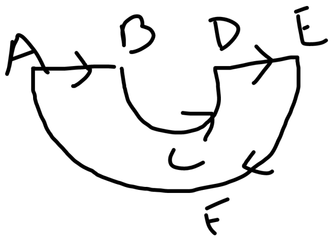
MSO202A: Assignment-IV Solutions

1. Evaluate

(a) $\int_C |z|\frac{z}{\bar{z}}\,dz$ where C is the clockwise oriented boundary of the part of the annulus $2 \leq |z| \leq 4$ lying in the third and fourth quadrants.



Soln:

$$\begin{aligned}\int_C |z|\frac{z}{\bar{z}}\,dz &= \int_{AB} |z|\frac{z}{\bar{z}}\,dz + \int_{BCD} |z|\frac{z}{\bar{z}}\,dz + \int_{DE} |z|\frac{z}{\bar{z}}\,dz + \int_{EFA} |z|\frac{z}{\bar{z}}\,dz \\ &= \int_{-4}^{-2} -x\,dx + \int_{\pi}^{2\pi} 2\frac{2e^{i\theta}}{2e^{-i\theta}}\,2ie^{i\theta}d\theta + \int_2^4 x\,dx + \int_0^{-\pi} 4\frac{4e^{i\theta}}{4e^{-i\theta}}\,4ie^{i\theta}d\theta \\ &= 6 + 8/3 + 6 - 32/3 = 4\end{aligned}$$

(b) $\int_C \frac{1}{\sqrt{z}}dz$ where C is the counterclockwise oriented semicircular part of the circle $|z| = 1$ in the lower half plane and $\sqrt{z}$ is defined such that $\sqrt{1} = -1$.

Soln: If $z = e^{i\theta}$, then $z^{1/2} = e^{i\theta/2+\pi i} = -e^{i\theta/2}$. Hence,

$$\int_C \frac{1}{\sqrt{z}}dz = \int_{\pi}^{2\pi} \frac{ie^{i\theta}}{-e^{i\theta/2}}d\theta = 2(1+i)$$

(c) $\int_C (z-a)^m dz$, where $m \in \mathbb{Z}$ and C is the semicircle $|z-a| = r$, $0 \leq \arg(z-a) \leq \pi$

Soln: If $m \neq -1$, then

$$\int_C (z-a)^m dz = \int_0^{\pi} r^m e^{in\theta} rie^{i\theta}d\theta = ir^{m+1}\int_0^{\pi} e^{i(m+1)\theta}d\theta = \frac{r^{m+1}}{m+1}\left((-1)^{m+1} - 1\right)$$

If $m = -1$, then

$$\int_C (z-a)^m dz = \int_0^{\pi} \frac{rie^{i\theta}}{re^{i\theta}}d\theta = \pi i$$

(d) $\int_C (z-a)^m dz$, where $m \in \mathbb{Z}$ and C is the circle $|z-a| = r$, $0 \leq \arg(z-a) \leq 2\pi$

Soln: If $m \neq -1$, then

$$\int_C (z-a)^m dz = 0$$

If $m = -1$, then

$$\int_C (z-a)^m dz = 2\pi i$$

2. Without actually evaluating the integral, prove that

(a) $|\int_\gamma \frac{dz}{z^2-1}| \leq \pi/3$, where $\gamma(t) = 2e^{it}$ for $0 \leq t \leq \pi/2$.

Soln: We have $L = \pi$ and $|z^2 - 1| \geq |z|^2 - 1 = 3$ and hence $|f(z)| \leq 1/3$

(b) $|\int_C \frac{dz}{z^2+1}| \leq 2\pi/(3-2\sqrt{2})$, where C is the circle $|z-1| = 1$.

