Problem Sheet I, PHY102A, part II

1) A particle moves in a central force potential in the (r, θ) plane. In this situation, the apsidal angle is defined as the angle subtended by two lines that join the centre to the perihelion and the next aphelion. Consider a circular orbit specified in the variable $u = 1/r$ by $u = u_0$. Say this orbit is perturbed slightly, so that $u = u_0 + \rho(\theta)$ with ρ small compared to u_0. This is called a near circular orbit.

a) Show that for a near circular orbit, $\rho(\theta)$ is an oscillatory function of θ and determine the frequency of oscillations. For this frequency to be a **constant**, What should be the form of the central potential ?
b) Explicitly calculate the apsidal angle for a near circular orbit, from your answer of part (a).

2) Suppose that the apsidal angle is constant for an arbitrary orbit, not necessarily near circular (only then the orbit might be closed, otherwise the perihelion or aphelion will precess). For positive energies, compute the apsidal angle for the energy of the particle in the limit $E \to \infty$. Compare this with the near circular orbit result in question 1(b) to determine the force law.

3) Do the same for negative energies, by computing the apsidal angle in the limit $E \to 0$. Determine the force law again by comparison to the near circular orbit of question 1(b).

The above exercise tells you that closed orbits in central forces are possible only for the Newtonian and the simple harmonic potentials ! This is called Bertrand's theorem and is a striking result in mechanics.