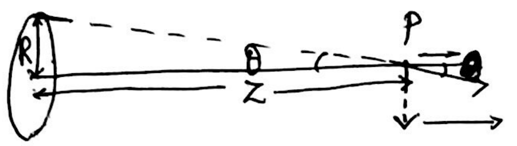
PHY 103A　　　　Assignment - 4

4.1

Magnitude of electric field due to a positive unit charge is given as;

$$f(r) = \frac{1}{r^{2+\epsilon}} \quad (0 < \epsilon << 1)$$

a) Field due to a uniformly charged ring:



This component will be cancelled out

Let charge per unit length $= \lambda$

So, Field at point P due to a small length dl will be

$$dE(r) = f(r)\,\lambda\, dl \cos\theta$$

$$= \frac{\lambda\, dl}{(\sqrt{R^2+z^2})^{2+\epsilon}} \cdot \frac{z}{\sqrt{R^2+z^2}}$$

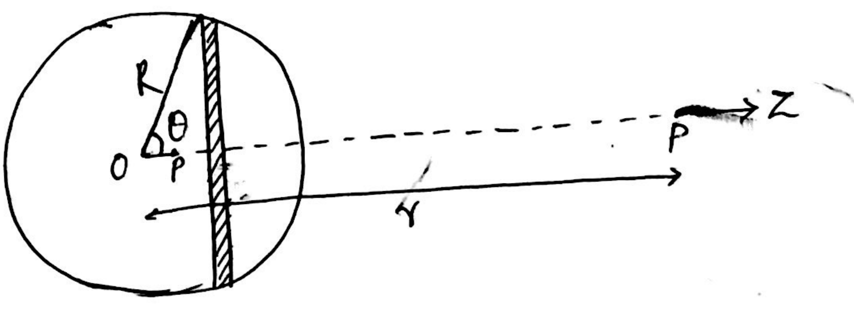
So, total field due to the ring will be

$$\vec{E}(r) = \int \frac{\lambda\, dl}{(\sqrt{R^2+z^2})^{2+\epsilon}} \cdot \frac{z}{\sqrt{R^2+z^2}}$$

$$= \frac{Q \cdot z}{(\sqrt{R^2+z^2})^{3+\epsilon}}$$

$$= \frac{Qz}{(R^2+z^2)^{\frac{3+\epsilon}{2}}}\,\hat{z}.$$

b) Electric field due to a uniformly charged shell:



Take a ring on the spherical shell as shown by shaded region in the figure.

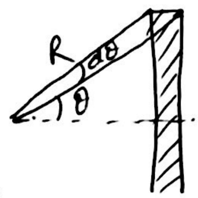
The radius of the ring $= R\sin\theta$

Let's assume σ be the surface charge density.

So, total charge due to the ring on the spherical shell $= \sigma$. area of the ring

circumference of the ring $= (2\pi R\sin\theta$

width ~~Circumference~~ of the ring $= \cancel{Rd\theta}\ R d\theta$



Area of the ring $= 2\pi R^2 \sin\theta \, d\theta$

So, total charge due to the ring $= 2\pi R^2 \sin\theta \, d\theta . \sigma$

Using relation from (a);

$$dE(r) = \frac{2\pi R^2 \sin\theta \, d\theta \, \sigma \, (r - R\cos\theta)}{\left[R^2\sin^2\theta + (r - R\cos\theta)^2\right]^{(3+\epsilon)/2}}$$

This relation is valid whether the point P is left or right to the ring. $(r - R\cos\theta)$ takes care of it.

So, total field

$$E(r) = 2\pi R^2 \sigma \int_0^{\pi} \frac{(r - R\cos\theta)\sin\theta \, d\theta}{\left[R^2\sin^2\theta + (r - R\cos\theta)\right]^{(3+\epsilon)/2}}$$

$$= 2\pi R^2 \sigma \int_1^{-1} \frac{(r - R\cos\theta)(-d(\cos\theta))}{\left[R^2\sin^2\theta + (r - R\cos\theta)\right]^{(3+\epsilon)/2}}$$

$$E(r) = 2\pi R^2 \sigma \int_{-1}^{1} \frac{(r - R\cos\theta)\, d(\cos\theta)}{\left[R^2 + r^2 - 2rR\cos\theta\right]^{(3+\epsilon)/2}}$$

Let's make a trick in solving this integral;

$$\frac{r - R\cos\theta}{\left[R^2 + r^2 - 2rR\cos\theta\right]^{(3+\epsilon)/2}} = -\frac{\partial}{\partial r}\left\{\frac{1}{(1+\epsilon)} \frac{1}{(R^2 + r^2 - 2rR\cos\theta)^{(1+\epsilon)/2}}\right\}$$

