Topics in Topology.

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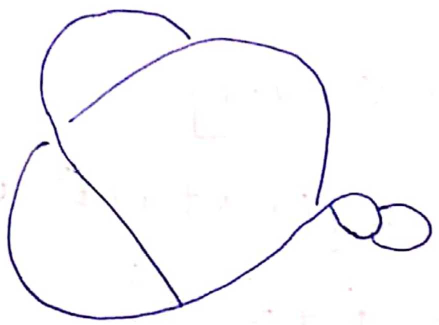
Knot theory - intro :-

Definition :- 1) A Kot K is an embedding of the circle S^1 in R^3; i.e., $f: S^1 \to R^3$ which is 1-1, onto and continous.

2). K_1, K_2 Knots are equivalent if K_1 Can be converted to K_2 by finitely.

Defn :- $c(K)$ = crossing numbers = min no. of crossing points over all diagramms of K.



$u(K)$ = unknotting = min no. of exchanged needed to unknot K.

$$c(K_1 \# K_2) = c(K_1) + c(K_2)$$

$$u(K_1 \# K_2) = u(K_1) + u(K_2)$$

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Knot polynomials:-

a very good method of ~~distinguish~~ distinguishing between "distinct" knots.

1) Alexander Polynomial (1928), used for about 50 yrs to distinguish

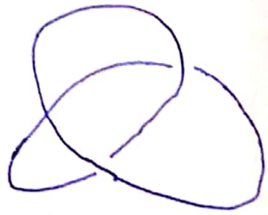
4_1 $\quad (-1)[-1+3-1] \longrightarrow t^{-1}[-1+3t-t^2]$.

$$\boxed{\Delta_{4_1}(t) = -t^{-1}+3-t}$$

$\Delta(K_{11}) = 1$

$\Delta_K(t) = \Delta_{K^*}(t)$

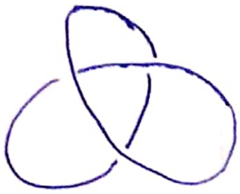
3_1



2) Jones polynomial

In 1984, discovered a new "knot polynomial"

3_1^*



$V_n'(t)$ jones polynomial

better than Alexander polynomial.

i) 3_1 $\quad (1)[1+0+1-1]$

$$V_{3_1} = t^1[1+0.t+1.t^2-t^3]$$

$$= t+t^3-t^4$$

$$V_{3_1^*} = \frac{1}{t}+\frac{1}{t^3}-\frac{1}{t^4}$$

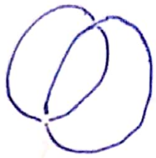
gives ~~a~~ quick proof that $V_{n_1} \neq V_{n_1}^*$

neither jones & Alexander explains complete theory

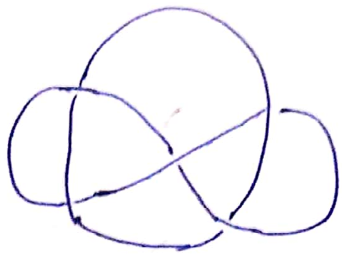
Links:-

Defn:- A Link is a finite ordered collection of knots that do not intersect each other.

components
↑
2_1^2
2 knots ← ↓ 1 link

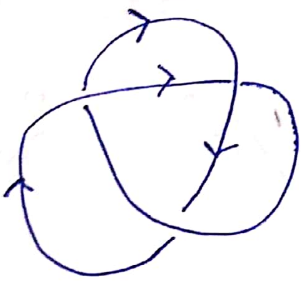


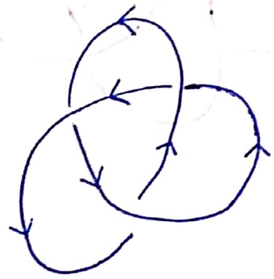
Hopf link



5_1^3

$6_1^3 \rightarrow$ Boromean Rings.





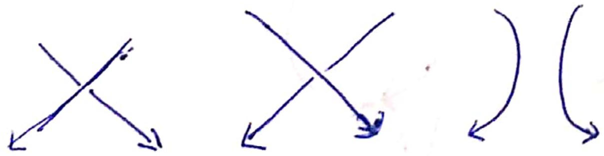
oriented knot is that we can choose it's orientatio

$\nabla_K(z)$

Alexander - Conway polynomial:- Given any oriented knot or Link., $\nabla_K(z)$ is got from 2 arrows

arrow 1 :- $\nabla_{0_1}(z) = 1$

arrow 2 :-



K^+ K^- K^0

Suppose knot diagram of K differ only at one crossing pt.

then $\boxed{\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\,\nabla_{K_0}(z)}$

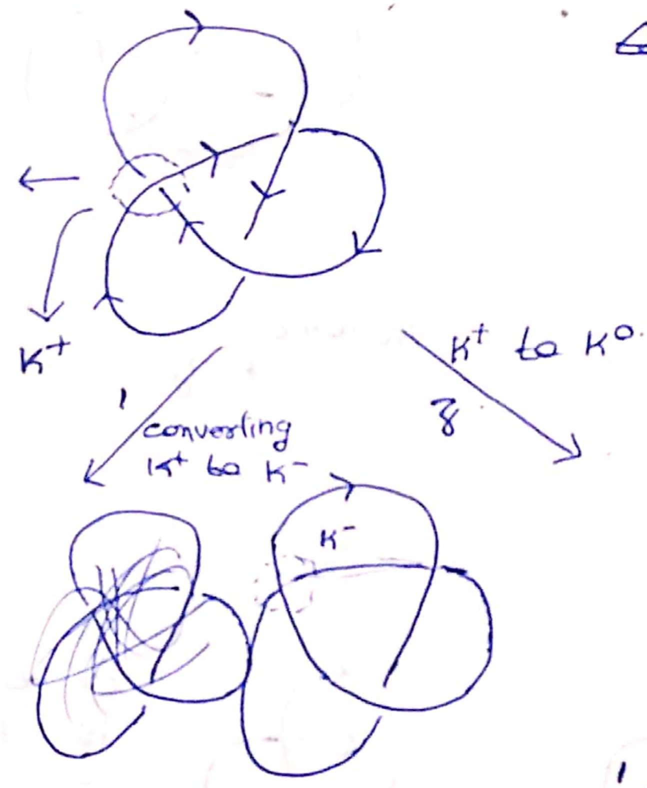
alexander polynomial : $\Delta_K(t) = \nabla_K(z)$ by subjection to $z = \sqrt{t} - \frac{1}{\sqrt{t}}$

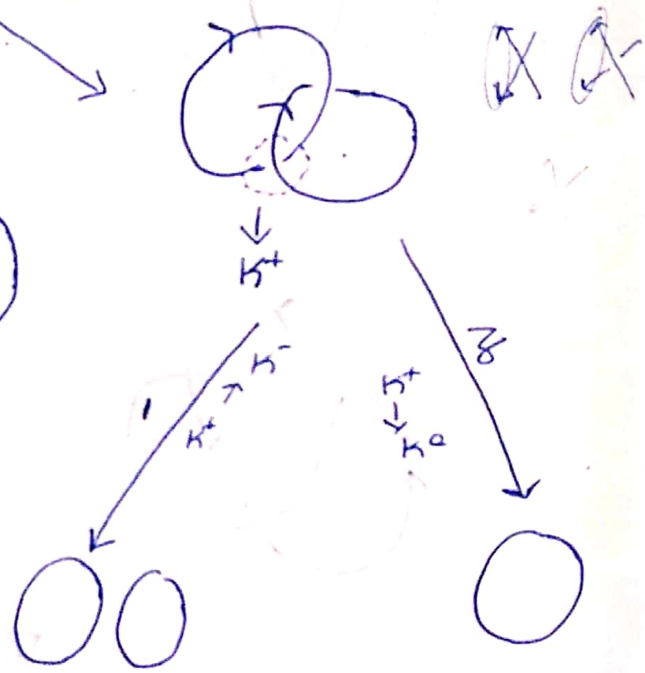
$\cancel{\Delta_{K^+}} -$

$\nabla_{K^+} - \nabla_{K^-} = z\,\nabla_{K^0}$

$\nabla_{K^+} = \nabla_{K^-} + z\,\nabla_{K^0}$

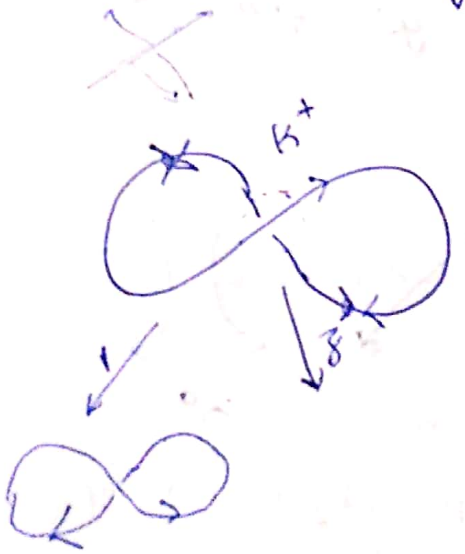
check out if it was K^+ or K^-

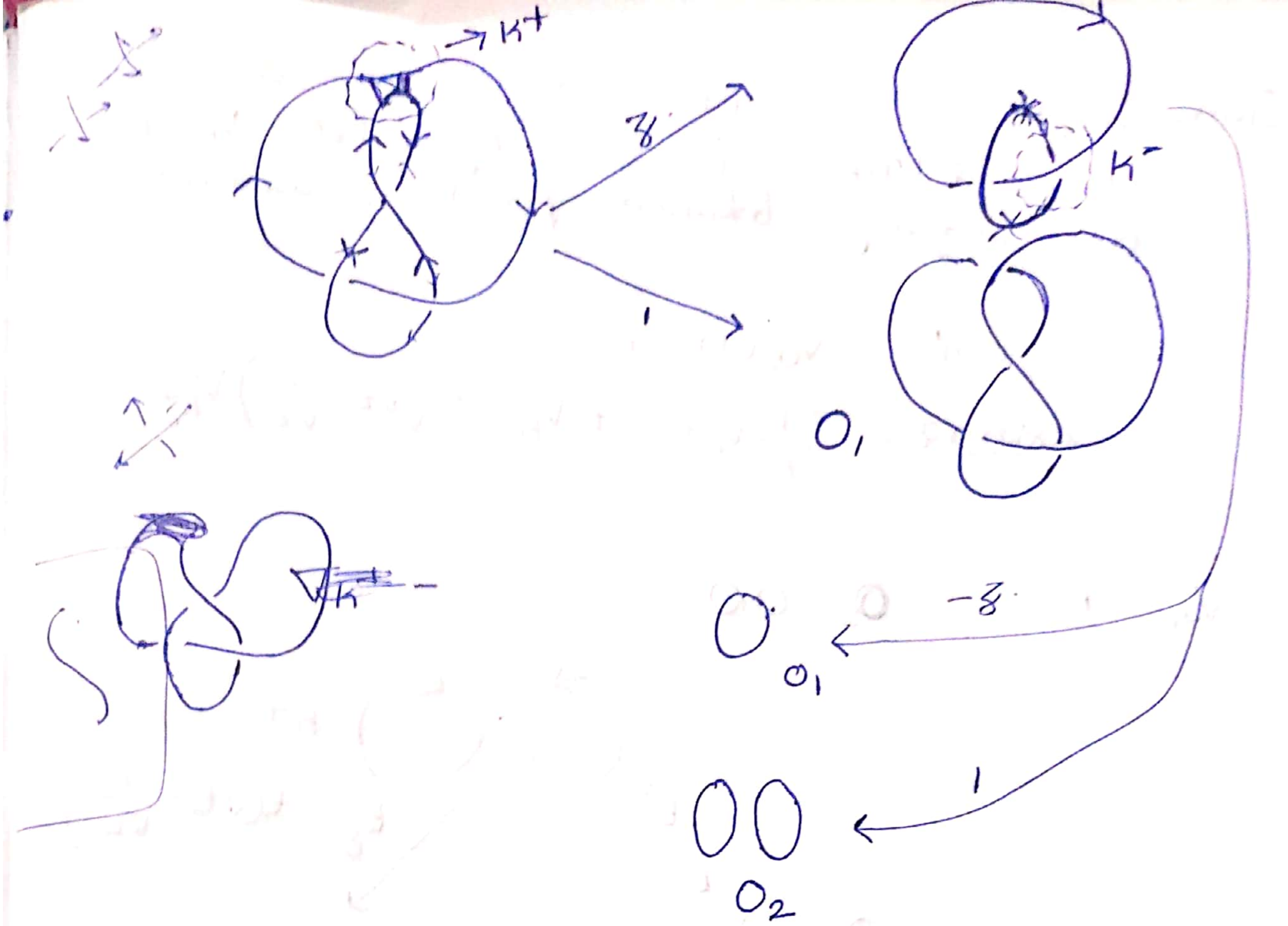




∇_{O_2} $\nabla_{3_1}(z) = 1\,\nabla_{0_1}(z) + z^2\,\nabla_{0_1}(z) + z \cdot 1\,\nabla_{0_2}(z)$.

H.W.





$$\nabla_{4_1}(z) = 1 - z^2 \nabla_{O_1} + z \cdot 1 \, \nabla_{O_2} \nearrow 0$$

$$= \boxed{\cancel{z0}}$$

$$z = \sqrt{t} - \frac{1}{\sqrt{t}}$$

$$\nabla_{4_1}(t) = \frac{-1}{t} + 3 - t.$$

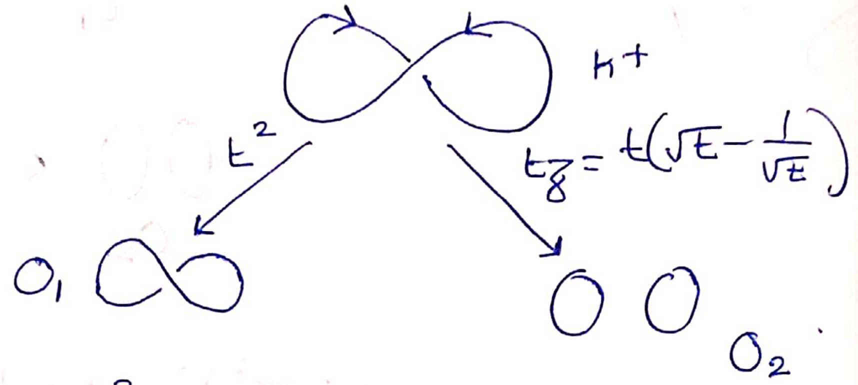
Home Work:- calculate Δ_K for all knots upto 6 crossing.

