Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment II

1. $\star$ Solve the following separable equations
(i) $y' = \frac{1+y^2}{1+x^2}$ (ii) $\sqrt{1-x^2}y' + \sqrt{1-y^2} = 0$.

We write the equation $y' = \frac{1+y^2}{1+x^2}$ *as* $\frac{y}{1+y^2}' = \frac{1}{1+x^2}$. *Hence* $\frac{d}{dx}\tan^{-1} y(x) = \frac{d}{dx}\tan^{-1} x$. *Integrating this equation, we get* $\tan^{-1} y(x) = \tan^{-1} x + c$ *where* c *is a constant. Hence* $y = \tan(\tan^{-1} x + c)$. *We simplify this to* $y = \frac{\tan\tan^{-1} x + \tan c}{1 - \tan(\tan^{-1} x)\tan c} = \frac{x + \tan c}{1 - x\tan c}$.
For the second problem we write the equation as $\frac{y'}{\sqrt{1-y^2}} = -\frac{1}{\sqrt{1-x^2}}$. *The one can easily solve it as* $y = \sin(c - \sin^{-1} t)$ *and this can be further simplified as* $y(x) = \sin c\sqrt{1-x^2} - \cos cx$ *where* c *is a constant.*

2. $\star$ Solve the following non-linear equations by converting them in to a separable equation.
(i) $xy' - y = \sqrt{x^2+y^2}$ (ii) $(x - \sqrt{xy})y' = y$ (iii) $y' + \frac{2}{x}y = \frac{3x^2y^2+6xy+2}{x^2(2xy+3)}$ with $y(2) = 2$.

To solve the differential equation $xy' - y = \sqrt{x^2+y^2}$, *we let* $y = ux$. *Then* $y' = u'x + u$ *and the equation is converted in to* $u'x^2 = xy' - y = \sqrt{x^2+y^2} = x\sqrt{1+u^2}$. *We can write this as a separable equation* $\frac{u'}{\sqrt{1+u^2}} = \frac{1}{x}$ *and this can be solved easily.*

For the equation $(x - \sqrt{xy})y' = y$, *the substitution* $y = ux$ *transforms the equation as* $(1 - \sqrt{u})(u'x + u) = u$. *We can simplify this and write it as* $\frac{1-\sqrt{u}}{u\sqrt{u}} = \frac{1}{x}$. *This equation can be solved foru easily by re-writing again as* $\frac{u'}{u^{\frac{3}{2}}} - \frac{u'}{u} = \frac{1}{x}$.

To solve the equation $y' + \frac{2}{x}y = \frac{3x^2y^2+6xy+2}{x^2(2xy+3)}$ *observe that* $y = \frac{1}{x^2}$ *is a solution of the homogeneous equation* $y' + \frac{2}{x}y = 0$. *Therefore to solve the non-homogeneous equation, we let* $y = \frac{u}{x^2}$. *This reduces the equation to* $\frac{u'}{x^2} = \frac{3x^2(u^2/x^4)+6x(u/x^2)+2}{x^2(2(u/x)+3)}$. *This is a homogeneous equation. We let* $u = vx$ *and simplify the equation to get* $v'x = \frac{(v+1)(v+2)}{2v+3}$. *This equation can be written as* $\left(\frac{1}{v+1} + \frac{1}{v+2}\right)v' = \frac{1}{x}$. *Solving this equation, we get* $(v+1)(v+2) = cx$. *Since* $y(2) = 2$, *we get* $u(2) = 8$ *and* $v(2) = 4$. *Therefore* $(v+1)(v+2) = 15x$. *Solving this equation for* v, *we get* $v = \frac{-3+\sqrt{1+60x}}{2}$ *and* $y = \frac{-3+\sqrt{1+60x}}{2x}$.

3. $\star$ Solve the following initial value problems and find the maximal interval on which the solution is defined.
(i) $x^2y' = y^2 + xy - x^2$ with $y(1) = 2$ (ii) $y' = -2x(y^2 - 3y + 2)$ with $y(0) = 3$.

In this problem too we make the substitiution $y = ux$ *and simplify the equation to* $\frac{u'}{u^2-1} = \frac{1}{x}$. *We can integrate this to get* $\ln|\frac{u-1}{u+1}| = \ln x^2 + c$. *The initial condition* $y(1) = 2$ *implies that* $u(1) = 2$ *and this shows that* $c = \ln\frac{1}{3}$. *Substituting this we get* $\frac{u-1}{u+1} = \frac{1}{3}x^2$. *Hence* $\frac{y-x}{y+x} = \frac{x^2}{3}$ *and we simplify this further to get* $y = \frac{x^3+3x}{3-x^2}$ *and the solution is defined for* $x^2 \neq 3$.

