Studies in Systems, Decision and Control 89

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Formal Methods for Discrete-Time Dynamical Systems

Springer

Chapter 6
Discrete-Time Dynamical Systems

In this chapter, we introduce the two classes of discrete-time dynamical systems that we will focus on in the rest of the book: piecewise affine control systems with polytopic parameter uncertainties and switched linear systems. As particular instantiations of the first class, we define autonomous systems, fixed parameter systems, and combinations of the above. By generalizing the ideas already presented in Sect. 1.2 and Example 1.8, we define embeddings of such systems into (infinite) transition systems. This enables formal definitions for their semantics and the use of abstractions to map analysis and control problems for such systems to verification and synthesis problems for finite transition systems, which were treated in Part II.

6.1 Piecewise Affine Systems

Let L be a finite index set and $\mathbf{X}_l, l \in L$ be a set of open, full dimensional polytopes[1] in $\mathbb{R}^N$, such that $\mathbf{X}_{l_1} \cap \mathbf{X}_{l_2} = \emptyset$ for all $l_1, l_2 \in L$, where $l_1 \neq l_2$.

Definition 6.1 (*PWA Control System*) A discrete-time, uncertain-parameter piecewise affine (PWA) control system $\mathscr{W}$ over $\mathbf{X} = \bigcup_{l \in L} \mathbf{X}_l$ is defined as:

$$\mathscr{W} \;:\; x(k+1) = A_l x(k) + B_l u(k) + c_l,\;\; x(k) \in \mathbf{X}_l,\;\; u(k) \in \mathbf{U},\;\; l \in L \qquad (6.1)$$

where, at each time step $k = 0, 1, \ldots, x(k) \in \mathbb{R}^N$ is the state of the system and $u(k)$ is the input restricted to a polytopic set $\mathbf{U} \subset \mathbb{R}^M$. Matrices $A_l \in \mathbf{P}_l^A$, $B_l \in \mathbb{R}^{N \times M}$, $c_l \in \mathbf{P}_l^c$ are the system parameters for mode $l \in L$, where parameters A_l and c_l for each $l \in L$ are restricted to polytopic sets $\mathbf{P}_l^A \subset \mathbb{R}^{N \times N}$ and $\mathbf{P}_l^c \subset \mathbb{R}^N$, respectively.

Note that, for technical reasons to become clear later, we assume that there is no parameter uncertainty in the control parameter matrix B_l.

[1] In the rest of the book, we assume polytopes are open and full dimensional, unless specifically mentioned otherwise. See Sect. A.1. This assumption is discussed in Sect. 6.3.

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C. Belta et al., *Formal Methods for Discrete-Time Dynamical Systems*,
Studies in Systems, Decision and Control 89, DOI 10.1007/978-3-319-50763-7_6

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In the following, we define several subclasses of PWA systems used later in the book, which are obtained by restricting Definition 6.1.

Definition 6.2 (*Fixed Parameter PWA Control System*) A fixed-parameter PWA control system is a PWA system (Definition 6.1) where, for all modes $l \in L$, the sets $\mathbf{P}_l^A$ and $\mathbf{P}_l^c$ are singletons, i.e., $A_l \in \mathbb{R}^{N \times N}$ and $c_l \in \mathbb{R}^N$.

In the particular case when $\mathbf{P}_l^c = \emptyset$, for all $l \in L$, the system from Definition 6.2 is called a *fixed parameter piecewise linear control system*. In other words, such a system is described by $x(k+1) = A_l x(k) + B_l u(k)$, $x(k) \in \mathbf{X}_l$, $u(k) \in \mathbf{U}$, $l \in L$.

Definition 6.3 (*Autonomous PWA System*) An autonomous PWA system is a PWA system (Definition 6.1) with input $u(k) = 0$ for all time steps $k = 0, 1, \ldots$ i.e.,

$$\mathscr{W} \ : \ x(k+1) = A_l x(k) + c_l, \ \ x(k) \in \mathbf{X}_l, \ \ l \in L. \tag{6.2}$$

Definition 6.4 (*Autonomous Fixed Parameter PWA System*) An autonomous, fixed-parameter PWA system is an autonomous PWA system (Definition 6.3) where, for all modes $l \in L$, the sets $\mathbf{P}_l^A$ and $\mathbf{P}_l^c$ are singletons, i.e., $A_l \in \mathbb{R}^{N \times N}$ and $c_l \in \mathbb{R}^N$.

In the particular case when $\mathbf{P}_l^c = \emptyset$, for all $l \in L$, the system from Definition 6.4 is called a *fixed parameter piecewise linear system*. In other words, such a system is described by $x(k+1) = A_l x(k)$, $x(k) \in \mathbf{X}_l$, $l \in L$.

Definition 6.5 (*Autonomous Additive Uncertainty PWA System*) An autonomous, additive uncertainty PWA system is an autonomous PWA system (Definition 6.3) where, for all modes $l \in L$, the set $\mathbf{P}_l^A$ is a singleton, i.e., $A_l \in \mathbb{R}^{N \times N}$ and $\mathbf{P}_l^c \subset \mathbb{R}^N$ is a polytopic parameter set.

In other words, only the vector component c_l is uncertain in Definition 6.5, while the matrix component A_l is fixed. Both the vector and matrix parameter components are fixed in Definition 6.4.

System $\mathscr{W}$ evolves along different affine dynamics in different regions of the continuous state space $\mathbf{X}$. When $\mathscr{W}$ is in a state $x(k) \in \mathbf{X}_l$ for some $l \in L$, we say that the system is in *mode* $l \in L$. Then, the next visited state $x(k+1)$ is computed according to the affine map of Eq. (6.1) with parameters A_l and c_l, specifying the dynamics of $\mathscr{W}$ in mode l. Starting from initial conditions $x(0) \in \mathbf{X}_{l_0}$ for some $l_0 \in L$ a trajectory of system $\mathscr{W}$ can be obtained by the following (numerical simulation) procedure:

i. Start from initial conditions $x(0) \in \mathbf{X}_{l_0}$, where the system is in mode l_0.
ii. Select parameters $A \in \mathbf{P}_{l_0}^A$ and $c \in \mathbf{P}_{l_0}^c$ from the allowed parameter sets.
iii. Select an input $u \in \mathbf{U}$ from the allowed input set.
iv. Apply the affine map of Definition 6.1 to compute the next state $x(1) = Ax(0) + B_{l_0} u + c$.
v. Find the mode $l_1 \in L$ of $\mathscr{W}$ such that $x(1) \in \mathbf{X}_{l_1}$.
vi. Repeat this procedure iteratively for each subsequent step.

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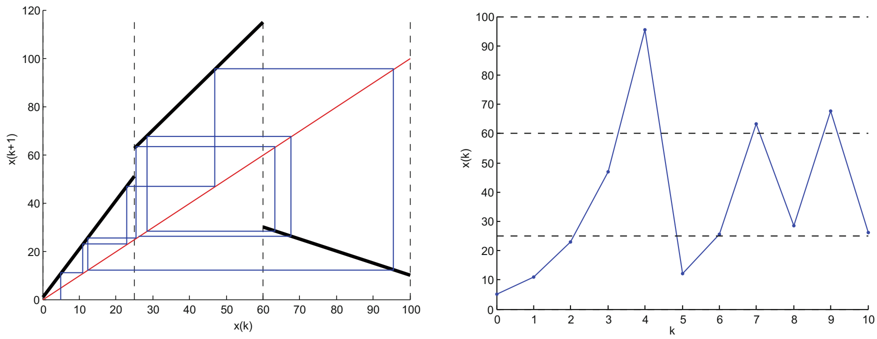

(a) Cobweb diagram of PWA system $\mathscr{W}$. (b) A trajectory produced by $\mathscr{W}$.

