MSO202A: Assignment-V

1. Find

   (a) Taylor series of the function $f(z) = 1/z^2$ in powers of $z-1$.

   Soln: We have

$$-\frac{1}{z} = -\frac{1}{1+(z-1)} = \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n$$

   Differentiating we find

$$\frac{1}{z^2} = \sum_{n=1}^{\infty}(-1)^{n+1}n(z-1)^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n$$

   (b) Laurent series of the function $f(z) = 1/z^2$ for $\{z : |z-1| > 1\}$.

   Soln: We have

$$-\frac{1}{z} = -\frac{1}{1+(z-1)} = -\frac{1}{z-1}\left(1+\frac{1}{z-1}\right)^{-1} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(z-1)^{n+1}}$$

   Differentiating we find

$$\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)\frac{1}{(z-1)^{n+2}}$$

2. (a) Find Laurent series of the function $f(z) = \frac{6z+8}{(2z+3)(4z+5)}$ in the region

   (i) $\{z \in \mathbb{C} : |z| < \frac{5}{4}\}$ (ii) $\{z \in \mathbb{C} : \frac{5}{4} < |z| < \frac{3}{2}\}$ (iii) $\{z \in \mathbb{C} : |z| > \frac{3}{2}\}$

   Soln: We have

$$f(z) = \frac{6z+8}{(2z+3)(4z+5)} = \frac{1}{2z+3} + \frac{1}{4z+5} = \frac{1}{3}\frac{1}{1+2z/3} + \frac{1}{5}\frac{1}{1+4z/5}$$

   (i) If $|z| < 5/4$, then $|z| < 3/2$ and hence

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{2^n}{3^{n+1}} + \frac{4^n}{5^{n+1}}\right)z^n$$

   (ii) If $\frac{5}{4} < |z| < \frac{3}{2}$, then

$$f(z) = \frac{1}{3}\frac{1}{1+2z/3} + \frac{1}{4z}\frac{1}{1+5/4z}$$

   Hence

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{2^n}{3^{n+1}}z^n + \frac{5^n}{4^{n+1}}\frac{1}{z^{n+1}}\right)$$

   (iii) If $|z| > \frac{3}{2}$, then

$$f(z) = \frac{1}{2z}\frac{1}{1+3/2z} + \frac{1}{4z}\frac{1}{1+5/4z} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{3^n}{2^{n+1}} + \frac{5^n}{4^{n+1}}\right)\frac{1}{z^{n+1}}$$

(b) Find Laurent series of the function $f(z) = \frac{1}{z^3 - z^4}$ in the region

(i) $\{z \in \mathbb{C} : 0 < |z| < 1\}$ (ii) $\{z \in \mathbb{C} : |z| > 1\}$

Soln: We have

(i) If $0 < |z| < 1$, then

$$f(z) = \frac{1}{z^3(1-z)} = \frac{1}{z^3}(1 + z + z^2 + z^3 + \cdots) = \sum_{n=-3}^{\infty} z^n$$

(i) If $|z| > 1$, then

$$f(z) = -\frac{1}{z^4(1 - 1/z)} = -\frac{1}{z^4}\sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=4}^{\infty} \frac{1}{z^n}$$

3. Find the Laurent series of the function $f(z) = \exp(z + \frac{1}{z})$ around $z = 0$. Hence, show that (for $n \geq 0$)

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta} \cos n\theta \, d\theta = \sum_{j=0}^{\infty} \frac{1}{(n+j)!j!}.$$

Soln: We have

$$e^z e^{1/z} = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{j=0}^{\infty} \frac{1}{j! z^j}\right) = \sum_{n=-\infty}^{\infty}\left(\sum_{k-j=n, k\geq 0, j\geq 0} \frac{1}{k!j!}\right) z^n$$

