Solution of Assignment #3

Problem: 3·1

$$T = r(\cos\theta + \sin\theta\cos\phi)$$

$$\vec{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

$$= (\cos\theta + \sin\theta\cos\phi)\hat{r} + (-\sin\theta + \cos\theta\cos\phi)\hat{\theta} + \frac{1}{\cancel{\sin\theta}}(-\cancel{\sin\theta}\sin\phi)\hat{\phi}$$

$$\nabla^2 T = \vec{\nabla}\cdot(\vec{\nabla}T) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\cos\theta + \sin\theta\cos\phi)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta(-\sin\theta + \cos\theta\cos\phi)\right]$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-\sin\phi)$$

$$= \frac{1}{r^2}\cdot 2r(\cos\theta + \sin\theta\cos\phi) + \frac{1}{r\sin\theta}(-2\sin\theta\cos\theta + \cos^2\theta\cos\phi$$

$$- \sin^2\theta\cos\phi) - \frac{1}{r\sin\theta}\cos\phi$$

$$= \frac{1}{r\sin\theta}\left[\cancel{2\sin\theta\cos\theta} + 2\sin^2\theta\cos\phi - \cancel{2\sin\theta\cos\theta} + \cos^2\theta\cos\phi - \sin^2\theta\cos\phi - \cos\phi\right]$$

$$= \frac{1}{r\sin\theta}\left[(\sin^2\theta + \cos^2\theta)\cos\phi - \cos\phi\right] = 0$$

CHECK:

$r\cos\theta = z$

$r\sin\theta\cos\phi = x$

$\Rightarrow T = x + z$

$\Rightarrow \nabla^2 T = 0$

Gradient theorem:

$$\int_a^b \vec{\nabla}T\cdot d\vec{l} = T(b) - T(a)$$

segment 1

$\theta = \frac{\pi}{2}$, $\phi = 0$, $r: 0 \to 2$, $d\vec{l} = dr\,\hat{r}$

$$\vec{\nabla}T\cdot d\vec{l} = (\cos\theta + \sin\theta\cos\phi)dr = (0+1)dr = dr$$

$$\int \vec{\nabla}T\cdot d\vec{l} = \int_0^2 dr = 2$$

Segment 2 $\quad \theta = \frac{\pi}{2}$, $r = 2$, $\phi : 0 \to \frac{\pi}{2}$, $d\vec{l} = r\sin\theta \, d\phi \, \hat{\phi} = 2 d\phi \, \hat{\phi}$

$$\vec{\nabla}T \cdot d\vec{l} = (-\sin\phi)(2d\phi) = -2\sin\phi \, d\phi$$

$$\int \vec{\nabla}T \cdot d\vec{l} = -\int_0^{\pi/2} 2\sin\phi \, d\phi = -2$$

Segment 3 $\quad r = 2$, $\phi = \frac{\pi}{2}$, $\theta : \frac{\pi}{2} \to 0$, $d\vec{l} = r\,d\theta \, \hat{\theta} = 2d\theta \, \hat{\theta}$

$$\vec{\nabla}T \cdot d\vec{l} = (-\sin\theta + \cos\theta\cos\phi)\, 2d\theta = -2\sin\theta \, d\theta$$

$$\int \vec{\nabla}T \cdot d\vec{l} = -\int_{\pi/2}^{0} 2\sin\theta \, d\theta = 2$$

So, total $\displaystyle\int_a^b \vec{\nabla}T \cdot d\vec{l} = 2 - 2 + 2 = \underline{\underline{2}}$

Also, $T(b) - T(a) = 2(1+0) - 0(\cdots) = \underline{\underline{2}}$

Problem! 3.2

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)$$

$$= \frac{1}{r^2}3r^2\cos\theta + \frac{1}{r\sin\theta}r.2\sin\theta\cos\theta + \frac{1}{r\sin\theta}r\sin\theta(-\sin\phi)$$

$$= 3\cos\theta + 2\cos\theta - \sin\phi = \underline{5\cos\theta - \sin\phi.}$$

$$\int(\vec{\nabla}\cdot\vec{V})d\tau = \int(5\cos\theta - \sin\phi)r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^R r^2 dr \int_0^{\pi/2}\Big[\underbrace{\int_0^{2\pi}(5\cos\theta - \sin\phi)d\phi}_{2\pi(5\cos\theta)}\Big]\sin\theta\, d\theta$$

$$= \frac{R^3}{3}\cdot(10\pi).\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= \frac{R^3}{3}\cdot 10\pi.\ \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = \frac{5\pi R^3}{3}.$$

