Department of Mathematics and Statistics
IIT Kanpur

Mid-Semester Examination Time: 2 Hours

You need to give proper reason(s) to get FULL MARKS

1. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 1 \end{bmatrix}$.

   (a) Obtain the RREF of A and use it to give a basis of $\mathcal{N}(A) = \textsc{Null Space}(A)$. 4 Marks

   Solution: $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -\frac{4}{7} \\ 0 & 1 & \frac{2}{7} \end{bmatrix}$ and basis of $\mathcal{N}(A)$ is $\left\{ \alpha \left[\frac{4}{7}, \frac{-2}{7}, 1 \right]^t \right\}$, for some $\alpha \neq 0$.
   $2 + 2$ Marks

   (b) Let $W = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0$. Determine the vector $[a, b, c]^t$ such that $W = \text{Column Space}(B) = \text{Col}(B)$. 3 Marks

   Solution: Consider $\begin{bmatrix} 1 & 2 & x \\ -1 & 0 & y \\ 3 & 1 & z \end{bmatrix}$. So, the required condition is $2(z + 3y) = x + y$ or equivalently, $x - 5y - 2z = 0$. Thus, $[a, b, c]^t = \alpha[1, -5, -2]^t$, for some $\alpha \neq 0$. 3 Marks

   (c) Determine a basis and dimension of $\mathcal{N}(A) \cap \text{Col}(B)$. 3 Marks

   Solution: As seen in the previous calculation, both the column vectors of B are orthogonal to the second row of A. So, $\left\{ \alpha \left[\frac{4}{7}, \frac{-2}{7}, 1 \right]^t \right\}$, for some $\alpha \neq 0$, is a basis of $\mathcal{N}(A) \cap \text{Col}(B)$. Hence, the dimension is 1.
   Marks to be given only when the student has found a basis of the intersection.
   3 Marks

2. Let $P = (3, 0, 2), Q = (1, 2, -1)$ and $R = (2, -1, 1)$ be three points in $\mathbb{R}^3$.

(a) Find the area of the triangle with vertices P, Q and R. 4 Marks

Solution: Note that $\overrightarrow{PQ} = (1, 2, -1) - (3, 0, 2) = (-2, 2, -3)$, $\overrightarrow{QR} = (2, -1, 1) - (1, 2, -1) = (1, -3, 2)$ and $\overrightarrow{RP} = (3, 0, 2) - (2, -1, 1) = (1, 1, 1)$. So, the triangle PQR is right angled at the vertex R. Hence,
$Area(PQR) = \frac{1}{2}\|\overrightarrow{QR}\| \cdot \|\overrightarrow{RP}\| = \frac{1}{2}\sqrt{14}\sqrt{3} = \frac{1}{2}\sqrt{42}.$ 4 Marks

Alternate:

$$\begin{aligned} \text{Area}(PQR) &= \frac{1}{2}\|\overrightarrow{PQ}\| \cdot \|\overrightarrow{QR}\| \text{ Sin}(\theta) = \frac{1}{2} \cdot \sqrt{17} \cdot \sqrt{14}\sqrt{1 - \frac{14^2}{17 \cdot 14}} \\ &= \frac{1}{2}\sqrt{42}. \end{aligned}$$

4 Marks

(b) Find a nonzero vector orthogonal to the plane of the triangle with vertices P, Q and R. 3 Marks

Solution: Note that we need to compute the normal vector of the plane containing the vectors $\overrightarrow{RP} = (1, 1, 1)$ and $\overrightarrow{QR} = (1, -3, 2)$. Or we need to find null space of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. It's basis is $\{\alpha[-5, 1, 4]^t\}$, for some $\alpha \neq 0$. So, any non-zero multiple of $[-5, 1, 4]^t$ will do. 3 Marks

(c) Determine all vectors $\mathbf{x}$ orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ with $\|\mathbf{x}\| = \sqrt{2}$. 3 Marks

Solution: As $\mathbf{x}$ is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, it is a multiple of $[-5, 1, 4]^t$. So, there are exactly two vectors $\alpha[-5, 1, 4]^t$, where $\alpha = \pm\frac{1}{\sqrt{21}}$. 3 Marks

3. Let $W = \{(x_1, x_2, \ldots, x_5) : x_1 + x_2 + x_3 + x_4 - 4x_5 = 0\}$ be a subspace of $\mathbb{R}^5$. Then, the set $\{\mathbf{u}_1 = [1,1,1,1,1]^t, \mathbf{u}_2 = [1,0,-1,0,0]^t, \mathbf{u}_3 = [1,-1,0,0,0]^t\}$ is a linearly independent set in W.

(a) Apply the Gram-Schmidt Orthogonalization process to the vectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$ (do not change the order of the vectors) to obtain an orthonormal set S such that $\mathrm{LS}(S) = \mathrm{LS}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. 3 Marks

Solution: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ then alternate solutions can be $S = \{\pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3\}$.
Clearly $\mathbf{v}_1 = \dfrac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \dfrac{1}{\sqrt{5}}[1,1,1,1,1]^t$, $\mathbf{v}_2 = \dfrac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \dfrac{1}{\sqrt{2}}[1,0,-1,0,0]^t$ and $\mathbf{v}_3 = \dfrac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \dfrac{1}{\sqrt{6}}[1,-2,1,0,0]^t$, where $\mathbf{w}_3 = \mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{v}_1\rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2\rangle \mathbf{v}_2$. 3 Marks

