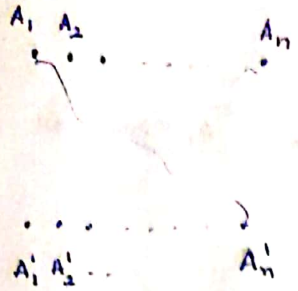
22/1/17

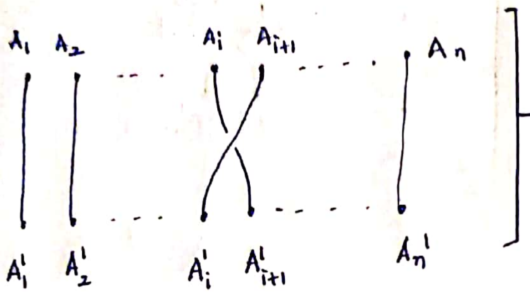
## Braid Groups and Braid Closures : $(B_n)$

$B_n \rightarrow$ braid grp on $n$-strings (1930's Artin).




Join these $A_1, A_2, \ldots A_n$ and $A_1', A_2', \ldots A_n'$ by $n$-strings ; moving monotone down and non-intersecting.

* Particular Braid $\rightarrow$ $\sigma_i, \sigma_i^{-1}$, namely elementary generators.




$\rightarrow \sigma_i$ ; $\sigma_i^{-1}$ is the mirror image of $\sigma_i$ (left above & right below).

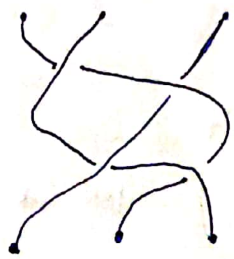
* Any Braid '$\alpha$' in $B_n$ can be written as a product of $\sigma_i$ and $\sigma_j^{-1}$ called the "**BRAID WORDS**".

Eg :- $B_2$

$\sigma_1 \circ \sigma_1 \circ \sigma_1$

In $B_2$ ; it can be $\sigma_1^n$ (or) $\sigma_1^{-m}$

$B_3$

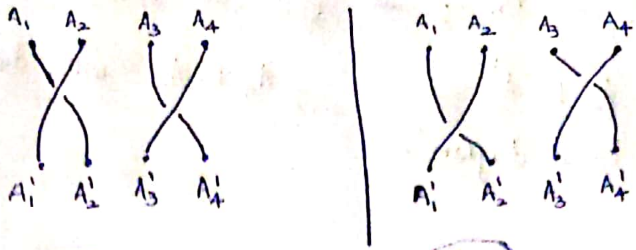


$\sigma_1 \circ \sigma_2^{-1} \circ \sigma_1 \circ \sigma_2^{-1}$

$\rightarrow \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \rightarrow$ Remember braid grps are not "Commutative".

* NOTE :- Braid Words are not unique

Eg :- $\boxed{B_4}$




$\sigma_1\sigma_3$ $\qquad\qquad$ $\sigma_3\sigma_1$

* $\alpha, \beta \in B_n$ are said to be equivalent $(\alpha \sim \beta)$ ; if we can go from '$\alpha$' to '$\beta$' by finitely many elementary knot moves.

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ ; if $|i-j| \geq 2$.

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Eg :- (i) $\sigma_1\sigma_2\sigma_1$

$\Rightarrow$

slide str 2.

$\sigma_2\sigma_1\sigma_2$.

Eg :- (i) $w_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4$

$w_2 = \sigma_2\sigma_1\sigma_2^2$.

$$w_1 = \sigma_1\sigma_2\underbrace{\sigma_4^{-1}\sigma_1}\,\underbrace{\sigma_2\sigma_4} = \sigma_1\sigma_2\sigma_1\sigma_4^{-1}\sigma_4\sigma_2$$

$$= \overbrace{\sigma_1\sigma_2\sigma_1}\sigma_2$$

$$= \cancel{\sigma_1\sigma_1\sigma}\ \sigma_2\sigma_1\sigma_2\sigma_2$$

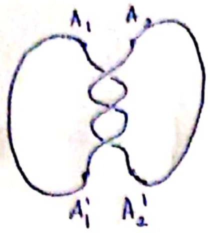
$$= \sigma_2\sigma_1\sigma_2^2$$

* **Braid Closure :**

Given a braid 'α' in $B_n$ ; we form its closure as follows ;

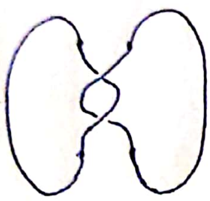
- Join $A_1$ to $A_1'$ ; $A_2$ to $A_2'$ ; . . . ; $A_n$ to $A_n'$ with large non-intersecting arcs outside the "Braid diagram".

