# Department of Mathematics & Statistics

**MTH-102A Ordinary Differential Equations**

Assignment VI

1. $\star$ Using the exapansion
$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$
show that the Legendre Polynomials $P_n(x)$ satisfy the following.
(i) $P_n(1) = 1$ (ii) $P_n(-1) = (-1)^n$
(iii) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
(iv) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

2. $\star$ Show that
(i) $\int_{-1}^{1} x^m P_n(x)dx = 0$ if $m < n$
(ii) $\int_{-1}^{1} x^m P_n(x)dx = 0$ if $m > n$ and $m-n$ is odd. What happens if $m-n$ is even?
(iii) $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$.

3. $\star$ The Bessel function $J_p(x)$, for any real number $p$, is defined as
$$J_p(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(x/2)^{2n+p}}{n!(n+p)!}.$$
Using this expression of $J_p(x)$, show that
(i) $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$ (ii) $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$
(iii) $J_p'(x) + \frac{p}{x}J_p(x) = J_{p-1}(x)$ (iv) $J_p'(x) - \frac{p}{x}J_p(x) = J_{p+1}(x)$.

4. $\star$ Show that, for every real number $p$, the Bessel function $J_p(x)$ has infinitely many positive zeros.

5. $\star$ For a given a real number $p$, we let $\lambda_n$ denote the positive zeros of the Bessel function $J_p(x)$. Show that
(i) $\int_0^1 xJ_p(\lambda_m x)J_p(\lambda_n x) = 0$ if $m \neq n$ and (ii) $\int_0^1 xJ_p(\lambda_m x)^2 = \frac{1}{2}\left[J_p'(\lambda_n)\right]^2 = \frac{1}{2}\left[J_{p+1}(\lambda_n)\right]^2$.

6. Let $p, q, r : (a,b) \to \mathbb{R}$ be continuous function. Show that any nontrivial solution $y$ of $y'' + py' + qy = 0$ has only finitely many zeros in any closed interval $[\alpha, \beta] \subseteq (a,b)$

7. Find the first four terms $a_n$ of the expansion $f(x) = \sum_{n\geq 0} a_n P_n(x)$, if
(i) $f(x) = x|x|$ for $|x| \leq 1$ and (ii) $f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1. \end{cases}$

8. Let $a, b, c, d$ be real numbers such that $ad - bc \neq 0$. Show that the zeros of the functions $a\sin x + b\cos x$ and $c\sin x + d\cos x$ are distinct and occur alternately.