Problem Set 2

Problems marked **(T)** are for discussions in Tutorial sessions.

1. **(T)** A square matrix P is called a *permutation matrix* if each row and column of P contains exactly one 1 and the rest of the entries are 0. Determine all the 3×3 permutation matrices. Now, Let

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Write down the permutation matrix P such that PA is upper triangular. Which permutation matrices P_1 and P_2 make $P_1 A P_2$ lower triangular?

Solution: To permute rows of A, left multiply by permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ yielding } PA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

$$\text{Right multiplication by } P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ permutes columns of } A \text{ yielding } AP_2 = \begin{bmatrix} 6 & 0 & 0 \\ 3 & 2 & 1 \\ 5 & 4 & 0 \end{bmatrix}.$$

Finally, rearrange rows of AP_2 by left multiplying

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ resulting is } P_1 A P_2 = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

2. **(T)** Decide if they are row-equivalent:

(a) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

Solution:
No, first one is singular and the second one is nonsingular.

(b) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$

Solution:

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix} \to R_2 \Rightarrow\Leftarrow R_3 \to \begin{pmatrix} 1 & 0 & 2 \\ 5 & -1 & 5 \\ 3 & -1 & 1 \end{pmatrix} \to R_2 - 5R_1, R_3 - R_1 \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -5 \\ 2 & -1 & -1 \end{pmatrix} \to$$

$$R_3 - R_2, -2R_2 \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix}.$$

(c) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 4 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \end{bmatrix}$

Solution:

No, two matrices must have the same size, in order to be row equivalent.

3. Supply two examples each and explain their geometrical meaning.

(a) Two linear equations in two variables with exactly one solution.

Solution:

$\left.\begin{array}{r} x+y=2 \\ x-y=0 \end{array}\right\}$. They represent two lines in $\mathbb{R}^2$ intersecting at a point.

(b) Two linear equations in two variables with infinitely many solutions.

Solution:

$\left.\begin{array}{r} x+y=2 \\ 2x+2y=4 \end{array}\right\}$. They represent the same line in $\mathbb{R}^2$.

(c) Two linear equations in two variables with no solutions.

Solution:

$\left.\begin{array}{r} x+y=2 \\ 2x+2y=1 \end{array}\right\}$. They represent two parallel lines in $\mathbb{R}^2$, no intersection.

(d) Three linear equations in two variables with exactly one solution.

Solution:

$\left.\begin{array}{r} x+y=2 \\ x-y=0 \\ 2x-y=1 \end{array}\right\}$. They represent three lines in $\mathbb{R}^2$ with a single point in common.

(e) Three linear equations in two variables with no solutions.

Solution:

$\left.\begin{array}{r} x+y=2 \\ x-y=0 \\ 2x+2y=1 \end{array}\right\}$. They represent three lines in $\mathbb{R}^2$ with no point in common.

4. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two distinct solutions of the system $A\mathbf{x} = \mathbf{b}$. Prove that there are infinitely many solutions to this system, by showing that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is also a solution, for each $\lambda \in \mathbb{R}$. Do you have a geometric interpretation?

5. Is it possible to have $RREF([A|\mathbf{b}]) = \left[\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$? Give reasons for your answer.

6. Give examples of three matrices of size 3×4 that are in Row Reduced Echelon Form (RREF) but are different from the one given above. Give examples of matrices that are not in RREF, specifying the reasons.

7. Let B be a square invertible matrix. Then, prove that the system $A\mathbf{x} = \mathbf{b}$ and $BA\mathbf{x} = B\mathbf{b}$ are row-equivalent.

   Solution:

   As B is invertible, there exists elementary matrices E_i's such that $B = E_1 E_2 \cdots E_k$. Thus, the system $BA\mathbf{x} = B\mathbf{b}$ is obtained from $A\mathbf{x} = \mathbf{b}$ by k elementary row operations.

   Conversely, as $B^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1}$ and inverse of elementary matrices are also elementary matrices, we do obtain $A\mathbf{x} = \mathbf{b}$ from $BA\mathbf{x} = B\mathbf{b}$ by k elementary row operations. Thus, the above two systems are row equivalent.

8. **[T]** Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have same solutions for every $\mathbf{b}$. Is it true that $A = C$?

