MSO 202

Th^m: If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = z_1 \neq 0$ implies $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < |z_1|$

Proof:

$$a_n z_1^n \to 0 \quad \text{as} \quad n \to \infty$$

$$|a_n z_1^n| \leq M_2 \quad \text{for } n \geq N$$

$$M_1 = \max \{ |a_1 z_1|, |a_2 z_1^2|, \ldots |a_{n-1} z_1^{n-1}| \}$$

$$M = \max \{ M_1, M_2 \}$$

$$\Rightarrow |a_n z_1^n| \leq M$$

$$\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n z_1^n| \left|\frac{z}{z_1}\right|^n \leq \sum M r^n \quad , r < 1$$

Hence Converges ... $\boxtimes$

Radius of Convergence: $R = \max \{ |z| : \sum a_n z^n \text{ converges} \}$

$\sum a_n z^n$ diverges at $z = z_1$ then $\sum a_n z^n$ diverges for $|z| > |z_1|$

Ratio test $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$ $\qquad$ $\cancel{R = \lim_{n\to\infty} |a_n|^{1/n}}$

Root test $R = \lim_{n \to \infty} (a_n)^{1/n}$

Ex: $a_n = 2^n$ $\quad n$ odd

$\qquad 3^n$ $\quad n$ even

$\lim_{n \to \infty} (a_n)^{1/n} \neq 0 \qquad \lim \sup (a_n)^{1/n} = 3$

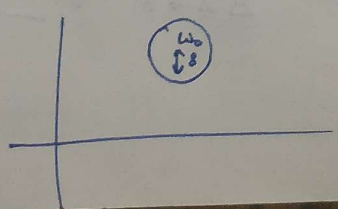
LIMIT:

$$f(z)$$

$$\lim_{z \to z_0} f(z) = \omega_0$$

if $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon)$ s.t.

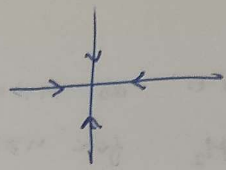
$|f(z) - \omega_0| < \epsilon \xrightarrow{\text{with}} 0 < |z - z_0| < \delta$



Ex:

We need to show: $f(z) = \frac{z}{\bar{z}}, \; z \neq 0$

$\lim_{z \to 0} f(z)$ does not exist



Along x-axis

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{x}{x} = 1$$

Along y-axis

$$\lim_{z \to 0} f(z) = \lim_{y \to 0} \frac{y}{-y} = -1$$

Hence $\lim_{z \to 0} f(z)$ does not exist

Continuity

$f(z)$ is cont. at $z = z_0$

if $\forall \; \epsilon > 0, \; \exists \; \delta(\epsilon)$ s.t $|f(z) - f(z_0)| < \epsilon$

$\forall \; z \in |z - z_0| < \delta$

Ex:

$$f(z) = \begin{cases} \dfrac{z^4}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Approach along x-axis

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{x^4}{x^4} = 1 \neq 0$$

Not cont. at $z = 0$.

Derivative

Derivative of a f^n f at $z = z_0$

f^n f is diff. at $z = z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists}$$

$$\lim_{z\to z_0} f(z)-f(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \lim_{z\to z_0}(z-z_0)$$

$$= 0.$$

Hence if diff. $\Rightarrow$ cont.

Ex: $f(z)=\bar{z}$

$$f'(0) = \lim_{\Delta z\to 0} \frac{f(0+\Delta z)-f(0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$\Delta z = \Delta x + i\,\Delta y$

Along x-axis, $f'(0) = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1$

Along y-axis, $f'(0) = \lim_{\Delta y\to 0} \frac{-\Delta y}{\Delta y} = -1$

Hence not diff.

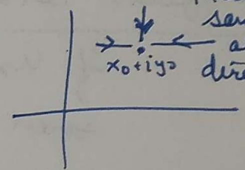
C-R Equations (Cauchy-Riemann Equations)

$f'(z_0)$ exists

$$f(z) = u(x,y) + i v(x,y)$$

$z_0 = x_0 + iy_0$
$\Delta z = h + ik$

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$



Must be same from all directions

$f(z_0) = u(x_0,y_0) + i\,v(x_0,y_0)$

$f(z_0+\Delta z) = u(x_0+h, y_0+k) + i\,v(x_0+h, y_0+k)$

$$f'(z_0) = \lim_{(h+ik)\to 0} \frac{\overbrace{u(x_0+h,y_0+k) - u(x_0,y_0)} + i\overbrace{\left(v(x_0+h,y_0+k) - v(x_0,y_0)\right)}}{h+ik}$$

~~lim~~

Along x-axis: $f'(z_0) = \lim_{h\to 0} \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} + i\lim_{h\to 0} \frac{v(x_0+h,y_0)-v(x_0,y_0)}{h}$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + i v_x$$

Along y-axis:

Illy, $f'(z_0) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -iu_y + v_y$

Limits must be same, $\Rightarrow u_x + iv_x = -iu_y + v_y$

$\Rightarrow \boxed{u_x = v_y \text{ and } v_x = -u_y}$ C-R equations

Ex: $f(z) = \bar{z} = x - iy$

$u = x, \quad v = -y$

$u_x = 1, \quad v_y = -1$ Hence non-diff.

Ex: $f(z) = |z|^2 = x^2 + y^2$

$u_x = 2x \quad v_x = 0$

$u_y = 2y \quad v_y = 0$

$u_x = v_y \quad \& \quad u_y = -v_x$

is satisfied only at $z = 0$.

So, if it is diff, it can only happen at $z = 0$

But we have to check diff at $z = 0$

Note: So, if CR Eq not satisfied $\Rightarrow$ f^n can't be diff.
But if satisfied, it doesn't gaurantee diff.

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z \cdot \Delta \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \Delta \bar{z} = 0.$$

~~Alongside if u_x, u_y, v_x, v_y are di continuous~~

Sufficient cond^n for diff.:

If u_x, u_y, v_x, v_y are continuous also along with C-R eq^{n}s then f is diff

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$z = (x+iy) = r(\cos\theta + i\sin\theta)$ $\qquad u(x,y) = u(r,\theta)$

$x = r\cos\theta$ $\qquad r^2 = x^2+y^2$ $\qquad f(z) = u + iv$

$y = r\sin\theta$

$$u_r = u_x \cdot \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r}$$

$$u_r = \cos\theta\, u_x + \sin\theta\, u_y$$

$$u_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y = -r v_r$$

$$v_r = \cos\theta\, v_x + \sin\theta\, v_y = \sin\theta\, u_x - \cos\theta$$

$$v_\theta = -r\sin\theta\, v_x + r\cos\theta\, v_y$$
$$= r\cos\theta\, u_x + r\sin\theta\, u_y$$
$$= r u_r$$

$$\boxed{\begin{array}{c} r u_r = v_\theta \\ r v_r = -u_\theta \end{array}}$$ C-R eq^{ns} in Polar form.

?

$$f'(z) = u_x + i v_x$$
$$= (u_r\cos\theta + v_r\sin\theta) + i(v_r\cos\theta - u_r\sin\theta)$$
$$= u_r e^{-i\theta} + i v_r \underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}}$$

$$\boxed{f'(z) = e^{-i\theta}(u_r + i v_r)}$$

Ex: $f(z) = \frac{1}{z^2} = \frac{1}{r^2 e^{i2\theta}} = \frac{1}{r^2}(\cos\theta - i\sin 2\theta)$ $\quad z \neq 0.$

$u(r,\theta) = \frac{\cos 2\theta}{r^2}$, $\quad v(r,\theta) = -\frac{\sin 2\theta}{r^2}$

$u_r = -\frac{2\cos 2\theta}{r^3}$, $\quad v_\theta = -\frac{2\cos 2\theta}{r^2}$

so, $r u_r = v_\theta$ ✓

$u_\theta = -\frac{2\sin 2\theta}{r^2}$, $\quad v_r = \frac{2\sin 2\theta}{r^3}$

so, $r v_r = -u_\theta$ ✓

Hence C-R eq^{ns} satisfy. Moreover, $u_\theta, u_r, v_\theta, v_r$ are cont everywhere except $z=0$, $f(z)$ is differentiable everywhere

Harmonic function

$\varphi(x,y)$ $\quad \varphi_{xx}, \varphi_{xy}, \varphi_{yx}, \varphi_{yy}$ exist and are cont.

& $\varphi_{xx}+\varphi_{yy}=0$. Then $\varphi(x,y)$ is harmonic f^n.

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi=0$$

$$\underline{\nabla^2\varphi=0}$$

Analytic f^n

$f(z)$ is analytic at $z=z_0$ if f is diff. in nbd of z_0.

$f(z)=u+iv$. | $\underline{u_x=v_y,\ u_y=-v_x}$ C-R eqns

if f is analytic, then all partial derivatives of u, v exist and are continuous.

$$u_{xx}+u_{yy}=\frac{\partial}{\partial x}(u_x)+\frac{\partial}{\partial y}(u_y)$$

$$=\frac{\partial}{\partial x}(v_y)+\frac{\partial}{\partial y}(-v_x)$$

$$=v_{yx}-v_{xy}=v_{xy}-v_{xy}=0.$$ Here u is harmonic.

So, $f(z)=u+iv$

$u \downarrow$ harmonic, $v \rightarrow$ harmonic.

here v is called harmonic conjugate of u.

Ex: $u(x,y)=y^3-3x^2y$

$u_x=-6xy \qquad u_y=3y^2-3x^2$

$u_{xx}=-6y \qquad u_{yy}=6y$

$u_{xx}+u_{yy}=0$.

v is harmonic conjugate.

$\Rightarrow$ $u+iv$ will be analytic f^n

$\Rightarrow$ $u+iv$ satisfies C-R eqns.

$\Rightarrow$ $u_x=v_y, \quad u_y=-v_x$

$v_x=3x^2-3y^2, \quad v_y=-6xy.$

$\Rightarrow v=x^3-3y^2x+g(y)$

$\because v_y=-6xy \Rightarrow \frac{d}{dy}g(y)=0. \Rightarrow g(y)=c.$

$\Rightarrow v=x^3-3y^2x+c.$

$\frac{\partial f}{\partial x}=3x^2$

$\Rightarrow f=x^3+g(y)$

So, $f(z) = y^3 - 3x^2y + i(x^3 - 3xy^2 + c)$

$= i[x^3 - 3xy^2 + i(3x^2y - y^3)] + c_1$

$= i(x+iy)^3 + c_1$

$\underline{f(z) = iz^3 + c_1}$

$f'(z) = 0$, $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$\Rightarrow u_x = 0$, $v_x = 0$

$u_y = 0$, $v_y = 0$

$\Rightarrow u = c_1$, $v = c_2$

$\Rightarrow f(z) = c_1 + ic_2 = c$

Power series and Analytic f^n

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ has Radius of convergence $R > 0$

$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$

$f'(z) = g(z)$

~~Both~~

To prove: i) $g(z)$ has Radius of convergence R.

ii) $f'(z) = g(z)$. , $|z| < R$

So, proving above 2 points will ~~prove~~ tell that ~~deriv~~ partial derivative of $f(z)$ exists inside Radius of convergence (i.e. it is differentiable) and $f'(z)$ has also Rad. of Conv. R. ~~$\Rightarrow g(z)$~~

Now, $g(z)$ is power series with R, so $g(z)$ also satisfies above 2 properties within $|z| < R$. and so on. $\Rightarrow f(z)$ is ~~diff~~ infinitely diff. $\Rightarrow f(z)$ is analytic f^n inside $R \Rightarrow$ <u>Any power series is analytic f^n within Radius of convergence</u>

Proof: i] Ratio test

$$R_g = \lim_{n\to\infty} \left| \frac{n a_n}{(n+1) a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R.$$

Ratio test is a little weaker.

Another test:

~~$R = \lim\sup_{n\to\infty} \frac{1}{|a_n|^{1/n}}$~~ $\underbrace{R = \lim\sup_{n\to\infty} |a_n|^{1/n}}_{\text{Root test}}$

To prove for $\sum n a_n z^{n-1}$.

$\hookrightarrow z$ s.t. $|z| = r < R$.

$\sum a_n z^n \longrightarrow$ conv for $|z|<R$, z. s.t. $|z|=s$.

let s s.t. $r<s<R$.

$$\sum |a_n z^n| = \sum |a_n| s^n \text{ converges}$$

$$|a_n| s^n \le M. \Rightarrow |a_n| \le \frac{M}{s^n}$$

For $\sum n a_n z^{n-1}$,

~~$\sum |a_n z^{n-1}| = \sum |a_n| r$~~

$$|n a_n z^{n-1}| = n|a_n| r^{n-1}$$

$$\le \frac{n}{r} \frac{M}{(s/r)^n} = \frac{M}{r} \frac{n}{(s/r)^n}$$

$$\Rightarrow \sum |n a_n z^{n-1}| \le \sum \frac{M}{r}\frac{n}{(s/r)^n} \le \frac{M}{r} \sum \frac{n}{(s/r)^n}$$

$$\limsup_{n\to\infty} \left(\frac{n}{(s/r)^n}\right)^{1/n} = \frac{1}{(s/r)} \lim_{n\to\infty} (n)^{1/n} = \frac{r}{s} < 1$$

Hence $\sum \frac{n}{(s/r)^n}$ converges.