Soln: Here $L = 2\pi$ and $|z^2+1| = |(z+i)||(z-i)|$. Now $|z-i|$ is the distance of a point on the circle from i and hence must be greater than or equal to the minimum distance between the point i and the circle. Thus $|(z-i)| \geq \sqrt{2}-1$ and same is for $|z+i|$. Hence $|z^2+1| \geq (\sqrt{2}-1)^2 = 3-2\sqrt{2}$

3. Let γ_1 be a semicircular path joining -1 and 1 with centre at 0 and γ_2 a rectangular path with vertices -1, $-1+i$, $1+i$ and 1. Find $\int_{\gamma_1} \bar{z}\, dz$ and $\int_{\gamma_2} \bar{z}\, dz$ (observe path dependence).

Soln:

$$\int_{\gamma_1} \bar{z}\, dz = \int_{\pi}^{0} e^{-i\theta} i e^{i\theta} d\theta = -\pi i$$

Let $A = -1, B = -1+i, C = 1+i$ and $D = 1$. Then

$$\int_{\gamma_2} \bar{z}\, dz = (\int_{AB} + \int_{BC} + \int_{CD})\bar{z}\, dz = \int_0^1 (-1-iy)idy + \int_{-1}^{1} (x-i)dx + \int_1^0 (1-iy)idy = -4i$$

4. Evaluate

$$(a) \int_{|z|=2} \frac{z}{z^2-1} dz \quad (b) \int_{|z|=2} \frac{z}{(z^2-1)^2} dz \quad (c) \int_{|z|=2} \frac{e^{2z}}{z(z+1)^4} dz$$

Soln: Let C_1 and C_2 be two small circles around $z = -1$ and $z = 1$.

(a)

$$\int_{|z|=2} \frac{z}{z^2-1} dz = \int_{C_1} \frac{z/(z-1)}{z+1} dz + \int_{C_2} \frac{z/(z+1)}{z-1} dz = 2\pi i \left(\frac{-1}{-2} + \frac{1}{2}\right)$$

Aliter:

$$\int_{|z|=2} \frac{z}{z^2-1} dz = \frac{1}{2}\left(\int_{|z|=2} \frac{1}{z-1} dz + \int_{|z|=2} \frac{1}{z+1} dz\right) = \frac{1}{2}\left(\int_{C_2} \frac{1}{z-1} dz + \int_{C_1} \frac{1}{z+1} dz\right) = \frac{1}{2}(2\pi i + 2\pi i)$$

(b)

$$\int_{|z|=2} \frac{z}{(z^2-1)^2} dz = \int_{C_1} \frac{z/(z-1)^2}{(z+1)^2} dz + \int_{C_2} \frac{z/(z+1)^2}{(z-1)^2} dz = \frac{2\pi i}{1!}\left[\left(\frac{z}{(z-1)^2}\right)'_{z=-1} + \left(\frac{z}{(z+1)^2}\right)'_{z=1}\right]$$

(c) Let C_1 and C_2 be two small circles around $z = -1$ and $z = 0$.

$$\int_{|z|=2} \frac{e^{2z}}{z(z+1)^4} dz = \int_{C_1} \frac{e^{2z}/z}{(z+1)^4} dz + \int_{C_2} \frac{e^{2z}/(z+1)^4}{z} dz = 2\pi i \left[\frac{1}{3!}\left(\frac{e^{2z}}{z}\right)^{(3)}_{z=-1} + \left(\frac{e^{2z}}{(z+1)^4}\right)_{z=0}\right]$$

5. Show that $\int_\gamma \frac{e^z}{z} dz = 2\pi i$, where $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$. Using this, evaluate

$$(a) \int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta \qquad (b) \int_0^{2\pi} e^{k\cos\theta} \sin(k\sin\theta)\, d\theta$$

Soln:

$$\int_\gamma \frac{e^z}{z} = 2\pi i (e_z)_{z=0} = 2\pi i$$

From (a)+i(b), we get

$$\int_0^{2\pi} e^{k\cos\theta} e^{ik\sin\theta} d\theta = \int_0^{2\pi} e^{ke^{i\theta}} d\theta = \int_{|z|=1} \frac{e^{kz}}{iz} dz = 2\pi$$

Hence integral in (a) is equal to 2π and that in (b) is zero.

6. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$. Find $\int_C P(z)/z^k \, dz$ where $C : |z| = R$ and $k \in \mathbb{N} \cup \{0\}$.

   Soln: If $k = 0$ or $k > n+1$, then the integral is zero. For $1 \le k \le n+1$

$$\int_C \frac{P(z)}{z^k} dz = \frac{2\pi i}{(k-1)!} (P(z))^{(k-1)}|_{z=0} = 2\pi i a_{k-1}$$

7. Let $C : |z| = 2$. Find the values of $\int_C z^n (1-z)^m \, dz$ for $m \in \mathbb{N} \cup \{0\}, n \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z}$

   Soln: Let $m \in \mathbb{N} \cup \{0\}$: Then $\int_C z^n (1-z)^m \, dz = 0$ for $n \in \mathbb{N} \cup \{0\}$. For $n \le 1$:

$$\int_C z^n (1-z)^m \, dz = \int_C \frac{(1-z)^m}{z^{-n-1+1}} \, dz = \frac{2\pi i}{(-n-1)!} \left((1-z)^n\right)^{(-n-1)}$$

   Other case can be considered in a similar way.

8. Evaluate the integral $\int_C \frac{dz}{z(z^2+1)}$ for all possible choice of the closed contour C that does not pass through $0, i, -i$.

   Soln:

   Case 1: C does not include any of the points $0, \pm i$. TheN $\int_C \frac{dz}{z(z^2+1)} = 0$.

   Case 2,3,4: C includes only the point $z = i$. TheN $\int_C \frac{dz}{z(z^2+1)} = 2\pi i (z/z+i)_{z=i} = -\pi i$. Similarly, we can calculate when C includes only the point $z = 0$ or i.

   Case 5,6,7: C includes only the points $z = 0$ and $z = i$. Let C_1 and C_2 be small circles around $z = 0$ and $z = i$. Then

$$\int_C \frac{dz}{z(z^2+1)} = \int_{C_1} \frac{dz/(z^2+1)}{z} + \int_{C_2} \frac{dz/(z(z+i))}{z-i} = 2\pi i \left[\left(\frac{1}{z^2+1}\right)_{z=0} + \left(\frac{1}{z(z+i)}\right)_{z=i} \right]$$

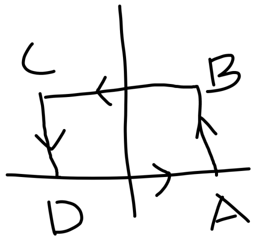
   Similarly, we can calculate when C includes only the point $z = 0, -i$ or $i, -i$.

   Case 8: C includes all the points $z = 0, i$ and $z = -i$. Let C_1, C_2 and C_3 be small circles around $z = 0, -i$ and $z = i$. Then

$$\int_C \frac{dz}{z(z^2+1)} = \int_{C_1} \frac{dz/(z^2+1)}{z} + \int_{C_2} \frac{dz/(z(z+i))}{z-i} + \int_{C_3} \frac{dz/(z(z-i))}{z+i} = \cdots (etc)$$

9. Show that $\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}$ for $\xi \in \mathbb{R}$ by integrating $f(z) = e^{-\pi z^2}$ along the lines of a rectangle with vertices $R, R + i\xi, -R + i\xi, -R$

Soln:



First consider $\xi > 0$. The vertices of the rectange are $A(R), B(R+i\xi), C(-R+i\xi)$ and $D(-R)$. We know that $\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1$. Since $f(z) = e^{-\pi z^2}$ is analytic, we have

$$(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}) f(z)\, dz = 0.$$

Now

$$\int_{DA} f(z)dz = \int_{-R}^{R} e^{-\pi x^2} dx \to 1 \quad \text{as } R \to \infty$$

$$\int_{AB} f(z)dz = \int_{0}^{\xi} e^{-\pi(R^2+2iRy-y^2)} i\, dy \implies |\int_{AB} f(z)dz| \leq e^{-\pi R^2} e^{\pi \xi^2} \xi$$

Hence this integeral tends to zero as $R \to \infty$ since ξ is fixed. Similarly, integral on CD is zwero. Now

$$\int_{BC} f(z)dz = \int_{R}^{-R} e^{-\pi(x+i\xi)^2} dx = -e^{-\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx$$

Hence as $R \to \infty$, $0 = 1 - e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx$

If $\xi < 0$, then we take the rectangle in the lower half.

