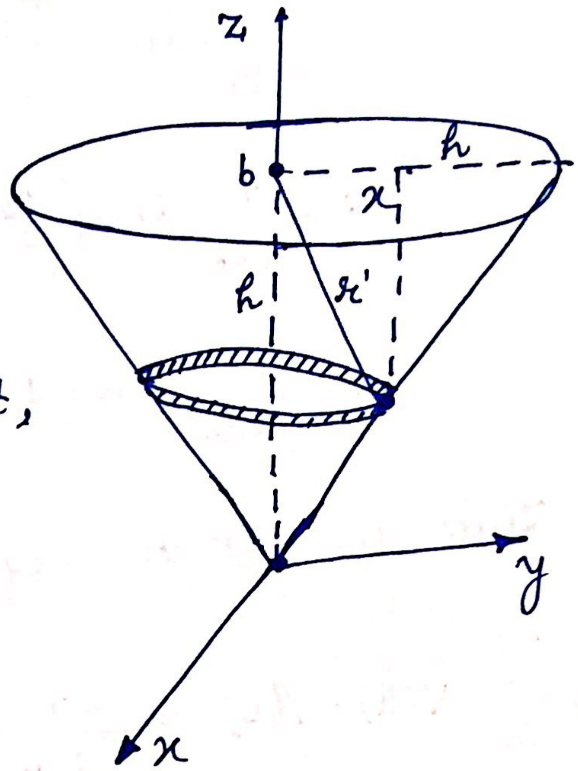
Solution of Problem Set #5

#5·1 The potential $V(\vec{a})$ at point $\vec{a}$ is given by



$$V(\vec{a}) = \frac{1}{4\pi\varepsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi x}{r}\right)dr$$

We have $x = r/\sqrt{2}$. Therefore we get,

$$V(\vec{a}) = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_0^{\sqrt{2}h}\left(\frac{1}{\sqrt{2}}\right)dr$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0}\frac{1}{\sqrt{2}}(\sqrt{2}h)$$

$$= \frac{\sigma h}{2\varepsilon_0}$$

The potential $V(\vec{b})$ at point $\vec{b}$ is given by

$$V(\vec{b}) = \frac{1}{4\pi\varepsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi x}{r'}\right)dr$$

We have $x = r/\sqrt{2}$ and $r' = \sqrt{h^2+r^2-\sqrt{2}hr}$. Therefore we get,

$$V(b) = \frac{1}{4\pi\varepsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi x}{r'}\right)dr$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0}\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}h}\left(\frac{r}{\sqrt{h^2+r^2-\sqrt{2}hr}}\right)dr$$

$$= \frac{\sigma}{2\sqrt{2}\varepsilon_0}\left[\sqrt{h^2+r^2-\sqrt{2}hr} + \frac{h}{\sqrt{2}}\ln\left(2\sqrt{h^2+r^2-\sqrt{2}hr}+2r-\sqrt{2}h\right)\right.$$

$$= \frac{\sigma}{2\sqrt{2}\varepsilon_0}\left[h + \frac{h}{\sqrt{2}}\ln(2h+2\sqrt{2}h-\sqrt{2}h) - h - \frac{h}{\sqrt{2}}\ln(2h-\sqrt{2}h)\right]$$

$$= \frac{\sigma}{2\sqrt{2}\,\varepsilon_0}\frac{h}{\sqrt{2}}\left[\ln(2h+\sqrt{2}h) - \ln(2h-\sqrt{2}h)\right]$$

$$= \frac{\sigma h}{4\varepsilon_0}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

$$= \frac{\sigma h}{4\varepsilon_0}\ln\left(\frac{(2+\sqrt{2})^2}{2}\right)$$

$$= \frac{\sigma h}{2\varepsilon_0}\ln(1+\sqrt{2})$$

Thus we get the required potential difference to be

$$V(a) - V(b) = \frac{\sigma h}{2\varepsilon_0}\ln\left[1 - \ln(1+\sqrt{2})\right]$$

5.2 (a) The electric potential is:

$$V(r) = A\frac{e^{-\lambda r}}{r}$$

Therefore, the electric field $\vec{E}(r)$ can be written as

$$\vec{E} = -\vec{\nabla}V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{r}$$

$$= -A\left[\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\right]\hat{r}$$

$$= Ae^{-\lambda r}(1+\lambda r)\frac{\hat{r}}{r^2}$$

(b) The corresponding charge density $\rho(r)$ can be calculated by using the differential form of Gauss's law

$$\rho = \varepsilon_0\vec{\nabla}\cdot\vec{E}$$

Using the product rule for divergence,

$$\vec{\nabla}\cdot(fA) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$$

we obtain,

$$\rho = \varepsilon_0 \vec{\nabla}\cdot E = \varepsilon_0 A e^{-\lambda r}(1+\lambda r)\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) + \varepsilon_0 A \frac{\hat{r}}{r^2}\cdot\vec{\nabla}\left[e^{-\lambda r}(1+\lambda r)\right]$$