This can be written as

$$e^z e^{1/z} = \sum_{n=-\infty}^{\infty}\left(\sum_{j\geq \max\{0,-n\}} \frac{1}{(j+n)!j!}\right) z^n = \sum_{n=-\infty}^{\infty} c_n z^n$$

Now for $n \geq 0$, we have

$$c_n = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{z+1/z}}{z^{n+1}}\, dz = \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta} e^{-in\theta}\, d\theta$$

Since c_n is real and $n \geq 0$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta} \cos n\theta \, d\theta = \left(\sum_{j=0}^{\infty} \frac{1}{(j+n)!j!}\right)$$

4. Is there a polynomial $P(z)$ such that $P(z)e^{1/z}$ is an entire function? Justify your answer.

Soln: Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$. Now $e^{1/z} = \sum_{j=0}^{\infty} \frac{1}{j! z^j}$. Now Laurent series of $P(z)e^{1/z}$ has terms of the form C_N/z^N where $C_N \neq 0$ and $N > 0$. To see this, note that

$$C_N = \frac{a_0}{N!} + \cdots + \frac{a_d}{(N+d)!}$$

Let r be smallest non-negative integer such that $a_r \neq 0$. Then

$$C_N = \frac{1}{(N+r)!}\left(a_r + \frac{a_{r+1}}{N+r+1} + \cdots + \frac{a_d}{(N+d)(N+d-1)\cdots(N+r+1)}\right)$$

For N large, $C_N \neq 0$ and hence the given function is not entire.

5. Which of the following singularities are removable/pole:

(i) $\frac{\sin z}{z^2-\pi^2}$ at $z = \pi$

(ii) $\frac{\sin \pi z}{(z-\pi)^2}$ at $z = \pi$

(iii) $\frac{z \cos z}{1-\sin z}$ at $z = \pi/2$

Soln: (i) $\frac{\sin z}{z^2-\pi^2} = -\frac{\sin(z-\pi)}{(z-\pi)(z+\pi)}$. Hence $z = \pi$ removable

(ii) $\frac{\sin z}{(z-\pi)^2} = -\frac{\sin(z-\pi)}{(z-\pi)^2}$. Hence $z = \pi$ simple pole

(iii) $1 - \sin z = 1 - \cos(z - \pi/2) = (z - \pi/2)^2 g(z)$, where $g(\pi/2) \neq 0$. Also $z \cos z = -z \sin(z - \pi/2) = (z - \pi/2)h(z)$ where $h(\pi/2) \neq 0$. Hence the required function has a simple pole at $z = \pi/2$.

6. Suppose f and g are two analytic functions in a neighbourhood of a point $z_0 \in \mathbb{C}$ such that $g(z_0) \neq 0$ and f has a simple zero at z_0. Prove that

$$\operatorname{Res}\left(\frac{g}{f} : z_0\right) = \frac{g(z_0)}{f'(z_0)}$$

Soln: Since $g(z_0) \neq 0$, g/f has a simple pole at z_0. Now

$$\operatorname{Res}\left(\frac{g}{f} : z_0\right) = \lim_{z \to z_0} (z - z_0)g(z)/f(z) = g(z_0)/f'(z_0)$$

7. Let f be analytic in a domain Ω and γ be a simple closed curve in Ω in the counterclockwise sense. Suppose z_0 is the only zero of f in the region enclosed by Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i m,$$

where m is the order of zero of f at z_0.

