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> Working problems is a crucial part of learning mathematics. No one can learn topology merely by poring over the definitions, theorems, and examples that are worked out in the text. One must work part of it out for oneself. To provide that opportunity is the purpose of the exercises.
>
> — *James R. Munkres*

Let G be a topological group with operation $\cdot$ and identity element x_0 . Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0 . If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operations $\star$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f \star e_{x_0}) \otimes (e_{x_0} \star g)$.]

(d) Show that $\pi_1(G, x_0)$ is abelian.

(a) Since $\cdot$, f and g are continuous, $f \otimes g$ is continuous and $(f \otimes g)(0) = (f \otimes g)(1) = x_0$, i.e. $f \otimes g \in \Omega(G, x_0)$. Moreover, since $\cdot$ is associative, so is $\otimes$, and further $x_0 \otimes f = f \otimes x_0 = f$, and the inverse of f is defined by $f^{-1}(s) = (f(s))^{-1}$.

(b) $[f] \otimes [g] = [f \otimes g]$ is well defined, because $F(s,t) \cdot G(s,t)$ is a homotopy between $F|_{t=0} \otimes G|_{t=0}$ and $F|_{t=1} \otimes G|_{t=1}$, and it satisfies group properties induced by $\otimes$.

(c) $[f] \otimes [g] = [f \star e_{x_0}] \otimes [e_{x_0} \star g] = [(f \star e_{x_0}) \otimes (e_{x_0} \star g)] \overset{?}{=} [f \star g] = [f] \star [g]$because

$$\begin{aligned} ((f \star e_{x_0}) \otimes (e_{x_0} \star g))(s) &= (f \star e_{x_0})(s) \cdot (e_{x_0} \star g)(s) = \\ &= \begin{cases} f(s) & s \in \left[0, \frac{1}{2}\right] \\ g(s) & s \in \left[\frac{1}{2}, 1\right] \end{cases} = \\ &= (f \star g)(s). \end{aligned}$$

(d) $[f] \star [g] = [f] \otimes [g] = [e_{x_0} \star f] \otimes [g \star e_{x_o}] = [g \star f] = [g] \star [f]$.