Eg : $B_2$




$\sigma_1^3$ - $3_1$ (or) $3_1^*$

$\sigma_1^2$ closure - Hoff link.

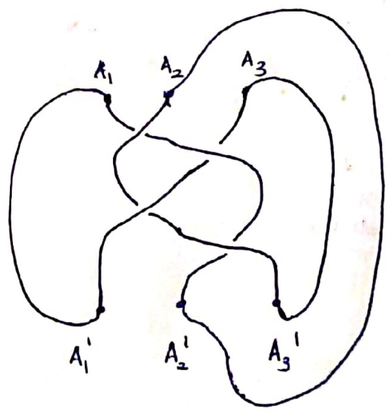


Hoff link.

In $B_3$ - $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$

↓

closure of this is $4_1$ .



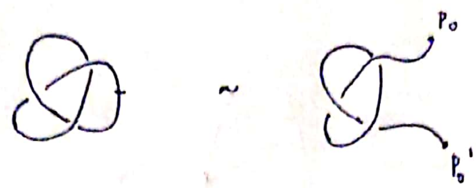

* **Alexander's Theorm :-** (1920's)

Any "knot" (or) "link" can be obtained as the closure of some braids "α" $[K \simeq \bar{\alpha}]$ .

Proof :-

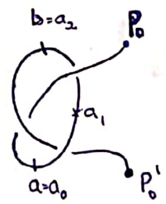
Suppose 'D' is the diagram of Knot 'k'.

Step-① :- We cut 'D' at some point $P_0$ (not a crossing pt.) and pull the loose ends apart $P_0, P_0'$.

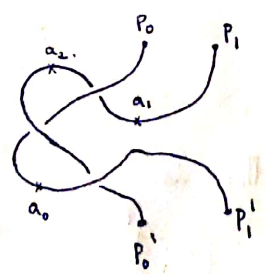



Step-② :- Let us assume that the remaining diagram will have atleast one "$\underbrace{\text{maxima}}_{b}$" and one "$\underbrace{\text{minima}}_{a}$".

- We assume that the strand $\overline{ab}$ intersects with the other crossing pts. at $a_1, a_2, \dots a_n = b$ such that $\overline{a_i a_{i+1}}$ intersects only at one crossing pt.




Step-③ : Replace the arc $\overline{a_0a_1}$ by larger arc such that all crossings remain as they were.

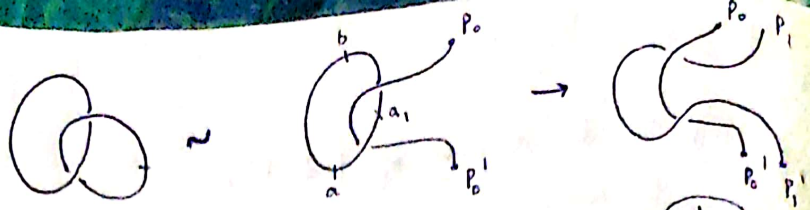



$\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}$.

Using the same method on $\overline{a_1a_2}, \overline{a_2a_3}, \dots, \overline{a_{n-1}a_n}$; we will get a braid '$\alpha$' whose ~~degree~~ closure is "k".

Eg :-

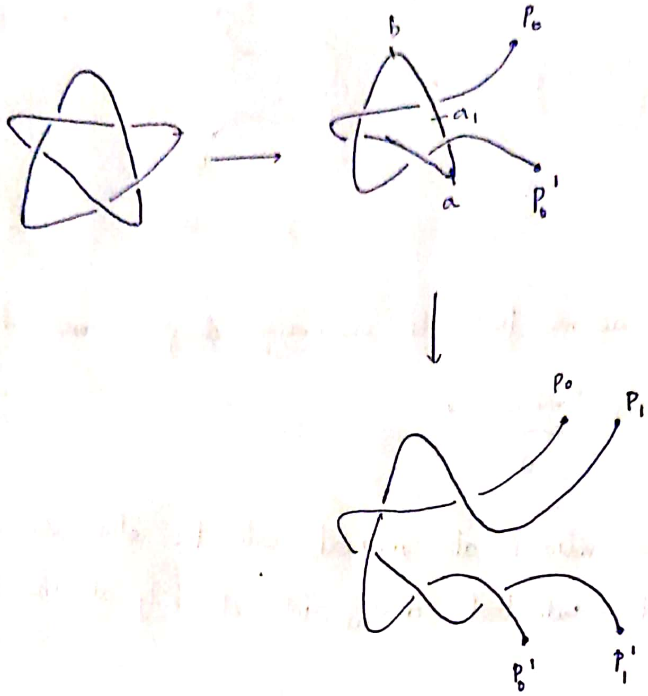
(i)