Next we use the properties of Dirac-delta function and the formula for gradient in spherical coordinates to get

$$\varepsilon_0 A e^{-\lambda r}(1+\lambda r)\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \varepsilon_0 A e^{-\lambda r}(1+\lambda r)4\pi\delta^3(\vec{r})$$
$$= \varepsilon_0 A 4\pi\delta^3(\vec{r})$$

$$\varepsilon_0 A \frac{\hat{r}}{r^2}\cdot\vec{\nabla}\left[e^{-\lambda r}(1+\lambda r)\right] = \varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\frac{\partial}{\partial r}\left[e^{-\lambda r}(1+\lambda r)\right]\hat{r}$$
$$= \varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\left[-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda\right]\hat{r}$$
$$= \varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\left[-\lambda^2 r e^{-\lambda r}\right]\hat{r}$$
$$= -\varepsilon_0 A\frac{\lambda^2}{r}e^{-\lambda r}$$

Therefore, we get the charge density $\rho(r)$ as

$$\rho = \varepsilon_0 A\left[4\pi\delta^3(\vec{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right]$$

(c) The total charge Q can now be calculated to be

$$Q = \int \rho\, d\tau$$
$$= \varepsilon_0 A 4\pi\int\delta^3(r)\,d\tau - \varepsilon_0 A\lambda^2\int\frac{e^{-\lambda r}}{r}4\pi r^2\,dr$$
$$= \varepsilon_0 A 4\pi - \varepsilon_0 A\lambda^2 4\pi\int r e^{-\lambda r}\,dr$$
$$= \varepsilon_0 A 4\pi - \varepsilon_0 A\lambda^2 4\pi\,(1/\lambda^2)$$
$$= 0$$

Therefore the total charge is zero.

#5·3 The potential $V(r)$ at ~~r~~ r due to the localized charge distribution is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{r\,\mathscr{r}}\,d\tau'$$

We note that the charge distribution has been respresented in the (r', θ', ϕ') coordinates. We take the Laplacian of the potential in (r, θ, ϕ) coordinates. Therefore, we get

$$\nabla^2 V(\vec{r}) = \nabla^2 \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \nabla^2 \frac{\rho(\vec{r}')}{\mathscr{r}}\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\left(\nabla^2 \frac{1}{\mathscr{r}}\right)d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\left(\vec{\nabla}\cdot\vec{\nabla}\frac{1}{\mathscr{r}}\right)d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\left(\vec{\nabla}\cdot \frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int -\rho(\vec{r}')\,4\pi\delta^3(\mathscr{r})\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int -\rho(\vec{r}')\,4\pi\delta^3(\vec{r}-\vec{r}')\,d\tau'$$

$$= \frac{-1}{\varepsilon_0}\rho(\vec{r})$$

Thus, we see that the given potential satisfies the Poisson's equation.

5.4 Electrostatic

(a) The electric field due to the two shells is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

for $(a < r < b)$, and $\vec{E}(\vec{r}) = 0$, otherwise.

The energy of this configuration is:

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_a^b \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

(b) Let's us first calculate the energy of the individual shells. The electric field due to the shell of radius a is

$$\vec{E}_a(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}, \quad \text{for } (r > a)$$

and $\vec{E}_a(\vec{r}) = 0$, otherwise.

The electric field due to the shell of radius b is

$$\vec{E}_b(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}, \quad \text{for } (r > b)$$

and $\vec{E}_b(\vec{r}) = 0$, otherwise.

Therefore, the energy of the first spherical shell is:

$$W_a = \frac{\varepsilon_0}{2}\int E_a^2 \, d\tau$$

$$= \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_0^\infty \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{q^2}{8\pi\varepsilon_0 a}$$

