Problem Set 3

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Draw and illustrate in $\mathbb{R}^2$.
   (a) $\mathbf{e}_1 + \{n\mathbf{e}_2 | n \in \mathbb{N}\}$.
   (b) $\mathbf{e}_1 + \{\alpha\mathbf{e}_2 | \alpha \in \mathbb{R}\}$.

2. In $\mathbb{R}^2$, Is $\{\alpha\mathbf{e}_1|\alpha \in \mathbb{R}\} + \{\alpha\mathbf{e}_2|\alpha \in \mathbb{R}\} = \mathbb{R}^2$? What about $\{\alpha\mathbf{e}_1|\alpha \in \mathbb{R}\} + \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\} = \mathbb{R}^2$?

3. In $\mathbb{R}^3$ prove that $\left\{\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} |\alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} |\alpha \in \mathbb{R}\right\} = \mathbb{R}^3$. Do you use Gauss-Jordan Elimination (GJE) method somewhere?

4. Let L_1 and L_2 be two nonparallel lines passing through origin in $\mathbb{R}^3$. What is $L_1 + L_2$?

5. **(T)** Let L_1 and L_2 be two skewed (non parallel, nonintersecting) lines in $\mathbb{R}^3$? What is $L_1 + L_2$?

6. **(T)** Fix a non-negative integer n and let $\mathbb{R}[x; n]$ be the set of polynomials with real coefficients and degree less than or equal to n. That is, $\mathbb{R}[x; n] = \left\{\sum_{i=0}^{n} c_i x^i : c_0, c_1, \cdots, c_n \in \mathbb{R}\right\}$. Show that $\mathbb{R}[x; n]$ is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

7. Show that the space of all real $m \times n$ matrices is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

8. Let $\mathbb{M}_n(\mathbb{R})$ be the set of all $n \times n$ real matrices. Then, from above we see that $\mathbb{M}_n(\mathbb{R})$ is a real vector space. Now, prove the following:
   (a) $\mathbb{S} = \{A \in \mathbb{M}_n(\mathbb{R}) : A^t = A$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.
   (b) Fix $A \in \mathbb{M}_n(\mathbb{R})$. Define $\mathbb{U} = \{B \in \mathbb{M}_n(\mathbb{R}) : AB = BA\}$. Then, $\mathbb{U}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.
   (c) Let $\mathbb{W} = \{a_0 I + a_1 A + \cdots + a_m A^m : m$ is a non-negative integer, $a_i \in \mathbb{R}\}$. Then, $\mathbb{W}$ is a subspace of $\mathbb{U}$.

9. In $\mathbb{R}$, consider the addition $x \oplus y = x + y - 1$ and $a.x = a(x - 1) + 1$. Show that $\mathbb{R}$ is a real vector space with respect to these operations with additive identity 1 (note that 0 is NOT the additive identity).

10. **(T)** Which of the following are subspaces of $\mathbb{R}^3$:
$$(a)\ \{(x, y, z) \mid x \geq 0\},\ (b)\ \{(x, y, z) \mid x + y = z\},\ (c)\ \{(x, y, z) \mid x = y^2\}.$$

11. Find the condition on real numbers a, b, c, d so that the set $\{(x, y, z) \mid ax + by + cz = d\}$ is a subspace of $\mathbb{R}^3$.

12. **(T)** Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace. Prove that one of the spaces $W_i,\ i = 1, 2$ is contained in the other.

13. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be n vectors from a vector space V over $\mathbb{R}$. Define **linear span** of this set of vectors as

$$\mathrm{LS}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} : c_1, c_2, \ldots, c_n \in \mathbb{R}\},$$

that is, the set of all linear combinations of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Show that $\mathrm{LS}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$ is a subspace of V.

14. **(T)** Show that $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} = \mathrm{LS}(\{(1, 0, 0, -1),\ (0, 1, 0, 1),\ (0, 0, 1, 1)\}$ and hence is a subspace of $\mathbb{R}^4$.