Let's see this differential,

$$\frac{\partial}{\partial r}\frac{1}{(r^2+R^2-2rR\cos\theta)^{(1+\epsilon)/2}} = \cancel{1+\epsilon} - \frac{(1+\epsilon)}{2}\cdot\frac{2(r-R\cos\theta)}{(r^2+R^2-2rR\cos\theta)^{\frac{(3+\epsilon)}{2}}}$$

$$= -\frac{(1+\epsilon)(r-R\cos\theta)}{(r^2+R^2-2rR\cos\theta)^{(3+\epsilon)/2}}$$

$$\frac{r-R\cos\theta}{[r^2+R^2-2rR\cos\theta]^{(3+\epsilon)/2}} = -\frac{\partial}{\partial r}\left[\frac{1}{(1+\epsilon)}\frac{1}{[r^2+R^2-2rR\cos\theta]^{(1+\epsilon)/2}}\right]$$

$$E(r) = \frac{-2\pi R^2\sigma}{(1+\epsilon)}\frac{\partial}{\partial r}\left[\int_{-1}^{1}\frac{d(\cos\theta)}{[R^2+r^2-2rR\cos\theta]^{(1+\epsilon)/2}}\right]$$

$$= \frac{-2\pi R^2\sigma}{(1+\epsilon)}\frac{\partial}{\partial r}\left\{-\frac{1}{2\pi R}\frac{2}{(1-\epsilon)}(R^2+r^2-2rR\cos\theta)^{(1-\epsilon)/2}\right\}_{-1}^{1}$$

$$= \frac{2\pi R\sigma}{(1-\epsilon^2)}\frac{\partial}{\partial r}\left\{\frac{|R-r|^{1-\epsilon}-(R+r)^{1-\epsilon}}{r}\right\}$$

For small ϵ

$$x^{1\pm\epsilon} = x \pm \epsilon x\ln x \qquad (\text{upto linear } \epsilon \text{ Terms})$$

$$E(r) = \frac{2\pi R\sigma}{1-\epsilon^2}\frac{\partial}{\partial r}\left\{\frac{|R-r|-\epsilon|R-r|\ln|R-r|-(R+r)+\epsilon(R+r)\ln(R+r)}{r}\right\}$$

$$= \frac{2\pi R\sigma}{1-\epsilon^2}\frac{\partial}{\partial r}\left\{\frac{|R-r|-(R+r)}{r}+\frac{\epsilon(R+r)\ln(R+r)-|R-r|\ln|R-r|}{r}\right\}$$

upto linear ϵ Terms;

$$E(r) = 2\pi R\sigma\frac{\partial}{\partial r}\left[\frac{|R-r|-(R+r)}{r}+\frac{\epsilon(R+r)\ln(R+r)-|R-r|\ln|R-r|}{r}\right.$$

Now we find field for $r<R$ and $r>R$

<u>$r<R$</u> $\quad |R-r| - (R+r) = R-r-R-r = -2r$

$$(R+r)\ln(R+r) - |R-r|\ln|R-r| = R\ln(R+r) + r\ln(R+r) - R\ln(R-r) + r\ln(R-r)$$

So, $$E(r) = 2\pi R\sigma \frac{\partial}{\partial r}\left[-\frac{2r}{r} + \epsilon\left\{\frac{R}{r}\ln(R+r) - \frac{R}{r}\ln(R-r) + \ln(R+r) + \ln(R-r)\right]\right.$$

$$= 2\pi R\sigma\epsilon\left[-\frac{R}{r^2}\ln(R+r) + \frac{R}{r(R+r)} + \frac{R}{r^2}\ln(R-r) + \frac{R}{r(R-r)} + \frac{1}{(R+r)} - \frac{1}{(R-r)}\right]$$

$$= 2\pi R\sigma\epsilon\left[\frac{R}{r(R+r)} + \frac{R}{r(R-r)} + \frac{1}{(R+r)} - \frac{1}{(R-r)} - \frac{R}{r^2}\ln\left(\frac{R+r}{R-r}\right)\right]$$

$$= 2\pi R\sigma\epsilon\left[\frac{R^2 - \cancel{rR} + R^2 + rR + \cancel{rR} - r^2 - \cancel{rR} - r^2}{r(R+r)(R-r)} - \frac{R}{r^2}\ln\left(\frac{R+r}{R-r}\right)\right]$$

$$= 2\pi R\sigma\epsilon\left[\frac{2}{r} - \frac{R}{r^2}\ln\left(\frac{R+r}{R-r}\right)\right]$$

$$\boxed{E(r) = \frac{4\pi R\sigma\epsilon}{r}\left[1 - \frac{R}{2r}\ln\left(\frac{R+r}{R-r}\right)\right]}$$

Thus the field is proportional to ϵ.