Hence $\sum n a_n z^{n-1}$ converges with R. ☒

Proof: ii]

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$\Rightarrow \left| \frac{f(z+h)-f(z)}{h} - f'(z) \right| < \epsilon \text{ for } |h|<\delta.$$

$$\frac{\sum_{n=0}^{\infty} a_n (z+h)^n - \sum_{n=0}^{\infty} a_n z^n}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$= \sum_{n=0}^{\infty} a_n \frac{(z+h)^n - z^n}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1}$$

$$= \sum_{n=0}^{\infty} a_n \left[\frac{(z+h)^n - z^n - nhz^{n-1}}{h} \right]$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{h} \left[\sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k - z^n - n z^{n-1} h \right]$$

$$= \sum_{n=0}^{\infty} a_n \cdot h \left(\sum_{k=2} \binom{n}{k} z^{n-k} h^{k-2} \right) \qquad k-2=j$$

$$= \sum_{n=0}^{\infty} a_n \cdot h \left(\sum_{j=0}^{n-2} \binom{n}{j+2} z^{n-j-2} h^j \right) \quad — ①$$

$$\binom{n}{j+2} = \frac{n(n-1)}{(j+2)(j+1)}\binom{n-2}{j} \le n(n-1)\binom{n-2}{j}$$

① ⟶ Taking mod.

$$① \le |h| \sum_{n=0}^{\infty} n(n-1)\,|a_n| \underbrace{\sum_{j=0}^{n-2}\binom{n-2}{j}|h|^j\,|z|^{n-2-j}}_{(|h|+|z|)^{n-2}}$$

$$= |h| \underbrace{\sum_{n=0}^{\infty} n(n-1)\,|a_n|\,|z+h|^{n-2}}$$

$$\underbrace{\sum_{n=0}^{\infty} n(n-1)\,a_n z^{n-2}}$$

has R (from(i))

so, converges when $|z|<R$.

$\Rightarrow |z|+|h|<R$.

so. it converge and has a finite value.

$\Rightarrow$ ① $\le$ $|h|\cdot$(finite value)

$\le \epsilon$,.

$\boxtimes$

Elementary f^n

def^n: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$ $\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

$= \infty$.

so, it is diff. everywhere

Hence it is analytic f^n everywhere.

$$(e^z)' = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{\lfloor n} = \sum_{k=0}^{\infty}\frac{z^k}{\lfloor k} = e^z.$$

$$e^{i\theta} = \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} = \left(1-\frac{\theta^2}{2!}+ - - \right) + i\left(\theta - \frac{\theta^3}{3!} + - - -\right)$$

$= \cos\theta + i\sin\theta$.

$g(z) = e^z . e^{\delta - z}$

$g'(z) = (e^z)'\,e^{\delta-z} + e^z\,(e^{\delta-z})' = e^z \cdot e^{\delta - z} - e^z . e^{\delta - z} = 0$

$g(z) = g(0) \Rightarrow g(z) = \underline{e^z . e^{\delta - z} = e^{\delta}}$

$\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

$(\cos z)' = \frac{ie^{iz} - ie^{-iz}}{2} = \frac{i^2(e^{iz}-e^{-iz})}{2i} = -\sin z$.

IIly $(\sin z)' = \cos z$.

$$\cosh z = \frac{e^{z}+e^{-z}}{2}$$

$$\sinh z = \frac{e^{z}-e^{-z}}{2}$$

$$(\cosh z)' = \sinh z \quad \& \quad (\sinh z)' = \cosh z.$$

$$\cos z = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2}$$

$$\underset{\cos(x+iy)}{\cos z} = \cos x \cosh y - i \sin \cancel{x}\, \sinh y$$

$$= u + iv.$$

$$\Rightarrow |\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$\Rightarrow |\cos z|^2 = \cos^2 x + \sinh^2 y$$

$|\cos z|$ is unbounded

or $\cos z$ is unbounded

bcoz, $\sinh^2 y$ approaches to ∞ as y increases.

||ly, ~~cos~~ $\sin z$ is unbounded.

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$$z = e^{i\theta} = e^{i(\theta) + 2k\pi i}$$

$$\sqrt{z} = e^{i\theta/2} e^{ik\pi} \qquad k=0,1$$

$$= \begin{cases} e^{i\theta/2} \\ -e^{i\theta/2} \end{cases}$$

$$\log z = w \Rightarrow w = u + iv$$

$$z = x+iy \qquad e^{w} = z$$

$$\boxed{\log z = \ln|z| + i \arg(z)}$$

$$\boxed{\begin{aligned} &\sin^{-1} z = w \\ \Rightarrow\ & e^{2iw} = iz + \sqrt{1-z^2} \end{aligned}} \longrightarrow \ ??$$

$$2iw = \log(iz+\sqrt{1-z^2}) \Rightarrow w = -\frac{i}{2}\log(iz+\sqrt{1-z^2}) = \sin^{-1} z$$

If arg = Arg

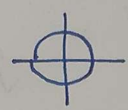
$\log z = \ln|z| + i\,\mathrm{Arg}\, z$

on −ve x-axis

$\log(-z) = \ln|-z| + i\pi$

$= \ln z + i\pi$

Discont. on −ve x-axis



If $0 \le \arg z < 2\pi$, discont. on +ve x-axis.

For obtaining cont., $S = \mathbb{C} \setminus \{(-\infty, 0]\}$

$\log z = \ln r + i\theta \qquad -\pi < \theta < \pi$

$= u + iv$

$\frac{\partial u}{\partial r} = \frac{1}{r}$, $\cancel{\frac{\partial u}{\partial \theta} = \theta}$ $\frac{\partial u}{\partial \theta} = 0$

$\frac{\partial v}{\partial r} = 0$, $\frac{\partial v}{\partial \theta} = 1$

$r u_r = 1 = v_\theta$ ✓

$-u_\theta = r v_r = 0$ ✓

And since $u_r, u_\theta, v_r, v_\theta$ are cont. everywhere in domain, so, $\log z$ is diff. in S.

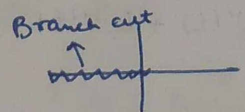
and hence $\log z$ is analytic f^n in domain S.

$(\log z)' = e^{-i\theta}(u_r + i v_r)$??

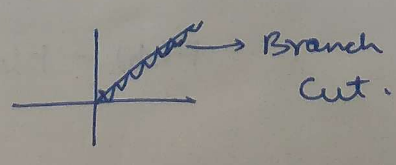
$= \frac{1}{re^{i\theta}} = \frac{1}{z}$

Branch cut → Removing that range from domain results in f^n being analytic in rest of domain

Ex: when arg = Arg



if $\pi/4 < \arg z < \frac{9\pi}{4}$



$\rightarrow \quad \sin^{-1} z = -\frac{i}{2} \log(iz + \sqrt{1-z^2})$

$(\sin^{-1} z)' = ?$

$\sin^{-1} z = w \Rightarrow z = \sin w \Rightarrow 1 = \cos w \dfrac{dw}{dz}$ $(\sin^{-1}z)'$

$\Rightarrow \dfrac{1}{\cos w} = \dfrac{1}{\sqrt{1-z^2}}$

$(\sin^{-1} z)' = \dfrac{1}{\sqrt{1-z^2}}$

✓ $\dfrac{dw}{dz} = \dfrac{1}{\cos w} = \dfrac{1}{\cos(\sin^{-1} z)}$

$(\sin^{-1}z)' = \dfrac{1}{\sqrt{1-z^2}}$

$\rightarrow \quad w = z^c = e^{c \log z}$

$(z^c)' = (e^{c \log z}) \dfrac{c}{z} = c\, z^{c-1}$

Complex Integral

$f(t) : [a,b] \rightarrow \mathbb{C}$

$\int_a^b f(t)\,dt \qquad f(t) = u(t) + i v(t)$

$= \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$

Ex: $\int_0^{\pi} e^{it} dt$

$= \int_0^{\pi} \cos t\,dt + i \int_0^{\pi} \sin t\,dt$

$= 2i$

For Real numbers,

if $F'(x) = f(x)$ | $\int_a^b f(t)\,dt = F(b) - F(a)$

Here if $U'(t) = u(t)$, $V'(t) = v(t)$

$u(t) + i v(t) = U'(t) + i V'(t) = \dfrac{d F(t)}{dt}$, $F(t) = U(t) + V(t)$

$\Rightarrow \int_a^b f(t)\,dt = U(b) - U(a) + i(V(b) - V(a))$

$= F(b) - F(a).$

Ex (contd): $\frac{d}{dt}(i \cdot e^{it}) = i^2 e^{it}$

$$= -e^{it}$$

$$\Rightarrow \quad e^{it} = \frac{d}{dt}(-i \cdot e^{it})$$

$$\int_0^{\pi} e^{it}\, dt = \cancel{\quad} \; -i\left[e^{it}\right]_0^{\pi} = 2i$$

The foll. relation also holds:

$$\left| \int_a^b \underset{\mathbb{R}\to\mathbb{C}}{f}(t)\, dt \right| \le \int_a^b |f(t)|\, dt$$

<u>Proof:</u> $R_0 e^{i\theta_0} = \int_a^b f(t)\, dt$

$$R_0 = \left| \int_a^b f(t)\, dt \right|$$

Real $\leftarrow R_0 = \int_a^b e^{-i\theta_0} f(t)\, dt$

$$= \int_a^b \mathrm{Re}\left(e^{-i\theta_0} f(t) \right) dt$$

$$\mathrm{Re}(z) \le |z|$$

$$\Rightarrow \quad R_0 \le \int_a^b |e^{-i\theta_0} f(t)|\, dt$$

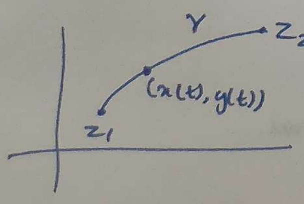
$$= \int_a^b |f(t)|\, dt$$

$$\Rightarrow \quad \left| \int_a^b f(t)\, dt \right| \le \int_a^b |f(t)|\, dt \qquad \boxtimes$$

What is meant by $\oint \int_a^b f(z)\, dz$?

For this, we have to understand Contour.

<u>Contour</u>



$\gamma : z(t)$

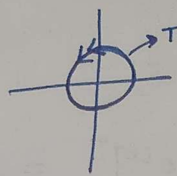
$z : [a, b] \to \mathbb{C}$ $\qquad z(t) = x(t) + i y(t)$

$z(a) = z_1 \;, \; z(b) = z_2$

Ex: $C : z(t) = e^{it}$ $\qquad 0 \le t \le 2\pi$

$$|z(t)| = 1$$

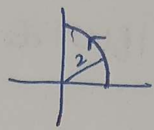
Ex: $C: z(t) = e^{2it}$ $\quad 0 \le t \le 2\pi$



Ex: $\gamma_1 : z(t) = 2e^{it}$ $\quad 0 \le t \le \pi/2$

& $\gamma_2 : z(t) = 2e^{2it}$ $\quad 0 \le t \le \pi/4$

Both represent



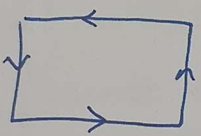
Smooth f^n: $x'(t)$, $y'(t)$ are not simultaneously zero.

$\Rightarrow z'(t) \neq 0$.

$z(t) = x(t) + i\,y(t)$ $\quad a \le t \le b$

is smooth if $z'(t)$ is cont. & $z'(t) \neq 0, a < t < b$.

Contour is curve which is collection of piecewise smooth f^ns.



Collection of 4 smooth curves

so, contour

Complex Integral: $\gamma : z(t) = x(t) + i\,y(t)$, $a \le t \le b$.

$$\int_\gamma f(z)\,dz = \int_a^b \underbrace{\boxed{f(z(t))\; z'(t)}}_{g(t)} dt$$

$$= \int_a^b f(z(\tau))\; z'(\tau)\, d\tau$$

★ For fixed curve, integral does not depend on parameterization

Ex: $\gamma_1 : z(t) = 2e^{it}$ $\quad 0 \le t \le \pi/2$

$\gamma_2 : z(t) = 2e^{2it}$ $\quad 0 \le t \le \pi/4$

Will have same complex Integral.

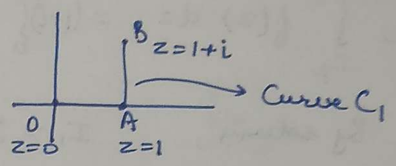
$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\; z'(t)\, dt$$

$$= \int_a^b (u(t) + i v(t)) \; (x'(t) + i y'(t)) \, dt$$

$$= \int_\gamma (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

Does not depend on parameter.

$$\int_0^{1+i} f(z)\, dz \longrightarrow \text{Path not defined}$$



$$\int_{C_1} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz$$

$$\cancel{\int_0^1 x\,dx + \int_0^1 (1+iy)\,dy}$$

$$= \int_0^1 f(t)\,dt + \int_0^1 f(1+it)\,dt$$

Ex1: $f(z) = \bar{z}$

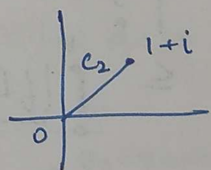
$$\Rightarrow \int_0^{1+i} \bar{z}\,dz = \int_{C_1} \cancel{f(z)}\, \bar{z}\,dz$$

$$= \int_0^1 x\,dt + \int_0^1 (1 - \bar{i}t)\,dt$$

$$= \frac{1}{2} + i\,(1 - i/2) = 1 + i$$

Suppose a different path.

