Solution of Assignment #1 (02/08/2016)

→ gradient at the top of the hill = 0.

Problem 1.1 (a) At the top of the ~~tob~~ hill, $\vec{\nabla} h(x,y) = 0$

$$\Rightarrow \frac{\partial}{\partial x} h(x,y)\hat{x} + \frac{\partial}{\partial y} h(x,y)\hat{y} = h(x,y).\frac{1}{60}.\left[(2y-6x-18)\hat{x} + (2x-8y+28)\hat{y}\right]$$

$$= 0$$

or,

$$\left.\begin{array}{l} 2y-6x-18=0 \\ 2x-8y+28=0 \end{array}\right] \Rightarrow x=-2,\ y=3.$$

So, the top of the hill is located at $(-2,3)$.

(b) Height of the hill = height of the top

$$= h(-2,3) = e^{11/12} \simeq 2.5 \text{ unit}.$$

(c) The steepest slope at any point is in the direction of the gradient $\vec{\nabla} h(x,y)$.

The gradient vector at (1,1) is $\vec{\nabla} h(x,y)\Big|_{1,1} = h(1,1).\frac{22}{60}(-\hat{x}+\hat{y})$.

The direction of this gradient vector is given by $\tan\theta = -1 \Rightarrow \theta = 135°.(-45°)$

So, the slope is steepest in the direction $\theta = 135°$. (

(d) The slope of $h(x,y)$ at (1,1) in the direction $\hat{n}$ ~~is~~, $(\hat{x}\cancel{/}+\hat{y}\cancel{/})$ is,

$$\vec{\nabla} h(1,1).n = h(1,1)\frac{22}{60}(-\hat{x}+\hat{y}).(\hat{x}+\hat{y}) = 0.$$

$$* \quad dT = (\vec{\nabla} T).(d\vec{l}) = |\vec{\nabla} T|.|d\vec{l}|.\cos\theta$$

for $\theta = 0$, $\quad dT = |\vec{\nabla} T||d\vec{l}|$

$\Rightarrow$ The gradient $\vec{\nabla} T$ points in the same direction as maximum slope.

(2)

#2 separation vector $\vec{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$

$$\& \quad \mathscr{r} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

(a) $\vec{\nabla}(\mathscr{r}^2) = \frac{\partial}{\partial x}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]\hat{x} + \frac{\partial}{\partial y}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]\hat{y}$

$$+ \frac{\partial}{\partial z}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]\hat{z}$$

$$= 2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z} = 2\vec{\mathscr{r}}.$$

(b) $\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \frac{\partial}{\partial x}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}\hat{x} + \frac{\partial}{\partial y}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}\hat{y}$

$$+ \frac{\partial}{\partial z}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}\hat{z}$$

$$= -\frac{1}{2}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-3/2} 2(x-x')\hat{x} - \frac{1}{2}\left[\cdots\right]^{-3/2} 2(y-y')\hat{y}$$

$$- \frac{1}{2}\left[\cdots\right]^{-3/2} 2(z-z')\hat{z}$$

$$= -\left[\cdots\right]^{-3/2}\left[(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\right] = -\frac{1}{\mathscr{r}^3}\vec{\mathscr{r}}$$

$$= -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

(c) General form of $\vec{\nabla}(\mathscr{r}^n)$.

$$\frac{\partial}{\partial x}(\mathscr{r}^n) = n.\mathscr{r}^{n-1}.\frac{\partial \mathscr{r}}{\partial x} = n\mathscr{r}^{n-1}.\left[\frac{1}{2}.\frac{1}{\mathscr{r}}.2\mathscr{r}_x\right] = n\mathscr{r}^{n-1}\hat{\mathscr{r}}_x$$

$$= n\mathscr{r}^{n-1}\left(\frac{\vec{\mathscr{r}}_x}{\mathscr{r}}\right)$$

so, in general,

$$\frac{\partial}{\partial y}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\left(\frac{\vec{\mathscr{r}}_y}{\mathscr{r}}\right)$$

$$\frac{\partial}{\partial y}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\left(\frac{\vec{\mathscr{r}}_z}{\mathscr{r}}\right)$$

$$\vec{\nabla}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\hat{\mathscr{r}}$$

