Problem Set 6

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Find the eigenvalues and corresponding eigenvectors of matrices

(a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$

Solution:

(a) $(1-\lambda)^2 - 4 = 0 \Rightarrow (\lambda-3)(\lambda+1) = 0 \Rightarrow \lambda_1 = 3,\ \lambda_2 = -1$. Also, $v_1 = [\ 1/2 \ \ 1\ ]^T,\ v_2 = [\ -1/2 \ \ 1\ ]^T$.

(b) $\lambda_1 = 0,\ \lambda_2 = -2,\ \lambda_3 = -3$ and $v_1 = [\ 0 \ \ -1 \ \ 1\ ]^T,\ v_2 = [\ -2 \ \ 1 \ \ 0\ ]^T,\ v_3 = [\ -1 \ \ 0 \ \ 1\ ]^T$.

2. Construct a basis of $\mathbb{R}^3$ consisting of eigenvectors of the following matrices

(a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

Solution:

(a) Eigenvalues are $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 4$ and eigenvectors are $\mathbf{v}_1 = [\ 1 \ \ 0 \ \ -1/2\ ]^T,\ \mathbf{v}_2 = [\ 0 \ \ 1 \ \ 0\ ]^T,\ \mathbf{v}_3 = [\ 1 \ \ 0 \ \ 2\ ]^T$. Since eigenvectors corresponding to different eigenvalues are linearly independent, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

(b) Similar to (a).

3. **(T)** This question deals with the following symmetric matrix A:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

One eigenvalue is $\lambda = 1$ with the line of eigenvectors $x = (c, c, 0)$.

(a) That line is the null space of what matrix constructed from A?

Solution: The eigenvectors of $\lambda = 1$ makes the null space of $A - I$.

(b) Find the other two eigenvalues of A and two corresponding eigenvectors.

Solution: A has trace 2 and determinant -2. So the two eigenvalues after $\lambda_1 = 1$ will add to 1 and multiply to -2. Those are $\lambda_2 = 2$ and $\lambda_3 = -1$. Corresponding eigenvectors are :

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

(c) The diagonalization $A = S\Lambda S^{-1}$ has a specially nice form because $A = A^t$. Write all entries in the three matrices in the nice symmetric diagonalization of A.

Solution: Every symmetric matrix has the nice form $A = Q\Lambda Q^t$ with an orthogonal matrix Q. The columns of Q are orthonormal eigenvectors.

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \text{-1} \end{bmatrix}.$$

4. Let A be an $n\times n$ invertible matrix. Show that eigenvalues of A^{-1} are reciprocal of the eigenvalues of A, moreover, A and A^{-1} have the same eigenvectors.

   Solution: $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{x} = \lambda A^{-1}\mathbf{x} \Rightarrow A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$ (Note that $\lambda \neq 0$ as A is invertible implies that $\det(A) \neq 0$).

5. Let A be an $n\times n$ matrix and α be a scalar. Find the eigenvalues of $A-\alpha I$ in terms of eigenvalues of A. Further show that A and $A-\alpha I$ have the same eigenvectors.

   Solution: If λ is an eigenvalue of $A-\alpha I$ with eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = (A-\alpha I)\mathbf{v} + \alpha\mathbf{v} = (\lambda+\alpha)\mathbf{v}.$$

   Thus, A and $A-\alpha I$ have same eigenvectors and eigenvalues of $A-\alpha I$ is $\mu-\alpha$ if μ is an eigenvalue of A.

6. **(T)** Let A be an $n \times n$ matrix. Show that A^t and A have the same eigenvalues. Do they have the same eigenvectors?

   Solution: Follows directly from $\det(A-\lambda I) = \det((A-\lambda I)^t) = \det(A^t - \lambda I)$. Eigenvectors are not same. Here is a counter example :

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

7. Let A be an $n \times n$ matrix. Show that:

   (a) If A is idempotent ($A^2 = A$) then eigenvalues of A are either 0 or 1.

   Solution: Let $A\mathbf{v} = \lambda\mathbf{v}$. Then $\lambda\mathbf{v} = A\mathbf{v} = A^2\mathbf{v} = \lambda^2\mathbf{v} \Rightarrow \lambda(\lambda-1)\mathbf{v} = \mathbf{0}$. Result follows.

   (b) If A is nilpotent ($A^m = \mathbf{0}$ for some $m \geq 1$) then all eigenvalues of A are 0.

   Solution: Let $A\mathbf{v} = \lambda\mathbf{v}$. Then $A^m\mathbf{v} = \lambda^m\mathbf{v}$. Now, $A^m = \mathbf{0} \Rightarrow \lambda^m = 0 \Rightarrow \lambda = 0$.

   (c) If $A^* = A$ then, the eigenvalues are all real.