~~For $r\to 0$, $E(r)\to 0$ & $r\to R$, $E(r)\to\infty$~~

~~So, the quantity in the bracket becomes 1~~

$\frac{r}{R} = x$

$\ln\left(\frac{1+x}{1-x}\right) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \left(-x - \frac{x^2}{2} - \cdots\right)$

$= 2x + \frac{2x^3}{3}$

$E(x) = \frac{4\pi R\sigma\epsilon}{x}\left[1 - \frac{1}{2x}\left(2x + \frac{2x^3}{3}\right)\right]$

$= -\frac{4\pi R\sigma\epsilon}{x}\cdot\frac{x^2}{3}$

$= -\frac{\epsilon Q x}{3R^3}$

So, $$E(r)\big|_{r\to 0} = \cancel{\frac{4\pi R\sigma\epsilon}{r}} = -\frac{\epsilon Q r}{3R^3}$$

So the field varies linearly with r.

$\underline{\underline{r > R}}$

$$|R-r| - (R+r) = r - R - R - r = -2R$$

$$(R+r)\ln(R+r) - |R-r|\ln|R-r| = (R+r)\ln(R+r) - (r-R)\ln(r-R)$$

So, the field is.

$$E(r) = 2\pi R\sigma \frac{\partial}{\partial r}\left[\frac{-2R}{r} + \epsilon\left(1+\frac{R}{r}\right)\ln(R+r) - \epsilon\left(1-\frac{R}{r}\right)\ln(r-R)\right]$$

$$= 2\pi R\sigma\left[\frac{2R}{r^2} + \epsilon\left(-\frac{R}{r^2}\right)\ln(R+r) + \epsilon\frac{\cancel{(r+R)}}{r}\frac{1}{\cancel{(r+R)}} - \epsilon\left(\frac{R}{r^2}\right)\ln(r-R) - \epsilon\frac{\cancel{(r-R)}}{r}\frac{1}{\cancel{(r-R)}}\right]$$

$$= 2\pi R\sigma\left[\frac{2R}{r^2} - \frac{\epsilon R}{r^2}\ln(r^2-R^2)\right]$$

$$= 4\pi R^2\sigma\left[\frac{1}{r^2} - \frac{\epsilon}{2r^2}\ln(r^2-R^2)\right]$$

$$\boxed{E(r) = \frac{Q}{r^2}\left[1 - \frac{\epsilon}{2}\ln(r^2-R^2)\right]}$$

$r \to R$, $E(r) \to \infty$

$r \to \infty$, $E(r) \to 0$

where Q is the total charge on the spherical shell.

<u>Question</u>: We should check if the answers are dimensionally correct or not.

Field should have dimension $\frac{Q}{r^2}$ form.

Also, for $r > R$ case, a dimensional quantity is appearing in ln function which should not be the case.

So, we have to find an alternative and better ~~approach~~ approach;

We had,

$$E(r) = \frac{2\pi R\sigma}{(1-\epsilon^2)}\frac{\partial}{\partial r}\left\{\frac{|R-r|^{1-\epsilon} - (R+r)^{1-\epsilon}}{r}\right\}$$

changing to dimensionless, $x = r/R$ we get

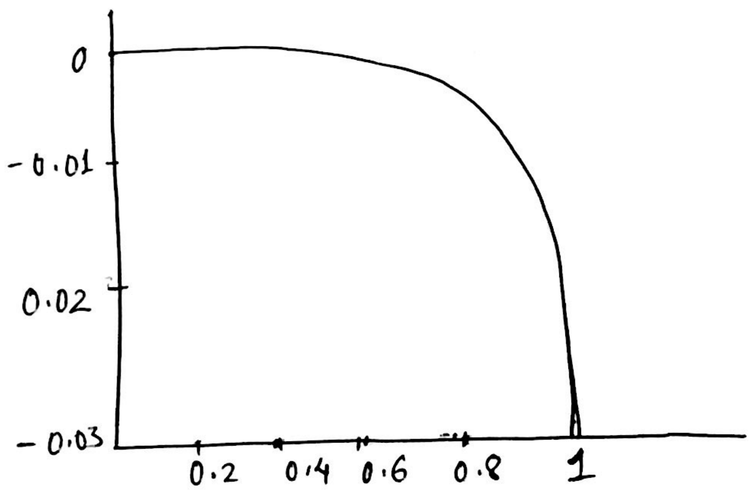
$$E(r) = \frac{2\pi R\sigma}{1-\epsilon^2} \frac{1}{R} \frac{\partial}{\partial x} \left\{ \frac{R^{1-\epsilon} |1-x|^{1-\epsilon} - R^{1-\epsilon} (1+x)^{1-\epsilon}}{Rx} \right\}$$