Jones Polynomial:- Suppose K is a oriented knot (or link) jones polynomial $V_K(t)$ is uniquely obtained from 2 axioms.

Axiom 1:- $V_{O_1}(t) = 1$.

Axiom 2:- $\frac{1}{t} V_{K^+} - t V_{K^-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K_0}$

$V_{O_1} = 1$; $O_2 = OO$

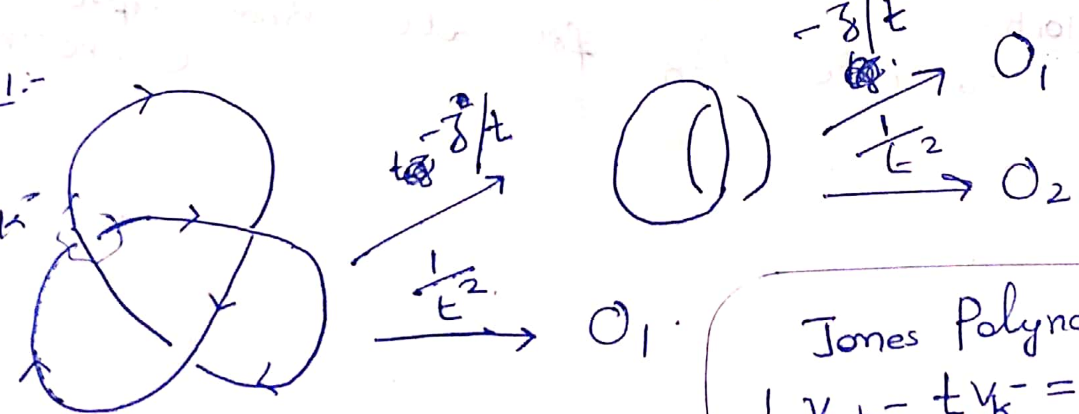


$$V_{O_1} = t^2 V_{O_1} + tz V_{O_2}$$

$$\frac{1-t^2}{t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)} = (-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right).$$

$$V_{O_2} = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right).$$

Eg 1:-

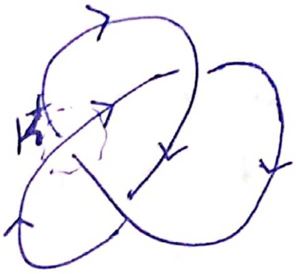


Jones Polynomial

$$\frac{1}{t} V_{K^+} - t V_{K^-} = z V_{K_0}$$

$$V_{K^+} = t^2 V_{K^-} + tz V_{K_0}$$

$$V_{K^-} = \frac{1}{t^2} V_{K^+} - \frac{z}{t} V_{K_0}$$



$\xrightarrow{tz}$

$\xrightarrow{tz} O_1$

$\xrightarrow{t^2} O_2$

$\xrightarrow{t^2} O_1$

$$V_{3_1} = t^2 V_{O_1} + t^2 z^2 V_{O_1} + t^3 z\, V_{O_2}$$

$$= t + t^3 - t^4.$$

Theorem :- Let K be an oriented knot or (link) and let K^* be it's mirror image with same orientation.

$$V_{K^*}(t) = V_K\left(\frac{1}{t}\right).$$

Proof :- Unravel K or K^* by the identical

$$K^+ \longrightarrow K^-$$

$$\longleftarrow$$

$K^+ \xrightarrow{t^2}$, $K^+ \xrightarrow{tz}$

$K^- \xrightarrow{\frac{1}{t^2}}$, $K^- \xrightarrow[-z/t]{}$.

$$t^2 \rightleftarrows 1/t^2$$

$$tz \rightleftarrows -z/t.$$

$$t \rightleftarrows 1/t.$$

$$tz = t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right).$$

$$-\frac{z}{t} = \frac{1}{t}\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right).$$

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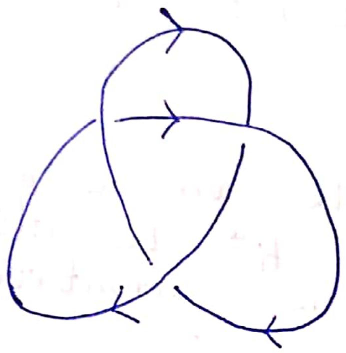
→ Tent completed first 800 Knots by hand.

① special type of knot is the "alternating knot".

Definition :- In alternating knot K, in a knot which has at atleast one diagram in which the over & under crossing alternative



$V_3(t) = t + t^3 - t^4$.

Max. degree = 4

min. degree = 1 $C(3_1) = 3$.

↑

K an alternating ~~polynomial~~ knot, $V_K(t)$ is Jones polynomial. Let max deg $V_K(t) = m$. min. degree $V_K(t) = n$, span $V_K(t) = m - n$.

Crossing number $c(K) = m - n$.

Tait conjecture :- Given an alternating knot, K all diagrams of K with no reducible crossing

Non-alternating knots.

$8_{19} \Rightarrow V_{8_{19}} = t^3 + t^5 + t^8$, $m - n = 5$;

span $V_K(t) < c(K)$.

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alexander poly 1928,

Jones poly 1984.

within 4 months of discovery of Jones $\uparrow$ polynomial, 6 people founded two variable Knot polynomial, which generalises both of them.

named as HOMFLY polynomial $P_K(v,z)$

$\rightarrow$ Let K be an oriented knot (or link) and D is a diagram of K

axiom 1:- $P_{o_1}(v,z) = 1$

axiom 2:- $\frac{1}{v} P_{D+}(v,z) - v P_{D-}(v,z) = z P_{D_0}(v,z)$

Special case:-

$\Rightarrow$ $v = 1$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$

$$P_K\left[1, \sqrt{t} - \frac{1}{\sqrt{t}}\right] = \Delta_K(t)$$

$\Rightarrow$ $v = t$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$

$$P_K\left(t, \sqrt{t} - \frac{1}{\sqrt{t}}\right) = V_K(t)$$

if $P_K(v,z) = 1$, $K = o_1$?

H.W
calculate $V_K(t) / P_K(v,z)$
for,
$3_1, 4_1, 5_1, 5_2, 6_1,$
$6_2, 6_3$
check span $= c(K)$

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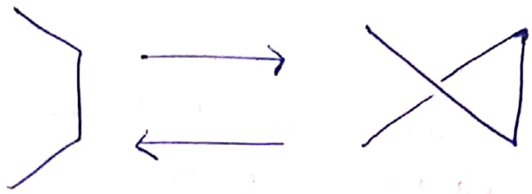
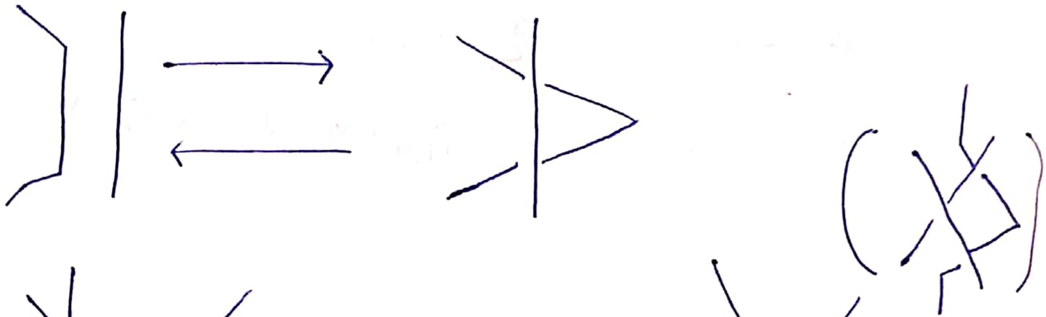
a given knot K ; is represented by many different knot diagram D. We would like to know what happens to D, if we perform elementary knot moves on K.

Reidemeister (1928) Moves :-

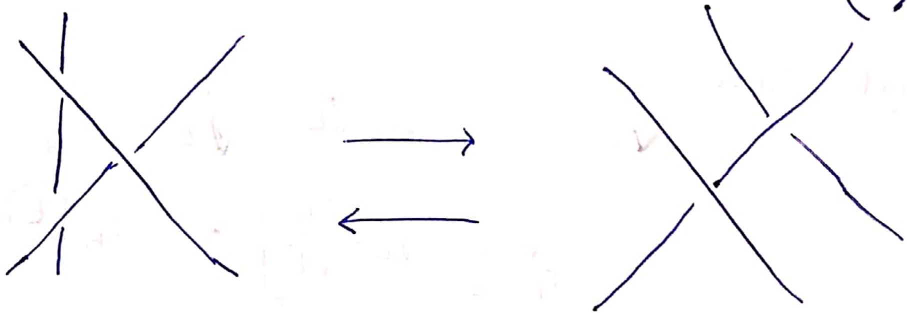
1) Ω_1



2) Ω_2



3) Ω_3



Definition:- If we can change a Diagram D to another diagram D' by performing 3 · R– moves finitely many times, then D is equivalent to D'

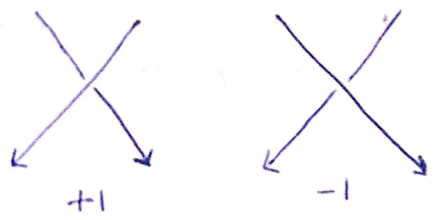
$D \Leftrightarrow D'$.

Theorem:- K, K' are two knots (or links)

D, D' are their diagrams, then.

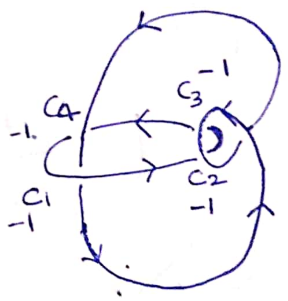
$K \cong K' \Leftrightarrow D \simeq D'$

Linking Number (2 comp links) :-

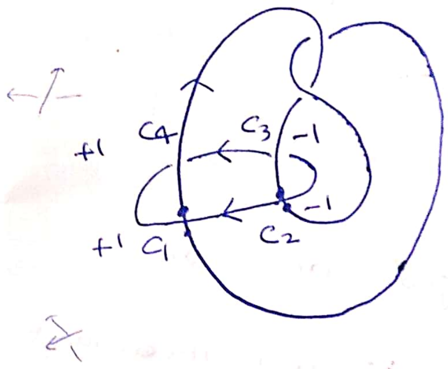


D is a diagram of a 2- component link,
$L = \{K_1, K_2\}$ Let
Crossing points of D between K_1 & K_2 be $C_1, C_2 \cdots C_m$

Linking $lk(K_1, K_2) = \frac{1}{2}\{\text{sign } C_1 + \text{sign } C_2 + \cdots + \text{sign } C_m\}$



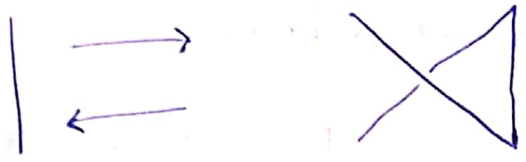
Linking number $= \frac{1}{2}\{-4\} = -2.$



Theorem :- $lk(K_1, K_2)$ is an " invariant".

proof:- <u>Strategy</u>:- We will show
$lk(K_1, K_2)$ is unchanged under the 3
R means $\Omega_1, \Omega_2, \Omega_3$.

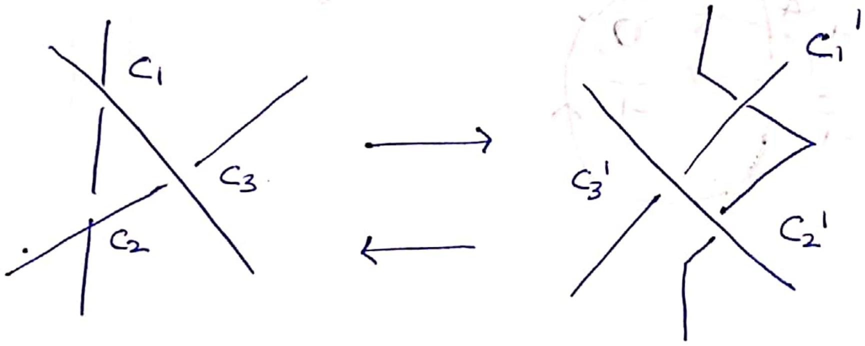
1) Ω_1



Ω_1 does not effect $lk(K_1, K_2)$ becoz it acts only on one component.

2) Ω_2

3) Ω_3



Sign c_1 = Sign c_2'

Sign c_2 = Sign c_1'.

Sign c_3 = Sign c_3'

$lk(K_1, K_2)$ ~~is~~ unchanged and on $\Omega_1, \Omega_2, \Omega_3$.

Hence $lk(K_1, K_2)$ is an invariant

Defn:- writhe $w(D)$ sum of all crossing points - Tait number not an invariant linking number (2 comp links).

$L = (K_1, K_2)$, $L^* = (K_1^*, K_2^*)$.

~~$lk(L) = -lk(L^*)$~~. $lk(L^*) = -lk(L)$.

$L = \{K_1, K_2, \ldots, K_n\}$

$lk = (K_i, K_j)$ $\frac{n(n-1)}{2}$ liking no's.

Total linking number. $= \sum lk(K_i, K_j)$
$1 \le i, j \le n$.

Definition:- a Knot K is said to be tricolourable, if we can colors, with 3 such that at each crossing points

0, 1, 2, R, B, Y.

1) A_K & A_L same color

2) A_r, A_s distinct colors or all 3 have same colors.

