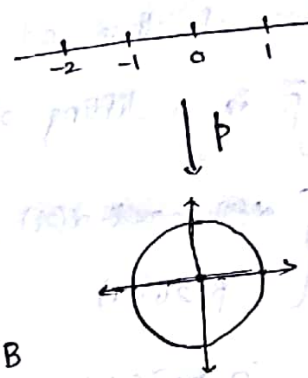
06/03/2018
Tuesday

* $p : \mathbb{R} \to S^1$ defined by

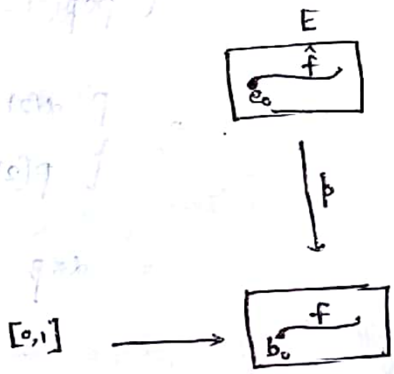
$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map.



* **Path-Lifting Lemma :**

Let $p : E \to B$ be a covering map s.t $p(e_0) = b_0$. Let $f : [0,1] \to B$ be a continuous map s.t $f(0) = b_0$, then $\exists$ unique $\hat{f} : [0,1] \to E$ s.t $\hat{f}(0) = e_0$ and $p \circ \hat{f} = f$



* **Homotopy Lifting Lemma :**

Let $H : I \times I \to B$ be a homotopy, $p : E \to B$ be a covering map s.t $p(e_0) = b_0$. then H has a lift $\tilde{H} : I \times I \to E$ s.t $\tilde{H}(0,0) = e_0$.

H is a path homotopy $\Rightarrow \tilde{H}$ is a path homotopy.

* **Theorem :**

Let $p : E \to B$ be a covering map s.t $p(e_0) = b_0$
Let f and g be two paths in B from b_0 to b_1, then $\tilde{f}$ and $\tilde{g}$ are their lifts to e_0.

If $f \simeq_p g \Rightarrow \tilde{f} \simeq_p \tilde{g}$ $\quad \tilde{f}(1) = \tilde{g}(1)$.

* **Theorem :**

$$\pi_1(S^1,(1,0)) \approx (\mathbb{Z},+)$$

Proof :

Let $\phi : \pi_1(S^1,(1,0)) \longrightarrow (\mathbb{Z},+)$

$$\phi[f] = \tilde{f}(1)$$

ϕ is onto

$$\phi(\cos 2\pi n x, \sin 2\pi n x) = n$$

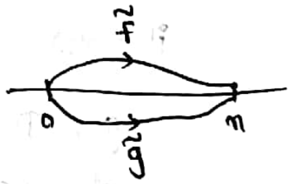
$$\hat{f}(\cos 2\pi n x, \sin 2\pi n x) = nx$$

$$\tilde{f}(1) = n.$$

ϕ is one-onto

$$\phi(f) = \phi(g) \Rightarrow \tilde{f}(1) = \tilde{g}(1)$$

$$\Rightarrow \tilde{f} \simeq_p \tilde{g}$$



$$\tilde{f}(0) = \tilde{g}(0) = 0$$

$$\tilde{f}(1) = \tilde{g}(1) = n$$

$$\tilde{f} * \bar{\tilde{g}} \simeq e \qquad (\mathbb{R} \text{ is Simply connected})$$

$$\Rightarrow \tilde{f} \simeq_p \tilde{g}$$

ϕ is a homeomorphism

$$\phi(m+n) = \phi(m) + \phi(n)$$

$\phi(f) = m$

$\phi(g) = n$

Construct a map

$$h(s) = \begin{cases} \tilde{f}(2s) & , 0 \le s \le \frac{1}{2} \\ m + \tilde{g}(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$\phi[f*g] = \widetilde{f*g}(1)$$

$$= m+n$$

$\Rightarrow$ ϕ is one-one, onto, homeomorphism

$$\Rightarrow \pi_1(S^1,(1,0)) \approx (\mathbb{Z},+)$$

⊛ Do Section 54 Q) 4, 5, 6, 8, 9 from Munkres book.

* $f : [0,1] \longrightarrow [0,1]$

then $\exists\ x_0 \in [0,1]$ s.t $f(x_0) = x_0$

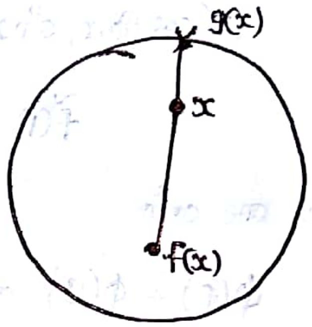
* <u>Theorem</u> : (Brouwer's Fixed point Theorem)

$f : D \longrightarrow D$ be a ~~constant~~ continuous map , $D = \{x^2+y^2 \leq 1\}$

then $\exists\ x_0$ in D st $f(x_0) = x_0$

<u>Proof</u> : Suppose not

i.e. $f(x) \neq x \quad \forall\ x \in D$



Join $f(x)$ to x by a straight line extended till it meets boundary circle S^1, call it $g(x)$

$g : D \longrightarrow S^1$

s.t $x \in S^1$, $g(x) = x$

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "id"'] & D \arrow[r, "g"] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "id_*"'] & \pi_1(D) \arrow[r, "g_*"] & \pi_1(S^1) \end{tikzcd}$$

$g_* \circ i_* = id$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & \{e\} \arrow[r, "g_*"] & \mathbb{Z} \end{tikzcd}$$

$\Rightarrow\Leftarrow$ Contradiction

$\Rightarrow f(x_0) = x_0$ for some $x_0 \in D$

08/03/2018
Thursday

* Theory of Braids (Murasugi book)

In 1930s', Emil Artin introduced the concept of a "mathematical braid" as a tool to study knots/algebra/cryptography

* Braid Def^n:

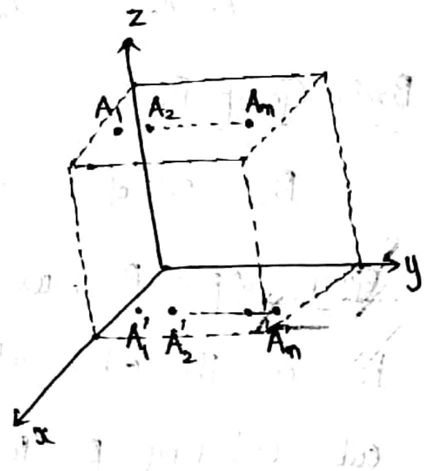
Let B be a cube

$$B = \{(x,y,z) \mid 0 \le x,y,z \le 1\}$$

Choose $A_1, A_2, \dots, A_n$, $A_1', A_2', \dots, A_n'$ as follows

$$A_1 = \left(\frac{1}{2}, \frac{1}{n+1}, 1\right), \dots\dots, A_n = \left(\frac{1}{2}, \frac{n}{n+1}, 1\right)$$

$$A_1' = \left(\frac{1}{2}, \frac{1}{n+1}, 0\right), \dots\dots, A_n' = \left(\frac{1}{2}, \frac{n}{n+1}, 0\right)$$

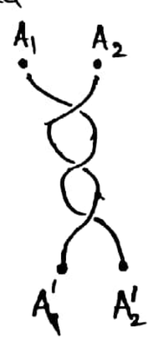


Join the points $A_1, A_2, \dots, A_n$ to $A_1', A_2', \dots, A_n'$ by means of n curves such that the curves don't intersect each other. These curves are called Strings.

Condition

If we take a plane P parallel to the xy plane, then P should intersect each string at one and only one point. Such, a collection of strings is an n-braid.

Eg: 2 Braid



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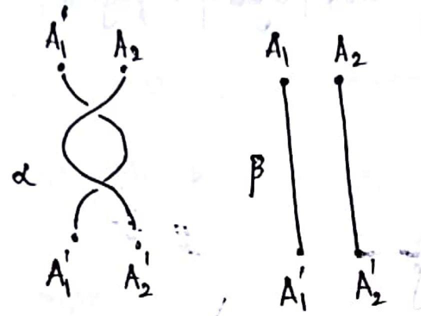
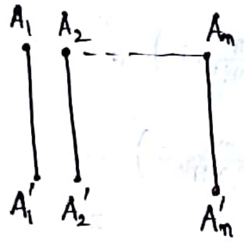
* Given two n-braids α and β. If by elementary knot moves we can go from α to β, then α is equivalent to β.

Eg:



(Equivalent)

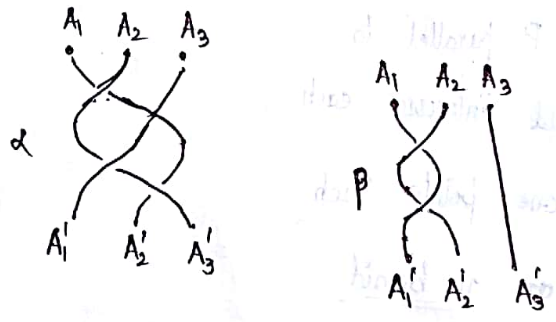
* The trivial n-braid is obtained by joining A_i to A_i', $i=1,2,\ldots,n$

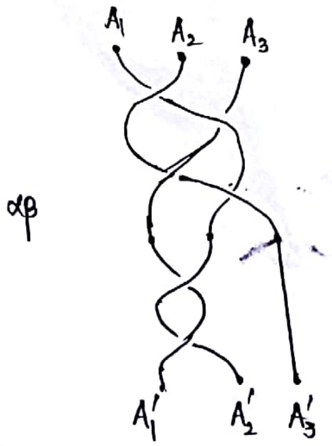


* <u>Braid Group B_n</u>:

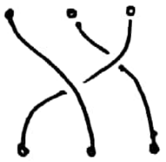
B_n - Set of all equivalence classes of n-braids

Given $\alpha, \beta \in B_n$, we define the product by gluing the base of the cube containing α to the top face of the cube containing β. This braid is called the product of α & β denoted $\alpha\beta$.



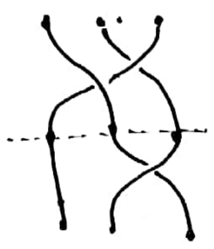


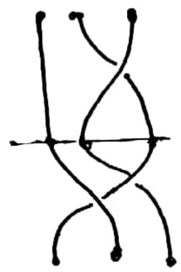
Eg:

α   β

$\alpha\beta$

$A_1 \to A_3'$
$A_2 \to A_2'$
$A_3 \to A_1'$





$\beta\alpha$

$A_1 \to A_2'$
$A_2 \to A_1'$
$A_3 \to A_3'$

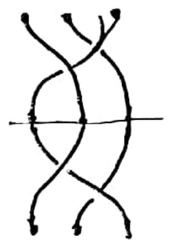
- In general, it is not true that $\alpha\beta = \beta\alpha$ i.e. $\alpha\beta$ and $\beta\alpha$ need not be equivalent braids.

- Braids are associative
 i.e. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

* Inverse of braid α

Consider the mirror image α^* of α.

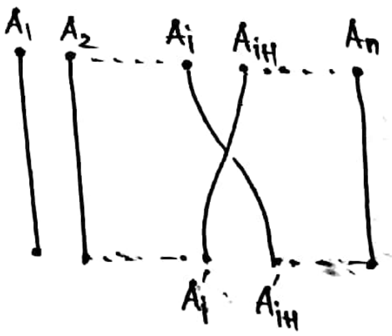
α



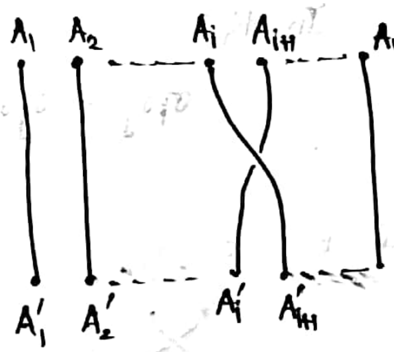
α^*

$\alpha\alpha^* = e = \alpha^*\alpha$

* Among the n-braids, we can create specific n-braids by connecting A_i to A_{i+1}' and A_{i+1} to A_i' and then connecting the remaining in A_j to A_j'



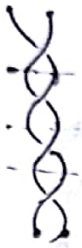
σ_i



σ_i^{-1}

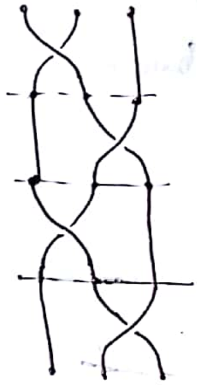
Given any n-braid $\alpha \in B_n$, we can express it as a finite product of $n-1$ generators.

Eg:

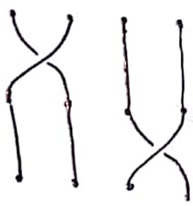


σ_1^4

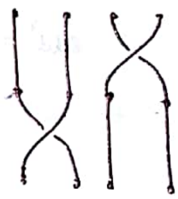
In B_3, $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$



In B_4, $\sigma_1\sigma_3$



$\sigma_3\sigma_1$



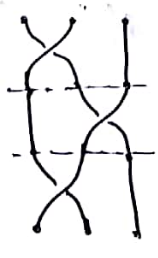
$\therefore \sigma_1\sigma_3$ and $\sigma_3\sigma_1$ are equivalent

NOTE:

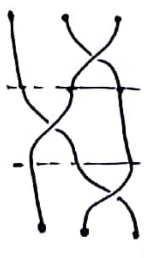
In B_n

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |j-i| \geq 2$$

Eg: In B_3, $\sigma_1\sigma_2\sigma_1$



$\sigma_2\sigma_1\sigma_2$



$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

* Fundamental relations of Braid group B_n

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j| \geq 2$

(2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $(i = 1, 2, \ldots, n-2)$

$$B_n = \left\{ \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \,\middle|\, \begin{matrix} (1) \\ (2) \end{matrix} \right\}$$

$$B_2 = \{\sigma_1\} \qquad \sigma_1^m \text{ or } \sigma_1^{-m}$$

$$B_3 = \{\sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$$

$$B_4 = \left\{ \sigma_1, \sigma_2, \sigma_3 \,\middle|\, \begin{matrix} \sigma_1\sigma_3 = \sigma_3\sigma_1 \\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \end{matrix} \right\}$$

* Eg:

$$\omega_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4$$

$$\omega_2 = \sigma_2\sigma_1\sigma_2^2$$

$$\begin{aligned} \omega_1 &= \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4 \\ &= \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_4\sigma_2 \\ &= \sigma_1\sigma_2\sigma_4^{-1}\sigma_4\sigma_1\sigma_2 \\ &= \sigma_1\sigma_2\sigma_1\sigma_2 \\ &= \sigma_2\sigma_1\sigma_2\sigma_2 \\ &= \sigma_2\sigma_1\sigma_2^2 \end{aligned}$$

$\Rightarrow$ ω_1 and ω_2 are equivalent.

* Given an n-braid α, we form its closure by connecting A_i to A_i' , $i = 1, 2, \ldots, n$ by non-intersecting arcs which are outside the braid α.

↳ Braid closure, $\bar{\alpha}$

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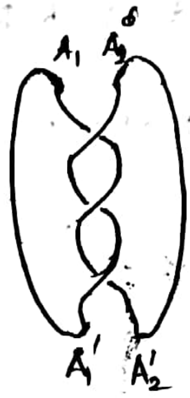
Eg: σ_1^3

$\overline{\sigma_1^3}$

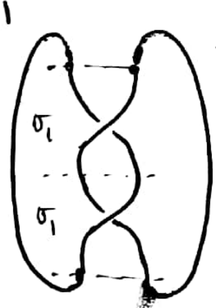


Knot

3_1 is closure of σ_1^3

3_1^* is closure of σ_1^{-3}

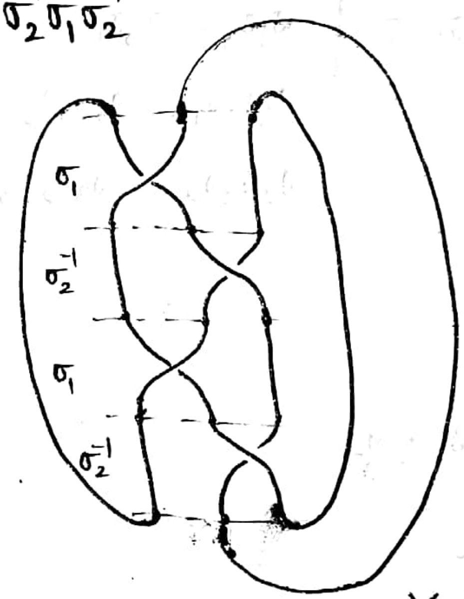
Eg: $\overline{\sigma_1^2}$



Link

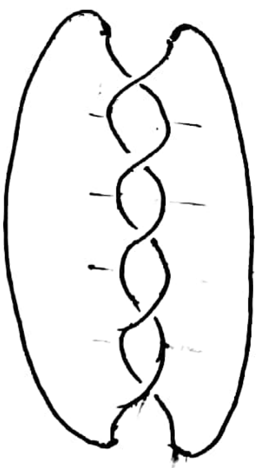
2_2^1

$\overline{\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}}$



4_1 knot

$\overline{\sigma_1^5}$



Knot

5_1

* Alexander's Theorem :

Any knot (or link) can be obtained as the closure of some n-braid α.