Fig. 6.1 A one dimensional PWA system is defined by the three regions of different dynamics, separated by *dashed lines* in (**a**). The parameters of the system in each mode are represented by the *black lines* in (**a**). Applying the PWA map iteratively is represented by the cobweb diagram from (**a**) and generates the trajectory of the system shown in (**b**). See Example 6.1 for additional details

The operating regions $\mathbf{X}_l, l \in L$ from Definition 6.1 of a PWA system are also considered as regions of interests of system $\mathscr{W}$. As we are only interested in trajectories of system $\mathscr{W}$ evolving in $\mathbf{X}$, we define an additional mode Out with trivial dynamics $x(k+1) = x(k)$ and region $\mathbf{X}_{\text{Out}} = \mathbb{R}^N \setminus \mathbf{X}$. Informally, a trajectory $w_{\mathbf{X}} = w_{\mathbf{X}}(1)w_{\mathbf{X}}(2)\ldots$ produces a word $w_L = w_L(1)w_L(2)\ldots$ such that $w_L(i)$ is the index of the region visited by state $w_{\mathbf{X}}(i)$, i.e., $w_{\mathbf{X}}(i) \in \mathbf{X}_{w_L(i)}$, and $w_L(i)$ is Out if $w_{\mathbf{X}}(i) \notin \mathbf{X}$. For example, trajectory $x(0)x(1)x(2)\ldots$ satisfying $x(0), x(1) \in \mathbf{X}_{l_1}$ and $x(2) \in \mathbf{X}_{l_2}$ for some $l_1, l_2 \in L$ produces word $l_1 l_1 l_2 \ldots$. After a short example, we formalize the semantics of PWA trajectories and their satisfaction of LTL formulas through an embedding into a transition system, which generalizes the one already introduced in Sect. 1.2.

Example 6.1 We define the fixed-parameter, autonomous, one dimensional ($N = 1$) piecewise affine system $\mathscr{W}$ shown schematically in Fig. 6.1a, which has three different modes ($L = \{1, 2, 3\}$). Regions $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ are open, full dimensional polytopes in $\mathbb{R}^1$, defined in V-representation (see Definition A.5) as the interiors of the convex hulls (Definition A.3) $\mathbf{X}_1 = \text{int}(\text{hull}(\{1, 25\}))$, $\mathbf{X}_2 = \text{int}(\text{hull}(\{25, 60\}))$ and $\mathbf{X}_3 = \text{int}(\text{hull}(\{60, 100\})$ (i.e., $V(\mathbf{X}_1) = \{1, 25\}$, $V(\mathbf{X}_2) = \{25, 60\}$ and $V(\mathbf{X}_3) = \{60, 100\}$ are the sets of vertices of closures of polytopes $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$, respectively). These polytopes define the regions of the state space of $\mathscr{W}$ where the system operates under different parameters (i.e., $\mathscr{W}$ is in a different mode in each region). They are represented in Fig. 6.1a, b by dashed lines, partitioning the state space $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2 \cup \mathbf{X}_3$.

The parameters of system $\mathscr{W}$ are defined as $A_1 = 2$, $b_1 = 1$, $A_2 = 1.5$, $b_2 = 25$ and $A_3 = -0.5$, $b_3 = 60$ and are represented as thick black lines in Fig. 6.1a.

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Selecting initial conditions $x(0) = 5$ allows us to generate a trajectory $w_{\mathbf{X}}$ of $\mathscr{W}$ by applying the update map iteratively (the map update is represented by the cobweb diagram of Fig. 6.1a). The initial state and the following 10 steps $w_{\mathbf{X}}(1) \ldots w_{\mathbf{X}}(11) = x(0) \ldots x(10)$ of this trajectory are represented in Fig. 6.1b. It is easy to see that in states $x(0), x(1), x(2), x(5) \in \mathbf{X}_1$, $x(3), x(6), x(8), x(10) \in \mathbf{X}_2$ and $x(4), x(7), x(9) \in \mathbf{X}_3$ the system is in mode 1,2 and 3, respectively. Then, the word produced by this trajectory is w_L where, for the fragment $w_L(1) \ldots w_L(11)$, we have $w_L(1) = w_L(2) = w_L(3) = w_L(6) = 1, w_L(4) = w_L(7) = w_L(9) = w_L(11) = 2$ and $w_L(5) = w_L(8) = w_L(11) = 3$ (i.e., $w_L = 1\,1\,1\,2\,3\,1\,2\,3\,2\,3\,2\,\ldots$).

Definition 6.6 (*Embedding Transition System for* $\mathscr{W}$) The embedding for PWA control system $\mathscr{W}$ (Definition 6.1) is a transition system (Definition 1.1) $T_{\mathscr{W}} = (X_{\mathscr{W}}, \Sigma_{\mathscr{W}}, \delta_{\mathscr{W}}, O_{\mathscr{W}}, o_{\mathscr{W}})$ with:

- $X_{\mathscr{W}} = \mathbb{R}^n$,
- $\Sigma_{\mathscr{W}} = \mathbf{U}$,
- $\delta_{\mathscr{W}}(x, u) = A_l x + B_l u + c_l$, if $x \in \mathbf{X}_l$,
- $O_{\mathscr{W}} = L \cup \{\text{Out}\}$,
- $o_{\mathscr{W}}(x) = l$ if and only if there exists $l \in L$ such that $x \in \mathbf{X}_l$ and $o_{\mathscr{W}}(x) = \text{Out}$ otherwise.

The embedding transition system from Definition 6.6 has an infinite number of states and inputs and is non-blocking. Furthermore, for the general class of uncertain-parameter PWA systems, the embedding transition system is non-deterministic. Indeed, given state $x \in \mathbf{X}$ and input $u \in \mathbf{U}$, multiple states can be reached in a single step through the dynamics defined in Definition 6.1, depending on the choice of parameters $A_l \in \mathbf{P}_l^A$ and $c_l \in \mathbf{P}_l^c$—the possible non-deterministic next-state choices are captured in the set $\delta_{\mathscr{W}}(x, u)$. In contrast, a fixed-parameter PWA system $\mathscr{W}$ (Definition 6.2) leads to a deterministic embedding transition system, since for each state $x \in \mathbf{X}$ and input $u \in \mathbf{U}$, only a single state $x' = A_l x + B_l u + c_l$ satisfies the dynamics of $\mathscr{W}$ and, therefore, $\delta_{\mathscr{W}}(x, u)$ is a singleton. Since the other PWA subclasses from Definitions 6.3, 6.4, and 6.5 are particular cases of the general PWA from Definition 6.1, the embedding $T_{\mathscr{W}}$ defined above applies with minor and obvious adjustments. For example, for the autonomous PWA systems from Definitions 6.3 and 6.5, the embeddings $T_{\mathscr{W}}$ are non-deterministic transition systems with no inputs, while for the PWA from Definition 6.4, the embedding is a deterministic transition system with no inputs.