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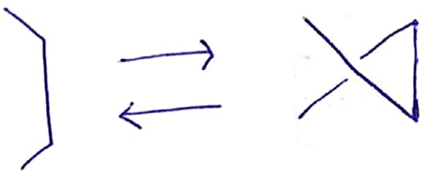
Reidemeister Moves:-

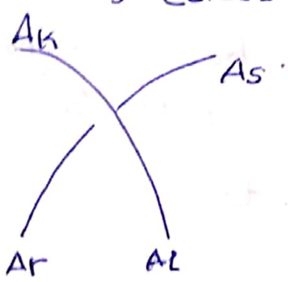
3 - R moves

1) Ω_1



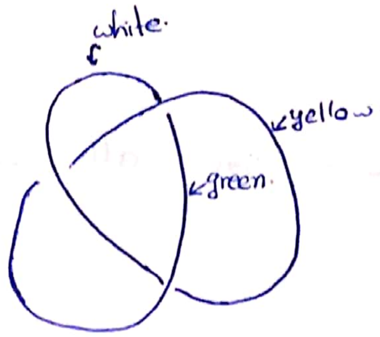
Tricolorability:-

suppose K has n crossing parts $P_1 \ldots \ldots P_n$ at each crossing part we assign 3 colors. Red, Blue, yellow etc.

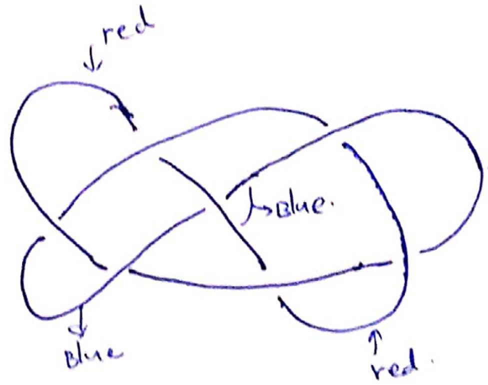


1.) Ak & Al has same color.

2.) Ak, Ar, As will have same color or all have diff. colors.



2)

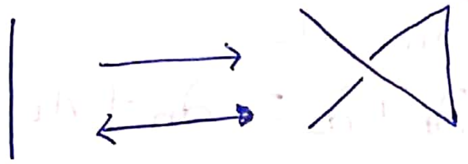


7_4

prop:- If there is a diagram D of K which
is 3 colorable, then every diagram D'
is 3 colorable, such a knot (or link) K
is 3 colorable.

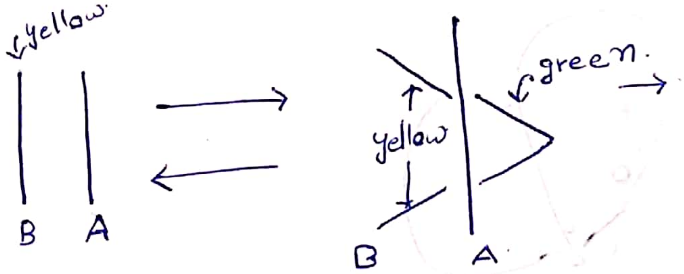
proof:- We will show tri colorability is
preserved under the 3 R-moves.

1) Ω_1

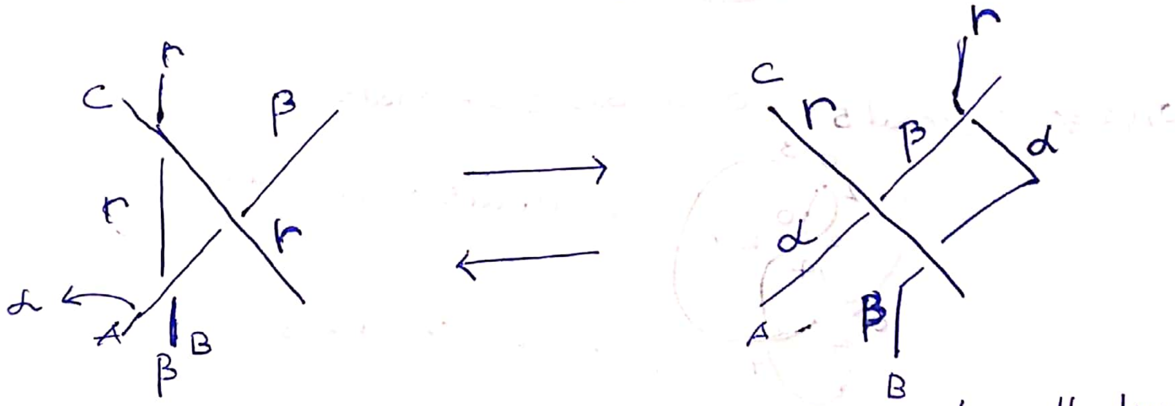


→ we will color each segment the same color.

2) Ω_2



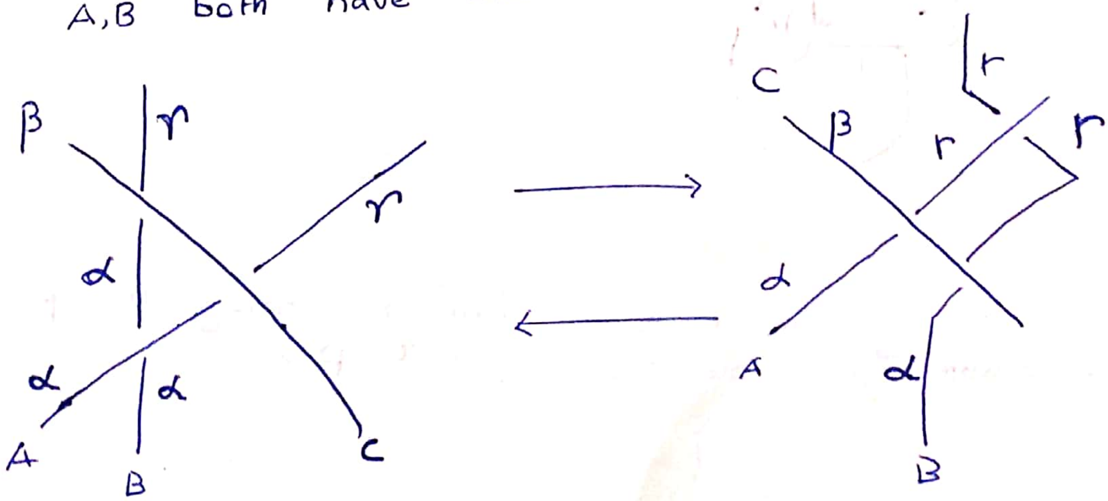
3).



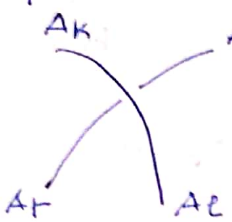
case 1:- if A, B, C have same color – trivially true.

case 2:- A, B, C diff colors α, β, γ
Y G white.

Case 3:- A, B both have color α, C have color β.



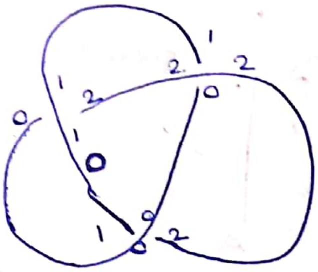
p-colorable, p a prime,



assign numbers 0, 1, 2 ... p−1 to A_K, A_l, A_r, A_s such that call them λ_K λ_L λ_r λ_s

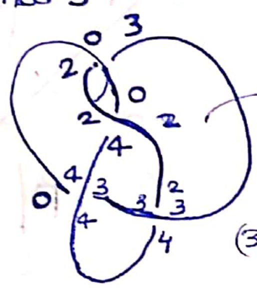
1) $\lambda_K = \lambda_L$

2) $\lambda_r + \lambda_s \equiv \lambda_K + \lambda_l \pmod{p}$.



$16 \equiv 1 \bmod 5$ $\quad a \equiv b \bmod c$ $\quad (a-b)/c$

0, 1, 2, 3, 4 ≡ mod 5 $\quad (0+0) \bmod 5 \equiv (2+3) \bmod 5$.



$(0+4) \bmod 5 \equiv (2+2) \bmod 5$

$(3+3) \bmod 5 \equiv (4+2) \bmod 5$

show 2 colorable.

0, 1 ≡ mod 2.

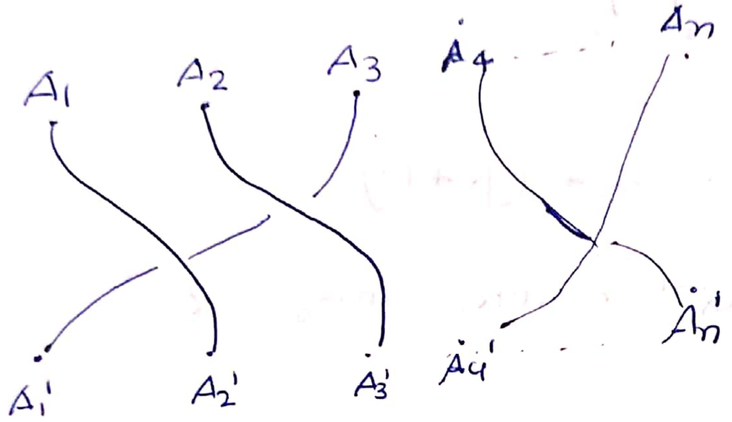
a given knot (or link) K may be p colorable for several different p's.

the coloring number

$K = \{P_1, P_2 \ldots \}$

Braid group B_n, $*$:-

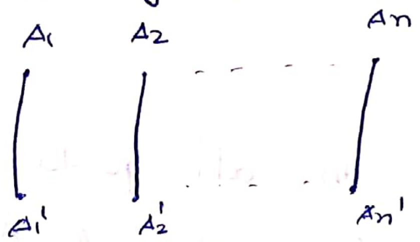
Take points $A_1, A_2, A_3 \ldots A_n$ in a row.
& points $A_1', A_2', A_3' \ldots A_n'$ directly below.



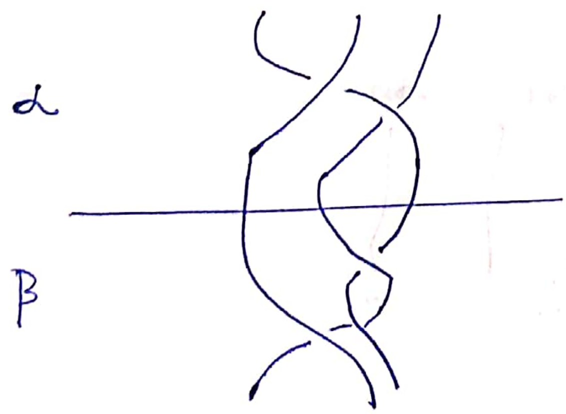
Connect $A_1, A_2 \ldots A_n$ to $A_1', A_2' \ldots A_n'$ by strings which donot intersect & move down.

claim:- B_n is a group

1) Id

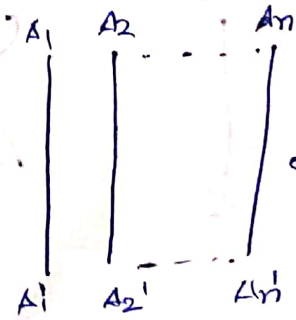


$\alpha * \beta \neq \beta * \alpha$.



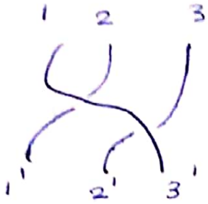
1) $\alpha * \beta$ is first do α, then perform β.

2) id



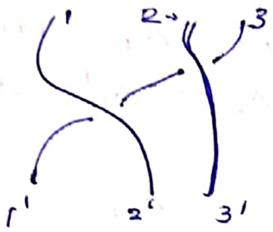
$\alpha * i = i * \alpha = \alpha$

$\alpha =$  $= \beta$

$\alpha * \beta =$ 

$1 \to 3'$
$2 \to 1'$
$3 \to 2'$

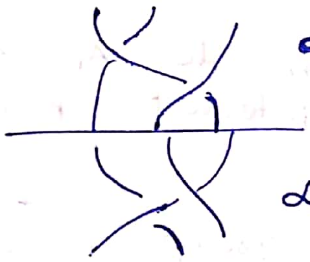
$\therefore \alpha * \beta \neq \beta * \alpha$

$\beta * \alpha =$ 

$1 \to 2'$
$2 \to 3'$
$3 \to 1'$

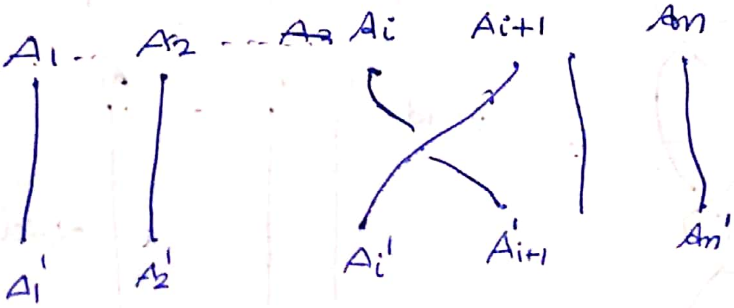
3). $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

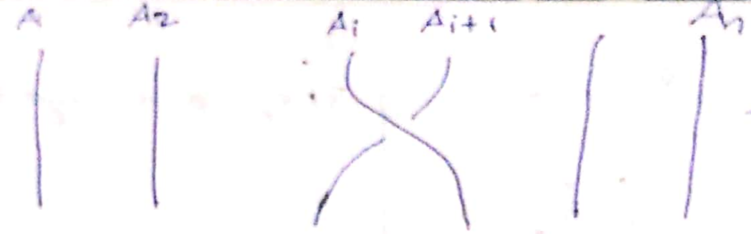
4). α^{-1} mirror image of α.



Consider σ_i

$A_1, A_2 \cdots A_n$ all go to $A_1' \ldots A_n'$
except $A_i \to A_{i+1}$ & $A_{i+1} \to A_i'$

σ_i 

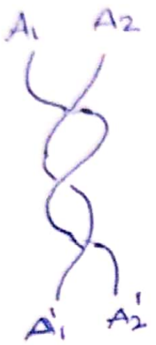
σ_i^{-1} 

→ braid.

22/1/19.

Any braid α in B_n can be written as a product of σ_i & σ_j^{-1}, Braid word.

