MSO202A: Assignment-II Solutions

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

1. If f is differentiable in an open set Ω, then

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \quad \text{and} \quad f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Soln: By definition

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \implies f'(z) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}(z)$$

and

$$f'(z) = \lim_{h\to 0}\frac{f(z+ih)-f(z)}{ih} \implies f'(z) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{ih} = -i\frac{\partial f}{\partial y}(z)$$

Subtracting and adding we get the desired results.

2. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there does not exist any point c on the line $x + y = 1$ joining z_1 and z_2 such that $f(z_1) - f(z_2) = (z_1 - z_2)f'(c)$, i.e., mean value theorem does not extend to complex plane.

Soln: Here $c = t + (1-t)i$ where $0 \le t \le 1$. Now if the relation is true, then

$$\frac{f(1)-f(i)}{1-i} = f'(c) \implies \frac{1+i}{1-i} = 3(t+(1-t)i)^2 \implies i = 3\big((2t-1) + 2it(1-t)\big)$$

This leads to $t = 1/2$ and $6t(1-t) = 1$. But $6t(1-t) = 3/2$, a contradiction.

3. Derive C-R equations in polar coordinates.

Soln: Let $z = re^{i\theta}$ and . Suppose that $z_0 \neq 0$ and $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. First, we consider $z \to z_0$ along $\theta = \theta_0$. Then

$$\begin{aligned} f'(z_0) &= \lim_{r\to r_0}\frac{f(re^{i\theta_0} - f(r_0e^{i\theta_0})}{re^{i\theta_0} - r_0e^{i\theta_0}} \\ &= e^{-i\theta_0}\lim_{r\to r_0}\frac{u(r,\theta_0) - u(r_0,\theta_0) + i(v(r,\theta_0) - v(r_0,\theta_0)}{r-r_0} \\ &= e^{-i\theta_0}\left[\lim_{r\to r_0}\frac{u(r,\theta_0)-u(r_0,\theta_0)}{r-r_0} + i\lim_{r\to r_0}\frac{v(r,\theta_0)-v(r_0,\theta_0}{r-r_0}\right] \end{aligned}$$

(Both limits in the last line exist since the limit in the first line does. Further, a complex function has a limit at a point iff its real and imaginary parts do). Hence, $f'(z_0) = e^{-i\theta_0}(u_r + iv_r)$.

Next we consider $z \to z_0$ along $r = r_0$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\theta\to\theta_0}\frac{f(r_0e^{i\theta} - f(r_0e^{i\theta_0})}{r_0e^{i\theta} - r_0e^{i\theta_0}} \\ &= \frac{1}{r_0}\lim_{\theta\to\theta_0}\frac{u(r_0,\theta) - u(r_0,\theta_0) + i(v(r_0,\theta) - v(r_0,\theta_0)}{e^{i\theta} - e^{i\theta_0}} \\ &= \frac{1}{r_0}\lim_{\theta\to\theta_0}\left\{\left[\frac{u(r_0,\theta)-u(r_0,\theta_0)}{\theta-\theta_0} + i\frac{v(r_0,\theta)-v(r_0,\theta_0)}{\theta-\theta_0}\right]\frac{\theta-\theta_0}{e^{i\theta}-e^{i\theta_0}}\right\} \end{aligned}$$

Now

$$\lim_{\theta \to \theta_0} \frac{e^{i\theta} - e^{i\theta_0}}{\theta - \theta_0} = -\sin\theta_0 + i\cos\theta_0 = ie^{i\theta_0}$$

Hence, $f'(z_0) = -ie^{-i\theta_0}(u_\theta + iv_\theta)/r_0$.

Comparing both we find $r_0\, u_r = v_\theta$ and $r_0 v_r = -u_\theta$. Further

$$f'(z_0) = e^{-i\theta_0}(u_r + iv_r) \quad \text{or} \quad f'(z_0) = \frac{e^{-i\theta_0}}{r_0}(v_\theta - iu_\theta)$$

(Note that C-R equations in polar form can also be derived from Cartesian form using change of coordinates)

4. Let $f : \mathbb{D} \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}$, $f(z) = f(w)$ whenever $|z| = |w|$. Using CR equations in polar coordinates, show that f is a constant function.