The equation $y' = -2x(y^2 - 3y + 2)$ can be written as $y' = -2x(y-2)(y-1)$. Solving this equation, we get $\frac{y-2}{y-1} = ce^{-x^2}$. The initial condition $y(0) = 3$ gives the value of c as $\frac{1}{2}$. Therefore $y = \frac{4-e^{-x^2}}{2-e^{-x^2}}$ and this is defined on $(-\infty, \infty)$.

4. $\star$ Show that the equations
(i) $(4x^3y^3+3x^2)+(3x^4y^2+6y^2)y' = 0$ and (ii) $(ye^{xy}\tan x + e^{xy}\sec^2 x) + (xe^{xy}\tan x)y' = 0$.
are exact and solve them.

For the first problem let $M = 4x^3y^3 + 3x^2$ and $N = 3x^4y^2 + 6y^2$. Then $\frac{\partial M}{\partial y} = 12x^2y = \frac{\partial N}{\partial x}$. This shows that the equation is exact.
Now we want to find a function $\varphi(x, y)$ such that $\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$. Integrating the equation $\frac{\partial \varphi}{\partial x} = M$ with respect to x, we get $\varphi(x, y) = x^4y^3 + x^3 + h(y)$ where h is a function of y alone. If we integrate the equation $\frac{\partial \varphi}{\partial y} = N$ with respect to y, we get $\varphi(x, y) = x^4y^3 + 2y^3 + g(x)$ and g is a function of x. Comparing these two equations we get $h(y) = 2y^3$ and the solution is $\varphi(x, y) = x^4y^3 + x^3 + 2y^3$. Hence $\varphi(x, y) = c$ is the general solution of the given equation.
For the second problem, we follow the same method as in the earlier one and find that $\frac{d}{dx}(e^{xy}\tan x) = M(x, y) + N(x, y)y'$ with obvious notation for M and N. Thus the general solution is $\varphi(x, y) = e^{xy}\tan x$.
Let us obeserve that in both the cases expressing y in a closed form is not possible.

5. $\star$Find an integrating factor of

   (a) $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) + (3x^2y^2 + 4y)y' = 0$,
   (b) $2xy^3 + (3x^2y^2 + x^2y^3 + 1)y' = 0$ and
   (c) $\left(3xy + 6y^2\right) + \left(2x^2 + 9xy\right)y' = 0$.

and solve them.

(a) *In the first equation, we let $M = 2xy^3 - 2x^3y^3 - 4x^2y^2 + 2x$ and $N = 3x^2y^2 + 4y$. Then $\frac{\partial M}{\partial y} = 6xy^2 - 6x^3y^2 - 8x^2y$ and $\frac{\partial N}{\partial x} = 3x^2y^2 + 4y$. Let us now compute $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$. In our case $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = -2x$ is a fubnction of x alone. Hence $\mu(x) = \exp(-\int 2x\,dx) = \exp(-x^2)$ is an integrating factor of the equation $M + Ny' = 0$ and multiplication by $\mu(x)$ yields the exact equation $\exp(-x^2)(M + Ny') = 0$. We can now appeal to our methods to solve an exact equation to find out that $\varphi(x, y) = \exp(-x^2)(x^2y^3 + 2y^2) - 1) = c$ is a general solution of $\exp(-x^2)\left[(2xy^3 - 2x63y^3 - 4xy^2 + 2x) + (3x^2y^2 + 4y)y'\right] = 0$ and it is also a general solution of $(2xy^3 - 2x63y^3 - 4xy^2 + 2x) + (3x^2y^2 + 4y)y' = 0$.*
(b) *For the second problem let M and N denote the standard functions. In this case, we see that $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = -\frac{2xy^3}{3x^2y^2+x^2y^3+1}$ is not a function of x alone. However $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = 1$ is a function of y-alone! Hence we can integrate to find that $\mu(y) = e^y$ is an integrating factor of the equation $2xy^3 + (3x^2y^2 + x^2y^3 + 1)y' = 0$. In this case the general solution is given by $\varphi(x, y) = e^y(x^2y^3 + 1) = c$.*
(c) *In this case $M = 3xy + 6y^2$ and $N = 2x^2 + 9xy$. Therefore $M_y - N_x = -x + 3y$, $\frac{M_y - N_x}{M} = \frac{-x+3y}{3xy+6y^2}$, and $\frac{N_x - M_y}{N} = \frac{x-3y}{2x^2+9xy}$. Therefore we can't hope to get integrating factors μ as a function of x or y alone and we look for functions such that $M_y - N_x = p(x)N - q(y)M$. That is $-x + 3y = p(x)(2x^2 + 9xy) - q(y)(3xy + 6y^2)$. Since the left hand side is a linear polynomial, we re-write the equation as $-x + 3y = xp(x)(2x + 9y) - yq(y)(3x + 6y)$. This will be an identity, if $xp(x) = A$ and $yq(y) = B$ such that $-x + 3y = A(2x + 9y) - B(3x + 6y)$. That is $-x + 3y = (2A - 3B)x + (9A - 6B)y$. Solving this, we get $A = B = 1$. Therefore $p(x) = \frac{1}{x}$ and $q(y) = \frac{1}{y}$.*