13/03/2018
Tuesday

* Recall :

Braid group, B_n

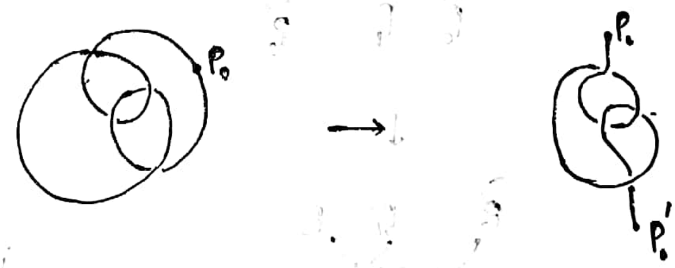
$$B_n = \left\{ \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{if } i = 1, 2, \ldots, n-2 \end{array} \right\}$$

* Alexander's Theorem :

Given any arbitrary knot (or link) K, it is equivalent to a knot formed from a braid closure $\bar{\alpha}$

Proof : (By construction)

Suppose D is a ~~knots~~ diagram of a knot (or) link



Step 1 :

Cut D at some point ~~P_0 is~~ P_0 (which is not a crossing point) and pull the loose ends apart.

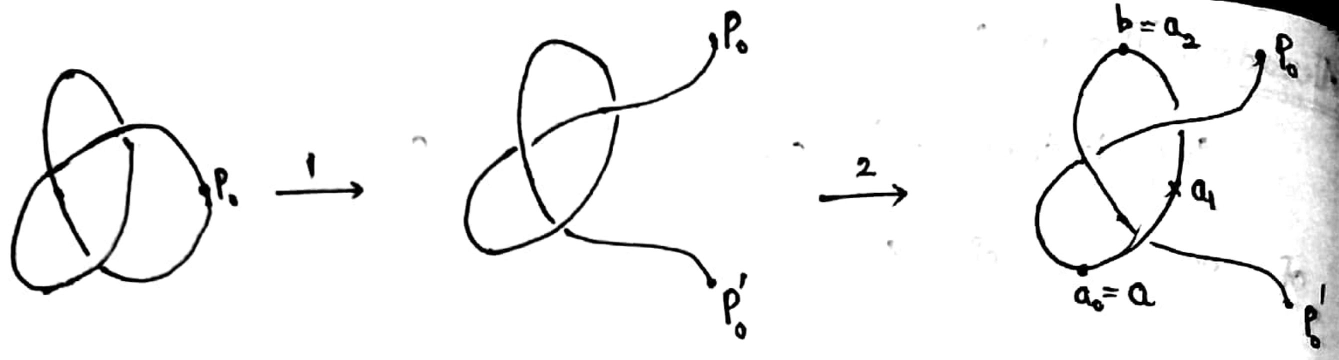
Step 2 :

Suppose the diagram has one local maxima b and one local minima a, We mark $n+1$ points on arc ab $(\widehat{ab})$

ie.

$$a = a_0 < a_1 < a_2 < \ldots < a_n = b$$

s.t the arc $\widehat{a_i a_{i+1}}$ intersects only one crossing point



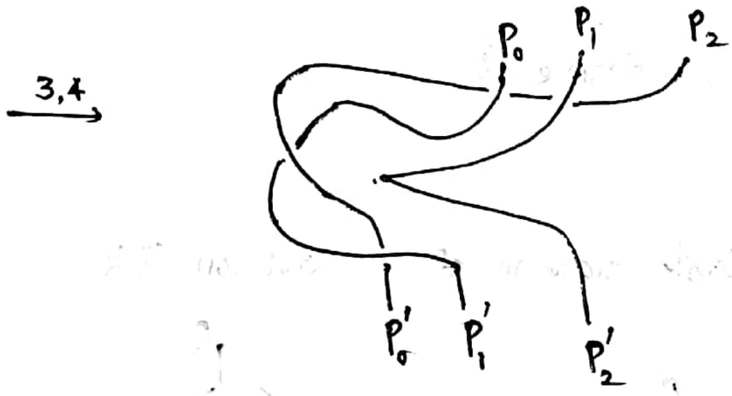
Step 3:

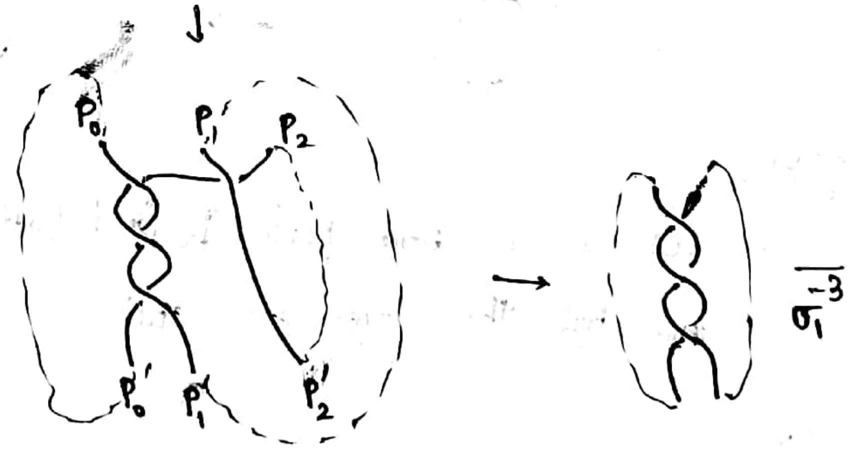
Replace the arc $\widehat{a_0a_1}$ by the much larger arcs $\widehat{a_0P_1'P_1a_1}$

Step 4:

By the same method, replace each arcs $\widehat{a_1a_2}, \widehat{a_2a_3}, \ldots, \widehat{a_{n-1}a_n}$ by the larger arcs $\widehat{a_1P_2P_2'a_2}, \ldots$

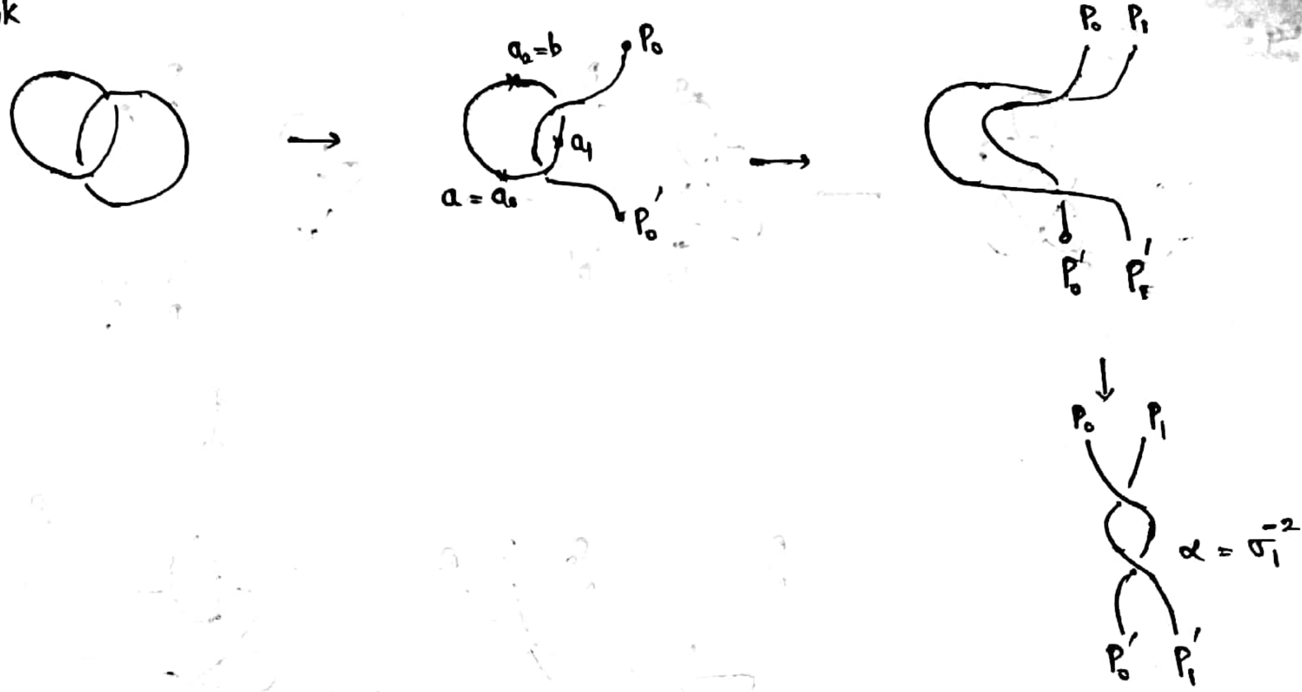
This will give a braid α s.t $\bar{\alpha} = K$



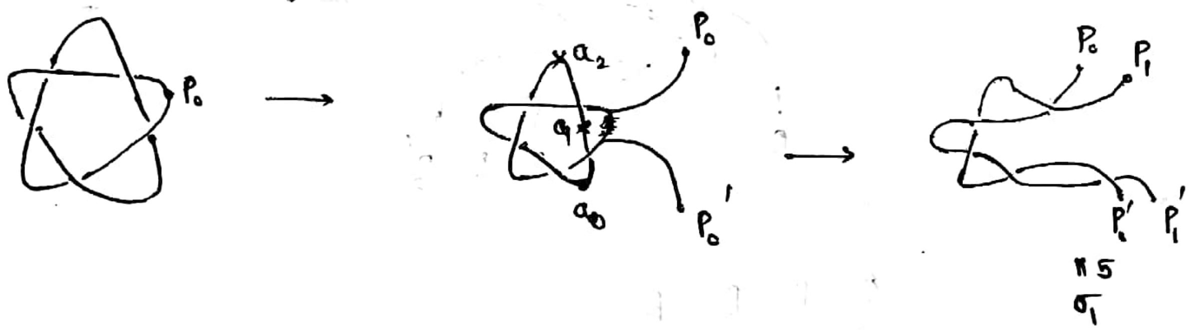


$\therefore$ $3_1 \simeq \sigma_1^{-3}$ closure

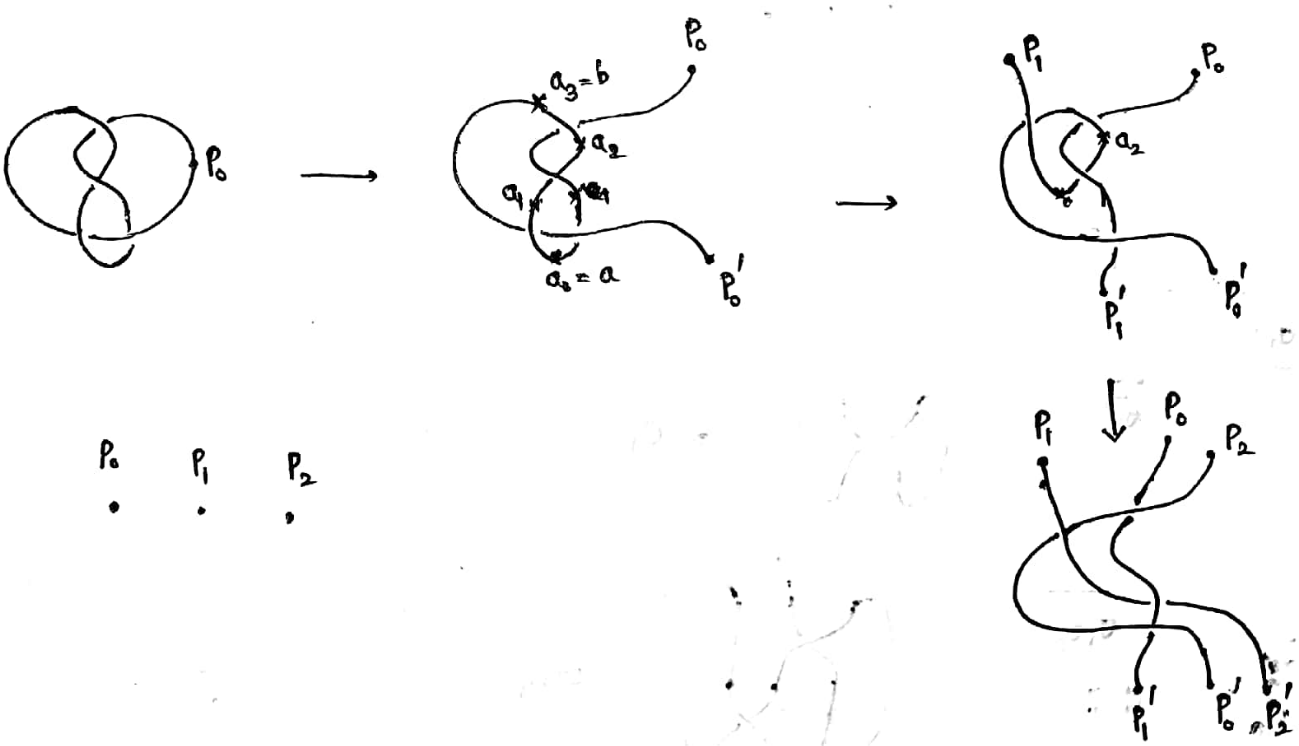
Eg: Hopf Link



Eg: 5_1



Eg: 4_1



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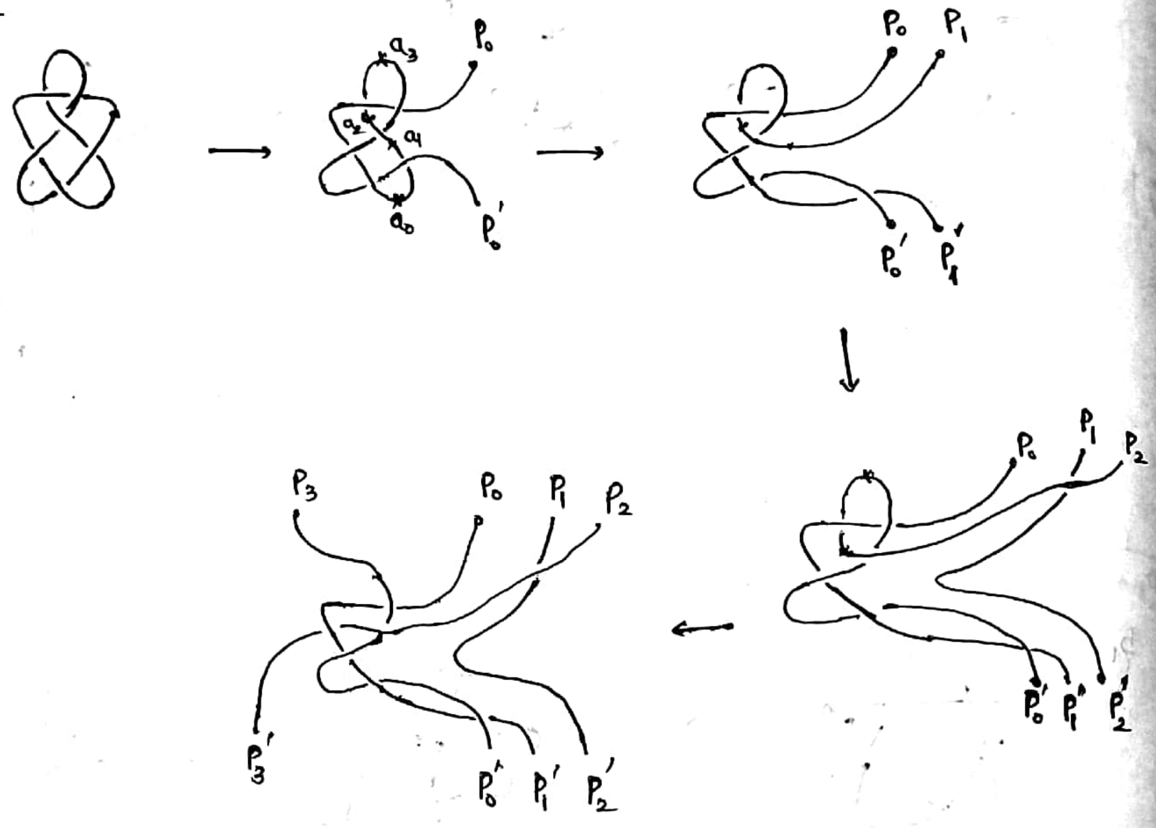
Eg:

6_2



$P_0 \quad P_1 \quad P_2 \quad P_3$

Eg:

σ_1^{-1}

$\cong O_1$

$\sigma_1^{-1}\sigma_2$

$\cong O_1$

$\sigma_1\sigma_2\sigma_3^{-1}$

$\cong O_1$

there distinct braids gives rise to same closure which is O_1 knot.