Only infinite words are produced by the embedding transition system for each of the system classes defined above and, therefore, LTL formulas over $L \cup \{\text{Out}\}$ can be interpreted over such words, leading to the following definition:

Definition 6.7 (*LTL satisfaction for* $\mathscr{W}$) Trajectories of a PWA system $\mathscr{W}$ (Definition 6.1) originating in a polytope $\mathbf{X}_0 \subseteq \mathbf{X}$ satisfy formula ϕ if and only if $T_{\mathscr{W}}(\mathbf{X}_0)$ satisfies ϕ (according to Definition 3.1).

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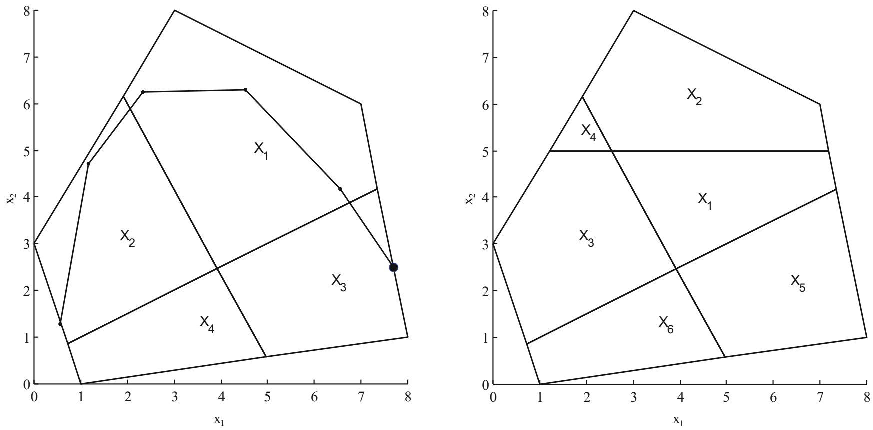

Fig. 6.2 A two dimensional PWA system $\mathscr{W}$ is defined by the four regions of different dynamics but trajectories of the system might leave the defined state space. In **a**, such a trajectory is shown evolving along the state space $\mathbf{X}$ of the system, where the initial state and subsequently visited states are represented by *circles*. The *larger circle* at the boundary of the outer polytope represents all states of index larger than 5. To formulate specifications over linear predicates that do not observe the initial set of polytopes, additional partitioning of the state space might be necessary as in (**b**). See Example 6.2 for additional details

Example 6.2 We define the two dimensional ($N = 2$) autonomous fixed-parameter PWA system $\mathscr{W}$ (Definition 6.4) shown schematically in Fig. 6.2a, which has four different modes ($L = \{1, \ldots, 4\}$). The state space of $\mathscr{W}$ is $\mathbf{X} = \mathbf{X}_1 \cup \ldots \cup \mathbf{X}_4$, where $\mathbf{X}_1, \ldots, \mathbf{X}_4$ are open, full dimensional polytopes determined by cutting hull({[1, 0], [8, 1], [7, 6], [3, 8], [0, 3]}) with hyperplanes $[1 - 2]x = -1$ and $[-1.83 - 1]x = -9.65$ (Fig. 6.2a). The parameters of the system in each mode are defined as

$$A_1 = \begin{bmatrix} 0.55 & -0.5 \\ 0.7 & 0.65 \end{bmatrix}, c_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.35 & -0.5 \\ 0.5 & 0.15 \end{bmatrix}, c_2 = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0.95 & -0.5 \\ 0.5 & 0.65 \end{bmatrix}, c_3 = \begin{bmatrix} 0.5 \\ -1.3 \end{bmatrix}, A_4 = \begin{bmatrix} 0.95 & -0.5 \\ 0.5 & 0.65 \end{bmatrix}, c_4 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \tag{6.3}$$

A trajectory $w_{X_\mathscr{W}} = w_{X_\mathscr{W}}(1)w_{X_\mathscr{W}}(2)\ldots$ of $\mathscr{W}$ is generated, starting from initial conditions $w_{X_\mathscr{W}}(1) = x(0) = [7.7, 2.5]$. After five steps, the trajectory exists $\mathbf{X}$ (i.e., $w_{X_\mathscr{W}}(6) = x(5) = [0.5489, 1.2808] \notin \mathbf{X}$). Since PWA systems were defined with trivial dynamics in mode Out (i.e., when a state outside the defined state space is visited) the trajectory remains in state $x(5)$ for all future times (i.e., for $k = 6, 7, 8, \ldots$ we have $w_{X_\mathscr{W}}(k) = x(5)$). The word produced by trajectory $w_{X_\mathscr{W}}$ is $w_{O_\mathscr{W}} = w_{O_\mathscr{W}}(1)w_{O_\mathscr{W}}(2)\ldots$, where $w_{O_\mathscr{W}}(1) = 3$, $w_{O_\mathscr{W}}(2) = w_{O_\mathscr{W}}(3) = w_{O_\mathscr{W}}(4) = 1$, $w_{O_\mathscr{W}}(5) = 2$ and $w_{O_\mathscr{W}}(k) = \text{Out}$ for $k = 6, 7, 8, \ldots$ (i.e., $w_{O_\mathscr{W}} = 3\,1\,1\,1\,2\,\text{Out}\,\text{Out}\ldots$).

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If we are interested in behaviors of the PWA system $\mathscr{W}$ where the second component of the state x reaches values above 5, we need to further partition the state of the system using hyperplane $[0, 1]x = 5$. This results in a system with 6 polytopic regions ($L = \{1, \ldots, 6\}$) denoted by $\mathbf{X}_1, \ldots, \mathbf{X}_6$ (see Fig. 6.2b). To specify that all trajectories eventually visit a state where the second component of x has a value above 5, we write the LTL formula $\Diamond(2 \vee 4)$. In other words, we require that trajectories eventually visit regions $\mathbf{X}_2$ or $\mathbf{X}_4$, where the above specification is satisfied.

6.2 Switched Linear Systems

Definition 6.8 (*Switched Linear System*) A discrete-time switched linear system over $\mathbf{X} \subset \mathbb{R}^N$ is defined as

$$\mathscr{S} \ : \ x(k+1) = A_{\gamma_k} x(k), \quad \gamma_k \in \Gamma, \tag{6.4}$$

where, at each time step $k = 0, 1, \ldots, x(k) \in \mathbb{R}^N$ is the state of the system, γ_k is the input that selects the active subsystem from a finite index set Γ, and $A_\gamma \in \mathbb{R}^{N \times N}$, for all $\gamma \in \Gamma$.

System $\mathscr{S}$ evolves along different linear dynamics, depending on the chosen value of γ_k from Γ at time k. Similar to the terminology for system $\mathscr{W}$ defined above, when $\mathscr{S}$ evolves along dynamics γ, we say that the system is in *mode* $\gamma \in \Gamma$. Starting from initial conditions $x(0) \in \mathbf{X}$ and initial mode $\gamma_0 \in \Gamma$, given a function $\gamma : \{0, 1, 2, \ldots\} \to \Gamma$, a trajectory of system $\mathscr{S}$ can be obtained by the following (numerical simulation) procedure:

i. Start from initial conditions $x(0) \in \mathbf{X}$ and initial mode $\gamma_0 \in \Gamma$.
ii. Apply the linear map from Eq. (6.4) to compute the next state $x(1) = A_{\gamma_0} x(0)$.
iii. Update the mode γ_1 according to function γ.
iv. Repeat this procedure iteratively for each subsequent step.

