MTH 102: Linear Algebra — *February 20, 2013*

Department of Mathematics and Statistics — Time: 2 hours

Indian Institute of Technology - Kanpur — Maximum Score: 60

Mid-Semester Examination

INSTRUCTIONS

i. Please write your **Name**, **Roll Number** and **Section** correctly on the answer booklet.
ii. Attempt each question on a new page and attempt all parts of a question at the same place.
iii. Attempt all questions. Each question carries 12 marks.
iv. **Please make a table on the front cover page indicating the question number and respective page number.**

1. Let A be a symmetric matrix of size 3×3 with the third column $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. It is also known that A has the cofactor $C_{23} = 1$ and

$$A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Write down the matrix A. Is A invertible? **[8+1]**

Solution: If $A_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$ are first two columns of A respectively, then from the given third column and the fact that $A^T = A$, we have (1 mark)

$$a_{31} = 1 \tag{1}$$

and (1 mark)

$$a_{32} = 0. \tag{2}$$

Now, from the condition $A \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}^T$, we also have

$$A_1 + A_2 + 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

which leads to equations (2 marks; 1 for each correct equation)

$$a_{11} + a_{12} = 1, \tag{3}$$

and

$$a_{21} + a_{22} = 1. \tag{4}$$

From the given cofactor, we have (1 mark)

$$1 = C_{23} = (-1)(a_{11}a_{32} - a_{31}a_{12}) = -(a_{11})(0) + (1)(a_{12}) \Rightarrow a_{12} = 1. \qquad (5)$$

Using (5) in (3), we get (1 mark) $a_{11} = 0$.
From $A^T = A$, we have (1 mark) $a_{21} = a_{12} = 1$,
and finally, from (4), we get (1 mark) $a_{22} = 0$.
Thus,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrix A is not invertible. One can conclude this from $\det(A) = 0$ (two identical columns or two identical rows) or $\mathrm{rank}(A) = 2$ or $\mathrm{N}(A) \neq \{0\}$, etc. (1 mark for this step.)

(b) Find the complete solution to **[3]**

$$Ax = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: We already know that a particular solution is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. (1 mark for this step.)

One can find a basis for the one dimensional null space of A, either directly by observation, or by solving the homogeneous equation. One such basis is (1 mark) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (any multiple of this vector also works).
Hence, the complete solution is (1 mark)

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

where c is an arbitrary real constant.

2. Let α with $|\alpha| < 1$ be an eigen-value of a 2×2 symmetric matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

where non-negative entries a, b and d of A satisfy $a + b = 1$ and $b + d = 1$.

(a) Rewrite the matrix A in terms of α. In other words, describe entries a, b, d of A as functions of α. (**Hint:** One method begins by finding the second eigen-value !) **[6]**

Solution: From equations $a + b = 1$ and $b + d = 1$, we have $a = d$. (1 mark)

One eigen-value, say λ_1, is given to be α, i.e., $\lambda_1 = \alpha$. Let λ_2 be the second eigen-value. Then,

$$\alpha + \lambda_2 = \lambda_1 + \lambda_2 = \mathrm{Tr}(A) = 2a, \tag{6}$$

and

$$\alpha\lambda_2 = \lambda_1\lambda_2 = \det(A) = a^2 - b^2 = (a+b)(a-b) = 2a - 1.$$

Using (6) in the above equation, we get, $(1-\alpha)\lambda_2 = 1 - \alpha \Rightarrow \lambda_2 = 1$. (2 marks for finding the second eigen-value)
From (6), we have (3 marks; 1 each for a, b and d.)

$$a = d = (1+\alpha)/2 \text{ and } b = 1 - a = (1-\alpha)/2.$$

Thus,

$$A = \begin{bmatrix} (1+\alpha)/2 & (1-\alpha)/2 \\ (1-\alpha)/2 & (1+\alpha)/2 \end{bmatrix}.$$

(b) Compute eigen-vectors of A. **[2]**

Solution: An eigen-vector corresponding to $\lambda_1 = \alpha$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (1 mark; any multiple also works) and one corresponding to $\lambda_2 = 1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (1 mark; any multiple also works).