$\sigma_1^{-1}\sigma_1^{-1}$

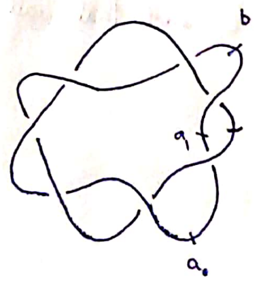
~~degree~~

(ii) $5_1$.




$\sigma_1^{5}$.

(iii)




after opening this is

$(\sigma_1^{-1})^7$

* "<u>Braid Index</u>" of a knot $b(K)$ is the min. no. of strings whose closure will give the ~~two~~ knot "<u>K</u>".

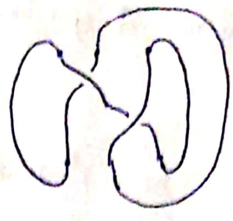
24/1/19

Eg :- $\overline{\sigma_1^{-1}}$ ; $\overline{\sigma_1^{-1}\sigma_2}$ ; $\overline{\sigma_1\sigma_2^{-1}}$ ; $\overline{\sigma_1\sigma_2\sigma_3^{-1}}$ ; $\overline{\sigma_1\sigma_2^{-1}\sigma_3^{-1}}$

(i)

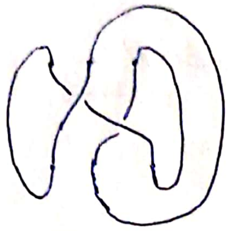
$\overline{\sigma_1^{-1}} = O_1$

(ii)



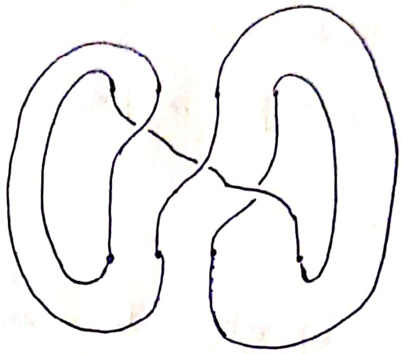
$\overline{\sigma_1^{-1}\sigma_2} = O_1$

(iii)



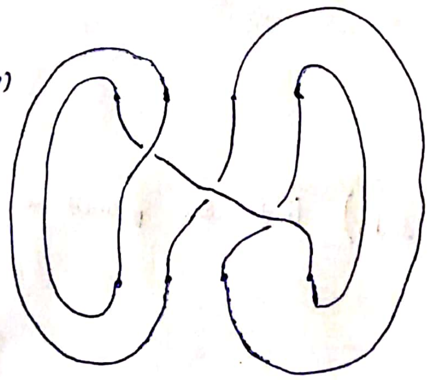
$\overline{\sigma_1\sigma_2^{-1}} = O_1$

(iv)



$\overline{\sigma_1\sigma_2\sigma_3^{-1}} = O_1$

(v)



$\overline{\sigma_1\sigma_2^{-1}\sigma_3^{-1}} = O_1$

* Here all are giving $O_1$ ; i.e different braids $\alpha, \beta$ are such that $\bar{\alpha} = \bar{\beta}$ .

* We want a relation '$\sim$' such that if $\alpha \sim \beta \Rightarrow \bar{\alpha} \simeq \bar{\beta}$ (Russian)

Given by Markov in 1935.

* **Def$^n$** : Suppose $B_\infty$ is the union of the grps $B_1, B_2, \ldots$

i.e; $B_\infty = \bigcup_{k\geq 1} B_k$.

on $B_\infty$ ; we perform 2 operations (called the Markov moves).

