Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment VI

1. $\star$ Using the exapansion

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

show that the Legendre Polynomials $P_n(x)$ satisfy the following.
(i) $P_n(1) = 1$ (ii) $P_n(-1) = (-1)^n$
(iii) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
(iv) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

Let $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$. If $x = 1$, then $\sum_{n=0}^{\infty} P_n(1)t^n = \frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$. This shows that $P_n(1) = 1$ for all $n \in \mathbb{N}$.
If $x = -1$, then we get $P_n(-1) = (-1)^n$.
Differentiate $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$ with respect to t to get

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Therefore

$$\begin{aligned}
\frac{x-1}{\sqrt{1-2xt+t^2}} &= (1-2xt+t^2)\sum_{n=0}^{\infty} P_n(x)t^n \\
(x-t)\left(\sum_{n=0}^{\infty} P_n(x)t^n\right) &= \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n = \sum_{n=0}^{\infty}(n+1)P_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1}.$$

Now we compare the coefficients to get the desired equality.
Now we differentiate $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$ with respect to x to get

$$\begin{aligned}
\frac{t}{1-2xt+t^2} &= (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n \\
t\left(\sum_{n=0}^{\infty} P_n(x)t^n\right) &= \sum_{n=0}^{\infty} P_n'(x)t^n - 2\sum_{n=0}^{\infty} xP_n'(x)t^{n+1} + \sum_{n=0}^{\infty} P_n'(x)t^{n+2}.
\end{aligned}$$

By comparing the coefficients, we get $P'_n(x) - 2xP'_{n-1}(x) + P'_{n-2}(x) = P_{n-1}(x)$ for all $n \geq 1$ and therefore

$$P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) - P_n(x) = 0. \tag{1}$$

Next we differentiate $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ to get

$$(n+1)P'_{n+1}(x) = (2n+1)\left[xP'_n(x) + P_n(x)\right] - nP'_{n-1}(x). \tag{2}$$

Eliminating $P'_{n+1}(x)$ from Equations (1) and (2), we get

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x).$$

2. $\star$ Show that
   (i) $\int_{-1}^{1} x^m P_n(x)dx = 0$ if $m < n$
   (ii) $\int_{-1}^{1} x^m P_n(x)dx = 0$ if $m > n$ and $m - n$ is odd. What happens if $m - n$ is even?
   (iii) $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$.

   Let $m < n$. Then

$$\begin{aligned} \int_{-1}^{1} x^m P_n(x) &= \frac{1}{2^n n!} \int_{-1}^{1} x^m \left(\frac{d}{dx}\right)^n (x^2-1)^n \\ &= \frac{1}{2^n n!} \left[x^m \left(\frac{d}{dx}\right)^{n-1} (x^2-1)^n \mid_{-1}^{1} - \int_{-1}^{1} \frac{d}{dx} x^m \left(\frac{d}{dx}\right)^{n-1} (x^2-1)^n \right] \\ &= -\frac{1}{2^n n!} \int_{-1}^{1} \frac{d}{dx} x^m \left(\frac{d}{dx}\right)^{n-1} (x^2-1)^n. \end{aligned}$$

   Repeated integration by parts proves the result.
   Let $m > n$ and $m - n$ odd. Then

$$\frac{1}{2^n n!} \int_{-1}^{1} x^m \left(\frac{d}{dx}\right)^n (x^2-1)^n = \frac{m(m-1)\cdots(m-n+1)}{2^n n!} \int_{-1}^{1} x^{m-n}(x^2-1)^n.$$

   Since $m - n$ is odd, the function x^{m-n} is odd and therefore $\int_{-1}^{1} x^m P_n(x)dx = 0$. Let us now assume that $m - n = 2k$ and let $x = \sin\theta$ to convert the integral $I = \frac{m(m-1)\cdots(m-n+1)}{2^n n!} \int_{-1}^{1} x^{m-n}(x^2-1)^n$ in to

$$\begin{aligned} I &= (-1)^n 2 \frac{m(m-1)\cdots(m-n+1)}{2^n n!} \int_0^{\pi/2} \sin^{2k}\theta \cos^{2n+1}\theta d\theta \\ &= (-1)^n 2 \frac{m(m-1)\cdots(m-n+1)}{2^n n!} I_{k,n}. \end{aligned}$$

   It is wasy to show that $I_{k,n} = \frac{2n}{2k+1} I_{k+1,n-1}$. By repeated integration by parts, we can show that $I_{k,n} = (-1)^n \frac{2n \cdot 2(n-1)\cdots 2\cdot 1}{(2k+1)(2k+3)\cdots(2[k+n]+1)}$.
   $\int_{-1}^{1} P_n^2(x) = \frac{2}{2n+1}$ was done in the class.