We are interested in studying trajectories of system $\mathscr{S}$ with respect to a finite set of semi linear sets $\mathbf{X}_l$, $l \in L$, where $\mathbf{X}_{l_1} \cap \mathbf{X}_{l_2} = \emptyset$, for any $l_1 \neq l_2$, and $\mathbf{X} = \cup_{l \in L} \mathbf{X}_l$ (see Appendix A.4 and Example 10.1). Informally, similar to the PWA system $\mathscr{W}$ presented above, a trajectory $w_{\mathbf{X}} = w_{\mathbf{X}}(1) w_{\mathbf{X}}(2) \ldots$ produces a word $w_L = w_L(1) w_L(2) \ldots$ such that $w_L(i)$ is the index of the region visited by state $w_{\mathbf{X}}(i)$, i.e., $w_{\mathbf{X}}(i) \in \mathbf{X}_{w_L(i)}$. As it will become clear in Chap. 10, because of the particular problem of interest, the trajectories of $\mathscr{S}$ always stay inside $\mathbf{X}$, and the extra mode Out is not necessary in this case. Also, as in Sect. 6.1, we informally think of $\mathbf{X}_l$, $l \in L$ as a partition of $\mathbf{X}$, and ignore behaviors on the boundaries of $\mathbf{X}_l$. This assumption is discussed in Sect. 6.3.

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Definition 6.9 (*Embedding Transition System for $\mathscr{S}$*) The embedding for the switched control system $\mathscr{S}$ (Definition 6.8) is a transition system $T_{\mathscr{S}} = (X_{\mathscr{S}}, \Sigma_{\mathscr{S}}, \delta_{\mathscr{S}}, O_{\mathscr{S}}, o_{\mathscr{S}})$ with:

- $X_{\mathscr{S}} = \mathbf{X}$,
- $\Sigma_{\mathscr{S}} = \Gamma$,
- $\delta_{\mathscr{S}}(x, \gamma) = A_{\gamma} x$,
- $O_{\mathscr{S}} = L$,
- $o_{\mathscr{S}}(x) = l, l \in L$ if and only if $x \in \mathbf{X}_l$.

The embedding transition system from Definition 6.9 has an infinite number of states and finitely many inputs. It is deterministic and non-blocking. It produces infinite words, and, therefore, LTL formulas over L can be interpreted over such words, leading to the following definition:

Definition 6.10 (*LTL satisfaction for $\mathscr{S}$*) Trajectories of a switched system $\mathscr{S}$ (Definition 6.8) originating in a region $\mathbf{X}_0 \subseteq \mathbf{X}$ satisfy LTL formula ϕ over L if and only if $T_{\mathscr{S}}(\mathbf{X}_0)$ satisfies ϕ (according to Definition 3.1).

Similar to the PWA system $\mathscr{W}$ treated in Sect. 6.1, we can define different types of embeddings for $\mathscr{S}$ as particular cases of the one defined above. With particular relevance to the verification Problem 10.2 treated in Chap. 10, an autonomous embedding transition system that captures the behavior of $\mathscr{S}$ under all possible switchings can be defined as $T_{\mathscr{S}}^A = (X_{\mathscr{S}}, \delta_{\mathscr{S}}^A, O_{\mathscr{S}}, o_{\mathscr{S}})$, where $X_{\mathscr{S}}$, $O_{\mathscr{S}}$, and $o_{\mathscr{S}}$ are as defined above and $\delta_{\mathscr{S}}^A(x) = \{A_{\gamma} x,\ \gamma \in \Gamma\}$.

6.3 Notes

Piecewise affine systems (PWA), i.e., systems that evolve along different discrete-time affine dynamics in different polytopic regions of the (continuous) state space are widely used as models in many areas. They can approximate nonlinear dynamics with arbitrary accuracy and are equivalent with several other classes of systems, including hybrid systems [82]. In addition, there exist efficient techniques for the identification of such models from experimental data, which include Bayesian methods, bounded-error procedures, clustering-based methods, mixed-integer programming, and algebraic geometric methods (see [64, 97] for a review).

We made some simplifying assumptions in our definition of the PWA system $\mathscr{W}$ and its embedding (Definitions 6.1 and 6.6), which is inspired from [5, 7, 138, 163] (see also [72, 74, 105, 106, 162, 180, 184, 185]). First, we defined the system on a set of open full dimensional polytopes, thus ignoring states where the dynamics are ambiguous (states on the boundaries between regions). This is enough for practical purposes, since only sets of measure zero are disregarded and it is unreasonable to assume that equality constraints can be detected in real-world applications. Trajectories starting and remaining in such sets are therefore of no interest. Trajectories

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starting in the interior of full-dimensional polytopes also cannot "vanish" in such zero-measure sets unless the dynamics of the system satisfy some special conditions, which are easy to derive but omitted. Furthermore, if such sets are of interest, the results presented throughout the book can be extended to more general case where the state space is partitioned into polytopes with some of the facets removed. In particular, facets are simply lower dimensional polytopes and the results can be extended by induction. Second, the semantics is defined over the polytopes $\mathbf{X}_l$, which are given a priori. However, arbitrary linear inequalities can be accommodated by including additional polytopes (as long as the region $l \in L$ visited at each step can be observed), in which case the system will have the same dynamics in several modes.

Switched systems have been extensively studied for more than fifty years. They have numerous applications in mechanical systems, automotive industry, power systems, aircraft and traffic control. Switched linear systems, as defined in this chapter, attracted most of the attention. Existing works focus on analysis of stability, controllability, reachability, and observability. Excellent overviews can be found in [125, 159]. Note that, for switched linear systems $\mathscr{S}$, we make the same simplifying assumptions as for the piecewise affine systems $\mathscr{W}$, and the limitations induced by these assumptions are similar.

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Chapter 7
Largest Satisfying Region

In this chapter, we develop a procedure that attempts to find the largest set of initial states from which an autonomous PWA system (Definition 6.3) satisfies an LTL formula over the set labeling the polytopes in its definition. The same problem was considered in Chap. 4 for a finite transition system and an LTL formula over its set of observations. Several methods were presented to find an exact solution to this problem. As expected, since PWA systems have infinitely many states, we are only able to find a subset of the largest satisfying region in this chapter. We formulate the problem for the general case of autonomous PWA systems with uncertain parameters, and we show that more efficient solutions can be found for the particular cases of autonomous PWA systems with fixed parameters and additive uncertainties. The problem that we consider in this chapter can be formally stated as follows:

Problem 7.1 (*Largest Satisfying Region for PWA Systems*) Given an autonomous PWA system $\mathscr{W}$ (Definition 6.3) and an LTL formula ϕ over $L \cup \{\text{Out}\}$, find the largest set of initial states from which all trajectories of $\mathscr{W}$ satisfy ϕ.

From Definition 6.7, solving Problem 7.1 involves working with the infinite embedding transition system $T_{\mathscr{W}}$ (Definition 6.6) and formula ϕ. In Chap. 3, we described LTL model checking as an algorithmic procedure for deciding whether a finite transition systems satisfies an LTL formula. Then, in Sect. 4.1, we used model checking to develop an analysis procedure for finite transition systems (Algorithm 3). Since the embedding $T_{\mathscr{W}}$ from Definition 6.6 is infinite, neither model checking, nor the analysis procedure from Algorithm 3 can be applied directly to solve Problem 7.1.

In Chap. 4, we also developed several methods for the analysis of potentially large transition systems through the construction and refinement of their quotients. In the following sections, we show that this theory can be extended to infinite transition systems such as $T_{\mathscr{W}}$, in order to address Problem 7.1.