(i) if $\beta \in B_n$ ; then $M_1$ - 1st Markov move.

conjugation - $\beta \to \gamma\beta\gamma^{-1}$ where $\gamma \in B_n$

(ii) $M_2$ - 2nd Markov move.

stabilization - $\beta \to \beta\sigma_n$ (or) $\beta\sigma_n^{-1}$ ; $\sigma_n \to$ (generator of $(n+1)$ braid).

* **Def$^n$** :- Suppose $\alpha, \beta \in B_\infty$ ; If we can transform '$\alpha$' into '$\beta$' by perform-ing the markov moves $M_1, M_2$ and their inverses ; finitely many times, then we say '$\alpha$' is markov equivalent to '$\beta$' and we write ;

$\alpha \sim_M \beta$.

* $\boxed{\text{Markov's Thm}}$ :- (1935)

Suppose '$K_1$' and '$K_2$' are oriented knots (or links) that are formed as closures of braids $\beta_1, \beta_2$ respectively ; then

$$K_1 \simeq K_2 \iff \beta_1 \sim_M \beta_2.$$

* **Eg** :- (i) $w_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_4$

$w_2 = \sigma_2^2\sigma_3\sigma_1\sigma_3^{-1}\sigma_4$

S.T. $w_1 \sim w_2$

$$w_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_4 = \sigma_2\cdot\sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_4$$
$$= \sigma_2^2\sigma_1\sigma_4 = \sigma_2^2\sigma_1\sigma_3\sigma_3^{-1}\sigma_4$$
$$= \sigma_2^2\sigma_3\sigma_1\sigma_3^{-1}\sigma_4 = w_2.$$

Eg :- (i) $\sigma_1^{-2}\sigma_2^{2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_2^{-2}\sigma_1^{-2}$

* **Braid Index b(k) :**

A knot (or) link can be formed from an infinite no. of braids but ∃ a braid which has the least no. of strings "$\alpha$". The no. of string of $\alpha = b(K)$ is the braid index of 'k'.

Recall : HOMFLY POLYNOMIAL $(P_k(v,z))$

$$\max.(v\text{-deg } P_k) - \min.(v\text{-deg } P_k) = v\text{-span}(P_k(v,z))$$

$$\boxed{b(k) \geq \frac{1}{2}\left[v\text{-span}(P_k(v,z))\right] + 1} \rightarrow (*)$$

for all knots
(*) is equality upto 10 crossings except $9_{42}, 9_{45}, 10_{132}, 10_{150}, 10_{156}$

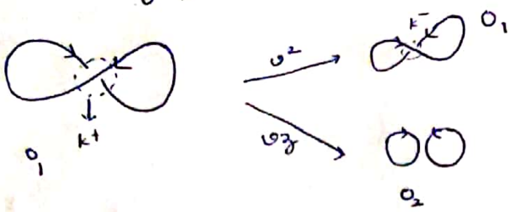
i.e

$$\boxed{b(k) = \frac{1}{2}\left[v\text{-span}(P_k(v,z))\right] + 1} \rightarrow (1)$$

Prove (1) for $3_1, 4_1$.

Sol: Calculate HOMFLY for $3_1$.

$$\frac{1}{v}P_{K^+} - vP_{K^-} = zP_{K_0}$$



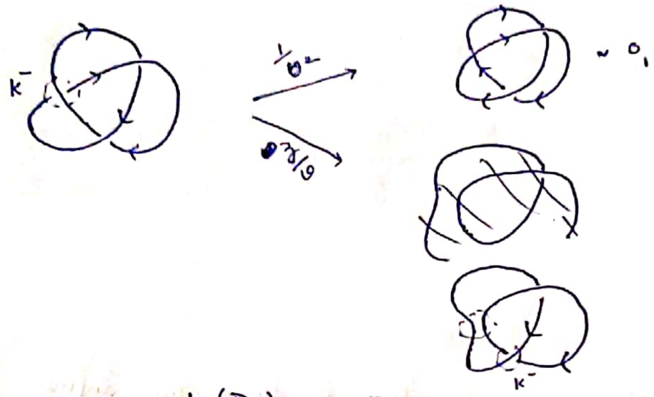

$$P_{K^+} = v^2 P_{K^-} + vz P_{K_0}$$

$$P_{O_1} = v^2 P_{O_1} + vz P_{O_2}$$

$$P_{O_1}(1-v^2) = vz P_{O_2}$$

$$\frac{1-v^2}{vz} = P_{O_2}$$

*




$$P_{K^-} = \frac{1}{v^2}(P_{O_1}) + \frac{z}{v}(P_{\text{Hoff}})$$

29/1/19

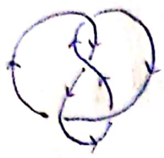
* Braids :