3. $\star$ The Bessel function $J_p(x)$, for any real number p is defined as

$$J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!(n+p)!}.$$

Using this expression of $J_p(x)$, show that
(i) $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$ (ii) $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$
(iii) $J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$ (iv) $J_p'(x) - \frac{p}{x} J_p(x) = J_{p+1}(x)$.

Since $J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!(n+p)!}$, it follows that

$$\begin{aligned} x^p J_p(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{2^p (x/2)^{2n+2p}}{n!(n+p)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2p}}{2^{2n+p} n!(n+p)!}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dx}[x^p J_p(x)] &= \sum_{n=0}^{\infty} (-1)^n \frac{2(n+p) x^{2n+2p-1}}{2^{2n+p} n!(n+p)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2p-1}}{2^{2n+p-1} n!(n+p-1)!} \\ &= x^p \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p-1}}{n!(p-1+n)!} \\ &= x^p J_{p-1}(x). \end{aligned}$$

(ii) is similar.
$\frac{d}{dx}\left(x^p J_p(x)\right) = p x^{p-1} J_p(x) + x^p J_p'(x)$. Therefore $p x^{p-1} J_p(x) + x^p J_p'(x) = x^p J_{-p}(x)$.
This shows that $J_p'(x) + \frac{p}{x} J_p(x) = J_{-p}(x)$.
(iv) is similar using (ii).

4. $\star$ Show that, for every real number p, the Bessel function $J_p(x)$ has infinitely many positive zeros.

The substitution $u(x) = \sqrt{x} y(x)$ converts the Bessel's equation $x^2 y'' + x y' + (x^2 - p^2) y = 0$ in to $u'' + \left[\frac{1-4p^2}{4x^2} + 1\right] u = 0$. The positive zeros of y are same as the positive zeros of u.
We will now show that u has infinitely many positive zeros. Since $1/x^2 \to 0$ as $x \to \infty$, there exists $M > 0$ such that $\frac{1-4p^2}{4x^2} > -\frac{3}{4}$ for all $x \geq M$. Therefore $\frac{1-4p^2}{4x^2} + 1 \geq 1/4$ for all $x \geq M$. We can now compare the solution of our equation with the equation $u'' + \frac{1}{4} u = 0$ and arrive at the result.

5. $\star$ For a given a real number p, we let λ_n denote the positive zeros of the Bessel function $J_p(x)$. Show that
(i) $\int_0^1 xJ_p(\lambda_m x)J_p(\lambda_n x) = 0$ if $m \neq n$ and (ii) $\int_0^1 xJ_p(\lambda_m x)^2 = \frac{1}{2}\left[J_p'(\lambda_n)\right]^2 = \frac{1}{2}\left[J_{p+1}(\lambda_n)\right]^2$.

Let λ and μ be two zeros of $J_p(x)$, the Bessel's function of first kind of order p. Let $u(x) := J_p(\lambda\, x)$ and $v(x) = J_p(\mu\, x)$ for $0 \leq x \leq 1$.
Then $u'(x) = \lambda\, J_p'(\lambda\, x)$ and $u''(x) = \lambda^2 J_p''(x)$. Therefore

$$\begin{aligned}\frac{1}{\lambda^2}u''(x) + \left(\frac{1}{\lambda\, x}\right)\frac{1}{\lambda}u'(x) + \left(1 - \frac{p^2}{\lambda^2 x^2}\right)u(x) &= J_p''(\lambda\, x) + \frac{1}{\lambda\, x}J_p'(\lambda\, x)\\ &\quad + \left(1 - \frac{p^2}{\lambda^2 x^2}\right)J_p(\lambda\, x)\\ &= 0.\end{aligned}$$

This shows that $u''(x) + \frac{1}{x}u'(x) + \left(\lambda^2 - \frac{p^2}{x^2}\right)u(x) = 0$.
Similarly $v''(x) + \frac{1}{x}v'(x) + \left(\mu^2 - \frac{p^2}{x^2}\right)u(x) = 0$. Now we mulitply, the first equation by the function v and the second equation by the function u, and subtract to get

$$\left(u''v - v''u\right) + \frac{1}{x}\left(u'v - uv'\right) + \left(\lambda^2 - \mu^2\right)uv = 0.$$

By clearing the factor x from the denominator, we can write the equation as

$$x\left(u''v - v''u\right) + \left(u'v - uv'\right) + \left(\lambda^2 - \mu^2\right)xuv = 0.$$