Q) When do braids α, β closure give the same knot $\bar{\alpha} = \bar{\beta}$?

* Markov moves :

Let $B_\infty = \bigcup_{n \geq 1} B_n$

We perform ~~of~~ two operations (Markov moves) on B_∞

(1) If $\beta \in B_n$, M_1 transforms β into n-braid $\underbrace{\gamma \beta \gamma^{-1}}_{\text{Conjugate of } \beta}$ where γ is some element of B_n. $\rightarrow$ Conjugation

(2) If $\beta \in B_n$, M_2 transforms β into either of the two $(n+1)$ braids $\beta\sigma_n$ (or) $\beta\sigma_n^{-1}$ $\rightarrow$ Stabilization

* Suppose α and β are elements of B_∞. If we can transform α to β by finitely many markov moves, then α is said to be Markov equivalent to β, $\alpha \sim_M \beta$.

* Markov's Theorem :

$$K_1 \cong K_2 \Leftrightarrow \beta_1 \sim_M \beta_2$$

* Braid Index : (also a knot invariant)

We define Braid index $b(K)$ = minimum string braid α st $\bar{\alpha} = K$

Eg: $b(3_1) = 2$

$b(4_1) = 3$

$b(6_2) = 3$

Exercise :

Find braids for $5_2, 6_1, 6_3$.

15/03/20 Thursday

* Markov moves

M1. $\beta \in B_n$ $\quad \beta \rightleftarrows \gamma\beta\gamma^{-1} \ (\in B_n)$ $\dashrightarrow$ Conjugation

M2. $\beta \in B_n$ $\quad \beta \rightleftarrows \beta\sigma_n$ (or) $\beta\sigma_n^{-1} \ (\in B_{n+1})$ $\dashrightarrow$ Stabilization.

* Determine braid closure

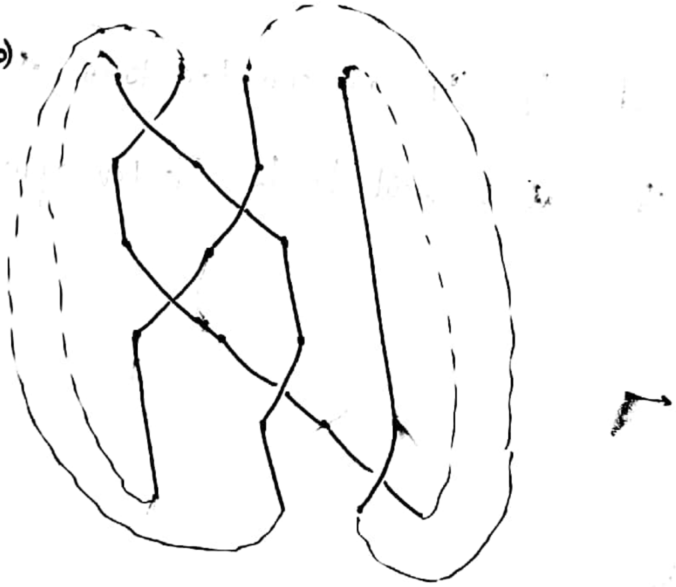
a) $\sigma_1^{-2}\sigma_2^{2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_2^{-2}\sigma_1^{2}$

b) $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3$

c) $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}$

Sol^n:

(b).



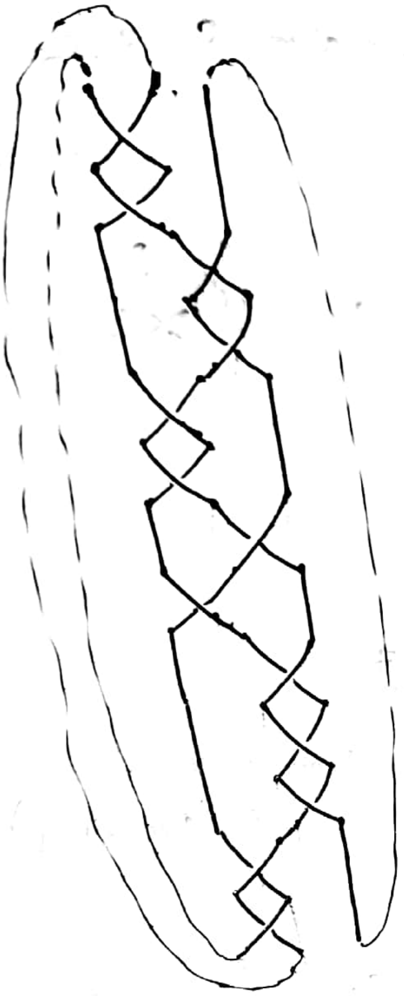
We can untwist

$\therefore \quad \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3 \sim_M \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2 \approx 4_1$

* Theorem :

$\beta_1 \sim_M \beta_2$ iff $\overline{\beta_1} = \overline{\beta_2}$

(a)



a) $\approx \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2 \approx 4_1$ (Figure 8).

* Show that the braid words $\omega_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_4$

and $\omega_2 = \sigma_2^2\sigma_3\sigma_1\sigma_3^{-1}\sigma_4$ have the same braid closure.

Sol^n:

$$\omega_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_4$$

$$\approx \cancel{\sigma_2^{-2}\sigma_1\sigma_2} \; \sigma_2(\sigma_2\sigma_1\sigma_2)\sigma_2^{-1}\sigma_4$$

$$\approx \sigma_2^2\sigma_1\sigma_4$$

$$\approx \sigma_2^2\sigma_1$$

$$\omega_2 = \sigma_2^2\sigma_3\sigma_1\sigma_3^{-1}\sigma_4$$

$$\approx \sigma_2^2\sigma_1\sigma_3\sigma_3^{-1}\sigma_4$$

By properties of B_5.

$$\approx \sigma_2^2\sigma_1\sigma_4$$

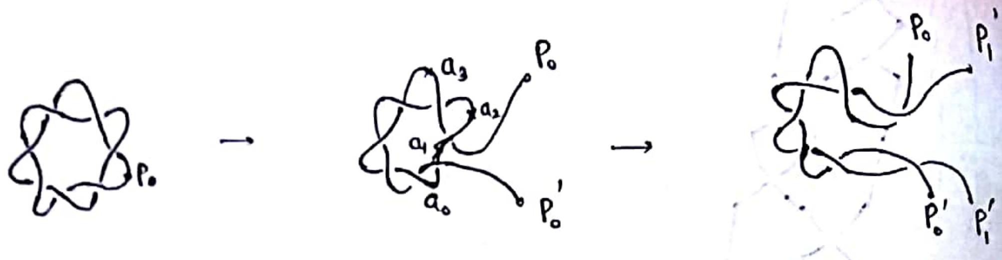
$$\approx \sigma_2^2\sigma_4$$

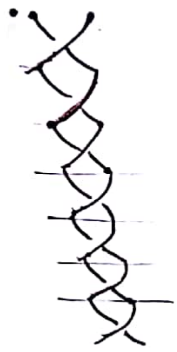
$$\Rightarrow \omega_1 \sim_M \omega_2$$

$$\Rightarrow \overline{\omega_1} = \overline{\omega_2}$$

* Use Alexander's Theorem to find a braid representation for $7_1, 6_3$.

Sol: a) 7_1



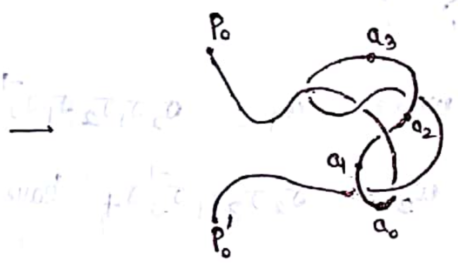


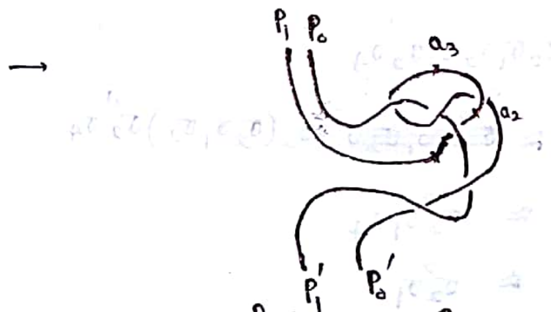
σ_1^7

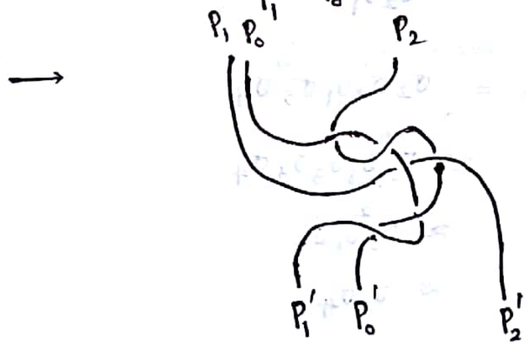
Braid index = 2

($\because$ Two strings)

b) 6_3







$\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1^{2}$

(No need to confuse with the answer given in the paper given (diagrams are)

All are markov equivalent

* Given a knot K, its Jones polynomial $V_K(t)$

max degree $V_K(t) = m$

min degree $V_K(t) = n$

V-span $V_K(t) = m-n$

$$\boxed{c(K) \geq m-n}$$

If K is Alternating knot, $c(K) = m-n$.

* <u>**HOMFLY Polynomial**</u>

$P_K(v,z)$

1. $K \simeq 0$, $V_{0_1}(v,z) = 1$

2. $\frac{1}{v} P_{K^+} - v P_{K^-} = z \cdot P_{K_0}$

- Given a knot K, its Homfly polynomial $P_K(v,z)$

- V-span $P_K(v,z)$ = max v. degree − min v. degree
- Braid index $b(K) \geq \frac{1}{2}\left(\text{V-span } P_K(v,z)\right) + 1$
- For knots of crossings ≤ 10 except for 9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}

$$b(K) = \frac{1}{2}\left(\text{V-span } P_K(v,z)\right) + 1$$

* <u>**Exercise**</u> :

Show $b(K) = \frac{1}{2}\left(\text{V-span } P_K(v,z)\right) + 1$ for 3_1, 4_1, Hopf Link.

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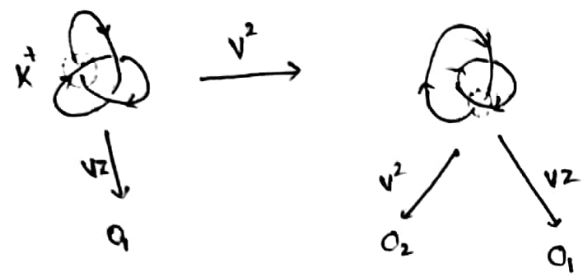
Sol^n:

3_1



$$P_{3_1}(v,z) = v^2\left(v^2 P_{O_2}(v,z) + vz\, P_{O_1}(v,z)\right) + vz\, P_{O_1}(v,z)$$

$$P_{O_2}(v,z) = \frac{1-v^2}{vz}$$

$$= v^4 P_{O_2}(v,z) + v^3 z\, P_{O_1}(v,z) + vz\, P_{O_1}(v,z)$$

$$= v^4 \frac{(1-v^2)}{vz} + v^3 z + vz$$

$$= \frac{v^3}{z} - \frac{v^5}{z} + v^3 z + vz$$

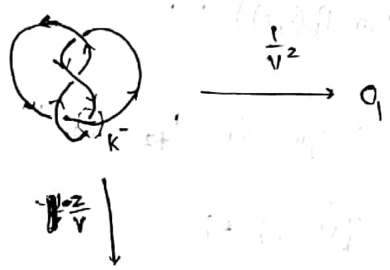
max v-degree = 5

min v-degree = 1

v-span $P_K(v,z)$ = 4

$\frac{1}{2}(4) + 1 = 3 = b(K)$

4_1



$$P_{4_1}(v,z) = \frac{1}{v^2}\cdot 1 - \frac{z}{v}\left[v^2\left(\frac{1-v^2}{vz}\right) + vz\cdot 1\right]$$

$$= \frac{1}{v^2} - \left((1-v^2) + z^2\right)$$

$$= \frac{1}{v^2} - 1 + v^2 - z^2$$

max v. degree = 2

min v. degree = -2

v-span = 4

$\frac{1}{2}$(v-span) + 1 = $\frac{1}{2}\times 4 + 1$ = 3 = b(K)

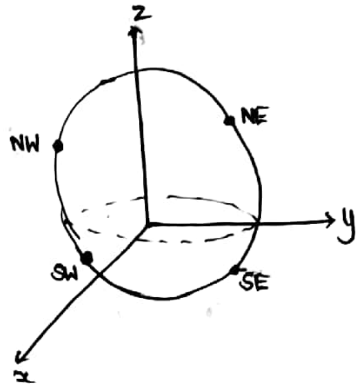
* **Rational knots :**

1960s' Conway

Consider the sphere S^2 ; fix 4 points on S^2

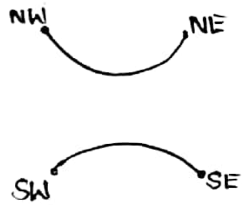
$NE = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $NW = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$SE = \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ $SW = \left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

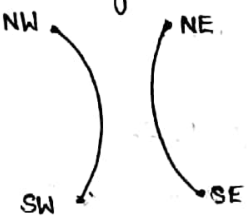


- A Tangle T consists of 2 curves, twisted in same way.

Trivial Tangle T(0)



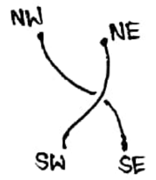
Trivial Tangle T(0,0)



Def^n:

Vertical Twist

When we exchange SW and SE

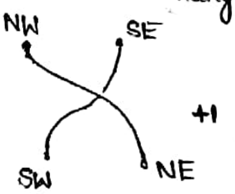


−1

+1

Horizontal Twist

When we exchange ~~NW and~~ NE and SE



+1

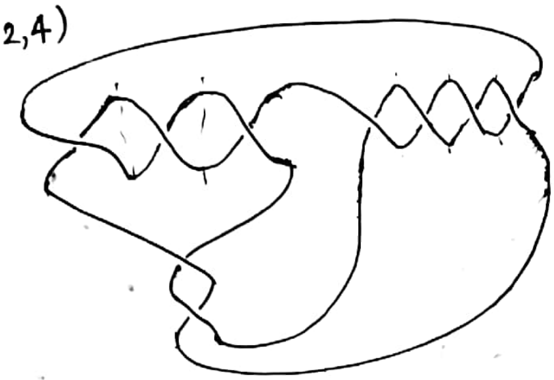
- We form a Tangle $T(a_1, a_2, \ldots, a_n)$ as follows

n odd. Start with T(0)

Peform a_1 horizontal twists
" a_2 vertical "
a_3 horizontal "
⋮
a_n horizontal "

Close it off by connecting NW to NE , SW to SE
↳ Rational knot.