First, we consider autonomous, fixed-parameter PWA systems (Definition 6.4) and autonomous, additive-uncertainty systems (Definition 6.5). For these systems, we show that the quotient construction and refinement procedures from Chap. 4

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C. Belta et al., *Formal Methods for Discrete-Time Dynamical Systems*,
Studies in Systems, Decision and Control 89, DOI 10.1007/978-3-319-50763-7_7

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are implementable, and use them to solve Problem 7.1 in Sect. 7.1. For general autonomous PWA systems with uncertain parameters (Definition 6.3), we develop a conservative procedure in Sect. 7.2.

7.1 PWA Systems with Fixed and Additive Uncertain Parameters

The following discussion applies to autonomous PWA systems with additive parameter uncertainty (Definition 6.5). All the results automatically apply to the subclass of autonomous fixed-parameter PWA systems (Definition 6.4). We will discuss the differences as appropriate.

We describe the construction of quotient $T_{\mathscr{W}}/_{\sim} = (X_{\mathscr{W}}/_{\sim}, \delta_{\mathscr{W},\sim}, O_{\mathscr{W}}, o_{\mathscr{W},\sim})$ through the construction of its sets of states and observations, and observation and transition maps. From the definition of the observational equivalence relation $\sim$ (Definition 1.2), induced by observation map $o_{\mathscr{W}}$ of $T_{\mathscr{W}}$ (Definition 6.6) and the definition of the quotient $T_{\mathscr{W}}/_{\sim}$ (Definition 1.3), the set of states $X_{\mathscr{W}}/_{\sim}$ of quotient $T_{\mathscr{W}}/_{\sim}$ is simply the set of observations $X_{\mathscr{W}}/_{\sim} = O_{\mathscr{W}} = L \bigcup \{\text{Out}\}$ of $T_{\mathscr{W}}/_{\sim}$, which is inherited from $T_{\mathscr{W}}$, and the observation map is identity. Given a state $l \in X_{\mathscr{W}}/_{\sim}$, where $l \neq \text{Out}$, the set of all equivalent states from l is

$$con(l) = \mathbf{X}_l. \tag{7.1}$$

In other words, each equivalence class is a polytope from the PWA system definition (Definition 6.1), while the explicit representation of the set $con(\text{Out}) = \mathbb{R}^N \setminus \mathbf{X}$ is not required for our methods.

In order to complete the construction of quotient $T_{\mathscr{W}}/_{\sim}$, we need to compute the transition function $\delta_{\mathscr{W},\sim}$. In Sect. 1.3, we showed that through Eq. (1.7), transitions of the quotient $T_{\mathscr{W}}/_{\sim}$ can be found by computing the set of successors of a region in $T_{\mathscr{W}}$ using the $Post()$ operation defined in Eq. (1.4)—given states $l_1, l_2 \in X_{\mathscr{W}}/_{\sim}$, there exists a transition from l_1 to l_2 (i.e., $l_2 \in \delta_{\mathscr{W},\sim}(l_1)$) if and only if the intersection $Post(con(l_1)) \cap con(l_2)$ is non-empty. From the computation of the set of equivalent states $con(l)$ for an equivalence class $l \in X_{\mathscr{W}}/_{\sim}$ given in Eq. (7.1), checking if a transition between states $l_1, l_2 \in X_{\mathscr{W}}/_{\sim}$ exists amounts to checking the non-emptiness of the intersection $Post(\mathbf{X}_{l_1}) \cap \mathbf{X}_{l_2}$. Formally, the computation of transitions in the quotient $T_{\mathscr{W}}/_{\sim}$ for any states $l_1, l_2 \in X_{\mathscr{W}}/_{\sim}$, where $l_1 \neq \text{Out}$ and $l_2 \neq \text{Out}$ is summarized as

$$l_2 \in \delta_{\mathscr{W},\sim}(l_1) \text{ if and only if } Post(\mathbf{X}_{l_1}) \cap \mathbf{X}_{l_2} \neq \emptyset. \tag{7.2}$$

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Given a polytope $\mathbf{X}_l$ for some $l \in L$, the set of successor states $Post(\mathbf{X}_l)$ is another polytope computable as[1]:

$$Post(\mathbf{X}_l) = A_l\mathbf{X}_l \oplus \mathbf{P}_l^c, \tag{7.3}$$

where $A_l\mathbf{X}_l$ is the image of polytope $\mathbf{X}_l$ through matrix A_l (see Appendix A.3) and "$\oplus$" denotes the Minkowski (set) sum (Definition A.7). Since $Post(\mathbf{X}_l)$ is a polytope, for any states $l_1, l_2 \in X_{\mathscr{W}}/_{\sim}$ the intersection $Post(\mathbf{X}_{l_1}) \cap \mathbf{X}_{l_2}$ is also a polytope and its non-emptiness can be checked easily using polyhedral operations. Note that, for the particular case of a fixed parameter PWA system, $\mathbf{P}_l^c$ in Eq. (7.3) is a singleton $c_l \in \mathbb{R}^N$.

Given a state $l \in X_{\mathscr{W}}/_{\sim}$, where $l \neq \text{Out}$, a transition from state l to state Out is assigned in accordance to Definition 6.6 as

$$\text{Out} \in \delta_{\mathscr{W},\sim}(l) \text{ if and only if } Post(\mathbf{X}_l) \nsubseteq \mathbf{X}, \tag{7.4}$$

which is also checked easily, since both $Post(\mathbf{X}_l)$ and $\mathbf{X}$ are polytopic sets. To complete the construction of $\delta_{\mathscr{W},\sim}$, transitions for state Out $\in X_{\mathscr{W}}/_{\sim}$ must be assigned but, from Definitions 6.1 and 6.6, it only has a transition to itself (i.e., $\delta_{\mathscr{W},\sim}(\text{Out}) = \{\text{Out}\}$).

Algorithm 14 $T_{\mathscr{W}}/_{\sim}$ =QUOTIENT($\mathscr{W}$) : Compute the quotient $T_{\mathscr{W}}/_{\sim}$ of an additive uncertainty PWA system $\mathscr{W}$

$X_{\mathscr{W}}/_{\sim} := L \bigcup \{\text{Out}\}$
$O_{\mathscr{W}} := X_{\mathscr{W}}/_{\sim}$
for all $l \in X_{\mathscr{W}}/_{\sim}$ **do**
  $o_{\mathscr{W},\sim}(l) := l$
  $\delta_{\mathscr{W},\sim} := \emptyset$
  if $Post(\mathbf{X}_l) \nsubseteq \mathbf{X}$ **then**
    $\delta_{\mathscr{W},\sim}(l) := \delta_{\mathscr{W},\sim}(l) \cup \{\text{Out}\}$
  end if
  for all $l' \in X_{\mathscr{W}}/_{\sim}$ **do**
    if $Post(\mathbf{X}_l) \cap \mathbf{X}_{l'} \neq \emptyset$ **then**
      $\delta_{\mathscr{W},\sim}(l) := \delta_{\mathscr{W},\sim}(l) \cup \{l'\}$
    end if
  end for
end for
$\delta_{\mathscr{W},\sim}(\text{Out}) := \{\text{Out}\}$
return $T_{\mathscr{W}}/_{\sim} = (X_{\mathscr{W}}/_{\sim}, \delta_{\mathscr{W},\sim}, O_{\mathscr{W}}, o_{\mathscr{W},\sim})$

The transition map of quotient $T_{\mathscr{W}}/_{\sim}$ is constructed using the computation described above, which completes the quotient's construction. The computation of

[1] In this chapter, we assume, for simplicity of presentation, that all matrices $A_l, l \in L$ are invertible. This assumption can be easily relaxed as discussed in Sect. 7.4. The technical details are included in Sects. A.3 and A.4.