B_2

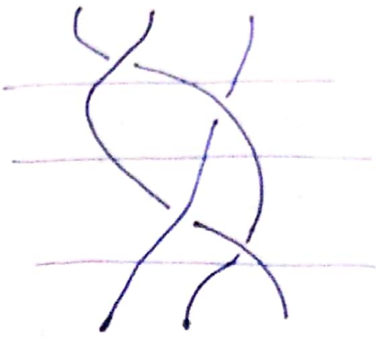


σ_1^3

σ_1^n or σ_1^{-m}

B_3

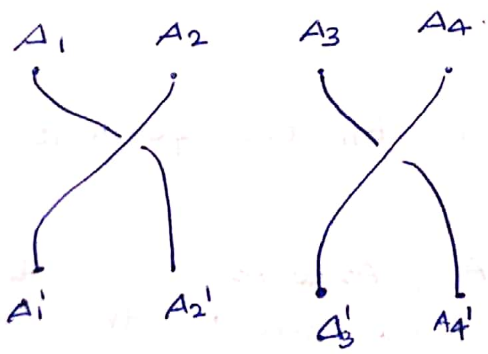
$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$



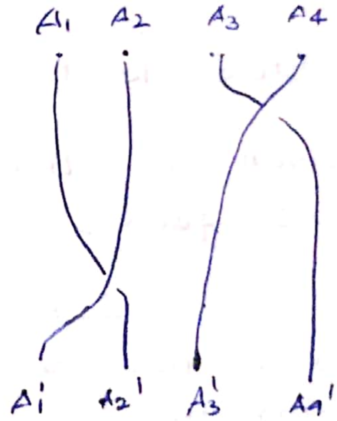
Note:- Braid words not unique

B_4

$\sigma_1\sigma_3$



$\sigma_3\sigma_1$



$$\boxed{\sigma_1\sigma_3 = \sigma_3\sigma_1}$$

$\alpha, \beta \in B_n$ then $\alpha \sim \beta$ if we can go from α to β by finitely many elementary Knot Moves.

$\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$.

$\sigma_1\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_2$.


They are infact same

2.) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

$$\omega_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4.$$

$$\omega_2 = \sigma_2\sigma_1\sigma_2^2$$

$$\omega_1 \sim \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_4\sigma_2 \sim \sigma_1\sigma_2\cancel{\sigma_4^{-1}}\cancel{\sigma_4}\sigma_1\sigma_2 \sim \sigma_1\sigma_2\sigma_1\sigma_2$$

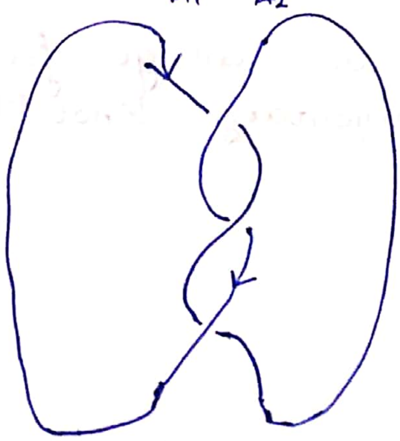
$$\sim \sigma_2\sigma_1\sigma_2\sigma_2 \sim \sigma_2\sigma_1\sigma_2^2.$$

Braid Closure:

Given a braid α in B_n we form it closure as follows.

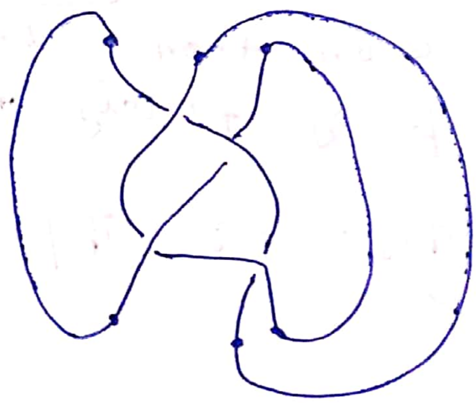
Join A_1 to A_1' ; A_2 to A_2' . . . . A_n to A_n' with large non-intersecting area outside the braid diagram.

A_1 A_2 σ_1^3



A_1' A_2'

B_3. $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$



$\rightarrow 4_1$

Alexander Theorem (1920): Any knot (or link) K can be obtained as the closure of some braid α

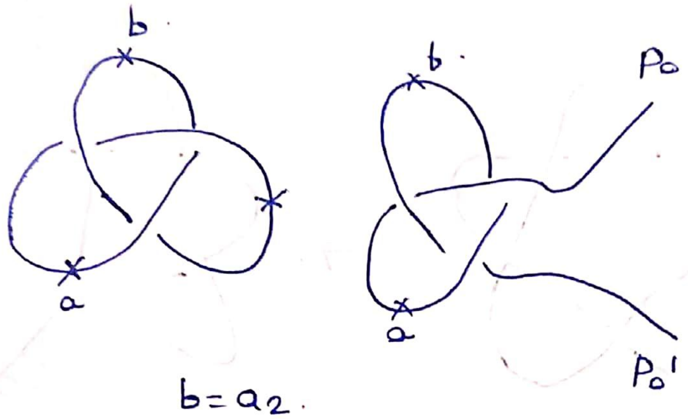
proof: Suppose D is a diagram of the knot K

step 1: We will cut D at a point P_0 (not a crossing point). pull the loose ends apart at P_0, P_0'

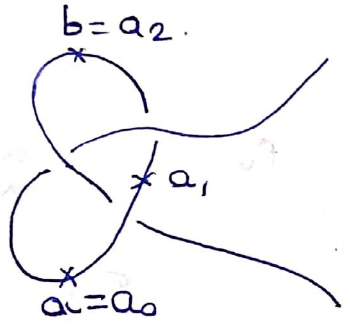
step 2:- The remaining diagram will have atleast one maxima ⓑ & one minima ⓐ.

we write ~~assume~~ that strand $\overline{ab}$ ~~intersects~~ ~~with the crossing~~ pts ~~at~~ as $a_1 . a_2 \ldots a_n = b$ such that $\overline{a_i a_{i+1}}$ intersects only at one crossing point

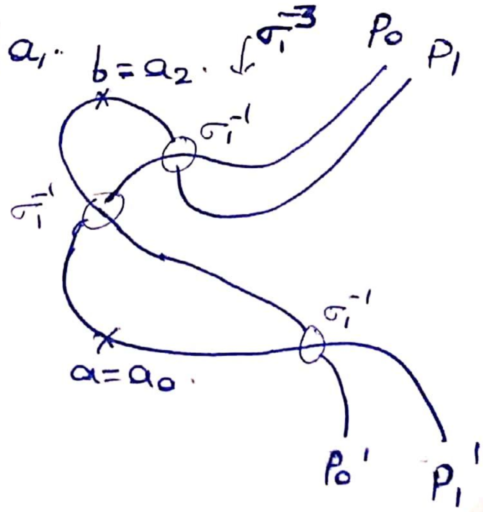
Step 1:



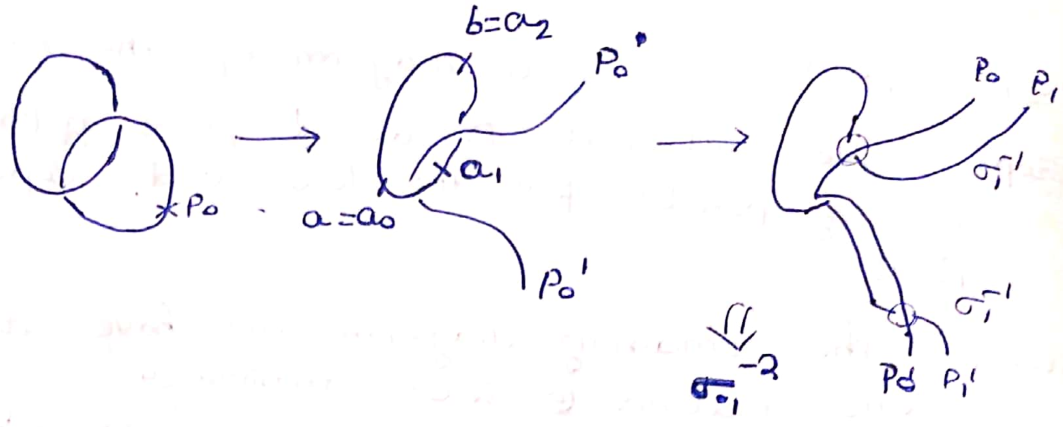
step 2:



step 3: Replace the $\overline{a_0 a_1}$ by the larger arc $a_0 P_1' P_1 a_1$. such that all crossings remain as ~~zero~~ they were.

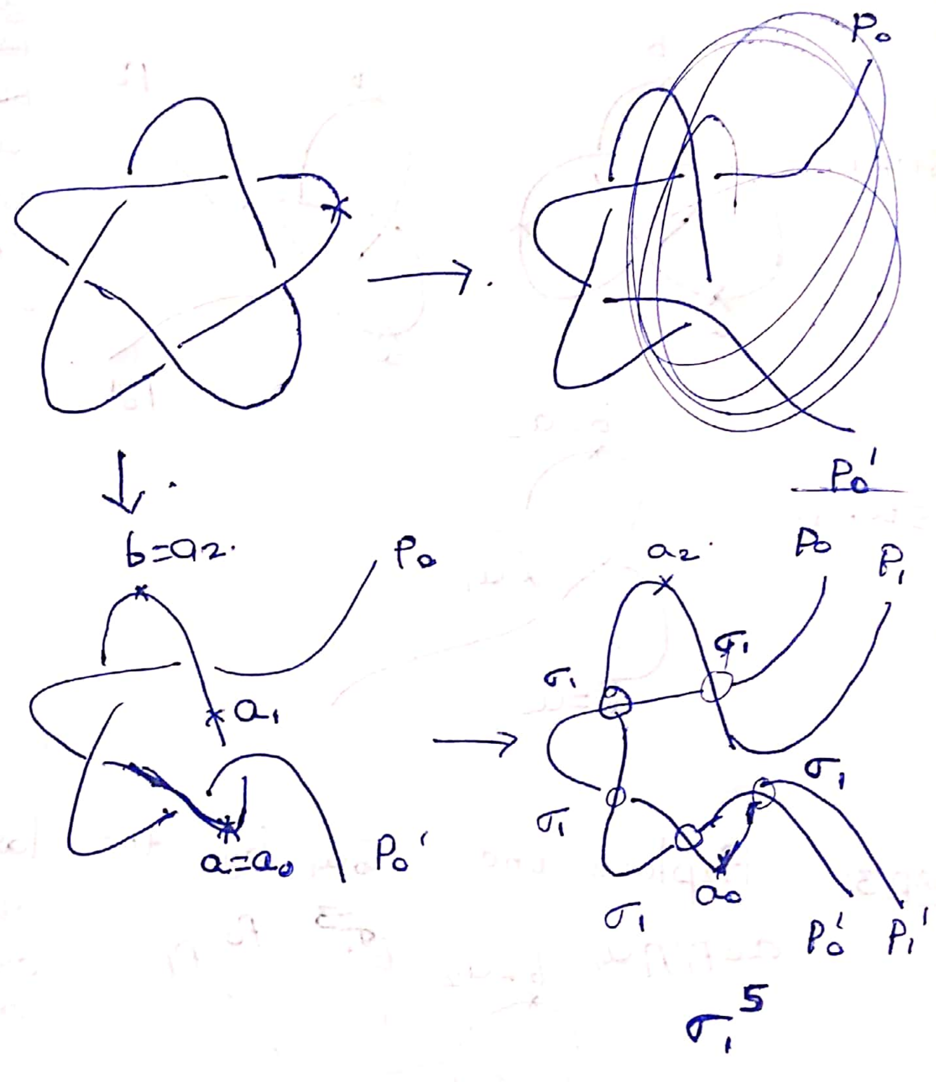


using Same Method on $\overline{a_1a_2}$; $\overline{a_2a_3}$; $\overline{a_{n-1}a_n}$; we will get a braid α where closure is K.

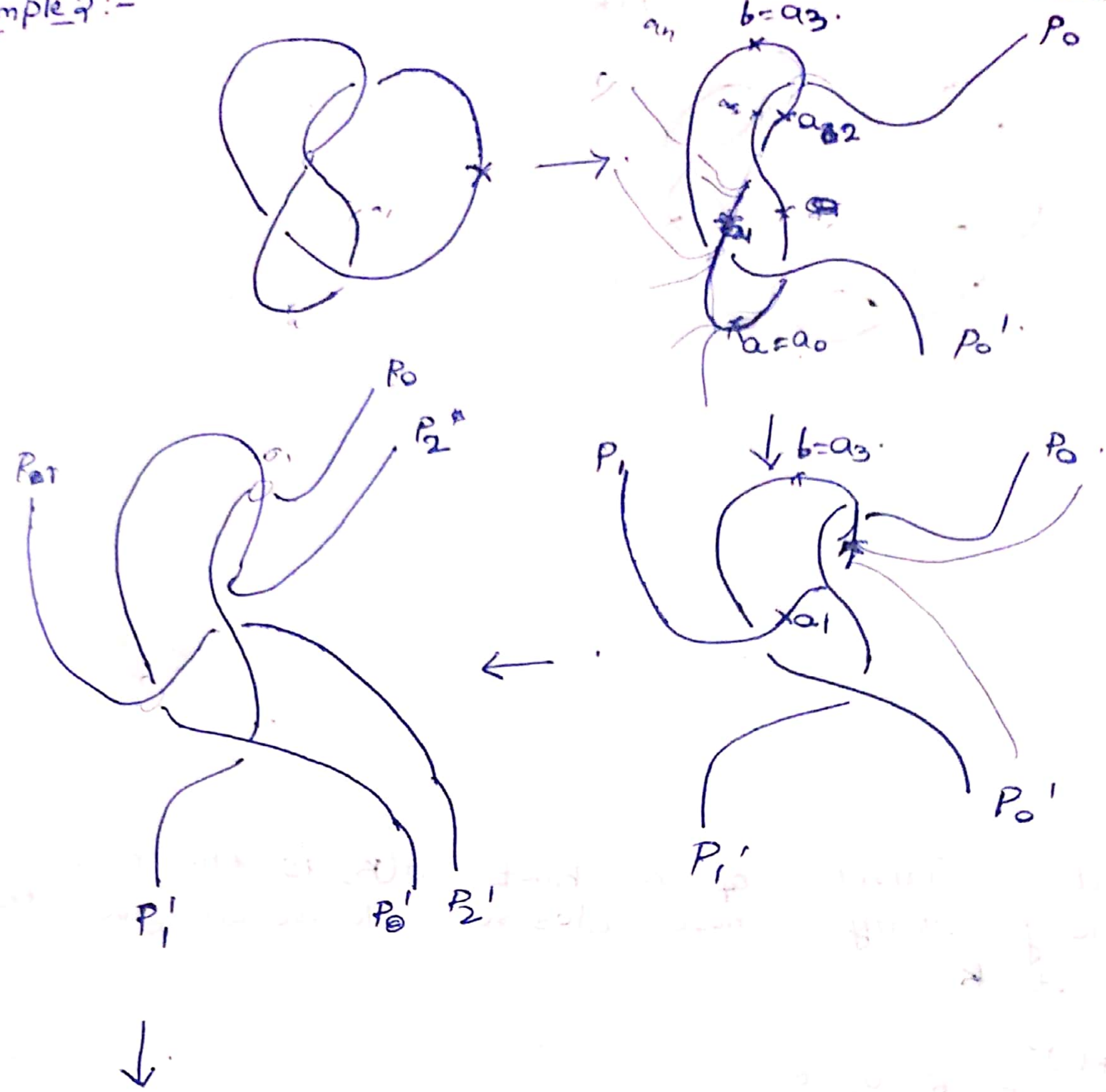


Example:-

5_1

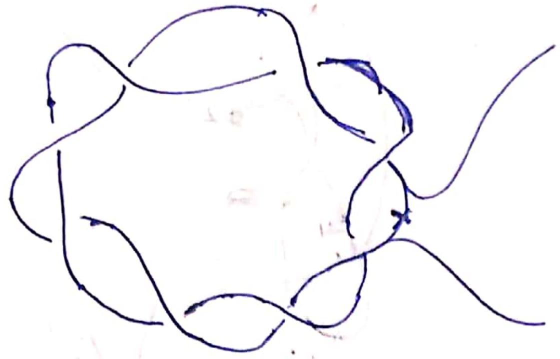


Example 2:-



H.W.