Soln: Let m be the order of zero of f and then $f(z) = (z - z_0)^m h(z)$ where $h(z_0) \neq 0$. Hence, there is a nbd $B_r(z_0)$ where $g(z) \neq 0$. Now in $B_r(z_0) \setminus \{z_0\}$, we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

Clearly, $g'(z)/g(z)$ is analytic in $B_r(z_0)$. Hence,

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_\gamma \frac{m}{z - z_0} dz = 2\pi i m$$

8. Find the isolated singularities and compute the residue of the functions

$$(i)\ \frac{e^z}{z^2 - 1} \qquad (ii)\ \frac{3z}{z^2 + iz + 2} \qquad (iii)\ \cot \pi z \qquad (iv)\ \frac{\pi \cot \pi z}{(z + 1/2)^2}$$

Soln: (i) Singularities are $z = \pm 1$. $\operatorname{Res}(f : 1) = \lim_{z \to 1} e^z/(z + 1) = e/2$ and $\operatorname{Res}(f : -1) = \lim_{z \to -1} e^z/(z - 1) = -e^{-1}/2$

(ii) $z^2 + iz + 2 = (z - i)(z + 2i)$. $\text{Res}(f : i) = \lim_{z\to i} 3z/(z + 2i) = 1$ and $\text{Res}(f : -2i) = \lim_{z\to -2i} 3z/(z - i) = 2$

(iii) $\sin \pi z = 0 \implies z = n$ where n is an integer. $\text{Res}(f : n) = \lim_{z\to n} \cos \pi z/ \sin \pi z = \cos \pi n/\pi \cos \pi n = 1/\pi$

(iv) Singularities $z = n, -1/2$ where n is an integer. At $z = n$, using previous question we get $\text{Res}(f : n) = 1/(n + 1/2)^2$. For $z = -1/2$ we have $\text{Res}(f : -1/2) = \frac{d}{dz}(\pi \cot \pi z)\big|_{z=-1/2} = -\pi^2/2$

9. Evaluate

$$(i) \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^{2n}}, \quad n \geq 1 \qquad (ii) \int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2 + a^2} dx \qquad (iii) \int_0^{\pi} \sin^{2n} \theta \, d\theta$$

Soln:

(i) Consider the function $f(z) = 1/(1+z^2)^n$ which has pole of order n at $z = \pm i$. Integrate over a contour that consists of real line from $-R$ to R and the semicircle C_R of radius R with centre at the origin. We take R large. Then

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \text{Res}(f, i)$$

Now

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi R}{(R^2 - 1)^n} \to 0 \quad \text{as} \quad R \to \infty$$

Hence

$$\int_{-R}^{R} f(x)dx = 2\pi i \text{Res}(f, i) = 2\pi i \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \frac{1}{(z+i)^n}\bigg|_{z=i} = \text{etc}$$

(ii) Consider $a > 0$ and let $f(z) = ze^{3iz}/(z^2 + a^2)$ which has simple poles at $z = \pm ai$. Take contour as in (i) and we get

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \text{Res}(f, ai)$$

$\Big($ Jordan's lemma: Consider a complex-valued continuous function f defined on a semicircular contour $C_R = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$ of radius R lying in the upper half plane centred at the origin. If the function is of the form $f(z) = e^{iaz}g(z)$ with positive parameter a, then

$$\left| \int_{C_R} f(z) \, dz \right| \leq \frac{\pi}{a} M_R \qquad \text{where} \qquad M_R := \max_{\theta\in[0,\pi]} \left| g(Re^{i\theta}) \right| \Big)$$

On C_R, using Jordan's lemma we get Hence

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi}{3} \frac{R}{R^2 - a^2} \to 0 \quad \text{as} \quad R \to \infty$$

Hence,

$$\int_{-\infty}^{\infty} \frac{xe^{i3x}}{x^2 + a^2} dx = 2\pi i \text{Res}(f, ai) = i\pi e^{-3a}$$

Hence

$$\int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2 + a^2} dx = \frac{\pi}{e^{3a}}$$