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$T_{\mathcal{W}}/_\sim$ is summarized in Algorithm 14, which is implementable using polyhedral operations on polytopes. Since the number of regions L of PWA system $\mathcal{W}$ is finite, $T_{\mathcal{W}}$ has a finite set of observations $O_{\mathcal{W}}$ and, as a result, the set of states $X_{\mathcal{W}}/_\sim$ of the quotient is also finite. This allows the application of model checking or analysis of $T_{\mathcal{W}}/_\sim$ through Algorithm 3 but the implementation of the more advanced analysis procedure from Chap. 4 requires additional operations, which will be discussed next.

Example 7.1 We apply Algorithm 14 to construct the quotient $T_{\mathcal{W}}/_\sim$ for the PWA system $\mathcal{W}$ defined in Example 6.2. Initially, the system had four regions (Fig. 6.2a) but additional partitioning of the state space was required to accommodate some specifications, resulting in a system with six regions denoted by $\mathbf{X}_1, \ldots, \mathbf{X}_6$ (Fig. 6.2b) with $L = \{1, \ldots, 6\}$. Therefore, the quotient $T_{\mathcal{W}}/_\sim$ has six states $X_{\mathcal{W}}/_\sim = \{1, \ldots, 6\}$ where, for each state $l \in X_{\mathcal{W}}/_\sim$, the set of equivalent states of $T_{\mathcal{W}}$ (and therefore $\mathcal{W}$) is given by $con(l) = \mathbf{X}_l$ as in Eq. (7.1).

To compute the transitions of $T_{\mathcal{W}}/_\sim$, we compute the set of successors $Post(\mathbf{X}_l)$ for each region $\mathbf{X}_l$ of $T_{\mathcal{W}}$ (see Fig. 7.1). Checking the non-emptiness of the intersection $Post(\mathbf{X}_{l_1}) \cap \mathbf{X}_{l_2}$ allows us to compute the transitions of $T_{\mathcal{W}}/_\sim$. Only the set of successors of region 6 is completely included within the defined state space $\mathbf{X}$ and, therefore, all other states have a transition to state Out (note that $Post(\mathbf{X}_1) \not\subset \mathbf{X}$, although this is not obvious from Fig. 7.1a). This leads to the inclusion of transitions $\delta_{\mathcal{W},\sim}(1) = \{1, 2, 3, \text{Out}\}$, $\delta_{\mathcal{W},\sim}(2) = \{2, 3, 4, \text{Out}\}$, $\delta_{\mathcal{W},\sim}(3) = \{3, 6, \text{Out}\}$, $\delta_{\mathcal{W},\sim}(4) = \{3, \text{Out}\}$, $\delta_{\mathcal{W},\sim}(5) = \{1, 2, 3, 5, 6, \text{Out}\}$ and $\delta_{\mathcal{W},\sim}(6) = \{1, 5, 6\}$. The resulting quotient $T_{\mathcal{W}}/_\sim$ is shown in Fig. 7.2, where the observations for each state are omitted but are clear from the state labels.

By embedding the PWA system $\mathcal{W}$ into an infinite transition system $T_{\mathcal{W}}$ (Definition 6.6), we reduced Problem 7.1 to Problem 4.1. However, since $T_{\mathcal{W}}$ was infinite, the analysis procedure outlined as Algorithm 3 in Chap. 4 could not be applied directly. So far, we showed that the quotient $T_{\mathcal{W}}/_\sim$ of the embedding $T_{\mathcal{W}}$ under the observational equivalence relation $\sim$ (Definition 1.2) can be constructed using polyhedral operations (Algorithm 14). Since $T_{\mathcal{W}}/_\sim$ is finite, this allows us to apply the analysis technique described in Sect. 1.3. However, as discussed there, such an approach leads to a conservative solution to Problem 7.1. In order to obtain less conservative results, bisimulation-based and formula-guided quotient refinement techniques were proposed in Sects. 4.3 and 4.5, respectively. Both methods were initialized by constructing a finite quotient such as $T_{\mathcal{W}}/_\sim$ but in addition required the implementation of a state refinement procedure.

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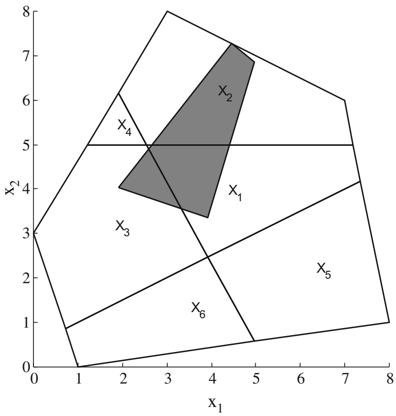
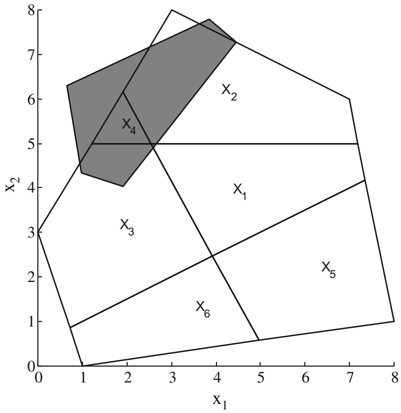
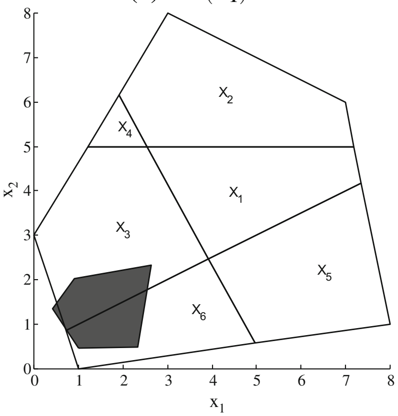


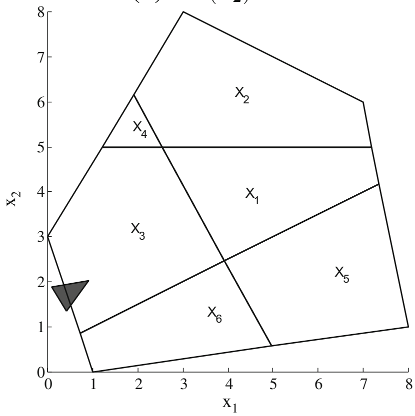
(a) *Post*($\mathrm{X}_\mathbf{1}$).



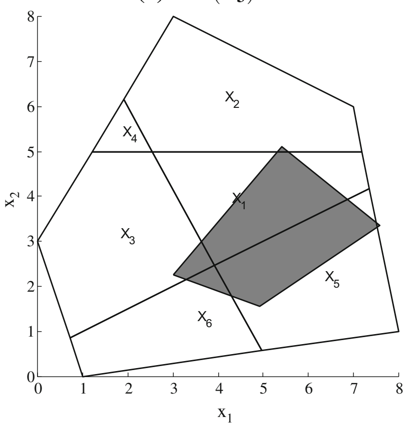
(b) *Post*($\mathrm{X}_\mathbf{2}$).