$$= \frac{2\pi\sigma}{(1-\epsilon^2) R^{\epsilon}} \frac{\partial}{\partial x} \left\{ \frac{|1-x|^{1-\epsilon} - (1+x)^{1-\epsilon}}{x} \right\}$$

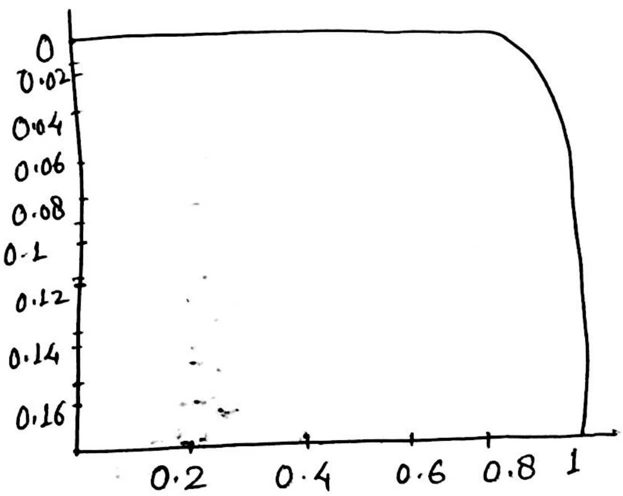
Now following the same steps as earlier, one would get

$$E(r) = \begin{cases} \frac{4\pi\sigma\epsilon}{R^{\epsilon} x} \left[1 - \frac{1}{2x} \ln\left(\frac{1+x}{1-x}\right) \right] & x < 1 \ (r < R) \\ \frac{Q}{R^{\epsilon} r^2} \left[1 - \frac{\epsilon}{2} \ln(x^2 - 1) \right] & x > 1 \ (r > R) \end{cases}$$

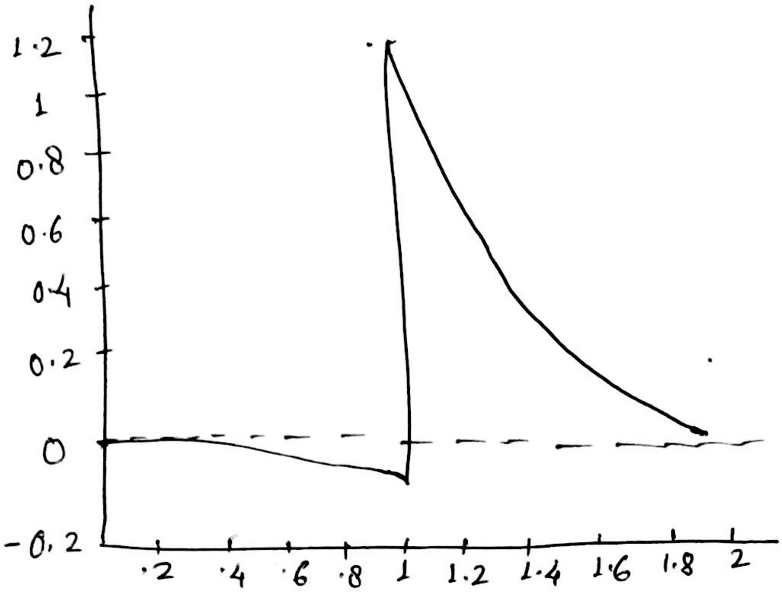
This is more accurate and quantitatively correct answer that we obtained earlier. Note that this is also dimensionally correct as field goes like $\frac{Q}{R^{(2+\epsilon)}}$ form and the log function does not have dimensional quantity as its argument.