Eg: $T(3,2,4)$



$T(-3,2,4)$

Remaining same.

n even

Start with $T(0,0)$

Perform a_1 vertical twists

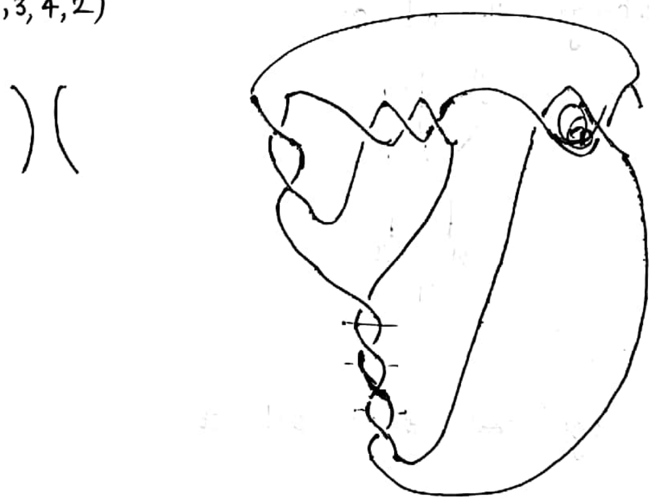
a_2 horizontal "

a_3 vertical "

⋮

a_n horizontal "

$T(2,3,4,2)$



$T(a_1, a_2, \ldots, a_n)$ closure corresponds to the

fraction $\frac{p}{q} = [a_n, a_{n-1}, \ldots, a_1]$ Continued

Eg: $[2,1,2,7] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{7}}} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{14+1}{7}}}$

$= \frac{59}{22}$

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$$[3,-3,-7] = 3 + \cfrac{1}{-3 + \cfrac{1}{-7}}$$

$$= 3 + \cfrac{1}{\cfrac{22}{-7}}$$

$$= 3 + \frac{-7}{22}$$

$$= \frac{66-7}{22}$$

$$= \frac{59}{22}$$

* **Theorem :**

There exists a correspondence between the set of rational numbers and the equivalence class of tangles, $T(a_1, a_2, \ldots, a_n) \simeq_{\text{equiv}} T(b_1, b_2, \ldots, b_m)$

iff $\underbrace{[a_n, a_{n-1}, \ldots, a_1]}_{\text{(continued fraction)}} = [b_m, b_{m-1}, \ldots, b_1]$

Eg:

3_1 corresponds to $T(3)$

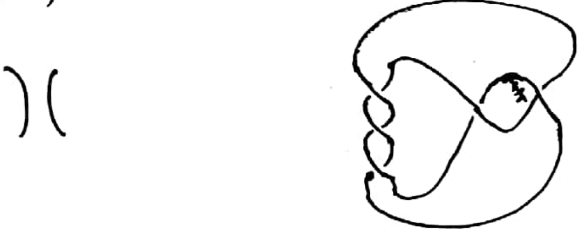
$T(2,2)$

$\approx 4_1$

$T(5)$

$\simeq 5_1$

$T(3,2)$



$\simeq 5_2$

If the number are positive or negative, then they are alternating knots.
(we get only alternating knots using Tangles)

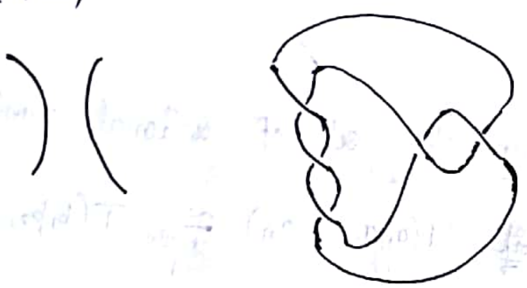
Eg: $T(7,2,1,2)$ -- 12 crossings

$[2,1,2,7] = \frac{59}{22}$

$T(-7,-3,3)$

$[3,-3,-7] = \frac{59}{22}$

- $T(3,2)$



$T(2,3)$

$[3,2] = 3 + \frac{1}{2}$

$= \frac{7}{2}$

$[2,3] = 2 + \frac{1}{3}$

$= \frac{7}{3}$

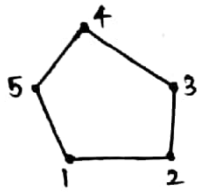
$T(5) = 5$

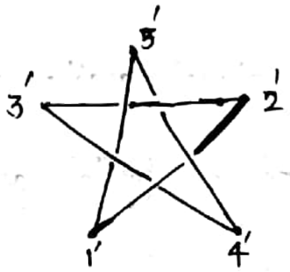
- $T(2)$

 $\simeq 2_1^2$

* Knots and Graphs

$G = \{V_G, E_G\}$





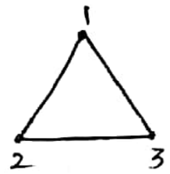
Embedding of a graph in $\mathbb{R}^3$ without self-intersection is a Spatial - Graph.

They are equivalent because they have a one-to-one correspondence.

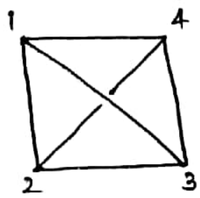
- The theory of spatial graph i.e. the embeddings of a graph in $\mathbb{R}^3$ is a generalization of knot theory.

- A graph is said to be complete if it has no loops and 2 distinct vertices are always the end-points of only one edge. K_m

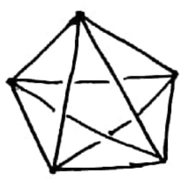
Eg: K_3



K_4



K_5

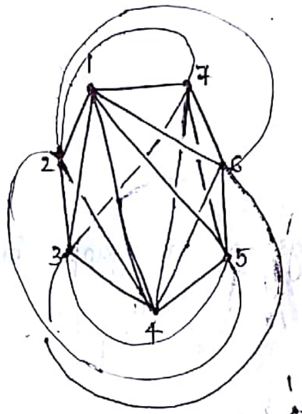


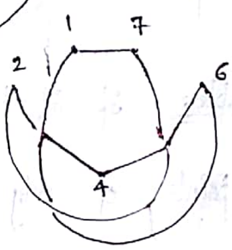
Ref: Journal Graph Theory 7 (1983) Pg 445-453

Bing's Conjucture :

Suppose K_m is a Complete graph with m veritices.

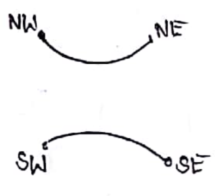
If $m \geq 7$, then whichever spatial graphs we use as a model for K_m, we will find a partial graph that represents a non-trivial knot.

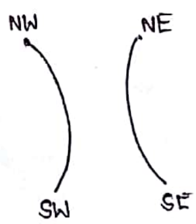




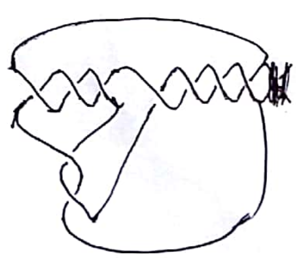
22/08/2018
Thursday

Recall : (Kunio Murasugi : Knot Theory & its Applications)
(Chapter - 9)





T(3,2,4)



Rational Knot

If we don't close, it is called Tangle.

Theorem:

If $T(a_1, a_2, \ldots, a_n) \simeq T(b_1, b_2, \ldots, b_m)$

$\Leftrightarrow \quad [a_n, a_{n-1}, \ldots, a_1] = [b_m, b_{m-1}, \ldots, b_1]$

(Holds only for Tangles)

Theorem:

Suppose K and K' are rational knots (or links) got ~~d~~ from closures of tangles having rational numbers $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$.

Then K and K' are equivalent iff

(i) $\alpha = \alpha'$, $\beta = \beta' \pmod{\alpha}$

(OR)

(ii) $\alpha = \alpha'$ and $\beta\beta' = 1 \pmod{\alpha}$

K is amphichiral if $\alpha = \alpha'$ and
$\beta' = -\beta$
$\beta\beta' = -\beta^2 \equiv 1 \pmod{\alpha}$
(or) $\beta^2 = -1 \pmod{\alpha}$

Eg:

$K = T(3,2) = 5_1$

$\left.\frac{p}{q}\right|_K = 2 + \frac{1}{3} = \frac{7}{3}$

$L = T(2,3)$

$3 + \frac{1}{2} = \frac{7}{2}$

$5_1^* = K^* = T(-3,-2)$

$[-2,-3] = -2 + \frac{1}{-3}$

$= \frac{7}{-3}$

$T(-2,-3)$.

$[-3,-2] = -3 + \frac{1}{-2}$

$= \frac{-7}{2}$.

$L = \frac{7}{2} = \frac{\alpha}{\beta}$

$K^* = \frac{7}{-3} = \frac{\alpha'}{\beta'}$

$\alpha = \alpha' = 7$

$\beta\beta' = -6 = 1 \pmod{7}$

$\Rightarrow L \approx K^*$

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Eg: 4_1 $T(2,2)$

$$[2,2] = 2+\frac{1}{2} = \frac{5}{2}$$

4_1^* $T(-2,-2)$

$$[-2,-2] = -2+\frac{1}{-2} = \frac{5}{-2}$$

$\alpha = 5 = \alpha'$

$\beta = 2,\ \beta' = -2$

$\beta\beta' = -4 \equiv 1 \pmod 5$

$\Rightarrow$ 4_1 is Amphichiral.

Eg: $T(1,1,1,1)$

$$\frac{p}{q} = 1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}$$

$$= 1+\cfrac{1}{1+\frac{1}{2}}$$

$$= 1+\frac{2}{3}$$

$$= \frac{5}{3}$$

$T(2,2)$

$$[2,2] = 2+\frac{1}{2} = \frac{5}{2}$$

$\alpha = 5 = \alpha'$

$\beta = 2,\ \beta' = 3$

$\beta\beta' = 6 \equiv 1 \pmod 5$

$\therefore$ $T(1,1,1,1) \approx T(2,2) \approx 4_1$

Scanned by CamScanner

* Amphichiral knots

4_1 — $T(2,2)$

6_3 — $T(2,1,1,2)$

8_3 — $T(4,4)$

8_9 — $T(3,1,1,3)$

8_{12} — $T(2,2,2,2)$

Q) Prove that $T(a,a)$ is Achiral ?

$$\frac{p}{q} = a + \frac{1}{a}$$

$$= \frac{a^2+1}{a}$$

$$K^* = T(-a,-a) =$$

$$\frac{p'}{q'} = -a + \frac{1}{-a}$$

$$= \frac{a^2+1}{-a}$$

$$p = p' = a^2+1$$

$$q = a, \quad q' = -a$$

$$qq' = -a^2 = -a^2-1+1$$

$$\equiv 1 \pmod{a^2+1}$$

- $T(a,b,b,a)$ is also Achiral
- $T(a_1, a_2, \ldots, a_n, a_n, a_{n-1}, \ldots, a_2, a_1)$ is also Achiral.

Palindromes

* Simple Homotopy Theory (Ref^n: Basic Topology by M.A. Armstrong)

Motivation:

Is $\mathbb{R}^n \underset{homeo}{\approx} \mathbb{R}^m$? $m \neq n$

Non-compact
Connected
⋮
$\pi_1(\mathbb{R}^n) = \{e\}$

$\pi_1(S^n) = \{e\}, n \geq 2$

$S^1 = \{(x,y) \mid x^2+y^2 = 1\}$

$S^2 = \{(x_1,x_2,x_3) \mid \sum_{i=1}^{3} x_i^2 = 1\}$

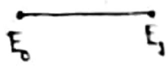
$S^n = \{(x_1,x_2,\dots,x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$

* Simplicial complexes

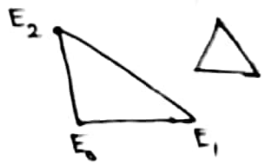
$\mathbb{R}^n$ $E_0 = (0,0,\dots,0)$
$E_1 = (1,0,\dots,0)$
$E_2 = (0,1,0,\dots,0)$
⋮
$E_n = (0,0,\dots,0,1)$

σ_0 – Zero - Simplex - E_0

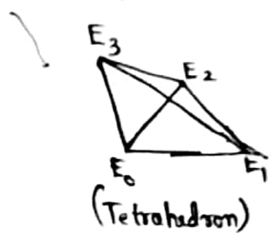
σ_1 – 1 – Simplex – Convex span of E_0, E_1



σ_2 – 2 - Simplex – Convex span of E_0, E_1, E_2



σ_3 – 3 – complex – " " " E_0, E_1, E_2, E_3



In general, consider vertices $v_0, v_1, \dots, v_n$ s.t

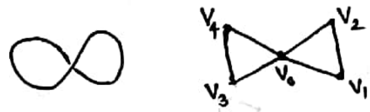
$v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly independent

then $v_0, v_1, v_2, \dots, v_n$

- Given a simplex $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_n)$,

  a face of σ is any proper subset of σ_i

- A simplicial complex K^n is a collection of simplexes which intersect in a common face.

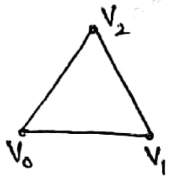


- A triangulation of a topological space X is a simplicial complex K st $K \underset{\text{homeo}}{\simeq} X$

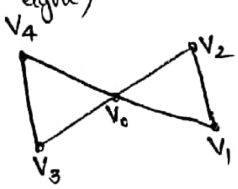
Eg:

1. S^1 Triangulation for S^1?

V_0V_1
V_1V_2
V_2V_0



2. $S^1 \vee S^1$ (Figure eight)



- K is a simplicial complex. Let α_0 = # of 0- simplexes (vertices)

  α_1 = # of 1- simplexes (edges

  α_2 = # of 2- simplexes (faces)

  $\vdots$

  α_n = # of n- simplexes.

Then, Euler Number $\chi(K) = \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 \dots$

$$= \sum_{i=0}^{n} (-1)^i \alpha_i$$

$\chi(K)$ is Topological invariant.

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Eg 1: ① $\chi(S^1) = 3-3$
$= 0$

$\chi(S^1 \vee S^1) = 5-6$
$= -1$

$\Rightarrow$ Figure eight knot is not homeomorphic to circle.

②

$\alpha_0 = 7$
$\alpha_1 = 9$
$\chi(K) = 7-9 = -2$

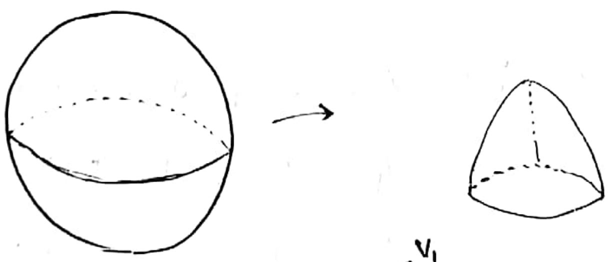
③ Vedge of n-circles $\underbrace{S^1 \vee S^1 \vee \dots \vee S^1}_{n \text{ times}} = G_n$

$\alpha_0 = 2n+1$
$\alpha_1 = 3n$

$\chi(G_n) = 2n+1-3n$
$= 1-n$

④

Ex: Triangulation of S^2

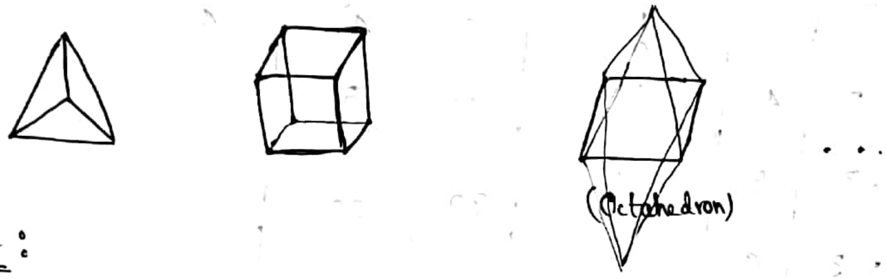


$\alpha_0 = 4$
$\alpha_1 = 6$
$\alpha_2 = 4$

$\chi(S^2) = 4-6+4$
$= 2$

* **Theorem:**

There exists only 5 regular polyhedrons homeomorphic to sphere.



(Octahedron)

Proof:

Let P be a polyhedron

\# of vertices $= V$

edges $= E$

faces $= F$

$$V - E + F = 2 \quad (\because \chi(S^2) = 2)$$

Suppose at every vertex there are n edges coming out

$$nV = 2E \quad (\text{Repetitions are not allowed})$$

~~$3 \times 8 = 24 / 2 = 12$~~

Suppose every faces has r edges

$$rF = 2E$$

$$V = \frac{2E}{n}$$

$$F = \frac{2E}{r}$$

$$V - E + F = 2$$

$$\frac{2E}{n} - E + \frac{2E}{r} = 2$$

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{r} = \frac{1}{E}$$

$$\frac{1}{n} + \frac{1}{r} - \frac{1}{2} = \frac{1}{E} \qquad n, r \geq 3.$$

Suppose $n, r \geq 4$

$$\frac{1}{E} = \frac{1}{n} + \frac{1}{r} - \frac{1}{2} \leq \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = 0$$

$\Rightarrow E \leq 0$

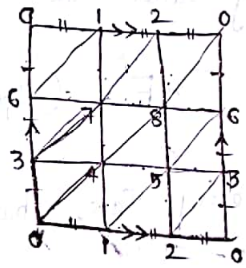
$\Rightarrow \Leftarrow$ Contradiction.