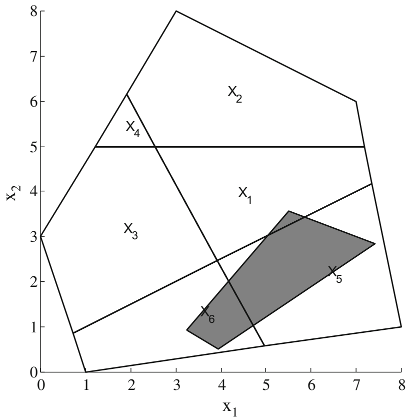
(c) *Post*($\mathrm{X}_\mathbf{3}$).



(d) *Post*($\mathrm{X}_\mathbf{4}$).



(e) *Post*($\mathrm{X}_\mathbf{5}$).



(f) *Post*($\mathrm{X}_\mathbf{6}$).

Fig. 7.1 Successor states (*shaded gray*) of different regions of PWA system $\mathscr{W}$ defined in Example 6.2 (Fig. 6.2b). See Example 6.2 for additional details

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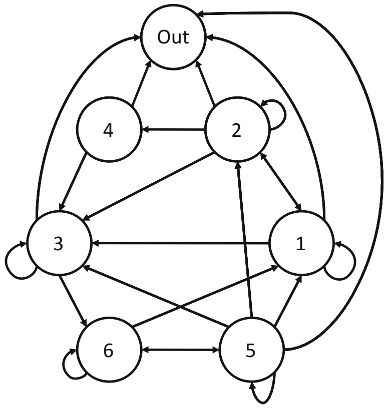

Fig. 7.2 Finite quotient $T_{\mathscr{W}}/_\sim$ of PWA system $\mathscr{W}$ defined in Example (6.2) (Fig. 6.2b). Observations of the states are omitted. See Example 7.1 for additional details

In the following, we focus on the implementation of the refinement procedure REFINE() (Algorithm 5, Chap. 4) and show that for autonomous additive uncertainty (and fixed parameter) PWA systems all its operations are computable through polyhedral operations. Specifically, as the embedding of a PWA system with additive parameter uncertainty is non-deterministic, we will refer to Algorithm 5. For the particular case of a PWA with fixed parameters, whose embedding is deterministic, the refinement procedure is described in Algorithm 6.

To implement function REFINE() for $T_{\mathscr{W}}$, given states $l_1, l_2 \in X_{\mathscr{W}}/_\sim$ such that $l_2 \in \delta_{\mathscr{W},\sim}(l_1)$ (i.e., l_2 is reachable from l_1 in $T_{\mathscr{W}}/_\sim$), we need to be able to construct a state l', such that $con(l') = con(l_1) \cap Pre(con(l_2))$ (see Algorithms 5 and 6). From Eq. (7.1), this computation reduces to the construction of a state l' where $con(l') = \mathbf{X}_{l_1} \cap Pre(\mathbf{X}_{l_2})$.

Under the invertibility assumption made earlier in this chapter, which, as stated, can be easily relaxed (see Sect. 7.4), this intersection is computable as

$$\mathbf{X}_{l_1} \cap Pre(\mathbf{X}_{l_2}) = \mathbf{X}_{l_1} \cap A_{l_1}^{-1}(\mathbf{X}_{l_2} \ominus \mathbf{P}_{l_1}^c), \tag{7.5}$$

where $\ominus$ denotes the Minkowski difference (Definition A.8). Note that while the $Pre()$ operation is applied to region $\mathbf{X}_{l_2}$, the parameters of region $\mathbf{X}_{l_1}$ are used for the computation, which is consistent with Definition 6.2. Using Eq. (7.5) to refine the states of $T_{\mathscr{W}}/_\sim$ and Eq. (7.3) to update its transitions wherever necessary (see Algorithms 5 and 6) allows the implementation of function REFINE() and all computation is performed using polyhedral operations.

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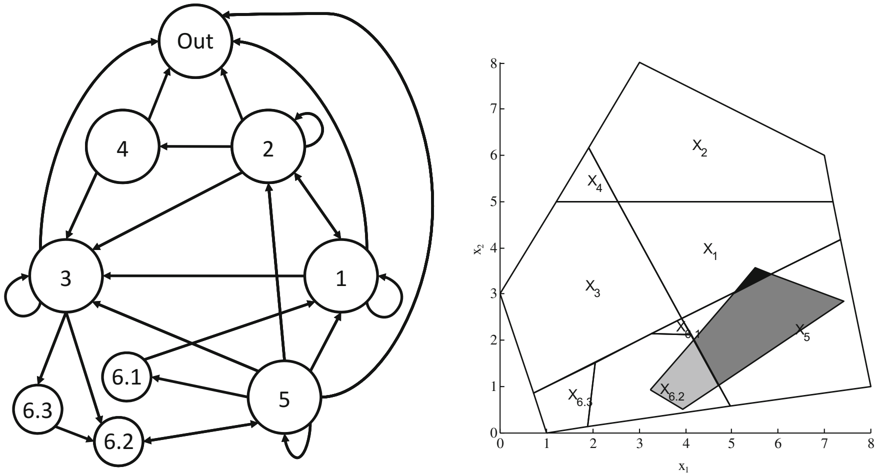

(a) Refined quotient $T_{\mathscr{W}}/_{\sim_r}$ (b) Equivalence classes $X_{\mathscr{W}}/_{\sim_r}$

Fig. 7.3 Refined quotient $T_{\mathscr{W}}/_{\sim_r}$ $=$REFINE($T_{\mathscr{W}}/_{\sim}, 6$) (**a**) and equivalence classes $X_{\mathscr{W}}/_{\sim_r}$ (**b**) of PWA system $\mathscr{W}$ from Example 6.2 (Fig. 6.2b). The successor states $Post(con(6.1))$ (dark gray), $Post(con(6.2))$ (medium gray) and $Post(con(6.3))$ (light gray) are also shown for the refined subsets $6.1, 6.2, 6.3 \in X_{\mathscr{W}}/_{\sim_r}$, where $con(6.1) \cup con(6.2) \cup con(6.3) = con(6)$ for state $6 \in X_{\mathscr{W}}/_{\sim}$. See Example 7.2 for additional details

Example 7.2 We apply function REFINE() (Algorithm 6) to refine the quotient $T_{\mathscr{W}}/_{\sim}$ (constructed in Example 7.1 and shown in Fig. 7.2) of PWA system $\mathscr{W}$ defined in Example 6.2 (Fig. 6.2b). We target refinement to state $6 \in X_{\mathscr{W}}/_{\sim}$ and construct the refined quotient $T_{\mathscr{W}}/_{\sim_r}$ $=$REFINE($T_{\mathscr{W}}/_{\sim}, 6$). State 6 has three successors in $X_{\mathscr{W}}/_{\sim}$ (i.e., $\delta_{\mathscr{W},\sim} = \{1, 5, 6\}$) and, therefore, refinement results in three subsets in $X_{\mathscr{W}}/_{\sim_r}$ denoted as 6.1, 6.2 and 6.3, where $con(6.1) \cup con(6.2) \cup con(6.3) = con(6)$. Each subset has only a single outgoing transitions in $T_{\mathscr{W}}/_{\sim_r}$ (see the sets of successors shown in Fig. 7.3b), which is implicitly induced through the refinement and incoming transitions are recomputed (see Algorithm 6). This results in the construction of the refined quotient $T_{\mathscr{W}}/_{\sim_r}$ shown in Fig. 7.3a.

