Solution of Assignment #2

#2·1 (a) $d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

so, $a = \int R^2 \sin\theta \, d\theta d\phi \, \hat{r} = 2\pi R^2 \int_0^{\pi/2} \sin\theta \, d\theta . \, \hat{z} \cos\theta$

$$= 2\pi R^2 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \, \hat{z} = 2\pi R^2 \hat{z} \left. \frac{\sin^2\theta}{2} \right|_0^{\pi/2} = \underline{\pi R^2 \hat{z}}$$

(b) According to the divergence theorem,

$$\oint \vec{v}.d\vec{a} = \int (\nabla.v) \, d\tau .$$

$\vec{v} = \vec{c}\,T$ where $\vec{c}$ is a constant vector.

so, $(\vec{\nabla}.\vec{v}) = (\vec{\nabla}.\vec{c}T) = T(\vec{\nabla}.\vec{c}) + \vec{c}.(\vec{\nabla}T)$

since $\vec{c}$ = constant, $\vec{\nabla}.\vec{c} = 0$.

$$(\vec{\nabla}.\vec{v}) = \vec{c}.(\vec{\nabla}T)$$

so, $$\oint T\,\vec{c}.d\vec{a} = \int \vec{c}.(\vec{\nabla}T)\, d\tau$$

since $\vec{c}$ is constant, it can be taken out of the integral,

$$\vec{c}.\oint T\, d\vec{a} = \vec{c}.\int (\vec{\nabla}T)\, d\tau$$

But $\vec{c}$ is any constant vector, it could be $\hat{x}, \hat{y}$ or $\hat{z}$, so each component of the integral on left equals corresponding component on the right.

so, $$\int T.d\vec{a} = \int \vec{\nabla}T \, d\tau$$

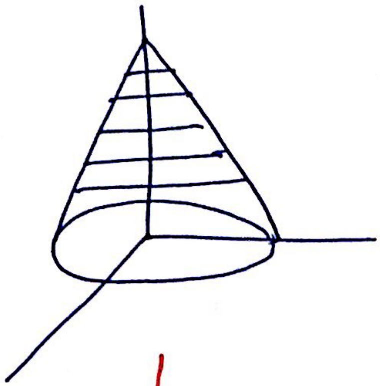
for $T=1$, $\vec{\nabla}T = 0 \Rightarrow \underline{\int d\vec{a} = 0}.$

(c) From (b) part, if $\vec{a}_1 \neq \vec{a}_2$ then if we put them together to make a closed surface,

$$\oint d\vec{a} = a_1 - a_2 \neq 0.$$

(d) Hint: one way to do it is to draw the cone subtended by the loop at the origin. Divide the conical surface up into infinitesimal triangular wedges, each with vertex at the origin and opposite side $d\vec{l}$, and exploit the geometrical interpretation of cross product

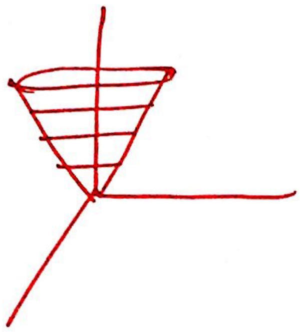
For one such triangle, $d\vec{a} = \frac{1}{2}(\vec{r} \times d\vec{l})$



[$\vec{r} \times d\vec{l}$ is the area of parallelogram, and the direction is perpendicular to the surface).

So for entire conical surface

$$\vec{a} = \frac{1}{2}\oint(\vec{r} \times d\vec{l})$$



(e) Let $T = \vec{c}\cdot\vec{r}$

& $\Delta T = \nabla(\vec{c}\cdot\vec{r}) = \vec{c}\times(\vec{\nabla}\times\vec{r}) + (\vec{c}\cdot\vec{\nabla})\vec{r}$

But $\vec{\nabla}\times\vec{r} = 0$ & $(\vec{c}\cdot\vec{\nabla})\vec{r} = \left(c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right)(x\hat{x}+y\hat{y}+z\hat{z})$

$$= c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = \vec{c}$$

From stokes' theorem, $\int(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint\vec{v}\cdot d\vec{l}$

Let $\vec{v} = \vec{c}T$, $\vec{\nabla}\times(\vec{c}T) = T(\vec{\nabla}\times\vec{c}) - \vec{c}\times(\vec{\nabla}T)$

$\downarrow$
Constant vector

$$= -c\times(\vec{\nabla}T)$$

so, $-\int(\vec{c}\times\vec{\nabla}T)\cdot d\vec{a} = \oint T\vec{c}\cdot d\vec{l}$

or, $\vec{c}\times(\vec{\nabla}T)\cdot d\vec{a} = c\cdot(\vec{\nabla}T\times d\vec{a})$

so, $-\int\vec{c}\cdot(\vec{\nabla}T\times d\vec{a}) = \oint\vec{c}\cdot T d\vec{l}$