Hemisphere surface: $d\vec{a} = R^2\sin\theta\, d\theta\, d\phi\, \hat{r}$, $r = R$, $\phi: 0\to 2\pi$, $\theta: 0\to\frac{\pi}{2}$

$$\int\vec{V}\cdot d\vec{a} = \int(r\cos\theta)R^2\sin\theta\, d\theta\, d\phi = R^3\int_0^{\pi/2}\underbrace{\sin\theta\cos\theta\, d\theta}_{\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}}\int_0^{2\pi}d\phi$$

$$= R^3.\frac{1}{2}.2\pi = \pi R^3$$

Flat bottom surface: $d\vec{a} = (dr).(r\sin\theta\, d\phi)\,\hat{\theta}$, $\theta = \frac{\pi}{2}$, $r: 0\to R$, $\phi: 0\to 2\pi$

$$d\vec{a} = r\, dr\, d\phi\, \hat{\theta}$$

$$\int\vec{V}\cdot d\vec{a} = \int r\sin\theta.\ r\, dr\, d\phi = \int_0^R r^2 dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}$$

So total, $\int\vec{V}\cdot d\vec{a} = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}.$

Problem: 3.3

(a) $\vec{\nabla}\cdot\vec{V} = \frac{1}{s}\frac{\partial}{\partial s}(s.s(2+\sin^2\phi)) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)$

$= \frac{1}{s}\, 2s(2+\sin^2\phi) + \frac{1}{s}.s(\cos^2\phi - \sin^2\phi) + 3$

$= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3$

$= 8$

(b) $\int(\vec{\nabla}\cdot\vec{V})d\tau = \int 8.\, s\,ds\,d\phi\,dz = 8.\int_0^2 s\,ds \int_0^{\pi/2} d\phi \int_0^5 dz$

$= 8.2.\frac{\pi}{2}.5 = 40\pi$

Surface integral have five parts:

top: $z=5$, $d\vec{a} = s\,ds\,d\phi\,\hat{z}$, $\vec{V}\cdot d\vec{a} = 3z\, s\,ds\,d\phi = 15\, s\,ds\,d\phi$

$\int\vec{V}\cdot d\vec{a} = 15\int_0^2 s\,ds\int_0^{\pi/2} d\phi = 15\pi$

$\int\sin^2 x\,dx$
$= -\frac{1}{2}\cos x\sin x + \frac{x}{2}$
$= \frac{x}{2} - \frac{\sin 2x}{4}$

bottom: $z=0$, $d\vec{a} = -s\,ds\,d\phi\,\hat{z}$, $\vec{V}\cdot d\vec{a} = -3z\,s\,ds\,d\phi = 0$

$\int\vec{V}\cdot d\vec{a} = 0$

back: $\phi = \frac{\pi}{2}$, $d\vec{a} = ds\,dz\,\hat{\phi}$, $\vec{V}\cdot d\vec{a} = s\sin\phi\cos\phi\, ds\,dz = 0$

$\int\vec{V}\cdot d\vec{a} = 0$

left: $\phi=0$, $d\vec{a} = -ds\,dz\,\hat{\phi}$, $\vec{V}\cdot d\vec{a} = -s\sin\phi\cos\phi\,ds\,dz = 0$

$\int\vec{V}\cdot d\vec{a} = 0$

front: $s=2$, $d\vec{a} = s\,d\phi\,dz\,\hat{s}$, $\vec{V}\cdot d\vec{a} = s(2+\sin^2\phi)\,s\,d\phi\,dz = 4(2+\sin^2\phi)\,d\phi\,dz$

$\int\vec{V}\cdot d\vec{a} = 4\int_0^{\pi/2}(2+\sin^2\phi)\,d\phi\int_0^5 dz = 4.(\pi+\frac{\pi}{4}).5 = 25\pi$