10. Show that $\int_{|z|=2} \frac{e^{az}}{z^2+1} dz = 2\pi i \sin a$

Soln:

$$\int_{|z|=2} \frac{e^{az}}{z^2+1} dz = \frac{1}{2i} \int_{|z|=2} \left(\frac{e^{az}}{z-i} - \frac{e^{az}}{z+i} \right) dz = \frac{1}{2i} 2\pi i (e^{ia} - e^{-ia}) = 2\pi i \sin a$$

11. Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\{z \in \mathbb{C} : z \neq 0\}$ and bounded on the set $\{z \in \mathbb{C} : |z| \leq 1/2\}$. Prove that $\int_{|z|=R} f(z)dz = 0$ for every $R > 0$.

Soln: Let $r < R$ and $r \leq 1/2$. Now

$$\int_{|z|=R} f(z)dz = \int_{|z|=r} f(z)dz \implies |\int_{|z|=R} f(z)dz| = |\int_{|z|=r} f(z)dz| \leq \sup_{|z|=r} |f(z)| 2\pi r \to 0 \quad \text{as } r \to 0$$

12. Show that $|\int_{|z|=R} \frac{\operatorname{Log} z}{z^2} dz| \leq 2\sqrt{2}\pi \frac{\ln R}{R}$, $R > e^{\pi}$.

Soln: We have $\operatorname{Log}(z) = \ln|z| + i\theta$ where $-\pi < \theta \leq \pi$. Hence on $|z| = R$, we have $\operatorname{Log}(z) = \ln R + i\theta$ and hence $|\operatorname{Log}(z)| = \sqrt{(\ln R)^2 + \theta^2} \leq \sqrt{(\ln R)^2 + \pi^2} \leq \sqrt{(\ln R)^2 + (\ln R)^2}$ using $R > e^{\pi}$. Thus, on $|z| = R$ we have $|\operatorname{Log}(z)| \leq \sqrt{2} \ln R$. Now we get the result using ML-inequality.

13. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function where $\mathbb{D}$ is the open unit disk. If $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$, then show that $2|f'(0)| \leq d$.

Soln: Let $r < 1$ and then

$$f'(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^2}dz$$

If $g(z) = f(-z)$, the g is analytic and

$$-f'(0) = g'(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{g(z)}{z^2}dz = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(-z)}{z^2}dz$$

Subtracting, we get

$$2f'(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z) - f'(-z)}{z^2}dz \implies 2|f'(0)| \leq \frac{d}{r} \quad \forall r < 1.$$

Taking limit of $r \to 1^-$, we get the result.

14. Prove Mean Value Theorem: Let Ω be an open set and $f : \Omega \to \mathbb{C}$ be an analytic function. Then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta$ for every $r > 0$ such that the open ball $B(z_0, r)$ is contained in Ω. Further show that if $f(z_0) = 0$ for some $z_0 \in \Omega$, then $\mathrm{Re}(f)$ takes both positive and negative values on the circle which is the boundary of $B(z_0, r)$ for every $r > 0$.

Soln: From $|z - z_0| = r$ we write $z = z_0 + re^{i\theta}$. Now

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

For $f(z_0) = 0$, we get

$$\int_0^{2\pi} \mathrm{Re}\ f(z_0 + re^{i\theta})d\theta = 0$$

and hence $\mathrm{Re}(f)$ takes both positive and negative values on the circle.

15. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leq A + B|z|^k$ for some $k \in \mathbb{N}$ where $A > 0, B > 0$. Show that f is a polynomial of degree at most k.

