(f(2s))$

$= h_*[g(2s-1))$ $\quad \frac{1}{2} \le s \le 1 = h(g(2s-1))$

$h : (X, x_0) \to (Y, h(x_0) = y_0)$

$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ satisfies 2 "functionality Properties"

1) $id : X \to X \Rightarrow id_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is id

2) $f : X \to Y$; $g : Y \to Z$ then $(g \circ f)_* = g_* \circ f_*$

$$\begin{tikzcd} X \arrow[rr, bend right, "\cancel{f \circ g}\ g \circ f"'] & Y & Z \end{tikzcd}$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "(g \circ f)_*"'] & \pi_1(Y, y_0) \arrow[r, "g_*"] & \pi_1(Z, z_0) \end{tikzcd}$$

Proof of 2)

Let $h \in \pi_1[X, x_0]$

$g_*(f_*[h]) = g \circ (f \circ h)$ They are same

$(g \circ f)_*[h] = (g \circ f) \circ h$

Thm If X is homeomorphism to Y, $h : X \to Y$

$\pi_1(X, x_0)$ is isomorphism to $\pi_1(Y, h(x_0) = y_0)$

Proof

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "id"'] & Y \arrow[r, "h^{-1}"] & X \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "h_*"] \arrow[rr, bend right, "id_* = id"'] & \pi_1(Y,y_0) \arrow[r, "h_*^{-1}"] & \pi_1(X,x_0) \end{tikzcd}$$

$h_*^{-1} \circ h_* = id$ ①

Similarly

$$\begin{tikzcd} \pi_1(Y,y_0) \arrow[r, "h_*^{-1}"] \arrow[rr, bend right, "id_* = id"'] & \pi_1(X,x_0) \arrow[r, "h_*"] & \pi_1(Y,y_0) \end{tikzcd}$$

$h_* \circ h_*^{-1} = id$ ②

① & ② $\Rightarrow$ $\pi_1(X,x_0) \cong \pi_1(Y,y_0)$

If two spaces are homeomorphic or Top. Equi. then their corresponding fund. grps are isomorphic

①