$z = x + iy$
$= (1+i)x$
$z = (1+i)t \quad , \; 0 \le t \le 1$



$$\int_{C_2} f(z)\,dz = \int_0^1 f(t(1+i))\,(1+i)\,dt$$

$$= (1+i) \int_0^1 f((1+i)t)\,dt$$

$f(z) = \bar{z}$

$$= (1+i) \int_0^1 t(1-i)\,dt$$

$$= 2 \int_0^1 t\,dt = \underline{1}$$

So, integral depends on the path.

So, $\int_0^{1+i} f(z)\,dz$ does not make sense without mentioning the path.

Ex2:

Suppose $f(z) = z^2$

$$I_1 = \int_{C_1} f(z)\,dz$$
$$= \int_0^1 x^2\,dx + \int (1+it)^2\, i\,dt$$

$$I_2 = \int_{C_2} f(z)\,dz = (1+i)\int_0^1 (1+i)^2 t^2\,dt$$

By solving, $I_1 = I_2$

So, here integral does not depend on path.

So, $f(z) = \bar{z}$ depends on path

& $f(z) = z^2$ does not depend on path

non-analytic analytic

ML-inequality

$(x(t), y(t)) \to$ Curve.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$
$$= |z'(t)|\,dt$$

$$L = \int_a^b |z'(t)|\,dt \quad , \quad M = \max_{a \le t \le b} f(z(t))$$

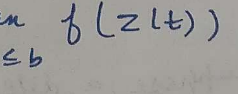
$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right|$$
$$\le \int_a^b \underbrace{|f(z(t))|}\,|z'(t)|\,dt$$
$$\le M \int_a^b |z'(t)|\,dt$$
$$= ML$$

So, $$\left|\int_\gamma f(z)\,dz\right| \le ML$$

Ex:

$$\int_\gamma \frac{z-3}{z^2+1}\,dz \qquad \gamma: z = 2e^{it} \quad 0 \le t \le \pi$$

$L = 2\pi$



$$|f(z)| = \frac{|z-3|}{|z^2+1|} \le \frac{|z|+3}{|z^2+1|} = \frac{5}{|z^2+1|}$$

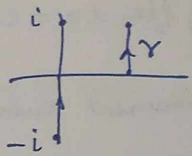
$|z_1+z_2| \geq |z_1|-|z_2| \Rightarrow |z^2+1| \geq |z^2|-1 = 3$

$\Rightarrow |f(z)| \leq \frac{5}{|z^2+1|} \leq \frac{5}{3}$

$\Rightarrow \left| \int_\gamma \frac{z-3}{z^2+1} dz \right| \leq 2\pi \times \frac{5}{3}$

Ex: $\int_\gamma \frac{dz}{z^2+1}$, $\gamma: z(t) = 1+it$, $0 \leq t \leq 1$

$L = 1$



$|f(z)| = \frac{1}{|z-i||z+i|}$

$\left.\begin{aligned} |z-i| &\geq 1 \\ |z+i| &\geq \sqrt{2} \end{aligned}\right] \; \underline{|z-i||z+i| \geq \sqrt{2}}$

$\Rightarrow |f(z)| \leq \frac{1}{\sqrt{2}} = M$

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$$\int_C f(z)\,dz = \int_a^b f(\eta(t))\,\eta'(t)\,dt$$

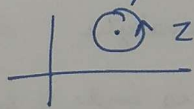
$$= \int_a^b f(\gamma(a+b-t))\,\gamma'(a+b-t)(-1)\,dt$$

$a+b-t = s$

$$= -\int_b^a f(\gamma(s))\,\gamma'(s)\,(-ds)$$

$$= -\int_a^b f(\gamma(s))\,\gamma'(s)\,ds = -\int_C f(z)\,dz$$

$\int_{|z-z_0|=r} \frac{dz}{(z-z_0)^n}$



By default → anti-clockwise

$z = \underbrace{z_0 + re^{it}}_{\gamma(t)}$ $\quad 0 \leq t \leq 2\pi$

$$= \int_0^{2\pi} \frac{1}{r^n e^{int}} rie^{it}\,dt$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)t}\,dt = \begin{cases} 2\pi i & n=1 \\ 0 & n \neq 1 \end{cases}$$

Domain: open and connected

$S_1 = \{z : |z| < 1\}$

$S_2 = \{z : 1 < |z| < 2\}$

Simply connected domain : Any curve drawn, ~~in~~ all points inside that curve must lie in the domain

$S = \{ z : 1 < |z| < 2 \}$ is not simply connected

Cauchy Integral Theorem

$f(z)$ is analytic in a $\overbrace{\text{simply connected domain}}^{SCD}$ D.
C is a simple closed contour lying inside D.

$$\int_C f(z)\, dz = 0$$

Cauchy - Goursat theorem (Extra assumption)

$f'(z)$ is cont in D.

Proof:

$$\left.\begin{aligned} f'(z) &= u_x + i v_x \\ f'(z) &= -i(u_y + i v_y) \end{aligned}\right] \text{C-R}$$

$$\int_C f(z)\, dz$$

$f(z) = u + iv$

$\gamma(t) = x(t) + iy(t)$

$$= \int_{\gamma(t)} f(\gamma(t))\, \gamma'(t)\, dt$$

$$= \int_{\gamma(t)} [(u x' - v y') + i (u y' + v x')] dt$$

$$= \int_C [(\underset{P}{u\, dx} - \underset{Q}{v\, dy}) + i (\underset{P}{u\, dy} + \underset{Q}{v\, dx})]$$

$$= \iint_A (-v_x - u_y)\, dA + i \iint_A (u_x - v_y)\, dA$$

$$= 0$$

$\boxed{\times}$

Green's Theorem

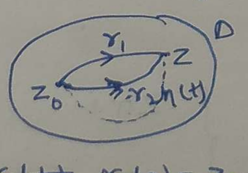
A

$$\int_L P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

1) $D \to SCD$, $f(z) \to$ analytic

$$\int_{z_0}^{z} f(z)\, dz \text{ is ind. of path}$$

Proof:



$\gamma_1(t) : 0 \le t \le 1$ ~~$\gamma_1(t)$~~ $\gamma_1(0) = z_0$, $\gamma_2(1) = z$

$\gamma_2(t) : 0 \le t \le 1$ $\gamma_2(0) = z_0$, $\gamma_2(1) = z$

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \eta(t) & \frac{1}{2} \le t \le 1 \end{cases}, \quad \eta(t) = \gamma_2(2(1-t))$$

$\Rightarrow \gamma$ is simple closed contour.

$$\Rightarrow \int_\gamma f(z)\,dz = 0$$

$$\Rightarrow \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt = 0$$

$$\Rightarrow \int_0^{1/2} f(\gamma(t))\,\gamma'(t)\,dt = -\int_{1/2}^1 f(\gamma(t))\,\gamma'(t)\,dt$$

$$\text{LHS} = \int_0^{1/2} f(\gamma_1(2t))\,\gamma_1'(2t)\,(2)\,dt$$

$$\overset{2t=s}{=} \int_0^1 f(\gamma_1(s))\,\gamma_1'(s)\,ds = \int_{\gamma_1} f(z)\,dz$$

$$\text{RHS} = -\int_{1/2}^1 f(\gamma_2(2(1-t)))\,\gamma_2'(2(1-t))\,(-2)\,dt$$

$2(1-t) = s$

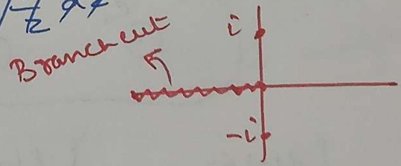
$$= \int_0^1 f(\gamma_2(s))\,\gamma_2'(s)\,ds = \int_{\gamma_2} f(z)\,dz$$

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz \qquad \boxtimes$$

Ex: $\displaystyle\int_0^{1+i} e^{z^2}\,dz \rightarrow$ ind. of path

$\hookrightarrow$ $\boxed{\text{Composition remains analytic}}$

Ex: $\cancel{\displaystyle\int_{-i}^{i} \log \frac{1}{z}\,dz}$ $\qquad$ $\cancel{\displaystyle\int_{-i}^{i} \frac{1}{z}\,dz}$



$$\int_{-i}^{i} \log z\,dz$$

Summary:

Th: $\displaystyle\oint_C f(z)\,dz = 0$

$\rightarrow \displaystyle\int_{z_0}^{z} f(z)\,dz$ is ind. of path

Theorem: $F(z) = \displaystyle\int_{z_0}^{z} f(z)\,dz$

$$\Rightarrow F'(z) = f(z).$$

Proof: $F(z) = \displaystyle\int_{z_0}^{z} f(w)\,dw$

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}$$

$$= \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw \right]$$

$$= \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(w)\,dw$$

Basically to show: $\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \epsilon$ as $\Delta z \to 0$.

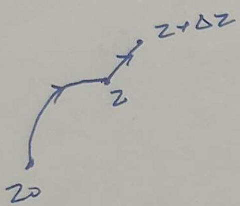
Now, $\int_{z}^{z+\Delta z} dw = \Delta z$

$$\Rightarrow f(z) = \int_{z}^{z+\Delta z} f(z)\,dw$$

As $\left| \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(w)\,dw - \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)\,dw \right|$

$$= \frac{1}{|\Delta z|} \left| \int_{z}^{z+\Delta z} (f(w) - f(z))\,dw \right| \le \frac{1}{\cancel{|\Delta z|}} \sup_{z \le w \le z+\Delta z} |f(w) - f(z)| \cdot \cancel{|\Delta z|}$$

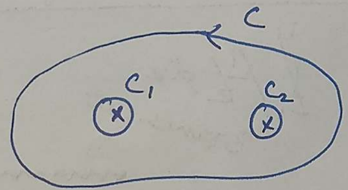
$$< \epsilon \quad \text{(bcoz } f^n \text{ is cont.)}$$



$\boxtimes$

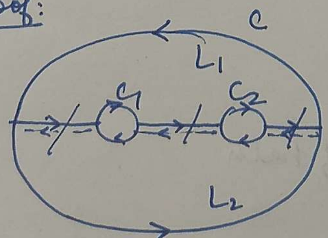
Extension of Cauchy Integral Th^m

$f(z)$ is analytic on C_1 & C_2 but not inside them.



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Outline of the proof:



$$\int_{L_1} f(z)\,dz = 0 \quad \& \quad \int_{L_2} f(z)\,dz = 0$$

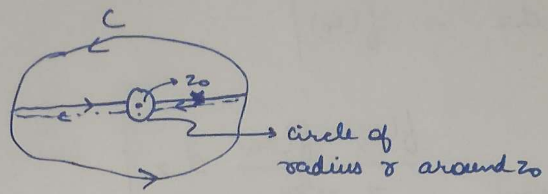
$$\Rightarrow \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 0$$

$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$

$\boxtimes$

→ Suppose $f(z)$ analytic in $D \setminus \{z_0\}$, curve C



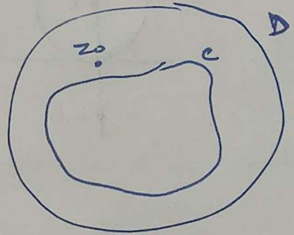
$$\int_C f(z)\,dz = \int_{|z-z_0|=r} f(z)\,dz$$

Ex: We know $\displaystyle\int_{|z-z_0|=r} \frac{dz}{z-z_0} = 2\pi i$

$$\int_C \frac{dz}{z-z_0} = \int_{|z-z_0|=r} \frac{dz}{z-z_0} = 2\pi i$$

C → any curve

→



$$\int_C \frac{f(z)}{z-z_0} = 0$$

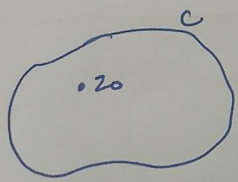
$\because$ $f(z)$ is analytic

& $\frac{1}{z-z_0}$ is analytic everywhere except at z_0.

& z_0 is outside C.

$\Rightarrow$ $\frac{f(z)}{z-z_0}$ is analytic inside C.