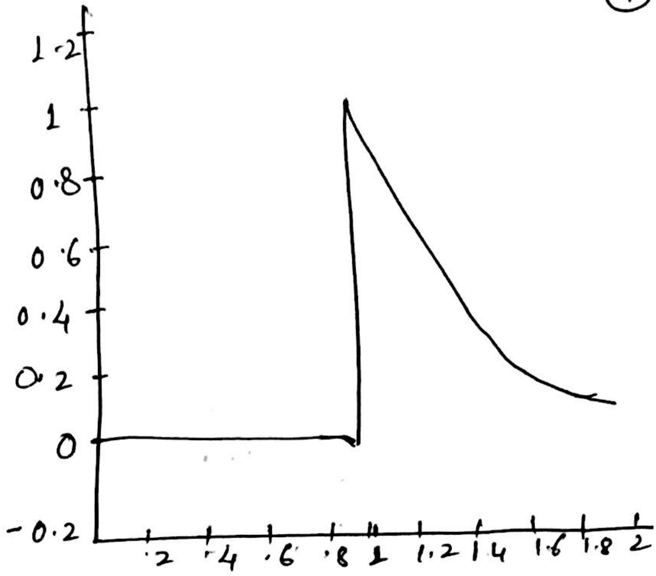
$R = 1, \ Q = 1, \ \epsilon = 0.01$



$R = 1, \ Q = 1, \ \epsilon = 0.05$

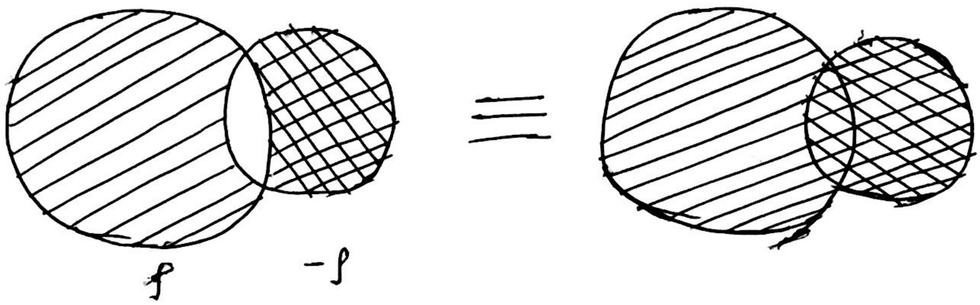


$R = 1,\ Q = 1,\ \epsilon = 0.05$



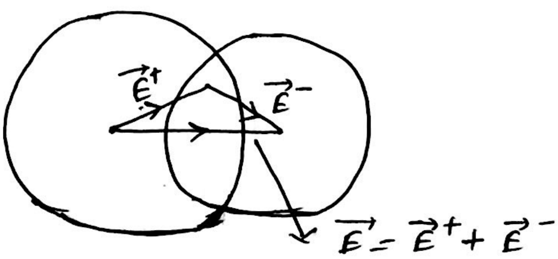
$R = 1,\ Q = 1,\ \epsilon = 0.01$

4.2



Electric field in a cavity formed by the intersection of two spheres is equivalent (electrically) to have one +vely charged sphere intersect with a -vely charged sphere. The intersection will then have zero charge making it equivalent to a cavity with no charge.

Now, from the principle of superposition, the electric field in the cavity will be given by the vector sum of the ~~fied~~ fields due to two spheres as shown in the figure;



$$E^+ \cdot A = \frac{Q}{4\pi\epsilon_0 r_1^2} \cdot 4\pi r_1^2$$

$$= \frac{Q}{\epsilon_0} = \frac{4\pi r_1^3 \rho}{3\epsilon_0} \qquad \rho = \frac{Q}{\frac{4}{3}\pi r^3}$$

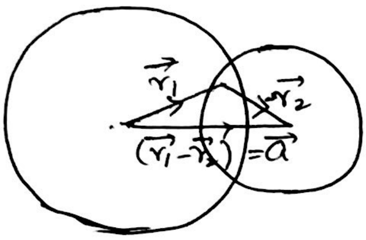
$$\Rightarrow \quad \vec{E}^+ = \frac{\rho \vec{r}_1}{3\epsilon_0}$$

Similarly, the field due to the negative charge is

$$\vec{E}^- = \frac{-\rho\vec{r}_2}{3\epsilon_0}$$

Where $\vec{r}_1$ & $\vec{r}_2$ are vectors from the center of respective spheres to the point in the cavity where the field is being calculated. So, the net field is

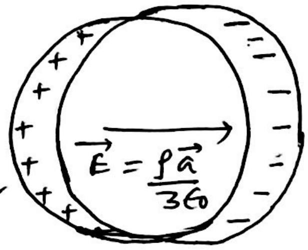
$$\vec{E} = \frac{\rho}{3\epsilon_0}(\vec{r}_1 - \vec{r}_2) \Rightarrow \vec{E} = \frac{\rho\vec{a}}{3\epsilon_0},$$ where $\vec{a}$ is the vector from the center of +vely charged sphere to the center of -vely charged sphere.