Similarly, the energy of the second spherical shell is

$$W_b = \frac{\varepsilon_0}{2} \int E_b^2 \, d\tau$$

$$= \frac{\varepsilon_0}{2} \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_b^\infty \left(\frac{1}{r^2}\right)^2 4\pi r^2 \, dr$$

$$= \frac{q^2}{8\pi\varepsilon_0 b}$$

The interaction energy of this system is therefore:

$$W_{int} = \varepsilon_0 \int \vec{E}_a \cdot \vec{E}_b \, d\tau$$

$$= W - W_a - W_b$$

$$= -\frac{q^2}{8\pi\varepsilon_0 b} - \frac{q^2}{8\pi\varepsilon_0 b}$$

$$= -\frac{q^2}{4\pi\varepsilon_0 b}$$

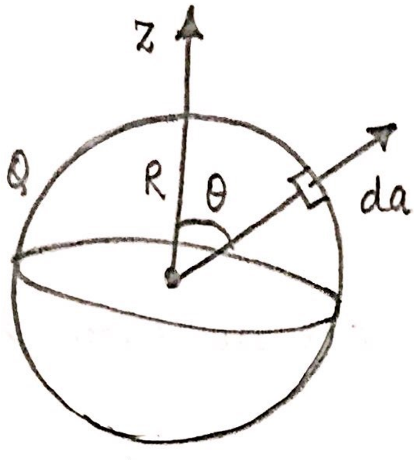
5.5 The electric field due to the metal sphere of radius R is given by $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}$, for $(r \geq R)$, and $\vec{E}(\vec{r}) = 0$, otherwise. From the symmetry of the problem, it is clear that the total electrostatic force on northern hemisphere will be in the z direction. Now the electrostatic force per unit area in the z direction at the area element da, as shown in figure a, is:

$$f_z = \sigma \vec{E}_{other} \cdot \hat{z} = \sigma \frac{\vec{E}(\vec{r})}{2} \cdot \hat{z}$$

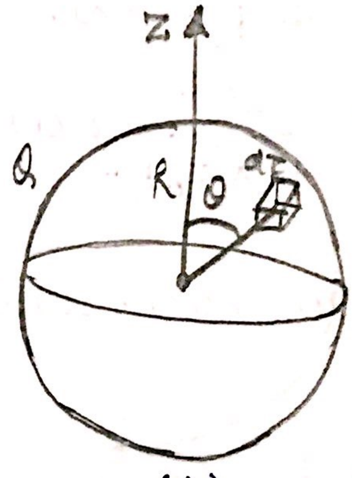
$$= \frac{Q}{4\pi R^2} \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \cos\theta$$

$$= \frac{Q^2}{32\pi^2 \varepsilon_0 R^4} \cos\theta$$

Therefore, the total repulsive force on the northern hemisphere is:



(a)



(b)

(ans).

$$F_z = \int f_z \, da = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \left(\frac{Q^2}{32\pi^2 \varepsilon_0 R^4} \cos\theta \right) R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{Q^2}{32\pi^2 \varepsilon_0 R^2} \, 2\pi \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= \frac{Q^2}{32\pi^2 \varepsilon_0 R^2} \, 2\pi \, \frac{1}{2}$$

$$= \frac{Q^2}{32\pi^{\cancel{2}} \varepsilon_0 R^2}$$

(b) The electric field inside a uniformly charged sphere of radius R and charge Q is given by $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q r}{R^3} \hat{r}$. From the symmetry of the problem, it is clear that the total electrostatic force on the northern hemisphere will be in the z direction. Now the electrostatic force per unit volume in the z direction on the volume element $d\tau$, as shown in figure b, is:

$$f_z = \rho \vec{E}(\vec{r}) \cdot \hat{z}$$

$$= \frac{3Q}{4\pi R^3} \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \cos\theta$$

$$= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} r\cos\theta$$

Therefore, the electrostatic force on the northern hemisphere is

$$F_z = \int f_z \, d\tau$$

$$= \int_0^R \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \left(\frac{3Q^2}{16\pi^2\epsilon_0 R^6} r\cos\theta \right) r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \int_0^R \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^3 \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \int_0^R r^3 dr \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \times \frac{R^4}{4} \times \frac{1}{2} \times 2\pi$$

$$= \frac{3Q^2}{64\pi\epsilon_0 R^2}$$

 # 5.6 Suppose that for a length L, the charge on the cylinder is Q and the charge on the outer cylinder is $-Q$. Using the Gaussian surface as shown in figure, it can be shown that the field in between the cylinder is $\vec{E}(\vec{s}) = \frac{Q}{2\pi\epsilon_0 L} \frac{1}{s} \hat{s}$.

The potential difference between the cylinder is therefore,

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot \vec{dl}$$

$$= -\frac{Q}{2\pi\epsilon_0 L} \int_a^b \frac{1}{s} ds.$$

$$= -\frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

We see that a is at a higher potential. So, we take the potential difference as $V = V(a) - V(b)$

$$= \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

The capacitance C of this configuration is therefore given by

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$