* Topology is "Rubber Sheet Geometry".

Properties of a space which remains invariant under ~~any~~ "continuous deformation"

* Two spaces X and Y are said to be topologically equivalent (or) homeomorphic if $\exists$ a cts fun^c $f : X \longrightarrow Y$ st f is 1-1, onto, cts, f^{-1} is cts

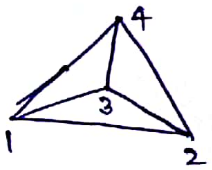
$$X \underset{\text{homeo}}{\cong} Y$$

* If $X \underset{\text{homeo}}{\simeq} Y \Rightarrow \mu(X) = \mu(Y)$

A topological invariant of a space X, $\mu(X)$ where $\mu(X)$ can be a number, group, ring or "captures" connectedness.

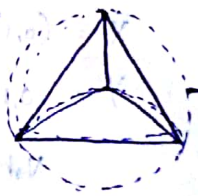
* 1750, Euler

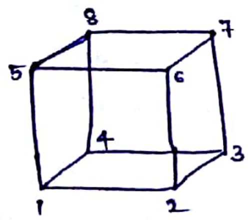
$$\chi(P) = v - e + f$$



v = 4
e = 6
f = 4

v − e + f = 2

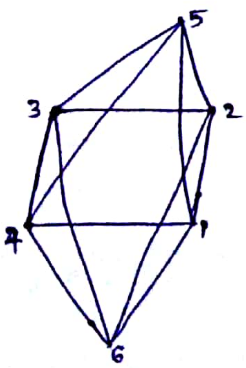




V = 8
e = 12
f = c

v − e + f = 2

All these polyhedrons are homeomorphic to S^2



V = 6
e = 12
f = 8

v − e + f = 2

$$P \underset{\text{homeo}}{\simeq} S^2$$

$\Rightarrow \chi(P)$ is a topological invariant

Q) Given spaces X and Y when is X homeomorphic to Y, $X \underset{\text{homeo}}{\simeq} Y$?

Eg:

1. $(0,1)$

$(0,2)$

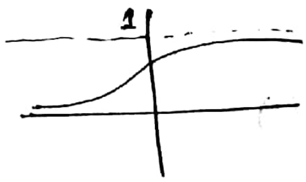
$f(x) = 2x$

homeomorphic

2. $X = (0,1)$

$Y = (-\infty, \infty)$

$y = \dfrac{e^x}{1+e^x}$



homeomorphic

3. $X = (0,1)$ → Not compact

$Y = [0,1]$ → Compact

Not homeomorphic

$\mu(X)$ = Compactness

4. $X = (0,1)$ → Connected

$Y = (0,1) \cup (1,2)$ → Not connected

Not homeomorphic

$\mu(X)$ = Connectedness

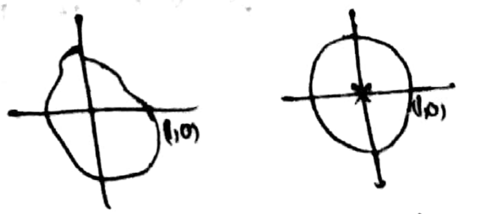
5. $X = \mathbb{R} \times \mathbb{R}$ → Simply connected

$Y = \mathbb{R} \times \mathbb{R} \setminus \{0,0\}$ → Not simply connected

Both not-compact and connected

Not homeomorphic

$\mu(X)$ = Simply connected



can't shrink

* Simply connected is a special case of Fundamental group of a space X

Topological invariant

$[\alpha]$

α loop based at x_0

* $\mathbb{R}$ & $\mathbb{R}^2$ Not homeomorphic

$\mathbb{R} \setminus$ any point $\rightarrow$ Not connected

$\mathbb{R}^2 \setminus$ any point $\rightarrow$ Connected

$\mathbb{R}$ ~~&~~ $\mathbb{R}^n$ $\rightarrow$ Not homeomorphic

Q) $\mathbb{R}^m \underset{homeo}{\simeq} \mathbb{R}^n$?

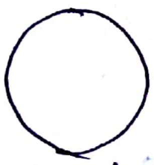
* <u>This course</u>

First half $\rightarrow$ Knot Theory.

* <u>Knot Theory</u> :

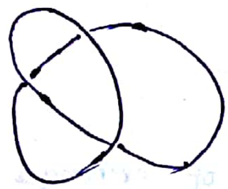
· A <u>knot</u> is a continuous map $f: S^1 \rightarrow \mathbb{R}^3 K$, f is 1–1, onto K, continuous.