$b(K) = \frac{1}{2}\left[v\text{-span}\left(P_K(v,z)\right)\right] + 1$

Axiom-(2) :

$\frac{1}{v} P_{D^+} - v P_{D^-} = z P_{D_0}$

$\boxed{P_{D^+} = v^2 P_{D^-} + vz P_{D_0}}$ $\boxed{P_{D^-} = \frac{1}{v^2} P_{D^+} - \frac{z}{v} P_{D_0}}$



By doing R-moves ;

$P_{4_1}(v,z) = v^2 z^2 + \frac{1}{v^2} - 1$

$\max(v\text{-deg}) = 2$

$\min(v\text{-deg}) = -2$

$v\text{-span} = 2-(-2) = 4$

$\frac{1}{2}(v\text{-span}) = 2$

$\boxed{b(K) = 3}$

* Rational Knots :

$S^2 = \{x^2+y^2+z^2 = 1\}$

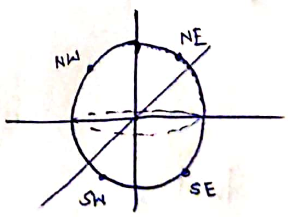
We fire 4 points on $S^2$

NE $= (0, 1/\sqrt{2}, 1/\sqrt{2})$

NW $= (0, -1/\sqrt{2}, -1/\sqrt{2})$

SE $= (0, 1/\sqrt{2}, -1/\sqrt{2})$

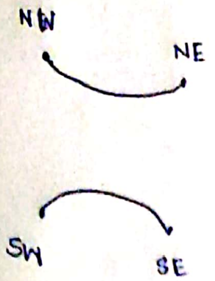
SW $= (0, -1/\sqrt{2}, -1/\sqrt{2})$

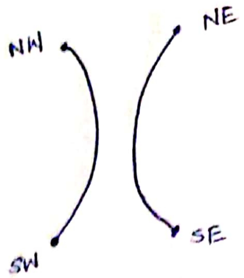



All 4 points lie on the y-z plane. $X=0$ .

*

- Trivial Tangle :

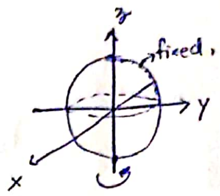


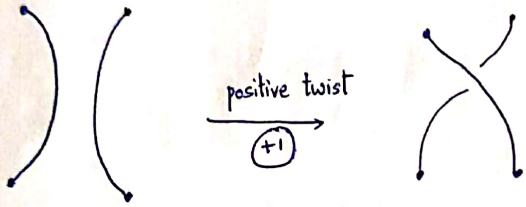



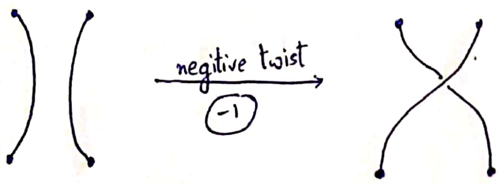

- O-type trivial triangle
- O-O type trivial tangle.

- Vertical Twist :

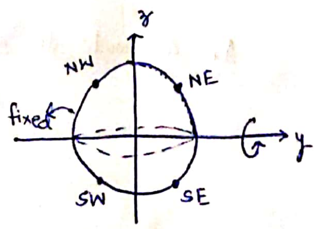
- Keep north hemisphere and south pole fixed. Rotate 180° about z-axis
Then SW, SE will exchange. position.




- 


- 


- Horizontal Twist :




Here NE, SE positions are exchanged.

* Rational Tangles : $T(a_1, a_2, \cdots a_n)$.

- Case-1 : If 'n' is odd.

