Problem Set 2

Problems marked **(T)** are for discussions in Tutorial sessions.

1. **(T)** A square matrix P is called a *permutation matrix* if each row and column of P contains exactly one 1 and the rest of the entries are 0. Determine all the 3×3 permutation matrices. Now, let $A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$. Write down the permutation matrix P such that PA is upper triangular. Which permutation matrices P_1 and P_2 make P_1AP_2 lower triangular?

2. **(T)** Decide if they are row-equivalent:

   (a) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 4 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \end{bmatrix}$

3. Supply two examples each and explain their geometrical meaning.

   (a) Two linear equations in two variables with exactly one solution.

   (b) Two linear equations in two variables with infinitely many solutions.

   (c) Two linear equations in two variables with no solutions.

   (d) Three linear equations in two variables with exactly one solution.

   (e) Three linear equations in two variables with no solutions.

4. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two distinct solutions of the system $A\mathbf{x} = \mathbf{b}$. Prove that there are infinitely many solutions to this system, by showing that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is also a solution, for each $\lambda \in \mathbb{R}$. Do you have a geometric interpretation?

5. Is it possible to have $RREF([A|\mathbf{b}]) = \left[\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$? Give reasons for your answer.

6. Give examples of three matrices of size 3×4 that are in Row Reduced Echelon Form (RREF) but are different from the one given above. Give examples of matrices that are not in RREF, specifying the reasons.

7. Let B be a square invertible matrix. Then, prove that the system $A\mathbf{x} = \mathbf{b}$ and $BA\mathbf{x} = B\mathbf{b}$ are row-equivalent.

8. **[T]** Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have same solutions for every $\mathbf{b}$. Is it true that $A = C$?

9. Find the coefficients a, b, c, d so that the graph of $y = ax^3 + bx^2 + cx + d$ passes through $(1, 2), (-1, 6), (2, 3)(0, 1)$.

10. Find matrices A and B with the given property or explain why you can not:

    (a) The only solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

    (b) The only solution to $B\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

11. Apply Gauss elimination to solve the system $2x+y+2z = 3$ $3x-y+4z = 7$ and $4x+3y+6z = 5$.

12. **(T)** Using Gauss Jordan method, find the inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

13. **(T)** Let $B_{n\times n}$ be a real skew-symmetric. Show that $I - B$ is nonsingular.

14. **(T)** For two $n \times n$ matrices A and B, show that $\det(AB) = \det(A)\det(B)$.

15. For an $n \times n$ matrix $A = [a_{ij}]$, prove that $\det(A) = \det(A^T)$.

16. Let A be an $n \times n$ matrix. Prove that

    (a) If $A^2 = \mathbf{0}$ then A is singular.

    (b) If $A^2 = A, A \neq I$ then A is singular.

17. Consider the system $A\mathbf{x} = \mathbf{b}$. Let $RREF([A|\mathbf{b}])$ be one of the matrices in Question 5. Now, recall the matrices A_j's, for $1 \leq j \leq 3$ (defined to state the Cramer's rule for solving a linear system), that are obtained by replacing the j-th column of A by $\mathbf{b}$. Then, we see that the above system has NO solution even though $\det(A) = 0 = \det(A_j)$, for $1 \leq j \leq 3$.

18. **(T)** Let A be an invertible square matrix with integer entries. Show that A^{-1} has integer entries if and only if $\det(A) = \pm 1$.

19. Let A be an $n \times n$ matrix. Prove that the following statements are equivalent:

    (a) $\det(A) \neq 0$.

    (b) A is invertible.

    (c) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

    (d) The row-reduced echelon form of A is I_n.

    (e) A is a product of elementary matrices.

    (f) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\underline{.}$

    (g) The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$.

19a $\Longrightarrow$ 19b

By, definition, whenever $\det(A) \neq 0$, $A^{-1} = \dfrac{C^t}{\det(A)}$, where C is the co-factor matrix.

19b $\Longrightarrow$ 19a

As A is invertible, $AA^{-1} = I_n$ and hence $\det(A)\det(A^{-1}) = \det(I_n) = 1$. Hence, $\det(A) \neq 0$.

