Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment VII

1. $\star$ Find the Laplace transform of the following functions.
(i) e^{at} for $a \neq 0$. (ii) $\cosh bt$. (iii) $e^{\lambda t} \cos \omega t$ for $\lambda, \omega \in \mathbb{R}$.
(iv) $\cos 2t + \sin 3t$ (v) $t^2 e^{3t} \sin 5t$.

   Some of these problems have been discussed. Let me indicate a proof for finding the Laplace transform of the function $x^2 e^{3x} \sin 5x$. Observe that $e^{-sx} x^k = (-1)^k \frac{d^k}{dx^k} e^{-st}$ for all $k \geq 1$. Therefore $\int_0^\infty e^{-st} t^2 e^{3t} \sin 5t dt = \int_0^\infty \frac{d^2}{dt^2} e^{-st} e^{3t} \sin 5t dt = \frac{d^2}{dt^2} \int_0^\infty e^{-st} e^{3t} \sin 5t dt$. This integral can be written as $\frac{d^2}{dt^2} \int_0^\infty e^{-(s-3)t} \sin 5t dt = \frac{d^2}{dt^2} \left(\frac{5}{(s-3)^2+5^2} \right)$.

2. $\star$ Find the Laplace transform of
(i) $f(t) = \begin{cases} e^{-t} & \text{for } 0 \leq t < 1 \\ e^{-2t} & \text{for } t \geq 1 \end{cases}$ (ii) $f(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ 2-t & \text{for } 1 \leq t \leq 2 \end{cases}$ and $f(t+2) = f(t)$.
(iii) $f(t) = |\sin t|$.

   For the first problem,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-(s+1)t} dt + \int_1^\infty e^{-(s+2)t} dt \\ &= -\frac{1}{s+1} e^{-(s+1)t}|_0^1 - \frac{1}{s+2} e^{-(s+2)t}|_1^\infty \\ &= \frac{1}{s} - \frac{1}{2} e^{-2t} - \frac{1}{3} e^{-3t}. \end{aligned}$$

   For the second problem, I have shown in the class if f is a continuous periodic function with period T, then $F(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$. In our case, the function f has period 2 and hence

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) dt.$$

   We will now evaluate $\int_0^2 e^{-st} f(t) dt$ as

$$\begin{aligned} \int_0^2 e^{-st} f(t) dt &= \int_0^1 e^{-st} t dt + \int_1^2 e^{-st} (2-t) dt \\ &= \frac{1}{s} \int_0^1 e^{-st} dt - \frac{1}{s} \int_1^2 e^{-st} dt \\ &= \frac{1}{s} \int_0^1 e^{-st} dt - \frac{e^{-s}}{s} \int_0^1 e^{-st} dt \\ &= \frac{1-e^{-s}}{s} \int_0^1 e^{-st} dt \\ &= \frac{(1-e^{-s})^2}{s^2}. \end{aligned}$$

   Hence $F(s) = \frac{(1-e^{-s})^2}{1-e^{-2s}} \frac{1}{s^2} = \frac{1-e^{-s}}{1+e^{-s}} \frac{1}{s^2}$ and this can also be written as $F(s) = \frac{1}{s^2} \tanh \frac{s}{2}$. For the next problem, observe that $f(t) = |\sin t|$ is a periodic function of period π and carrying out the computations as in the earlier problem, we get that

$$F(s) = \frac{1+e^{-\pi s}}{1-e^{-\pi s}} \frac{1}{s^2+1} = \frac{1}{s^2+1} \coth \frac{s}{2}.$$

3. $\star$ Show that
(i) $L(\int_0^t f(\tau) d\tau) = \frac{1}{s} L(f)$ (ii) $L(\frac{1}{t} f(t)) = \int_s^\infty F(r) dr$.

In all these problems, our standing assumption is that f is of exponential order. That is $|f(t)| \leq Me^{ct}$ for some $M > 0$ and $c \in \mathbb{R}$. I have not gone in to the technical details. First problem is simple integration by parts.

$$
\begin{aligned}
L\left(\int_0^t f(\tau)d\tau\right) &= \int_0^\infty e^{-st}\int_0^t f(\tau)d\tau \\
&= -\frac{1}{s}\int_0^\infty \left(\int_0^t f(\tau)d\tau\right)\frac{d}{dt}(e^{-st})dt \\
&= -\frac{1}{s}\left[e^{-st}\int_0^t f(\tau)d\tau|_0^\infty - \int_0^\infty e^{-st}f(t)dt\right] \\
&= \frac{1}{s}F(s).
\end{aligned}
$$

In the second problem we need to assume that $\lim_{t\to 0}\frac{f(t)}{t}$ exists. Let $F(s) = L(f)$ and $F_1(s) = L(\frac{f(t)}{t})$. Then

$$
\begin{aligned}
F_1(s) &= \int_0^\infty e^{-st}\frac{f(t)}{t}dt \qquad \text{and} \\
F_1'(s) &= -\int_0^\infty e^{-st}t\frac{f(t)}{t}dt \\
&= -F(s).
\end{aligned}
$$

If we now intgrate this equation from s to ∞ and observe that $\lim_{s\to\infty} F_1(s) = 0$, we get $F_1(s) = \int_s^\infty F(r)dr$.