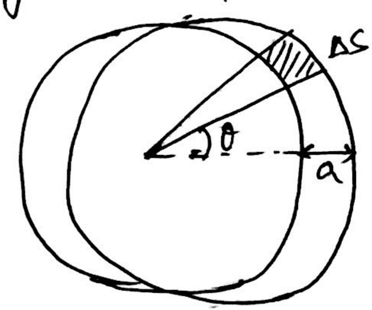


Thus, in any point inside the cavity, electric field is constant and equal to $\left(\frac{\rho\vec{a}}{3\epsilon_0}\right)$.

#4.3 Surface charge density on a spherical surface can be generated by displacing a +vely charged surface very slightly with respect to a -vely charged sphere of the same size. Their intersection will then form a cavity ~~rest~~ representing the inside of the sphere and field there will be a constant.

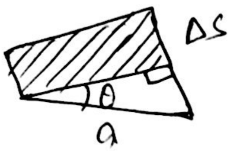
In the limit of $\vec{a} \to 0$ and $\rho \to \infty$, So that their product remains a constant, we will have a perfect surface charge density. Surface charge can be calculated as follows;





Consider a small volume element (shaded region) of base area ΔS at an angle θ from the displacement direction.

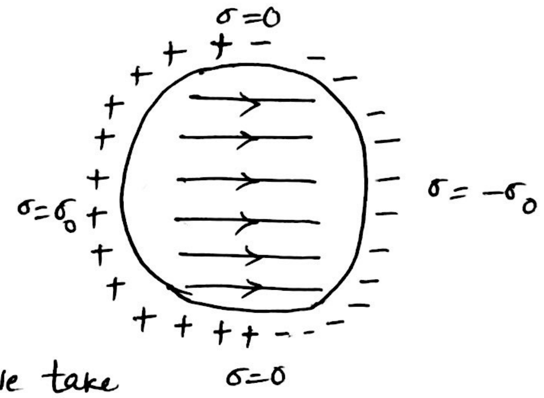
The volume of this element is;



$$\Delta V = \Delta S \times a \cos\theta$$
$$= \Delta S\, a \cos\theta$$

So, the charge in it is $\Delta q = -\rho \Delta V$
$$= -\Delta S\, a\, \rho \cos\theta$$

and it gives a surface charge density

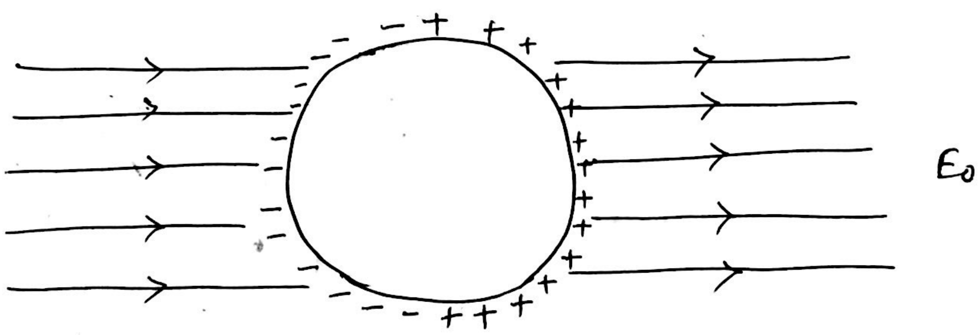


$$\sigma(\theta) = \frac{\Delta q}{\Delta S} = -a\rho\cos\theta$$
$$= -\sigma_0 \cos\theta$$

where $\sigma_0 = a\rho$ = Constant even when we take the limit $a \to 0$ & $\rho \to \infty$ and the corresponding electric field is

$$|\vec{E}| = \frac{\rho a}{3\epsilon_0} = \frac{\sigma_0}{3\epsilon_0}$$

We now use this result to find the induced charge density on a sphere when it is put in a uniform electric field.



The induced charge density should be such that it produces an exactly opposite field in the sphere so that the net field inside is zero. This gives

$$E_0 = \frac{\sigma_0}{3\epsilon_0} \Rightarrow \sigma_0 = 3\epsilon_0 E_0$$

Thus the ~~field~~ induced surface charge density will be

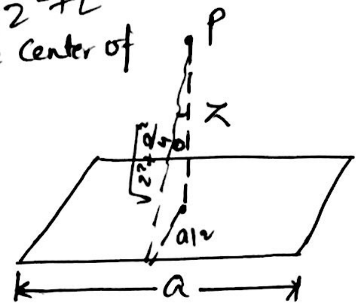
$$\boxed{\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta}$$

4.4

As electric field due to a straight line segment of length 2L carrying a uniform line charge λ is given by

$$E = \frac{1}{4\pi\epsilon_0}\cancel{\frac{2\lambda}{z}}\;\frac{2\lambda L}{z\sqrt{z^2+L^2}}\,\hat{z}$$

where z is the distance of the point P from the center of the wire.