Eg:



Braid :- Inside of a Knot $b(K)$ is the minimum no. of strings whose closure closure will give the Knot K.

H.W:-
for 5_2, 6_1, 6_2, 6_3

for

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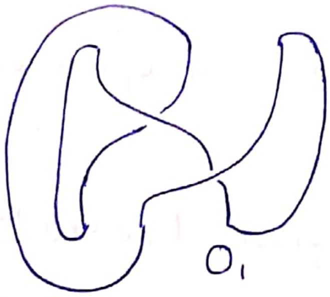
1. $\overline{\sigma_1^{-1}}$; $\overline{\sigma_1^{-1}\sigma_2}$; $\overline{\sigma_1\sigma_2^{-1}}$, $\overline{\sigma_1\sigma_2\sigma_3^{-1}}$; $\overline{\sigma_1\sigma_2^{-1}\sigma_3^{-1}}$,

(i)

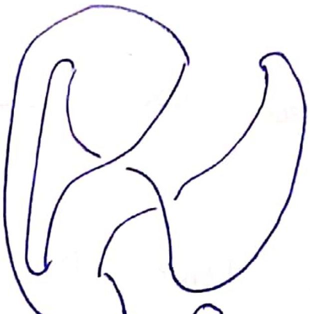
O_1

(ii)



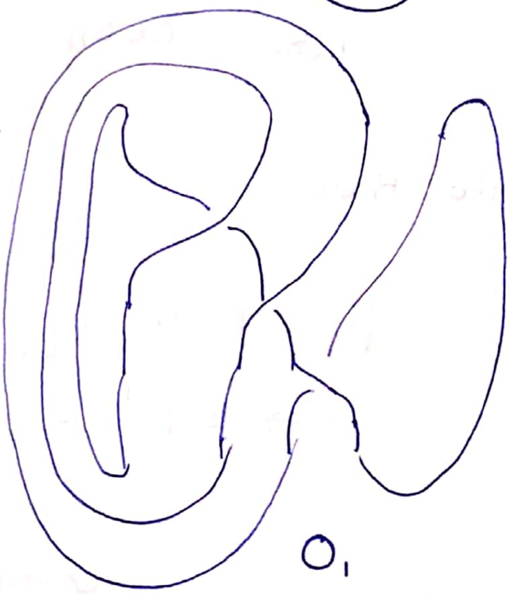
O_1

(iii)



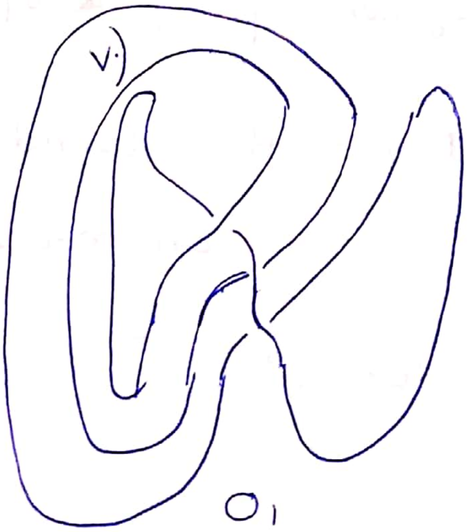
O_1

iv).



O_1

v.)



O_1

all give O_1 $\rightarrow$ different braids α, β ~~using~~ giving $\bar{\alpha} = \bar{\beta}$.

$\rightarrow$ We want '$\sim$' such that if $\alpha \sim \beta$.

$\Rightarrow$ $\bar{\alpha} \simeq \bar{\beta}$. (Russian).

answered by Markov in 1935.

Defn :- Suppose B_∞ is the union of the groups $B_1, B_2, \ldots$

$$\text{i.e,} \quad B_\infty = \bigcup_{k \geq 1} B_k$$

on B_∞, we perform 2 operations (called the Markov Moves)

1) if $\beta \in B_n$, then M_1 - first Markov Move.

Conjugation :- $\beta \rightarrow \gamma \beta \gamma^{-1}$ where $\gamma \in B_n$.

2) M_2 — the second Markov Rule.

β an n-braid

stabilization :-

$$\left.\begin{aligned} &\beta \rightarrow \beta \sigma_n \quad (\text{or}) \\ &\beta \rightarrow \beta \sigma_n^{-1} \end{aligned}\right\} n \rightarrow n+1$$

Defn :- Suppose $\alpha, \beta \in B_\infty$, If we can transform α into β by performing the Markov moves M_1, M_2 & their inverses; finitely many time, we will say α is Markov equivalent to β & we write $\alpha \sim_M \beta$

Markov's Theorem (1935) :- Suppose K_1 & K_2 are oriented Knot (or links) formed on closure of braids β_1 & β_2 respectively.

Then $K_1 \simeq K_2 \iff \beta_1 \sim_M \beta_2$.

~~An complete~~

S.T $\bar{\omega}_1 = \bar{\omega}_2$

Example:-

$\omega_1 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_4$

$\omega_2 = \sigma_2^{2} \sigma_3 \sigma_1 \sigma_3^{-1} \sigma_4 \quad = \sigma_2^{2} \sigma_1 \sigma_3 \sigma_3^{-1} \sigma_4 \to \sigma_2^{2} \sigma_1 \sigma_4$

Sol:- $\omega_1 = \sigma_2 \sigma_2 \sigma_1 \sigma_2 \sigma_2^{-1} \sigma_4$

$= \sigma_2^{2} \sigma_1 \sigma_2 \sigma_2^{-1} \sigma_4$

$= \sigma_2^{2} \sigma_1 \sigma_4$

$\therefore \bar{\omega}_1 = \bar{\omega}_2$

which knot these 3 braid represent?

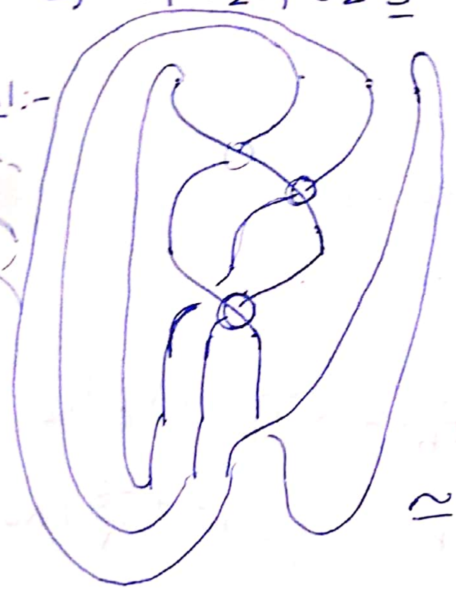
Ex2:-

a) $\sigma_1^{-2} \sigma_2^{2} \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_2^{-2} \sigma_1^{2}$

b) $\sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_3 \sim_M \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2$.

c) $\sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \sim_M \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2$.

Sol:-



$\simeq 4_1$

a) $\omega_1 \to \sigma_1 \omega_1 \sigma_1^{-1} \to \sigma_1^{2} \omega_1 \sigma_1^{-2} \to \sigma_2^{-2} (\sigma_1^{2} \omega \sigma_1^{-2}) \sigma_2^{2}$

Braid Index $b(K)$:

A knot (link) can be formed from an infinite no. of braids $\exists$ a braid which has the least no. of string α.

The no. of string of $\alpha = b(K)$ is the braid index of K.

Recall — HOMFLY polynomial $P_K(v, z)$.

max v-deg P_K $-$ min v-deg P_K = v-span $P_K(v,z)$

$$b(K) \geq \frac{1}{2}\left(v\text{-span } P_K(v,z)\right) + 1 \quad (A) \quad (*)$$

$(A) \to$ is an equality for all knots up to 10 crossings except 9_{42}, 9_{45}, 10_{132}, 10_{150}, 10_{156}.

$$\boxed{b(K) = \frac{1}{2}(v\text{-span } p) + 1} \quad *$$

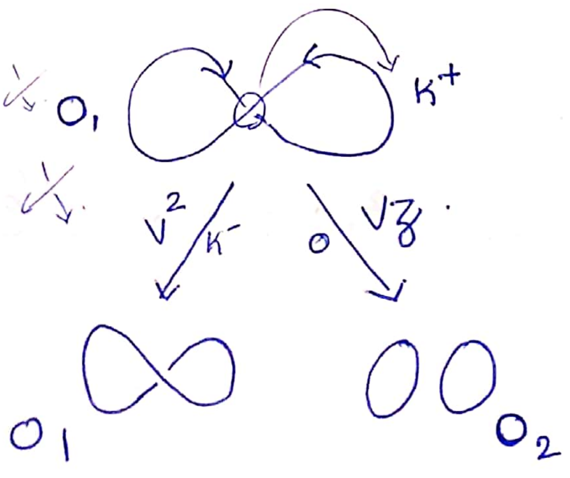
prove $*$ for $3_1, 4_1$.

Sol:- Calculate HOMFLY for 3_1.

$$\frac{1}{v} P_{K^+} - v P_{K^-} = z P_{K_0}$$

$$P_{K^+} = v^2 P_{K^-} + vz P_{K_0}.$$

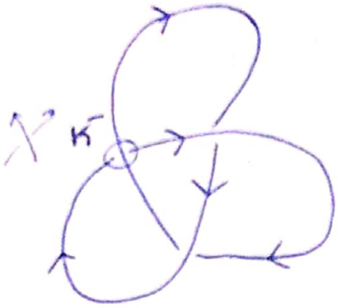
$$P_{K^-} = \frac{1}{v^2} P_{K^+} + \frac{-z}{v} P_{K_0}.$$



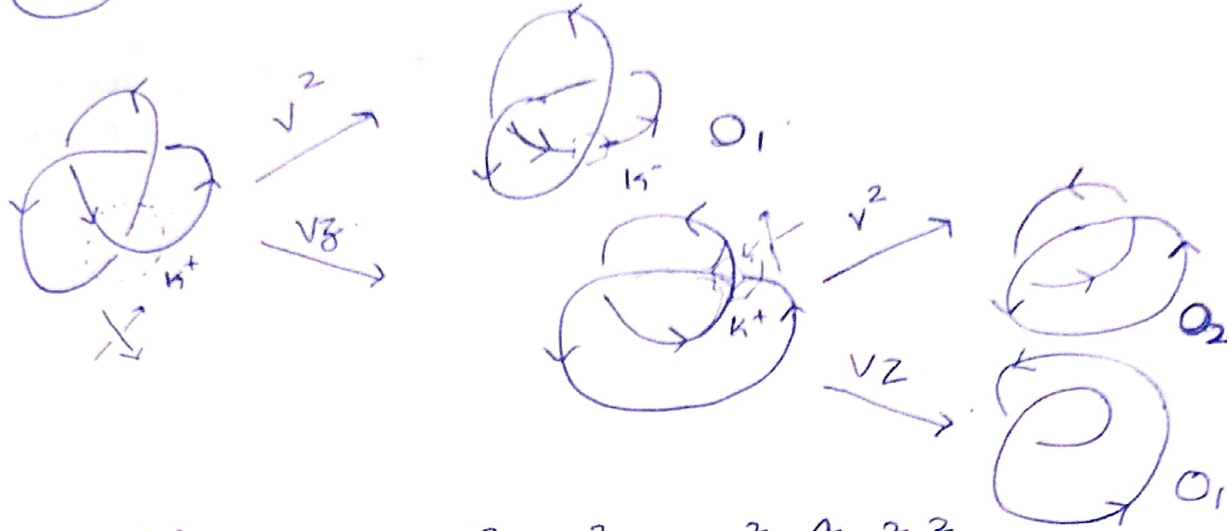
$$P_{O_1} = v^2 P_{O_1} + vz P_{O_2}$$

$$1 = v^2 + vz P_{O_2}$$

$$\boxed{\frac{1-v^2}{vz} = P_{O_2}}$$



$v^2 O_1 + vz$



$$\Rightarrow \left(\frac{1-v^2}{\cancel{vz}}\right) v^2 \cancel{vz} + (vz)^2 + v^2 = 2v^2 - v^4 + v^2 z^2$$

$$\frac{1}{2}(v\text{-span}+1) = \frac{1}{\cancel{2}}(\cancel{2}) + 1$$

$$= 2$$

$$\boxed{b(K) = 2}$$

<u>H.W:-</u> check $(*)$ for 4_1, 5_1.