   Solution: First, as $Ax = b$ and $Cx = b$ have same solutions, A and C have same shapes, that is, same number of rows and columns, as well as same null space (follows from taking $b = 0$).

   Now, if we take b to be the first column of A then $x = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^t$ solves $Ax = b$ and therefore also solves $Cx = b$ which in turn imply that the first columns of A and C are same. Same argument holds for other columns as well. Thus $A = C$.

9. Find the coefficients a, b, c, d so that the graph of $y = ax^3 + bx^2 + cx + d$ passes through $(1, 2), (-1, 6), (2, 3)(0, 1)$.

   Solution:

   This is as good as solving $a+b+c = 1, a-b+c = 5, 4a+2b+c = 1$. Apply Gaussian elimination.

10. Find matrices A and B with the given property or explain why you can not:

    (a) The only solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

    Solution: Any A satisfying the given equation has to be a 3×2 matrix. The linear system has a unique solution when rank of A is 2 and $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^t \in \mathrm{col}(A)$. Among many possibilities, one such A is

    $$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}.$$

    (b) The only solution to $B\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

    Solution: Any B satisfying the given equation has to be a 2×3 matrix which implies that the null space of B, $N(B)$, can not be trivial and hence we either have an infinitely many solutions (when $[0\ 1]^t$ lies in the column space of B) or no solution. Thus, finding a B that exhibits a unique solution is not possible.

11. Apply Gauss elimination to solve the following system

$$\begin{aligned} 2x + y + 2z &= 3 \\ 3x - y + 4z &= 7 \\ 4x + 3y + 6z &= 5 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 3 & -1 & 4 & 7 \\ 4 & 3 & 6 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - (3/2)R_1} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & -5/2 & 1 & 5/2 \\ 4 & 3 & 6 & 5 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & -5/2 & 1 & 5/2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + (2/5)R_2} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & -5/2 & 1 & 5/2 \\ 0 & 0 & 12/5 & 0 \end{array}\right]$$

We can thus obtain the solution to the given linear system by solving the equivalent system

$$\begin{aligned} 2x + y + 2z &= 3 \\ (-5/2)y + z &= 5/2 \\ (12/5)z &= 0 \end{aligned}$$

The solution is $x = 2, y = -1$ and $z = 0$.

12. **(T)** Using Gauss Jordan method, find the inverse of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - (2/3)R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & 0 & -5/3 & -2/3 & -2/3 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow (-3/5)R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right]$$

$$\xrightarrow[R_1 \leftarrow R_1 - 2R_3]{R_2 \leftarrow R_2 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/5 & -4/5 & 6/5 \\ 0 & -3 & 0 & -6/5 & 9/5 & -6/5 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right] \xrightarrow{R_2 \leftarrow (-1/3)R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/5 & -4/5 & 6/5 \\ 0 & 1 & 0 & 2/5 & -3/5 & 2/5 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/5 & 2/5 & 2/5 \\ 0 & 1 & 0 & 2/5 & -3/5 & 2/5 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right]$$

Thus, the inverse is $\begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$.

13. **(T)** Let $B_{n\times n}$ be a real skew-symmetric. Show that $I - B$ is nonsingular.

Solution:

Let if possible $I - B$ be singular. Then, the system $(I - B)\mathbf{x} = \mathbf{0}$ has a non-trivial solution, say $\mathbf{x}_0 \neq \mathbf{0}$. Hence, $B\mathbf{x}_0 = \mathbf{x}_0$. Also, $\mathbf{x}_0^t B\mathbf{x}_0 \in \mathbb{R}$ and hence

$$\mathbf{x}_0^t B\mathbf{x}_0 = (\mathbf{x}_0^t B\mathbf{x}_0)^t = \mathbf{x}_0^t B^t\mathbf{x}_0 = -\mathbf{x}_0^t B\mathbf{x}_0.$$

Thus, $\mathbf{x}_0^t B\mathbf{x}_0 = 0$. But, $0 = \mathbf{x}_0^t B\mathbf{x}_0 = \mathbf{x}_0^t(B\mathbf{x}_0) = \mathbf{x}_0^t\mathbf{x}_0$ and hence $\mathbf{x}_0 = \mathbf{0}$.