(c) Write A^{2013} in its simplest form, that is, write all entries of the matrix A^{2013}. **[4]**

Solution: Let, S be the eigen-vector matrix and Λ be the eigen value matrix. Then

$$\Lambda = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \; S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

As $A^{2013} = S\Lambda^{2013}S^{-1}$, we have (2 marks)

$$A^{2013} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^{2013} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, in the simplest form, (2 marks)

$$A^{2013} = \begin{bmatrix} (1+\alpha^{2013})/2 & (1-\alpha^{2013})/2 \\ (1-\alpha^{2013})/2 & (1+\alpha^{2013})/2 \end{bmatrix}.$$

3. Suppose q_1, q_2, q_3, q_4 are orthonormal vectors in $\mathbb{R}^5$.

(a) Find the length of the vector $v = q_1 - 2q_2 + 3q_3 - 4q_4$. **[2]**

Solution: By orthogonality, $\|v\|^2 = \|q_1 - 2q_2 + 3q_3 - 4q_4\|^2 = 1 + 4 + 9 + 16 = 30$. Therefore, the length of v, given by $\|v\|$, is equal to $\sqrt{30}$. (2 marks)

(b) What five vectors does Gram-Schmidt process produce when it orthonormalizes the vectors q_1, q_2, q_3, q_4, u, where u is any vector in $\mathbb{R}^5$ outside the span of q_1, q_2, q_3 and q_4? **[5]**

Solution: q_1, q_2, q_3, q_4 and

$$q_5 = \frac{u - (q_1^T u)q_1 - (q_2^T u)q_2 - (q_3^T u)q_3 - (q_4^T u)q_4}{\|u - (q_1^T u)q_1 - (q_2^T u)q_2 - (q_3^T u)q_3 - (q_4^T u)q_4\|}.$$

(4 marks for the correct numerator (1 each for removing components along q_1, q_2, q_3 and q_4) and 1 mark for normalizing.)

(c) If u in part (b) is the vector v in part (a), why does Gram-Schmidt process break down? Find a non-zero vector in the nullspace of the 5×5 matrix **[2+3]**

$$A = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & v \end{bmatrix}.$$

Solution: Gram-Schmidt fails because v is a linear combination of q_i's : 5 vectors are not linearly independent. (2 marks) A vector in $N(A)$ is $\begin{bmatrix} 1 & \text{-2} & 3 & \text{-4} & \text{-1} \end{bmatrix}^T$. (3 marks)

4. Suppose A is an $m \times n$ matrix with rank$(A) = n$ and satisfy $A = QR$ where the $m \times n$ matrix Q has orthonormal columns and the matrix R is an upper-triangular matrix.

(a) Is it true that R is a square invertible matrix? Explain your answer. **[2]**

Solution: The matrix R is clearly an $n \times n$ matrix (follows from shapes of matrices A and Q) (1 mark). Note that $N(R) \subseteq N(QR) = N(A) = \{0\}$. Thus, $N(R) = \{0\}$ and hence R is invertible (1 mark; any other correct reasoning is acceptable as well).

(b) Explain why column spaces of A and Q are the same. Then use this fact to show that the projection matrix P that projects vectors in $\mathbb{R}^m$ orthogonally onto the column space of A is given by **[2+3]**

$$P = QQ^T.$$

Solution:

Reasoning for C(A)=C(Q):

Clearly, $C(A) = C(QR) \subseteq C(Q)$ (1 mark). Also, $\dim(C(A)) = \dim(C(Q)) = n$ (1 mark).. Thus, $C(Q) = C(A)$.

Alternate reasoning for C(A)=C(Q):

Clearly, $C(A) = C(QR) \subseteq C(Q)$ (1 mark). Need to show that $C(Q) \subseteq C(A)$. Let $x \in C(Q)$. Then there is a vector $c \in \mathbb{R}^n$ such that $x = Qc$. Now, $x = Qc = QRR^{-1}c = A(R^{-1}c)$ imply that $x \in C(A)$. This shows that $C(Q) \subseteq C(A)$ (1 mark) and therefore $C(Q) = C(A)$.

Method 1 for $P = QQ^T$:

In general, the projection matrix P that projects vectors in $\mathbb{R}^m$ orthogonally onto the column space of A is given by $P = A(A^T A)^{-1}A^T$ (1 mark). However, as $C(A) = C(Q)$, it is equivalent to find a projection matrix P that projects vectors in $\mathbb{R}^m$ orthogonally onto the column space of Q (1 mark). Thus, (1 mark for using $Q^T Q = I$ to get the final expression.)

$$P = Q(Q^T Q)^{-1}Q^T = QQ^T \text{ (as } Q^T Q = I).$$

Method 2 for $P = QQ^T$:
We need to find P such that, for all $b \in \mathbb{R}^m$,

i. $Pb \in C(A)$, and
ii. $b - Pb \perp C(A)$.

