Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment IV

1. $\star$ Using the method of variation of parameters find a particular solution of

   (a) $x^2y'' - 2xy' + 2y = x^{\frac{9}{2}}$.

   (b) $y'' + 3y' + 2y = \frac{1}{1+e^x}$.

   The functions $y_1(x) = x$ and $y_2(x) = x^2$ are solutions of the homogeneous equation $x^2y'' - 2xy' + 2y = 0$. Hence to find a general solution, we set $y = u_1y_1x + u_2y_2 = u_1x + u_2x^2$ where

$$\begin{aligned} u_1'x + u_2'x^2 &= 0 \qquad \textit{and} \\ u_1' + 2u_2'x &= \frac{x^{\frac{9}{2}}}{x^2} = x^{\frac{5}{2}}. \end{aligned}$$

   The first equation shows that $u_1' = -u_2'x$ and we substitute this in the second equation to get $u_2'x = x^{\frac{5}{2}}$. So $u_2' = x^{\frac{3}{2}}$ and $u_1' = -x^{\frac{5}{2}}$. Integrating these two equations and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7}x^{\frac{7}{2}} \qquad \textit{and} \qquad u_2 = \frac{2}{5}x^{\frac{5}{2}}.$$

   Therefore $y = u_1x + u_2x^2 = -\frac{2}{7}x^{\frac{7}{2}}x + \frac{2}{5}x^{\frac{5}{2}}x^2 = \frac{4}{35}x^{\frac{9}{2}}$ is a particular solution of the equation. The general solution is $y = c_1x + c_2x^2 + \frac{4}{35}x^{\frac{9}{2}}$.

   We will now find general solution of $y'' + 3y' + 2y = \frac{1}{1+e^x}$.
   The characteristic polynomial of the complementary equation $y'' + 3y' + 2y = 0$ is $r^2 + 3r + 2 = (r+2)(r+1)$. So $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$. Now we look for a particular solution of the form $y = u_1e^{-x} + u_2e^{-2x}$ where

$$u_1'e^{-x} + u_2'e^{-2x} = 0 \qquad \textit{and} \;\; -u_1'e^{-x} - 2u_2'e^{-2x} = \frac{1}{1+e^x}.$$

   Adding these two equations yields $-u_2'e^{-2x} = \frac{1}{1+e^x}$. So $u_2' = -\frac{e^{2x}}{1+e^x}$ and we substitute this in the first equation to get $u_1' = -u_2'e^{-x} = \frac{e^x}{1+e^x}$. Integrating these two equations and taking the constants of integration to be zero, we get $u_1 = \ln(1+e^x)$ and $u_2 = \ln(1+e^x) - e^x$. Therefore a particular solution $y = (\ln(1+e^x))\,e^{-x} + [\ln(1+e^x) - e^x]\,e^{-2x}$ and we can write this as $y = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) - e^{-x}$. Since the last term e^x is also a solution of the homogeneous equation we can drop it and write the general solution as $y = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) + c_1e^{-x} + c_2e^{-2x}$

2. $\star$ Using the method of undetermined coefficients find a particular solution of

   (a) $y'' - 3y' + 2y = e^{3x}(x^2 + 2x - 1)$.

   (b) $y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x$.

   We write a solution y as $y = e^{3x}u$. Then $y' = 3y + e^{3x}u'$ and $y'' = 9e^{3xu} + 6e^{3x}u' + e^{3x}u''$. We substitute this in the equation $y'' - 3y' + 2y = e^{3x}(x^2 + 2x - 1)$ to obtain $e^{3x}(u'' - 3u' + 2u) = e^{3x}(x^2 + 2x - 1)$. Hence $u'' - 3u' + 2u = x^2 + 2x - 1$. Let us write the undetermined function u as a polynomial $u(x) = A_0 + A_1x + A_2x^2$.

Then $u' = A_1 + 2A_2x$ *and* $u'' = 2A_2$. *Now we substitute this in the equation* $u'' - 3u' + 2u = x^2 + 2x - 1$ *and compare the coefficients to get*

$$2A_2 = 1, \qquad 2A_1 - 6A_2 = 2 \qquad \textit{and} \qquad 2A_2 - 3A_1 + 2A_0 = -1.$$

Solving these equations we get $A_2 = \frac{1}{2}$, $A_1 = \frac{5}{2}$ *and* $A_0 = \frac{19}{4}$. *Hence a particular solution is* $y = e^{3x}\left(\frac{19}{4} + \frac{5}{2}x + \frac{1}{2}x^2\right)$.

To solve $y'' + 3y' + 2y = (16 + 20x)\cos x + 10 \sin x$.
We write a particular solution as $y = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x$. *Then*

$$y' = (A_1 + B_0 + B_1x)\cos x + (B_1 - A_0 - A_1x)\sin x$$

and

$$y'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x.$$

Therefore

$$\begin{aligned} y'' + 3y' + 2y &= (A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x)\cos x \\ & \quad (B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x)\sin x. \end{aligned}$$

Comparing the coefficients we get

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10 \end{aligned}$$

Solving the first two equations yields $A_1 = 2$ *and* $B_1 = 6$. *Substituting these values in the other two equations and solving we get* $A_0 = 1$, $B_0 = -1$. *Thus a particular solution is* $y = (1 + 2x)\cos x - (1 - 6x)\sin x$.