19b $\Longrightarrow$ 19c

As A is invertible, $A^{-1}A = I_n$. Let $\mathbf{x}_0$ be a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Then,

$$\mathbf{x}_0 = I_n\mathbf{x}_0 = (A^{-1}A)\mathbf{x}_0 = A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{0}$ is the only solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

19c $\Longrightarrow$ 19d

Let $\mathbf{x}^t = [x_1, x_2, \ldots, x_n]$. As $\mathbf{0}$ is the only solution of the linear system $A\mathbf{x} = \mathbf{0}$, the final equations are $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. These equations can be rewritten as

$$\begin{array}{rcl}
1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + \cdots + 0\cdot x_n & = & 0 \\
0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 + \cdots + 0\cdot x_n & = & 0 \\
0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + \cdots + 0\cdot x_n & = & 0 \\
\vdots & = & \vdots \\
0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + \cdots + 1\cdot x_n & = & 0.
\end{array}$$

That is, the final system of homogeneous system is given by $I_n \cdot \mathbf{x} = \mathbf{0}$. Or equivalently, the row-reduced echelon form of the augmented matrix $[A \;\; \mathbf{0}]$ is $[I_n \;\; \mathbf{0}]$. That is, the row-reduced echelon form of A is I_n.

19d $\Longrightarrow$ 19e

Suppose that the row-reduced echelon form of A is I_n. Then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_1E_2\cdots E_kA = I_n. \tag{1}$$

Now, the matrix E_j^{-1} is an elementary matrix and is the inverse of E_j for $1 \leq j \leq k$. Therefore, successively multiplying Equation (1) on the left by $E_1^{-1}, E_2^{-1}, \ldots, E_k^{-1}$, we get

$$A = E_k^{-1}E_{k-1}^{-1}\cdots E_2^{-1}E_1^{-1}$$

and thus A is a product of elementary matrices.

19e $\Longrightarrow$ 19b

Suppose $A = E_1E_2\cdots E_k$; where the E_i's are elementary matrices. As the elementary matrices are invertible and the product of invertible matrices is also invertible, we get the required result.

19b $\Longrightarrow$ 19f

Observe that $\mathbf{x}_0 = A^{-1}\mathbf{b}$ is the unique solution of the system $A\mathbf{x} = \mathbf{b}$.

19f $\Longrightarrow$ 19g

The system $A\mathbf{x} = \mathbf{b}$ has a solution and hence by definition, the system is consistent.

19g $\Longrightarrow$ 19b

For $1 \leq i \leq n$, define $\mathbf{e}_i = (0, \ldots, 0, \underbrace{1}_{i^{\text{th}} \text{ position}}, 0, \ldots, 0)^t$, and consider the linear system $A\mathbf{x} = \mathbf{e}_i$. By assumption, this system has a solution, say $\mathbf{x}_i$, for each i, $1 \leq i \leq n$. Define a matrix $B = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$. That is, the i^{th} column of B is the solution of the system $A\mathbf{x} = \mathbf{e}_i$. Then

$$AB = A[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = [A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n] = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] = I_n.$$

Therefore, the matrix A is invertible.

20. Let A be an $n \times n$ matrix. Then prove that $\det(A) = 0$ if and only if the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

21. Let A be an $n \times n$ matrix. Then the two statements given below cannot hold together.

    (a) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

    (b) The system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

22. **(T)** A real square matrix A is said to be orthogonal if $A^T A = AA^T = I$. Show that if A is orthogonal then $\det(A) = \pm 1$.

23. Let $A = [a_{ij}]$ be an invertible matrix and let $B = [p^{i-j} a_{ij}]$, for some $p \neq 0$. Find the inverse of B and also find $\det(B)$.

24. Suppose the 4×4 matrix M has 4 equal rows all containing a, b, c, d. We know that $det(M) = 0$. The problem is to find by any method

$$det(I + M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

25. **(T)** For a complex matrix $A = [a_{ij}]$, let $\bar{A} = [\,\overline{a_{ij}}\,]$ and $A^* = \bar{A}^T$. Show that $\det(\bar{A}) = \det(A^*) = \overline{\det A}$. Therefore if A is Hermitian (that is, $A^* = A$) then its determinant is real.

26. The numbers $1375, 1287, 4191$ and 5731 are all divisible by 11. Prove that 11 also divides the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 5 & 7 & 1 & 1 \end{bmatrix}.$$

27. Compute determinant of $\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix}$.