Department of Mathematics & Statistics

MTH-102A Ordinary Differential Equations

Assignment I

1. $\star$ Classify the following differential in to linear and non-linear differential equations. Further specify the order of each of these equations.
(i) $y' = \sin x$ (ii) $y' = x^2 + y^2$ (iii) $y'' + p(x)y' + q(x)y = r(x)$
(iv) $y' = -\frac{x}{y}$ (v) $y' = \sin(xy)$ (vi) $y' + by^2 + ay = 0$.
In problem (iii) p, q and r are continuous functions on some open interval in $\mathbb{R}$ and a, b are constants in problem (vi).

   (a) The equations (i) and (iii) are linear and the other four equations are non-linear.

   (b) All equations except (iii) are first order and (iii) is second order.

2. $\star$ Show that the function $y(x) = \frac{x^2}{3} + \frac{1}{x}$ is a solution of the differential equation $xy' + y = x^2$ in $(-\infty, 0) \cup (0, \infty)$.

   The function $y(x) = \frac{x^2}{3} + \frac{1}{x}$ is defined in $(-\infty, 0) \cup (0, \infty)$ and its derivative $y'(x) = \frac{2x}{3} - \frac{1}{x^2}$. Hence $xy' + y = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \frac{x^2}{3} + \frac{1}{x} = x^2$.

3. $\star$ Find all solutions of the differential equation $y^{(n)} = e^{2x}$ for $n \geq 2$; here $y^{(n)}$ is the n-th order derivative of y.

   Let y be a function satisfying the equation $y^{(n)} = e^{2x}$. Then $\frac{d}{dx}y^{(n-1)} = y^{(n)} = e^{2x}$. By integrating, we get $y^{(n-1)} = \frac{e^{2x}}{2} + k_1$ where k_1 is a constant.
   If $n-1 \geq 2$, we integrate again to get $y^{(n-2)} = \frac{e^{2x}}{2^2} + k_1 x + k_2$ where k_2 is a constant. By repeated integration we see that $y(x) = \frac{e^{2x}}{2^n} + k_1\frac{x^{n-1}}{(n-1)!} + k_2\frac{x^{n-2}}{(n-2)!} + \cdots + k_{n-1}x + k_n$ where k_i's are constants.

4. $\star$ Solve the following initial value problems.
(i) $y' = -xe^x$ such that $y(0) = 1$.
(ii) $y' = x\sin(x^2)$ such that $y(\sqrt{\frac{\pi}{2}}) = 1$.
(iii) $y' + y = \frac{e^{-x}\tan x}{x}$ such that $y(1) = 0$.
(iv) $y'' = xe^{2x}$ such that $y(0) = 7$ and $y'(0) = 1$.

   (a) Let y be a solution of $y' = -xe^x$ with $y(0) = 1$. Since $\frac{dy}{dx} = -xe^x$, integrating from 0 to x, we get $y(x) = (1-x)e^x + k$ where k is a constant. It is easy to see that $k = 0$, if $y(0) = 1$. Thus $y(x) = (1-x)e^x$ is the requoired solution.

   (b) Let y be a function such that $y'(x) = x\sin(x^2)$. That is $y' = -\frac{d}{dx}\left(\cos(x^2)\right)$. If we now integrate we get $y(x) = -\frac{1}{2}\cos(x^2) + k$ for a constant k in $\mathbb{R}$. The condion $y(\sqrt{\frac{\pi}{2}}) = 1$ shows that $k = 1$ and $y(x) = 1 - \frac{1}{2}\cos(x^2)$ is the solution of $y' = x\sin(x^2)$ such that $y(\sqrt{\frac{\pi}{2}}) = 1$.

   (c) The general solution is $y(x) = e^{-x}\left(c + \int_1^x \frac{tant}{t}dt\right)$. The initial condition $y(1) = 0$, shows that the solution is $y(x) = e^{-x}\int_1^x \frac{tant}{t}dt$ and this integral can't be written in closed form.

   (d) The integral $\int xe^{2x} = \frac{2x-1}{4}e^{2x} + k_1$ and the condition $y'(0) = 1$ shows that $k_1 = \frac{5}{4}$. Therefore $y'(x) = \frac{2x-1}{4}e^{2x} + \frac{5}{4}$. Integrating this again, we get $y(x) = \frac{x-1}{4}e^{2x} + \frac{5}{4}x + k_2$. Since $y(0) = 7$, it follows that $k_2 = \frac{29}{4}$ and $y(x) = \frac{(x-1)e^{2x}+5x+29}{4}$.