$$\frac{\partial \mathscr{r}}{\partial x} = \frac{\partial}{\partial x}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2} = \frac{1}{2}\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \cdot 2(x-x')\hat{x}$$

$$= \frac{1}{2}.\frac{1}{\mathscr{r}}.2\vec{\mathscr{r}}_x$$

#3 After rotation,

$$\bar{y} = y\cos\phi + z\sin\phi \quad \text{—(1)}$$

& $$\bar{z} = -y\sin\phi + z\cos\phi. \quad \text{—(2)}$$

(1)× $\sin\phi$

$$\Rightarrow \quad \begin{aligned} \bar{y}\sin\phi &= y\cos\phi\sin\phi + z\sin^2\phi \\ \bar{z}\cos\phi &= -y\sin\phi\cos\phi + z\cos^2\phi \end{aligned}$$

$$\Rightarrow \bar{y}\sin\phi + \bar{z}\cos\phi = z.$$

similarly, (2× sinφ) → $\bar{y}\cos\phi - \bar{z}\sin\phi = y.$

So, $\dfrac{\partial y}{\partial \bar{y}} = \cos\phi, \; \dfrac{\partial y}{\partial \bar{z}} = -\sin\phi, \; \dfrac{\partial z}{\partial \bar{y}} = \sin\phi, \; \dfrac{\partial z}{\partial \bar{z}} = \cos\phi.$

So, $$\overline{(\vec{\nabla} f)_y} = \frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \bar{y}}$$

$$= \cos\phi\,(\vec{\nabla} f)_y + \sin\phi\,(\vec{\nabla} f)_z$$

& $$\overline{(\vec{\nabla} f)_z} = \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \bar{z}} = -\sin\phi.\,(\vec{\nabla} f)_y + \cos\phi\,(\vec{\nabla} f)_z$$

So, $\vec{\nabla} f$ also transforms as vector.

#4 To the fixed observer, unit vectors $\hat{i}, \hat{j}, \hat{k}$ also change with time. So,

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

$$\Rightarrow \frac{d\vec{A}}{dt} = \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k} + A_1 \frac{d\hat{i}}{dt} + A_2 \frac{d\hat{j}}{dt} + A_3 \frac{d\hat{k}}{dt}.$$

$$\Rightarrow \left.\frac{d\vec{A}}{dt}\right|_f = \left.\frac{d\vec{A}}{dt}\right|_m + A_1 \frac{d\hat{i}}{dt} + A_2 \frac{d\hat{j}}{dt} + A_3 \frac{d\hat{k}}{dt}.$$

$\frac{d\hat{i}}{dt}$ is perpendicular to $\hat{i}$ & lies in the plane of $\hat{j}$ & $\hat{k}$. So,

$$\frac{d\hat{i}}{dt} = \alpha_1 \hat{j} + \alpha_2 \hat{k} \quad = \alpha_1 \hat{j} + \alpha_2 \hat{k}$$

$$\frac{d\hat{j}}{dt} = \alpha_3 \hat{k} + \alpha_4 \hat{i} \quad = \alpha_3 \hat{k} - \alpha_1 \hat{i}$$

$$\frac{d\hat{k}}{dt} = \alpha_5 \hat{i} + \alpha_6 \hat{j} \quad = -\alpha_2 \hat{i} - \alpha_3 \hat{j}$$

$A \cdot A =$ Constant

$\frac{d}{dt}(A \cdot A) = A \cdot \frac{dA}{dt} + \frac{dA}{dt} \cdot A = 0$

$= 2A \cdot \frac{dA}{dt} = 0$

$\Rightarrow A \cdot \frac{dA}{dt} = 0.$

$\Downarrow$

A & $\frac{dA}{dt}$ are perpendicular if $\left|\frac{dA}{dt}\right| \neq 0$

$$\Rightarrow \hat{i} \cdot \hat{j} = 0 \Rightarrow \hat{i} \cdot \frac{d\hat{j}}{dt} + \frac{d\hat{i}}{dt} \cdot \hat{j} = 0.$$