Replacing ~~a~~ L with $a/2$ and z with $\sqrt{z^2+(a/2)^2}$ in the above expression (distance from center of one ~~side~~ edge to P),

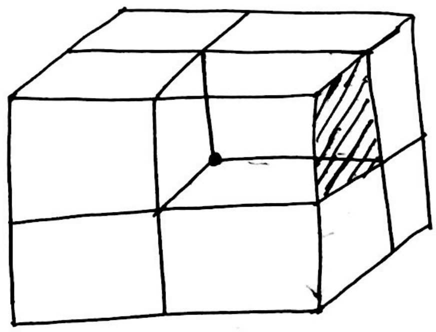
field of one edge is;

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{\lambda a}{\sqrt{z^2+\frac{a^2}{4}+\frac{a^2}{4}}\sqrt{z^2+\frac{a^2}{4}}}$$

There are 4 sides, and contribution will be due to vertical components only, so multiplying with $4\cos\theta = \frac{4z}{\sqrt{z^2+\frac{a^2}{4}}}$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{4\lambda a z}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}\,\hat{z}}$$

4.5



Thinking of the cube as one of 8 surrounding the charge. Each of the 24 squares which make up the surface of the larger cube gets the same flux as every other one.

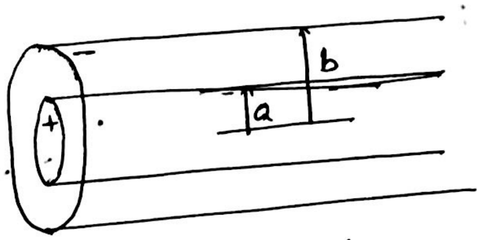
So;

$$\int_{\text{One face}} \vec{E}\cdot d\vec{a} = \frac{1}{24}\int_{\substack{\text{whole}\\ \text{large cube}}} \vec{E}\cdot d\vec{a}$$

As, $\int_{\text{Whole large Cube}} \vec{E}\cdot\vec{da} = \frac{q}{\epsilon_0}$ · by Gauss's law

So, $$\boxed{\int_{\text{One face}} \vec{E}\cdot\vec{da} = \frac{1}{24}\frac{q}{\epsilon_0}}$$

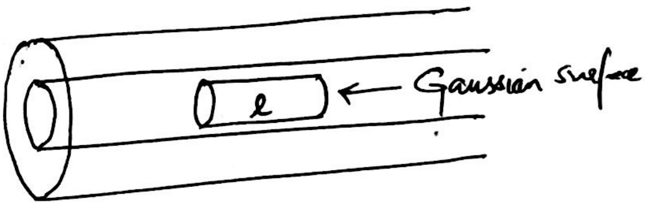
4.6



Given is a long coaxial cable having uniform volume charge density ρ on the inner cylinder and uniform surface charge density on the outer cylindrical shell.

Electric field calculation at different positions;

(i) Inside the cylinder $(s<a)$



Using Gauss law;

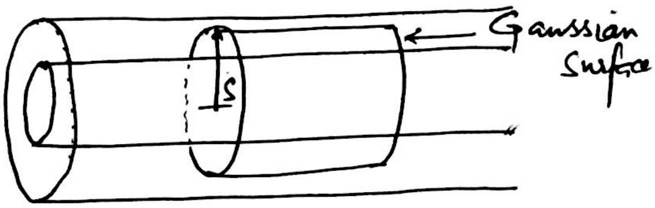
$$\oint \vec{E}\cdot\vec{da} = E.\,2\pi s.l$$