$K \subset \mathbb{R}^3$



Unknot
(Trivial knot)

3_1



Trefoil knot

4_1

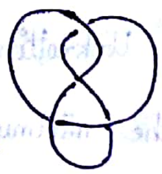


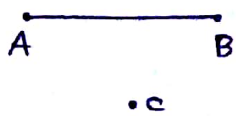
Figure eight knot
Listing knot

Tait Conjuctures $\rightarrow$ 3 conjuctures

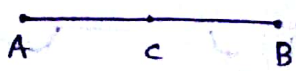
Q) Given knots K_1 and K_2. When is K_1 equivalent to K_2.

* <u>Elementary Knot Move</u> :

1.



$\rightleftarrows$



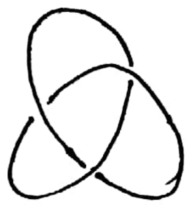
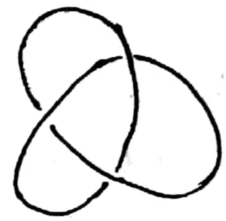
2.

$\rightleftarrows$

Def^n :

Given knots K_1 and K_2. K_1 is said to be equivalent to K_2 if we can convert K_1 to K_2 by finitely many elementary knot moves.

Eg:

3_1   3_1^*

$3_1 \not\simeq 3_1^*$

Knot invariants :

$\mu(K)$ is a quantity s.t

if $K_1 \underset{\text{equiv}}{\simeq} K_2$ then $\mu(K_1) = \mu(K_2)$

Def^n :

1. $C(K)$ "Crossing Number"
   is the minimum number of crossings over all possible knot diagrams.

2. $u(K)$ " Unknotting Number"
   is the minimum number of exchanges needed to unknot the knot.

   $u(3_1) = 1$

Def^n :

Given knots K_1 and K_2, we can form their connected sum $K_1 \# K_2$

3_1^*

3_1

$3_1 \# 3_1^*$

Q) Is $C(K_1 \# K_2) = C(K_1) + C(K_2)$? → Big open problem Kawby 1.65

Q) Is $u(K_1 \# K_2) = u(K_1) + u(K_2)$? → Big open problem Kawby 1.69

* Alexander, a topologist discovered a knot invariant which was a polynomial called Alexander polynomial.

Eg: For Trefoil ~~pol~~ knot

$(-1)\,[1 -1 +1]$

$t^{-1}(1 - t + t^2)$

$t^{-1} - 1 + t$ → Alexander polynomial for 3_1

09/01/2018
Tuesday

* Defn:

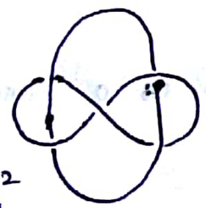
A Link is a finite ordered collection of knots that donot intersect each other.

Eg:

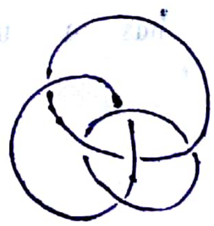
O_2 (Trivial link)

2_1^2 Hopf Link

4_1^2



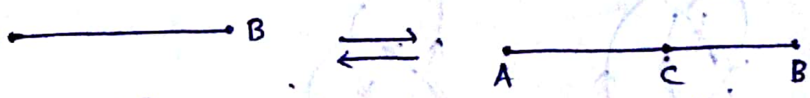
5_1^2 (Whitehead link)



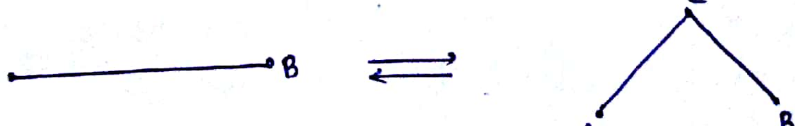
6_1^3 Borromean Rings.

Elementary Knot moves

1.



2.



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* **Def^n:**

Two links $L = \{K_1, K_2, \ldots, K_n\}$ and $L' = \{K_1', K_2', \ldots, K_m'\}$ are equivalent if $m = n$ and if we can convert L to L' by finitely many elementary knot moves.

* **Def^n:**

A link invariant $\mu(L)$ is a quantity s.t if $L \underset{equi}{\simeq} L'$ then $\mu(L) = \mu(L')$.