3. $\star$ Let $p, q : (a, b) \to \mathbb{R}$ be two continuous functions. Let y_1 and y_2 be two solutions of the differential equation $y'' + py' + qy = 0$ in (a, b). Show that the solutions y_1 and y_2 are linearly dependent if any of the following conditions hold.

   (a) $y_1(x_0) = y_2(x_0) = 0$ at some point x_0 in (a, b).

   (b) y_1 and y_2 attain an extremum at same point x_0 in (a, b).

   The functions y_1 *and* y_2 *vanish at the same point* x_0 *in* (a, b). *Hence the wronskian* $W(y_1, y_2)(x_0) = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) = 0$. *Hence the solutions* y_1 *and* y_2 *are linearly dependent.*
   For the second part, since both the functions attain extremum at same point x_0, *it follows that* $y_1'(x_0) = 0 = y_2'(x_0)$. *As in the first part of the problem, the wronskian* $W(y_1, y_2)(x_0) = 0$. *As a consequence the solutions* y_1 *and* y_2 *are linearly dependent.*

4. $\star$ Let y_1 and y_2 be two linearly independent soltuions of the differential $y'' + py' + qy = 0$ where p and q are as in the earlier problem. Let x_1 and x_2 be two points in (a, b) such that $y_1(x_1) = 0 = y_1(x_2)$. Show that there exists a point z in (a, b) such that $x_1 < z < x_2$ and $y_2(z) = 0$.

   Without loss of generality, we assume that $y_1(x_1) = 0 = y_1(x_2)$ *and* $y_1(x) > 0$ *in* (x_1, x_2). *Therefore* $y_1'(x_1) > 0 > y_1'(x_2)$. *Let us observe that* $y_2(x_1) \neq 0 \neq y_2(x_2)$.

The wronskian $W(y_1, y_2)(x_i) = -y_1'(x_i)y_2(x_i)$ for $i = 1, 2$. By Abel's formula, $W(y_1, y_2)(x_2) = W(y_1, y_2)(x_1)\exp(-\int_{x_1}^{x_2} p(s)ds) = -y_1'(x_1)y_2(x_1)\exp(-\int_{x_1}^{x_2} p(s)ds)$. If $y_2(x_1)$ is positive, then $-y_1'(x_2)y_2(x_2) = W(y_1, y_2)(x_1) < 0$. Since $y_1'(x_2) < 0$, it follows that $y_2(x_2) < 0 < y_2(x_1)$. Hence there exists z in (x_1, x_2) such that $y_2(z) = 0$.

5. $\star$ Let $y_1, y_2 : (a, b)$ be two twice differentiable functions such that $W(y_1, y_2)(x) \neq 0$ for all points x in (a, b). Show that there exists two functions $p, q : (a, b) \to \mathbb{R}$ such that y_1 and y_2 are two linearly independent solutions of $y'' + py' + qy = 0$.

   This is analogous to finding the equation of a plane conatining two linearly independent vectors.
   We expand the determinant $\begin{vmatrix} y & y' & y'' \\ y_1 & y_1' & y_1'' \\ y_2 & y_2' & y_2'' \end{vmatrix} = 0$ to obtain the differential equation $Wy'' - W'y' + (y_1'y_2'' - y_1''y_2')y = 0$, where W denotes the wronskian of the two functions y_1 and y_2. Since $W(x) \neq 0$ for all x in (a, b), we can re-write this equation as $y'' - \frac{W'}{W}y' + \frac{(y_1'y_2'' - y_1''y_2')}{W}y = 0$. It is now easy to see that y_1 and y_2 are two linearly independent solutions of this equation.

6. $\star$ Let a, b, c be three positive real numbers and let y be a solution of the differential equation $ay'' + by' + cy = 0$. Show that $\lim_{n\to+\infty} y(x) = 0$.

   The characteristic equation is $ar^2 + br + c = 0$ and the roots are $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If $b^2 - 4ac > 0$ then $0 < \sqrt{b^2 - 4ac} < b$. Therefore the root $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} < 0$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} < 0$. As a consequence the two linearly independent solutions $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$ both tend to zero as $x \to +\infty$.
   If $b^2 - 4ac = 0$, then two linearly independent solutions are e^{rx} and xe^{rx} where $r = -\frac{b}{2a}$. It is easy to see that both the solutions tend to zero as $x \to +\infty$.
   If $b^2 - 4ac < 0$, then $y_1(x) = e^{rx}\cos\mu x$ and $y_2(x) = e^{rx}\sin\mu x$ are two linearly independent solutions of the equations; here $r = -\frac{b}{2a}$ and $\mu = \frac{4ac-b^2}{2a}$. Since cos *and* sin *are bounded functions, it can be seen that the two solutions tend to zero as $x \to +\infty$.*
   In each of the cases we have shown two linearly independent tend to zero as $x \to +\infty$. Hence the result is true for every solution.