So, putting $\vec{c}$ outside the integral & comparing all the coordinates

$$\int \vec{\nabla} T \times d\vec{a} = - \oint T d\vec{l}$$

So,

$$\oint T d\vec{l} = \oint (\vec{c}\cdot\vec{r}) d\vec{l} = - \int (\vec{\nabla} T) \times d\vec{a} = - \int \vec{c} \times d\vec{a}$$

$$= - \vec{c} \times \int d\vec{a} = - \vec{c} \times \vec{a} = \underline{\underline{\vec{a} \times \vec{c}}}$$

Problem! $\underline{\underline{2 \cdot 2}}$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r^2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, 4r^2\cos\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(r^2\tan\theta)$$

$$= \frac{1}{r^2} 4r^3\sin\theta + \frac{1}{r\sin\theta} 4r^2(\cos^2\theta - \sin^2\theta) = \frac{4r}{\sin\theta}\left[\sin^2\theta + \cos^2\theta - \sin^2\theta\right]$$

$$= 4r\frac{\cos^2\theta}{\sin\theta}.$$

So,

$$\int (\vec{\nabla}\cdot\vec{V})\, d\tau = \int 4r \frac{\cos^2\theta}{\sin\theta}(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$$= \int_0^R 4r^3 dr \int_0^{\pi/6}\cos^2\theta\, d\theta \int_0^{2\pi} d\phi = (R^4)\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/6} \cdot (2\pi)$$

$$= \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$$

Surface consists of two parts!

(i) $\underline{\text{the ice-cream:}}$ $r = R$, $\phi: 0 \to 2\pi$, $\theta: 0 \to \pi/6$

$$d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$\vec{V}\cdot d\vec{a} = (R^2 \sin\theta)(R^2 \sin\theta\, d\theta\, d\phi) = R^4 \sin^2\theta\, d\theta\, d\phi$$

$$\int \vec{V}\cdot d\vec{a} = R^4 \int_0^{\pi/6} \sin^2\theta\, d\theta \int_0^{2\pi} d\phi = (R^4).(2\pi).\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$$

$$= \frac{\pi R^4}{6}\left(\pi - \frac{3\sqrt{3}}{2}\right)$$

(ii) <u>The cone:</u> $\theta = \frac{\pi}{6}$, $r: 0 \to R$, $\phi: 0 \to 2\pi$

$$d\vec{a} = r \sin\theta\, d\phi\, dr\, \hat{\theta} = \frac{\sqrt{3}}{2} r\, dr\, d\phi\, \hat{\theta}$$

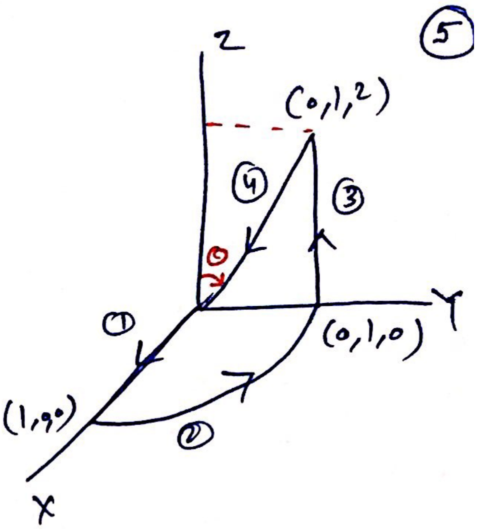
$$\vec{V}\cdot d\vec{a} = \frac{\sqrt{3}}{2} . r\, dr\, d\phi . \frac{1}{2} . 4r^2 = \sqrt{3}\, r^3 dr\, d\phi$$

$$\int \vec{V}\cdot d\vec{a} = \sqrt{3} \int_0^R r^3 dr \int_0^{2\pi} d\phi = \frac{\sqrt{3}}{2}\pi R^4$$

So $$\int_{total} \vec{V}\cdot d\vec{a} = \frac{\pi R^4}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3}\right] = \frac{\pi R^4}{12}\cdot(2\pi + 3\sqrt{3})$$

Problem: 2-3

Start at the origin!