4. $\star$ Find the inverse Laplace transform of the following functions.
(i)$F(s) = \frac{2+3s}{(s^2+1)(s+2)(s+1)}$ (ii) $F(s) = \frac{3s^2+2s+1}{(s^2+1)(s^2+2s+3)}$.

We write $F(s) = \frac{2+3s}{(s^2+1)(s+2)(s+1)}$ as

$$\frac{2+3s}{(s^2+1)(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

and we simplify this as

$$A(s+1)(s^2+1) + B(s+2)(s^2+1) + (Cs+D)(s+1)(s+2) = 2+3s.$$

By letting

- $s = -2$, we get $-5A = -4$,
- $s = -1$, we get $2B = -1$ and
- $s = 0$, we get $A + 2B + 2D = 2$.

Finally by comparing the coefficient of s^3, we get $A + B + C = 0$. Solving for A, B, C and D, we get $A = \frac{4}{5}$, $B = \frac{-1}{2}$, $C = \frac{-3}{10}$ and $D = \frac{11}{10}$. Therefore

$$
\begin{aligned}
F(s) &= \frac{4}{5}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s+1} - \frac{1}{10}\frac{3s-11}{s^2+1} \\
&= \frac{4}{5}L(e^{-2t}) - \frac{1}{2}L(e^{-t}) - \frac{1}{10}L(3\cos t - 11\sin t)
\end{aligned}
$$

For the next function, we can show, by doing the computations exactly as above that it is the Laplace transform of the function $f(t) = \frac{6}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}(6\cos t + 7\sin t)$.

5. $\star$ Solve the following initial value problems.
(i)$2y'' + 3y' + y = 8e^{-2t}$, $y(0) = -4$ and $y'(0) = 2$.
(ii)$y'' + y = \sin 2t$, $y(0) = 0$ and $y'(0) = 1$.

Let $Y(s)$ denote the Laplace transform of a function y. If y is differentiable, then $L(y') = sY(s) - y(0)$ and $L(y'') = s^2Y(s) - y'(0) - sy(0)$. If we substitute the values of the initial values, then we get $(2s^2 + 3s + 1)Y(s) - (2(2-4s) + 3(-4))$. Therefore

$$\begin{aligned}
Y(s) &= \frac{1}{2s^2+3s+1}\left[L(e^{-2t}) - 8(s+1)\right] \\
&= \frac{1}{(2s+1)(s+1)}\left[\frac{8}{s+2} - 8(s+1)\right] \\
&= \frac{4}{3}\frac{1}{s+1/2} - \frac{8}{s+1} + \frac{8}{3}\frac{1}{s+2} \\
&= \frac{4}{3}L(e^{-t/2}) - 8L(e^{-t}) + \frac{8}{3}L(e^{-2t}).
\end{aligned}$$

For the next equation, we get $Y(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{s}{s^2+1} = L\left(\frac{4\sin\, t - \sin 2t)}{3}\right)$.

6. Using the unit step function find the $L(f)$ if $f(t) = \begin{cases} \sin\, t & \text{if } 0 \leq t \leq \frac{\pi}{2} \\ \cos\, t - 3\sin\, t & \text{if } \frac{\pi}{2} \leq t < \pi \\ 3\cos\, t & \text{if } t \geq \pi \end{cases}$.

7. Find the inverse Laplace transform of $\frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{2}{s}\right] + e^{-4s}\left[\frac{4}{s^3} + \frac{1}{s}\right]$.

8. Solve the following initial value problems:

   . (i) $y'' + y = f$ where $f(t) = \begin{cases} \sin\, t & \text{if } 0 \leq t < \frac{\pi}{2} \\ \cos t & \text{if } \frac{\pi}{2} \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases}$ and $y(0) = 2$, $y'(0) = -1$.

   (ii) $y'' - 4y' + 4y = f$ where $f(t) = \begin{cases} e^{2t} & \text{if } 0 \leq t < 2 \\ -e^{2t} & \text{if } t \geq 2 \end{cases}$ and $y(0) = 0$, $y'(0) = -1$.

9. Using convolution method solve the equation $y(t) = 1 + 2\int_0^t e^{-2(t-\tau)}y(\tau)d\tau$.

10. Solve the initial value problem
    $y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t-1), \qquad y(0) = -3$ and $y'(0) = 2$.