Soln: Need to show $f^{(k+l)}(0) = 0$ for $l = 1, 2, \cdots$. This is equivalent to $a_{k+l} = 0$ $(l = 1, 2, \cdots)$ in the power series exapnasion of $f(z)$ around $z = 0$.

Now

$$f^{(k+l)}(0) = \frac{(k+l)!}{2\pi i}\int_{|z|=R} \frac{f(z)}{z^{k+l+1}}dz \implies |f^{(k+l)}(0)| \leq \frac{(k+l)!}{2\pi}\frac{A + BR^k}{R^{k+l+1}}2\pi R \to 0 \quad \text{as } R \to \infty$$

16. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$. Show that f is constant.

Soln: Since $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$, we must have $|f(z)|/|z| < 1$ for $|z| > R$ for sufficeintly large R. Also, since $\{z : |z| \leq R\}$ is closed and bounded, we have $|f(z)| \leq A$ on it. Combining we get $|f(z)| \leq A + |z|$ and hence by Q.16, $f(z) = a + bz$. Since $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$, we must have $b = 0$ and hence $f(z)$ is a constant.

17. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the image of the function has to necessarily meet the real axis and imaginary axis.

Soln: Let image of f does not meet real axis. Then either Im $f > 0$ or Im $f < 0$. Assume that Im $f > 0$. Now take $g(z) = e^{if(z)}$. Then $g(z)$ is an entire function and $|g(z)| = e^{-\text{Im } f} < 1$. Thus, $g(z)$ is a constant (by Liouville's theorem) and hence $f(z)$ must be a constant.

18. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $f(0) = 0$. Show that (a) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$, (b) If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$ or $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $f(z) = cz$ for all $z \in \mathbb{D}$.

Soln: (a) Let $g(z) = f(z)/z$ for $z \neq 0$. Then g can be expressed as a power series since $f(0) = 0$. Hence, $g(z)$ is analytic. Now for $0 < r < 1$, we have on $|z| = r$, $|g(z)| < 1/r$ since $|f(z)| < 1$ ($f : \mathbb{D} \to \mathbb{D}$). Taking $r \to 1$ we get $|g(z)| \leq 1$ and hence $|f(z)| \leq |z|$. Also, we have $g(0) = f'(0)$ and from $|g(z)| \leq 1$ we get $|f'(0)| \leq 1$

(b) If $|f(z_0)| = |z_0|$, then g [as defined in (a)] has maximum inside D and hence by maximum principle, $|g(z)|$ is a constant. Hence, $|f(z)| = |z|$ or $f(z) = cz$ where $|c| = 1$. If $|f'(0)| = 1$, then maximum of g occurs at $z = 0$ and same argument holds.

19. Let $f_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$ be analytic functions such that $f_1(a_n) = f_2(a_n)$ for a bounded sequence of complex numbers. Show that the functions are same.

Soln: Let $g(z) = f_1(z) - f_2(z)$. Then $g(a_n) = 0$. Since $\{a_n\}$ is a bounded sequence, it has a convergent subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to a$ as $k \to \infty$. Then $g(a) = 0$ and hence zeros of g has a limit point. Hence $g \equiv 0 \implies f_1(z) \equiv f_2(z)$

20. Find the maximum of the function $|f|$ on $\overline{\mathbb{D}}$ (closed unit disk) for (a) $f(z) = z^2 - z$ and (b) $f(z) = \sin z$.

Soln: Need to look only on the boundary $|z| = 1$.

(a) Here $|f(z)| = |z||z - 1| \leq 2$ and $f(-1) = 2$. Hance max. is 2.

(b) Here

$$|\sin z| = \left| \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \right| \leq \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} = \frac{1}{2}\left(e - \frac{1}{e}\right)$$

Also, since $2i \sin z = e^{iz} - e^{-iz}$ we have for $z = e^{i\theta}$ at $z = i$ ($\theta = \pi/2$)

$$2|\sin i| = |e^{-1} - e| \implies |\sin i| = \frac{1}{2}\left(e - \frac{1}{e}\right)$$

Hence, maximum is $(e - e^{-1})/2$