$\Rightarrow n, r$ both cannot be ≥ 4

$\Rightarrow$ Either $n = 3$ (or) $r = 3$

	n	r	$\frac{1}{n}+\frac{1}{r}-\frac{1}{2}$	E	$V = \frac{2E}{n}$	$F = \frac{2E}{r}$
Corresponds to Tetrahedron $\longrightarrow$	3	3	$\frac{1}{6}$	6	4	4
Cube $\longrightarrow$	3	4	$\frac{1}{12}$	12	8	
Icosahedron $\longrightarrow$	3	5	$\frac{1}{3}+\frac{1}{5}-\frac{1}{2} = \frac{10+6-15}{30}$	30	20	
No sol^n $\longrightarrow$	3	6	$\frac{1}{3}+\frac{1}{6}-\frac{1}{2} = \frac{2+1-3}{6}$			
$r \geq 6$ — No sol^n						
Octahedron $\longrightarrow$	4	3		12		
$\longrightarrow$	5	3		30		

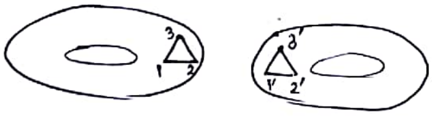
Ex: Triangulation of Torus



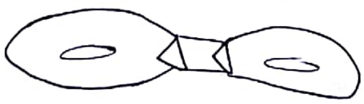
$$\alpha_0 = \cancel{8}\,9$$
$$\alpha_1 = 27$$
$$\alpha_2 = 18$$

$$\chi(\text{Torus}) = 9 - 27 + 18 = 0$$

*



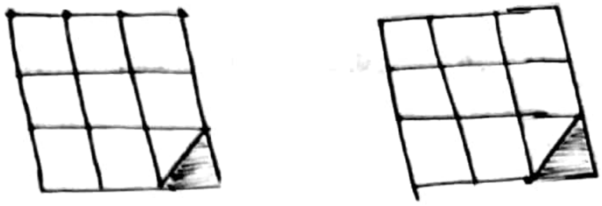
Cut two triangles and glue both torous.



$T \# T = T_2$ — connected sum.

T_2 ~ Double Torus.

Calculate $\chi(T_2)$



$\alpha_0' = 2\alpha_0 - 3$ $(\cancel{-3+}\ -3-3+3)$ vertices

$\alpha_1' = 2\alpha_1 - 3$ $(-3-3+3)$ edges

$\alpha_1' = 2\alpha_2 - 2$ (-2) faces

$$\chi(T_2) = \alpha_0' - \alpha_1' + \alpha_1'$$
$$= \underline{\underline{-2}}$$

<u>Exercise</u>:

T_2 T

$T_2 \# T$

T_3

$\chi(T_3) = ?$

<u>Ans</u>: -4

<u>Exercise</u>:

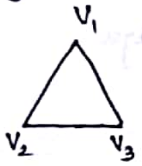
$\chi(T_3)$, $\chi(T_4)$, $\chi(T_g)$

* Simplicial complexes :

K^n a simplicial complex, consists of simplexes st any two simplexes intersect in a common face

Eg:

S^1



$\chi = 0$

∞

$\chi = -1$

S^2

$\chi(S^2) = 2$

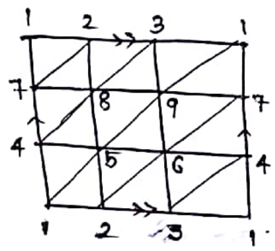
- Euler Number, $\chi(K^n) = \alpha_0 - \alpha_1 + \alpha_2 \ldots\ldots \alpha_n$

$$= \sum_{i=0}^{n} (-1)^n \alpha_i$$

- $\chi(K)$ is a topological invariant

Eg:

Torus



$\alpha_0 = 9$

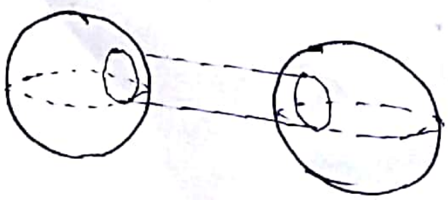
$\alpha_1 = 27$

$\alpha_2 = 18$

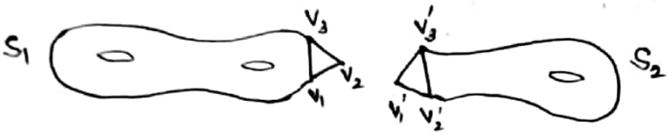
$\chi(T) = 0$

* Defn :

Let S_1, S_2 be any 2-dimensional surfaces (or) simplicial complexes The connected sum, $S_1 \# S_2$ is formed by removing a disc of radius r from S_1 and S_2 and gluing along common boundary



- Since S_1 and S_2 are simplicial complexes, we can remove 2-simplexes from S_1 and S_2, v_1, v_2, v_3 and v_1', v_2', v_3' and glue it along boundary. v_1v_2 to $v_1'v_2'$, v_2v_3 to $v_2'v_3'$ and v_3v_1 to $v_3'v_1'$



Let S_1 has u_1 vertices, e_1 edges, f_1 faces

" S_2 " u_1' " , e_1' " , f_1' faces

$S_1 \# S_2$ has $u_1 + u_1' - 3$ vertices

$e_1 + e_1' - 3$ edges

$f_1 + f_1' - 2$ faces

$$\chi(S_1 \# S_2) = (u_1 + u_1' - 3) - (e_1 + e_1' - 3) + (f_1 + f_1' - 2)$$

$$= (u_1 - e_1 + f_1) + (u_1' - e_1' + f_1') - 2$$

$$\boxed{\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2}$$ — Generalized formula.

Eg:

$$\underset{T_2}{\chi(S_1 \# S_2)} = \chi(S_1) + \chi(S_2) - 2$$

$$= 0 + 0 - 2$$

$$= -2$$

$$\chi(T_3) = \chi(S_1) + \chi(S_2) - 2$$

$$= 0 - 2 - 2$$

$$= -4$$

$$\therefore \chi(T_n) = \chi(T_{n-1}) + \chi(T) - 2$$

$$= \chi(T_{n-1}) - 2$$

$$= \chi(T_{n-2}) - 2 - 2$$

$$\vdots$$

$$= \chi(T) - (n-1)2$$

$$\boxed{\chi(T_n) = 2 - 2n}$$

(Maasey – Algebraic Topology book)

* **Classification Theorem : (Mobeius)**

Any compact orientable 2-dimensional surface which has no boundary is either the sphere S^2 or torous T or connected sum of n torus $T_n = \underbrace{T \# T \# \dots \# T}_{n}$.

* **Simplicial Homology Groups :**

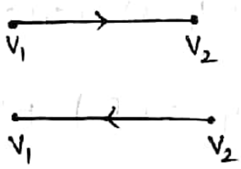
Motivation :

- Is $\mathbb{R}^n \underset{\text{homeo}}{\simeq} \mathbb{R}^m$? $m \neq n$
- Is $S^n \underset{\text{homeo}}{\simeq} S^m$? $m \neq n$

- Orientation of simplexes

0 – simplex •

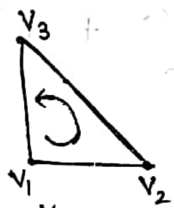
1 – simplex



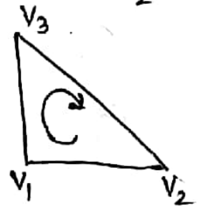
$V_1V_2 = \sigma_1$

$V_2V_1 = -\sigma_1$

2 – simplex



$V_1V_2V_3 = \sigma_2$



$V_1V_3V_2 = -\sigma_2$

$\text{sgn}\,\pi = \pm 1$ (Sign of Permutation)

$V_{\pi(1)}V_{\pi(2)}V_{\pi(3)} = (\text{sign}\,\pi)(V_1V_2V_3)$

even no. of permutations/transpositions $\to +1$

odd no of permutation/transposition $\to -1$

$\sigma_2 = V_1V_2V_3$

$V_1V_3V_2 = -\sigma_2$ ($\because$ odd no. of transpositions)

- $\sigma_i = v_0 v_1 \dots v_i$ oriented i-simplex

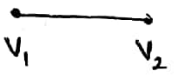
Let π be a permutation of $0, 1, 2, \dots, i$

σ_i has two orientations $\langle\, +\sigma_i,\ -\sigma_i$

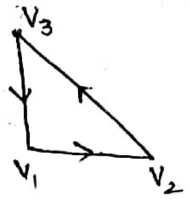
$$v_{\pi(0)} v_{\pi(1)} \dots v_{\pi(i)} = \operatorname{sgn}(v_0 v_1 v_2 \dots v_i)$$

- Boundary of an i-simplex (∂)

- v $\qquad \partial(v) = 0$

- 

$$\partial(v_1 v_2) = v_2 - v_1$$

- 

$$\partial(v_1 v_2 v_3) = v_1 v_2 + v_2 v_3 + v_3 v_1$$

- $\sigma_i = v_0 v_1 v_2 \dots \dots v_i$

$$\partial(\sigma_i) = \sum_{j=0}^{i} (-1)^j\, v_0 v_1 \dots \hat{v}_j \dots v_i$$

where $\hat{v}_j$ means v_j is deleted

$$\partial(\sigma_2) = \sum_{j=0}^{2} (-1)^j\, v_0 v_1 .. \hat{v}_j .. v_i$$

$$= v_1 v_2 - v_0 v_2 + v_0 v_1$$

$$= v_0 v_1 + v_1 v_2 + v_2 v_0$$

- $C_i(K^n)$ - i^{th} chain group of K^n

Let $\sigma_i^1, \sigma_i^2, \dots \dots, \sigma_i^{\alpha(i)}$ be the i-simplexes of K^n

$$C_i(K^n) = \lambda_1 \sigma_i^1 + \lambda_2 \sigma_i^2 + \dots + \lambda_{\alpha(i)} \sigma_i^{\alpha(i)} \quad \text{where } \lambda_i \in \mathbb{Z}$$

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• **Claim :**

$C_i(K)$ is an Abelian group.

Suppose $c_i = \lambda_1\sigma_i^1 + \lambda_2\sigma_i^2 + \dots\dots + \lambda_{\alpha(i)}\sigma_i^{\alpha(i)}$

$c_i' = \lambda_1'\sigma_i^1 + \lambda_2'\sigma_i^2 + \dots\dots + \lambda_{\alpha(i)}'\sigma_i^{\alpha(i)}$

$c_i + c_i' = (\lambda_1+\lambda_1')\sigma_i^1 + (\lambda_2+\lambda_2')\sigma_i^2 + \dots\dots + (\lambda_{\alpha(i)}+\lambda_{\alpha(i)}')\sigma_i^{\alpha(i)}$

. Identity $\to$ $\lambda_i = 0 \ \forall i$ (Integer addition)

. Inverse $\to$ $-c_i = (-\lambda_1, -\lambda_2, \dots, -\lambda_{\alpha(i)})$

. Integer adition is associative

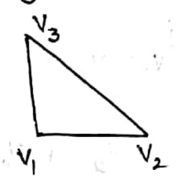
. Abelian group.

• **Claim :**

$\partial : C_{i+1}(K) \longrightarrow C_i(K)$

is a Group homomorphism.

Suppose $K = S^1$



$\partial(vw) = w - v$

$c_1 = v_1v_2 + v_2v_3 + v_3v_1$

$\partial(c_1) = \partial(v_1v_2) + \partial(v_2v_3) + \partial(v_3v_1)$

$= v_2 - v_1 + v_3 - v_2 + v_3 - v_1$

$= 0$

$\Rightarrow$ Circle has no boundary.

• **Claim :** $\partial(-\sigma_i) = -\partial(\sigma_i)$

$\sigma_i = v_0v_1v_2\dots\dots v_i$

$\partial(\sigma_i) = \sum_{j=0}^{i} (-1)^j v_0v_1\dots\hat{v}_j\dots v_i$

$-\sigma_i = v_1v_0v_2\dots\dots v_i$

$$\partial(-\sigma_i) = \sum_{j=0}^{i} (-1)^j V_1 V_0 V_2 \dots \hat{V}_j \dots V_i$$

$$= V_0V_2 \dots V_i - V_1V_2 \dots V_i + V_1V_0V_3 \dots V_i \dots\dots$$

$$= -V_1V_2 \dots V_i + V_0V_2 \dots V_i - V_0V_1V_3 \dots V_i$$

$$= -(V_1V_2 \dots V_i - V_0V_2 \dots V_i + V_0V_1V_3 \dots V_i \dots)$$

$$= -\partial(\sigma_i)$$

$$\partial(\sigma_i + (-\sigma_i)) = \partial(\sigma_i) - \partial(\sigma_i)$$

$$= 0$$

$\partial : C_i \longrightarrow C_{i-1}$ is defined linearly over $\mathbb{Z}$

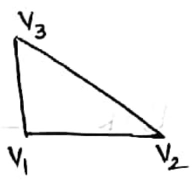
$$\partial[\lambda_1\sigma_i^1 + \lambda_2\sigma_i^2 + \dots + \dots] = \lambda_1 \partial\sigma_i^1 + \lambda_2 \partial\sigma_i^2 + \dots\dots$$

Group homomorphism.

- K^n

$$C^n(K) \xrightarrow{\partial_n} C^{n-1}(K) \xrightarrow{\partial_{n-1}} C^{n-2}(K) \longrightarrow \dots \longrightarrow C^1(K) \xrightarrow{\partial_0} C^0(K) \longrightarrow 0$$

$$\partial_{i+1} \circ \partial_i = 0$$

$\sigma_2 =$ 

$$\partial\sigma_2 = V_1V_2 + V_2V_3 + V_3V_1$$

$$\partial \circ \partial\sigma_2 = \partial(V_1V_2 + V_2V_3 + V_3V_1)$$

$$= \partial(V_1V_2) + \partial(V_2V_3) + \partial(V_3V_1)$$

$$= V_2 - V_1 + V_3 - V_2 + V_1 - V_3$$

$$\boxed{\partial(\partial\sigma_2) = 0}$$

$\sigma_3 =$ 

$$\sigma_3 \simeq D(3) \rightarrow x_1^2 + x_2^2 + x_3^2 \leq 1$$

$$\partial\{\partial(\sigma_3)\} = \partial(S^2)$$

$$= 0$$

- $\sigma_i = v_0 v_1 v_2 \ldots\ldots v_i$

$$\partial \sigma_i = \sum_{j=0}^{i} (-1)^j v_0 v_1 \ldots \hat{v}_j \ldots v_i$$

$$\partial \circ \partial \sigma_i = \sum_{j=0}^{i} (-1)^j (-1)^k v_0 v_1 \ldots \hat{v}_k \ldots \hat{v}_j \ldots v_i$$
$$+ \sum_{j=0}^{i} (-1)^j (-1)^{k-1} v_0 \ldots \hat{v}_j \ldots \hat{v}_k \ldots v_i$$

$$= 0$$

- $$\cdots \longrightarrow C_{i+1}(K) \xrightarrow{\partial_{i+1}} C_i(K) \xrightarrow{\partial_i} C_{i-1}(K) \rightarrow \cdots$$

$\partial_{i+1} \circ \partial_i = 0$

$Z_i = \text{Ker}\, \partial_i$ = subgroup of i-cycles

$B_i = \text{Image}\, \partial_{i+1}$ = subgroup of i-boundaries.

$B_i \subset Z_i$

$$\boxed{i^{th} \text{ Homology group } H_i = \frac{Z_i}{B_i}}$$

- Let K^n be path-connected, $H_0(K) = \mathbb{Z}$

$$C_0 \xrightarrow{\partial_0} 0$$

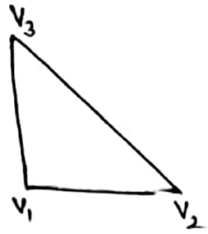
$$Z_0 = \{ \sigma_0^1, \sigma_0^2, \ldots, \sigma_0^{\alpha(0)} \}$$

$$\sigma_0^i - \sigma_0^j = \partial(\text{path})$$

$$[\sigma_0^i] = [\sigma_0^j]$$

Eg:

(1.) S^1 (Circle)



$C_1 = \{V_1V_2, V_2V_3, V_3V_1\}$

$C_0 = \{V_1, V_2, V_3\}$

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Two homology groups H_0, H_1

$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}$

$= V_1V_2 + V_2V_3 + V_3V_1$

Homework:

Compute homology groups for

(i.) Figure 8

(ii.)