* Crossing Number $C(L)$ 1) $C(L_1 \# L_2) = C(L_1) + C(L_2)$?

Unknotting Number $u(L)$ 2) $u(L_1 \# L_2) = u(L_1) + u(L_2)$?

(Knots are a part of Links)

Eg: 6_1^3, 6_2^3, 6_3^3 (See Colin Adams Book)

* **Def^n:**

A Knot K is said to be a prime Knot if whenever $K = K_1 \# K_2$ either $K_1 \simeq O_1$ or $K_2 \simeq O_1$.

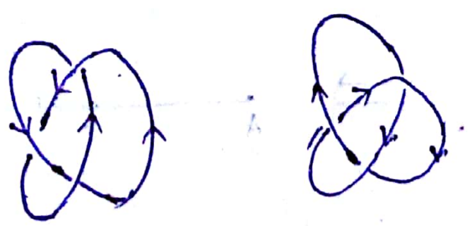
* **Theorem:**

Any knot K has a unique decomposition as a connected sum of prime knots.

* **Def^n:**

An oriented knot K is a knot with a choice of orientation.

Eg:



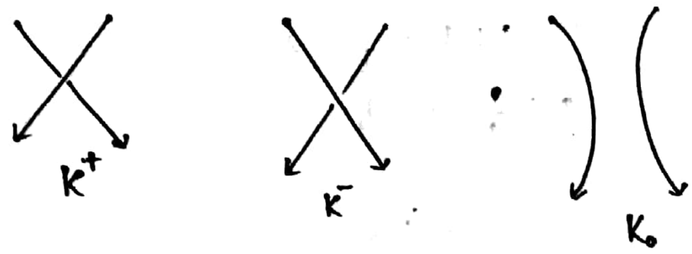
Def^n: (Alexander Polynomial)

Given an oriented knot (link) K, we define the Alexander-Conway polynomial $\nabla_K(z)$ as

Axiom 1:

If $K \simeq O_1$ (trivial), $\nabla_K = 1$

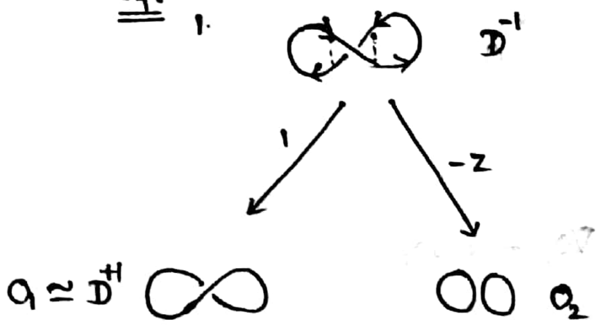
Axiom 2:



$$\nabla_{K^+}(z) - \nabla_{K^-}(z) = z\nabla_{K_0}(z)$$

Suppose K differs only at one crossing as K_+, K_-, K_0

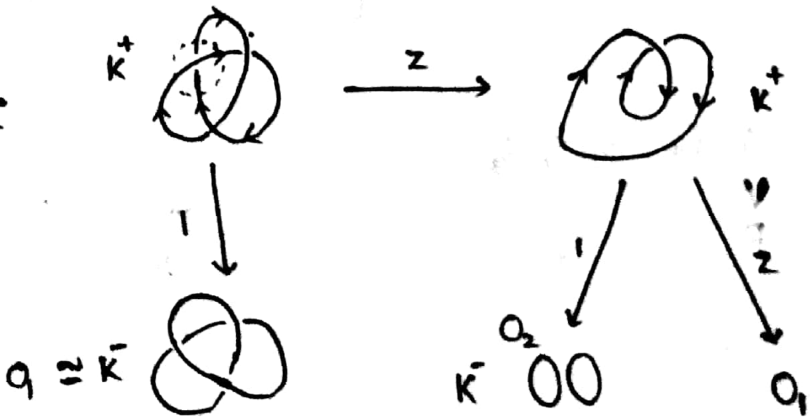
Eg: 1.



$$\nabla_{O_1} = 1\,\nabla_{O_1} - z.\nabla_{O_2}$$
$$\Rightarrow \nabla_{O_2} = 0.$$

2.

