MSO202A: Assignment-III Solutions

1. Determine all $z \in \mathbb{C}$ for which the following series converge absolutely.

$$(a) \sum \frac{z^n}{n^2} \qquad (b) \sum \frac{z^n}{n!} \qquad (c) \sum \frac{1}{n!}\frac{1}{z^n} \qquad (d) \sum \frac{1}{2^n}\frac{1}{z^n}$$

Soln: (a) Here $\lim_{n\to\infty} |a_n|/|a_{n+1}| = 1$. Taking $|z| = 1$, we find the series to be convergent too. Hence the series converges for $|z| \leq 1$.

(b) $\lim_{n\to\infty} |a_n|/|a_{n+1}| = \infty$ and hence it converges for all $z \in \mathbb{C}$.

(c) Take $w = 1/z$ where $z \neq 0$. Using (b), we see that it converges for all $w \in \mathbb{C}$ and hence it converges for all $z \in \mathbb{C}\backslash\{0\}$.

(d) Take $w = 1/z$ where $z \neq 0$. Now $\lim_{n\to\infty} |a_n|/|a_{n+1}| = 2$. Hence, it converges for $|w| < 2$. For $|w| = 2$, the n-th term is $e^{in\theta}$ which does not go to zero as $n \to \infty$. Hence, the series converges for $|z| > 1/2$.

2. Let $a_n = \frac{(-1)^n}{\sqrt{n}} + i\frac{1}{n^2}$ for $n = 1, 2, 3, \cdots$. Show that the series $\sum a_n$ converges but it does not converge absolutely.

Soln: We know that $\sum a_n$ converges iff $\sum x_n$ and $\sum y_n$ converges where $a_n = x_n + iy_n$. Now $\sum x_n = \sum(-1)^n/\sqrt{n}$ converges due to alternating series test and $\sum y_n = \sum 1/n^2$ converges. Hence $\sum a_n$ converges. Further, $|a_n|^2 = 1/n + 1/n^4 \geq 1/n$. Clearly, $\sum |a_n|^2$ diverges and hence $\sum |a_n|$ also diverges. (If $\sum |a_n|$ converges, then $|a_n| \leq M$ for all n and $\sum |a_n|^2 \leq M \sum |a_n|$ then converges too.)

3. The following series $\sum z^n$, $\sum z^n/n$ and $\sum z^n/n^2$ have radius of convergence 1. Show that the series

   (a) $\sum z^n$ does not converge for any z such that $|z| = 1$,

   (b) $\sum z^n/n$ converges for all z for which $z \neq 1$ and $|z| = 1$ and

   (c) $\sum z^n/n^2$ converges for all z such that $|z| = 1$.

Soln: Use of ratio test gives radius of convergence to be 1.
(a) For $|z| = 1$, we have $\sum e^{in\theta}$ and the n-th term does not go to zero as $n \to \infty$, Hence, it does not converge for $|z| = 1$.
(b) (Dirichlet test: Suppose that the partial sums of the series $\sum a_n$ are uniformly bounded (although the series $\sum a_n$ may not converge). Then for any sequence $\{b_n\}$ that is of bounded variation and converges to zero, the series $\sum a_n b_n$ converges. In particular, the series $\sum a_n b_n$ converges if $\{b_n\}$ is a monotone sequence of real numbers approaching zero.)

Here for $z = 1$, the series becomes $\sum 1/n$ which diverges. Let $z \neq 1$ and $|z| = 1$, then $z = e^{i\theta}$ where $0 < \theta < 2\pi$. Then the series becomes $\sum e^{in\theta}/n = \sum \cos n\theta/n + i \sum \sin n\theta/n$. Now

$$\sum_{n=1}^{m} \cos n\theta = \text{Re}\left(e^{i\theta}\frac{1 - e^{im\theta}}{1 - e^{i\theta}}\right)$$

Hence,

$$\left|\sum_{n=1}^{m} \cos n\theta\right| \leq \frac{|1 - e^{im\theta}|}{|1 - e^{i\theta}|} \leq \frac{2}{|1 - e^{i\theta}|} = \frac{1}{\sin\theta/2}$$

Further $\{1/n\}$ is monotone and $1/n \to 0$ as $n \to \infty$. Hence, $\sum \cos n\theta/n$ and similarly $\sum \sin n\theta/n$ converge by Dirichlet test. Hence, $\sum z^n/n$ converges for all z for which $z \neq 1$ and $|z| = 1$.
(c) Here for $|z| = 1$, the series converges absolutely and hence the series converges.