Since $\int p = \ln|x|$ and $\int q = \ln|y|$, we let $P = x$ and $Q = y$. Hence $\mu(x,y) = P(x)Q(y) = xy$ is an integrating factor of the given differential equation and by multiplying with the function xy, we get the equation $(3x^2y^2 + 6xy^3) + (2x^3y + 9x^2y^2)y' = 0$ is exact. The function $F(x,y)x^3y^2 + 3x^2y^3 = c$ gives the general solution of the equation.

6. $\star$ Solve the initial value problem $y' + 2xy = -e^{-x^2}\left(\frac{3x+2ye^{x^2}}{2x+3ye^{x^2}}\right)$ with $y(0) = -1$.

   The function $y_1 = e^{-x^2}$ is a solution of $y' + 2xy = 0$. Therefore we let $y = ue^{-x^2}$ to be a solution of the given equation and get $u'e^{-x^2} = -e^{-x^2}\left(\frac{3x+2u}{2x+3u}\right)$. So $u' = -\left(\frac{3x+2u}{2x+3u}\right)$ and $(3x+2u) + (2x+3u)u' = 0$. This is an exact equation. Now we need to find a function $F(x,u)$ such that $F_x = 3x + 2u$ and $F_u = 2x + 3u$. Integrating $F_x = 3x + 2u$ with respect to x, we get $F(x,u) = \frac{3x^2}{2} + 2xu + \varphi(u)$. Since $F_u = 2x + 3u$, we get $2x + 3u = F_u = 2x + \varphi'(u)$. Therefore $\varphi'(u) = 3u$ and hence $\varphi(u) = \frac{3u^2}{2}$. This shows that $F(x,u) = \frac{3x^2}{2} + 2xu + \frac{3u^2}{2} = c$ is a general solution of $(3x+2u) + (2x+3u)u' = 0$. The initial condition $y(0) = -1$ shows that $u(0) = -1$. Hence $c = \frac{3}{2}$ and $3x^2 + 4xu + 3u^2 = 3$ is an implicit solution for $(3x+2u) + (2x+3u)u' = 0$. Solving for u, we get $u = -\left(\frac{2x+\sqrt{9-5x^2}}{3}\right)$ and therefore $y = -e^{-x^2}\left(\frac{2x+\sqrt{9-5x^2}}{3}\right)$.

7. $\star$ Find the Picard iterates of $y' = y$ with $y(0) = 1$.

   The first iterate of the equation is $y_1(x) = y_0(x) + \int_0^x y_0(s)ds = 1 + x$. Substituing this in the second iterate $y_2(x) = y_1(0) + \int_0^x y_1(s)ds$, we get $y_2(x) = 1 + x + \frac{x^2}{2}$. By induction one can show that the n-th iterate y_n is $y_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$.

8. $\star$ Find the first three Picard iterates of $y' = 1 + y^3$ with $y(1) = 1$.

   For the equation $y' = 1 + y^3$ with $y(1) = 1$, the first iterate is $y_1(x) = y_0(1) + \int_1^x (1 + y_0(s))ds = 1 + \int_1^x (1+1)ds = 1 + 2(\frac{x^2}{2} - \frac{1}{2}) = x^2$. The second iterate is $y_2(x) = y_1(1) + \int_1^x (1 + y_1^3(s))ds = 1 + \int_1^x (1 + s^6)ds = x + \frac{x^7}{7} - \frac{1}{7}$. Leave it as exercise for the students to find the third iterate.

