Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment III

1. $\star$ Applying Euler's method with step size $h = 0.1$, find the approximate values for the solution of the differential equation
$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$
$x = 0.1, 0.2, 0.3$ and compare these values with the values of the solution $y = e^{4x} + e^{-3x}$ at these points.

   Do the same problem with improved Euler's method.

2. $\star$ Apply Euler's method with step size $h = 0.1$, $h = 0.05$ and $h = 0.025$ to the differential equation
$$y' - 2y = \frac{x}{1+y^2}, \qquad y(1) = 7$$
to find approximate values $x = 1, 1.1, 1.2, 1.3$.

3. $\star$ Use improved Euler's method to find the approximate values for the solution of the initial value problem
$$y' = -2y^2 + xy + x^2 \;\; y(0) = 1$$
with step size $h = 0.1$ and $h = 0.05$.

4. $\star$ Verify that $y_1 = x^2$ and $y_2 = \frac{1}{x^2}$ are a set of fundamental solutions of the differential equation $x^2y'' + xy' - 4y = 0$ in $(-\infty, 0) \cup (0, \infty)$.

   Find the solution y if (i) $(y(1), y'(1)) = (2, 0)$ and (ii) $(y(-1), y'(-1)) = (2, 0)$.

   Observe that $y_1(x) = x^2$ is a solution on the whole of $\mathbb{R}$.

5. $\star$ Let $p, q : (a, b) \to \mathbb{R}$ be two continuous functions. Let $\{y_1, y_2\}$ be a fundamental set of solutions of the differential equations $y'' + py' + qy = 0$ in (a, b). Let y be the solution of the differential equation $y'' + py' + qy = 0$ with initial condition $y(x_0) = k_0$ and $y'(x_0) = k_1$. Show that $y = c_1y_1 + c_2y_2$ where $c_1 = \frac{y_2'(x_0)k_0 - y_2(x_0)k_1}{W(y_1,y_2)(x_0)}$ and $c_2 = \frac{y_1(x_0)k_1 - y_1'(x_0)k_0}{W(y_1,y_2)(x_0)}$.

6. $\star$ In the following problems, use the method of reduction of order to find a solution y_2 that is not a constant multiple of the solution y_1.
   (a) $y'' - 2ay' + a^2y = 0, \qquad y_(x) = e^{ax}$.
   (b) $4x^2 \sin x y'' - 4x(x \cos x + \sin x)y' + (2x \cos x + 3 \sin x)y = 0, \qquad y_(x) = x^{\frac{1}{2}}$.

7. Use Euler's method to find the approximate value of the following initial value problems.
   (a) $y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \qquad y(1) = 1; \;\; h = 0.1, h = 0.05$ at $x = 1.0, 1.1, 1.2, 1.3, 1.4$.
   Compare these approximate values with the values of the exact solution $y = \frac{1}{3x^2}(9 \ln x + x^3 + 2)$.
   (b) $(3y^2 + 4y)y' + 2x + \cos x = 0, \qquad y(0) = 1; \;\; h = 0.1h = 0.05$ at $x = 0, 0.1, 0.2, 0.3, 0.4$

8. In the exercises given below, use improved Euler's method to find the approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$ where x_0 is the initial point and $i = 1, 2, 3$.
   (a) $y' = 2x^2 + 3y^2 - 2, \qquad y(2) = 1; \quad h = 0.05$.
   (b) $y' = y + \sqrt{x^2 + y^2}, \qquad y(0) = 1; \quad h = 0.1$.

(c) $y' + x^2 y = \sin(xy)$ $\qquad y(1) = \pi; \qquad h = 0.2.$

9. Let $p, q : (a, b) \to \mathbb{R}$ be two continuous functions and y_1 be a solution of the differential equation $y'' + py' + qy = 0$ in (a, b). Let $y_2(x) = ky_1(x)$ for all $x \in (a, b)$ and k is a constant. Show that $W(y_1, y_2) \equiv 0$ in (a, b).

10. Let $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ for all $x \in \mathbb{R}$. Show that

    (a) the two functions y_1 and y_2 are linearly independent in any interval (a, b) such that $a < 0 < b$,

    (b) the wronskian $W(y_1, y_2) \equiv 0$ in $\mathbb{R}$ and

    (c) the two functions y_1 and y_2 can't be solutions of an ordinary differential equation $y'' + py' + qy = 0$.

11. Let y_1 and y_2 be two solutions of $x^2y'' + xy' + (x^2 - n^2)y = 0$ in $(0, \infty)$ with $(y_1(0), y_1'(0)) = (1, 0)$ and $(y_2(0), y_2'(0)) = (0, 1)$. Compute $W(y_1, y_2)$.