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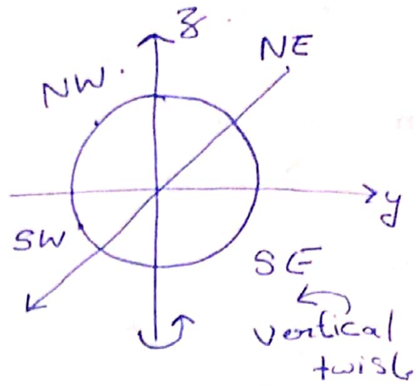
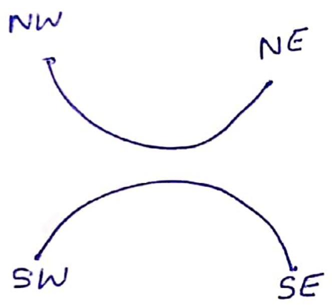
Rational Knots:-

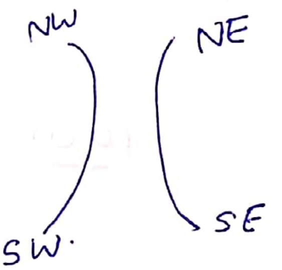
$$S^2 = \{x^2+y^2+z^2=1\}.$$



Trivial Tangles:



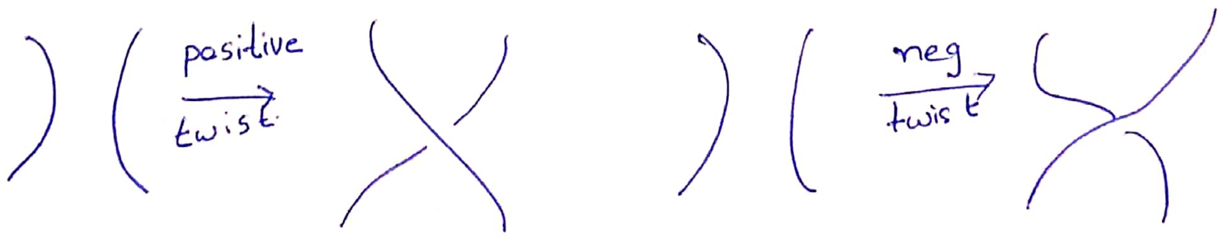
O-type trivial tangle.



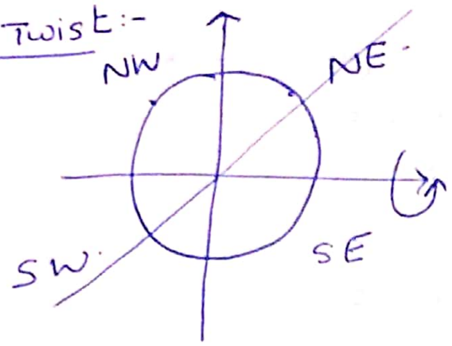
0-0 type trivial tangle.

Vertical Twist:

Keep north hemisphere and south pole fixed and rotate 180° about the z-axis SW & SE will exchange portion.



Horizontal Twist :-



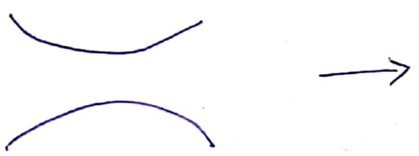
NE & SE exchange portions.

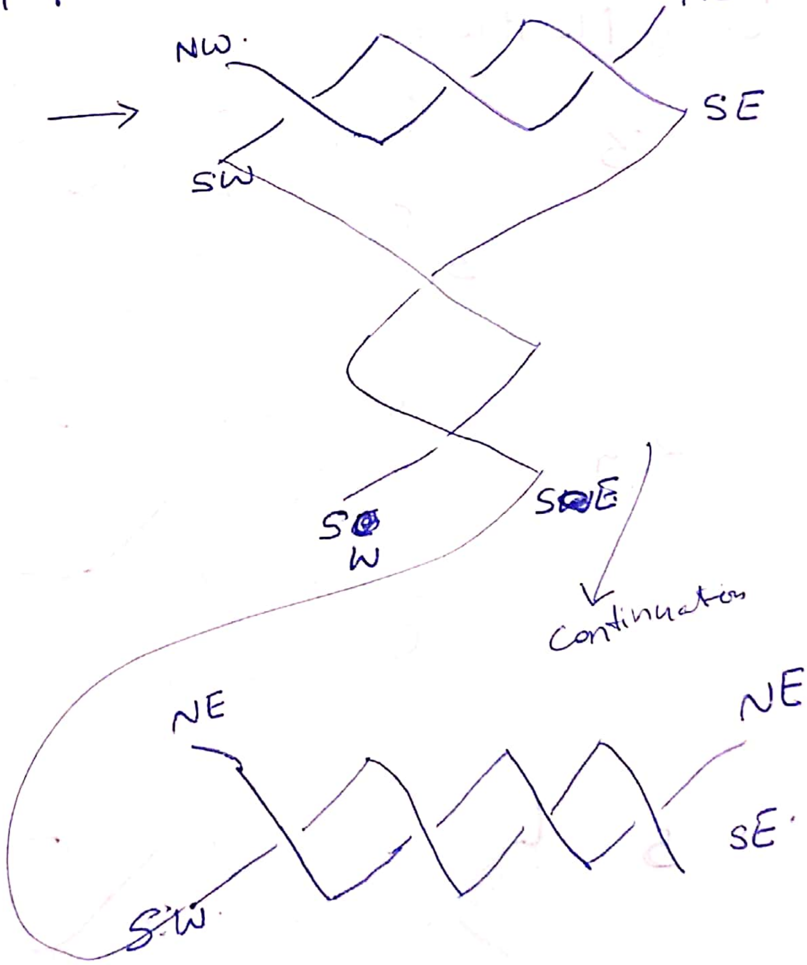
Rational Tangles :-

$$T(a_1, a_2, a_3 \ldots a_n).$$

if n is odd, start with $T(0)$ perform a_1 horizontal twists, then perform a_2 vertical twists and etc.,

Eg :- $T(3, 2, 4)$ perform horizontal twists 3 times ~~& vertical &~~





Connect NW to NE
SW to SE

by non-interesting arcs. This will give a knot or link. $T(a_1 \ldots a_n)$.

Case 2: n even.

Start with trivial tangle $T(0,0)$

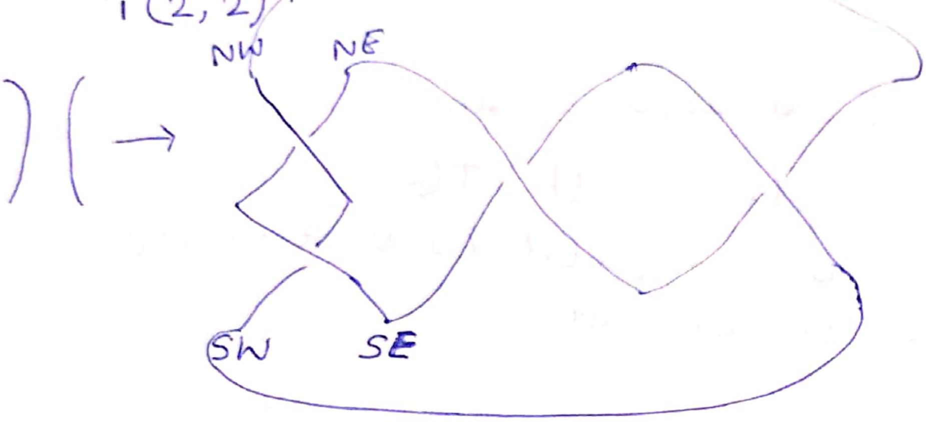
perform a_1, vertical twist

a_2 hori. twist

⋮

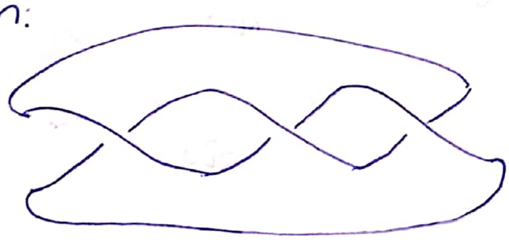
a_n horizontal twist

$T(2,2)$.



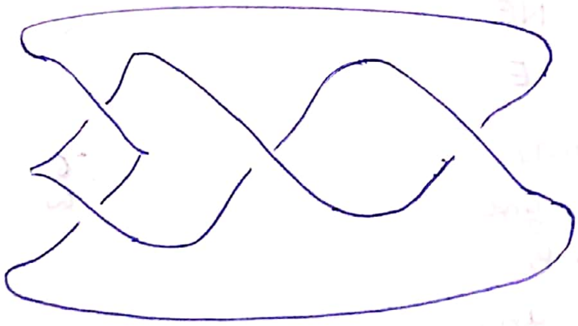
Conway Notation:

3

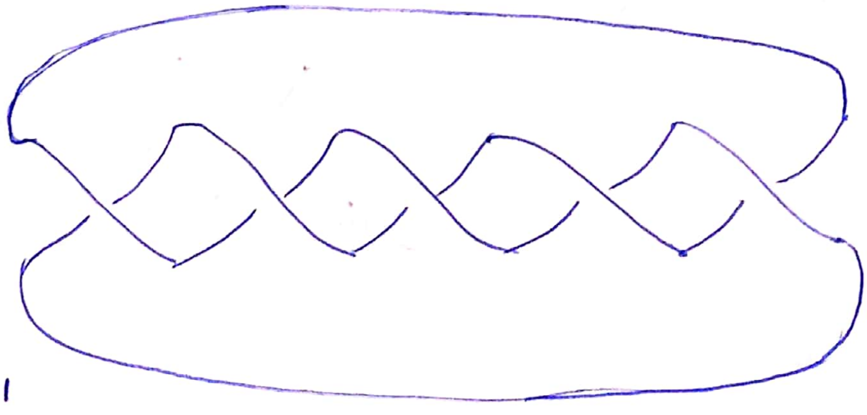


3_1

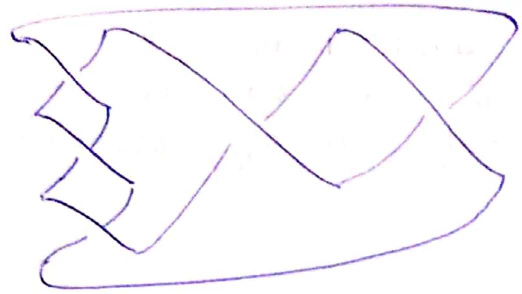
2 2.



5



5_1



only Knots that cannot be obtained in this manner are (up to 8 crossings)

$8_{21}, 8_{20}, 8_{19}, 8_{18}, 8_{17}, 8_{16}, 8_{15}, 8_{10}$ & 8_5.

let $T(a_1, a_2 \ldots a_n)$ be an n-tangle consider the associated rational number.

$$[a_n, a_{n-1} \ldots a_2, a_1] = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2}}}.$$

Eg:- $[2, 2] = 2 + \frac{1}{2} = 5/2.$

② $T[7, 2, 1, 2] =$ $\cancel{7 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}}$

$[2, 1, 2, 7] \Rightarrow$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{7}}}$$

$$= 59/22$$

③ $T[-7, -3, 3]$.

$$[3, -3, -7] = 3 + \cfrac{1}{-3 - \cfrac{1}{7}} = \frac{59}{22}$$

* Tangles $T(a_1, \dots a_n)$ and $T(b_1 \dots b_m)$ are said to be equivalent, if we can convert one to the other by finitely many elementary knot moves.

$T(a_1, \dots a_n)$ corresponds to the fraction.

$$\frac{p}{q} = [a_n, a_{n-1} \dots a_2, a_1].$$

*theorem :-

if $T(a_1 \dots a_n) \simeq T(b_1 \dots b_n)$ then their corresponding fractions are equal

$$[a_n, a_{n-1}, \dots a_2, a_1] = [b_n, b_{n-1} \dots b_2, b_1]$$

The converse also holds

$T(-7, -3, 3)$ $T(7, 2, 1, 2)$

Thm :- Suppose K, K' are rational knots having rational number $\frac{\alpha}{\beta}$ & $\frac{\alpha'}{\beta'}$ respectively.

K and K' are equal, if and only iff

i) $\alpha = \alpha'$; $\beta \equiv \beta' \pmod{\alpha}$

ii) $\alpha = \alpha'$; $\beta\beta' \equiv 1 \pmod{\alpha}$.

$$K = \frac{\alpha}{\beta}, \qquad K^* = \frac{-\alpha}{\beta} = \frac{\alpha}{-\beta}$$

$$K \simeq K^*$$

$$-\beta^2 \equiv 1 \pmod{\alpha}.$$

eg:- $[2, 2] = 5/2 \cdot \left(\frac{\alpha}{\beta}\right)$

$[-2, -2] = -2 - \frac{1}{2}$

$= -5/2 \equiv \frac{5}{-2} \cdot \left(\frac{\alpha'}{\beta'}\right)$

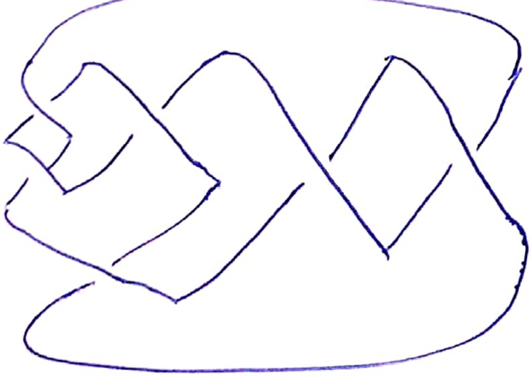
$\alpha = \alpha' = 5$, $\beta = 2$, $\beta' = -2$.

$\beta\beta' = -4 \equiv 1 \pmod{5}$.

eg ii). $6_3 = T(2, 1, 1, 2)$.

$$[2, 1, 1, 2] = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

$$= 13/5 \left(\frac{\alpha}{\beta}\right).$$



$T(-2, -1, -1, -2)$

$$[-2, -1, -1, -2] = -2 + \cfrac{1}{-1 + \cfrac{1}{-1 + \cfrac{1}{-2}}}$$

$$= -13/5 \equiv \frac{13}{-5} \left(\alpha'/\beta'\right).$$

$\alpha = \alpha' = 13$.