14. **(T)** For two $n \times n$ matrices A and B, show that $\det(AB) = \det(A)\det(B)$.

Solution: First, suppose that A is singular. Then, AB is singular as well. We therefore have, $det(A) = 0$ and $det(AB) = 0$ which leads to $det(AB) = 0 = det(A)det(B)$.

Now we assume that A is non-singular. Recall that, for a non-singular square matrix, the reduced row echelon form is the identity matrix, I. In other words, there exist elementary matrices $E_s, E_{s-1}, \ldots, E_2, E_1$ such that

$$A = E_s E_{s-1} \ldots E_2 E_1 I.$$

We therefore have

$$AB = E_s E_{s-1} \ldots E_2 E_1 B.$$

Thus the problem of showing $det(AB) = det(A)det(B)$ reduces to showing

(a) $det(E_{ij}B) = det(E_{ij})det(B)$.
(b) $det(E_i(c)B) = det(E_{ij}(c))det(B)$.
(c) $det(E_{ij}(c)B) = det(E_{ij}(c))det(B)$.

Now, for the proof, we use the defining properties 1, 2 and 3 discussed in class. We have

(a) $det(E_{ij}B) = -det(B)$ (property 2) $= det(E_{ij})detB$ (using $det(E_{ij}) = -det(I) = -1$, follows from properties 2 and 1).
(b) $det(E_i(c)B) = c\ det(B)$ (property 3a) $= det(E_i(c))det(B)$ (using $det(E_i(c)) = c\ det(I) = c$, follows from properties 3a and 1).
(c) $det(E_{ij}(c)B) = det(B)$ (property 6) $= det(E_{ij}(c))det(B)$ (using $det(E_{ij}(c)) = 1$, follows from property 6 on the identity matrix).

The result now follows

$$\begin{aligned} det(AB) &= det(E_s E_{s-1} \ldots E_2 E_1 B) = det(E_s)det(E_{s-1} \ldots E_2 E_1 B) = \cdots \\ &= det(E_s)det(E_{s-1}) \ldots det(E_2)det(E_1 B) \\ &= det(E_s)det(E_{s-1}) \ldots det(E_2)det(E_1)det(B) \\ &= det(E_s)det(E_{s-1}) \ldots det(E_2 E_1)det(B) = \cdots \\ &= det(E_s E_{s-1} \ldots E_2 E_1)det(B) = det(A)det(B). \end{aligned}$$

15. For an $n \times n$ matrix $A = [a_{ij}]$, prove that $\det(A) = \det(A^T)$.

    Solution:

    Recall that, a square matrix can be reduced to an upper form using elementary row operations. In other words, there exist elementary matrices $E_s, E_{s-1}, \ldots, E_2, E_1$ such that

    $$E_s E_{s-1} \ldots E_2 E_1 A = U.$$

    where U is an upper triangular matrix and each E_k, $1 \leq k \leq s$ is either an elementary matrix E_{ij} or $E_{ij}(c)$ for some i, j and c. We therefore have

    $$EA = U$$

    with $\det(E) = \det(E^T) = \pm 1$. We also have $\det(U) = \det(U^T)$ for the upper triangular matrix U. Thus

    $$\det(A) = \frac{\det(U)}{\det(E)},$$

    and

    $$A^T E^T = U^T$$

    yields

    $$\det(A^T) = \frac{\det(U^T)}{\det(E^T)} = \frac{\det(U)}{\det(E)} = \det(A).$$

16. Let A be an $n \times n$ matrix. Prove that

    (a) If $A^2 = \mathbf{0}$ then A is singular.

    Solution: $0 = \det(A^2) = (\det A)^2 \Rightarrow \det A = 0.$

    (b) If $A^2 = A, A \neq I$ then A is singular.

    Solution: If A is nonsingular, then it is invertible. Then $A = A^{-1}A^2 = A^{-1}A = I$.

17. Consider the system $A\mathbf{x} = \mathbf{b}$. Let $RREF([A|\mathbf{b}])$ be one of the matrices in Question 5. Now, recall the matrices A_j's, for $1 \leq j \leq 3$ (defined to state the Cramer's rule for solving a linear system), that are obtained by replacing the j-th column of A by $\mathbf{b}$. Then, we see that the above system has NO solution even though $\det(A) = 0 = \det(A_j)$, for $1 \leq j \leq 3$.

