Section 52: Problem 3 Solution

> Working problems is a crucial part of learning mathematics. No one can learn topology merely by poring over the definitions, theorems, and examples that are worked out in the text. One must work part of it out for oneself. To provide that opportunity is the purpose of the exercises.
>
> — *James R. Munkres*

Let x_0 and x_1 be points of the path-connected space X . Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1 , we have $\hat{\alpha} = \hat{\beta}$.

For every α, β , $\hat{\alpha} = \hat{\beta} \Leftrightarrow$ for every α, β and f , a loop based at x_0 , $\hat{\alpha}([f]) = \hat{\beta}([f]) \Leftrightarrow [\overline{\alpha}] \star [f] \star [\alpha] = [\overline{\beta}] \star [f] \star [\beta] \Leftrightarrow [f] \star [\alpha] \star [\overline{\beta}] = [\alpha] \star [\overline{\beta}] \star [f] \Leftrightarrow [f] \star [\alpha \star \overline{\beta}] = [\alpha \star \overline{\beta}] \star [f]$. Note that $\alpha \star \overline{\beta}$ is an arbitrary loop based at x_0 that passes through x_1 . So, if the group of the path homotopy classes of loops based at x_0 is abelian, then the right hand side expression holds for arbitrary α , β and f , therefore, for every α, β , $\hat{\alpha} = \hat{\beta}$. Vice versa, if for every α, β , $\hat{\alpha} = \hat{\beta}$, then we have shown that the group is commutative when at least one of the terms is a path homotopy class of a loop passing through x_1 . So, take arbitrary $[f], [g] \in \pi_1(X, x_0)$ and take any path α from x_0 to x_1 . Then, $g \star \alpha \star \overline{\alpha}$ is a loop based at x_0 passing through x_1 . Then, $[f] \star [g \star \alpha \star \overline{\alpha}] = [g \star \alpha \star \overline{\alpha}] \star [f]$, but $[g \star \alpha \star \overline{\alpha}] = [g]$.