→



If z_0 lies inside C,

~~$f(z_0$~~ $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

Cauchy Integral formula: If z_0 lies inside C,

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\,dz}$$

Proof:

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\,dz \quad , \ r \text{ is very small.} \quad —(1)$$

Now $\displaystyle\frac{1}{2\pi i} \int_C \frac{dz}{z-z_0} = 1 \Rightarrow \frac{1}{2\pi i} \int_C \frac{f(z_0)\,dz}{z-z_0} = f(z_0)$

$$\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z_0)\,dz}{z-z_0} \quad —(2)$$

From ① & ②,

$$\left| \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz - f(z_0) \right|$$

$$= \left| \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right|$$

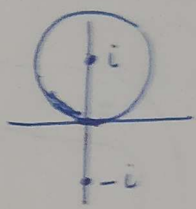
$$\leq \frac{1}{2\pi} \sup_{|z-z_0|=r} |f(z)-f(z_0)| \cdot \frac{1}{r} \cdot 2\pi r \qquad \text{(M-L ineq.)}$$

$$= \sup_{|z-z_0|=r} |f(z)-f(z_0)| < \epsilon$$

$$\Rightarrow \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz = f(z_0) \qquad \boxtimes$$

Ex:

$$\int_{|z-i|=1} \frac{e^z}{z^2+1}\, dz$$



$$= \int_{|z-i|=1} \frac{\overbrace{\left(\frac{e^z}{z+i}\right)}^{\to f(z)}}{(z-i)}\, dz$$

$$= 2\pi i\, f(i)$$

$$= 2\pi i\, \frac{e^i}{2i} = \pi e^i$$

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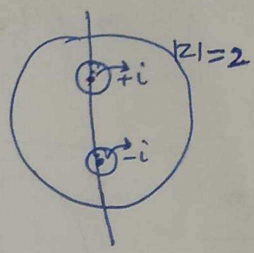
Cauchy Integral formula

$$\oint_C f(z)\, dz = 0$$

$$\int_{|z|=2} \frac{e^z}{z^2+1}\,dz$$

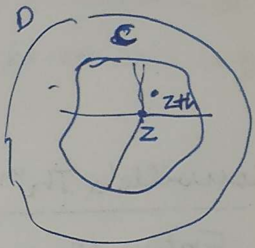


$$= \oint_{C_1} \frac{e^z}{z^2+1}\,dz + \oint_{C_2} \frac{e^z}{z^2+1}\,dz$$

$$= \oint_{C_1} \frac{\left(\frac{e^z}{z+i}\right)}{z-i}\,dz + \oint_{C_2} \frac{\left(\frac{e^z}{z-i}\right)}{z-(-i)}\,dz$$

$$= 2\pi i\left(\frac{e^i}{2i} + \frac{e^{-i}}{-2i}\right)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}}\,dw$$



Proof for n is non-trivial

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^2}\,dw$$

Proof: $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

Since h can be taken small, z+h lies inside C.

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i h} \oint_C \left\{ \frac{f(w)}{(w-(z+h))} - \frac{f(w)}{w-z} \right\} dw$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(w)\,dw}{(w-(z+h))(w-z)}$$

Now $\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z)^2}\,dw = I$

$$I = \frac{1}{2\pi i}\left[\oint_C \frac{f(w)\,dw}{(w-z)(w-z-h)} - \frac{f(w)\,dw}{(w-z)^2} \right]$$

$$= \frac{h}{2\pi i} \oint_C \frac{f(w)\,dw}{(w-z)^2(w-z-h)}$$

$$\alpha = \min_{w\in C} |w-z| \Rightarrow |w-z| \geq \alpha \ \forall\, w \in C$$

$$\alpha \leq |w-z| = |w-z-h+h| \leq |w-z-h| + |h|$$

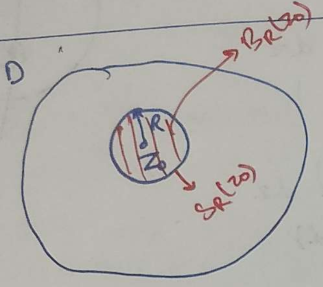
$$\Rightarrow |w-z-h| \geq \alpha - |h|$$

Choose h s.t. $|h| < \alpha/2 \Rightarrow |w-z-h| \geq \alpha/2$

$$\Rightarrow |I| \le \frac{|h|}{2\pi} \frac{M}{d^2 \cdot d/2} \cdot L \to 0 \quad \text{as } h \to 0.$$

$$\Rightarrow f'(z) = \frac{1!}{2\pi i} \oint \frac{f(w)\,dw}{(w-z)^2} \qquad \boxtimes$$

Th^m:



$B_R \to$ disc
$S_R \to$ circle

$B_R(z_0) \subseteq D$

$|f(z)| \le M$ in $S_R(z_0)$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{S_R(z_0)} \frac{f(w)\,dw}{(w-z_0)^{n+1}}$$

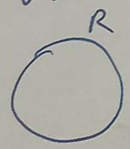
$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \frac{M}{R^{n+1}} \cdot 2\pi R$$

Estimated value — $|f^{(n)}(z_0)| \le n! \cdot \dfrac{M}{R^n}$

Liouville's Th^m:

Entire f^n: analytic at any $z \in \mathbb{C}$

$\to$ If f is bounded and entire f^n in entire complex plane, f is constant.



$$\underbrace{|f'(z)| \le \frac{M}{R}}_{\text{from th}^m \text{ above.}} \Rightarrow f'(z) = 0 \quad (\because R \text{ can be as large as possible since } f^n \text{ is entire})$$

$\Rightarrow f$ is a constant

~~f must be bounded else we can't say f is~~

Ex: (From Assign): To prove $\sin z$ is unbdd.

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

e^{iz} and e^{-iz} are entire f^n

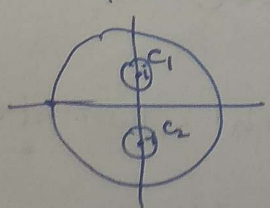
$\Rightarrow \sin z$ is entire f^n.

$\Rightarrow$ If $\sin z$ is bdd $\Rightarrow \sin z = $ const. (From Liouville's Th^m)
false

$\Rightarrow \sin z$ is unbdd.

Ex:

$$\oint_{|z|=2} \frac{e^z}{(z^2+1)^3}\,dz = \oint_{C_1} \frac{e^z}{(z^2+1)^3}\,dz + \oint_{C_2} \frac{e^z}{(z^2+1)^3}\,dz$$



$$= \oint_{C_1} \frac{\frac{e^z}{(z+i)^3}}{(z-i)^3}\,dz + \cdots$$

Consider $\cancel{f=}$ $f(z) = \frac{e^z}{(z+i)^3}$

& $(z-i)^3 = (z-z_0)^{n+1}$

$$\Rightarrow \oint_{C_1} \frac{\frac{e^z}{(z^2+i)^3}}{(z-i)^3} dz = \oint_{C_1} \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{2!} \left(\frac{e^z}{(z+i)^3} \right)^{(2)} \Bigg|_{z=i}$$

$\frac{2\pi i}{n!} f^{(n)}(z_0)$

~~IIIly~~

$$\Rightarrow \oint_{|z|=2} \frac{e^z}{(z^2+1)^3} dz = \frac{2\pi i}{2!} \left(\frac{e^z}{(z+i)^3} \right)^{(2)} \Bigg|_{z=i} + \frac{2\pi i}{2!} \left(\frac{e^z}{(z-i)^3} \right)^{(2)} \Bigg|_{z=-i}$$

Morera's Theorem

Let $f(z)$ be cont. in ~~domain D~~ simply connected domain D

and $\oint_C f(z)\,dz = 0$ for every simple closed contour in D

$\Rightarrow$ $f(z)$ is analytic in D

Argument: $\int_{z_0}^{z} f(w)\,dw$ is independent of the path.

$\Rightarrow$ $F(z) = \int_{z_0}^{z} f(w)\,dw$

well-defined & $F'(z) = f(z)$

$\Rightarrow$ F exists at each $z \in D$

$\Rightarrow$ F'' exists for $\forall z \in D$ or f' exists $\forall z \in D$

$\Rightarrow$ f is analytic

ge as
is entire)

Th^{m})

Th^{m}:

$P(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0$

$a_i \in \mathbb{C}$, $i = 0, \ldots, n-1$

$\exists z_1 \in \mathbb{C}$ s.t. $P(z_1) = 0$

$\downarrow$

$P(z) = (z-z_1) \underbrace{P_1(z)}_{\deg n-1}$

and so on,

$P(z) = (z-z_1) \cdots (z-z_n)$

~~Proof~~ To prove: Every poly. of degree $n \geq 1$ must have at least one root

Proof: suppose $P(z) \neq 0$ $\forall z \in \mathbb{C}$.

$P(z)$ is polynomial $\Rightarrow$ $P(z)$ is analytic

& $\because P(z) \neq 0 \Rightarrow \frac{1}{P(z)}$ is analytic everywhere $\Big] \Rightarrow \frac{1}{P(z)}$ is entire

If $\frac{1}{P(z)}$ is proved bdd $\Rightarrow \frac{1}{P(z)}$ will be const. $\Rightarrow$ Contradiction

$P(z) = z^n + a_{n-1}z^{n-1} + \cdots \cdots + a_0$

Consider $|z| > R$

$\Rightarrow \dfrac{P(z)}{z^n} = 1 + \dfrac{a_{n-1}}{z} + \cdots + \dfrac{a_0}{z^n}$

$\left|\dfrac{P(z)}{z^n}\right| \geqslant 1 - \left|\dfrac{a_{n-1}}{z} + \cdots \dfrac{a_1}{z^{n-1}} + \dfrac{a_0}{z^n}\right|$

$\geqslant 1 - \left(\left|\dfrac{a_{n-1}}{z}\right| + \left|\dfrac{a_{n-2}}{z^2}\right| + \cdots \left|\dfrac{a_0}{z^n}\right| \right)$

$A = \max\limits_{0 \le i \le n-1} \{ |a_i| \}$

$\Rightarrow \dfrac{|P(z)|}{|z|^n} \geqslant 1 - \dfrac{A}{|z|}\left(1 + \dfrac{1}{|z|} + \cdots \dfrac{1}{|z|^{n-1}} \right)$

$\geqslant 1 - \dfrac{A}{|z|}\underbrace{\left(1 + \dfrac{1}{|z|} + \cdots + \dfrac{1}{|z|^{n-1}} + \dfrac{1}{|z|^n} + \cdots \infty\right)}$

since $|z| > R \Rightarrow \dfrac{1}{|z|} < \dfrac{1}{R}$ Consider $R > 2$.

$= 1 - \dfrac{A}{|z|}\left(\dfrac{1}{1 - \frac{1}{|z|}}\right) = 1 - \dfrac{A}{|z| - 1}$

$\dfrac{|P(z)|}{|z|^n} \geqslant 1 - \dfrac{2A}{|z| + |z| - 2}$

$\geqslant 1 - \dfrac{2A}{|z|}$ $\qquad |z| \geqslant 2$

$\geqslant \dfrac{1}{2}$ $\qquad 1 - \dfrac{2A}{|z|} \geqslant \dfrac{1}{2} \Rightarrow \dfrac{2A}{|z|} \le \dfrac{1}{2}$

$\Rightarrow |z| \geqslant 4A$

$\Rightarrow \dfrac{|P(z)|}{|z|^n} \geqslant \dfrac{1}{2}$ $\qquad R = \max\{2, 4A\}$

$\Rightarrow \dfrac{1}{|P(z)|} \le \dfrac{2}{|z|^n}$

$\Rightarrow \dfrac{1}{|P(z)|} \to 0$ as $|z| \to \infty$.

So, $\dfrac{1}{P(z)}$ is bounded outside $|z| = R$

& since $|z| \le R$ is bdd. space, so, $\dfrac{1}{P(z)}$ must have some max. value ($\because P(z) \neq 0$) $\Rightarrow \dfrac{1}{P(z)}$ is bdd in entire complex plane & since it is analytic, By Liouville's Th^m, $\dfrac{1}{P(z)}$ is const. $\longrightarrow$ Contradiction $\Rightarrow P(z) = 0$ for some $z \in \mathbb{C}$. ☒

Th^m:

$|f(z_0)| \geq |f(z)| \quad \forall\, z \in B_r(z_0)$

then $f(z)$ is const. in $B_r(z_0)$

Proof: $0 < \rho < r$

$C_\rho : |z - z_0| = \rho \Rightarrow z(\theta) = z_0 + \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{z - z_0}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}}\, \rho i e^{i\theta}\, d\theta$$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta \longrightarrow \text{Mean value theorem.}$$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \underbrace{|f(z_0 + \rho e^{i\theta})|}_{\leq |f(z_0)|}\, d\theta \leq |f(z_0)|$$

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, d\theta \quad \text{— I}$$

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \quad \text{— II}$$

Using ① & ②,

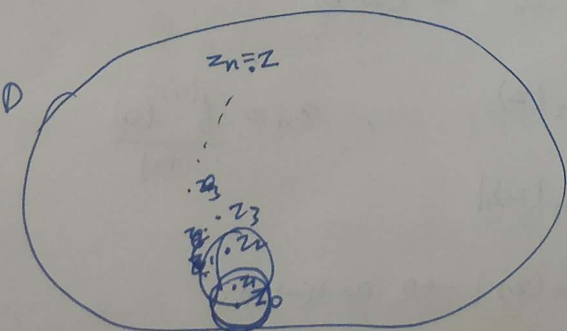
$$\int_0^{2\pi} \left(|f(z_0)| - |f(z_0 + \rho e^{i\theta})| \right) d\theta = 0$$

$$\Rightarrow |f(z_0)| = |f(z_0 + \rho e^{i\theta})| \quad \forall\, \theta \in [0, 2\pi]$$

$$\Rightarrow |f(z_0)| = |f(z)| \quad \forall\, z \in B_r(z_0)$$

$\boxtimes$

Th^m:



A non-const. analytic f^n can't have

4/7/18

$z \in \{z : |z| < R\}$

Let $|z| = r$

and consider a circle $C_0 : |z| = r_0$ where $r < r_0 < R$

$$f(z) = \frac{1}{2\pi i} \oint_{C_0} \frac{f(w)\,dw}{w - z} \quad \text{---} (1) \qquad |w| = r_0,\ |z| = r,\ \frac{r}{r_0} < 1$$

$$1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

$$\frac{1}{1-q} = 1 + q + \cdots + \frac{q^n}{1-q}, \quad q = z/w$$

$$\frac{1}{1-\frac{z}{w}} = 1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots + \frac{z^{n-1}}{w^{n-1}} + \frac{z^n}{w^n(1-z/w)}$$

$$\frac{1}{w-z} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \cdots + \frac{z^{n-1}}{w^n} + \frac{z^n}{w^n(w-z)}$$

Substituting in (1),

$$f(z) = \frac{1}{2\pi i} \oint_{C_0} \frac{f(w)\,dw}{w-z}$$

$$= \frac{1}{2\pi i}\Big[\underbrace{\oint_{C_0} \frac{f(w)}{w}dw + z\oint \frac{f(w)}{w^2}dw + \cdots + z^{n-1}\oint \frac{f(w)}{w^n}dw}_{f(0) + z f'(0) + \frac{f''(0)}{2!}z^2 + \cdots}\Big] + \rho_n(z)$$

where $\rho_n(z) = \frac{1}{2\pi i} z^n \oint_{C_0} \frac{f(w)\,dw}{w^n(w-z)}$

$\Rightarrow f(z) = \sum_{n=0}^{n-1} a_n z^n + \rho_n(z), \quad a_n = \frac{f^{(n)}(0)}{n!}$

$$\left| f(z) - \sum_{n=0}^{n-1} a_n z^n \right| = |\rho_n(z)|$$

So, we have to show $|\rho_n(z)| \to 0$ as $n \to \infty$.

$$|S_n(z)| \le \frac{1}{2\pi}\left(\frac{r}{r_0}\right)^n \frac{M}{r_0 - r}\, 2\pi r_0 \quad , \frac{r}{r_0}<1$$

$\left(\frac{r}{r_0}\right)^n \to 0$ as $n\to\infty$

Hence $|S_n(z)| \to 0$ as $n\to\infty$.

$$\Rightarrow \boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad , a_n = \frac{f^{(n)}(z_0)}{n!}}$$

Hence if $f(z)$ is analytic inside a circle, then it can be expressed as power series.
This power series must be unique.

Ex:

$f(z) = \frac{1}{1-z}$. To find taylor series around $z=0$.

$$= \sum_{n=0}^{\infty} a_n z^n \quad , \quad a_n = \frac{f^{(n)}(0)}{n!}$$

$$f'(z) = \frac{1}{(1-z)^2} \quad , f''(z) = \frac{2}{(1-z)^3} , \qquad f^{(n)}(0) = n!$$

$$\Rightarrow a_n = 1$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} z^n \quad \text{which is true.}$$

Ex:

$f(z) = \sin z$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Ex: $f(z) = \sin^3 z$

$$\sin 3z = 3\sin z - 4\sin^3 z$$

$$\Rightarrow f(z) = \frac{1}{4}(3\sin z - \sin 3z)$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(3z^{2n+1} - 3^{2n+1} z^{2n+1}\right)$$

Ex:

$f(z) = \log z$ ar $z=1$

$$= \sum_{n=0}^{\infty} a_n (z-1)^n$$

$$f(z) = \log z, \quad f'(z) = \frac{1}{z}, \ f''(z) = -\frac{1}{z^2}, \ f^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{z^n}$$

$$\Rightarrow f^{(n)}(1) = (-1)^n (n-1)!$$

$$a_n = \frac{(-1)^{n-1}}{n}$$

$$\Rightarrow \log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots \quad , \text{valid for } |z-1|<1$$

Ex: $f(z) = \log(1+z)$

$$= \log Z = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (Z-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n \quad \text{where } |Z-1|<1$$

or $|z|<1$

Ex: (Another method)

$$\log z = \sum_{n=0}^{\infty} a_n (z-1)^n$$

Diff.,

$$\frac{1}{z} = \sum_{n=1}^{\infty} n a_n (z-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-1)^n$$

$$\frac{1}{z} = \frac{1}{1+\underbrace{(z-1)}_{Z}} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

$$(n+1) a_{n+1} = (-1)^n$$

$$\Rightarrow a_n = \frac{(-1)^{n-1}}{n}, \quad n = 1, 2, \ldots$$

$$a_0 = f(1) = 0.$$

So, $\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$

Zeroes of an Analytic f^n

z_0 is a zero of f if $f(z_0) = 0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$f(z_0) = a_0 = 0$$

Case I:

$a_n = 0 \ \forall n$

$f(z) \equiv 0$

Case II:

$\exists m$ s.t. $a_i = 0$, $0 \le i < m$

& $a_m \neq 0$

$$f(z) = a_m (z-z_0)^m + a_{m+1}(z-z_0)^{m+1} \ldots = (z-z_0)^m . g(z)$$

$$f(z) = (z-z_0)^m g(z)$$

$$g(z) = \sum_{i=0}^{\infty} a_{m+i} (z-z_0)^i$$

$$g(z_0) = a_m \neq 0.$$

Choose $B_\epsilon(z_0)$ s.t. $g(z) \neq 0$ for $z \in B_\epsilon(z_0)$

$\Rightarrow f(z) \neq 0$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$.

so, analytic f^n can't have zeroes in nbd of each other - ie roots of analytic f^n are isometric.

Not applicable to real f^{ns}.

$\underline{\text{Th}^m}$: $\{z_n\} \to z_0$, $|z_n - z_0| < \epsilon \ \forall n \geq N$

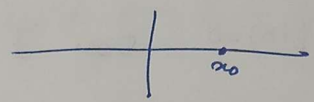
$f(z_0) = 0$ Q $\nexists \{z_n\}$ s.t $f(z_n) = 0$ s.t $\{z_n\} \to z_0$

Ex: To prove: $\sin^2 z + \cos^2 z = 1$

$f(z) = \sin^2 z + \cos^2 z - 1$

$f(x_0) = \sin^2 x_0 + \cos^2 x_0 - 1 = 0.$

$\Rightarrow f(x_0) = 0$



<To prove this we need another result>

Identity Theorem

$f(z)$ is analytic ~~at z_0~~ in $|z - z_0| < r$ & $f(z_0) = 0$

then either of ~~the~~ the foll. hold

1) $f(z) = 0 \quad \forall \ z \in |z - z_0| < r$

2) $f(z) \neq 0 \quad \forall \ z \in |z - z_0| < r \setminus \{z_0\}$

Uniqueness Theorem

$f(z)$ is analytic in domain D

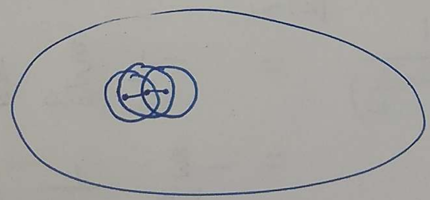
$f(z_0) = 0$

& $\exists \{z_n\}$ where $z_n \neq z_0$, $f(z_n) = 0$ s.t $\{z_n\} \to z_0$

then $f(z) = 0$ in D.

Proof outline

D



Since series converges to z_0, there would always be a nbd of kind $f(z) = 0 \ \forall \ z \in |z - z_0| < r$.

By considering such circles extending upto z , $f(z) = 0$. $\boxtimes$

Ex (contd.): $f(x_0) = 0.$

$\{x_n\} = \left\{x_0 - \frac{1}{n}\right\}$ $f(x_n) = 0$

$\Rightarrow f(z) = 0.$

Ex: $f(z) = x^2|x|$

$$= \begin{cases} x^3, & x>0 \\ -x^3, & x<0 \end{cases}$$

$g(z) = f(z) - z^3$

$x_0 > 0 \quad g(x_0) = x_0^3 - x_0^3 = 0$

$f(z) = -z^3 \neq f(z) = z^3$

Ex: $f(0)=0$. Does $\exists$ a f^n s.t. $f(1/n) = \sin(1/n)$

$g(z) = f(z) - \sin z$

$g(1/n) = 0 \qquad \{\frac{1}{n}\} \to 0$ as $n \to \infty$

$g(0) = 0$

$\Rightarrow g(z) = 0 \Rightarrow f(z) = \sin z.$

Th^m: Minimum Principle:

Consider D, s.t. $f(z) \neq 0 \ \forall z \in D$, then min. value occurs ~~at~~ on the boundary.

Proof: $g(z) = \dfrac{1}{f(z)}$ is analytic & max occurs on boundary. $\boxtimes$

$\rightarrow \quad \dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n, \quad |z|<1$

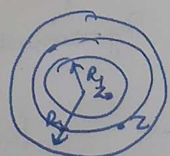
$|z|>1$

$$= \frac{-1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} \qquad \frac{1}{|z|}<1 \Rightarrow |z|>1$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\sum_{n=1}^{\infty} \frac{1}{z^n}$$

Lorentz series

$f(z)$ analytic in $R_1 < |z-z_0| < R_2$



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \quad , n = 1, 2 \ldots$$

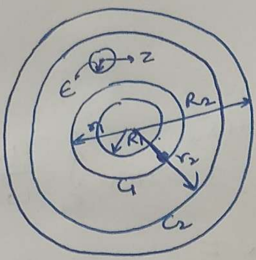
$C \rightarrow$ closed contour & must contain $|z| = R_1$.

$m = -n$

$$b_{-m} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{m+1}} dz$$

$$\Rightarrow \quad f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad , \quad c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

Proof:



$|z| = r$. $\frac{f(\omega)}{\omega - z}$ is analytic on C_1, C_2, C_ϵ.

$$\oint_{C_2} \frac{f(\omega)}{\omega - z} d\omega = \oint_{C_1} \frac{f(\omega)}{\omega - z} d\omega + \oint_{C_\epsilon} \frac{f(\omega)}{\omega - z} d\omega$$

$$\frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{\omega - z} d\omega = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{\omega - z} d\omega + \underbrace{\frac{1}{2\pi i} \oint_{C_\epsilon} \frac{f(\omega)}{\omega - z} d\omega}_{\Downarrow \; f(z)}$$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \left[\overbrace{\oint_{C_2} \frac{f(\omega)}{\omega - z} d\omega}^{g(z)} + \overbrace{\oint_{C_1} \frac{f(\omega)\,d\omega}{z - \omega}}^{h(z)} \right]$$

for C_2, $\left|\frac{z}{\omega}\right| < 1$, for C_1, $\left|\frac{\omega}{z}\right| < 1$

$$g(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)\,d\omega}{\omega - z} = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{(Using Power series)}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

hence, $g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n = \underbrace{\frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{(z-z_0)^{n+1}} dz}_{\displaystyle \| \;\; \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz}$

~~For h(z), we have to do some manipulations~~

6/7/18 (L8)

On C_1, $|\omega| = r_1 < r$, $q = \omega / z$

$$\frac{1}{1-q} = 1 + q + \cdots + q^{n-1} + \frac{q^n}{1-q}$$

$$\frac{1}{z-\omega} = \sum_{k=0}^{n-1} \frac{\omega^k}{z^{k+1}} + \frac{\omega^n}{z^n (z-\omega)}$$

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{z-\omega} d\omega = \cancel{\frac{1}{2\pi}} \frac{1}{2\pi i} \sum_{k=0}^{n-1} \oint_{C_1} \frac{\omega^k}{z^{k+1}} f(\omega)\,d\omega + \underbrace{\frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)\,\omega^n}{z^n (z-\omega)} d\omega}_{\cancel{f_n(z)} \; \sigma_n(z)}$$

$\oint \quad |\sigma_n(z)| \leq \frac{1}{2\pi}\left(\frac{r_1}{r}\right)^n \frac{M}{r-r_1} \cdot 2\pi r_1 \longrightarrow 0 \quad \text{as } n \to \infty$

$$\Rightarrow \quad h(z) = \sum_{k=0}^{\infty} \frac{d_k}{z^{k+1}} \qquad \begin{array}{l} k+1=n \\ d_{n-1} = b_n \end{array}$$

$$= \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

$$d_k = \frac{1}{2\pi i} \oint_{C_1} w^k f(w)\,dw$$

$$k = n-1 \Rightarrow \quad b_n = \frac{1}{2\pi i} \oint_{C_1} w^{n-1} f(w)\,dw = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w^{-n+1}}\,dw$$

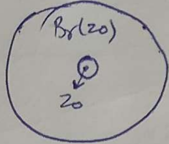
$$\left(\begin{array}{c} \text{for} \\ z_0 \neq 0 \end{array}\right) \quad = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

hence, $h(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ where $b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$

$\boxtimes$

~~Ex:~~ $\cancel{f(z) = e^{1/z}}$

$\Rightarrow$ If a f^n $f(z)$ is analytic in $B_r(z_0) \setminus \{z_0\}$, then that f^n follows Lorentz series ($\because$ Annulus will be formed)



$\Rightarrow$ If f^n $f(z)$ is analytic inside whole $B_r(z_0)$, $b_n = \frac{1}{2\pi i} \oint_C f(w) w^{n-1} dw$
$= 0.$

It is no longer Lorentz series, but normal power series.

