Problem Set 6

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Find the eigenvalues and corresponding eigenvectors of matrices

   (a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$

2. Construct a basis of $\mathbb{R}^3$ consisting of eigenvectors of the following matrices

   (a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

3. **(T)** This question deals with the following symmetric matrix A:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

   One eigenvalue is $\lambda = 1$ with the line of eigenvectors $x = (c, c, 0)$.

   (a) That line is the null space of what matrix constructed from A?

   (b) Find the other two eigenvalues of A and two corresponding eigenvectors.

   (c) The diagonalization $A = S\Lambda S^{-1}$ has a specially nice form because $A = A^t$. Write all entries in the three matrices in the nice symmetric diagonalization of A.

4. Let A be an $n \times n$ invertible matrix. Show that eigenvalues of A^{-1} are reciprocal of the eigenvalues of A, moreover, A and A^{-1} have the same eigenvectors.

5. Let A be an $n \times n$ matrix and α be a scalar. Find the eigenvalues of $A - \alpha I$ in terms of eigenvalues of A. Further show that A and $A - \alpha I$ have the same eigenvectors.

6. **(T)** Let A be an $n \times n$ matrix. Show that A^t and A have the same eigenvalues. Do they have the same eigenvectors?

7. Let A be an $n \times n$ matrix. Show that:

   (a) If A is idempotent ($A^2 = A$) then eigenvalues of A are either 0 or 1.

   (b) If A is nilpotent ($A^m = \mathbf{0}$ for some $m \geq 1$) then all eigenvalues of A are 0.

   (c) If $A^* = A$ then, the eigenvalues are all real.

   (d) If $A^* = -A$ then, the eigenvalues are either zero or purely imaginary.

   (e) Let A be a unitary matrix ($AA^* = I = A^*A$). Then, the eigenvalues of A have absolute value 1. It follows that if A is real orthogonal then the eigenvalues of A have absolute value 1. Give an example to show that the conclusion may be false if we allow **complex orthogonal**.

8. **(T)** Suppose that $A_{5\times 5}^{15} = \mathbf{0}$. Show that there exists a unitary matrix U such that U^*AU is upper triangular with diagonal entries 0.

9. **(T)** Suppose that $A_{17\times 17}^{29} = \mathbf{0}$. Show that $A^{17} = \mathbf{0}$.

10. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is NOT diagonalizable.

11. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable.

12. Show that Hermitian, Skew-Hermitian and unitary matrices are normal.

13. Suppose that $A = A^*$. Show that rank$A$ = number of nonzero eigenvalues of A. Is this true for each square matrix? Is this true for each square symmetric complex matrix?

14. Show that $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ is diagonalizable. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

15. Let $A = \begin{bmatrix} 7 & -5 & 15 \\ 6 & -4 & 15 \\ 0 & 0 & 1 \end{bmatrix}$. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix and hence calculate A^6.

16. Consider the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ 1 & d & e \\ 0 & 1 & f \end{bmatrix}.$$

Determine the entries a, b, c, d, e, f so that:

- the top left 1×1 block is a matrix with eigenvalue 2;
- the top left 2×2 block is a matrix with eigenvalue 3 and -3;
- the top left 3×3 block is a matrix with eigenvalue 0, 1 and -2.

17. **NOT for mid-sem or end-sem**

    (a) Find the eigenvalues and eigenvectors (depending on c) of

    $$A = \begin{bmatrix} 0.3 & c \\ 0.7 & 1-c \end{bmatrix}.$$

    For which value of c is the matrix A not diagonallizable (so $A = S\Lambda S^{-1}$ is impossible)?

    (b) What is the largest range of values of c (real number) so that A^n approaches a limiting matrix A^∞ as $n \to \infty$?

    (c) What is that limit of A^n (still depending on c)? You could work from $A = S\Lambda S^{-1}$ to find A^n.