Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment IV

1. $\star$ Using the method of variation of parameters find a particular solution of
   (a) $x^2y'' - 2xy' + 2y = x^{\frac{9}{2}}$.
   (b) $y'' + 3y' + 2y = \frac{1}{1+e^x}$.
2. $\star$ Using the method of undetermined coefficients find a particular solution of
   (a) $y'' - 3y' + 2y = e^{3x}(x^2 + 2x - 1)$.
   (b) $y'' + 3y' + 2y = (16 + 20x)\cos x + 10 \sin x$.
3. $\star$ Let a, b, c be three positive real numbers and let y be a solution of the differential equation $ay'' + by' + cy = 0$. Show that $\lim_{n\to+\infty} y(x) = 0$.
4. $\star$ Let $p, q : (a, b) \to \mathbb{R}$ be two continuous functions. Let y_1 and y_2 be two solutions of the differential equation $y'' + py' + qy = 0$ in (a, b). Show that the solutions y_1 and y_2 are linearly dependent if any of the following conditions hold.
   (a) $y_1(x_0) = y_2(x_0)$ at some point x_0 in (a, b).
   (b) y_1 and y_2 attain an extremum at same point x_0 in (a, b).
5. $\star$ Let y_1 and y_2 be two linearly independent soltuions of the differential $y'' + py' + qy = 0$ where p and q are as in the earlier problem. Let x_1 and x_2 be two points in (a, b) such that $y_1(x_1) = 0 = y_1(x_2)$. Show that there exists a point z in (a, b) such that $x_1 < z < x_2$ and $y_2(z) = 0$.
6. $\star$ Let $y_1, y_2 : (a, b)$ be two twice differentiable functions such that $W(y_1, y_2)(x) \neq 0$ for all points x in (a, b). Show that there exists two functions $p, q : (a, b) \to \mathbb{R}$ such that y_1 and y_2 are two linearly independent solutions of $y'' + py' + qy = 0$.
7. Find the wronskian W of a given set $\{y_1, y_2\}$ of solutions of
   (a) $y'' + 3(x^2 + 1)y' - 2y = 0$ given that $W(y_1, y_2)(\pi) = 0$.
   (b) $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ given that $W(0) = 1$. (This is *Legendre's equation*).
8. Verify that $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutuons of $y'' - 2y' + y = 0$ on $(-\infty, \infty)$. Further find the solution y with the initial conditions $y(0) = 7$ and $y'(0) = 4$.
9. Let $p, q : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Show that $\sin(x^2)$ can't be a solution of the differential equation $y'' + py' + qy = 0$.
10. Using the method of undetermined coeeficients find a particular solution of
    (a) $y'' - 7y' + 12y = 5e^{4x}$.
    (b) $y'' - 3y' + 2y = e^{-2x}\left(2\cos 3x - (34 - 150x)\sin 3x\right)$.
11. For each of the following set of functions $\{y_1, y_2\}$ given below find a differential equation $y'' + py' + qy = 0$ such that the set is a fundamental set of solutions, where p and q are continuous functions on the domain of defintion of y_1 and y_2.
    (a) $\{x^2 - 1, x^2 + 1\}$.
    (b) $\{x, e^{2x}\}$.

(c) $\{\frac{1}{x-1}, \frac{1}{x+1}\}$.

12. Find the solution of

    (a) $y'' + y = 1$ with $(y(0), y'(0)) = (2, 7)$.

    (b) $y'' - 2y' + y = x^2 - x - 3$ with $(y(0), y'(0)) = (-2, 1)$.

13. Solve the equation

    (a) $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{\frac{x}{2}}$.

    (b) $y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}$.