Problem Set 7

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation with $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

   Let $v \in \mathbb{R}^3$. What is $T(\mathbf{v})$? Is T one-one, as a function? Is T onto?

2. Change $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ in the above. What is $T(\mathbf{v})$? Is T one-one? Is T onto?

3. Can we ever find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ which is onto?

4. Find out $[\mathbf{v}]_{\mathcal{B}}$, where $\mathcal{B}$ is an ordered basis:

   (a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Solution:** $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$.

   (b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Solution:** $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

   (c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Solution:** $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

   (d) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$. **Solution:** $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$.

   (e) $\mathcal{B} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Solution:** $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

5. Find out $\mathbf{v}$ given $[\mathbf{v}]_{\mathcal{B}}$, where $\mathcal{B}$ is an ordered basis:

   (a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Solution:** $\mathbf{v} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$.

   (b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Solution:** $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$.

(c) $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. **Solution:** $\mathbf{v} = \begin{bmatrix}3\\4\\5\end{bmatrix}$.

(d) $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\}$, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\1\\2\end{bmatrix}$.

(e) $\mathcal{B} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

6. Give three linear transformations from $\mathbb{R}^3$ to $\mathbb{W} = \{\mathbf{w} : w_1 - w_2 + w_3 - w_4 + w_5 = 0\}$. Give their coordinate matrices w.r.t the ordered bases $\mathcal{B}_1 = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$, $\mathcal{B}_2 = \left\{ \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$ on $\mathbb{R}^3$ and some ordered basis of $\mathbb{W}$.

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ as $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x - y + z\\x + 2z\end{bmatrix}$. Find

   (a) a basis of Range (T),

   (b) rank (T),

   (c) a basis for $\mathcal{N}(T)$, and

   (d) $\dim(\mathcal{N}(T))$.

8. **(T)** Find all linear transformations from $\mathbb{R}^n \longrightarrow \mathbb{R}$.

   Solution: Let $T(\mathbf{e}_i) = \alpha_i, \quad 1 \le i \le n$. So

$$T(\mathbf{x}) = T\left(\sum_{i=1}^{n} x_i \mathbf{e}_i\right) = \sum_{i=1}^{n} x_i T(\mathbf{e}_i) = \sum_{i=1}^{n} \alpha_i x_i = \langle \mathbf{x}, [\alpha_1, \ldots, \alpha_n]^t \rangle.$$

   Also, given any $T : \mathbb{R}^n \longrightarrow \mathbb{R}$, linear, then we know the images of $\mathbf{e}_i$, for $1 \le i \le n$, the basis vectors. That is, there exists $\beta_i \in \mathbb{R}$ such that $T(\mathbf{e}_i) = \beta_i$, for $1 \le i \le n$. Thus,

$$T(\mathbf{x}) = \sum_{i=1}^{n} \beta_i x_i = \langle \mathbf{x}, [\beta_1, \ldots, \beta_n]^t \rangle.$$

9. Let $\mathbf{v} \in \mathbb{R}^n$ and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis of $\mathbb{R}^n$. Form a matrix $B = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n]$. Is $B[\mathbf{e}_1]_{\mathcal{B}} = \mathbf{e}_1$? What is $B\left[[\mathbf{e}_1]_{\mathcal{B}}, \ldots, [\mathbf{e}_n]_{\mathcal{B}}\right]$? Show that B is invertible and $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.
   Solution:

   Let $\mathbf{e}_1 = \sum\limits_{i=1}^{n} \alpha_i \mathbf{v}_i$. Then, $[\mathbf{e}_1]_{\mathcal{B}} = \begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{bmatrix}$. So, $B[\mathbf{e}_1]_{\mathcal{B}} = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n]\begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{bmatrix} = \sum\limits_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{e}_1$.

Hence, we see that $B[\mathbf{e}_i]_{\mathcal{B}} = \mathbf{e}_i$, for all $i, 1 \le i \le n$. Also,

$$B\left[[\mathbf{e}_1]_{\mathcal{B}}, \ldots, [\mathbf{e}_n]_{\mathcal{B}}\right] = [B[\mathbf{e}_1]_{\mathcal{B}}, \ldots, B[\mathbf{e}_n]_{\mathcal{B}}] = [\mathbf{e}_1, \ldots, \mathbf{e}_n] = I.$$

Thus, $B^{-1} = [[\mathbf{e}_1]_{\mathcal{B}}, \ldots, [\mathbf{e}_n]_{\mathcal{B}}]$. Further, $[\mathbf{e}_i]_{\mathcal{B}} = B^{-1}\mathbf{e}_i$, for all $i, 1 \le i \le n$ implies that $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.

10. Let V, W be vector spaces and let $L(V, W)$ be the vector space of all linear transformations from V to W. Show that $\dim L(V, W) = \dim V . \dim W$.

    Solution: Let $\dim(V) = m$ and $\dim(W) = n$. Then, there is a one-to-one correspondence between $L(V, W)$ and the vector space of $n \times m$ matrices. Therefore, $\dim(L(V, W) = mn = dim(V)dim(W)$.

11. **(T)** Show that a linear transformation is one-one if and only if null-space of $\mathcal{N}(T)$ is $\{0\}$.

    Solution: $\mathcal{N}(T) \neq \{\mathbf{0}\} \Rightarrow$ there is an $\mathbf{x} \in \mathcal{N}(T)$, $\mathbf{x} \neq \mathbf{0} \Rightarrow T(\mathbf{x}) = T(\mathbf{0}) \Rightarrow T$ is not one-one. If $\mathcal{N}(T) = \{\mathbf{0}\}$ then $T(\mathbf{x}) = T(\mathbf{y}) \Rightarrow T(\mathbf{x} - \mathbf{y}) = \mathbf{0} \Rightarrow \mathbf{x} - \mathbf{y} \in \mathcal{N}(T) \Rightarrow \mathbf{x} = \mathbf{y}$.

12. Describe all 2×2 orthogonal matrices. Prove that action of any orthogonal matrix on a vector $\mathbf{v} \in \mathbb{R}^2$, is either a rotation or a reflection about a line.

    Solution: As A preserves length, there is a $\theta \in [0, 2\pi)$ such that $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $(\sin^2\theta + \cos^2\theta = 1)$. Since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have, $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \perp A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore, $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ or $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$, which further implies that

    $$A = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation by an angle } \theta,} \quad or \quad A = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}}_{\text{reflection about a line of inclination } \theta/2.}$$

13. **(T)** Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $n \ge 2$, with $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$. Prove that there exist an orthogonal matrix A such that $A(\mathbf{v}) = \mathbf{w}$. Prove also that A can be chosen such that $\det(A) = 1$.
    (*This is why orthogonal matrices with determinant one are called rotations.*))

    Solution: As composition and inverse of orthogonal matrices are orthogonal, it is enough to choose $\mathbf{v} = \mathbf{e}_1$. So we need an orthogonal matrix A such that $A(\mathbf{e}_1) = \mathbf{w}$. Find an orthonormal basis of $\mathbb{R}^n$, say $\{\mathbf{w}, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ by Gram Schmidt. Let $A = \begin{bmatrix} \mathbf{w} & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix}$, that is, first column of A is $\mathbf{w}$ and so on. Then A is orthogonal and $det(A) = \pm 1$. If $det(A) = -1$, then multiply the second column with -1.