G_n n Circles touching at a point

$H_0(G_n) = \mathbb{Z}$

$H_1(G_n) = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}}_{n \text{ times}}$

(iii.) S^2

(iv.) Torus, T

(v.) T_n

03/04/2018

Tuesday

* **Recall**

* **Simplicial Homology Group**

Let K^n be a finite dimensional simplicial complex of dim n

Let $C(K)$ denote the chain complex of K^n as follows

$$C_n(K^n) \xrightarrow{\partial_n} C_{n-1}(K^n) \xrightarrow{\partial_{n-1}} C_{n-2}(K^n) \longrightarrow \dots\dots \longrightarrow C_1(K^n) \xrightarrow{\partial_1} C_0(K^n) \xrightarrow{\partial_0} 0$$

$$\dots \longrightarrow C_{i+1}(K^n) \xrightarrow{\partial_{i+1}} C_i(K^n) \xrightarrow{\partial_i} C_{i-1}(K^n) \longrightarrow \dots$$

If K^n has i-simplexes

$$\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^{\alpha(i)}$$

$$C_i = \left\{ \lambda_1\sigma_i^1 + \lambda_2\sigma_i^2 + \ldots + \lambda_{\alpha(i)}\sigma_i^{\alpha(i)} \;\middle|\; \lambda_j \in \mathbb{Z} \right\}$$

- $\sigma_i = V_0V_1\ldots\ldots V_i$
- $\partial\sigma_i = \sum_{j=0}^{i} (-1)^j V_0\ldots\hat{V_j}\ldots V_i$
- $\partial(-\sigma) = -\partial\sigma$
- $\partial(\sigma + (-\sigma)) = 0$
- C_i is an Abelian group
- ∂_i is a Group homomorphism.
- $\partial_i \circ \partial_{i+1} = 0$

Let $Z_i = \text{Ker}\,\partial_i$

= Group of i-cycles of K^n

B_i = Image of ∂_{i+1}

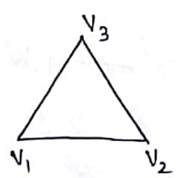
= Group of i-boundaries of K^n

$B_i \subset Z_i$ (subgroup)

$$H_i = \frac{Z_i}{B_i}$$

= i^{th} homology group of K^n (those i-cycles which are not i-boundaries)

Eg: S^1



$C_1 = \left\{ \lambda_1(V_1V_2) + \lambda_2(V_2V_3) + \lambda_3(V_3V_1) \;\middle|\; \lambda_i \in \mathbb{Z} \right\}$

$C_0 = \{ \lambda_1V_1 + \lambda_2V_2 + \lambda_3V_3 \mid \lambda_i \in \mathbb{Z}$

Calculate homology groups of simplicial complex associated with circle?

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

If K^n is connected, path connected, then $H_0(K^n) \simeq \mathbb{Z}$

$H_0(S^1) = \mathbb{Z}$

$H_i(S^1) = 0 \quad \forall \ i \geq 2$

Group of one-cycles

$$Z_1 = \operatorname{Ker} \partial_1$$
$$= \{ x \in Z \mid \partial(x) = 0 \}$$
$$= \{ \lambda(v_1v_2 + v_2v_3 + v_3v_1) \mid \lambda \in \mathbb{Z} \}$$

$$Z_1(S^1) \simeq \mathbb{Z}$$

$$B_1 = \text{Image } \partial_{i+1}$$
$$= \{0\}.$$

$$H_1 = \frac{Z_1}{B_1} = Z_1 \simeq \mathbb{Z}$$

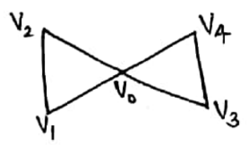
$\therefore$

$H_0(S^1) = \mathbb{Z}$

$H_1(S^1) = \mathbb{Z}$

$H_i(S^1) = 0 \quad \forall \ i \geq 2.$

Eg: Figure 8



$H_0(8) = ?$

$H_1(8) = ?$

$H_0(8) = \mathbb{Z}$ ($\because$ path connected)

$H_1(8) = \mathbb{Z} \times \mathbb{Z}$ ($\because$ Two cycles)

$$Z_1 = \operatorname{Ker} \partial_1$$

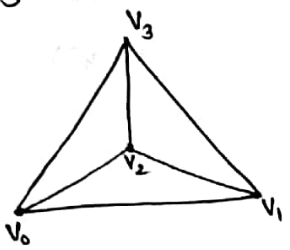
$$\lambda_1(v_0v_1) + \lambda_2 v_1v_2 + \lambda_3 v_2v_0 = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3.$$

$$\mu_1 v_0v_3 + \mu_2 v_3v_4 + \mu_3 v_4v_0 = 0$$

$$\mu_1 = \mu_2 = \mu_3$$

Eg: Sphere S^2



$H_0(S^2) = \mathbb{Z}$ ($\because$ S^2 is path-connected)

$H_1(S^2) = 0$

$H_2(S^2) = \mathbb{Z}$

$$\partial(\lambda(V_0V_1 + V_1V_2 + V_2V_0)) = 0$$

$$\partial(V_0V_1V_2) = V_0V_1 + V_1V_2 + V_2V_0$$

For two cycles

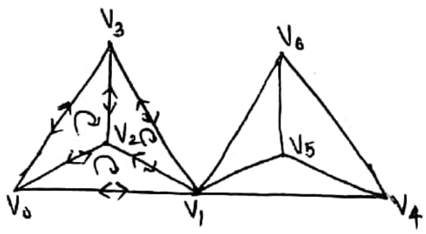
$$Z_2 = \lambda\{V_0V_3V_2 + V_2V_3V_1 + V_0V_2V_1 + V_3V_0V_1\}$$

$$= \mathbb{Z}$$

$$B_2 = 0$$

$$\therefore H_2 = \mathbb{Z}$$

Eg: $S^2 \vee S^2$



$$H_2(S^2 \vee S^2) = \mathbb{Z} \times \mathbb{Z}$$

$$H_1(S^2 \vee S^2) = 0$$

$$H_0(S^2 \vee S^2) = \mathbb{Z}$$

* Given any simplicial homology group K^n

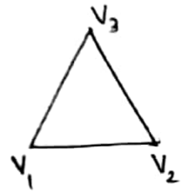
$$H_i(K^n) = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}}_{\beta_i} \oplus \underbrace{T}_{\text{Torsion sub-group}}$$

$\beta_i \downarrow$ i^{th} Betti-number

* Euler - Poincare Theorem :

$$\chi = \sum_{i=0}^{n} (-1)^i \alpha_i = \sum_{j=0}^{n} (-1)^j \beta_j$$

Eg: 1. Verify above formula for circle S^1



$\alpha_0 = 3$

$\alpha_1 = 3$

$\alpha_2 = 0$

$\chi(S^1) = 3 - 3 + 0$
$= 0$

$H_0(S^1) = \mathbb{Z}$

$H_1(S^1) = \mathbb{Z}$

$H_i(S^1) = \cancel{\mathbb{Z}}\ 0 \quad \forall\ i \geq 2$

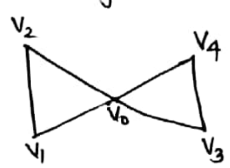
$\beta_0 = 1$

$\beta_1 = 1$

$\beta_i = 0 \quad \forall \quad i \geq 2$

$\sum_{j=0}^{n} (-1)^j \beta_j = 1 - 1 + 0$
$= 0$

2. Verify for figure 8



$\alpha_0 = 5$

$\alpha_1 = 6$

$\alpha_2 = 0$

$\chi(S^1 \vee S^1) = 5 - 6 + 0$
$= -1$

$H_0(S^1 \vee S^1) = \mathbb{Z}$

$H_1(S^1 \vee S^1) = \mathbb{Z} \times \mathbb{Z}$

$H_i(S^1 \vee S^1) = 0 \quad \forall\ i \geq 2$

$\beta_0 = 1$

$\beta_1 = 2$

$\beta_i = 0 \quad \forall\ i \geq 2$

$\sum_{j=0}^{n} (-1)^j \beta_j = 1 - 2 + 0$
$= \underline{\underline{-1.}}$

(3) For $\underbrace{S^1 \vee S^1 \ldots \vee S^1}_{n}$

$\alpha_0 = \cancel{3n+1} \; 2n+1$

$\alpha_1 = 3n$

$\alpha_2 = 0$

$\chi = 2n+1 - 3n$

$= 1-n$

$H_0(S^1 \vee \ldots \vee S^1) = \mathbb{Z}$

$H_1(S^1 \vee \ldots \vee S^1) = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n}$

$H_i(\;) = 0 \quad \forall\, i \geq 2$

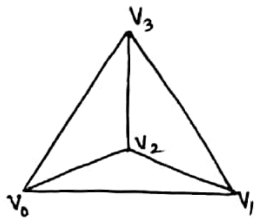
$\beta_0 = 1$

$\beta_1 = n$

$\beta_i = 0 \quad \forall\, i \geq 2$

$\sum_{j=0}^{n} (-1)^j \beta_j = 1-n$

(4) For S^2



$\alpha_0 = 4$

$\alpha_1 = 6$

$\alpha_2 = 4$

$\chi(S^2) = 4-6+4$

$= 2$

$H_0(S^2) = \mathbb{Z}$

$H_1(S^2) = 0$

$H_2(S^2) = \mathbb{Z}$

$\beta_0 = 1$

$\beta_1 = 0$

$\beta_2 = 1$

$\sum_{j=0}^{n} (-1)^j \beta_j = 1-0+1$

$= 2$

(5) For $S^2 \vee S^2$

$H_0(S^2 \vee S^2) = \mathbb{Z}$ $\quad \beta_0 = 1$

$H_1(S^2 \vee S^2) = 0$ $\quad \beta_1 = 0$

$H_2(S^2 \vee S^2) = \mathbb{Z} \times \mathbb{Z}$ $\quad \beta_2 = 2$

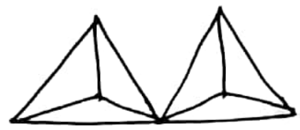
$\sum_{j=0}^{m} (-1)^j \beta_j = 1-0+2$

$= 3$

$\alpha_0 = 7$

$\alpha_1 = 12$

$\alpha_2 = 8$



$$\chi(S^2 \vee S^2) = 7 - 12 + 8$$
$$= 3$$

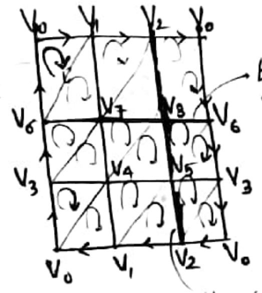
Eg: Torus.

$H_0(T) = \mathbb{Z}$

$H_1(T) =$

$H_2(T) = \mathbb{Z}$

For $\mathbb{Z}_2$



$\alpha_0 = 9$

$\alpha_1 = 27$

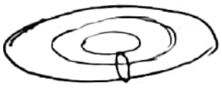
$\alpha_2 = 18$

$\chi(T) = 9 - 27 + 18$
$= 0$

∂(18 2-simplexes oriented ~~anti~~ clockwise) = 0

$Z_2 = \mathbb{Z}$

$\therefore H_2 = \mathbb{Z}$



Two cycles

In fact $H_1(T) = \mathbb{Z} \times \mathbb{Z}$ $\quad$ ($\because \chi(T) = 0$
$1 - x + 1 = 0$
$x = 2$
$\therefore \beta_1 = 2$)

Eg:

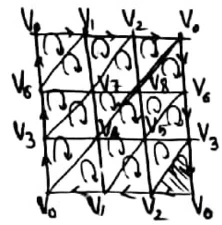
$T \# T = T_2$

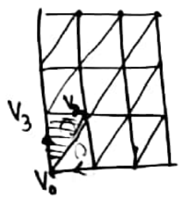
$\chi(T_2) = -2$

$H_0(T_2) = \mathbb{Z}$

$H_1(T_2) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

$H_2(T_2) = \mathbb{Z}$





∂(17 2-complexes oriented $\curvearrowright$) $= -(V_2V_3 + V_3V_0 + V_0V_2)$
$= V_2V_0 + V_0V_3 + V_3V_2$

∂(17 2-complexes oriented $\curvearrowleft$) $= V_3V_0 + V_0V_2 + V_2V_3$

4 cycles

If $z_2 = z_1 + \partial(\ )$

then $[z_2] = [z_1]$

homology group of z_2

05/04/2018
Thursday

* Euler – Poincare Theorem :

K^n, simplicial complex of dimension n,

$\alpha_i :=$ # of i-simplexes of K^n

$\beta_j :=$ Rank free subgroup of $H_j(K^n)$

$$\chi(K^n) = \sum_{i=0}^{n} (-1)^{n} \alpha_i = \sum_{j=0}^{n} (-1)^{j} \beta_j$$

Verified for

S^1, Fig 8, $G_n = \underbrace{S^1 \vee S^1 \vee \ldots \vee S^1}_{n}$

S^2, T, T_2

Eg: Verify for $T_g = \underbrace{T \# T \ldots \# T}_{g}$

$\chi(T_g) = 2 - 2g$

$H_0 = \mathbb{Z}$

$H_1 = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}}_{2g}$

$H_2 = \mathbb{Z}$

$\beta_0 = 1$

$\beta_1 = 2g$

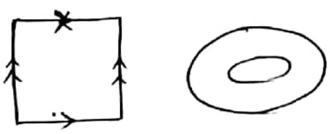
$\beta_2 = 1$

$$\sum_{j=0}^{n} (-1)^{j} \beta_j = 1 - 2g + 1 = 2 - 2g.$$

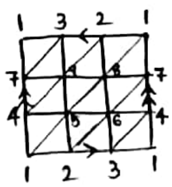
* Non-orientable surface : ↳ Eg: Mobius Strip

Klein Bottle K

All surfaces in $\mathbb{R}^3$ are orientable



} Torus.



↳ Klein bottle

It actually lies in $\mathbb{R}^4$

$\chi(K) = 0$ $\quad \alpha_0 = 9$, $\alpha_1 = 27$, $\alpha_2 = 18$

(Exactly same as torus but the directions are different)

$H_0(K) = \mathbb{Z}$

$H_1(K) = \mathbb{Z} \times \mathbb{Z}_2$

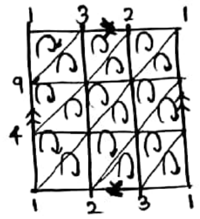
$H_2(K) = 0$

$\beta_0 = 1$

$\beta_1 = 1$

$\beta_2 = 0$

$$\sum_{j=0}^{n} (-1)^j \beta_j = 1 - 1 + 0 = 0$$



∂(18 2-simplexes oriented ↻) $= 13 + 32 + 21 + 13 + 32 + 21$

$= 2\{13 + 32 + 21\}$

$z_1 = 13 + 32 + 21$

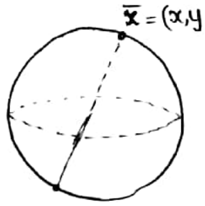
$2z_1 = \partial(\ ) = 0$

$Z_2 \simeq \mathbb{Z}_2$

$z_2 = 14 + 49 + 91$ remains unchanged.