   Solution: Let $(\lambda, \mathbf{x})$ be an eigenpair. Then

$$\lambda\mathbf{x}^*\mathbf{x} = \mathbf{x}^*(\lambda\mathbf{x}) = \mathbf{x}^*(A\mathbf{x}) = \overline{(\mathbf{x}^*A\mathbf{x})^*} = \overline{\mathbf{x}^*A^*\mathbf{x}} = \overline{\mathbf{x}^*A\mathbf{x}} = \overline{\lambda\mathbf{x}^*\mathbf{x}} = \overline{\lambda}\mathbf{x}^*\mathbf{x}.$$

   Hence, the required result follows.

(d) If $A^* = -A$ then, the eigenvalues are either zero or purely imaginary.

Solution: Proceed as in the above problem.

(e) Let A be a unitary matrix ($AA^* = I = A^*A$). Then, the eigenvalues of A have absolute value 1. It follows that if A is real orthogonal then the eigenvalues of A have absolute value 1. Give an example to show that the conclusion may be false if we allow **complex orthogonal**.

Solution: Let $(\lambda, \mathbf{x})$ be an eigenpair of A. Then

$$\|\mathbf{x}\|^2 = \mathbf{x}^*\mathbf{x} = \mathbf{x}^*(A^*A)\mathbf{x} = (\mathbf{x}^*A^*)(A\mathbf{x}) = (A\mathbf{x})^*(A\mathbf{x}) = (\lambda\mathbf{x})^*(\lambda\mathbf{x}) = \mathbf{x}^*\overline{\lambda}\lambda\mathbf{x} = |\lambda|^2\|\mathbf{x}\|^2.$$

So $|\lambda|^2 = 1$. For counter example, take $A = \begin{bmatrix} \sqrt{2} & i \\ -i & \sqrt{2} \end{bmatrix}$.

8. **(T)** Suppose that $A_{5\times 5}^{15} = \mathbf{0}$. Show that there exists a unitary matrix U such that U^*AU is upper triangular with diagonal entries 0.

   Solution: There exists U unitary such that $U^*AU = T$, upper triangular with $\text{diag}(T) = \{\lambda_1, \ldots, \lambda_5\}$. Hence T^{15} has diagonal entries $\lambda_1^{15}, \ldots, \lambda_5^{15}$. As $0 = U^*A^{15}U = T^{15}$ we see that $\lambda_i^{15} = 0$. So, $\lambda_i = 0$ for all i.

9. **(T)** Suppose that $A_{17\times 17}^{29} = \mathbf{0}$. Show that $A^{17} = \mathbf{0}$.

   Solution: There exists U unitary such that $U^*AU = T$, upper triangular with $\text{diag}(T) = \{\lambda_1, \ldots, \lambda_{17}\}$. As $A^{29} = \mathbf{0}$, it follows that $\lambda_i = 0$. So, $A = UTU^*$, $A^2 = UT^2U^*, A^3 = UT^3U^*$ and so on. Also, verify that as T is upper triangular with zeroes on the diagonal, we must have $T^{17} = \mathbf{0}$. So, the result follows.

   Alternate: As each eigenvalue of A is 0, the characteristic polynomial, namely $p_A(x) = x^{17}$. So, by Cayley Hamilton theorem, $A^{17} = \mathbf{0}$.

10. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is NOT diagonalizable.

11. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable.

12. Show that Hermitian, Skew-Hermitian and unitary matrices are normal.

13. Suppose that $A = A^*$. Show that $\text{rank}A =$ number of nonzero eigenvalues of A. Is this true for each square matrix? Is this true for each square symmetric complex matrix?

    Solution: By spectral theorem, there exists U, unitary such that $U^*AU = D$, diagonal. Since U is invertible, we see that $\text{rank}A = \text{rank}U^*AU = \text{rank}D =$ number of nonzero entries of $D =$ eigenvalues of A.

    It is not true for general square matrices, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here $\text{rank}A = 1$, whereas both eigenvalues are 0.

    It is not true for a general complex symmetric matrix, consider $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$. Here $\text{rank}A = 1$, whereas both eigenvalues are 0 (as $\det A = 0, \text{tr}A = 0$).

14. Show that $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ is diagonalizable. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

Solution: $det(A - \lambda I) = (1-\lambda)(3-\lambda)^2$. Therefore, eigen-values are 1 and 3. The eigen spaces (null space of $A - \lambda I$), are given by $E_1 = \{\mathbf{x} : A\mathbf{x} = \mathbf{x}\} = \{(x_1, x_2, x_3) : x_2 = x_1,\ x_3 = -2x_1,\ x_1 \in \mathbb{R}\} = \mathrm{LS}(\{(1,1,-2)\})$ and $E_3 = \{(x_1, -x_1, x_3) : x_1,\ x_3 \in \mathbb{R}\} = \mathrm{LS}(\{(1,-1,0),\ (0,0,1)\})$. Clearly, $\{(1,1,-2),\ (1,-1,0),\ (0,0,1)\}$ are linearly independent and hence A is diagonalizable.

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

15. Let $A = \begin{bmatrix} 7 & -5 & 15 \\ 6 & -4 & 15 \\ 0 & 0 & 1 \end{bmatrix}$. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix and hence calculate A^6.