$\beta = 5$, $\beta' = -5$.

$5 \times (-5) \equiv 1 \pmod{13}$.

$-25 \equiv 1 \pmod{13}$.

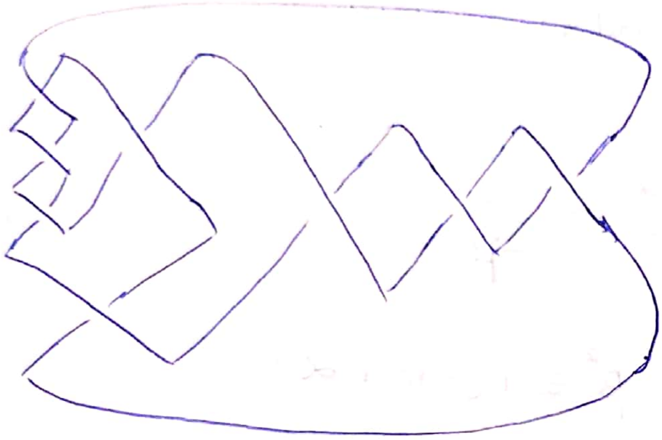
3) 8_9 $\quad T(3,1,1,3)$ $\qquad$ $T(-3,-1,-1,-3)$

$$[3,1,1,3] = 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}$$

$$= 25/7 \;(\alpha/\beta)$$

$$[-3,-1,-1,-3] = -3+\cfrac{1}{-1+\cfrac{1}{-1+\cfrac{1}{-3}}}$$

$$= \frac{-25}{7} \equiv \frac{25}{-7}\;(\alpha'/\beta')$$



$\alpha = \alpha' = 25.$

$\beta = 7, \; \beta' = -7$

$\beta \cdot \beta' \equiv 1 \mod (\alpha)$

$-49 \equiv 1 \mod (25).$

Verify for $T(n,1,1,n)$.

$$[n,1,1,n] = n+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{n}}} = n+\frac{n+1}{2n+1}$$

$$= \frac{2n^2+2n+1}{2n+1}$$

$$= \frac{2n^2+2n+1}{2n+1}\;(\alpha/\beta).$$

$$\beta^2 = (2n+1)^2$$

$$= 4n^2+4n+1$$

$\therefore \quad \beta^2 \equiv 1 \mod (\alpha).$

$4_1 = T(1, 1, 1, 1)$.

$$[1, 1, 1, 1] = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}$$

$$= 5/3.$$

$T(2, 2)$

$[2, 2] = 5/2$.

$\frac{\alpha}{\beta} \cong \frac{\alpha'}{\beta'}$ i) $\alpha = \alpha'$

ii) $\beta\beta' \equiv 1 \bmod (\alpha)$

$\alpha = \alpha' = 5$.

$2 \cdot 3 \equiv 1 \bmod (5)$

$6 \equiv 1 \bmod (5)$.

Tangles. $T(1,1,1,1)$ & $T(2,2)$ represent same knot or link.

31/1/19.

— Munkres – Topology
2^{nd} Edition

Fundamental Group (Topological invariant):

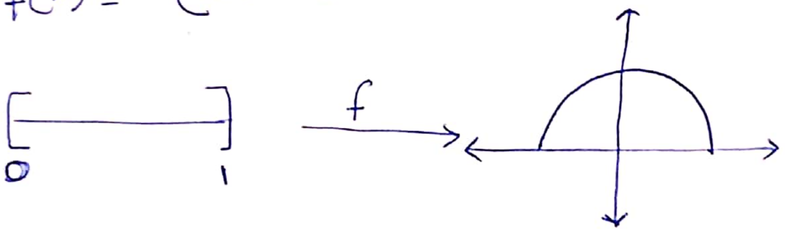
Defn (Homology): Let f, f' are cont. maps of the space X (subset of $\mathbb{R}^n$) into the space Y (subset of $\mathbb{R}^m$) we say that f is homotopic to f'. if there is a cont. map.

$$H : X \times I \to Y \text{ s.t } H(x,0) = f(x).$$

(H is called the homotopy b/w f & f'). $H(x, t)$ & $H(x,1) = f'(x)$

$f : [0,1] \to \mathbb{R}^2$

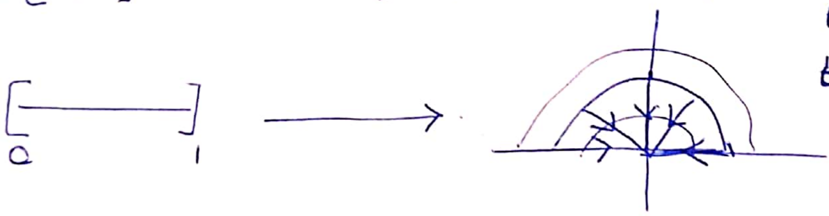
$f(x) = (\cos \pi x, \sin \pi x)$.



$H \quad I \times I \to \mathbb{R}^2$

& $H(x,t) = (1-t) f(x)$

$f' : [0,1] \to \mathbb{R}^2$; $f'(x) = (0,0)$.



$t=0 \quad H(x,0) = f(x)$.

$t=1/4 \quad H(x,1/4) = \frac{f(x) \times 3}{4}$.

$t=1/2 \quad H(x,1/2) = f(x)/2$.

$t=3/4 \quad H(x,3/4) = f(x)/4$

$t=1 \quad H(x,1) = 0$.

→ Homotopy is an equivalent relation on the set of cont. functions $f : X \to Y$

Reflexive :- $f \sim f$

$$H : X \times I \to Y$$
$$H(x,t) = f(x)$$

Symmetry :- if $f \sim f'$ then $f' \sim f$

If $f \sim f' \Rightarrow \exists\ H : X \times I \to Y$ at $H(x,0) = f(x)$
$H(x,t) = f'(x)$

now consider
$H' : X \times I \to Y$ at $H'(x,t) = H(x, 1-t)$

$$H'(x,0) = H(x,1) = f'(x)$$
$$H'(x,1) = H(x,0) = f(x).$$

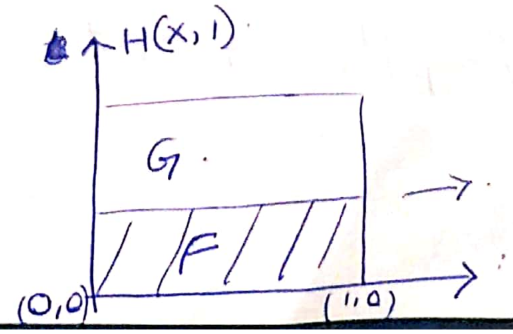
Transitive :-
if $f \sim f'$ & $f' \sim f''$
then $f \sim f''$

let
$$\left.\begin{array}{l} F(x,0) = f(x) \\ F(x,1) = f'(x) \end{array}\right\} f \sim f'.$$

$$\left.\begin{array}{l} G(x,0) = f'(x) \\ G(x,1) = f''(x) \end{array}\right\} f' \sim f''.$$

Define $H : X \times I \to Y$

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1. \end{cases}$$



Defn:- (path Homology).

path ; $f : [0,1] \to X$; f is cont.

&

$f(0) = x_0$ starting pt of path.
$f(1) = x_1$ ending pt of path.

Let $f, f' [0,1] \to X$ be cont. ~~points~~ paths in X

Such that $f(0) = f'(0) = X_0$

&

$f(1) = f'(1) = X_1$.

$\Rightarrow$ f is said to be path homotopic ~~of~~ to f'
if $\exists$ $H : \cancel{X} I \times I \to X$ Such that
$H(x,0) = f(x)$ $H(x,1) = f'(x)$.

s.t. $H(0,t) = \cancel{f} X_0$ $H(1,t) = X_1$.

$\rightarrow$ $f \sim_p f'$, f is path homotopic to f'.

$\rightarrow$ $\sim_p$ is an equivalence reln,

Reflexive.-

$f \sim_p f$. $H(x,t) = f(x)$.

Symmetry:-

if $f \sim_p f'$ then $f' \sim_p f$.

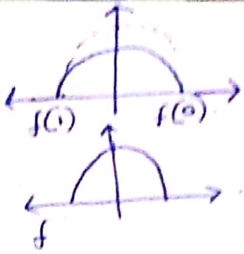
$H'(x,t) = H(x, 1-t)$.

Transitive:-

$f \sim_p f'$, $f' \sim_p f''$ then $f \sim_p f''$

Ex:-

$$f(x) = (\cos\pi x, \sin\pi x)$$
$$f'(x) = (\cos\pi x, 2\sin\pi x)$$

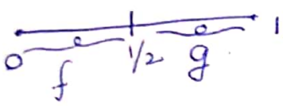


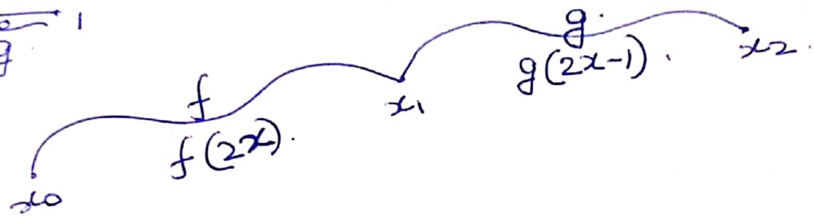
$$H(x,t) = (1-t)f(x) + tf'(x).$$

Defn: If f is a path in X from x_0 to x_1 & g is a path in X from x_1 to x_2.
we define the composition

$$f * g = h(x) = \begin{cases} f(2x) & 0 \le x \le 1/2 \\ g(2x-1) & \frac{1}{2} \le x \le 1. \end{cases}$$

h.



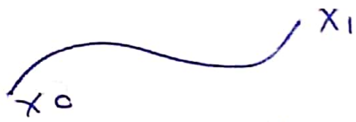


Defn:- $f * g$ is well defined on path equivalent classes and

i). associativity $[f] * ([g] * [h]) = ([f] * [g]) * [h]$

2) Right & left identity.

$[f] * [e_{x_1}] = [f]$ & $[e_{x_0}] * [f] = [f]$.



$e_{x_0} : I \to X_0$

$e_{x_1} : I \to X_1$.

iii). Given $f: I \to X$.
s.t $f(0) = x_0$, $f(1) = x_1$.

inverse of $\bar{f} = f(1-x)$.

iii). inverse $[f] * [\bar{f}] \simeq e_{x_0}$

$$[\bar{f}] * [f] = [e_{x_1}].$$

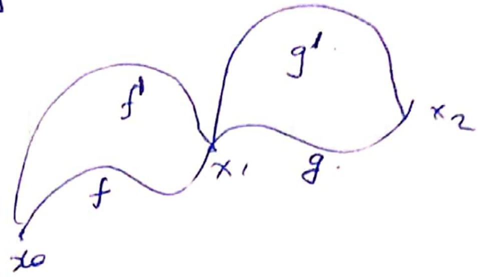
Thm :- $*$ is well defined

proof :- $[f] * [g] = [f * g]$.

To show

if $f' \sim_p f$ $g' \sim_p g$.

then $f * g \simeq_p f' * g'$.



$H : I \times I \to X$	$G : I \times I \to X$.
$H(x, 0) = f(x)$	$G(x, 0) = g(x)$
$H(x, 1) = f'(x)$.	$G(x, 1) = g'(x)$.

$F : I \times I \to X$.

$$F(x, t) = \begin{cases} H(2x, t), & 0 \le x \le 1/2 \\ G(2x-1, t), & 1/2 \le x \le 1. \end{cases}$$

$F(x, 0) = f(2x)$, $0 \le x \le 1/2$.

$= g(2x-1)$ $\frac{1}{2} \le x \le 1$.

(Munkers. Sec. 51
1, 2, 3)

5/2/19

prob: Given spaces X & Y, let [X, Y] denote the set of homology of classes of cont. maps of X into Y. Let $I = [0, 1]$ S.T for any X, thet set [X, I] has a single element.

$h \in [X, Y]$
$h : X \to Y$;
$h' : X \to Y$, $h \sim h'$
$[h] = [h']$.

Sol:-

$f : X \to I$ s.t $f(x) = 0 \; \forall x \in X$.

claim: For any $g : X \to I$ such that it is continous will be homotopic to $f(x)$.

Define, $H(x, t) = (1-t) f(x) + t g(x)$.

$H(x, 0) = f(x)$,

$H(x, 1) = g(x)$

$H(x, t)$ is continous as f & g are continous.

Since, this is a convex combination of f & g, $H(x, t) \in I$ for any $x \in X$ s.t $\in [0, 1]$.

b). show that Y is path connected, the set [I, Y] has a single element.

Sol:- $f_1, f_2 \quad I \to Y$ $\qquad H : I \times I \to Y$.

let $f_2(X) = f_2(0)$

let $H(X, t) = f_2((1-t)x)$

$H(X, 0) = f_2(x)$.

$H(X, 1) = f_2(0) = \cancel{x_0}\, C_0$

$H'(X, t) = f_1((1-t)x)$.

$H'(X, 0) = f_1(X)$

$H(X, 1) = f_1(0) = C_0'$

$H'' : I \times I \to Y$.

$H''(X, t) = r(t)$.

$H''(X, 0) = r(0) = C_0$

$H''(X, 1) = r(1) = C_0'$

Let r be path connected from C_0 to C_0' $\qquad r(0) = C_0$, $r(1) = C_0'$.

prob 3. A space X is contractable if the identity map $i: X \to X$ is a homotype to a cont. map

a) show that $I: [0,1]$ and R are contractable.

Sol:-

$$f(x) = x \qquad x \in [0,1]$$

$$H(x,t) = f((1-t)x).$$

$$H(x,0) = f(x)$$

$$H(x,1) = f(0) = c_0 \quad \text{for } [0,1].$$

$$g(x) = x \qquad x \in R.$$

$$H'(x,t) = g[(1-t)x].$$

$$H'(x,0) = g(x)$$

$$H'(x,1) = g(0) = c_0 \quad \text{for } R.$$

b) show if X is contractable; X is path connected.