(b) Extend the set S to form an orthonormal basis of W. 3 Marks

Solution: Note that $\mathbf{u}_4 = [0,0,0,4,1]^t$ is an element of W and does not belong to $\mathrm{LS}(S)$. Hence, $\mathbf{w}_4 = \mathbf{u}_4 - \langle \mathbf{u}_4, \mathbf{w}_4\rangle \mathbf{w}_4 = [-1,-1,-1,3,0]^t$ and thus, $\mathbf{v}_4 = \pm\dfrac{1}{2\sqrt{3}}[1,1,1,-3,0]^t$. So, the required orthonormal basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. 3 Marks

(c) Determine the orthogonal projection of the vector $[1,0,1,0,1]^t$ on $\mathrm{LS}(S) = \mathrm{LS}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. 4 Marks

Solution: Let $\mathbf{x} = [1,0,1,0,1]^t$. Then, the projection vector is given by $\sum\limits_{i=1}^{3} \langle \mathbf{x}, \mathbf{v}_i\rangle \mathbf{v}_i$ which equals

$$\frac{3}{5}[1,1,1,1,1]^t + \frac{0}{2}[1,0,-1,0,0]^t + \frac{2}{6}[1,-2,1,0,0]^t = \frac{1}{15}[14,-1,14,9,9]^t.$$ 4 Marks

Alternate: Projection matrix equals $\sum\limits_{i=1}^{3} \mathbf{v}_i \mathbf{v}_i^t = \dfrac{1}{15}\begin{bmatrix} 13 & -2 & -2 & 3 & 3 \\ -2 & 13 & -2 & 3 & 3 \\ -2 & -2 & 13 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$ 3 Marks

Getting the projection vector $\frac{1}{15}[14,-1,14,9,9]^t$. 1 Mark

4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$. If 6 is one of the eigenvalues of A, then determine

(a) the characteristic polynomial of A. 2 Marks

Solution: The characteristic polynomial is given by $p(x) = x^3 - 6x^2 - 3x + 18$ or $p(x) = -x^3 + 6x^2 + 3x - 18$. 2 Marks

(b) the other eigenvalues of A. 2 Marks

Solution: The other eigenvalues are roots of $x^2 - 3$ and hence they are $\pm\sqrt{3}$. 2 Marks

(c) all the eigenvectors of A. $2 + 2 + 2$ Marks

Solution: The eigenpairs, for $\alpha \neq 0$, are

- $(6, \alpha[1, 1, 1]^t)$, 2 Marks
- $(\sqrt{3}, \alpha[-1 + \sqrt{3}, -1 - \sqrt{3}, 2]^t)$ or $(1.732, \alpha[.732, -2.732, 2]^t)$, 2 Marks
- $(-\sqrt{3}, \alpha[-1 - \sqrt{3}, -1 + \sqrt{3}, 2]^t)$ or $(-1.732, \alpha[-2.732, .732, 2]^t)$. 2 Marks

5. Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ be a real symmetric matrix with eigenvalues $1, 0$ and 3. Also, let the eigenvectors corresponding to 1 and 0 be $[1,1,1]^t$ and $[1,-1,0]^t$, respectively. Then, determine the values of a, c and f. 10 Marks

Solution:

- The condition 0 is an eigenvalue of A with eigenvector $[1,-1,0]^t$ implies $a = b = d$ and $c = e$. 3 Marks
- The condition 1 is an eigenvalue with eigenvector $[1,1,1]^t$ with the above condition implies $2a + c = 1, 2c + f = 1$. 3 Marks
- $\text{Trace}(A) = 2a + f = 4$ as $\text{Trace}(A) =$Sum of eigenvalues. 1 Mark
- Solving for a, c and f gives $a = \frac{5}{6}, c = -\frac{2}{3}$ and $f = \frac{7}{3}$. $1+1+1$ Marks

Alternate: As A is symmetric, there exists a unitary matrix U such that $U^*AU = D = Diagonal(\lambda_1, \lambda_2, \lambda_3)$. Thus, $A = UDU^*$.

Here, two vectors $[1,1,1]^t$ and $[1,-1,0]^t$ imply that the third eigenvector will be $[1,1,-2]$. 3 Marks

Let $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$ 3 Marks

Correct values of a, c and f 4 Marks.

If $U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$, then give 3 marks. As, due to NOT normalizing the columns of U, the student cannot get the correct answer and he loses 4 Marks.

6. Let $\mathbb{V}$ denote the vector space of all real polynomials of degree less than or equal to 3 and $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3 + a_2x + a_1x^2 + a_0x^3.$$

(a) Determine the matrix of the linear transformation, say A, with respect to the ordered basis $\mathcal{B} = \{1, 1+x, x^2+x^3, x^3\}$. 4 Marks

Solution: By definition

$$\begin{aligned} A = T[\mathcal{B},\mathcal{B}] &= \big[[T(1)]_{\mathcal{B}}, [T(1+x)]_{\mathcal{B}}, [T(x^2+x^3)]_{\mathcal{B}}, [T(x^3)]_{\mathcal{B}}\big] \\ &= \big[[x^3]_{\mathcal{B}}, [x^2+x^3]_{\mathcal{B}}, [T(1+x)]_{\mathcal{B}}, [T(1)]_{\mathcal{B}}\big] && \text{2 Marks} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. && \text{2 Marks} \end{aligned}$$

(b) Determine the characteristic polynomial of A. 3 Marks

Solution: $p(x) = x^4 - 2x^2 + 1$ 3 Marks

(c) Now, verify the Cayley-Hamilton Theorem (Hint: $A^2 = I$). 3 Marks

Solution: Statement of Cayley-Hamilton Theorem 1 Marks
Note that $A^2 = I$ and hence

$A^4 - 2A^2 + I = I - 2I + I = \mathbf{0}$. 2 Marks

Alternate: Even if NOT stated correctly, but correctly verified, give 3 Marks.