$$= \frac{Q_{enclosed}}{\epsilon_0} = \frac{1}{\epsilon_0}\rho\pi s^2 l$$

$$\Rightarrow E.\,2\pi s l = \frac{1}{\epsilon_0}\cdot\rho\pi s^2 l$$

$$\Rightarrow \boxed{\vec{E} = \frac{\rho s}{2\epsilon_0}\hat{s}}$$

(ii) Between the cylinders $(a<s<b)$

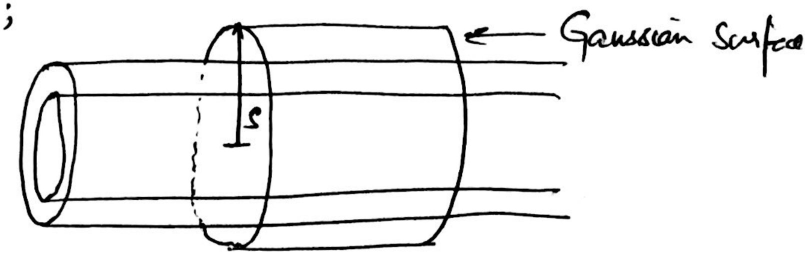


$$\oint \vec{E}\cdot\vec{da} = E.2\pi s l = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$= \frac{1}{\epsilon_0}\rho\pi a^2 l$$

$$\Rightarrow \boxed{\vec{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{s}}$$

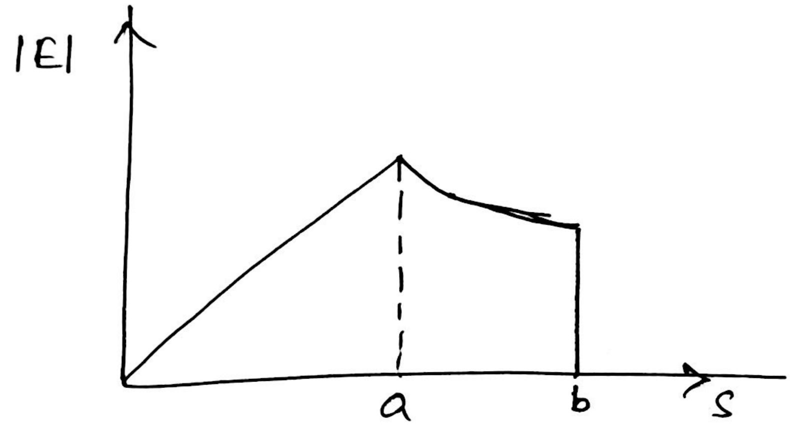
(iii) Outside the cable $(s>b)$;



$$\oint \vec{E}\cdot\vec{da} = \frac{1}{\epsilon_0} Q_{enclosed}$$

as $Q_{enclosed} = 0$

$$E \cdot 2\pi s l = 0 \Rightarrow \boxed{E = 0}$$



4.7

For a uniformly charged sphere, Electric field can be given as

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \qquad \text{Outside the sphere } (r>R)$$

and $$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3} r\hat{r}. \qquad \text{Inside the sphere } (r<R)$$

Potential at a point r can be given as

$$V(r) = -\int_\infty^r \vec{E}\cdot\vec{dl}$$

For $r>R$; $$V(r) = -\int_\infty^r \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)\vec{dr}$$

$$= \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{r}\right)\Big|_\infty^r = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

for $r<R$, $$V(r) = -\int_\infty^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)\vec{dr} - \int_R^r \left(\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\vec{r}\right)\vec{dr}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right]$$

$$V(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3-\frac{r^2}{R^2}\right)$$

Calculating the gradient of V for each region,

For $r>R$, $\vec{\nabla}V = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r}$

$$= \frac{-q}{4\pi\epsilon_0}\cdot\frac{1}{r^2}\hat{r}$$

So, $\vec{E} = -\vec{\nabla}V = \frac{q}{4\pi\epsilon_0}$

$$\Rightarrow \boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\cdot\frac{1}{r^2}\hat{r}}$$

For $r<R$, $\vec{\nabla}V = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3-\frac{r^2}{R^2}\right)\hat{r}$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{r} = -\frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{r}$$

So, $\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{r}}$

4.8 Electric field due to a infinitely long straight wire at a ~~point~~ distance s is given as

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{s}$$

where $\lambda \rightarrow$ Line charge density

As the wire is infinitely long, the refence point can not be chosen at ∞.

Let's ~~the~~ chose the reference at $s = a$.

So, $$V(s) = -\int_a^s \left(\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\right) \vec{ds}$$

$$\boxed{V(s) = \frac{-1}{4\pi\epsilon_0} 2\lambda \ln\left(s/a\right)}$$

Taking the gradient;

$$\vec{\nabla} V = \frac{-1}{4\pi\epsilon_0} 2\lambda \frac{\partial}{\partial s}\left[\ln\left(s/a\right)\right]\hat{s}.$$

$$\vec{\nabla} V = -\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{s} = -\vec{E}$$

Thus, it gives the correct field.