4. Find the radius of convergence of the power series $\sum a_n(z-a)^n$ for which

   (a) $a_n = r^n/n^p$ where r and p are two positive real numbers

   (b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$

   (c) $a_n = \frac{1}{2^n-1}$

   Soln: Use of ratio test: (a) $1/r$ (b) 1 (c) 2

5. Find the radius of convergence of the following power series

   (a) $\sum 2nz^n$

   (b) $\sum n! z^{2n+1}$

   (c) $\sum (-1)^n \frac{z^{2n}}{(2n)!}$

   Soln: (a) Let $z^{2n} = w$, then the series becomes $2nw^n$ for which $R = \lim_{n\to\infty} |a_n|/|a_{n+1}| = 1$. Hence, radius of convergence of the original series is also 1.
   (b) Let $z^{2n} = w$, then the series becomes $z \sum n! w^n$ for which $R = \lim_{n\to\infty} |a_n|/|a_{n+1}| = 0$. Hence, the radius of convergence of the original series is also 0.
   (c) Let $z^{2n} = w$, then the series becomes $z \sum (-1)^n \frac{w^n}{(2n)!}$ for which $R = \lim_{n\to\infty} |a_n|/|a_{n+1}| = \infty$. Hence, the radius of convergence of the original series is also ∞.

6. If R_1 and R_2 are the radii of convergence of the series $\sum a_n z^n$ and $\sum b_n z^n$ respectively, then show that $R \geq \min\{R_1, R_2\}$ is the radius of convergence of the series $\sum (a_n + b_n) z^n$.

   Soln: Let $S = \min\{R_1, R_2\}$. If $|z| < S$, then $|z| < R_1$ and $|z| < R_2$. Hence, $\sum a_n z^n$ and $\sum b_n z^n$converge absolutely for $|z| < S$. Now for $|z| < S$, we have

$$\sum |(a_n + b_n)z^n| \leq \sum (|a_n| + |b_n|)|z|^n.$$

   Thus, $\sum (a_n + b_n)z^n$ converges absolutely for $|z| < S$. Thus the raduis of convergence R for $\sum (a_n + b_n)z^n$ must satisfy $R \geq S$. If $R_1 = R_2$, it may be possible that $R > R_1$ (choose $b_n = -a_n$), otherwise $R = S$.

7. Show that $\sum_{n=0}^{\infty} (n+1)^2 z^n = \frac{1+z}{(1-z)^3}$ for $|z| < 1$.

   Soln: We have for $|z| < 1$:

$$\sum_{n=0} z^{n+1} = \frac{z}{1-z} \implies \sum_{n=0} (n+1)z^n = \frac{d}{dz}\left(\frac{z}{1-z}\right) = \frac{1}{(1-z)^2}.$$

   Hence, for $|z| < 1$,

$$\sum_{n=0} (n+1)z^{n+1} = \frac{z}{(1-z)^2} \implies \sum_{n=0} (n+1)^2 z^n = \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{1+z}{(1-z)^3}.$$

8. Find i^i and $\cosh(\operatorname{Log} 4)$. (Log stands for the principal branch of the logarithm)

Soln: We have

$$i^i = e^{i \log i} = e^{i(i\pi/2 + i2n\pi)} = e^{-(2n\pi + \pi/2)}$$

Here $\operatorname{Log} 4 = \ln 4$ and hence

$$\cosh \operatorname{Log} 4 = \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{1}{2}(4 + 1/4) = 17/8$$

9. For $z_1, z_2 \in G = \{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$, is it always true that $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$? Find the conditions on z_1 and z_2 so that the equality holds.

Soln: Not true. For example, take $z_1 = z_2 = -i$. Then $z_1 z_2 = -1 \implies \operatorname{Log}(z_1 z_2) = i\pi$. But $\operatorname{Log}(z_1) = \operatorname{Log}(z_2) = -i\pi/2$ and hence $\operatorname{Log} z_1 + \operatorname{Log} z_2 = -i\pi$.