$\nabla_{K^+} = \nabla_{K^-} + z\nabla_{K^0}$



$$\nabla_{3_1}(z) = 1 + z^2 \qquad \nabla_{O_1} = 1, \quad \nabla_{O_2} = 0.$$

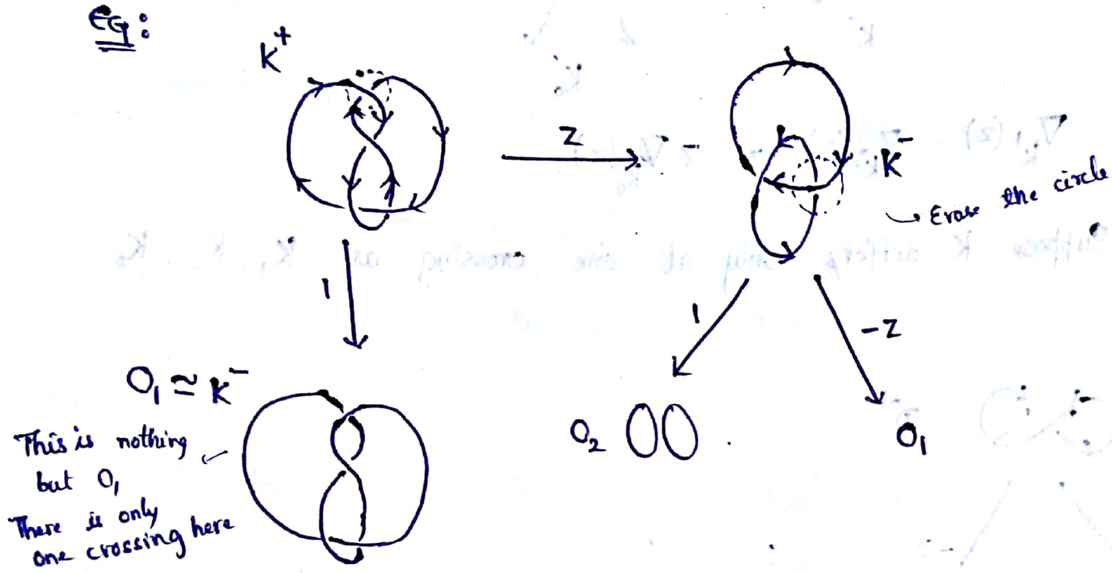
$$\boxed{\text{Alexander polynomial } \Delta_K(t) = \nabla\left(z = \sqrt{t} - \frac{1}{\sqrt{t}}\right)}$$

Alex polynomial $\Delta_{3_1}(t) = 1 + \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2$

$$= 1 + t + \frac{1}{t} - 2$$

$$= \frac{1}{t} - 1 + t$$

Eg:



$$\nabla_{4_1}(z) = 1 \cdot \nabla_{O_1} + z \cdot \nabla_{O_2} - z^2 \nabla_{O_1}$$

$\nabla_{O_1} = 1$
$\nabla_{O_2} = 0$

$$= 1 - z^2$$

Alex polynomial $\Delta_{4_1}(t) = 1 - \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2$

$$= 1 - \left(t + \frac{1}{t} - 2\right)$$

$$= -\frac{1}{t} + 3 - t$$

Homework :

Calculate Δ for $5_1, 5_2, 6_3$
" ∇ " "
" P " "

* <u>Def^n</u> : (Jones polynomial)

Suppose K is an oriented knot (or link). Then the Jones polynomial $V_K(t)$ is uniquely defined from 2 axioms

<u>Axiom 1</u>

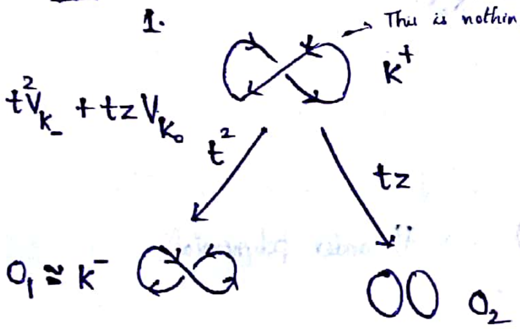
If $K \simeq O_1$ then $V_K(t) = 1$

<u>Axiom 2</u>

$$\frac{1}{t} V_{K_+} - t V_{K_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K_0}$$

<u>Eg</u> :

1.



