Problem Set 4

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Determine whether the following sets of vectors are linearly independent or not
   (a) $\{(1,0,0),(1,1,0),(1,1,1)\}$ of $\mathbb{R}^3$
   (b) $\{(1,0,0,0),(1,1,0,0),(1,2,0,0),(1,1,1,1)\}$ of $\mathbb{R}^4$
   (c) $\{(1,0,2,1),(1,3,2,1),(4,1,2,2)\}$ in $\mathbb{R}^4$.

2. Find a maximal linearly independent subset of
$$S = \left\{ \begin{bmatrix}1\\2\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\-3\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\\-1\\0\\1\end{bmatrix} \right\}.$$
   Find another. And another. Do they have the same cardinality?

3. Give 2 bases for the trace 0 real symmetric matrices of size 3×3. Extend these bases to bases of the real matrices of size 3×3.

4. Consider $\mathbb{W} = \{\mathbf{v} \in \mathbb{R}^6 : \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0, \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = 0, \mathbf{v}_5 + \mathbf{v}_6 = 0\}$. Supply a basis for $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

5. Let M be the vector space of all 2×2 matrices and let $A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, B = \begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}$.
   (a) Give a basis of M.
   (b) Describe a subspace of M which contains A and does not contain B.
   (c) True (give a reason) or False (give a counter example) : If a subspace of M contains A and B, it must contain the identity matrix.

6. **[T]** Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be a basis of the finite dimensional vector space $\mathbb{V}$. Let $\mathbf{v}$ be any non zero vector in $\mathbb{V}$. Show that there exists $\mathbf{w}_i$ such that if we replace $\mathbf{w}_i$ by $\mathbf{v}$ then we still have a basis.

7. Show that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent if and only if $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ is linearly independent.

8. **(T)** Show that $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{R}^n$ are linearly independent if and only if $A\mathbf{u}_1, \dots, A\mathbf{u}_k$ are linearly independent for any invertible matrix $A_{n\times n}$.

   That is, suppose we have an $n \times n$ invertible matrix A and consider the map $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(\mathbf{x}) = A\mathbf{x}$. Then, '$\mathbf{u}_1, \dots, \mathbf{u}_k$ are linearly independent if and only if their images are also linearly independent'.

9. Show that $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{V}$ is linearly independent if and only if $\sum\limits_{i=1}^{k} a_{i1}\mathbf{u}_i, \ldots, \sum\limits_{i=1}^{k} a_{ik}\mathbf{u}_i$ are linearly independent for any invertible matrix $A_{k\times k}$. This means: In $\mathrm{LS}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ are linearly independent if and only if the vectors $\mathbf{w}_j = \sum\limits_{i=1}^{k} a_{ij}\mathbf{u}_i$ (which are nothing but some linear combinations of $\mathbf{u}_i$'s given by the matrix A) are linearly independent.

10. **(T)** If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ is a basis for a vector space $\mathbb{V}$, then show that any set of n vectors in $\mathbb{V}$ with $n > d$, say $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, is linearly dependent.

11. Suppose $\mathbb{V}$ is a vector space of dimension d. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a set of vectors from $\mathbb{V}$. Then show that S does not span $\mathbb{V}$ if $n < d$.

12. **(T)** Determine if the set $T = \{1, x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2\}$ spans the vector space of polynomials with degree 4 or less.

13. Let $\mathbb{W}$ be a proper subspace of $\mathbb{V}$.

    (a) Show that there is a subspace $\mathbb{U}$ of $\mathbb{V}$ such that $\mathbb{W} \cap \mathbb{U} = \{\mathbf{0}\}$ and $\mathbb{U} + \mathbb{W} = \mathbb{V}$.

    (b) Show that there is no subspace $\mathbb{U}$ such that $\mathbb{U} \cap \mathbb{W} = \{\mathbf{0}\}$ and $\dim \mathbb{U} + \dim \mathbb{W} > \dim \mathbb{V}$.

14. **(T)** Describe all possible ways in which two planes (passing through origin) in $\mathbb{R}^3$ could intersect.

15. Construct a matrix with the required property or explain why this is impossible:

    (a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}$.

    (b) Column space has basis $\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}\right\}$, null-space has basis $\left\{\begin{bmatrix}3\\1\\1\end{bmatrix}\right\}$. What if $\begin{bmatrix}3\\1\\1\end{bmatrix}$ belongs to the null space (but not necessarily forms a basis)?

    (c) The dimension of null-space is one more than the dimension of left null-space.

    (d) Left null-space contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.

16. Suppose A is a 3 by 4 matrix and B is a 4 by 5 matrix with $AB = \mathbf{0}$. Show that

$$\mathrm{rank}(A) + \mathrm{rank}(B) \leq 4.$$

17. **(T)** Let A be an m by n matrix and B be an n by p matrix with $\mathrm{rank}(A) = \mathrm{rank}(B) = n$. Show that $\mathrm{rank}(AB) = n$.