True if $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$

10. Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Hence prove that cos function is not bounded in $\mathbb{C}$. Also, find the zeros of $\cos z$.

Soln: We have

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \implies |\cos z|^2 = (\cos z)(\overline{\cos z}) = \frac{1}{4}(e^{iz} + e^{-iz})(e^{-i\bar{z}} + e^{i\bar{z}})$$

Simplifying, we find

$$4|\cos z|^2 = e^{-2y} + e^{i2x} + e^{-i2x} + e^{2y} \implies |\cos z|^2 = \frac{1}{2}(\cos 2x + \cosh 2y) = \cos^2 x + \sinh^2 y$$

Note that $\sinh^2 y \geq (e^{2y} - 2)/4$ and hence $\sinh^2 y \to \infty$ as $y \to \infty$. Hence, cos function is unbounded.

Now $\cos z = 0 \implies \cos x = 0, \sinh y = 0 \implies y = 0,\ x = (n + 1/2)\pi, \quad n \in \mathbb{Z} \implies z = (n + 1/2)\pi, \quad n \in \mathbb{Z}$

11. Show that $\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$.

Soln: Using

$$\sin(z_1 + z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \frac{e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}}{2i}$$

we can show that $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$. Similarly, $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. Thus

$$\tan(z_1 + z_2) = \frac{\sin(z_1 + z_2)}{\cos(z_1 + z_2)} = \frac{\sin z_1 \cos z_2 + \cos z_1 \sin z_2}{\cos z_1 \cos z_2 - \sin z_1 \sin z_2} = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

12. Show that $\sin \bar{z}$ and $\cos \bar{z}$ are not analytic functions on any domain.

Soln: We know $\sin z = (e^{iz} - e^{-iz})/2i$ and hence (writing $z = x + iy$)

$$\sin \bar{z} = u(x, y) + iv(x, y) = \frac{e^{y}e^{ix} - e^{-y}e^{-ix}}{2i} = \frac{e^{y}e^{ix} - e^{-y}e^{-ix}}{2i} = \sin x \cosh y - i \cos x \sinh y$$

Hence, $u = \sin x \cosh y$ and $v = -\cos x \sinh y$. Now $u_x = \cos x \cosh y$ and $v_y = -\cos x \cosh y$ and $u_x \neq v_y$. Since, the CR equations are not satisfied anywhere, $\sin \bar{z}$ is not an analytic function on any domain.

Similar resonings holds for $\cos \bar{z}$ too.

13. Find all solutions z of (a) $\cos z = 2$ (b) $\sin\theta \sin z = 1$ where $\theta \in \mathbb{R}$ (c) $|\cot z| = 1$

Soln: (a) Here

$$\cos z = 2 \implies e^{2iz} - 4e^{iz} + 1 = 0 \implies e^{iz} = 2 \pm \sqrt{3} \implies iz = \log(2 \pm \sqrt{3})$$

Thus

$$iz = \ln(2 \pm \sqrt{3}) + i2k\pi \implies z = -i\ln(2 \pm \sqrt{3}) + 2\pi k, \quad k \in \mathbb{Z}.$$

(b) Note that $\theta \neq 0$. Now from $\sin(x+iy) = \text{cosec}\ \theta$, we get $\sin x \cosh y = \text{cosec}\ \theta$ and $\cos x \sinh y = 0$. If $\sinh y = 0 \implies y = 0$, then $\sin x = \text{cosec}\ \theta$ which has no solution unless $x = \pm\pi/2$. If $\theta \neq \pm\pi/2 \implies \cos x = 0 \implies x = (k+1/2)\pi,\ k \in \mathbb{Z}$. If $\sin\theta > 0$, then $x = 2m\pi + \pi/2$, $m \in \mathbb{Z}$ and hence $\cosh y = \text{cosec}\ \theta \implies e^y = \tan\theta/2$ or $\cot\theta/2$ or $e^y = \tan(\theta/2 + n\pi)$ or $\cot(\theta/2 + n\pi)$. Hence $y = \pm\ln(\tan(2n\pi + \theta)/2)$. Thus, for $\sin\theta > 0$:

$$z = (2m + 1/2)\pi \pm i\ln(\tan(2n\pi + \theta)/2), \qquad m, n \in \mathbb{Z}$$