Start with trivial tangle $T(0)$ ; then

Perform '$a_1$' horizontal twists

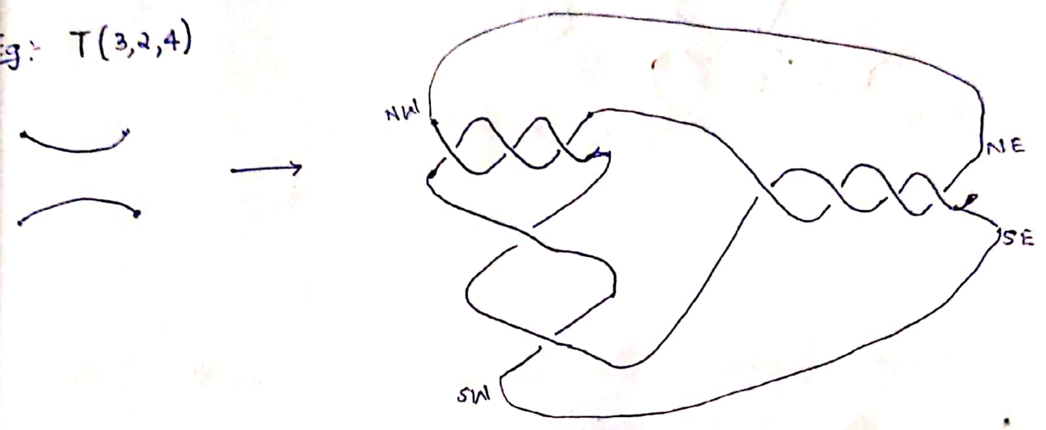
'$a_2$' vertical twists

'$a_3$' horizontal twists

⋮

$a_n$ horizontal twists.

Eg:- $T(3,2,4)$




* Connect NW to NE and SW to SE by non-intersecting arcs.

This will give a knot (or) link $T(a_1, a_2, \cdots a_n)$

- Case-2 : If 'n' is even.

Start with the trivial tangle $T(0{-}0)$ ; then

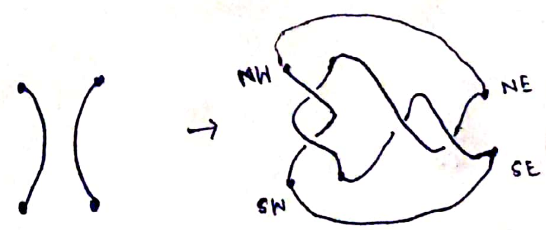
Perform '$a_1$' vertical twists;

'$a_2$' horizontal twists ;

⋮

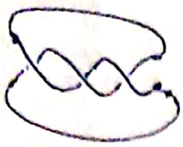
'$a_n$' horizontal twists

Eg:- $T(2,2)$



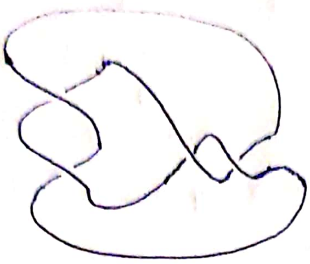
* Conway Notation :

(i) 3 - (3 horizontal twists)



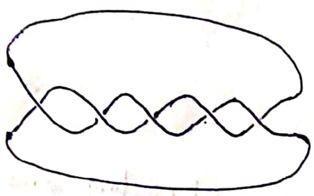
Trefoil knot - $3_1$.

(ii) 2 2



$4_1$

(iii) 5



$5_1$

* Only knots not obtained in this way (upto 8 crossings)

$8_{21}, 8_{20}, 8_{19}, 8_{18}, 8_{17}, 8_{16}, 8_{15}, 8_{10}, 8_5$.

* Let $T(a_1, a_2, \ldots a_n)$ be an $n$-tangle.

Consider the associated rational number

$$[a_n, a_{n-1}, \ldots a_1] = \left( a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\ddots}}} \right)$$

Eg :- (i) $[2,2] = 2 + \frac{1}{2} = \frac{5}{2}$.

(ii) $T[7,2,1,2] = [2\ 1\ 2\ 7] = \frac{59}{22}$.

(iii) $T[-7,-3,3] = [3,-3,-7] = \frac{59}{22}$.

* Tangles $T(a_1, a_2, \dots a_n)$ and $T(b_1, b_2, \dots b_m)$ are equivalent; if we can convert one to other by finitely many elementary knot moves.

* $T(a_1, a_2, \dots a_n)$ corresponds to the fraction '$p/q$' $= [a_n, a_{n-1}, \cdots a_1]$

* Theorm :- If $T(a_1, a_2, \dots a_n) \simeq T(b_1, b_2, \dots b_m)$; then the corresponding fractions have to be equal; i.e

$$[a_n, a_{n-1}, \dots a_1] = [b_m, b_{m-1}, \dots b_1]$$ and the converse also holds.

* $T(-7,-3,3)$ and $T(7,2,1,2)$; these are equivalent as Tangles.