& $\hat{i} \cdot \frac{d\hat{j}}{dt} = \alpha_4$ & $\hat{j}\frac{d\hat{i}}{dt} = \alpha_1 \Rightarrow \alpha_4 = -\alpha_1$

similarly & $\alpha_6 = -\alpha_3$

$\alpha_5 = -\alpha_2.$

So,
$$A_1 \frac{d\hat{i}}{dt} + A_2 \frac{d\hat{j}}{dt} + A_3 \frac{d\hat{k}}{dt} = (-\alpha_1 A_2 - \alpha_2 A_3)\hat{i} + (\alpha_1 A_1 - \alpha_3 A_3)\hat{j} + (\alpha_2 A_1 + \alpha_3 A_2)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha_3 & -\alpha_2 & \alpha_1 \\ A_1 & A_2 & A_3 \end{vmatrix}.$$

If we choose, $\alpha_3 = \omega_1$, $-\alpha_2 = \omega_2$ & $\alpha_1 = \omega_2$, then the determinant is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \vec{\omega} \times \vec{A} \qquad \left[\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}\right]$$

$$\Rightarrow \left.\frac{d\vec{A}}{dt}\right|_f = \left.\frac{d\vec{A}}{dt}\right|_m + (\vec{\omega} \times \vec{A}).$$

#5 $\vec{V} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ (5)
$\vec{V} = (-4x-3y+az)\hat{i} + (bx+3y+5z)\hat{j} + (4x+cy+3z)\hat{k}$

(a) $\text{curl}\,\vec{V} = \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$= (c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k}$

$= (c-5)\hat{i} + (a-4)\hat{j} + (b+3)\hat{k}$

$\boxed{a=4,\ b=-3,\ c=5}$

This equals zero when $a=4$, $b=2$ & $c=-1$.

(b) Assume $\vec{V} = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

So, $\frac{\partial\phi}{\partial x} = x+2y+4z$, $\frac{\partial\phi}{\partial y} = 2x-3y-z$, $\frac{\partial\phi}{\partial z} = 4x-y+2z$

Integrating them,
$$\left.\begin{aligned} \phi &= \frac{x^2}{2} + 2xy + 4zx + f(y,z) \\ \phi &= 2xy - \frac{3y^2}{2} - yz + g(x,z) \\ \phi &= 4xz - yz + z^2 + h(x,y). \end{aligned}\right\}$$

Comparing above three tells that there will be a common value of ϕ if we choose,

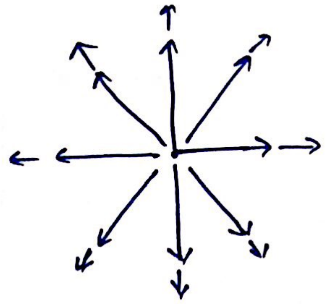
$$f(y,z) = -\frac{3y^2}{2} + z^2$$
$$g(x,z) = \frac{x^2}{2} + z^2$$
$$h(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

So, $\underline{\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz.}$

Note that we can also add any constant to ϕ. In general if $\vec{\nabla}\times\vec{V}=0$, then we can find ϕ so that $\vec{V} = \vec{\nabla}\phi$. A vector field $\vec{V}$ which can be derived from a scalar field ϕ so that $\vec{V} = \vec{\nabla}\phi$ is called conservative vector field and ϕ is called the scalar potential.

Also note that conversely if $\vec{V} = \vec{\nabla}\phi$, then $\vec{\nabla}\times\vec{V} = 0$.

#6 $\vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$



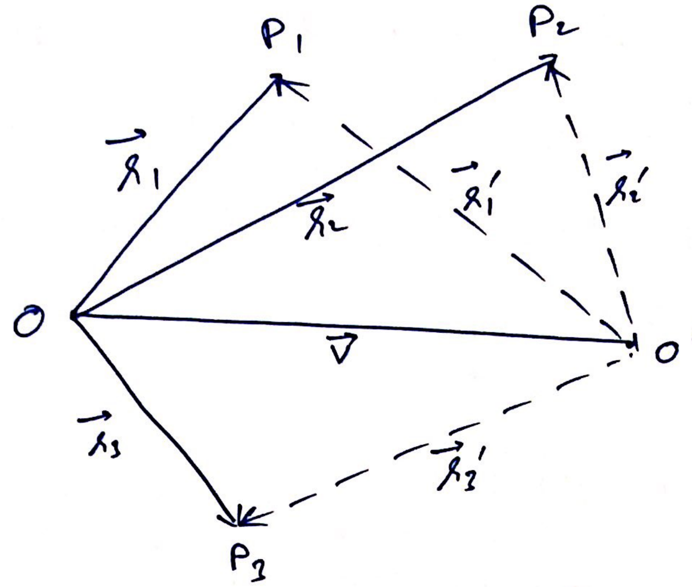
$$= \frac{\partial}{\partial x}\left[x\,(x^2+y^2+z^2)^{-3/2}\right] + \frac{\partial}{\partial y}\left[y\,(x^2+y^2+z^2)^{-3/2}\right] + \frac{\partial}{\partial z}\left[z\,(x^2+y^2+z^2)^{-3/2}\right]$$