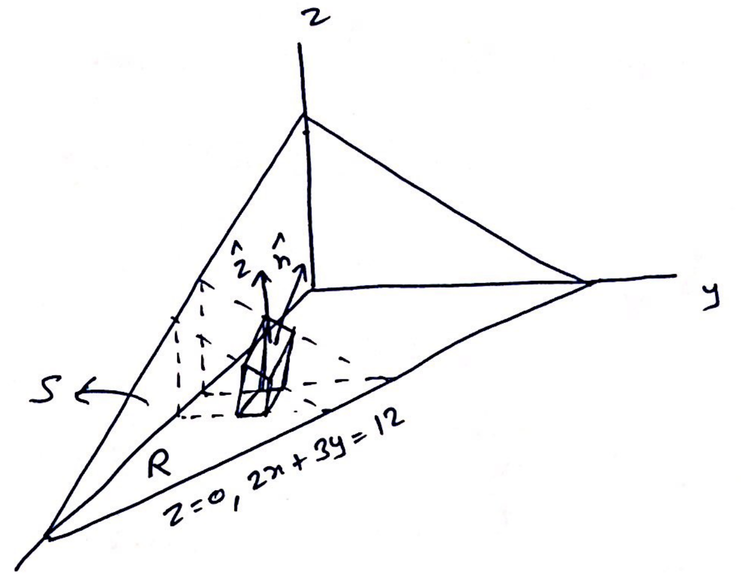
So, $\oint \vec{v}\cdot d\vec{a} = 15\pi + 25\pi = \underline{\underline{40\pi}}$

(C). $\vec{\nabla}\times\vec{v} = \left[\frac{1}{s}\frac{\partial}{\partial\phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right]\hat{s} + \left[\frac{\partial}{\partial z}(s(2+\sin^2\phi)) - \frac{\partial}{\partial s}(3z)\right]\hat{\phi}$

$$+\frac{1}{s}\left[\frac{\partial}{\partial s}(s^2\sin\phi\cos\phi) - \frac{\partial}{\partial\phi}(s(2+\sin^2\phi))\right]\hat{z}$$

$$= \frac{1}{s}\left[2s\sin\phi\cos\phi - s.2\sin\phi\cos\phi\right]\hat{z} = 0$$

<u>Problem 3.4</u> $\iint_S \vec{A}\cdot\vec{n}\, ds$



$\vec{A} = 18z\hat{x} - 12\hat{y} + 3y\hat{z}$

The surface S, and its projection on xy plane R is shown in the figure.

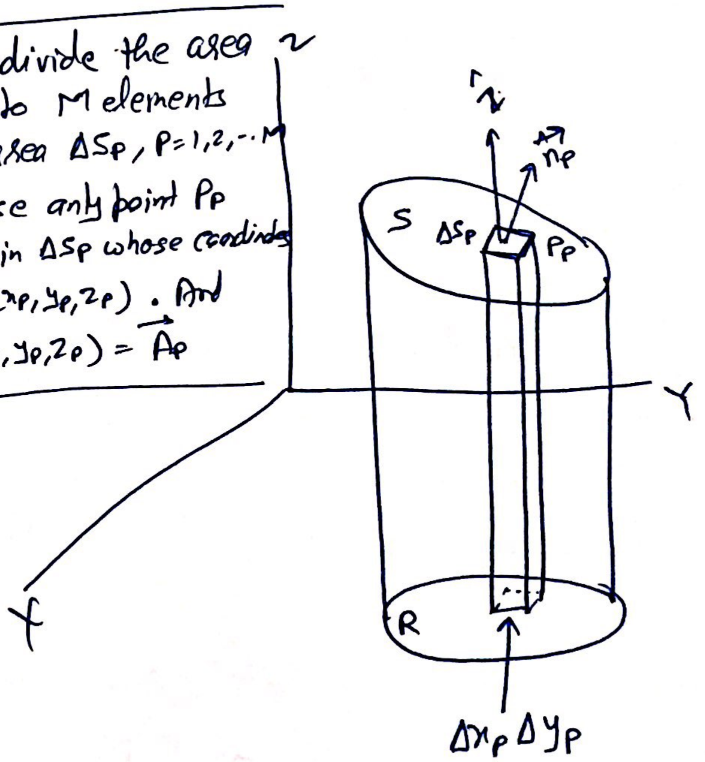
Surface integral $\iint_S \vec{A}\cdot\vec{n}\, ds$

Can be written in terms of sums as:

$$\sum_{P=1}^{M} \vec{A}_P.\vec{n}_P\, \Delta S_P$$

where $\vec{A}_P\cdot\vec{n}_P$ is the normal component of A_P at P_P.

