Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment V

1. $\star$ Show that the substitution $x = e^t$ transforms the Euler's equation $ax^2y'' + bxy' + cy = 0$ for $x > 0$, in to constant coefficient differential equation.

   Let $x = e^t$ and $Y(t) = y(x = e^t)$. Then

$$Y'(t) := \frac{dY}{dt} = \frac{dy}{dx}\frac{dx}{dt} = e^t y'(x) = xy'(x)$$

   and

$$Y''(t) = \frac{d^2Y}{dt^2} = \frac{d}{dt}\left(e^t y'(x)\right) = e^t y'(x) + e^t\frac{d}{dt}(y'(x)) = e^t y'(x) + e^{2t}y''(x) = xy'(x) + x^2y''(x).$$

   Hence $x^2y''(x) = Y''(t) - xy'(x) = Y''(t) - Y'(t)$ and we substitute in the equation $ax^2y''(x) + bxy'(x) + cy(x) = 0$ to get

$$\begin{aligned} 0 &= ax^2y''(x) + bxy'(x) + cy(x) \\ &= a(Y''(t) - Y'(t)) + bY'(t) + cY(t) \\ &= aY''(t) + (b-a)Y'(t) + cY(t). \end{aligned}$$

2. $\star$ Find the power series

   (a) in x for the general solution of $(1 + 2x^2)y'' + 6xy' + 2y = 0$

   (b) in $x - 1$ for the general solution of $(2 + 4x - 2x^2)y'' - 12(x-1)y' - 12y = 0$.

   First problem done in the class.
   We will now do the second one: $(2 + 4x - 2x^2)y'' - 12(x-1)y' - 12y = 0$. This equation can be written as $(4 - 2(x-1)^2)y'' - 12(x-1)y' - 12y = 0$.
   In the class we have shown that for the equation

$$(1 + \alpha(x - x_0)^2)y''(x) + \beta(x - x_0)y' + \gamma\, y = 0$$

   the general solution is of the forem $y = \sum_{n=0}^{\infty} a_n(x-1)^n$ where

$$a_{n+2} = -\frac{p(n)}{(n+1)(n+2)}a_n \qquad \text{for } n \geq 0$$

   and

$$p(n) := \alpha\, n(n-1) + \beta\, n + \gamma.$$

   To apply this formula we write the equation as

$$(1 - \frac{(x-1)^2}{2})y''(x) - 3(x-1)y'(x) - 3y(x) = 0.$$

   Hence $\alpha = -\frac{1}{2}$ and $\beta = -3 = \gamma$ and

$$p(n) = -\frac{1}{2}n(n-1) - 3n - 3 = -\frac{(n+2)(n+3)}{2}.$$

   Now we leave it to show that

$$a_{2m} = \frac{2m+1}{2^m}a_0, \qquad \text{and} \qquad a_{2m+1} = \frac{m+1}{2^m}a_1 \qquad \text{for} \qquad m \geq 0.$$

3. $\star$ Find the power series in $x - x_0$ for the general solution of the differential equations

   (a) $y'' - y = 0, \quad x_0 = 3.$

(b) $(1-4x+2x^2)y''+10(x-1)y'+6y=0, \qquad x_0=1.$

Let $y=\sum_{n=0}^{\infty} a_n(x-3)^n$. Then

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n(x-3)^{n-2} - \sum_{n=0}^{\infty} a_n(x-3)^n \\ &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-3)^n - \sum_{n=0}^{\infty} a_n(x-3)^n \\ &= \sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2}-a_n\right](x-3)^n. \end{aligned}$$

Thus we get

$$a_{n+2}=\frac{1}{(n+1)(n+2)}a_n.$$

For the second problem: Let $t=x-1$. Then the equation can be written as

$$(1-2t^2)y''-10ty'-6y=0.$$

We solve this as in problem 2. If $y=\sum_{n=0}^{\infty} a_n t^n$ is the general solution of the equation, then

$$a_{n+2}=-\frac{p(n)}{(n+1)(n+2)}a_n$$

where $p(n)=-2n(n-1)-10n-6=-2(n+1)(n+3)$. Therefore $a_{n+2}=2\frac{n+3}{n+2}a_n$. Using this we can show that

$$a_{2m}=\frac{1}{m!}\left[\prod_{j=1}^{m-1}(2j+3)\right]a_0 \qquad\qquad a_{2m+1}=4^m\frac{(m+1)!}{\prod_{j=1}^{m-1}(2j+3)}a_1.$$