   Soln: Let $z = re^{i\theta}$. From the given condition, it is clear that $f(z)$ is idependent of θ. Let $f(z) = u(r, \theta) + iv(r, \theta)$. Now from $u_r = v_\theta/r$ and $-u_\theta/r = v_r$, we get $u_r = v_r = 0$. Thus, u and v are constants in $\mathbb{D}\backslash\{0\}$. Thus, f is a constant in $\mathbb{D}\backslash\{0\}$ and by continuity f is constant in $\mathbb{D}$.

5. Show that the function $f(z) = \bar{z}$ is not differentiable at any point of $\mathbb{C}$. Find the points of differentiability of the function $f_a(z) = (z - a)\mathrm{Re}(z - a)$ for a given $a \in \mathbb{C}$.

   Soln: Using CR equations, we can show that $f(z) = \bar{z}$ is not differentiable at any $z \in \mathbb{C}$.

   For $a = a_1 + ia_2$, we have $f_a(z) = u(x, y) + iv(x, y)$ where $u(x, y) = (x - a_1)^2$ and $v(x, y) = (x - a_1)(y - a_2)$. Hence, $u_x = 2(x - a_1)$, $v_y = (x - a_1)$, $u_y = 0$ and $v_x = (y - a_2)$. Hence, CR equations are satisfied only at $z = a$. Further, u_x, u_y, v_x and v_y are continuous at $z = a$. Hence, the function is differentiable at $z = a$.

6. Let U be an open set and $f : U \to \mathbb{C}$ be a differentiable function. Let $\bar{U} = \{\bar{z} : z \in \mathbb{C}\}$. Show that $g : \bar{U} \to \mathbb{C}$ defined by $g(z) := \overline{f(\bar{z})}$ is differentiable on $\bar{U}$.

   Soln: From definition, we have, for $z_0 \in \bar{U}$:

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} \qquad \text{where } z \in \bar{U}$$

   Thus $\bar{z}, \bar{z}_0 \in U$ and we get

$$g'(z_0) = \lim_{z \to z_0} \frac{\overline{f(\bar{z})} - \overline{f(\bar{z}_0)}}{z - z_0} = \lim_{z \to z_0} \overline{\left(\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}\right)} = \overline{\left(\lim_{z \to z_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}\right)} = \overline{f'(\bar{z}_0)}.$$

   Hence, the result. $\Big($Here we have used the fact that $\lim_{z \to z_0} \overline{g(z)} = \overline{\lim_{z \to z_0} g(z)}$.$\Big)$

7. Show that the following functions satisfy CR equations at $z = 0$, but they are not differentiable at $z = 0$.

   (a)

$$f(z) = \begin{cases} \frac{z^5}{|z|4} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0. \end{cases}$$

Soln:

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$
$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$v_y(0,0) = \lim_{h\to 0} \frac{v(0,h)-v(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Hence CR equations are satisfied at $z = 0$. For differentiability

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = 1 \quad \text{along } z = x \qquad \text{and } \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = -1 \quad \text{along } z = x+ix.$$

Hence, the function is not differentiable at $z = 0$.

(b) $f(z) = \sqrt{|xy|}$

Soln:

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$v_y(0,0) = \lim_{h\to 0} \frac{v(0,h)-v(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

Hence CR equations are satisfied at $z = 0$. For differentiability

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = 0 \quad \text{along } z = x \qquad \text{and } \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \frac{1}{1+i} \quad \text{along } z = x+ix,\ x > 0.$$

Hence, the function is not differentiable at $z = 0$.

8. Let Ω be an open connected subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a differentiable function. Show that $f = u + iv$ is constant for any of the following conditions:
(a) either of the functions u or v is constant

Soln: If u is constant, then $u_x = u_y = 0$ and CR equations give $v_x = v_y = 0$. Thus, v is constant and hence $f = u + iv$ is constant.