* Theorm :- Suppose $K_1$ and $K_2$ are rational knots having rational no.s $\alpha/\beta$ and $\alpha'/\beta'$ respectively. $K_1$ and $K_2$ are equivalent iff

(i) $\alpha = \alpha'$ ; $\beta \equiv \beta' (\text{mod } \alpha)$

(ii) $\alpha = \alpha'$ ; $\beta\beta' \equiv 1 (\text{mod } \alpha)$.

* Consider mirror images; $K = \frac{\alpha}{\beta} \to [a_n, a_{n-1}, \cdots a_1]$

$K^* = -\frac{\alpha}{\beta} \to [-a_n, -a_{n-1}, \cdots, -a_1]$

A knot is said to be "achiral" (or) "amphichiral"; if the knot and its mirror images are equivalent.

i.e; if $\boxed{-\beta^2 \equiv 1 (\text{mod } \alpha)}$

Then, a knot is "amphichiral".

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Eg :-

(i) $[2,2] = 2+\frac{1}{2} = \frac{5}{2}\left(\frac{\alpha}{\beta}\right)$

(ii) $[-2,-2] = -2-\frac{1}{2} = \frac{-5}{2}\left(\frac{\alpha'}{\beta'}\right)$

$\beta\beta' = -4 \equiv 1 \pmod 5$.

So, [2,2] closure is a achiral.

(iii) $G_3 : T[2,1,1,2]$

$[2,1,1,2] = 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}} = \frac{13}{5}\left(\frac{\alpha}{\beta}\right)$.

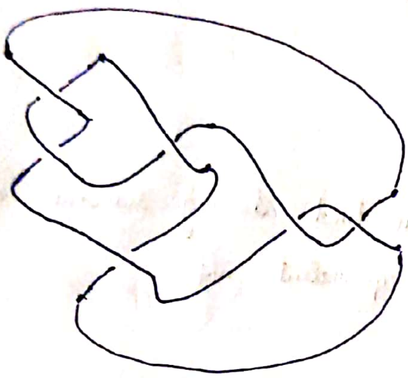
$T[-2,-1,-1,-2]$

$= [-2,-1,-1,-2]$

$= -2+\cfrac{1}{-1+\cfrac{1}{-1+\cfrac{1}{-2}}} = \frac{-13}{5}$.

$-\beta^2 = -25 \equiv 1 \pmod{13}$

$\therefore$ '$G_3$' is a chiral.



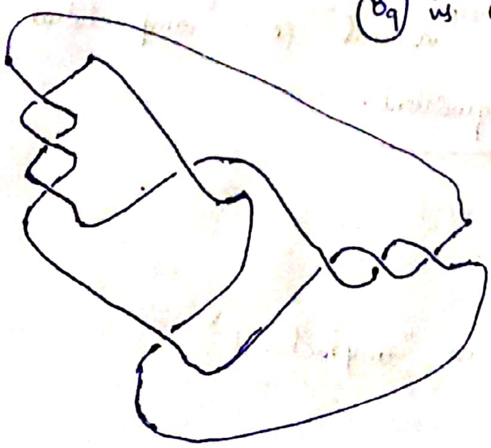
(iv) $8_9$ $T[3,1,1,3]$ ; $T[-3,-1,-1,-3]$

$= 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}} = \frac{25}{7}\left(\frac{\alpha}{\beta}\right)$

$= -3+\cfrac{1}{-1+\cfrac{1}{-1+\cfrac{1}{-3}}} = -\frac{25}{7}$

$-\beta^2 = -49 \equiv 1 \pmod{25}$.

$8_9$ is achiral.



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* Verify this for $T[n,1,1,n]$.

$$[n,1,1,n] : n + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{n}}} = \frac{(2n+1)n + (n+1)}{2n+1} = \frac{2n^2+2n+1}{2n+1} \left(\frac{\alpha}{\beta}\right).$$

$$\begin{aligned} -\beta^2 = -(2n+1)^2 &= -(4n^2+4n+1) \\ &= -(4n^2+4n+2)+1 \\ &= -2(2n^2+2n+1)+1 \equiv 1 \bmod (2n^2+2n+1). \end{aligned}$$

(v) $4_1$

$T[1,1,1,1]$

$$T[2,2] = 2 + \frac{1}{2} = \frac{5}{2} \cdot \left(\frac{\alpha'}{\beta'}\right).$$

$$[1,1,1,1] : 1 + \cfrac{1}{1+\cfrac{1}{1+1}} = \frac{5}{3} \left(\frac{\alpha}{\beta}\right)$$