Note that the notation is abused in this example and in the rest of this chapter. As we assumed that all the polytopes are open, the equality $con(6.1) \cup con(6.2) \cup con(6.3) = con(6)$ does not hold precisely. Indeed, $con(6)$ contains some facets of $con(6.1)$, $con(6.2)$, and $con(6.3)$, which are not contained in $con(6.1) \cup con(6.2) \cup con(6.3)$. More discussions are included in Sect. 7.4.

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After a state $l \in X_{\mathscr{W}}/_{\sim}$ is refined into states l_1 and l_2 such that $con(l_1) \cup con(l_2) = con(l)$, the computation from Eqs. (7.3) and (7.5) can be applied to the subsets l_1 and l_2. This enables the iterative refinement of the quotient $T_{\mathscr{W}}/_{\sim}$ and allows the implementation of the bisimulation algorithm (Algorithm 1) and the analysis methods described in Sects. 4.3 (Algorithm 7) and 4.5 (Algorithm 8) for PWA systems.

A termination condition based on the sizes of equivalence classes was also proposed in Chap. 4 for the analysis procedures discussed there. To determine if a state $l \in X_{\mathscr{W}}/_{\sim}$ is "large enough" to undergo additional refinement, we compute the radius of the largest sphere inscribed in polytope $con(l)$ and apply the refinement procedure only if it is larger than a certain predefined limit ε. In other words, we apply the refinement procedure to state l only if $r(\mathbf{X}_l) > \varepsilon$, where $r(\mathbf{X}_l)$ is the radius of the Chebyshev ball of $\mathbf{X}_l$ (see Definition A.9 in the Appendix).

Example 7.3 We apply the analysis method from Sect. 4.3 summarized as Algorithm 7 to identify satisfying and violating regions of PWA system $\mathscr{W}$ defined in Example 6.2 and shown in Fig. 6.2a. We are interested in testing whether trajectories of the system keep reaching values over 5 in the second component x_2 of the system state x. Therefore, we introduce additional partitions to the states space of the system as shown in Fig. 6.2b and formulate the specification as the LTL formula $\phi = \Box\Diamond(2 \vee 4)$, requiring that states from regions $\mathbf{X}_2$ and $\mathbf{X}_4$ are visited infinitely often. Furthermore, we want to guarantee that trajectories of the system remain within the defined state space $\mathbf{X}$ and therefore augment the specification as $\phi' = \Box\Diamond(2 \vee 4) \wedge \Box\neg\text{Out}$.

We set $\varepsilon = 0.1$ as the limit on the states from $X_{\mathscr{W}}/_{\sim_r}$ that can undergo refinement, which guarantees the termination of the analysis procedure. While only an under-approximation of the largest satisfying and strictly violating regions of $T_{\mathscr{W}}$ (and therefore $\mathscr{W}$) is obtained as discussed in Chap. 4, most of the system's state space is characterized as satisfying or violating (see Fig. 7.4a).

All trajectories originating in the regions shown in dark and medium gray in Fig. 7.4a violate specification ϕ'—trajectories originating in the dark gray region leave the defined state space $\mathbf{X}$, while trajectories originating in the light gray region oscillate but do not visit states where the values of the second component x_2 are above 5. However, all trajectories originating in the satisfying region shown in light gray in Fig. 7.4a are guaranteed to satisfy the specification.

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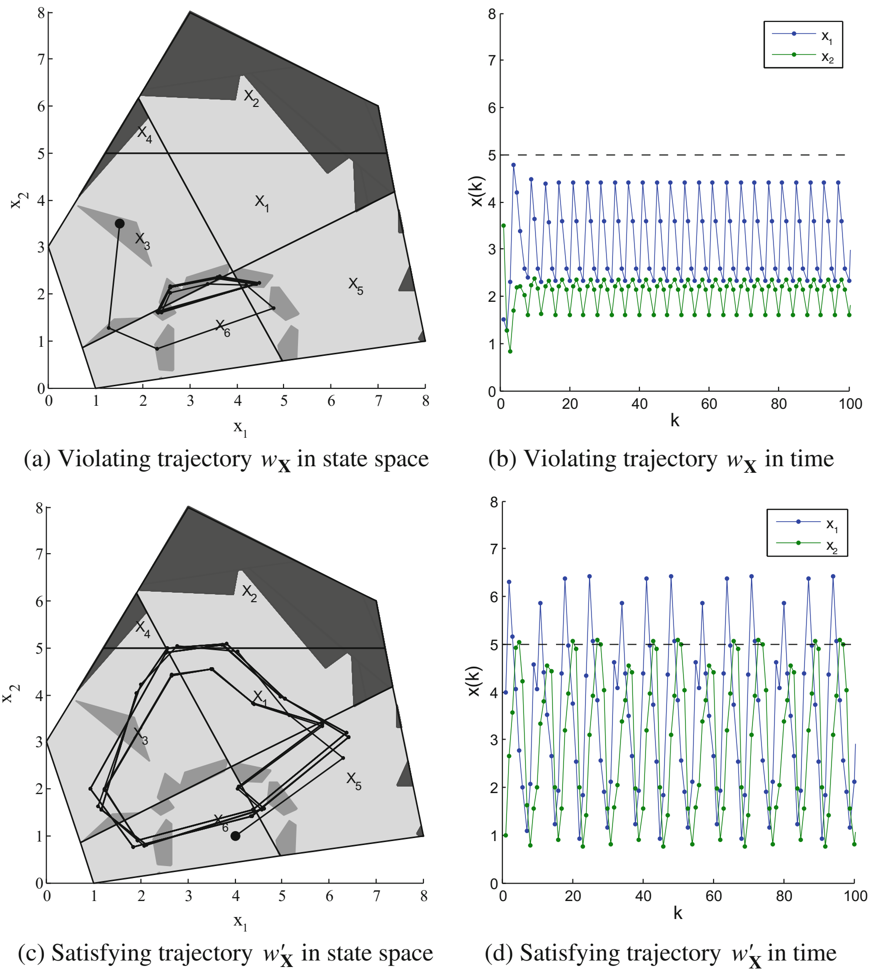

(a) Violating trajectory $w_{\mathbf{X}}$ in state space

(b) Violating trajectory $w_{\mathbf{X}}$ in time

(c) Satisfying trajectory $w'_{\mathbf{X}}$ in state space

(d) Satisfying trajectory $w'_{\mathbf{X}}$ in time

Fig. 7.4 Satisfying (*light gray*) and violating (*medium gray*) regions of PWA system $\mathscr{W}$ defined in Example 6.2 (Fig. 6.2b) for specification "$\Box\Diamond(2 \vee 4) \wedge \Box\neg\mathrm{Out}$" were identified using the analysis procedure described in Sect. 4.3 (Algorithm 7). Trajectories of $\mathscr{W}$ originating in the region shown in dark gray leave the defined state space of the system and therefore are also violating. A violating trajectory $w_{\mathbf{X}}$ (**a**) and (**b**) and a satisfying trajectory $w'_{\mathbf{X}}$ (**c**) and (**d**) were obtained by initializing $\mathscr{W}$ in the violating or satisfying regions (initial conditions are shown as large *circles*). See Example 7.3 for additional details

Note that for an autonomous, fixed-parameter PWA system $\mathscr{W}$ (Definition 6.4), the embedding $T_{\mathscr{W}}$ is deterministic, which allows the application of the more efficient refinement strategies from Algorithm 6. The computation from Eq. (7.5) is also sufficient to implement refinement strategies for autonomous, additive uncertainty PWA systems (Definition 6.5) through Algorithm 5. Refinement strategies for more general autonomous, uncertain parameters systems are discussed in the following Sect. 7.2.

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