9. Solve the equation $y' = \frac{ax+by+h}{cx+dy+k}$ where a, b, c, d, h and k are constants.

   Let us afirst assume that $ad - bc \neq 0$. We use the transformation $x = X + \alpha$ and $y = Y + \beta$ where α and β are real numbers such that $a\alpha + b\beta + h = 0$ and $c\alpha + d\beta + k = 0$ to reduce the equation to $Y' = \frac{aX+bY}{cX+dY}$. The right hand side is homogeneous and hence we use the transformation $Y = ZX$ to re write the equation as $Z'X + Z = \frac{a+bZ}{c+dZ}$. This equation now written as $(c+dZ)(Z'X+Z) = a + bZ$. Let us now observe that this can be written as $\left((cz + dz^2)X\right)' = a + bZ$. This can be easily solved.

10. Show that the separable equation $-y + (x + x^6)y' = 0$ can be converted to an exact equation by multiplying with an integrating factor.

    By multiplying with an integral factor $\mu(x,y) = \frac{1}{y(x+x^6)}$, the equation can be written as a separable equation $\frac{1}{y}y' = \frac{1}{x+x^6}$. This canbe solved easily.

11. Let a, b, c, $d \in \mathbb{R}$ be such that $ad - bc \neq 0$ and m, $n \in \mathbb{R}$. Show that the equation $(ax^my + by^{n+1}) + (cx^{m+1} + dxy^n)y' = 0$ has an integrating factor of the form $x^\alpha y^\beta$.

    This is compuatational and leave it as exercise for the students.

12. Construct the first two Picard iterates of $y' = (x^2 + y^2)$ with $y(0) = 1$.

The first iterate is $y_1(x) = y(0) + \int_0^x (s^2 + y(0)^2)ds = 1 + \frac{x^3}{3} + x = 1 + x + \frac{x^3}{3}$ *and the second iterate is* $y_2(x) = 1 + \int_0^x (x^2 + (1 + s + \frac{s^3}{3})^2 ds$. *Leave it as an exercise for the students to compute this integral.*

13. Construct the Picard iterates of $y' = 2x(y + 1)$ with $y(0) = 0$ and show that $y(x) \to e^{x^2} - 1$.

$y_1(x) = y(0) + \int_0^x 2s(y(0) + 1)ds = \int_0^x 2sds = x^2$
$y_2(x) = y(0) + \int_0^x 2s(y_1(s) + 1) = \int_0^x (2s^3 + 2s)ds = x^2 + \frac{(x^2)^2}{2}$
$y_3(x) = \int_0^x 2s(x^2 + \frac{(x^2)^2}{2} = \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3!}$. *Inductively assume that* $y_n(x) = \sum_{k=1}^{n} \frac{(x^2)^k}{k!}$. *Then* $y_{n+1}(x) = y(0) + \int_0^x 2s(\sum_{k=1}^{n} \frac{(x^2)^k}{k!} + 1) = \sum_{k=1}^{n+1} \frac{(x^2)^k}{k!}$. *This proves that the picard iterates* y_n *converge to* $\exp(x^2) - 1$ *and* $y(x) = \exp(x^2) - 1$ *is the solution of the IVP* $y' = 2x(y + 1)$ *with* $y(0) = 0$.

14. Show that the solution y of $y' = x^2 + e^{-y^2}$ with $y(0) = 0$ exists for $0 \leq x \leq \frac{1}{2}$ and $|y(x)| \leq 1$ for $0 \leq x \leq \frac{1}{2}$.

This is easy.

15. Show that $W := \{y : \mathbb{R} \to \mathbb{R} : y$ is a solution of $y' + py = 0\}$ is a vector space; here $p : \mathbb{R} \to \mathbb{R}$ is a continuous function. What is the dimension of W?

Easy.

16. Solve the given Berboulli equation.
(i) $7xy^6y' - 2y^7 = -x^2$ (ii) $x^2y' + 2y = 2e^{\frac{1}{x}}y^{\frac{1}{2}}$.

Let $w = y^7$. *Then* $w' = 7y^6y'$ *and the equation van be written as* $w'x - 2w = x^2$. *Now it is very easy to solve.*

The second one is easy.

17. Miscellaneous Problems. Solve the following equations.
(i) $y' + \frac{x}{1+x^2}y = 1 - \frac{x^3}{1+x^4}y$ (ii) $y' = k(a - y)(b - y)$ where a, $b > 0$
(iii) $y' = -y\sqrt{x}\sin x$.