4. $\star$ Find $a_0,\ldots,a_n$ for at least 7 in the power series $y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ for the soltuion of the initial value problems

   (a) $y''+(x-3)y'+3y=0, \qquad y(3)=-2, \quad y'(3)=3.$

   (b) $(4x^2-24x+37)y''+y=0, \qquad y(3)=4, \quad y'(3)=-6.$

   For the first problem $p(n)=n+3$. Hence

   $$a_{n+2}=-\frac{n+3}{(n+1)(n+2)}a_n.$$

   Now we use the relation $a_0=y(3)=-2$ and $a_1=y'(3)=3$ to compute the coefficients a_n's for $n\geq 2$.
   For the second problem, we let $t=x-3$ and write the equation as $(1+4t^2)y''+y=0$. In this case $p(n)=4n(n-1)+1=(2n-1)^2$. Therefefore

   $$a_{n+2}=-\frac{(2n-1)^2}{(n+1)(n+2)}a_n$$

   and we use this relation to determine the other coefficients.

5. $\star$ Find a fundamental set of Frobenius solutions of

   $$x^2(3+x)y''+5x(x+1)y'-(1-4x)y=0.$$

   Let $y=\sum_{n=0}^{\infty} a_n x^{n+r}$ be a Frobenius solution of the differential equation $x^2(3+x)y''+5x(x+1)y'-(1-4x)y=0$.
   The indicial polynomial for this equation is $F(r)=3r(r-1)+5r-1=(3r-1)(r+1)$ and the zeros of the indicial polynomial are $r_1=\frac{1}{3}$ and $r_2=-1$. In this case $r_1-r_2=\frac{4}{3}$ and hence we have two linearly independent Frobenius solutions which can be written as

   $$y_1=x^{\frac{1}{3}}\sum_{n=0}^{\infty} a_n(\frac{1}{3})x^n \qquad \text{and} \qquad y_2=x^{-1}\sum_{n=0}^{\infty} a_n(-1)x^n$$

form a fundamental set of Frobenius solutions. Using the relation between the coefficients, we can show that

$$a_n(r) = -\frac{n+r+1}{3n+3r-1}a_{n-1}(r) \qquad \text{for } n \geq 1.$$

First notice that $a_0(r) = 1$. For $r = \frac{1}{3}$, it follows that

$$\begin{aligned} a_n(\frac{1}{3}) &= -\frac{3n+4}{9n}a_{n-1}(\frac{1}{3}) \\ &= (-1)^n \frac{\prod_{j=1}^{n}(3j+4)}{9^n n!} \quad \text{for } n \geq 0. \end{aligned}$$

Thus $y_1 = x^{\frac{1}{3}} \sum_{n=0}^{\infty} (-1)^n \frac{\prod_{j=1}^{n}(3j+4)}{9^n n!} x^n$.
Now setting $r = -1$, yields

$$a_n(-1) = -\frac{n}{3n-4}a_{n-1}(-1), \qquad \text{for } n \geq 1.$$

So

$$a_n(-1) = (-1)^n \frac{n!}{\prod_{j=1}^{n-1}(3j-4)}$$

and

$$y_2(x) = x^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{\prod_{j=1}^{n-1}(3j-4)} x^n.$$

6. Find a fundamental set of Frobenius solutions of

   (a) $4x^2y'' + x(7+2x+4x^2)y' - (1-4x-2x^2)y = 0$,

   (b) $x^2(5+x+10x^2)y'' + x(4+3x+8x^2)y' + (x+36x^2)y = 0$,

   (c) $2x^2y'' + x(3+2x)y' - (1-x)y = 0$, and

   (d) $x^2(8+x)y'' + x(2+3x)y' + (1+x)y = 0$.