P^2 – Projective Plane

↳ another non-orientable surface

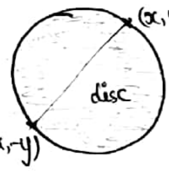


$\bar{x} = (x, y, z)$

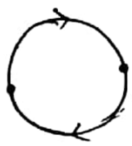
$-\bar{x} = (-x, -y, -z)$

$(x, y, z) \sim (-x, -y, -z)$

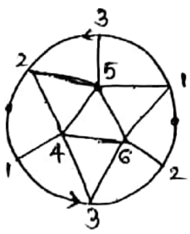
P^2 is



(x, y), disc, $(-x, -y)$



Triangulation for P^2



$\alpha_0 = 6$

$\alpha_1 = 15$

$\alpha_2 = 10$

$\chi(P^2) = 6 - 15 + 10 = 1$

$H_0(P^2) = \mathbb{Z}$

$H_1(P^2) = \mathbb{Z}_2$

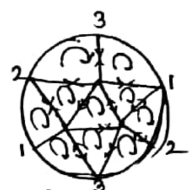
$H_2(P^2) = 0$

$\beta_0 = 1$

$\beta_1 = 0$

$\beta_2 = 0$

$$\sum_{j=0}^{n} (-1)^j \beta_j = 1 - 0 + 0 = 1$$



$\partial(\ ) = 12 + 23 + 31 + 12 + 23 + 31$

$= 2(12 + 23 + 31)$

$\simeq \mathbb{Z}_2$

$\chi(P^2 \# P^2) = \chi(P^2) + \chi(P^2) - 2 = 0$

$P^2 \# P^2 = K$ (Klein bottle)

* **Classification Theorem:**

(For 2-dimensional surfaces without boundary);

Any 2-dimensional compact surface without boundary is

S, T (or) T_g (OR) P^2 (or) $\underbrace{P^2 \# P^2 \# \cdots \# P^2}_{g\ n}$

* **Proof:** (of Euler-Poincare Theorem)

We will use simplicial homology with rational coefficients $H_i(K, \mathbb{Q})$

$$C_i(K^n, \mathbb{Q}) = r_1\sigma_i^1 + r_2\sigma_i^2 + \cdots + r_{\alpha(i)}\sigma_i^{\alpha(i)} \quad \text{where } r_i \in \mathbb{Q}$$

Addition as before; forms a finite dimensional vector space over $\mathbb{Q}$ with dim $\alpha(i)$

$$\partial(r_1\sigma_i^1 + r_2\sigma_i^2 + \cdots + r_{\alpha(i)}\sigma_i^{\alpha(i)}) = r_1\partial(\sigma_i^1) + r_2\partial(\sigma_i^2) + \cdots + r_{\alpha(i)}\partial(\sigma_i^{\alpha(i)})$$

$$C_n(K,\mathbb{Q}) \longrightarrow C_{n-1}(K,\mathbb{Q}) \longrightarrow \cdots \longrightarrow C_{i+1}(K,\mathbb{Q}) \xrightarrow{\partial} C_i(K,\mathbb{Q}) \xrightarrow{\partial} C_{i-1}(K,\mathbb{Q}) \to \cdots \to C_0(K,\mathbb{Q}) \to 0$$

$$\partial_i \circ \partial_{i+1} = 0 \qquad \cancel{\partial\circ\partial = 0}$$

$\operatorname{Ker} \partial_i = Z_i$ = Vector subspace of rational i-cycles

$\operatorname{Im} \partial_{i+1} = B_i$ = " " " " i-boundaries

$$B_i \subset Z_i$$

$$H_i(K, \mathbb{Q}) = \frac{Z_i(K,\mathbb{Q})}{B_i(K,\mathbb{Q})}$$

There is no torsion element in $\mathbb{Q}$ (vector spaces)

Lemma:

β_q is the dimension of $H_q(K, \mathbb{Q})$

Proof:

Choose a minimum sets of generators

$[z_1], [z_2], \ldots, [z_{\beta_q}]$ $[w_1], \ldots, [w_{\gamma_q}]$ w — represent torsion.

for ~~$H_q(K,\mathbb{Q})$~~ $H_q(K, \mathbb{Z})$

A q-cycle with integer coefficients can be thought of as a q-cycle with rational coefficients

Thus, determines an element of $H_q(K,\mathbb{Q})$

$\{z\}$

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Suppose

$\frac{a_1}{b_1}\sigma_1 + \cdots + \frac{a_s}{b_s}\sigma_s$ is a rational q-cycle $a_i, b_i \in \mathbb{Z}$

$= \frac{1}{b_1 b_2 \cdots b_s} \times$ (A cycle with integer coefficients)

$= \frac{1}{b_1 b_2 \cdots b_s} \times$ (A linear combination of z_i's and w_i's)

$\{z_1\}, \{z_2\}, \ldots \{z_{\beta_q}\}, \{w_1\}, \ldots \{w_{\gamma_q}\}$

span $H_q(K, \mathbb{Q})$

Suppose a torsion element $[w] \in H_q(K)$ of order m

$$mw = \partial(\underbrace{\quad}_{\text{boundary}})$$

In rational coefficients

$$w = \frac{1}{m}\partial(\quad) = \{w\} = 0$$

$\Rightarrow$ All torsion elements are being eliminated in rational coefficients.

Therefore,

$H_q(K, \mathbb{Q})$ is generated by $\{z_1\}, \{z_2\}, \ldots, \{z_{\beta_q}\}$

<u>Want to show</u> $\{z_1\}, \ldots, \{z_{\beta_q}\}$ remain linearly independent over $\mathbb{Q}$

Suppose not

$$\frac{a_1}{b_1}z_1 + \frac{a_2}{b_2}z_2 + \cdots + \frac{a_q}{b_q}z_{\beta_q} = \partial(\quad) = 0$$

(zero means boundary of something)

$a_i, b_i, m_i, n_i \in \mathbb{Z}$

$$= \partial\left(\frac{m_1}{n_1}c_1 + \cdots + \frac{m_p}{n_p}c_p\right) \rightarrow (*)$$

Multiply $(*)$ by the product of all denominators.

$b_1 b_2 \cdots b_q \cdot n_1 n_2 \cdots n_p$

$\Rightarrow$ A linear combination of z_i's with integer coefficients is boundary of $q+1$ chain with integer coefficients

$\Rightarrow\Leftarrow$ Contradiction.

ie. $H_q(K, \mathbb{Q})$ has the basis $\{z_1\}, \ldots, \{z_{\beta_q}\}$

$\dim = \beta_q$.

Proof: (of Theorem (E-P Theorem))

$$\dim C_i(K,\mathbb{Q}) = \alpha_i = \#\text{ of } i\text{-simplexes of } K$$

$$\Sigma (-1)^i \dim C_i(K,\mathbb{Q}) = \Sigma (-1)^i \alpha_i = \chi(K)$$

Instead choose a different basis for each $C_i(K,\mathbb{Q})$ as follows

$C_n(K,\mathbb{Q})$ $\quad \dim Z_n(K,\mathbb{Q}) = \beta_n$

Select a basis $\{z_1^n\}, \{z_2^n\}, \dots, \{z_{\beta_n}^n\}$

Extend the basis to the basis for C_n as

$$C_n(K,\mathbb{Q}) = \left[\{z_1^n\}, \{z_2^n\}, \dots, \{z_{\beta_n}^n\}, \{c_1^n\}, \dots, \{c_{\gamma_n}^n\}\right]$$

$$= \left[\{\partial c_1^n\}, \{\partial c_2^n\}, \dots, \{\partial c_{\gamma_n}^n\} \{z_1^{n-1}\} \dots \{z_{\beta_{n-1}}^{n-1}\}\{c_1^{n-1}\} \dots \{c_{\gamma_{n-1}}^{n-1}\}\right]$$

$$= \{\partial c_1^{n-1}\} \dots \{\partial c_{\gamma_{n-1}}^{n-1}\}\{z_1^{n-2}\} \dots \{z_{\beta_{n-2}}^{n-2}\}\{c_1^{n-2}\} \dots \{c_{\gamma_{n-2}}^{n-2}\}$$

$$\dim C_n(K,\mathbb{Q}) = \beta_n + \gamma_n$$

$$\dim C_{n-1}(K,\mathbb{Q}) = \gamma_n + \beta_{n-1} + \gamma_{n-1}$$

$$\dim C_{n-2}(K,\mathbb{Q}) = \gamma_{n-1} + \beta_{n-2} + \gamma_{n-2}$$

$$\chi(K^n) = \sum_{i=0}^{n} (-1)^i \dim C_i(K,\mathbb{Q})$$

$$= (-1)^n(\beta_n + \cancel{\gamma_n}) + (-1)^{n-1}(\cancel{\gamma_n} + \beta_{n-1} + \gamma_{n-1}) + (-1)^{n-2}(\cancel{\gamma_{n-1}} + \beta_{n-2} + \gamma_{n-2})$$

$$+ \dots + \cancel{\gamma_1} + \beta_0$$

$$= \sum_{i=0}^{n} (-1)^i \beta_i \qquad (\text{By pairwise } \text{~~canc~~ cancellation of } \gamma_i \text{ terms})$$

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Homework

Verify formula for $T_g = \underbrace{T \# T \# \dots \# T}_{g \text{ times}}$

05/04/20.. Thursday

10/04/20.. Tuesday

* Simplicial Homology Groups

Eg: (1) $S^1 \amalg S^1$

$H_0 = \mathbb{Z} \times \mathbb{Z}$

$H_1 = \mathbb{Z} \times \mathbb{Z}$

(2) $X = K_1 \cup K_2 \cup \dots \cup K_n$

Each K_i path connected

$K_i \cap K_j = \phi$

$H_0(X) = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}}_{n \text{ times}}$

- We will prove that for the n-sphere S^n

$$S^n = \left\{ (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

$H_0(S^n) = \mathbb{Z}$

$H_i(S^n) = 0$ if $i \neq 0, n$

$H_n(S^n) = \mathbb{Z}$

* **Theorem:**

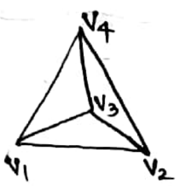
$\{H_i(X)\}$ are Topological invariant.

* **Cone complex**

$z^2 = x^2 + y^2$, $0 \le z \le 1$

Cone on a circle

$C(S^1)$



$\alpha_0 = 4$

$\alpha_1 = 6$

$\alpha_2 = 3$

For

$H_0(S^1) = \mathbb{Z}$

$H_1(S^1) = \mathbb{Z}$

$\chi(C) = 4 - 6 + 3$
$= 1$

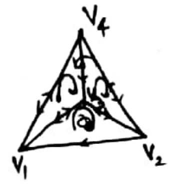
$H_0(C) = \mathbb{Z}$ ($\because$ path connected)

$H_1(C) = 0$

$H_i(C) = 0 \quad \forall\ i \geq 2$

$\beta_0 = 1$

$\beta_i = 0 \ , \ i \geq 1$



Boundary of
3 2-simplexes $= V_1V_2 + V_2V_3 + V_3V_1$

$Z_1(C) = B_1(C)$

$\Rightarrow H_1(C) = \dfrac{Z_1(C)}{B_1(C)} = \{0\}$.

* Def^n:

Given a simplicial complex $K^n \subset \mathbb{R}^n$,

Let $v = (0,0,\ldots,0,1) \in \mathbb{R}^{n+1}$.

We construct the cone simplex cK on K as follows

If A is a k-simplex σ_k in $\mathbb{R}^n$; $V_0V_1V_2\ldots V_k$

then $V_0V_1V_2\ldots V_kV$ forms a $k+1$ simplex in $\mathbb{R}^{n+1}$

called the join of σ_k with v

Def^n:

Given a simplicial complex $K^n \subset \mathbb{R}^n$, the cone simplex cK consists of

(i) All the simplexes of K

(ii) The extra vertex V

(iii) The join of each k simplex of K with v.

Eg: σ_0 $\quad\quad$ $V\sigma_0 = 1$-simplex

V_0V_1 $\quad\quad$ $c(V_0V_1) = \underbrace{V_0V_1V_2}_{2\text{-simplex}}$

Eg:

$K = \sigma_{k-1} = V_0V_1 \ldots V_{k-1}$ ↪ k-1 simplex

$cK = \sigma_k = V_0V_1 \ldots V_{k-1}V$ ↪ k simplex

* **Theorem:**

If cK is any cone simplex

$H_0(cK) = \mathbb{Z}$

$H_i(cK) = \phi \quad i \geq 1$

Proof:

A cone is always connected and path connected

$\therefore H_0(cK) \simeq \mathbb{Z}$

Suppose $q > 0$

Define a homomorphism

$$d : C_q(K) \longrightarrow C_{q+1}(K)$$

If $\sigma = (V_0V_1 \ldots V_q)$ is a simple on K

define $d(V_0V_1 \ldots V_q) = VV_0 \ldots V_q$

If $\sigma = VV_0 \ldots V_{q-1}$, $d(\sigma) = 0$

Claim:

$d\partial + \partial d : C_i \longrightarrow C_i$

$d\partial + \partial d = Id$

Proof:

(1) If $\sigma = V_0V_1 \ldots V_q$ in K

$\partial d\sigma = \partial\{VV_0V_1 \ldots V_q\}$

$= (-1)^0 V_0V_1 \ldots V_q + \sum_{i=0}^{q} (-1)^{i+1} VV_0 \ldots \hat{V}_{i+1} \ldots V_q$

$= \sigma - d\partial\sigma$

$\Rightarrow \partial d\sigma + d\partial\sigma = \sigma$

(2) If $\sigma = VV_0 \ldots V_{q-1}$

$\partial d\sigma = 0$

$\partial\sigma = V_0V_1 \ldots V_{q-1} + \sum_{i=0}^{q-1} (-1)^{i+1} VV_0 \ldots \hat{V}_{i+1} \ldots V_{q-1}$

$d\partial\sigma = VV_0V_1 \ldots V_{q-1} + 0$

$= \sigma$

$\therefore \partial d\sigma + d\partial\sigma = \sigma$

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We have proved that

$$d\partial + \partial d : C_i(cK) \longrightarrow C_i(cK) \text{ is } Id$$

$$d\partial + \partial d(\sigma_i) = \sigma_i$$

Suppose z_i is a cycle in cK

$$\{d\partial + \partial d\}[z_i] = [z_i]$$

$$\underbrace{\{d\partial\}[z_i]}_{0} + \partial\{d[z_i]\} = [z_i]$$

$$\partial\{d[z_i]\} = [z_i]$$

$\Rightarrow z_i$ is a boundary.

(Since, every cycle is becoming a boundary)

IF $i \geq 1$,

$$H_i(cK) \simeq \phi$$

$$\begin{cases} H_0(cK) \simeq \mathbb{Z} \\ H_i(cK) \simeq \phi, \quad i \geq 1 \end{cases}$$

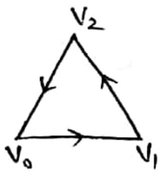
- σ_0 - 0 simplex $\quad c(\sigma_0) = \sigma_1$

$c(\sigma_k) = \sigma_{k+1}$

<u>Eg:</u>

- $\sigma_2 = V_0V_1V_2$

$S^1 = \partial(\sigma_2)$



- $\sigma_3 = V_0V_1V_2V_3$

$S^2 = \partial(\sigma_3)$

- $S^n = \partial(\sigma_{n+1})$

* Any simplex σ_i is a cone

$$H_0(\sigma_i) = \mathbb{Z}$$

$$H_j(\sigma_i) = \phi, \quad j \geq 1$$

* $S^n = \partial(\sigma_{n+1})$

All i-simplex of S^n and σ_{n+1} are identical, except σ_{n+1} has one extra $(n+1)$ simplex

$$C_i(S^n) \cong C_i(\sigma_{n+1}) \text{ if } 0 \le i \le n$$

$$\Rightarrow H_i(S^n) \cong H_i(\sigma_{n+1}) \text{ for } 0 \le i \le n-1$$

$$\Rightarrow H_i(S^n) = 0 \text{ if } 1 \le i \le n-1$$

$$H_n(S^n) = \frac{Z_n(S^n)}{B_n(S^n)}$$

$$B_n(S^n) = 0$$

$$\Rightarrow H_n(S^n) = Z_n(S^n) \quad \text{Just the cycles.}$$

$$Z_n(S^n) = Z_n(\sigma_{n+1})$$

$$H_n(\sigma_{n+1}) = 0$$

$$H_n(\sigma_{n+1}) = \frac{Z_n(\sigma_{n+1})}{B_n(\sigma_{n+1})}$$

$$B_n(\sigma_{n+1}) = \mathbb{Z}$$

$$\Rightarrow Z_n(\sigma_{n+1}) = B_n(\sigma_{n+1}) = \mathbb{Z}$$

$$\Rightarrow Z_n(S^n) = \mathbb{Z}$$

If $S^n = \{(x_0, x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$

$$H_0(S^n) = \mathbb{Z}$$

$$H_i(S^n) = 0, \text{ if } i \ne 0, n$$

$$H_n(S^n) = \mathbb{Z}$$

- If $f : K \longrightarrow L$, then $\exists\, f_* : H_i(K) \longrightarrow H_i(L)$ s.t

  (i) $\underset{(id)}{i} : K \longrightarrow K \Rightarrow i_* : H_i \longrightarrow H_i$ is id and $i_* = id$

- If $f : K \longrightarrow L$, $g : L \longrightarrow P$, then $g \circ f : K \longrightarrow P$

  $f_* : H_i(K) \longrightarrow H_i(L)$, $g_* : H_i(L) \longrightarrow H_i(P)$

$$(g \circ f)_* = g_* \circ f_*$$

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* **Brouwer's Fixed point Theorem for D^n :**

$$D^n = \{(x_1, x_2, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 \leq 1\}$$

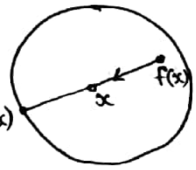
If $f : D^n \to D^n$ any continuous map, then f has atleast one fixed point at x_0, $f(x_0) = x_0$

Proof:

Suppose $f : D^n \to D^n$ has no fixed point

$x \neq f(x)$

Join $f(x)$ to x by a straight line extend till it meets the boundary S^n, call it $g(x)$



If $x \in S^n$, $g(x) = x$

$$\begin{tikzcd} S^n \arrow[r, "i"] \arrow[rr, bend right, "id"'] & D^n \arrow[r, "g"] & S^n \end{tikzcd}$$

i, inclusion map

$$\begin{tikzcd} H_n(S^n) \arrow[r, "i_*"] \arrow[rr, bend right, "id_* = id"'] & H_n(D^n) \arrow[r, "g_*"] & H_n(S^n) \end{tikzcd}$$

$H_n(S^n) = \mathbb{Z}$

$H_n(D^n) = 0$,

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & 0 \arrow[r, "g_*"] & \mathbb{Z} \end{tikzcd}$$

$g_* \circ i_* \simeq 0 \neq id$

$\Rightarrow f(x_0) = x_0$ for some $x_0 \in D^n$

* A topological space K^n has a fixed point property if any continuous map $f : K \to K$ has atleast one fixed point.

