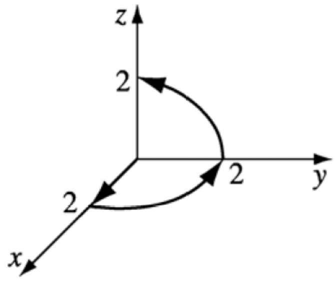
Problem 3.1: Compute the gradient and Laplacian of the function

$T = r(\cos\theta + \sin\theta\cos\phi)$

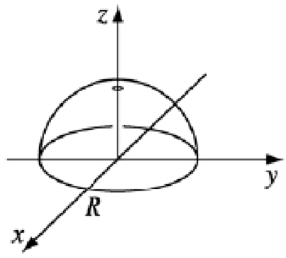
in spherical coordinates. Check the laplacian by converting T to Cartesian coordinates. Test the gradient theorem for this function from (0,0,0) to (0,0,2) along the path shown in figure.



Problem 3.2: Compute the divergence of the function

$\mathbf{v} = r\cos\theta\,\hat{\mathbf{r}} + r\sin\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\cos\phi\,\hat{\boldsymbol{\phi}}$.

Check the divergence theorem for this function using volume as the inverted hemispherical bowl of radius R resting on the xy plane and centred at the origin.

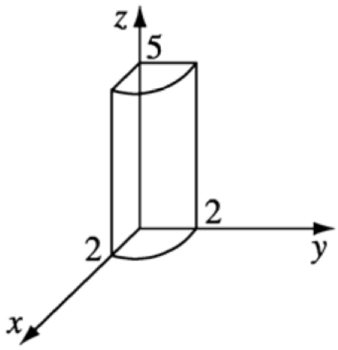


Problem 3.3: (a) Find the divergence of the function

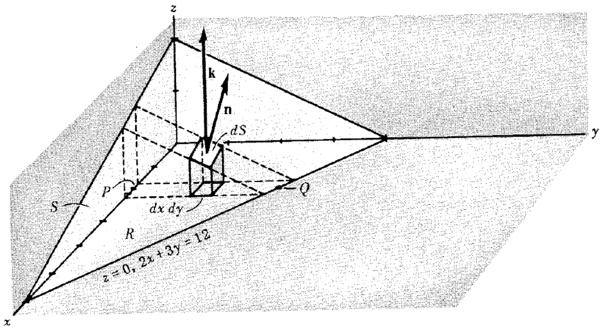
$\mathbf{v} = s(2+\sin^2\phi)\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\hat{\mathbf{z}}$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) as shown in the figure.

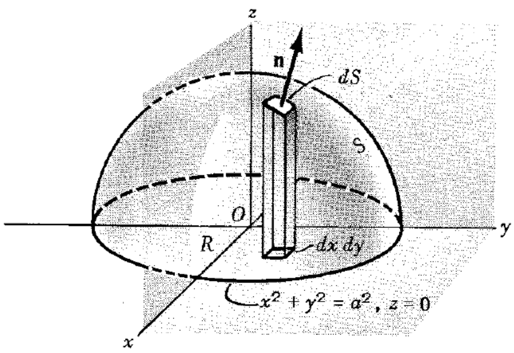
(c) Find the curl of **v**.



Problem 3.4: Evaluate $\iint_S \mathbf{A}.\mathbf{n}\,dS$ where $\mathbf{A} = 18z\hat{\mathbf{x}} - 12\hat{\mathbf{y}} + 3y\hat{\mathbf{z}}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant as shown in the figure.



Problem 3.5: Suppose $\mathbf{F} = y\hat{\mathbf{x}} + (x - 2xz)\hat{\mathbf{y}} - xy\hat{\mathbf{z}}$. Evaluate $\iint_S (\nabla \times \mathbf{F}).\mathbf{n}\,dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane as shown in the figure.



Problem 3.6: Let S be a closed surface and $\mathbf{r}$ denote the position vector of any point (x, y, z) measured from an origin O. Prove that

$$\iint_S \frac{\mathbf{n}.\mathbf{r}}{r^3} dS$$

is equal to (a) zero if O lies outside S; (b) 4π if O lies inside S.

Problem 3.7: A fluid of density $\rho(x, y, z, t)$ moves with velocity $\mathbf{v}(x, y, z, t)$. If there are no sources or sinks, prove that

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

where $\mathbf{J} = \rho\mathbf{v}$.