18. **(T)** Let A be an invertible square matrix with integer entries. Show that A^{-1} has integer entries if and only if $\det(A) = \pm 1$.

    Solution: If $\det(A) = \pm 1$, then entries of A^{-1} are integers as $A^{-1} = \frac{1}{\det(A)} C^T$, where C is the matrix of co-factors. If entries of A^{-1} are integers then $1/\det(A) = \det(A^{-1}) \in \mathbb{Z}$. As $\det(A) \in \mathbb{Z}$, the result follows.

19. Let A be an $n \times n$ matrix. Prove that the following statements are equivalent:

    (a) $\det(A) \neq 0$.

    (b) A is invertible.

(c) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The row-reduced echelon form of A is I_n.

(e) A is a product of elementary matrices.

(f) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every .

(g) The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$.

19a $\Longrightarrow$ 19b

By, definition, whenever $\det(A) \neq 0$, $A^{-1} = \dfrac{C^t}{\det(A)}$, where C is the co-factor matrix.

19b $\Longrightarrow$ 19a

As A is invertible, $AA^{-1} = I_n$ and hence $\det(A)\det(A^{-1}) = \det(I_n) = 1$. Hence, $\det(A) \neq 0$.

19b $\Longrightarrow$ 19c

As A is invertible, $A^{-1}A = I_n$. Let $\mathbf{x}_0$ be a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Then,

$$\mathbf{x}_0 = I_n\mathbf{x}_0 = (A^{-1}A)\mathbf{x}_0 = A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{0}$ is the only solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

19c $\Longrightarrow$ 19d

Let $\mathbf{x}^t = [x_1, x_2, \ldots, x_n]$. As $\mathbf{0}$ is the only solution of the linear system $A\mathbf{x} = \mathbf{0}$, the final equations are $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. These equations can be rewritten as

$$\begin{array}{rcl}
1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \cdots + 0 \cdot x_n & = & 0 \\
0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + \cdots + 0 \cdot x_n & = & 0 \\
0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + \cdots + 0 \cdot x_n & = & 0 \\
\vdots & = & \vdots \\
0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \cdots + 1 \cdot x_n & = & 0.
\end{array}$$

That is, the final system of homogeneous system is given by $I_n \cdot \mathbf{x} = \mathbf{0}$. Or equivalently, the row-reduced echelon form of the augmented matrix $[A \;\; \mathbf{0}]$ is $[I_n \;\; \mathbf{0}]$. That is, the row-reduced echelon form of A is I_n.

19d $\Longrightarrow$ 19e

Suppose that the row-reduced echelon form of A is I_n. Then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_1E_2 \cdots E_kA = I_n. \tag{1}$$

Now, the matrix E_j^{-1} is an elementary matrix and is the inverse of E_j for $1 \leq j \leq k$. Therefore, successively multiplying Equation (1) on the left by $E_1^{-1}, E_2^{-1}, \ldots, E_k^{-1}$, we get

$$A = E_k^{-1}E_{k-1}^{-1} \cdots E_2^{-1}E_1^{-1}$$

and thus A is a product of elementary matrices.

19e $\Longrightarrow$ 19b

Suppose $A = E_1 E_2 \cdots E_k$; where the E_i's are elementary matrices. As the elementary matrices are invertible and the product of invertible matrices is also invertible, we get the required result.

19b $\Longrightarrow$ 19f

Observe that $\mathbf{x}_0 = A^{-1}\mathbf{b}$ is the unique solution of the system $A\mathbf{x} = \mathbf{b}$.

19f $\Longrightarrow$ 19g

The system $A\mathbf{x} = \mathbf{b}$ has a solution and hence by definition, the system is consistent.

19g $\Longrightarrow$ 19b

For $1 \le i \le n$, define $\mathbf{e}_i = (0, \ldots, 0, \underbrace{1}_{i\text{th position}}, 0, \ldots, 0)^t$, and consider the linear system $A\mathbf{x} = \mathbf{e}_i$. By assumption, this system has a solution, say $\mathbf{x}_i$, for each i, $1 \le i \le n$. Define a matrix $B = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$. That is, the i^{th} column of B is the solution of the system $A\mathbf{x} = \mathbf{e}_i$. Then

$$AB = A[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = [A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n] = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] = I_n.$$

Therefore, the matrix A is invertible.