$V_{K_+} = t^2 V_{K_-} + tz V_{K_0}$

$$V_{K_+}(t) = t^2 V_{K_-}(t) + tz V_{O_2}(t)$$

$$1 = t^2 \cdot 1 + t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{O_2}$$

$$V_{O_2} = \frac{1 - t^2}{t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)} = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)$$

$$V_{K_+} = t^2 V_{K_-} + tz V_{K_0}$$

$$V_{K_-} = \frac{1}{t^2} V_{K_+} - \frac{z}{t} V_{K_0}$$

2. $V_K(t)$ for 4_1

$$\begin{tikzcd} 4_1 \arrow[r, "tz"] \arrow[d, "t^2"'] & K^- \arrow[dl, "\frac{1}{t^2}"'] \arrow[dr, "-\frac{z}{t}"] & \\ O_1 & O_2 & O_1 \end{tikzcd}$$

$$V_{4_1}(t) = t^2 V_{O_1}(t) + \frac{z}{t} V_{O_2}(t) - z^2 V_{O_1}(t)$$

$$= t^2 \cancel{+} - \frac{1}{t}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) - \cancel{t}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2$$

$$= t^2 - t + 1 - \frac{1}{t} + \frac{1}{t^2}$$

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* **Open problem**

Given a knot K, if $V_K(t) = 1$ is K the unknot?

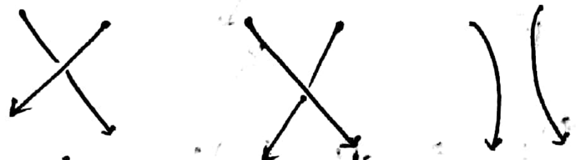
* **HOMFLY Polynomial:**

Let K be an oriented knot (or link). Then the 2-variable HOMFLY Polynomial is uniquely determined by

Axiom 1

If $K \simeq O_1$, $P_K(v,z) = 1$

Axiom 2



K^+ $\quad K^-$ $\quad K_0$

$$\frac{1}{v} P_{K^+}(v,z) - v P_{K^-}(v,z) = z \cdot P_{K_0}(v,z)$$

NOTE:

1. $v = 1$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$

$$P_K\left(1, \sqrt{t} - \frac{1}{\sqrt{t}}\right) = \Delta_K(t) \hookrightarrow \text{Alexander polynomial}$$

2. $v = t$; $z = \sqrt{t} - \frac{1}{\sqrt{t}}$

$$P_K\left(t, \sqrt{t} - \frac{1}{\sqrt{t}}\right) = V_K(t) \hookrightarrow \text{Jones polynomial}$$

$\hookrightarrow$ Alexander & Jones polynomial are knot invariants.

* **Basic question in Knot Theory**

Given any two arbitrary knots K_1, K_2, is K_1 equivalent to K_2 (or) not ?

If $K_1 \simeq K_2$, got to show equivalent.

If $K_1 \not\simeq K_2$, then we need to prove it by finding some knot invariant $\mu(K)$ s.t $\mu(K_1) \neq \mu(K_2)$.

$$\left[\begin{array}{l} \text{If } K_1 \simeq K_2 \Rightarrow \mu(K_1) = \mu(K_2) \\ \text{But converse doesn't hold.} \end{array} \right]$$

→ "Complete Knot invariant" $\mu(K)$ s.t

$$\mu(K_1) = \mu(K_2) \Rightarrow K_1 \simeq K_2$$

So far, no complete knot invariant is known.

* **Symmetry** :

Given a knot K, we can form its mirror image K^* by interchanging all crossings.

- If $K \simeq K^*$, we say K is "Achiral" (or) "Amphichiral".

3_1  $\overset{?}{\simeq}$  3_1^*

$c(3_1) = c(3_1^*) = 3$

$u(3_1) = u(3_1^*) = 1$

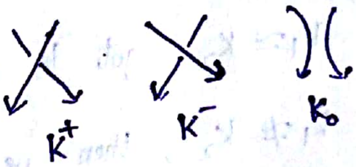
$$\Delta_{3_1} = \Delta_{3_1^*} = \frac{1}{t} - 1 + t$$

*** Theorem :**

If a knot K is equivalent to its mirror image K^*, then we have $V_{K^*}(t) = V_K(\frac{1}{t})$

Proof :