Ex: (From last class)

f is analytic, $f(0) = 0$, Is it possible that $f(1/n) = \sin(1/n)$

$g(z) = f(z) - \sin z$

$g(1/n) = 0 \qquad \{\frac{1}{n}\} \to 0 \in \mathbb{C}$

$g(0) = 0$

$\{z_n\} \to z_0$

$f(z_0) = f\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} f(z_n) = 0. \quad \Big|$ Take $z_n = \frac{1}{n}$

So, $g(1/n)=0$, $g(0)=0$ $\Rightarrow g \equiv 0$ (Using Uniqueness Th^m)
$f(z) = \sin z$

Ex: If $f(n) = \sin(1/n)$ in $\mathbb{C}\setminus\{0\}$ then above can't be shown ($\because$ limit pt. is not part of domain)

Ex: $D := \{z : |z| < 1\}$

i) $f(1/n) = 1/n^2$

$g(z) = f(z) - z^2$ $\qquad g(1/n) = 0 \quad \{\frac{1}{n}\} \to 0 \in D$

$\Rightarrow g(1/n+1) = 0$

$\Rightarrow g(z) = 0.$

$f(z) = z^2$

(ii) $f\left(1-\frac{1}{n}\right) = \left(1-\frac{1}{n}\right)^2$

$\left\{1-\frac{1}{n}\right\} \to 1 \to$ Limit point is ~~is not~~ not in D.

Ex: (From Last Class)

$g(z) = \sin^2 z + \cos^2 z - 1$

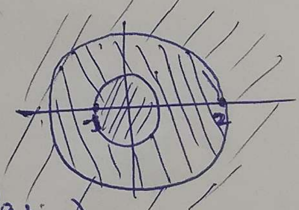
$g(\pi/2) = 0$ $\qquad \left\{\frac{\pi}{2} + \frac{1}{n}\right\} \to \pi/2 \in$ Domain.

$g\left(\frac{\pi}{2} + \frac{1}{n}\right) = 0$

$\Rightarrow g(z) = 0$ (Uniqueness theorem)

Ex: Find lorentz series of $\frac{3}{2+z-z^2} = f(z)$

$$f(z) = \frac{3}{(2-z)(1+z)} = \frac{1}{2-z} + \frac{1}{1+z}$$



$f(z)$ is analytic in all 3 regions (except on boundaries)

i) $|z| < 1$, $\quad f(z) = (1+z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$
$\Rightarrow \left|\frac{z}{2}\right| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$ [Here it is taylor series]

(Series will be unique, so any method can be used)

(ii) $1 < |z| < 2$

$$f(z) = \frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{1}{2} \cdot \frac{1}{\left(1-z/2\right)}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

iii) $|z| > 2$

$$f(z) = \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n} - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \left((-1)^n - 2^n\right)$$

$\Rightarrow$ In Lorentz Series,

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

$\underline{\underline{n=1}}$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz$$

$$\Rightarrow \oint_C f(z) dz = 2\pi i \boxed{b_1} \rightarrow \text{Residue of } f \text{ at } z_0$$

$\underline{\underline{\text{Ex:}}}$

$$\int_{|z|=1} e^{1/z} dz = 2\pi i\, b_1$$

$$e^{1/z} = 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \cdots$$

$\Rightarrow$ $\underline{b_1 = 1}$

$$\Rightarrow \int_{|z|=1} e^{1/z} dz = 2\pi i$$

$\underline{\text{Ex:}}$

$$\int_{|z|=1} e^{1/z^2} dz = 2\pi i\, b_1$$

$$e^{1/z^2} = 1 + \frac{1/z^2}{1!} + \cdots$$

$\Rightarrow b_1 = 0 \quad \Rightarrow \int_{|z|=1} e^{1/z^2} dz = 0.$

Singularity :

$f(z)$ has isolated singularity at $z = z_0$

if $f(z)$ is analytic in $B_r(z_0) \setminus \{z_0\}$

ie $f(z)$ is singular at z_0 but not its nbd.

So, now we can have annulus, $0 < |z - z_0| < r$ where f is analytic

And hence Lorentz series.

3 types of Isolated Singularity

1) Removable $\Rightarrow b_n = 0 \ \forall n$.
2) Pole of order $m \Rightarrow \exists m$ s.t $b_m \neq 0$ & $b_k = 0 \ \forall \ k > m$
3) Essential $\Rightarrow$ does not truncate, goes on till infinity

1) Removable example: $f(z) = \dfrac{\sin z}{z}$, $z \neq 0$

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) \qquad |z| > 0$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots \qquad z \neq 0$$

If we define $f(0) = 1$, Then $f(z)$ becomes analytic

2) Pole of order m

$\exists m$ s.t $b_m \neq 0$ & $b_k = 0 \ \forall \ k > m$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}$$

Example: $f(z) = \dfrac{\sin z}{z^4}$, $z \neq 0$

$$= \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= \frac{1}{z^3} - \frac{1}{z(3!)} + \frac{z}{5!} + \cdots$$

So, $z = 0$ is pole of order 3.

3) Essential example: $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}$

Zero of order m of an analytic f^n

z_0 is a zero of f if $f(z_0)=0$

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

and $f^{(m)}(z_0) \neq 0$

$$a_0 = f(z_0) = 0$$
$$a_1 = f'(z_0) = 0$$

$$f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n$$
$$= (z-z_0)^m \sum_{n=m}^{\infty} a_n (z-z_0)^{n-m}$$
$$= (z-z_0)^m \sum_{k=0}^{\infty} a_{m+k} (z-z_0)^k$$
$$= (z-z_0)^m h(z)$$

$$h(z) = \sum_{k=0}^{\infty} a_{m+k} (z-z_0)^k$$

$$h(z_0) = a_m \neq 0$$

So, Def^n: z_0 is a zero of order m iff

$$f(z) = (z-z_0)^m h(z), \quad h(z_0) \neq 0.$$

Proof: $\rightarrow$ is proved

$\leftarrow$ to prove

Given $f(z) = (z-z_0)^m h(z)$

$$h(z) = \sum_{k=0}^{\infty} b_k (z-z_0)^k$$

$$f(z) = \sum_{k=0}^{\infty} b_k (z-z_0)^{k+m}$$
$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

$$c_0 = c_1 = \cdots = c_{m-1} = 0$$
$$c_{m+k} = b_k, \quad k = 0, 1, \ldots$$
& $c_m \neq 0$.

Since Power series is unique,

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

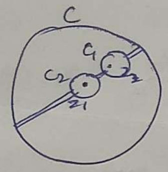
$\Rightarrow$ $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \ldots$ ⊠

Ex: Evaluate $\displaystyle\int_{|z|=1} \frac{\sin z}{z^4}\,dz = 2\pi i \cdot b_1$

$= 2\pi i\left(-\frac{1}{6}\right)$

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \cdots\right)$$

Suppose if you have more than one singularity,



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$

$$= 2\pi i \cdot \mathrm{Res}(f; z_1) + 2\pi i \cdot \mathrm{Res}(f; z_2)$$

($\mathrm{Res}(f; z_1)$ is residue of f at z_1)

Ex: $\displaystyle\oint_{|z|=2} \underbrace{\frac{\sin z}{z^3(z-1)}}_{f(z)}\,dz = 2\pi i\left(\mathrm{Res}(f:0) + \mathrm{Res}(f:1)\right)$

Near ~~at~~ $z=0$,

$$f(z) = -\frac{1}{z^3}\left(z - \frac{z^3}{3!} + \cdots\right)\left(1 + z + z^2 \cdots\right)$$

$$\cancel{= -\frac{1}{z^2}(1} = \left(-\frac{1}{z^2} + \frac{1}{3!} + \frac{z^2}{5!} + \cdots\right)\left(1 + z + z^2 + \cdots\right)$$

$b_1 = -1$

Near $z=1$

$$f(z) = \frac{\sin(z-1+1)}{(z-1)(1+z-1)^3} = \left[\frac{\sin(z-1)\cos 1 + \cos(z-1)\sin 1}{(z-1)}\right](1+(z-1))^{-3}$$

Sometimes, it becomes complicated to calculate b_1, so there are formulas.

9/07/18 (L9)

Ex: $\displaystyle\int_{|z|=1} \frac{e^z}{z^3(z-1)}\,dz = \oint_{C_1} \frac{e^z}{z^3(z-1)}\,dz + \oint_{C_2} \cdots = 2\pi i\left(\mathrm{Res}(f:0) + \mathrm{Res}(f:1)\right)$

$= 2\pi i\left(\frac{-5}{2}\right) + 2\pi i e$

Formula:

$\dfrac{e^z}{z^3(z-1)}$ $z=0$ pole of order 3

$z=1$ is simple pole

~~$z=0$~~, $z=1$ simple pole

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots \Rightarrow b_1 = \lim_{z\to z_0}(z-z_0)f(z)$$

Using the formula in above example,

~~Res (f:1)~~

$$\mathrm{Res}(f:1) = \lim_{z\to 1} (z-1)\frac{e^z}{z^3(z-1)} = e.$$

So, for pole of order m.

$$f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$b_m + b_{m-1}(z-z_0) + \cdots + b_2(z-z_0)^{m-2} + b_1(z-z_0)^{m-1} + \cdots = (z-z_0)^m f(z)$$

$$\Rightarrow b_1 = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$$

In above example,

$$\frac{e^z}{z^3(z-1)}$$

$$\mathrm{Res}(f:0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\cancel{z^3}\frac{e^z}{\cancel{z^3}(z-1)}\right)$$

$$= -\frac{5}{2}$$

Pole of Order m

$$f(z) = \frac{1}{(z-z_0)^m}\left(b_m + b_{m-1}(z-z_0) + \cdots\right)$$

$$= \frac{g(z)}{(z-z_0)^m}, \quad \cancel{g(z_0) = b_m \neq 0} \quad g(z_0) = b_m \neq 0.$$

Ex: (contd)

$$f(z) = \frac{e^z}{z^3(z-1)} = \frac{g(z)}{z^3}$$

$$g(z) = \frac{e^z}{z^3}$$

$$f(z) = \frac{\frac{e^z}{z^3}}{z-1} = \frac{g(z)}{z-1} \qquad \cancel{g(z)=}\ g(1) \neq 0$$

$z=1$ is simple pole

$f(z), g(z)$

$f(z_0) = g(z_0) = 0$

$g'(z_0) \neq 0$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \frac{f'(z_0)}{g'(z_0)}$$

If $f(z)$ has a simple pole at z_0.

$f(z) = \frac{g(z)}{(z - z_0)}$ $g(z)$ is ~~analy~~

Suppose $f(z) = \frac{g(z)}{h(z)}$

$g(z)$ & $h(z)$ are analytic at z_0

z_0 is zero of ~~order~~ order m for h

& z_0 ———— n for g

$h(z) = (z - z_0)^m h_1(z)$, $h_1(z_0) \neq 0$

$g(z) = (z - z_0)^n g_1(z)$, $g_1(z_0) \neq 0$.

$f(z) = \frac{1}{(z - z_0)^{m-n}} f_1(z)$

$f_1(z_0) \neq 0$.

$m > n$, pole $m - n$

$m \leq n$ z_0 is removable ~~sin~~ identity

$\Rightarrow \mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$

$= \lim_{z \to z_0} \left((z - z_0) \frac{g(z)}{h(z)} \right)$

$\left[\begin{array}{l} g(z_0) \neq 0 \\ \& \ h(z_0) = 0 \\ \& \ h'(z_0) \neq 0 \end{array} \right]$

$$= \lim_{z\to z_0} \lim_{z\to w} \frac{g(z)}{\left(\frac{h(z)-h(z_0)}{z-z_0}\right)}$$

$$= \frac{g(z_0)}{h'(z_0)}$$

Ex: $f(z) = \frac{1+z^3}{\sin z} = \frac{g(z)}{h(z)}$

$h(0) = 0$

$\cancel{g}\ h'(0) \neq 0$

$g(0) \neq 0.$

$\text{Res}(f; 0) = \frac{1}{1} = 1.$

$\rightarrow \quad f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$b_n = 0 \quad \forall\ n$

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad 0 < |z-z_0| < K.$

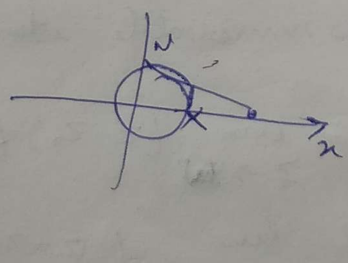
Ex: $f(z) = \frac{1-\cos z}{z^2}, \quad z \neq 0$

$$= \frac{1 - (1 + \frac{z^2}{2!} + \frac{z^4}{4!} - -)}{z^2}$$

$$= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots, \quad z \neq 0.$$

$\rightarrow \lim_{z\to z_0} f(z) = \infty$

$= \frac{1}{\lim_{z\to z_0} f(z)} = 0$



$\lim_{z\to\infty} f(z) = w_0 \equiv \lim_{z\to z_0} f\left(\frac{1}{z}\right) = w_0.$

$\lim_{z\to\infty} f(z) = \infty \equiv \lim_{z\to 0} \frac{1}{f(1/z)} = 0.$

→ $f(z)$ z_0 removable singularity

⇒ $f(z)$ is analytic in nbd of z_0, & is bdd.