Eg:

1. S^1 Does circle has fixed point property ?

$f(x,y) = (-x,-y)$

↳ Doesn't have fixed point property.

2. S^2 , $f(\vec{x}) = -(\vec{x})$ → No fpp.

3 S^n → No fpp.

4. Torus

↳ No fpp

* **Lefschetz Fixed point Theorem :**

If K^n is a simplicial complex s.t $H_i(K, \mathbb{Q})$ is

$H_0(K, \mathbb{Q}) = \mathbb{Q}$

$H_i(K, \mathbb{Q}) = 0$

then K has a fixed point property.

- $H_i(cK) = \mathbb{Z}$ if $i = 0$

$= 0$ if $i \neq 0$

$\Rightarrow$ cK has fpp

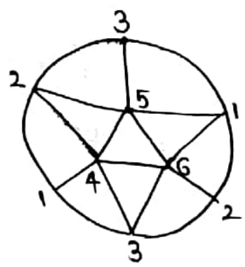
L.FPT includes B.FPT

- P^2

$H_0(P^2) = \mathbb{Z}$

$H_1(P^2) = \mathbb{Z}_2$

$H_2(P^2) = 0$



$H_1(P^2, \mathbb{Q}) = 0$

$H_0(P^2, \mathbb{Q}) = \mathbb{Q}$

$\Rightarrow$ P^2 will have fpp

12/04/2018
Thursday

* Recall :

(1) $c(K) \rightarrow$ Cone on a simplicial complex K^n

$$H_0(cK) = \mathbb{Z}$$
$$H_i(cK) = 0 \quad \forall \; i \geq 1$$

(2) Using (1) for $S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$

$$H_0(S^n) = \mathbb{Z}$$
$$H_n(S^n) = \mathbb{Z}$$
$$H_i(S^n) = 0 \quad \text{if } i \neq 0, n$$

Q) Is S^n homeomorphic to S^m ?

* If $X^n \underset{\text{homeo}}{\cong} Y^n$, then $H_i(X) \cong H_i(Y)$ for each $0 \leq i \leq n$.

i.e. Homology group is topological invariant.

Ans: S^n homeomorphic to S^m iff $n = m$.

Q) Is $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^m$?

* Deformation Retract : (Munkres book Section 58)

If A is a subspace of topological space X, $A \subset X$

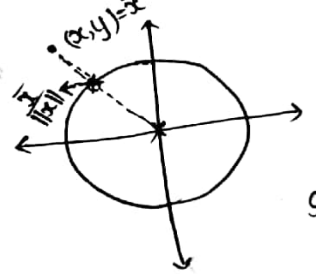
A is said to be a strong deformation retract of X if

$\exists$ a homotopy $H : X \times I \longrightarrow X$ s.t

$$H(x,0) = x, \quad H(x,1) \in A$$
$$H(a,t) = a$$

Eg:

(1) $\mathbb{R}^2 \setminus \{(0,0)\} \supset S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$



$$H : \mathbb{R}^2 \setminus \{(0,0)\} \times I \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$

$$H(\bar{x}, t) = (1-t)\bar{x} + t \frac{\bar{x}}{\|\bar{x}\|}$$

Straight line homotopy

$$H(\bar{x}, 0) = \bar{x}$$
$$H(\bar{x}, 1) = \frac{\bar{x}}{\|\bar{x}\|} \in S^1$$

If $\|\bar{x}\| = 1$ then $H(\bar{x}, t) = \bar{x}$

* **Propⁿ:**

If A is a strong deformation retract of X, then

(i) $\pi_1(A, x_0) \underset{\text{isom}}{\cong} \pi_1(X, x_0)$

(ii) $H_i(A) \underset{\text{isom.}}{\cong} H_i(X)$, $0 \le i \le n$

Eg:

1.' Strong deformation retract of $\mathbb{R}$

$$\mathbb{R} \supset \{0\}$$

$\mathbb{R}^n \rightarrow$ origin

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n \rightarrow$ Origin is the deformation retract

Let $\bar{x} \in \mathbb{R}^n$

$$H(\bar{x}, t) = (1-t)\bar{x} + t\underbrace{(0, \ldots, 0)}_{n}$$

$$H(\bar{x}, 0) = \bar{x}$$

$$H(\bar{x}, 1) = \bar{0}$$

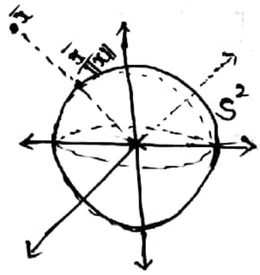
* $\pi_1(\mathbb{R}^n) = \pi_1(\text{Origin}) = e$

~~$H_i(\mathbb{R}^n)$~~

$\pi_1(\mathbb{R}^2 \setminus 0) = \pi_1(S^1) = \mathbb{Z}$

Eg:

(2.) $\mathbb{R}^3 \setminus \{(0,0,0)\} \supset S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$



$H : \mathbb{R}^3 \setminus \{(0,0,0)\} \times I \longrightarrow \mathbb{R}^3 \setminus \{(0,0,0)\}$

$$H(\bar{x}, t) = (1-t)\bar{x} + t\frac{\bar{x}}{\|x\|}$$ ← Connecting $\bar{x}, \frac{\bar{x}}{\|x\|}$

Straight line homotopy.

$$H(\bar{x}, 0) = \bar{x}$$

$$H(\bar{x}, 1) = \frac{\bar{x}}{\|x\|} \in S^2$$

$$\pi_1(\mathbb{R}^3 \setminus \{(0,0,0)\}) = \pi_1(S^2)$$

$$H_i(\mathbb{R}^3 \setminus 0) = H_i(S^2)$$

(3) $\mathbb{R}^{n+1} \setminus \{(0,0,..,0)\} \supset S^n = \left\{ (x_1,x_2,..,x_{n+1}) \,\middle|\, \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$

$$H : \mathbb{R}^{n+1} \setminus \{(0,0,..,0)\} \times I \longrightarrow \mathbb{R}^{n+1} \setminus \{(0,0,..,0)\}$$

$\bar{x} \in \mathbb{R}^{n+1} \setminus 0$ $\quad H(\bar{x},t) = (1-t)\bar{x} + t\dfrac{\bar{x}}{\|x\|}$

$$H(\bar{x},0) = \bar{x}$$

$$H(\bar{x},1) = \frac{\bar{x}}{\|x\|} \in S^n$$

$$\pi_1(\mathbb{R}^{n+1} \setminus 0) = \pi_1(S^n)$$

* **Theorem:**

$\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ iff $n = m$.

Proof:

By method of contradiction.

Suppose $\mathbb{R}^n$ homeo to $\mathbb{R}^m$

$\Rightarrow \exists$ a cts map $h : \mathbb{R}^n \to \mathbb{R}^m$ s.t

h is one-one, onto, h^{-1} continuous.

Let us say $h(\underbrace{0,..,0}_{\substack{n \\ (\text{origin})}}) = p \in \mathbb{R}^m$.

Consider the restriction

$$h' = h|: \mathbb{R}^n \setminus \{0,0,..,0\} \longrightarrow \mathbb{R}^n \setminus p.$$

$h' = h|_{\mathbb{R}^m \setminus \{0,..,0\}}$ is one-one, onto, continuous, & inverse also continuous.

i.e. h' is also homeomorphism.

$$H_i(\mathbb{R}^m \setminus \{0,0,..,0\}) \cong H_i(\mathbb{R}^n \setminus \{0,..,0\})$$

$$\| \qquad\qquad\qquad \|$$

$$H_i(S^{m-1}) \qquad\qquad H_i(S^{n-1})$$

p can be any point.

$$H_i(S^{n-1}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & 1 \le i \le n-2 \\ \mathbb{Z} & i = n-1 \end{cases} \qquad H_i(S^{m-1}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & 1 \le i \le m-2 \\ \mathbb{Z} & i = m-1 \end{cases}$$

$H_i(S^{m-1}) = H_i(S^{n-1})$ iff $m-1 = n-1$

iff $m = n$.

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* If $X \underset{\text{homeo}}{\cong} Y$, then $\pi_1(X) \overset{\text{isom}}{\cong} \pi_1(Y)$

$H_i(X) \overset{\text{isom}}{\cong} H_i(Y)$

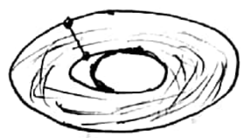
However, Converse, doesn't hold.

Eg: Section 58, Munkres

#2)

s.d.r point, circle, fig 8

(a) Solid torus $D \times S^1$



s.d.r $\rightarrow S^1$

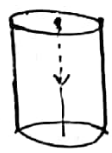
$\pi_1(D \times S^1) = \pi_1(S^1) = \mathbb{Z}$

$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}$

$H_i = 0$

(c) Cylinder $S^1 \times I$



$H(\bar{x}, y, z, t) = (x, y, z(1-t))$

$H(\bar{x}, 0) = \bar{x}$

$H(x, 1) = (x, y) \in S^1$

$\pi_1(S^1 \times I) = \pi_1(S^1) = \mathbb{Z}$

$H_i(S^1 \times I) = H_i(S^1)$

$$= \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}, & i = 1 \\ 0, & i \neq 0, 1 \end{cases}$$

(b) $\mathbb{R}^3 \setminus$ z-axis (Use vertical retraction.)

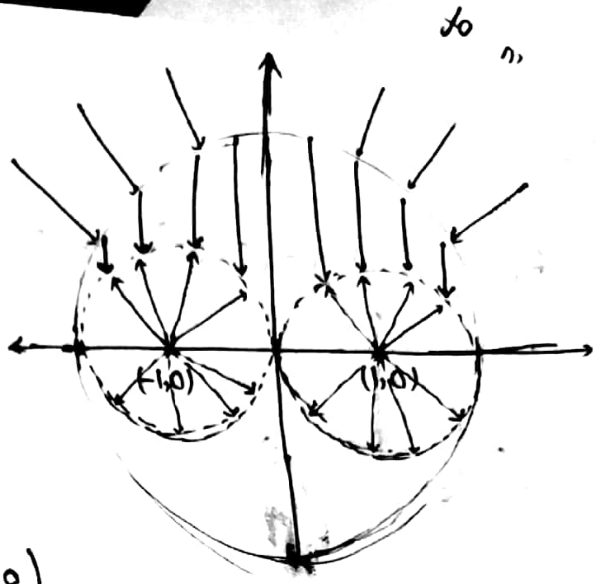
$\mathbb{R}^3 \setminus$ z-axis $\longrightarrow \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow S^1$

$H(x, y, z, t) = (x, y, z(1-t))$

$H(x, y, z, 0) = (x, y, z)$

$H(x, y, z, 1) = (x, y, 0)$

(d) Doubly punctured plane



i. Radial retraction ↘

ii. Vertical retraction

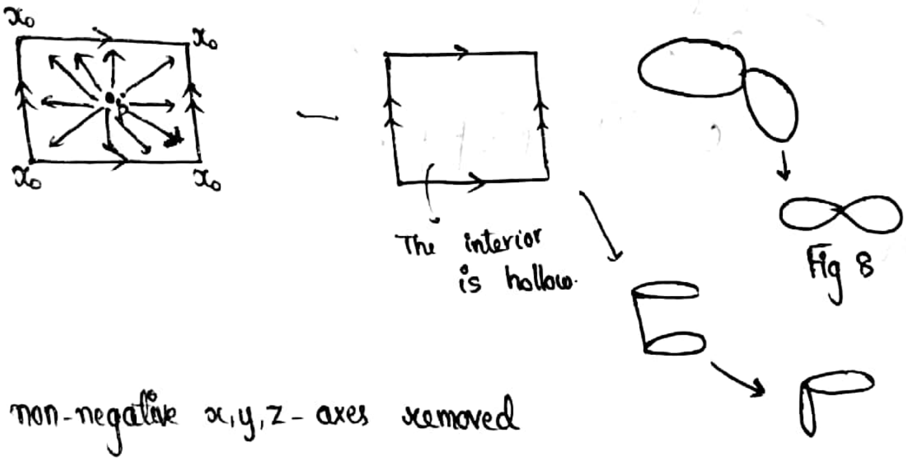
(iii.) Radial retraction ↖

$$\pi_1(\mathbb{R}^2 \setminus p,q) = \pi_1(\text{Fig } 8)$$

(e) $T \setminus \{p\}$
↓
Torus

$T = S^1 \times S^1$

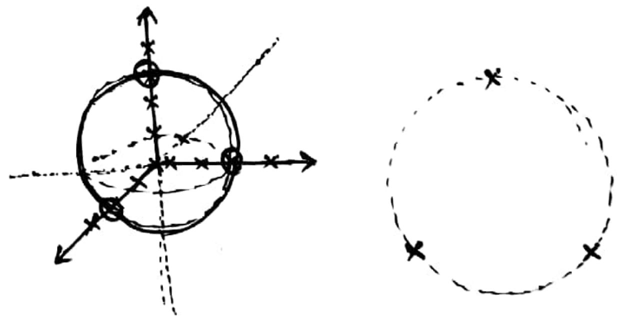
s.d.r is Fig 8



(f) $\mathbb{R}^3$ with non-negative x, y, z - axes removed

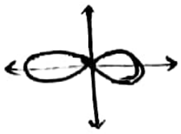
Hint :- use radial retraction.

$$H(\bar{x}, t) = (1-t)\bar{x} + t\frac{\bar{x}}{\|x\|}$$



$$S^2 \setminus \{(0,0,1)\} \approx \mathbb{R}^2$$

$$S^2 \setminus \{(0,0,1), (0,1,0), (1,0,0)\} \approx \mathbb{R}^2 \setminus p,q$$



sdr is Fig 8.

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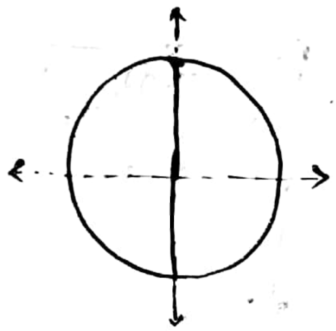
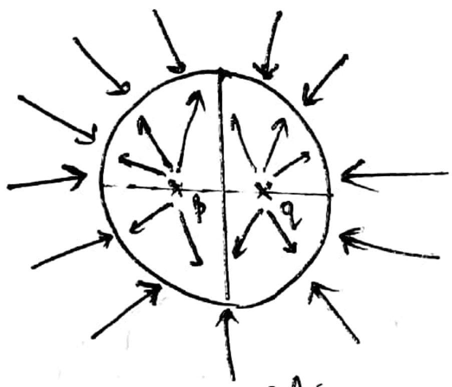
* Θ- space

$$\Theta = S^1 \cup (0 \times [-1,1]) \subset \mathbb{R}^2$$



sdr is Fig 8.



$$\Theta \xrightarrow{sdr} \mathbb{R}^2 \setminus p, q \longrightarrow \infty$$

Exercise: Section 58

Pb 5)