This can be written as $[x(u'v - uv')]' = \left(\mu^2 - \lambda^2\right)xuv$. We integrate this equation from 0 to 1 to get

$$0 = \left(u'(1)v(1) - u(1)v'(1)\right) = \left(\mu^2 - \lambda^2\right)\int_0^1 xu(x)v(x)dx.$$

If $\lambda \neq \mu$, then $\int_0^1 xu(x)v(x)dx = 0$. This proves the first part.
To prove the other part, we integrate the equation from 0 to x to get

$$x\left(u'v - uv'\right) = \left(\mu^2 - \lambda^2\right)\int_0^x tu(t)v(t)dt.$$

We re write the equation as

$$x\left(\lambda\, J_p'(\lambda\, x)J_p(\mu\, x) - \mu\, J_p(\lambda\, x)J_p'(\mu\, x)\right) = \left(\mu^2 - \lambda^2\right)\int_0^x tJ_p(\lambda\, t)J_p(\mu\, t)dt.$$

We differentiate this equation w.r.t λ to get

$$x\left(J_p'(\lambda x)J_p(\mu x) + x\lambda J_p''(\lambda x)J_p(\mu x) - x\lambda J_p'(\lambda x)J_p'(\mu x)\right)$$
$$= -2\lambda \int_0^x tJ_p(\lambda t)J_p(\mu t)dt + \left(\mu^2 - \lambda^2\right) \int_0^x t^2 J_p'(\lambda t)J_p(\mu t)dt.$$

If we let $\lambda = \mu$ and $x = 1$, then we get

$$\begin{aligned}\left(J_p'(\lambda)J_p(\mu) + x\lambda J_p''(\lambda)J_p(\mu) - \lambda J_p'(\lambda)^2\right) &= -2\lambda \int_0^1 xJ_p^2(\lambda x)dx \\ &\quad + \left(\lambda^2 - \lambda^2\right) \int_0^1 x^2 J_p'(\lambda x)J_p(\lambda x)dx.\end{aligned}$$

Since $J_p(\mu) = 0$, we get

$$\int_0^1 xJ_p^2(\lambda x)dx = \frac{1}{2}\left[J_p'(\lambda)\right]^2.$$

6. $\star$ Let $p, q : (a, b) \to \mathbb{R}$ be two continuous functions. Show that any non-trivial solution y of the differential equation $y'' + py' + qy = 0$ has only finitely many zeros in any subinterval $[\alpha, \beta]$ of (a, b).

If possible let $[\alpha, \beta]$ be a subinterval of (a, b) and $x_1, x_2, \cdots$ be a sequence of zeros of the solution y in $[\alpha, \beta]$. Since $[\alpha, \beta]$ is compact, there exists a subsequence (x_{n_k}) of (x_n) converging to a point x in $[\alpha, \beta]$. Since y is continuous, it follows that $y(x) = \lim_{k\to\infty} y(x_{n_k}) = 0$. Further, since $y(x_{n_k}) - y(x) = 0$ for all k, we see that

$$y'(x) = \lim_{k\to\infty} \frac{y(x_{n_k}) - y(x)}{x_{n_k} - x} = 0.$$

Hence by uniqueness theorem we conclude that $y \equiv 0$ in (a, b).

7. Find the first four terms a_n of the expansion $f(x) = \sum_{n\geq 0} a_n P_n(x)$, if

(i) $f(x) = x|x|$ for $|x| \leq 1$ and (ii) $f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1. \end{cases}$

Let $f(x) = x|x|$ for $|x| \leq 1$. Then

$$\begin{aligned} an &= \frac{2n+1}{2} \int_{-1}^1 f(x)P_n(x)dx \\ &= \frac{2n+1}{2} \int_{-1}^0 x^2 P_n(x)dx - \frac{2n+1}{2} \int_0^1 x^2 P_n(x)dx. \end{aligned}$$

If $n > 2$, then $a_3 = 0 = a_4 = \cdots\cdots$. Therefore we need to compute only a_0, a_1 and a_2 and it is easy.

For the function $f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1. \end{cases}$ we apply the formula to get the coefficients $a_0 = 1/4$, $a_1 = 1/2$ and $a_2 = 5/16$.

8. Let a, b, c, d be real numbers such that $ad - bc \neq 0$. Show that the zeros of the functions $a \sin x + b \cos x$ and $c \sin x + d \cos x$ are distinct and occur alternately.

   The condition $ad - bc \neq 0$ means that the two functions are fundamental set of solutions of the equation $y'' + y = 0$. It is now easy to complete the proof.