As $Pb \in C(A) = C(Q)$, we can assume that $Pb = Qc$ for some $c \in \mathbb{R}^n$ (1 mark). Also, we require $0 = Q^T(b - Pb) = Q^T(b - Qc)$ (1 mark). Thus, $Q^Tb = Q^TQc = c$ (1 mark). Therefore, $Pb = Qc = QQ^Tb$ and hence the projection matrix $P = QQ^T$.

(c) Let

$$A = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solve $Ax = b$ in the least-squares sense for $b = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$.

(**Hint:** Solve $Q^TAx = Q^Tb$!) **[5]**

Solution: The linear system $Q^TAx = Q^Tb$ is equivalent to the upper triangular system $Q^TQRx = Q^Tb$. (2 marks)

$$Rx = Q^Tb \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 & 2/5 & 4/5 \\ -2/5 & 1/5 & -4/5 & 2/5 \\ -4/5 & 2/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}. \qquad (7)$$

Solving equation (7) using back-substitution yields $x = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (3 marks; 1 for each correct component.).

5. For each of the following statements, determine if it is always true. If so, answer TRUE and explain why it is always true; otherwise answer FALSE and provide a counter example. **Note that no credit will be given to a correct guess without any explanation or followed by an incorrect justification.**

(a) There is no orthogonal matrix Q where one of its eigen-values is 0. **[2]**

Solution: The statement is TRUE. Every orthogonal matrix is non-singular as columns are orthonormal and hence independent. Matrices with non-zero determinant can not have a zero eigen-value (det = product of eigen-values !) (2 marks for a correct explaination.)

(b) If $A^5 = 0$, where 0 is a 5×5 matrix with all entries equal to zero, then $A = 0$. **[2]**

Solution: The statement is FALSE. A counter example is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(1 mark for a counter-example and 1 mark for showing why that counter-example works.) Easy to check that $A^2 = 0$ and therefore $A^5 = 0$ but $A \neq 0$.

(c) There is no 3×3 matrix A of the form

$$A = \begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}$$

that satisfies $A^2 = 2A$. **[4]**

Solution: The statement is TRUE. Suppose there is such a matrix, then all its eigen values are either 0 or 2. (1 mark for this observation and 1 mark for explaning why this is true.) This is because if λ is an eigen-value of A with a non-zero eigen-vector x, then

$$A^2 = 2A \Rightarrow A^2x = 2Ax \Rightarrow \lambda^2 x = 2\lambda x \Rightarrow \lambda(\lambda - 2)x = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = 2.$$

But the sum of eigen-values of A is $\mathrm{Tr}(A) = 3$ (1 mark for this observation.) which can not be obtained by adding 0's and 2's. (1 mark for this observation.)

(d) If A be a 2×2 matrix, then $N(A^2) = N(A^3)$. **[4]**

Solution: The statement is TRUE.
Clearly, $N(A^k) \subseteq N(A^{k+1})$ and therefore $\dim(N(A^k)) \leq \dim(N(A^{k+1}))$ for all $k \geq 0$.
Note that $\dim(N(A))$ is either 0, 1 or 2.
If $\dim(N(A)) = 0$, that is, if A is invertible, then A^2 and A^3 are also invertible and hence $N(A^2) = \{0\} = N(A^3)$. (1 mark for this observation.)
If $\dim(N(A)) = 2$, then $\dim(N(A)) = \dim(N(A^2)) = \dim(N(A^3)) = 2$. (1 mark for this observation.)
If $\dim(N(A)) = 1$, then either $\dim(N(A^2)) = 1$ or $\dim(N(A^2)) = 2$. If $\dim(N(A^2)) = 2$, then $\dim(N(A^2)) = \dim(N(A^3)) = 2$ and long with $N(A^2) \subseteq N(A^3)$ imply $N(A^2) = N(A^3)$. (1 mark for this observation.) Otherwise $\dim(N(A)) = \dim(N(A^2)) = 1$ along with $N(A) \subseteq N(A^2)$ imply that $N(A) = N(A^2)$. This in turn imply that $N(A^3) = N(A^2)$. (1 mark for this observation.)