15. Suppose S and T are two subspaces of a vector space V. *Define* the **sum**

$$S + T = \{\mathbf{s} + \mathbf{t} : \mathbf{s} \in S, \mathbf{t} \in T\}.$$

Show that $S + T$ satisfies the requirements for a vector space. Moreover, $LS(S \cup T) = S + T$.

16. **(T)** Find all the subspaces of $\mathbb{R}^2$.

17. **(T)** Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, with $a_{ij} \in \mathbb{C}$. Then, we define the following 4 fundamental subspaces:

   (a) The column space of A is defined as

$$\mathrm{col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{C}^n\} = \mathrm{LS}(A(:, 1), \ldots, A(:, n)) = \mathrm{LS}\left(\left\{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}\right\}\right)$$

   (b) The column space of A^* is defined as

$$\mathrm{col}(A^*) = \mathrm{LS}(A^*(1, :), \ldots, A^*(m, :)) = \{A^*\mathbf{x} : \mathbf{x} \in \mathbb{C}^m\}.$$

   (c) The null space of A is defined as

$$\text{Null Space}(A) = \mathcal{N}(A) = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\}.$$

   (d) The null space of A^* is defined as

$$\text{Null Space}(A^*) = \mathcal{N}(A^*) = \{\mathbf{x} \in \mathbb{C}^m : A^*\mathbf{x} = \mathbf{0}\}.$$

   Important: In case the matrix A has real entries, the spaces col(A^*) and Null Space(A^*) are called the row-space of A and the left-null space of A, respectively

Now, determine the above 4 mentioned fundamental spaces for the following matrices.

(i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$ (iii) $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

(iv) Suppose B and C are two $m \times n$ matrices and $S = \text{col}(B)$ and $T = \text{col}(C)$, then $S + T$ is a column space of what matrix M?

18. Construct a matrix whose column space contains $[\ 1\ \ 1\ \ 1\ ]^T$ and whose null space is the line of multiples of $[\ 1\ \ 1\ \ 1\ \ 1\ ]^T$.

19. **(T)** Suppose A is an m by n matrix of rank r.

   (a) If $A\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$, what is the column space of A?

   (b) In part (a), what are all equations or inequalities that must hold between the numbers m, n and r?

   (c) Give a specific example of a 3 by 2 matrix A of rank 1 with first row $[\ 2\ \ 5\ ]$. Describe the column space, $\text{col}(A)$, and the null space $N(A)$ completely.

   (d) Suppose the right side $\mathbf{b}$ is same as the first column in your example (part c). Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

20. Suppose the matrix A has row reduced echelon form R :

$$A = \begin{bmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ & (row & 3) & \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) What can you say immediately about row 3 of A?

   (b) What are the numbers a and b?

   (c) Describe all solutions of $R\mathbf{x} = \mathbf{0}$. Which among row spaces, column spaces and null spaces are the same for A and for R.

21. **(T)** Suppose that A is a 3×3 matrix. What relation is there between the null space of A and the null space of A^2? How about the null space of A^3?

22. Suppose R (an $m \times n$ matrix) is in row reduced echelon form $\begin{pmatrix} I_r & F \\ 0 & 0 \end{pmatrix}$, with r non-zero rows and first r pivot columns. Describe the column space and null space of R.

23. **(T)** Let $W_1 = \text{span}\left\{[1\ 1\ 0]^T, [-1\ 1\ 0]^T\right\}$ and $W_2 = \text{span}\left\{[1\ 0\ 2]^T, [-1\ 0\ 4]^T\right\}$. Show that $W_1 + W_2 = \mathbb{R}^3$. Give an example of a vector $v \in \mathbb{R}^3$ such that v can be written in two different ways in the form $v = v_1 + v_2$, where $v_1 \in W_1, v_2 \in W_2$.