$$= (x^2+y^2+z^2)^{-3/2} + x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2x$$
$$+ (x^2+y^2+z^2)^{-3/2} + y\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2y$$
$$+ (x^2+y^2+z^2)^{-3/2} + z\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2z$$

$$= 3r^{-3} - 3r^{-5}(x^2+y^2+z^2) = 3r^{-3} - 3r^{-3} = 0.$$

This conclusion is surprising because from the diagram the vector field is diverging away from the origin. The reason is that $\vec{\nabla}\cdot\vec{V}=0$ everywhere except at the origin. But it is difficult to calculat at the origin because the function blows up.

#7



From the figure, $\vec{r}_1 = \vec{v} + \vec{r}_1'$, $\vec{r}_2 = \vec{v} + \vec{r}_2'$, $\vec{r}_3 = \vec{v} + \vec{r}_3'$

& given that $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$.

$$\Rightarrow a_1 r_1 + a_2 r_2 + a_3 r_3 = a_1(v + r_1') + a_2(v + r_2') + a_3(v + r_3')$$

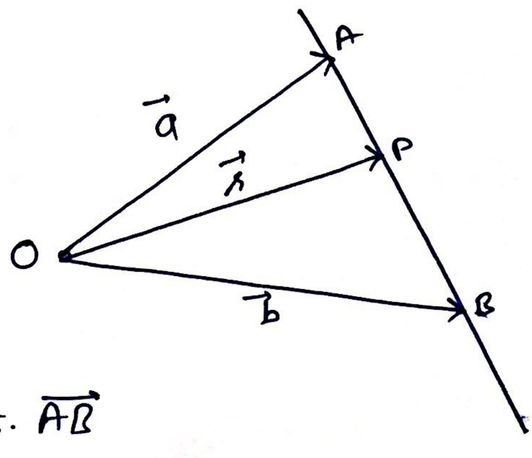
$$= (a_1 + a_2 + a_3)v + a_1 r_1' + a_2 r_2' + a_3 r_3' = 0.$$

The result $a_1 r_1' + a_2 r_2' + a_3 r_3' = 0$ will hold if and only if

$$(a_1 + a_2 + a_3).v = 0 \Rightarrow \underline{\underline{a_1 + a_2 + a_3 = 0}}$$

#8

From figure,



$\vec{OA} + \vec{AP} = \vec{OP} = \vec{a} + \vec{AP} = \vec{r}$

$\Rightarrow \vec{AP} = \vec{r} - \vec{a}$

similarly, $\vec{AB} = \vec{b} - \vec{a}$

since AP & AB are collinear, $\vec{AP} = t.\vec{AB}$

$\Rightarrow \vec{r} - \vec{a} = t(\vec{b} - \vec{a})$

or, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) \Rightarrow \vec{r} = (1-t)\vec{a} + t\vec{b}$

If the equation is written $(1-t)\vec{a} + t\vec{b} - \vec{r} = 0$, the sum of the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{r}$ is $1-t+t-1 = 0$. Hence the point P is always on the line joining A and B and does not depend on the choice of origin O.

#9

$$\vec{r}_1 = 2\hat{i} + 4\hat{j} - 5\hat{k} \quad , \quad \vec{r}_2 = -\hat{i} - 2\hat{j} + 3\hat{k}$$

Resultant $\vec{R} = \vec{r}_1 + \vec{r}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

$$R = |\vec{R}| = |\hat{i} + 2\hat{j} - 2\hat{k}| = \sqrt{9} = 3.$$

Then a unit vector parallel to $\vec{R}$ is $\dfrac{\vec{R}}{R} = \dfrac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}$

$$\vec{u} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$$

CHECK: $\left|\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1.$