Recall : $\frac{1}{t}V_{K^+} - tV_{K^-} = zV_{K_0} \longrightarrow (1)$



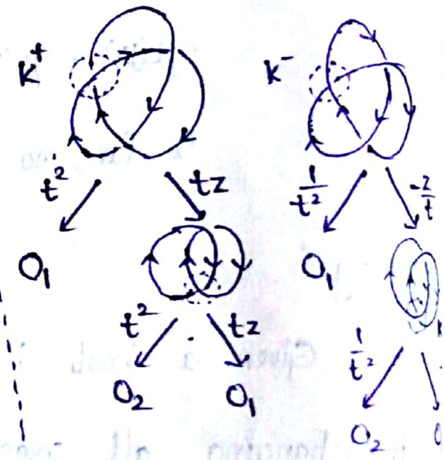
$z = \sqrt{t} - \frac{1}{\sqrt{t}}$

$V_{K^+} = t^2 V_{K^-} + tzV_{K_0} \longrightarrow (2)$

$V_{K^-} = \frac{1}{t^2}V_{K^-} - \frac{z}{t}V_{K_0} \longrightarrow (3)$

Suppose D is a regular diagram of knot K
and D^* " " " " " K^*

If the skein tree diagram of D is R we form the skein tree diagram of D^* in the "same manner" - where we perform a skein tree operation at a crossing c on D, we perform the same operation on D^*



At any crossing in D & D^*, coefficients will differ precisely as follows

$t^2 \longrightarrow \frac{1}{t^2}$; $tz \longrightarrow -\frac{z}{t}$ $\quad \frac{1}{t}(\frac{1}{\sqrt{t}} - \sqrt{t})$

$t(\sqrt{t} - \frac{1}{\sqrt{t}}) \longrightarrow -\frac{1}{t}(\sqrt{t} - \frac{1}{\sqrt{t}})$

D and D^* coefficients are exactly replacing t by $\frac{1}{t}$

$\therefore V_{K^+}(t) = V_{K^-}(t)$

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Eg: 1. $V_{3_1} = t + t^3 - t^4$

$V_{3_1^*} = \frac{1}{t} + \frac{1}{t^3} - \frac{1}{t^4}$

$\Rightarrow 3_1 \not\simeq 3_1^*$

"Converse is False"

(From 9_{42} it is false)

2. $V_{4_1} = V_{4_1^*}$

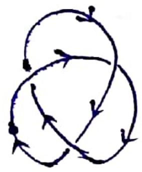
$\therefore 4_1 \simeq 4_1^*$

$\rightarrow$ Palindromic symmetry - $V_K(t) = V_K(\frac{1}{t})$

* Def^n:

A knot K is said to be Alternative if it has atleast one diagram whose the + and - signs alternate.

Eg: 1.



3_1 is an alternative knot

2. 8_{21} is not alternative knot

8_{19} " " " "

8_{20} " " " "

} only not alternative knots in the given paper.

* Theorem:

Let K be an alternating knot and $V_K(t)$ is its Jones polynomial

Let max. degree $V_K(t) = m$, min. degree $V_K(t) = n$

then span $V_K(t) = m - n = C(K)$, crossing number

$\rightarrow$ For a non-alternating knot, span $V_K(t) < C(K)$

Eg: $K = 8_{24}$

span $V_{8_{24}}(t) = 7 - 1$

$= 6 < 8$

$K = 8_{19}$

span $V_{8_{19}}(t) = 8 - 3$

$= 5 < 8$

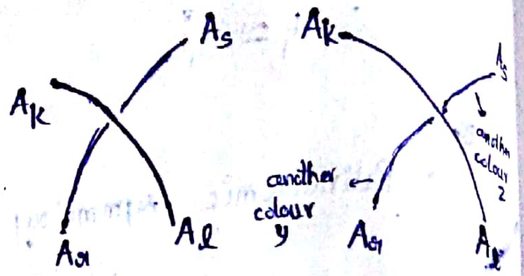
* Classical Knot invariants :

Suppose K has n crossing points $P_1, P_2, \dots$

It will divide K into $2n$ segments.

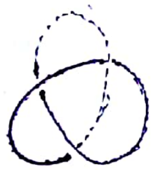
At each crossing point assign three colours R, B, Y s.t

1. A_k & A_l have same colour
2. A_k, A_r, A_s all have same colour or different colours.



Tricolourability : (classical knot invariant)

A regular diagram which can have all 3 colours assigned is "tricolourable".



→ 3 colours (Tricolourable)

- Does 8_1 has a tricolouration ? Yes.

* Def^n:

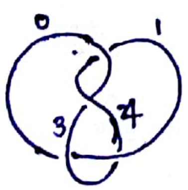
Let p be a prime.