(iii)

$$\begin{aligned}
\int_0^{\pi} \sin^{2n}\theta \, d\theta &= \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta \, d\theta \\
&= \frac{1}{2(2i)^{2n}} \int_{|z|=1} \frac{(z^2-1)^{2n}}{z^{2n}} \frac{dz}{iz} \\
&= \frac{(-1)^n}{2^{2n+1} i} \int_{|z|=1} \frac{(z^2-1)^{2n}}{z^{2n+1}} dz \\
&= \frac{(-1)^n}{2^{2n+1} i} 2\pi i \ \mathrm{Res}\left(\frac{(z^2-1)^{2n}}{z^{2n+1}}, 0\right) \\
&= \frac{(-1)^n}{2^{2n+1}} 2\pi \frac{1}{(2n)!} \left(\frac{d^{2n}}{dz^{2n}}(z^2-1)^{2n}\right)_{z=0} \\
&= \frac{(-1)^n \pi}{2^{2n}(2n)!} \binom{2n}{n} (-1)^n (2n)! = \frac{\pi}{2^{2n}} \frac{(2n)!}{(n!)^2}
\end{aligned}$$

10. Compute the following integrals

$$(i) \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \qquad (ii) \int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \qquad (iii) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx, \quad 0 < a < 1.$$

Soln: (i) Consider $f(z) = e^{iz}/z$ which has a simple pole at $z = 0$ on the real axis. Consider a contour C which consists of $C_R : Re^{i\theta}$, $0 \le \theta \le \pi$, semicircle $C_\rho : \rho e^{i\theta}$, $0 \le \theta \le \pi$ and the line segments $[-R, -\rho]$ and $[\rho, R]$ traversed in the anticlockwsie sense where $\rho < R$. Since the function is analytic inside the contour C, we have $\int_C f(z)dz = 0$. Using Jordan's lemma

$$\left| \int_{C_R} \frac{e^{iz}}{z} dz \right| \le \frac{\pi}{R} \to 0 \quad \text{as} \quad R \to \infty$$

Also,

$$\int_{C_\rho} \frac{e^{iz}}{z} dz = \int_{C_\rho} \left(\frac{1}{z} + g(z)\right) dz = -\pi i,$$

where $g(z)$ is analytic and its integral vanish due to bounded $g(z)$ on C_ρ and $\rho \to 0$.

Taking limit $R \to \infty$ and $\rho \to 0$ we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

(ii) Take contour like in (i) and $f(z) = (e^{iaz} - e^{ibz})/z^2$ for which $z = 0$ is a simple pole. Now

$$\left| \int_{C_R} f(z) dz \right| \le \frac{1}{R} \int_0^{\pi} (e^{-aR\sin\theta} + e^{-bR\sin\theta}) d\theta \to 0 \quad \text{as} \quad R \to \infty$$

Also, [as in (i)]

$$\int_{C_\rho} f(z) dz = -\pi i \mathrm{Res}(f, 0) = -i\pi i(a-b)/2,$$

Taking limit $R \to \infty$, $\rho \to 0$ and real part we get

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \pi \frac{b-a}{2}$$

(iii) Let $f(z) = e^{az}/e^z + 1$ which has a simple pole at $z = \pi i$. Tale rectangular contour C on the upper plane with vertices at $R, R + 2\pi i, -R + 2\pi i$ and R and we must have

$$\int_C f(z) dz = 2\pi i \text{ Res}(f, z = \pi i)$$

Hence,

$$I_1 + I_2 + I_3 + I_4 = -2\pi i e^{a\pi i}$$

where $I_1 = \int_{-R}^{R} f(x) dx$ is the integral along $y = 0$. Now along $y = 2\pi$:

$$I_3 = \int_R^{-R} \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i} + 1} dx = -e^{2\pi a i} I_1$$

Now along $x = R$, we have

$$I_2 = \int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy} + 1} i dy \implies |I_2| \leq 2\pi \frac{e^{aR}}{e^R - 1} \to 0 \quad \text{as} \quad R \to \infty \qquad (\text{since } a < 1)$$

Similarly I_4 along $x = -R$ is also zero. Thus, taking $R \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} = \frac{-2\pi i e^{a\pi i}}{1 - e^{2\pi a i}} = \frac{\pi}{\sin \pi a}$$