If $\sin\theta < 0$, then $x = (2m+1)\pi + \pi/2$ and $\cosh y = -\text{cosec}\ \theta = \text{cosec}\ (\pi + \theta) \implies e^y = \tan\theta/2$. Hence, $e^y = \tan(\theta + \pi)/2$ or $\cot(\theta + \pi)/2$ or $e^y = \tan(\theta/2 + n\pi + \pi/2)$ or $\cot(\theta/2 + n\pi + \pi/2)$. Hence $y = \pm\ln(\tan(2n\pi + \pi + \theta)/2)$. Thus, for $\sin\theta < 0$:

$$z = (2m + 1 + 1/2)\pi \pm i\ln(\tan(2n\pi + \pi + \theta)/2), \qquad m, n \in \mathbb{Z}$$

Note that both the solutions can be combined to arrive at

$$z = (m + 1/2)\pi \pm i\ln\tan(n\pi + \theta)/2$$

where m and n are integers. Further, m and n are both even or odd depending on $\sin\theta > 0$ or $\sin\theta < 0$.

(c) Given $|\cot z| = 1$. Clearly, $z \neq 0$. Now

$$|\tan z| = 1 \implies |\sin z| = |\cos z| \implies |e^{iz} + e^{-iz}| = |e^{iz} - e^{-iz}| \implies |e^{2iz} + 1|^2 = |e^{2iz} - 1|^2$$

Thus

$$(e^{2iz} + 1)(e^{-2i\bar{z}} + 1) = (e^{2iz} - 1)(e^{-2i\bar{z}} - 1) \implies e^{2iz} = -e^{-2i\bar{z}} \implies e^{4ix} = -1$$

$$\implies 4ix = i(\pi + 2n\pi) \implies z = \frac{n\pi}{2} + \frac{\pi}{4} + iy, \quad y \in \mathbb{R}, \quad n \in \mathbb{Z}$$

14. Express in the form $a + ib$: (a) $\log \text{Log i}$ (b) $(-3)^{\sqrt{2}}$ (c) i^{-i}

Soln: (a) $\log \text{Log i} = \log(i\pi/2) = \ln\pi/2 + i(\pi/2 + 2n\pi)$

(b) $(-3)^{\sqrt{2}} = e^{\sqrt{2}\log -3} = e^{\sqrt{2}(\ln 3 + i(2n+1)\pi)} = e^{\sqrt{2}\ln 3}\left(\cos\sqrt{2}(2n+1)\pi + i\sin\sqrt{2}(2n+1)\pi\right) = 3^{\sqrt{2}}\left(\cos\sqrt{2}(2n+1)\pi + i\sin\sqrt{2}(2n+1)\pi\right)$

(c) $i^{-i} = e^{-i\log i} = e^{-i(i\pi/2 + 2n\pi)} = e^{\pi/2 + 2n\pi}$

15. Show that (a) $\sin^{-1} z = -i\log(iz + \sqrt{1 - z^2})$ (b) $\cot^{-1} z = \frac{i}{2}\log(z - i)/(z + i)$ (c) $\cosh^{-1} z = \log(z + \sqrt{z^2 - 1})$

Soln: (a) $\sin^{-1} z = w \implies z = \sin w \implies (e^{iw} - e^{-iw}) = 2iz \implies e^{iw} = iz + (1 - z^2)^{1/2}$. Since z is a complex variable, $(1 - z^2)^{1/2}$ is the complex square-root function. This is a multi-valued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign.

(b) $\cot^{-1} z = w \implies z = \cos w / \sin w \implies (e^{iw} + e^{-iw})/(e^{iw} - e^{-iw}) = z/i \implies e^{2iw} = (z+i)/(z-i) \implies 2iw = \log(z+i)/(z-i)$

(c) $\cosh^{-1} z = w \implies w = \cosh z \implies e^{w} = z + (z^2 - 1)^{1/2}$