$\alpha = \alpha'$

$$\beta \cdot \beta' = 3 \cdot 2 = 6 \equiv \underline{1 (\text{mod } 5)}$$

So; Tangles $[1,1,1,1]$ and $T[2,2]$ represent same knot (or) link

31/1/19

* <u>Fundamental</u> <u>Group</u> (Topological Invariant) :

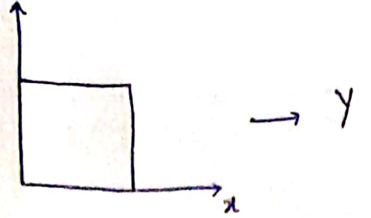
* <u>Homotopy</u> :

Let $X, Y$ be 2 topological spaces; and 2 continuous maps $f, g : X \to Y$ are called "homotopic" if there is a cont. map $H : X \times I \to Y$ such that;

$$H\big|_{X \times \{0\}} = f \quad \text{and} \quad H\big|_{X \times \{1\}} = g.$$

* Here 'H' is in b/w 'f' and 'g'.

$X \times [0,1]$
if $X$ - unit circle.
then, $X \times I$
$\Downarrow$




$(x,t) \in X \times I$

$H(x,t) \in Y$

$H(x,0) = f(x)$

$H(x,1) = g(x)$.

* Claim : Homotopy is an equivalence relation on the set of cont. fns ; $\boxed{f : X \to Y}$

* Reflexive : $f \sim f$ ; since $H(x,s) = f(x)$ is a homotopy b/w 'f' & 'f'.

* Symmetric : If $f \sim f'$ ; then $f' \sim f$.

i.e $\exists\ H : X \times I \to Y$ s.t. $H(x,0) = f(x)$
$H(x,1) = f'(x)$

Now, consider $H' : X \times I \to Y$ s.t. ;
$H'(x,t) = H(x,1-t)$.
then as $H$ is cts
$H'$ is also cts.

and
$H'(x,0) = H(x,1) = f'(x)$
$H'(x,1) = H(x,0) = f(x)$

$\therefore$ '$H'$' gives a homotopy from $f'$ to $f$.

* Transitive : If $f \sim g$ & $g \sim k$ ; then $f \sim k$.

i.e $\exists\ H, H' : X \times I \to Y$ s.t.
$H(x,0) = f(x)$ and $H'(x,0) = g(x)$ and both are
$H(x,1) = g(x)$ $\quad H'(x,1) = k(x)$ cts.

*


* Now, consider $H'' : X \times I \to Y$ as ;

$$H''(x,t) = \begin{cases} H(x,2t) & ; \ 0 \le t \le \frac{1}{2} \\ H'(x, 2t-1) & ; \ \frac{1}{2} < t \le 1 \end{cases}$$

now ; $H''$ is a homotopy from '$f$' to '$k$'.

So ; $f \sim k$.

* Def$^n$ : [Path Homotopy]

$\boxed{\text{Path}}$ : $f : [0,1] \to X$ ; '$f$' is cont.

$f(0) = x_0$ is the starting point.

$f(1) = x_1$ is the ending point.


$\boxed{\text{Path}}$ $\boxed{\text{Homotopy}}$ :

Let $f, f' : [0,1] \to X$ be cont. paths in $X$ s.t. $f(0) = f'(0) = x_0$ and $f(1) = f'(1) = x_1$ ; '$f$' is said to be path homotopic to '$f'$' if $\exists$ H

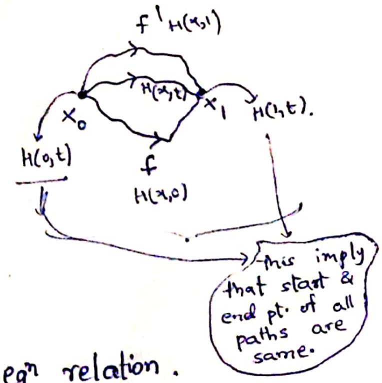
$H : I \times I \to X$ such that ;

$H(x,0) = f(x)$

$H(x,1) = f'(x)$

$H(0,t) = x_0$

$H(1,t) = x_1$




* NOTE : Path homotopy ($\sim_p$) is also an eq$^n$ relation.