$\lim_{z\to z_0} f(z)$ has finite values.

z_0 is a pole of order m, $f(z) = \frac{g(z)}{(z-z_0)^m}$, $g(z_0) \neq 0$.

$\lim_{z\to z_0} \frac{1}{f(z)} = 0$

so, $\lim_{z\to z_0} f(z) = \infty.$

Theorem $f(z)$ has z_0 essential singularity

then, For any $w_0 \in \mathbb{C}$

$\exists\, z \in 0 < |z-z_0| < r$

s.t $|f(z) - w_0| < \epsilon.$

Proof: Let $B_r^*(z_0) = \{z : |z-z_0| < r\}$ $\forall z \in B_r^*(z_0)$

s.t $|f(z) - w_0| \geq \epsilon.$

→ $g(z) = \frac{1}{f(z) - w_0}$

$|g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{\epsilon}.$

$g(z)$ is bounded

⇒ z_0 is removable sing.

$g(z)$ is analytic for suitable value of $g(z_0)$.

2 cases:

Case I: $g(z_0) \neq 0$

$$g(z) = \frac{1}{f(z) - w_0}$$

$$f(z) - w_0 = \frac{1}{g(z)}$$

Removable sing. for $f(z)$

$$f(z) = w_0 + \frac{1}{g(z)} \Rightarrow f(z_0) = w_0 + \frac{1}{g(z_0)}$$ Contradiction

Case 2: $g(z) = 0 \Rightarrow \exists\, m$ s.t. $g^{(m)}(z_0) \neq 0$.

$$\therefore\ f(z) = w_0 + \frac{1}{g(z)}$$

$\Rightarrow$ z_0 is pole of order m

Contradiction

Hence $\exists\, z \in |z - z_0| < r$ s.t. $|f(z) - w_0| < \epsilon$ $\boxtimes$

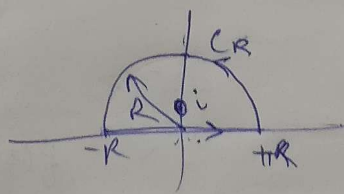
Assignment-5

Q $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_{0}^{R} \frac{dx}{1+x^2}$$

$$\int_{-R}^{R} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{-R}^{R}$$

$$f(z) = \frac{1}{1+z^2}$$



$$\int_{-R}^{R} f(z)\, dz + \int_{C_R} f(z)\, dz = 2\pi i \operatorname{Res}(f; i)$$

$$\int_{-R}^{R} \frac{dx}{1+x^2} + \int_{C_R} f(z)\, dz = 2\pi i \operatorname{Res}(f, i)$$

$$f(z) = \frac{1}{(z+i)(z-i)}$$

$$\text{Res}(f, i) = \lim_{z \to i} (z-i)\, f(z)$$

$$= \frac{1}{z+i} = \frac{1}{2i} \quad \text{— (1)}$$

Using (1),

$$\int_{-R}^{R} \frac{dx}{1+x^2} + \int_{C_R} f(z)\,dz = 2\pi i \frac{1}{2i}$$

$$= \pi.$$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \max_{z \in C_R} |f(z)| \cdot \pi R.$$

on C_R, $f(z) = \frac{1}{R^2 e^{2i\theta}+1} \leq \frac{1}{R^2-1}$

$$\Rightarrow \left| \int_{C_R} f(z)\,dz \right| \leq \frac{\pi}{R^2-1} \quad \text{— (2)}$$

Using (2),

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} + 0 = \pi$$

11/7/18 (L10)

Q $\sum_{j=0}^{\infty} \frac{1}{j!(j+n)!} = c_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{z+1/z}}{z^{n+1}}\, dz$

$$= \frac{1}{2\pi i} \int_{\theta=0}^{2\pi} \frac{e^{2\cos\theta}}{e^{i(n+1)\theta}} i e^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} e^{\cos\theta} e^{-in\theta}\, d\theta$$

Q- $P(z)e^{1/z}$ entire?

$P(z) = a_0 + a_1 z + \cdots + a_d z^d$

$P(z) \cdot e^{1/z}$

$= (a_0 + a_1 z + \cdots a_d z^d)\left(\sum_{j=0}^{\infty} \frac{1}{j!\, z^j}\right)$

$= (\ ) + \sum \frac{C_N}{z^N}$ where $C_N \neq 0$, $N > 0$

→ These terms will make it non-entire

$C_N = \frac{a_0}{N!} + \frac{a_1}{(N+1)!} + \cdots + \frac{a_d}{(N+d)!}$

Let $a_r \neq 0$ & $a_0 = a_1 = \cdots a_{r-1} = 0$.

$\Rightarrow C_N = \frac{a_r}{(N+r)!} + \cdots + \frac{a_d}{(N+d)!}$

$= \frac{1}{(N+r)!}\left(a_r + \frac{a_{r+1}}{(N+r+1)} + \cdots \frac{a_d}{(N+d)(N+d-1)\cdots(N+r+1)}\right)$

If N is large.

$\Rightarrow C_N = \frac{a_r}{(N+r)!}$, $a_r \neq 0 \Rightarrow \underline{C_N \neq 0}$

Hence not entire.

Q

$\int_{|z|=2} \frac{dz}{(z-3)(z^8-1)}$ → 8 residues need to be calculated

so, there is another method.

Method :

$f(z)$ → all singularities lie inside finite area of complex plane.

Ex: $f(z) = \frac{1}{\sin z}$, $z = n\pi$ → not finite area of complex plane

Let C be that curve inside which all singularities of $f(z)$ lie.

then $\int_C f(z)\,dz = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} f(1/z), 0\right)$

Ex: (Contd):

$$\int_{|z|=2} \frac{dz}{(z-3)(z^8-1)}$$

$$\int_{|z|=4} \frac{dz}{(z-3)(z^8-1)} = \oint 2\pi i \, Res\left(\frac{1}{z^2} f(1/z), 0\right)$$

$$\frac{1}{z^2} f(1/z) = \frac{\cancel{1/z}\; z^7}{(1-3z)(1-z^8)}$$

$$= z^7 (1-3z)^{-1} (1-z^8)^{-1}$$

$$Res\left(\frac{1}{z^2} f(1/z), 0\right) = 0.$$

$$\Rightarrow \int_{|z|=4} \frac{f(z)}{(z-3)(z^8-1)} = 0$$

$$\Rightarrow \int_{|z|=2} \frac{dz}{(z-3)(z^8-1)} + \oint_{\underset{|z-3|=r}{\downarrow}{C_1}} \frac{dz}{(z-3)(z^8-1)} = 0$$

$$\Rightarrow \int_{|z|=2} \frac{dz}{(z-3)(z^8-1)} = -2\pi i \, Res(f(z), 3)$$

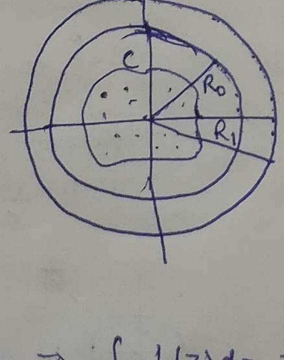
$$= -2\pi i \lim_{z \to 3} (z-3) f(z)$$

$$= \frac{-2\pi i}{3^8 - 1}$$

Proof: $\left[\oint_C f(z)dz = 2\pi i \, Res\left(\frac{1}{z^2} f(1/z), 0\right)\right]$

$f(z)$ is analytic in $\overbrace{R < |z| < \infty.}^{\text{annulus.}}$

$\Rightarrow$ Lorentz series.



$R_1 > R_0$

$$f(z) = \oint \sum_{n=-\infty}^{\infty} C_n z^n$$

$$C_{-1} = \frac{1}{2\pi i} \int_{|z|=R_1} f(z)dz \Rightarrow \underbrace{\int_{|z|=R_1} f(z)dz}_{= \int_C f(z)dz} = 2\pi i \, C_{-1}$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\, C_{-1}$$

Now,

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n z^n \qquad R_0 < |z| < \infty$$

$$f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} C_n \frac{1}{z^n} \qquad 0 < |z| < \frac{1}{R_0}$$

$$\Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} C_n \frac{1}{z^{n+2}} \qquad 0 < |z| < \frac{1}{R_0}$$

$$= (\ ) + \underbrace{\frac{C_{-1}}{z}}_{\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right),\, 0\right)} + (\ \ )$$

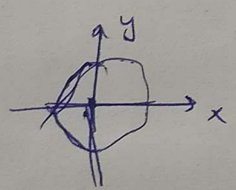
$$\Rightarrow \int_C f(z)\,dz = 2\pi i\, C_{-1}$$

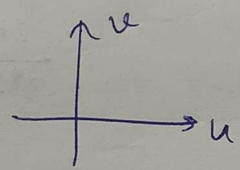
$$= 2\pi i \,\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) \qquad \boxtimes$$

Topic: Linear Fractional Transformation

$$\omega = f(z) = u + iv$$

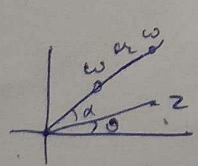
$$\omega = \frac{az+b}{cz+d} = f(z) \qquad ad-bc \neq 0$$





$$w = Az$$

$$w = |A| e^{i\alpha} \cdot r e^{i\theta} = r|A|\, e^{i(\theta+\alpha)}$$



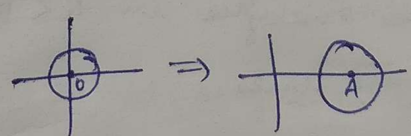
$$w = \frac{1}{z}$$

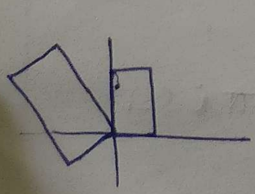
$$w = \frac{\bar{z}}{|z|^2}$$

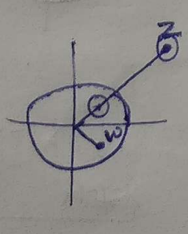
Suppose $|z| = r$

$$\Rightarrow |w| = \frac{1}{r}$$

$$w = z + A$$







$$w = f(z) = \frac{az+b}{cz+d} = \text{(Composition of above 3 functions)}$$

Ex: $f(z)$ has an isolated zero of order 3

$\frac{1}{f(z)}$ & find residue.

$$f(z) = (z-z_0)^3 (a_3 + a_4(z-z_0) + \cdots)$$
$$= (z-z_0)^3 g(z)$$

$$g(z_0) = a_3 = \frac{f^{(3)}(z_0)}{3!} \neq 0.$$

$$\frac{1}{f(z)} = \frac{h(z)}{(z-z_0)^3}, \quad h(z_0) = \frac{1}{g(z_0)} \neq 0, \quad h(z) \text{ is analytic.}$$

↳ Pole of order 3.

$$\operatorname{Res}\left(\frac{1}{f(z)}, z_0\right) = \lim_{z \to z_0} \frac{1}{2!} \frac{d^2}{dz^2} (z-z_0)^3 \frac{1}{f(z)}$$
$$= \frac{1}{2!} \lim_{z \to z_0} \frac{d^2}{dz^2} h(z)$$
$$= \frac{1}{2!} h''(z_0)$$

$h(z) = \frac{1}{g(z)}$

$h'(z) = -\frac{g'}{g^2}$

$h''(z) = \frac{2g'}{g^3} - \frac{g''}{g^2}$

$\Rightarrow h''(z_0) = \frac{2g'(z_0)}{g^3(z_0)} - \frac{g''(z_0)}{g^2(z_0)}$

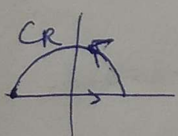
$$g(z) = a_3 + a_4(z-z_0) + a_5(z-z_0)^2 + \cdots$$

$$h''(z_0) = 2\frac{g'(z_0)}{g^3(z_0)} - \frac{g''(z_0)}{g^2(z_0)} = \frac{2a_4}{a_3^3} - \frac{2!\,a_5}{a_3^2}$$

$$a_3 = \frac{f^3(z_0)}{3!}, \quad a_4 = \frac{f^4(z_0)}{4!}, \quad a_5 = \frac{f^5(z_0)}{5!}$$

$$\Rightarrow \operatorname{Res}\left(\frac{1}{f(z)}, z_0\right) = \frac{a_4 a_3 - a_5}{a_3^3}$$ with a_3, a_4, a_5

Ex: (From last class) $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$. we did $f(z) = \frac{1}{1+z^2}$



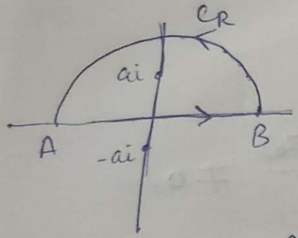
$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty$$

But sometimes this technique will not work

Ex: (Counter Example for above method)

$$\int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2+a^2}\, dx$$

$$f(z) = \frac{z\, e^{i3z}}{z^2+a^2}$$ Singularity $\Rightarrow z = ai,\ z = -ai$