5. $\star$ A model for the growth of population of a species assumes that
(i) if the population is small, the rate of growth is nearly proportional to the size of the population;
(ii) but if the population becomes too large, the growth becomes negative.
Let $P(t)$ denote the population at time t, a the reproduction rate of the population and N represent the ideal population. Thus we get the following differential equation

$$P'(t) = aP(1 - \frac{P}{N}).$$

Assuming that the population at time $t = 0$ is $P(0) = p_0$, show that the population at time t is

$$P(t) = \frac{Np_0e^{at}}{N - p_0 + p_0e^{at}}.$$

First we write the equation

$$P'(t) = aP(1 - \frac{P}{N})$$

as

$$\frac{NP'}{P(N-P)} = a.$$

Using partial fraction, we can write this as $\left(\frac{1}{P} + \frac{1}{N-P}\right) P'(t) = a$. *Therefore* $\frac{d}{dt} \ln\left|\frac{P}{N-P}\right| = a$. *Integrating this equation from* 0 *to* t, *we get* $\ln\left|\frac{P(t)(N-P_0)}{P_0(N-P)}\right| = at$. *Simplifying this we get,*

$$P(t) = \frac{Np_0e^{at}}{N - p_0 + p_0e^{at}}.$$

6. $\star$ A model for the spread of epidemics assumes that the number of people infected changes at a rate proportional to the product of number of people infected and the number of people whoe are susceptible, but not infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then $S - I$ is the number of susceptible people, but not infected at time t. Thus we get the following differential equation

$$I'(t) = rI(S - I)$$

where r is a positive constant. Assuming that the number of people infected at time $t = 0$ is $I(0) = I_0$, show that the number of people infected at time t is

$$I = \frac{SI_0}{I_0 + (S - I_0)e^{-rSt}}.$$

Notice that $\lim_{t\to\infty} I(t) = S$. Thus this model predicts that all the susceptible people eventually become infected.

First we write the equation $I'(t) = rI(S - I)$ *as* $\left(\frac{1}{I} + \frac{1}{S-I}\right) I'(t) = rS$. *This can again be written as* $\frac{d}{dt} \ln\left|\frac{I}{S-I}\right| = rS$. *Consequently* $\ln\left|\frac{I}{S-I}\right| = rSt + k$ *and* $\left|\frac{I}{S-I}\right| = e^k e^{rSt}$. *The initial condtion gives us that* $e^k = \frac{I_0}{S-I_0}$. *Hence the solution is* $\frac{I}{S-I} = \frac{I_0}{S-I_0}e^{rSt}$ *and we simplify this to get* $I = \frac{SI_0}{I_0+(S-I_0)e^{-rSt}}$.

7. Show that $y(x) = \begin{cases} e^x - 1 & \text{if } x \geq 0 \\ 1 - e^{-x} & \text{if } x < 0 \end{cases}$ is a solution of $y'(x) = |y(x)| + 1$ on $(-\infty, \infty)$.

8. Find all solutions of

(a) $y' = -\frac{x}{y}$.

(b) $y' + ay - by^2 = 0$ where a and b are constants.

(c) $x^2yy' = (y^2 - 1)^{\frac{3}{2}}$.

9. Find a solution of $y' + y^2 + a = 0$ for a constant a. Find the largest interval on which the solution is defined.

10. Find all solutions of the differential equation $y' = 2xy^2$. Find the largest subset of $\mathbb{R}$ on which the solution is defined.

11. Solve the differential equation $y' = \frac{1}{2}x(1 - y^2)$.

12. Show that $y(x) = x \cos x$ is the solution of $y' = \cos x - y \tan x$ such that $y(\frac{\pi}{4}) = \frac{\pi}{4\sqrt{2}}$.

13. Let c_1 and c_2 be two real numbers. Show that the function $y(x) = (c_1 + c_2x)e^{-x} + 2x - 4$ is a solution of $y'' + 2y' + y = 2x$ in $\mathbb{R}$.

14. In Exercise 5, let us modify the logistic model to take in to account harvesting of the population. Let us assume that the population is harvested at the constant rate h. Therefore the differential equation becomes

$$P'(t) = aP(1 - \frac{P}{N}) - h.$$

Find the general solution of this differential equation.