Sub divide the area S into M elements of area ΔS_P, $P=1,2,\ldots M$. Choose any point P_P with in ΔS_P whose coordinates are (x_P, y_P, z_P). And $\vec{A}(x_P, y_P, z_P) = \vec{A}_P$



The projection of ΔS_P on the xy plane is

$$|(\vec{n}_P\,\Delta S_P).\hat{z}| = |\vec{n}_P.\hat{z}|\,\Delta S_P$$

$$= \Delta x_P\,\Delta y_P$$

So, $\Delta S_P = \dfrac{\Delta x_P\,\Delta y_P}{|\vec{n}_P.\hat{z}|}$

So, $$\sum_{p=1}^{M} \vec{A}_p \cdot \vec{n}_p \, \Delta S_p = \sum_{p=1}^{M} \vec{A}_p \cdot \vec{n}_p \frac{\Delta x_p \, \Delta y_p}{|\vec{n}_p \cdot \hat{z}|}$$ [M = total no. of elements]

So, in the limits $M \to \infty$,

$$\iint_S \vec{A} \cdot \vec{n} \, ds = \iint_R \vec{A} \cdot \vec{n} \frac{dx\,dy}{|\vec{n} \cdot \hat{z}|}$$

Here, in the given problem,

vector perpendicular to the surface $2x + 3y + 6z = 12$ is

$$\vec{\nabla}(2x + 3y + 6z) = 2\hat{x} + 3\hat{y} + 6\hat{z}$$

So, $$\vec{n} = \frac{2\hat{x} + 3\hat{y} + 6\hat{z}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{6}{7}\hat{z}$$

So, $\vec{n} \cdot \hat{z} = \frac{6}{7}$ & $\frac{dx\,dy}{|\vec{n} \cdot \hat{z}|} = \frac{7}{6} dx\,dy$

[$2x + 3y + 6z = 12$
$36z + 18y = 72 - 12x$]

Also, $$\vec{A} \cdot \vec{n} = (18z\hat{x} - 12\hat{y} + 3y\hat{z}) \cdot \left(\frac{2}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{6}{7}\hat{z}\right) = \frac{36z - 36 + 18y}{7}$$

$$= \frac{36 - 12x}{7}$$

$$\iint_S \vec{A} \cdot \vec{n} \, ds = \iint_R \vec{A} \cdot \vec{n} \frac{dx\,dy}{|\vec{n} \cdot \hat{z}|} = \iint_R \left(\frac{36 - 12x}{7}\right) \frac{7}{6} dx\,dy = \iint_R (6 - 2x)\,dx\,dy$$

From equation of S, $$z = \frac{12 - 2x - 3y}{6}.$$

To evaluate this integral, first keep x fixed and integrate w.r.t. y from $y = 0$ to $y = \frac{12 - 2x}{3}$, & then integrate w.r.t. x from $x = 0$, $x = 6$.

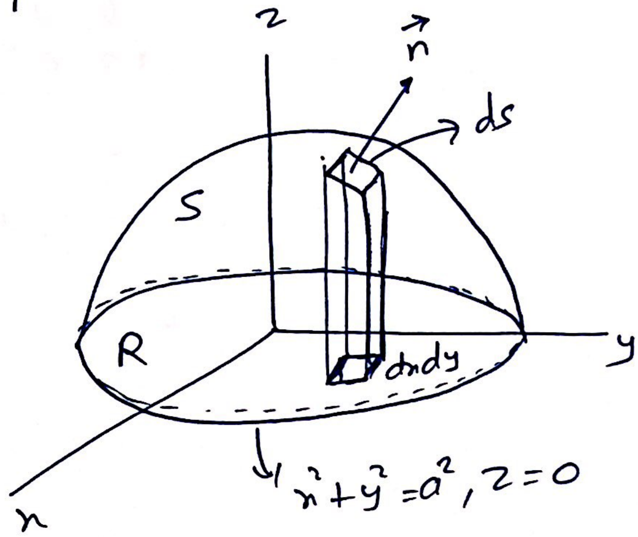
So, $$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(6-2x)\,dy\,dx = \int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx$$

$$= \underline{\underline{24}}$$

Problem 3.5

$$\vec{F} = y\hat{x} + (x-2xz)\hat{y} - xy\hat{z}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & (x-2xz) & -xy \end{vmatrix} = x\hat{x} + y\hat{y} - 2z\hat{z}$$



normal to $x^2+y^2+z^2=a^2$ is,

$$\nabla(x^2+y^2+z^2) = 2x\hat{x} + 2y\hat{y} + 2z\hat{z}$$

so, unit normal

$$\vec{n} = \frac{2x\hat{x}+2y\hat{y}+2z\hat{z}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{a}$$