$R > a$

$$\int_{AB+C_R} f(z)\,dz = 2\pi i \,\text{Res}(f(z), ai)$$

$$\text{Res}(f, ai) = \lim_{z\to ai} (z-ai) f(z)$$

$$= \frac{e^{-3a}}{2}$$

$$\Rightarrow \int_{AB+C_R} f(z)\,dz = 2\pi i \frac{e^{-3a}}{2}$$

$$\int_{AB} f(z)\,dz = \int_{-R}^{R} \frac{x\, e^{i3x}}{x^2+a^2}\, dz.$$

$$\left|\int_{C_R} f(z)\,dz\right| = \left|\int_{\theta=0}^{\pi} \frac{Re^{i\theta}\, e^{i3R(\cos\theta + i\sin\theta)}}{R^2 e^{i2\theta}+a^2} \cancel{dz}\, Re^{i\theta} i\, d\theta\right|$$

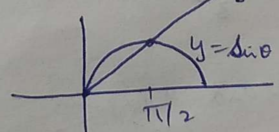
$$\le \frac{R^2}{R^2-a^2} \int_0^{\pi} e^{-3R\sin\theta}\, d\theta$$

$$= c. \underbrace{\frac{R^2}{R^2-a^2}}$$

does not $\to 0$ as $R\to\infty$.

$$\int_0^{\pi} e^{-3R\sin\theta}\, d\theta = 2\int_0^{\pi/2} e^{-3R\sin\theta}\, d\theta$$

$\sin\theta,\ \frac{2\theta}{\pi}$



$\frac{2\theta}{\pi} \le \sin\theta$

in 0 to $\frac{\pi}{2}$

$\Rightarrow -\frac{2\theta}{\pi} \geq -\sin\theta$

$$\int_0^{\pi} e^{-3R\sin\theta}\, d\theta \le \int_0^{\pi} e^{-3R\left(\frac{2\theta}{\pi}\right)}\, d\theta = \frac{c}{3R}$$

$\Rightarrow -\frac{2\theta}{\pi}\cdot\alpha \geq -\sin\theta\cdot\alpha.$

One extra R in denominator

Hence $\to 0$ as $R\to\infty$

Jordan's Lemma

$f(z)$ $\quad\quad C_R : \{Re^{i\theta}, 0 \le \theta \le \pi\}$

$f(z) = e^{iaz} \cdot g(z)$

for above example, $a = 3$
$g(z) = \frac{z}{z^2+a^2}$

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{\pi}{a} M_R$$

$$M_R = \max_{0 \le \theta \le \pi} |g(Re^{i\theta})|$$

For our case, $|g(Re^{i\theta})| = \left| \frac{Re^{i\theta}}{R^2 e^{2i\theta} + a^2} \right| \le \frac{R}{R^2 - a^2}$

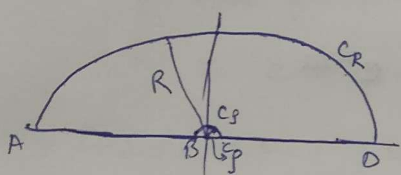
$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{\pi}{3} \frac{R}{R^2-a^2} \to 0 \text{ as } R \to \infty.$$

Ex:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

$f(z) = \frac{e^{iz}}{z}$ → Singularity at $z = 0$
↓
Simple pole.

The foll. technique will work only for simple pole.



$$\left(\int_{AB} + \int_{C_\rho} + \int_{CD} + \int_{C_R} \right) = 0$$

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{\pi}{1} \cdot \frac{1}{R} \to 0 \text{ as } R \to \infty.$$

$$\Rightarrow \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx + \int_{C_\rho} + \int_{\rho}^{R} \frac{e^{ix}}{x}\,dx = 0.$$

$$\int_{C_\rho} \frac{e^{iz}}{z}\,dz = \int_{C_\rho} \left(\frac{1}{z} + g(z) \right) dz$$

$$\left| \int_{C_\rho} g(z)\,dz \right| \le \max_{z \in C_\rho} |g(z)| \cdot \pi\rho \to 0 \text{ as } \rho \to 0.$$

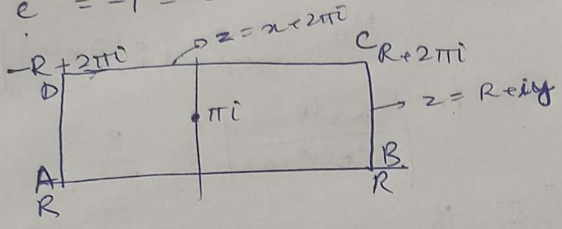
$$\Rightarrow \int_{C_\rho} \frac{e^{iz}}{z}\,dz = \int_{C_\rho} \frac{1}{z}\,dz = \int_{\theta=\pi}^{0} \frac{\rho i e^{i\theta}}{\rho e^{i\theta}}\,d\theta = -i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - \pi i = 0 \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^{x}+1}\,dx \qquad 0<a<1$

$$f(z) = \frac{e^{az}}{e^{z}+1}$$

Singularity: $e^{z} = -1 = e^{\pi i} \Rightarrow z = 2n\pi i + \pi i, \; n \in \mathbb{Z}$



$$\int_{AB+BC+CD+DA} f(z)\,dz = 2\pi i \,\operatorname{Res}(f, \pi i)$$

$$\operatorname{Res}(f,\pi i) = \lim_{z\to\pi i} (z-\pi i)\frac{e^{az} \to g}{e^{z}+1 \to h} \qquad \begin{array}{l} g(z_0)\neq 0 \\ \& \; h(z_0)=0 \\ h'(z_0)\neq 0. \end{array}$$

$$= \frac{e^{\pi i a}}{e^{\pi i}} = -e^{\pi i a}$$

$$\Rightarrow \int_{AB+BC+CD+DA} f(z)\,dz = -2\pi i\, e^{\pi i a}$$

$$\int_{AB} = \int_{-R}^{R} \frac{e^{ax}}{e^{x}+1}\,dx = I_1$$

$$\int_{CD} f(z)\,dz = -\int_{DC} f(z)\,dz = -\int_{-R}^{R} \frac{e^{a(x+2\pi i)}}{e^{x}+1}\,dx$$

$$= -e^{-2\pi i a} I_1$$

$$\left|\int_{BC} f(z)\,dz\right| = \left|\int_{0}^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}+1}\, i\,dy\right|$$

$$\leq \frac{e^{aR}}{e^{R}-1}\cdot 2\pi \qquad 0<a<1$$

$\to 0$ as $R\to\infty$. ($\because$ e^{R} advances faster than e^{aR})

$$\int_{DA} f(z)\,dz = \left|\int_{0}^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}\, i\,dy\right|$$

$$\leq \frac{e^{-aR}}{1-e^{-R}}\cdot 2\pi = \frac{e^{R}\, e^{(1-a)R}}{e^{R}-1}\, 2\pi$$

$\to 0$ as $R\to\infty$.

$$\Rightarrow \quad I_1 - e^{-2\pi i a} I_1 = -e^{\pi i a}(2\pi i)$$

$$\Rightarrow \quad I_1 = \frac{-2\pi i\, e^{\pi i a}}{1-e^{-2\pi i a}} = \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} = \frac{\pi}{\sin \pi a}$$

Rouche's Theorem

f & g are not zero on C.

$$|f(z)| > |f(z)-g(z)| \qquad z \in C$$

$$|f(z)| > 0$$

$$\cancel{f(z)}\, |g(z)| = |f(z) - (f(z)-g(z))|$$
$$\geq |f(z)| - |f(z)-g(z)|$$
$$> 0.$$

Then no. of zeroes of f and g is same inside curve C.

(No time for proof :P)

Ex: $g(z) = z^4 - 7z - 1$

Inside $|z| = 2$, how meny zeroes of $g(z)$ are there?

$f(z) = z^4$ $\qquad$ $|f(z) - g(z)| = |7z+1|$

$$\leq 7|z|+1 = 15 < 16 = |f(z)|$$

$\Rightarrow$ $\cancel{f(z)}$ $|f(z)| > |f(z) - g(z)|$

$\therefore$ Rouche's Th^m satisfied.

$f(z) = z^4$ has 4 zeroes inside $|z| = 2$

$\Rightarrow$ $g(z)$ has 4 zeroes inside $|z| = 2$.

Short notes (MSO202B)

Date ___/___/___

L6

→ $\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

→ $f^{n}(z) = \frac{n!}{2\pi i} \oint_{c} \frac{f(\omega)}{(\omega - z)^{n+1}}\,d\omega$

→ Cauchy Integral formula → If z_0 lies inside c,

$$f(z_0) = \frac{1}{2\pi i} \oint_{c} \frac{f(z)}{z - z_0}\,dz$$

→ $|f(z)| \le M$ in $S_R(z_0)$ {circle}

$$|f^{n}(z_0)| \le \frac{n!}{2\pi} \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n! M}{R^N}$$

→ Liouville's Th^m : If f^n is bdd & analytic, f is const.

$$|f'(z)| \le \frac{M}{R} \Rightarrow f'(z) = 0$$

→ Morera's Th^n : $f(z) \to$ cont. & $\oint_{c} f(z)\,dz = 0 \;\forall$ SCC in D

$\Rightarrow f(z)$ is analytic in D.

→ entire $f^n \to$ analytic in complex plane (entire)

→ If $|f(z_0)| \geq |f(z)| \;\forall z \in B_r(z_0)$, then $f(z)$ is const. in $B_R(z_0)$

(applicable to any shape)

L7

→ $f(z)$ is analytic inside a circle, then it can be expressed as a unique power series :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad , \quad a_n = \frac{f^n(z_0)}{n!}$$

Power Series to remember*

$\log z = \sum a_n (z-1)^n \qquad \frac{1}{1-z} = \sum z^n$

$\sin z = \sum (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

Page No.

↳ If $f(z)$ is analytic, it can't have zeroes in vicinity of each other (roots are isometric)

↳ Identity Th^m: $f(z)$ is analytic in $|z-z_0|<r$ & $f(z_0)=0$
then either $f(z)=0 \ \forall z \in |z-z_0|<r$
or $f(z) \neq 0 \ \forall z \in |z-z_0|<r$

↳ Uniqueness Th^m: $f(z)$ analytic in D & $f(z_0)=0$
$\Rightarrow f(z)=0$ in D $\{f(z_n)=0$ s.t. $z_n \to z\}$

↳ Minimum Principle: $f(z) \neq 0 \ \forall z \in D$, min. value occurs at boundary

↳ Lorentz Th^m: $f(z)$ is analytic in $R_1 < |z-z_0| < R_2$

$$f(z) = \sum a_n (z-z_0)^n + \sum \frac{b_n}{(z-z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (\text{due to } n)$$

or $f(z) = \sum_{-n}^{n} c_n (z-z_0)^n$; $c_n = \frac{1}{2\pi i} \oint_c \frac{f(z)dz}{(z-z_0)^{n+1}}$

L8

↳ If $f(z)$ is analytic inside $B_R(z_0) \Rightarrow$ Lorentz series
↳ has to be in domain.
↳ $\oint f(z)dz = 2\pi i \, b_1$ (residue of f at z_0)
↳ Singularity (isolated)
- Removable $\Rightarrow b_n = 0 \ \forall n$
- Pole of order m $\Rightarrow$ s.t. $b_m \neq 0$ & $b_k = 0 \ \forall k > m$
- essential $\Rightarrow$ doesn't truncate

at z_0 not analytic but near at it.

Saathi

Date ___/___/___

↳ Zero (of order m) : $f(z) = (z-z_0)^m h(z)$; $h(z_0) \neq 0$

L9

↳ $\oint f(z)dz = 2\pi i (b_1 + b_2 \ldots)$

↳ formula → pole of order m

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left(f(z)(z-z_0)^m \right)$$

↳ $\frac{0}{0}$ form : $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

↳ $f(z) = \frac{g(z)}{h(z)}$ (analytic at z_0)

z_0 is zero of order m & n for h & g

$$f(z) = \frac{(z-z_0)^n g_1(z)}{(z-z_0)^m h_1(z)}$$

If $m > n$ pole of order $m-n$

$m \leq n$ z_0 is removable identity

↳ $\text{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0) f(z)$ if $h(z_0) = 0$ $h'(z_0) \neq 0$, $g(z_0) \neq 0$

$$\text{Res}(f : z_0) = \frac{g(z_0)}{h'(z_0)}$$

↳ $f(z)$ has z_0 essential singularity then for any $w_0 \in \mathbb{C}$

$\exists z \in 0 < |z-z_0| < r$ s.t. $|f(z) - w_0| < \epsilon$

L10

↳ All singularities lie inside the complex plane.

$$\int_C f(z)dz = 2\pi i \,\text{Res}\left(\frac{1}{z^2} f(1/z), 0\right) = 2\pi i \,\text{Res}(h(z), 0)$$

↳ Jordan lemma

$f(z) = e^{iaz} g(z) \Rightarrow \left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{a} M_R$ $M_R = \max_{0 \leq \theta \leq \pi} |g(Re^{i\theta})|$ $C_R : \{Re^{i\theta}, 0 \leq \theta \leq \pi\}$

$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$

Date ___/___/___

↳ Linear fractional Transformation $w = \frac{az+b}{cz+d} = f(z)$

↳ Rouche's Theorem:

f & g are not zero on C

$$|f(z)| > |f(z) - g(z)|$$

$$|f(z)| > 0$$

Then no. of zeroes of f & g is same inside curve C.