(b) $|f(z)|$ is constant for all $z \in \Omega$

Soln: This gives $u^2 + v^2 =$ constant. If the constant is zero, then nothing to prove. Let $u^2 + v^2$ be a nonzero constant. Now we have $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. Applying CR equations we get $uu_x - vu_y = 0$ and $vu_x + uu_y = 0$. Solving we get $u_x = u_y = 0$ and hence $v_x = v_y = 0$. Thus u and v and hence $f = u + iv$ are constants.

(c) if there exists an $\alpha \in \mathbb{R}$ such that $f(z) = |f(z)|e^{i\alpha}$ for all $z \in \Omega$

Soln: Let $g(z) = e^{-i\alpha} f(z) = |f(z)|$. Then $g(z)$ is differentiable and Im($g(z)$)=0. Hence, $g(z)$ must be a constant and so $f(z)$ is also constant.

9. Show that the function $f(z) = (2-x^2-y^2)(x-iy)$ has derivative only on the points of the circle $x^2+y^2=1$.

Soln: Here $u = (2-x^2-y^2)x$ and $v = -(2-x^2-y^2)y$. Now $u_x = 2-3x^2-y^2$, $u_y = -2xy$, $v_x = 2xy$ and $v_y = -2+x^2+3y^2$. Clearly u_x, u_y, v_x, v_y are continuous and they satisfy C-R equations provided $u_x = v_y$ and $u_y = -v_x$. The last relation is satisfied automatically and the first relation gives $x^2+y^2=1$.

10. Does there exist an analytic function $f(z) = u+iv$ where $u(x,y)$ is given by (a) x^2y (b) $e^x\cos(x-y)$ (c) $e^x \sin y$

Soln: (a) $u = x^2y$ and (b) $u = e^x\cos(x-y)$ are not harmonic and hence conjugate v does not exist.

(c) $u = e^x \sin y$ is harmonic and then $f(z) = u+iv$ is analytic. Then $v_y = u_x = e^x \sin y$ and $v_x = -u_y = -e^x \cos y$. From the first, we get $v(x,y) = -e^x\cos y + g(x)$ and from the second, $g'(x) = 0$ and thus $v(x) = -e^x \cos y+$ consnt. Hence, $f(z) = -ie^z+$ constant.

11. If $f(z)$ is an analytic function, then show that $\nabla^2|f(z)|^2 = 4|f'(z)|^2$.

Soln: Let $\phi(x,y) = |f(z)|^2 = u^2+v^2$. Now $\phi_x = 2uu_x + 2vv_x$ and

$$\phi_{xx} = 2(u_x^2+v_x^2) + 2(uu_{xx}+vv_{xx}), \quad \phi_{yy} = 2(u_y^2+v_y^2)+2(uu_{yy}+vv_{yy})$$

Since u and v are harmonic, we get $\phi_{xx}+\phi_{yy} = 4|f'(z)|^2$

12. Find the domain in which the function $f(z) = |\text{Re } z^2| + i|\text{Im } z^2|$ is analytic.

Soln: Let $z = re^{i\theta} \implies z^2 = r^2e^{2i\theta}$. Hence, $|\text{Re } z^2| = r^2|\cos 2\theta|$ and $|\text{Im } z^2| = r^2|\sin 2\theta|$. Hence,

$$f(z) = \begin{cases} z^2 & \text{for} \quad 0<\theta<\pi/4,\ \pi<\theta<5\pi/4 \\ -\bar{z}^2 & \text{for} \quad \pi/4<\theta<\pi/2,\ 5\pi/4<\theta<3\pi/2 \\ -z^2 & \text{for} \quad \pi/2<\theta<3\pi/4,\ 3\pi/2<\theta<7\pi/4 \\ \bar{z}^2 & \text{for} \quad 3\pi/4<\theta<\pi,\ 7\pi/4<\theta<2\pi \end{cases}$$

Hence, the function is analytic in the domain $0<\theta<\pi/4$, $\pi<\theta<5\pi/4$, $\pi/2<\theta<3\pi/4$, $3\pi/2<\theta<7\pi/4$. Note that along the rays $\theta = 0, \pi/4, \cdots$, either real or imaginary part of f is zero and hence C-R equations are not satisfied there.