(1) $\theta = \frac{\pi}{2}$, $\phi = 0$, $r: 0 \to 1$

$d\vec{l} = dr\,\hat{r}$

$\vec{v} \cdot d\vec{l} = (r\cos^2\theta)(dr) = 0$

$\int \vec{v} \cdot d\vec{l} = \underline{\underline{0}}$

(2) $r = 1$, $\theta = \frac{\pi}{2}$, $\phi: 0 \to \frac{\pi}{2}$

$d\vec{l} = r\sin\theta\, d\phi\, \hat{\phi}$

$\vec{v} \cdot d\vec{l} = (3r)(r\sin\theta\, d\phi) = 3\,d\phi$, $\int \vec{v} \cdot d\vec{l} = 3\int_0^{\pi/2} d\phi = \underline{\underline{\frac{3\pi}{2}}}$

(3) $\phi = \frac{\pi}{2}$, $r\sin\theta = y = 1$, so, $r = \frac{1}{\sin\theta}$, $dr = -\frac{1}{\sin^2\theta}\cos\theta\, d\theta$

$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$\theta: \frac{\pi}{2} \to \theta_0 \left(\equiv \tan^{-1}\left(\frac{1}{2}\right)\right)$

$\vec{v} \cdot d\vec{l} = (r\cos^2\theta)(dr) - (r\cos\theta\sin\theta)(r\,d\theta)$

$= \left(\frac{1}{\sin\theta} \cdot \cos^2\theta\right)\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta$

$= -\frac{\cos\theta}{\sin^3\theta}\,d\theta$

So, $\int \vec{v} \cdot d\vec{l} = -\int_{\pi/2}^{\theta_0} \frac{\cos\theta}{\sin^3\theta}\,d\theta = \left.\frac{1}{2\sin^2\theta}\right|_{\pi/2}^{\theta_0} = \underline{\underline{2}}$

(4) $\theta = \theta_0$, $\phi = \frac{\pi}{2}$, $r: \sqrt{5} \to 0$

$d\vec{l} = dr\,\hat{r}$

$\vec{v} \cdot d\vec{l} = (r\cos^2\theta)(dr) = \frac{4r}{5}\,dr$

$\int \vec{v} \cdot d\vec{l} = \frac{4}{5}\int_{\sqrt{5}}^{0} r\,dr = \underline{\underline{-2}}$

So total, $\oint \vec{v}\cdot d\vec{l} = 0 + \frac{3\pi}{2} + 2 - 2 = \underline{\underline{\frac{3\pi}{2}}}$

Now, $\int (\vec{\nabla}\times\vec{v})\cdot d\vec{a}$

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta . 3r) - \frac{\partial}{\partial\phi}(-r\sin\theta\cos\theta)\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos^2\theta) - \frac{\partial}{\partial r}(r3r)\right]$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(-r\, r\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right]\hat{\phi}$$

$$= 3\cot\theta\,\hat{r} - 6\,\hat{\theta}$$

(i) Back face: $d\vec{a} = -r\,dr\,d\theta\,\hat{\phi}$, $(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 0 \Rightarrow \int(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 0.$

(ii) Bottom face: $d\vec{a} = -r\sin\theta\,dr\,d\phi\,\hat{\theta}$

$(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = +6r\sin\theta\,dr\,d\phi.$, $\theta = \frac{\pi}{2} \Rightarrow (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 6r\,dr\,d\phi.$

$$\int(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int_0^1 6r\,dr \int_0^{\pi/2} d\phi = 6.\frac{1}{2}.\frac{\pi}{2} = \underline{\underline{\frac{3\pi}{2}}}$$

Problem: 2·4

$\int(\vec{\nabla}\cdot\vec{v})\,d\tau = \oint \vec{v}\cdot d\vec{a}$

$$(\vec{\nabla}\cdot\vec{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi + \frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi)$$

$$= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta$$

$$\int(\vec{\nabla}\cdot\vec{v})\,d\tau = \int(4r\cos\theta)\cdot r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_0^R r^3 dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2} d\phi$$

$$= R^4\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi R^4}{4}$$

Surface consists of four parts:

① Curved: $d\vec{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$, $r=R$

$$\vec{v}\cdot d\vec{a} = (R^2\cos\theta)(R^2\sin\theta\,d\theta\,d\phi)$$

$$\int\vec{v}\cdot d\vec{a} = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi = R^4\left(\frac{1}{2}\right)\cdot\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4}$$

② Left: $d\vec{a} = -r\,dr\,d\theta\,\hat{\phi}$, $\phi = 0$, $\vec{v}\cdot d\vec{a} = (r^2\cos\theta\sin\phi)(r\,dr\,d\theta) = 0$

$$\int\vec{v}\cdot d\vec{a} = 0$$

③ Back: $d\vec{a} = r\,dr\,d\theta\,\hat{\phi}$; $\phi = \pi/2$, $\vec{v}\cdot d\vec{a} = (-r^2\cos\theta\sin\phi)(r\,dr\,d\theta)$

$$= -r^3\cos\theta\,dr\,d\theta$$

$$\int\vec{v}\cdot d\vec{a} = \int_0^R r^3 dr\int_0^{\pi/2}\cos\theta\,d\theta = -\left(\frac{R^4}{4}\right)(+1) = -\frac{R^4}{4}$$