20. Let A be an $n \times n$ matrix. Then prove that $\det(A) = 0$ if and only if the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

21. Let A be an $n \times n$ matrix. Then the two statements given below cannot hold together.

    (a) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

    (b) The system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

22. **(T)** A real square matrix A is said to be orthogonal if $A^T A = AA^T = I$. Show that if A is orthogonal then $\det(A) = \pm 1$.

    Solution: Since $AA^T = I$, we have $\det(AA^T) = \det(I) = 1$. Now,

    $$1 = \det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = (\det(A))^2.$$

    Thus, $\det(A) = \pm 1$.

23. Let $A = [a_{ij}]$ be an invertible matrix and let $B = [p^{i-j} a_{ij}]$, for some $p \neq 0$. Find the inverse of B and also find $\det(B)$.

    Solution: Note that $B = DAD^{-1}$, where D is a diagonal matrix with $d_{ii} = p^i$. Hence, $B^{-1} = DA^{-1}D^{-1}$. Also,

    $$\det(B) = \det(DAD^{-1}) = \det(D) \cdot \det(A) \cdot \det(D^{-1}) = \det(A).$$

24. Suppose the 4×4 matrix M has 4 equal rows all containing a, b, c, d. We know that $det(M) = 0$. The problem is to find by any method

    $$det(I + M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

Solution: Subtracting row 1 from rows 2, 3 and 4, we get

$$det(I+M) = \begin{vmatrix} 1+a & b & c & d \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}.$$

Now, adding columns 2, 3 and 4 to column 1, we get

$$det(I+M) = \begin{vmatrix} 1+a+c+d & a & b & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Thus, $det(I+M) = 1+a+b+c+d$.

25. **(T)** For a complex matrix $A = [a_{ij}]$, let $\bar{A} = [\ \overline{a_{ij}}\ ]$ and $A^* = \bar{A}^T$. Show that $\det(\bar{A}) = \det(A^*) = \overline{\det A}$. Therefore if A is Hermitian (that is, $A^* = A$) then its determinant is real.

    Solution: The first equality follows directly from the statement in Problem 4:

$$\det(\bar{A}) = \det\left((\bar{A})^T\right) = \det(A^*).$$

The equality $\det(\bar{A}) = \overline{\det A}$ follows from the determinant formula:

$$\det(\bar{A}) = \sum_{\sigma \in S_n} (sgn\ \sigma)\ \overline{a_{1\sigma(1)}}\ \overline{a_{2\sigma(2)}} \cdots \overline{a_{n\sigma(n)}} = \overline{\sum_{\sigma \in S_n} (sgn\ \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}} = \overline{\det(A)}.$$

26. The numbers $1375, 1287, 4191$ and 5731 are all divisible by 11. Prove that 11 also divides the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 5 & 7 & 1 & 1 \end{bmatrix}.$$

Solution: Adding $1000 \times Row_1$, $100 \times Row_2$, $10 \times Row_3$ to Row_4, we have

$$\begin{vmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 5 & 7 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 1375 & 1287 & 4191 & 5731 \end{vmatrix}.$$

Since Row_4 is divisible by 11, the determinant is divisible by 11.

27. Compute determinant of $\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix}$. .

Solution: We give the solution for the general case. Let

$$A_n = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

If $n = 2$, $\det(A_2) = x_2 - x_1$. We will prove that

$$\det(A_n) = \prod_{i<j} (x_j - x_i).$$

Assume the result for $n-1$ and define

$$F(x) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x & x^2 & \cdots & x^{n-1} \end{vmatrix}.$$

Then F is a polynomial of degree $n-1$ with roots $x_1, x_2, \ldots, x_{n-1}$. So, $F(x) = c \prod_{i=1}^{n-1} (x - x_i)$ where c is coefficient of x^{n-1} which is clearly $\det(A_{n-1})$. Therefore,

$$F(x) = \det(A_{n-1}) \prod_{i=1}^{n-1} (x - x_i).$$

The result follows for n as

$$\det(A_n) = F(x_n) = \det(A_{n-1}) \prod_{i=1}^{n-1} (x_n - x_i).$$