7. Find the wronskian W of a given set $\{y_1, y_2\}$ of solutions of
   (a) $y'' + 3(x^2 + 1)y' - 2y = 0$ given that $W(y_1, y_2)(\pi) = 0$.
   (b) $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ given that $W(0) = 1$. (This is *Legendre's equation*).

   Since the wronskian at a point is zero, it is zero everywhere. Hence the wronskian is identically zero in the first problem.
   In the second problem Wronskian is

$$W(x) = W(0)\exp\left(-\int_0^x \frac{-2s}{1-s^2}ds\right) = \exp\left(-\ln(1-x^2)\right) = \frac{1}{1-x^2}.$$

8. Verify that $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutuons of $y'' - 2y' + y = 0$ on $(-\infty, \infty)$. Further find the solution y with the initial conditions $y(0) = 7$ and $y'(0) = 4$.

   Easy. The solution is $y(x) = (7 - 3x)e^x$.

9. Let $p, q : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Show that $y(x) = \sin(x^2)$ can't be a solution of the differential equation $y'' + py' + qy = 0$.

Easy. Apply uniqueness theorem: $y(0) = 0$ *and* $y'(0) = 0$.

10. Using the method of undetermined coeeficients find a particular solution of

    (a) $y'' - 7y' + 12y = 5e^{4x}$.

    (b) $y'' - 3y' + 2y = e^{-2x}\,(2\cos 3x - (34 - 150x)\sin 3x)$.

General solution of $y'' - 7y' + 12y = 5e^{4x}$.
The charactristic equation of the homogeneous part is $r^2 - 7r + 12 = (r-4)(r-3) = 0$. *Hence* $y = e^{4x}$ *is a solution of the homogeneous equation. Hence we look for a solution of the form* $y = ue^{4x}$ *where* u *is the function to be determined. We differentiate* y *to obtain*

$$y' = 4e^{4x}\,u + u'e^{4x} \qquad \text{and} \qquad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x}.$$

We substitute this in the equation and simplify to get $u'' + u' = 5$. *One can check by inspection that* $u = 5x$ *is a particular solution of this equation, so* $y = 5xe^{4x}$ *is a partcular solution of* $y'' - 7y' + 12y = 5e^{4x}$. *Therefore* $y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$ *is the general solution.*
Now we find the general solution of $y'' - 3y' + 2y = e^{-2x}\,(2\cos 3x - (34 - 150x)\sin 3x)$.
Let $y = ue^{-2x}$. *Then*

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

if $u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x$. *Since* $\cos 3x$ *and* $\sin 3x$ *are not solutions of the complementary equation* $u'' - 7u' + 12u = 0$, *we look for a particular solution of the form* $u_p = (A_0 + A_1x)\cos 3x + (B_0 + B_1x)\sin 3x$ *for the equation* $u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x$. *Now*

$$\begin{aligned} u_p' &= (A_0 + 3B_0 + 3B_1x)\cos 3x + (B_1 - 3A_0 - 3A_1x)\sin 3x \text{ and} \\ u_p'' &= (-9A_0 + 6B_1 - 9A_1x)\cos 3x - (9B_0 + 6A_1 + 9B_1x)\sin 3x. \end{aligned}$$

So

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= \left[3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x\right]\cos 3x \\ &\quad + \left[21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x\right]\sin 3x. \end{aligned}$$

Now we compare the coefficients of $\cos 3x$, $\sin 3x$, $x\cos 3x$ *and* $x\sin 3x$ *to get*

$$\begin{aligned} 3A_1 - 21B_1 &= 0 \\ 21A_1 + 3B_1 &= 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &= 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &= -34. \end{aligned}$$

Solving these equations we get $A_0 = 1$, $A_1 = 7$, $B_0 = -2$ *and* $B_1 = 1$. *Hence* $u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$ *is a particuar solution. Therefore* $y_p = e^{-2x}\left[(1 + 7x)\cos 3x - (2 - x)\sin 3x\right]$ *is a particular solution of the we started with.*

11. For each of the following set of functions $\{y_1, y_2\}$ given below find a differential equation $y'' + py' + qy = 0$ such that the set is a fundamental set of solutions, where p and q are continuous functions on the domain of defintion of y_1 and y_2.

    (a) $\{y_1(x) = x^2 - 1, y_2(x) = x^2 + 1\}$.

(b) $\{y_1(x) = x, y_2(x) = e^{2x}\}$.

(c) $\{y_1(x) = \frac{1}{x-1}, y_2(x) = \frac{1}{x+1}\}$.

Easy computations !

12. Find the solution of

(a) $y'' + y = 1$ with $(y(0), y'(0)) = (2, 7)$.

(b) $y'' - 2y' + y = x^2 - x - 3$ with $(y(0), y'(0)) = (-2, 1)$.

Easy computations.

13. Solve the equation $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{\frac{x}{2}}$.

Find a particular solution y_1 *for* $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3$ *and* y_2 *for* $y'' + 2y' + 10y = e^{\frac{x}{2}}$ *and add them.*
Solving for y_1 *and* y_2 *is easy.*