Solution: $det(A - \lambda I) = (\lambda-1)^2(\lambda-2)$. Therefore, eigen-values are 1 and 2. $E_1 = \{(x_1, x_2, x_3) : 6x_1 - 5x_2 + 15x_3 = 0\} = \mathrm{LS}(\{(1,0,-6/15),\ (0,1,1/3)\})$. $E_2 = \{(x_1, x_1, 0) : x_1 \in \mathbb{R}\} = \mathrm{LS}(\{(1,1,0)\})$. For

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -\frac{6}{15} & \frac{1}{3} & 0 \end{bmatrix},$$

we have

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore

$$A^6 = S^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^6 \end{bmatrix} S.$$

16. Consider the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ 1 & d & e \\ 0 & 1 & f \end{bmatrix}.$$

Determine the entries a, b, c, d, e, f so that:

- the top left 1×1 block is a matrix with eigenvalue 2;
- the top left 2×2 block is a matrix with eigenvalue 3 and -3;
- the top left 3×3 block is a matrix with eigenvalue 0, 1 and -2.

Solution: Let A_i denote the top left $i \times i$ block of A. The matrix A_1 is the matrix $[a]$. Since a is the only eigenvalue of this matrix, we conclude that $a = 2$.

We now move onto determining the entries of the matrix A_2 : $A_2 = \begin{bmatrix} 2 & b \\ 1 & d \end{bmatrix}$.

Since the sum of the eigenvalues of A_2 is 0 by hypothesis, and it is also equal to the trace of A_2, we obtain that $2 + d = 0$ or $d = -2$. Moreover the product of the eigenvalues of A_2 is -9 by hypothesis, and it is qual to the determinant of A_2. Thus we have

$$-9 = 2d - b = -4 - b$$

and we deduce that $b = 5$ and therefore $A_2 = \begin{bmatrix} 2 & 5 \\ 1 & \text{-2} \end{bmatrix}$.

Finally, consider $A = A_3$. Again, the sum of the eigenvalues of A is -1 and it is also equal to the trace of A. We deduce that $f = -1$. We still need to determine the entries c and e of A and we have

$$A = \begin{bmatrix} 2 & 5 & \text{c} \\ 1 & \text{-2} & \text{e} \\ 0 & 1 & \text{-1} \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$-\lambda^3 - \lambda^2 + (e + 9)\lambda + c - 2e + 9.$$

We know that the roots of this polynomial must be 0, 1 and -2. Setting $\lambda = 0$ and $\lambda = 1$, we obtain

$$\begin{aligned} c - 2e + 9 &= 0 \\ -1 - 1 + (e + 9) + c - 2e + 9 &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} c - 2e &= -9 \\ c - e &= -16. \end{aligned}$$

Thus $c = -7$ and $e = 9$ and we conclude

$$A = \begin{bmatrix} 2 & 5 & \text{-7} \\ 1 & \text{-2} & 9 \\ 0 & 1 & \text{-1} \end{bmatrix}.$$

17. **NOT for mid-sem or end-sem**

(a) Find the eigenvalues and eigenvectors (depending on c) of

$$A = \begin{bmatrix} 0.3 & c \\ 0.7 & 1 - c \end{bmatrix}.$$

For which value of c is the matrix A not diagonallizable (so $A = S\Lambda S^{-1}$ is impossible)?

Solution: Eigen values are $\lambda = 1$ and $\lambda = 0.3 - c$. The eigenvector for $\lambda = 1$ is in the null space of

$$A - I = \begin{bmatrix} \text{-0.7} & \text{c} \\ 0.7 & \text{-c} \end{bmatrix}$$

so

$$\mathbf{x}_1 = \begin{bmatrix} c \\ 0.7 \end{bmatrix}.$$

Similarly, the eigenvector for $\lambda = 0.3 - c$ is in the null space of

$$A - (0.3 - c)I = \begin{bmatrix} c & c \\ 0.7 & 0.7 \end{bmatrix}$$

so

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ \text{-1} \end{bmatrix}.$$

A is not diagonalizable when its eigen values are equal : $1 = 0.3 - c$ or $c = -0.7$.

(b) What is the largest range of values of c (real number) so that A^n approaches a limiting matrix A^∞ as $n \to \infty$?

Solution:

$$A^n = S\Lambda^n S^{-1} = S \begin{bmatrix} 1 & 0 \\ 0 & (0.3 - c)^n \end{bmatrix} S^{-1}.$$

This approaches a limit if $|0.3 - c| < 1$. We could write that out as $-0.7 < c < 1.3$.

(c) What is that limit of A^n (still depending on c)? You could work from $A = S\Lambda S^{-1}$ to find A^n.

Solution: The eigen vectors are in S. As $n \to \infty$, the smaller eigen value λ_2^n goes to zero, leaving

$$\begin{aligned} A^\infty &= \begin{bmatrix} c & 1 \\ 0.7 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.7 & -c \end{bmatrix} /(c + 0.7) \\ &= \begin{bmatrix} c & c \\ 0.7 & 0.7 \end{bmatrix} /(c + 0.7). \end{aligned}$$