Sol:- suppose X be contracted, to some point $x_0 \in X$. There exist a homotype H, such that for any $x \in X$, $H(x,0) = x$, $H(x,1) = x_0$. $(H: X \times I \to X)$

Define $\Gamma_x(t) = H(x,t)$ then Γ_x is a path joining x to x_0, for any $x_1, x_2 \in X$ we have Γ_{x_1}, Γ_{x_2} joining them to x_0,

Now $\Gamma_{x_1} * \Gamma_{x_2}^{-1}$ is path joining x_1 & x_2.

c) if Y is contractable; then for any X, the set $[X,Y]$ has a single element.

Sol:- let $f: X \longrightarrow Y$

Given Y contractable, $\Rightarrow \exists\, H: Y \times I \to Y$.

$$H(y,0) = y$$

$$H(y,1) = y_0$$

$G \cancel{X \to} I$

$G : X \times I \longrightarrow Y$

$G(x, t) = H(f(x), t)$.

$G(x, 0) = f(x)$.

$G(x, 1) = e_{y_0}$

$e_{y_0} : X \to Y$.

$e_{y_0}(x) \to y_0$.

d). If X is contractable and Y is path connected then $[X, Y]$ has a single connected.

Sol:- Given X is contractable, $\Rightarrow \exists\ H : X \times I \to X$

$H(x, 0) = x$

$H(x, 1) = x_0$.

claim $f \sim g$ where $g(x) = f(x_0)$ $\forall x \in X$

consider $H(x, t) = f[H(x, t)]$, $H'(x, 0) = f(x)$

$H'(x, 1) = f(x_0)$.

$f \sim g$. consider $f' \sim g'$, where $g'(x) = f'(x_0)$ $\forall x \in X$.

Consider path from $f(x_0)$ to $f'(x_0)$.

$e_{f(x_0)} \sim e_{f'(x_0)}$.

$$E = \frac{-m_e q^4}{8h^2 \epsilon_0^2} \qquad E = \frac{-m_e q^4}{8h^2 \epsilon_0^2}$$

$N_A >> n_D >> N_i$

$$E = -\frac{m_e q^4}{8h^2 \epsilon_0^2} \times \frac{1}{\left(1 + \exp\left[\frac{E - E_F}{kT}\right]\right)}$$

$n = N_D$

$p = \frac{n_i^2}{N_D}$

$n + N_A^- = p + N_D^+$ $\qquad np = n_i^2$

$p = \frac{n_i^2}{n}$

$N_A >> N_D >> n_i$

$p = N_A$

$N = \frac{n_i^2}{N_A}$

$$\frac{a}{\sqrt{h^{1/2}\,l}}$$

$n^2 + nN_A^- = n_i^2 + nN_D^+$

$n_i = \sqrt{N_c N_v}\, e^{\left[-\frac{E_g}{2kT}\right]}$

$n = n_i \exp\left(\right.$

$n\lambda r$ $\qquad \boxed{n\lambda = d\sin\theta}$

$K = \sqrt{\frac{2mE}{\hbar}}$

5 4 2

$\frac{1}{5}, \frac{1}{4}, \frac{1}{2}$

$\boxed{4, 5, 10}$

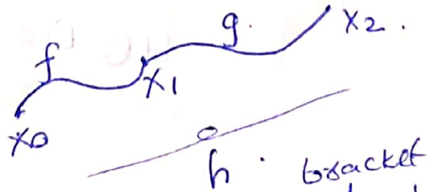
$$\frac{K^2 \hbar}{2m_e} = E$$

7/1/19. $\qquad$ 628..

Defn:- Let f be a path in X from x_0 to x_1; and g ~~be~~ a path from x_1 to x_2, we define a composition $f * g$ given by the part h, as.

$$h(s) = \begin{cases} f(2s) & \text{for } 0 \le s \le 1/2 \\ g(2s-1) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

✓.



→ well defined on path homotopy classes

$[f]$ ← bracket here denotes path equivalence classes of f.

$$[f] * [g] = [f * g].$$

claim:- $(*)$ satisfies groupoid properties

$[f] * [g]$ defined only for paths f, g

s.t $f(1) = g(0)$

Thm:- $*$ is a well-defined homtopy classes and satisfies

1) associativity. $[f] * ([g] * [h]) \cong ([f] * [g]) * [h]$.

2) Identity,

Let e_x denote constant path from $I \to X$;

$e(s) = x$.

then $[f] * [e_{x_1}] = [f]$;

$[e_{x_0}] * [f] = [f]$.

③ We define $\bar{f}(s) = f[1-s]$;

$\bar{f}$ ↓ reverse of f.

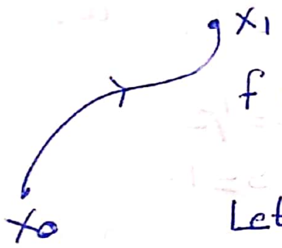
then $[f] * [\bar{f}] = [e_{x_0}]$

$[\bar{f}] * [f] = [e_{x_1}]$.

$$\boxed{f * \bar{f} = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ \bar{f}(2(1-s)) & \frac{1}{2} \le s \le 1 \end{cases}}$$

proof of ③:-

We will prove $f * \bar{f} \simeq_p e_{x_0}$.



f reverse is same ~~map~~ path going backward.

Let $H(s,t) : I \times I \to X$ defined by

$$H(s,t) = \begin{cases} f(2ts), & 0 \le s \le 1/2. \\ f(2t(1-s)), & \frac{1}{2} \le s \le 1. \end{cases}$$

$H(s,1) = f * \bar{f}$

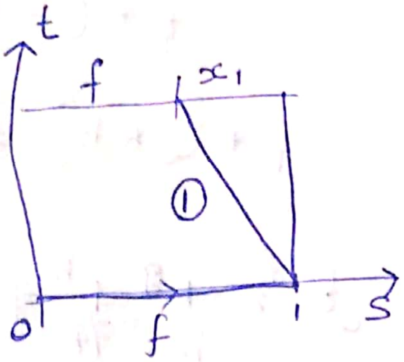
$$H(s, 1/2) = \begin{cases} f(s) & 0 \le s \le 1/2 \\ f(1-s) & 1/2 \le s \le 1. \end{cases}$$

$H(s, 0) = e_{x_0}$.

proof of 2 :-

We will prove $f \simeq_p f * e_{x_1}$

$$f * e_{x_1} = f(2s) \quad 0 \le s \le 1/2.$$
$$= x_1 \quad 1/2 \le s \le 1.$$



slope of Line ① $= \frac{1}{-1/2} = -2$.

Eqn. $t = -2s + c$.

$t=0, s=1, \quad c=2 \quad \therefore \Rightarrow t = -2s+2$

$$s = \frac{2-t}{2}.$$

$f : [0,1] \to X$.

$f' : [0, \alpha] \to X$. $\quad f'(s) = f\left(\frac{s}{\alpha}\right)$.

$H : I \times I \to X$.

$$H(s,t) = \begin{cases} f\left(\frac{s}{(2-t/2)}\right) & 0 \le s \le \frac{2-t}{2}. \\ x_1 & \frac{2-t}{2} \le s \le 1 \end{cases}$$

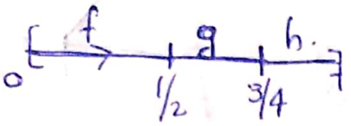
$$H(s,0) = f(s) \quad 0 \le s \le 1.$$

$$H(s,1) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ x_1 & \frac{1}{2} \le s \le 1 \end{cases}$$

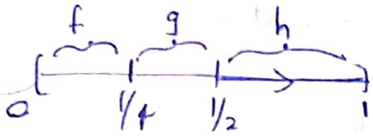
proof 1 :-

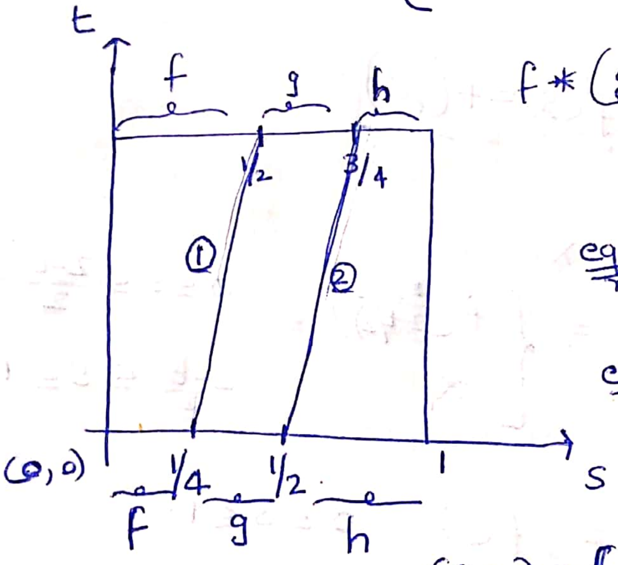
$$f * (g * h) \simeq_p (f * g) * h$$

$$f*(g*h) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(4(s-1/2)) & \frac{1}{2} \le s \le 3/4 \\ h(4(s-3/4)) & \frac{3}{4} \le s \le 1 \end{cases}$$



$$(f*g)*h = \begin{cases} f(4s) & 0 \le s \le 1/4 \\ g(4(s-\frac{1}{4})) & \frac{1}{4} \le s \le 1/2 \\ h(2(s-\frac{1}{2})) & \frac{1}{2} \le s \le 1 \end{cases}$$





$f*(g*h)$

$(f*g)*h$

eq^n of ① $t = 4s-1$

$\Rightarrow s = \left(\frac{t+1}{4}\right)$

eq^n of ② $t = 4s-2$

$s = \left(\frac{t+2}{4}\right)$

$$\boxed{\begin{array}{l} f : [0,1] \to X \\ f' : [0,\alpha] \to X \\ f'(s) = f(s/\alpha) \end{array}}$$

To show $(f*g)*h \simeq_p f*(g*h)$

the homotopy. $H : I \times I \to X$

$$H'(s,t) = \begin{cases} f\left(\dfrac{s}{\left(\frac{t+1}{4}\right)}\right) & 0 \le s \le \dfrac{t+1}{4} \\ g\left(4\left(s - \frac{t+1}{4}\right)\right) & \dfrac{t+1}{4} \le s \le \dfrac{t+2}{4} \\ h\left(\left(s - \frac{t+2}{4}\right) \times \dfrac{4}{2-t}\right) & \dfrac{t+2}{4} \le s \le 1 \end{cases}$$

$\cancel{\left(s - \frac{t+2}{4}\right) = \frac{2-t}{4}}$

$\cancel{s = \frac{2-t}{4} + \frac{t}{4}}$

Defn. Let $\alpha : [0,1] \to X$ be a continous ~~path~~ map such that $\alpha(0) = x_0 = \alpha(1)$.

based at x_0

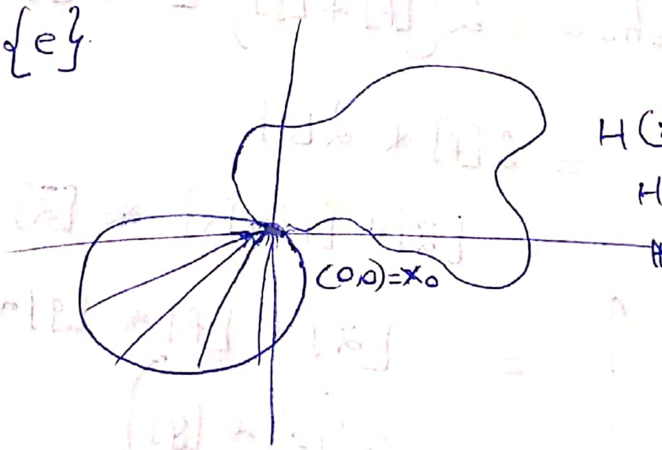
$\equiv$ 

Defn: X a space, $x_0 \in X$ Let α be a loop in X based at x_0; & let $[\alpha]$ be its path homotopy class of α.

$$[\alpha] * [\beta] = [\alpha * \beta] \text{ is a group.}$$

called the fundamental grp. $\pi_1(X, x_0)$ based at x_0.

Eg:- $\pi_1(\mathbb{R}^2, 0) = \{e\}$



$H(x,t) = (1-t)f(x)$

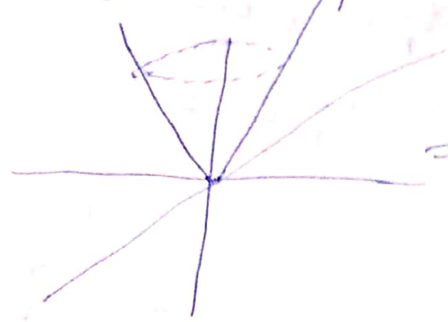
$H(x,0) = f(x)$.

$H(x,1) = 0$.

- C any convex subset of $\mathbb{R}^n$, $x_0 \in C$.

$\pi_1(C, x_0)$

D: $z^2 = x^2 + y^2 \subset \mathbb{R}^3 / z \geq 0$



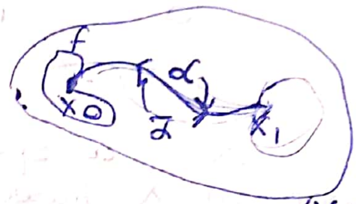
straight line homotype

$\Rightarrow$ choose base point wisely.

Thm:- Let X be path connected $x_0, x_1 \in X$ then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

proof:-

let α be a path from x_0 to x_1.



we define a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

Define: $\hat{\alpha}[f] = [\bar{\alpha}] * [f] * [\alpha]$

$[f] = \pi_1[X, x_0]$

$\hat{\alpha}$ is a group homomorphisom

To show $\hat{\alpha}([f] * [g]) = \hat{\alpha}[f] * \hat{\alpha}[g]$

$$= \hat{\alpha}[f] * \hat{\alpha}[g]$$

$$= [\bar{\alpha}] * [f] * \cancel{[\alpha]} * \cancel{[\bar{\alpha}]} * [g] * [\alpha]$$

$$= [\bar{\alpha}] * [f] * [g] * [\alpha]$$

$$= \hat{\alpha}([f] * [g])$$

$\hat{\alpha} \ \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$(\hat{\bar{\alpha}}) \cdot \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$

prove that $\hat{\alpha} \cdot (\hat{\bar{\alpha}}) =$ identity

and

$(\hat{\bar{\alpha}}) \cdot (\hat{\alpha}) = id.$

Section # 52
1, 2, 3, 4, 5