The projection of S on xy plane is the region R bounded by the circle $x^2+y^2=a^2, z=0$, then,

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,ds = \iint_R (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,\frac{dx\,dy}{|\vec{n}\cdot\hat{z}|}$$

$$= \iint_R (x\hat{x}+y\hat{y}-2z\hat{z})\cdot\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{a}\right)\cdot\frac{dx\,dy}{z/a}$$

$$= \iint_R \frac{x^2+y^2-2z^2}{a}\cdot\frac{a}{z}\cdot dx\,dy = \iint_R \frac{x^2+y^2-2a^2+2(x^2+y^2)}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$= \int_{x=-a}^{+a} \int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\, dy\, dx \qquad \left[z=\sqrt{a^2-x^2-y^2}\right]$$

$$x = r\cos\phi, \quad y = r\sin\phi, \quad dx\,dy = r\,dr\,d\phi$$

$$\Rightarrow \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\, r\,dr\,d\phi = \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \frac{3(r^2-a^2)+a^2}{\sqrt{a^2-r^2}}\, r\,dr\,d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \left[-3r\sqrt{a^2-r^2} + \frac{a^2 r}{\sqrt{a^2-r^2}}\right] dr\,d\phi$$

$$= \int_{\phi=0}^{2\pi} \left[(a^2-r^2)^{3/2} - a^2\sqrt{a^2-r^2}\right]_{r=0}^{a} d\phi$$

$$= \int_{\phi=0}^{2\pi} (a^3-a^3)\, d\phi = 0$$

Problem: 3.6

(a) By divergence theorem;

$$\iint_S \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds = \iiint_V \vec{\nabla}\cdot\frac{\vec{r}}{r^3}\,dV$$

But $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)=0$ everywhere within V provided $r\neq 0$ in V, that is

provided O is outside of V and thus outside of S. Then,

$$\iint_S \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds = 0.$$

(b) If O is inside S, surround O by a small sphere s of radius a.

Let τ denote the region bounded by S and s. Then by divergence theorem,

$$\iint_{S+s} \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds = \iint_S \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds + \iint_s \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds = \iiint_\tau \vec{\nabla}\cdot\frac{\vec{r}}{r^3}\,dV = 0$$

Since $r\neq 0$ in τ, thus

$$\iint_S \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds = -\iint_s \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds$$

Now on s, $r=a$, $\vec{n}=-\frac{\vec{r}}{a}$, $\frac{\vec{n}\cdot\vec{r}}{r^3} = \frac{(-\vec{r}/a)\cdot\vec{r}}{a^3} = -\frac{\vec{r}\cdot\vec{r}}{a^4} = -\frac{a^2}{a^4} = -\frac{1}{a^2}$

So, $\iint_S \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds = -\iint_s \frac{\vec{n}\cdot\vec{r}}{r^3}\,ds = \iint_s \frac{1}{a^2}\,ds = \frac{1}{a^2}\iint_s ds = \frac{4\pi a^2}{a^2} = \underline{\underline{4\pi}}$

Problem: 3.7 Consider an arbitrary surface enclosing a volume V of the fluid. At any time, the mass of fluid within V is,

$$M = \iiint_V \rho\, dv$$

The time rate of increase of this mass,

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\iiint_V \rho\, dv = \iiint_V \frac{\partial \rho}{\partial t}\, dv$$

the mass of fluid per unit time leaving V is

$$\iint_S \rho \vec{v}\cdot\vec{n}\, ds$$

and the time rate of increase in mass is therefore,

$$-\iint_S \rho\vec{v}\cdot\vec{n}\, ds = -\iiint_V \vec{\nabla}\cdot(\rho v)\, dV$$

by divergence theorem.

Then,

$$\iiint_V \frac{\partial \rho}{\partial t}\, dv = -\iiint_V \nabla\cdot(\rho\vec{v})\, dV$$

or,

$$\iiint_V \left(\nabla\cdot(\rho\vec{v}) + \frac{\partial \rho}{\partial t}\right) dv = 0.$$

so,

$$\nabla\cdot J + \frac{\partial \rho}{\partial t} = 0 \qquad \text{where } \vec{J} = \rho\vec{v}$$