(4) Bottom: $d\vec{a} = r\sin\theta \, dr \, d\phi \, \hat{\theta}$, $\theta = \frac{\pi}{2}$

$$\vec{v}\cdot d\vec{a} = (r^2\cos\phi)(r\,dr\,d\phi)$$

$$\int \vec{v}\cdot d\vec{a} = \int_0^R r^3 dr \int_0^{\pi/2} \cos\phi \, d\phi = \frac{R^4}{4}$$

Total: $\oint \vec{v}\cdot d\vec{a} = \frac{\pi R^4}{4} + 0 - \frac{R^4}{4} + \frac{R^4}{4} = \frac{\pi R^4}{4}$

Problem 2.5 I WILL DISCUSS IT IN CLASS. See on page 10

Problem: 2.6 $T = r(\cos\theta + \sin\theta\cos\phi)$

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

$$= (\cos\theta + \sin\theta\cos\phi)\hat{r} + (-\sin\theta + \cos\theta\cos\phi)\hat{\theta} + \frac{1}{\sin\theta}(-\sin\theta\sin\phi)\hat{\phi}$$

$$\nabla^2 T = \vec{\nabla}\cdot(\vec{\nabla}T)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\cos\theta + \sin\theta\cos\phi)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta(-\sin\theta + \cos\theta\cos\phi)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-\sin\phi)$$

$$= \frac{1}{r^2}\cdot 2r(\cos\theta + \sin\theta\cos\phi) + \frac{1}{r\sin\theta}(-2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi) - \frac{1}{r\sin\theta}\cos\phi$$

$$= \frac{1}{r\sin\theta}\left[2\sin\theta\cos\theta + 2\sin^2\theta\cos\phi - 2\sin\theta\cos\theta + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi - \cos\phi\right]$$

$$= \frac{1}{r\sin\theta}\left[(\sin^2\theta + \cos^2\theta)\cos\phi - \cos\phi\right] = 0.$$

Check:

$r\cos\theta = z.$

$r\sin\theta\cos\phi = x$

So, $T = x + z$, $\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} = 2$

$\Rightarrow \nabla^2 T = 0$, $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

Gradient theorem: $\int_a^b \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$

Segment 1: $\theta = \frac{\pi}{2}$, $\phi = 0$, $r: 0 \to 2$, $d\vec{l} = dr\,\hat{r}$

$\vec{\nabla} T \cdot d\vec{l} = (\cos\theta + \sin\theta\cos\phi)\,dr = (0+1)\,dr = dr$

$\int \vec{\nabla} T \cdot d\vec{l} = \int_0^2 dr = 2$

Segment 2: $\theta = \frac{\pi}{2}$, $r = 2$, $\phi: 0 \to \frac{\pi}{2}$, $d\vec{l} = r\sin\theta\, d\phi\,\hat{\phi} = 2\,d\phi\,\hat{\phi}$

$\vec{\nabla} T \cdot d\vec{l} = (-\sin\phi)(2\,d\phi) = -2\sin\phi\, d\phi$

$\int (\vec{\nabla} T)\cdot d\vec{l} = \int_0^{\pi/2} 2\sin\phi\, d\phi = -2 = 2\cos\phi\Big|_0^{\pi/2}$

Segment 3: $r = 2$, $\phi = \frac{\pi}{2}$, $\theta: \frac{\pi}{2} \to 0$, $d\vec{l} = r\,d\theta\,\hat{\theta} = 2\,d\theta\,\hat{\theta}$

$(\vec{\nabla} T)\cdot d\vec{l} = (-\sin\theta + \cos\theta\cos\phi)(2\,d\theta) = -2\sin\theta\, d\theta$

$\int (\vec{\nabla} T)\cdot d\vec{l} = \int_{\pi/2}^{0} 2\sin\theta\, d\theta = 2\cos\theta\Big|_{\pi/2}^{0} = 2$

Total: $\int_a^b (\vec{\nabla} T)\cdot d\vec{l} = 2 - 2 + 2 = \underline{\underline{2}}$

Also, $T(b) - T(a) = 2(1+0) - 0(\;) = \underline{\underline{2}}$

Problem: 2·7 See solution of Problem # 3·2 of Problem set # 3.

Problem # 2·5

(a) $\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}\,')$

Check: $\int \rho(\vec{r})\,d\tau = q\int \delta^3(\vec{r}-\vec{r}\,')\,d\tau = q$

(b) $\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{a}) - q\,\delta^3(\vec{r})$

(c) $\rho(\vec{r})$ will of the form $\rho(\vec{r}) = A\,\delta(\vec{r}-\vec{R})$.

To determine the constant A, we require

$$Q = \int \rho\, d\tau = \int A\,\delta(r-R)\,4\pi r^2\,dr = A\cdot 4\pi R^2$$

$$\Rightarrow A = \frac{Q}{4\pi R^2}$$

$$\Rightarrow \rho(\vec{r}) = \frac{Q}{4\pi R^2}\cdot \delta(\vec{r}-\vec{R}).$$