

Problem Set 7

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be a linear transformation with $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Let $v \in \mathbb{R}^3$. What is $T(\mathbf{v})$? Is T one-one, as a function? Is T onto?

2. Change $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ in the above. What is $T(\mathbf{v})$? Is T one-one? Is T onto?

3. Can we ever find a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ which is onto?

4. Find out $[\mathbf{v}]_{\mathcal{B}}$, where \mathcal{B} is an ordered basis:

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

(d) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$

(e) $\mathcal{B} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

5. Find out \mathbf{v} given $[\mathbf{v}]_{\mathcal{B}}$, where \mathcal{B} is an ordered basis:

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

$$(d) \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$(e) \mathcal{B} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

6. Give three linear transformations from \mathbb{R}^3 to $\mathbb{W} = \{\mathbf{w} : w_1 - w_2 + w_3 - w_4 + w_5 = 0\}$. Give their coordinate matrices w.r.t the ordered bases $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ on \mathbb{R}^3 and some ordered basis of \mathbb{W} .

7. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ as $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y + z \\ x + 2z \end{bmatrix}$. Find

- (a) a basis of $\text{Range}(T)$,
- (b) $\text{rank}(T)$,
- (c) a basis for $\mathcal{N}(T)$, and
- (d) $\dim(\mathcal{N}(T))$.

8. **(T)** Find all linear transformations from $\mathbb{R}^n \rightarrow \mathbb{R}$.

9. Let $\mathbf{v} \in \mathbb{R}^n$ and $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis of \mathbb{R}^n . Form a matrix $B = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$. Is $B[\mathbf{e}_1]_{\mathcal{B}} = \mathbf{e}_1$? What is $B[[\mathbf{e}_1]_{\mathcal{B}}, \dots, [\mathbf{e}_n]_{\mathcal{B}}]$? Show that B is invertible and $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.

10. **(T)** Show that a linear transformation is one-one if and only if null-space of $\mathcal{N}(T)$ is $\{0\}$.
11. Describe all 2×2 orthogonal matrices. Prove that action of any orthogonal matrix on a vector $\mathbf{v} \in \mathbb{R}^2$, is either a rotation or a reflection about a line.
12. **(T)** Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $n \geq 2$, with $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$. Prove that there exist an orthogonal matrix A such that $A(\mathbf{v}) = \mathbf{w}$. Prove also that A can be chosen such that $\det(A) = 1$.
(This is why orthogonal matrices with determinant one are called rotations.)