A knot K is p-colourable if we can assign the integers $0, 1, 2, \dots, p-1$ s.t

1. $\lambda_k = \lambda_l$
2. $\lambda_r + \lambda_s \equiv \lambda_k + \lambda_l \pmod{p}$

(We don't have to use all the colours. We can use atleast 2)

eg:



→ 5 - colourable

$1+0 = 3+3 \pmod 5$

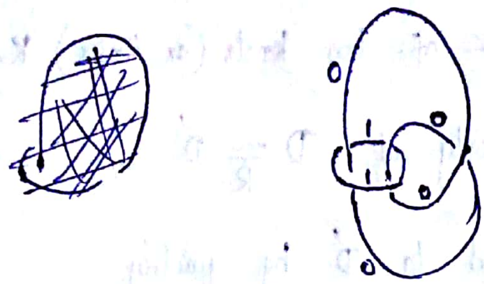
$0+0 = 4+1 \pmod 5$

$3+0 = 4+4 \pmod 5$

$1+1 = 4+3 \pmod 5$

NOTE : Atleast 2 distinct numbers should be used.

→ Links are 2-colourable

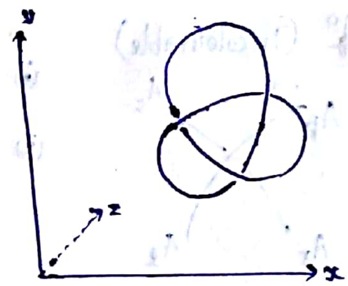


16/01/2018
Tuesday

* Two knots K_1 & K_2 are equivalent (or equal) if they differ by a finite sequence of elementary knot moves.

- A knot $K \subset \mathbb{R}^3$

  The representation of K on 2-dim plane is called a Knot diagram D



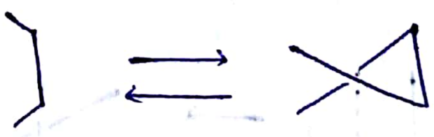
* Reidemeister (1926) – German Topologist

discovered the 3 $\underbrace{\text{Reidemeister moves}}_{\text{R-moves}}$ on a knot diagram

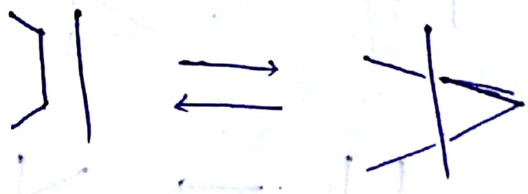
* <u>3 R-moves on a knot diagram D :</u>

First R-move

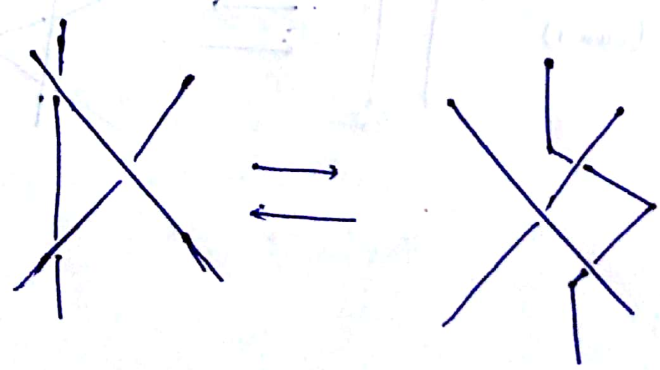
Ω_1



Ω_2



Ω_3



* **Theorem : (Reidemeister)**

Suppose D and D' are diagrams of two knots (or links) K and K' respectively. Then, $K \underset{equi}{\simeq} K'$ if and only if $D \underset{R}{\simeq} D'$.

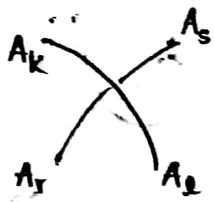
$D \underset{R}{\simeq} D'$ means D can be changed to D' by finitely many 3 R-moves.

* **Theorem :**

Tricolourability is a knot invariant

Proof :

Def^n (Tricolourable)

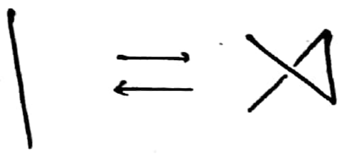


(i) A_k and A_l have same colour

(ii) A_k, A_r and A_s all have same colour or all have different colour.

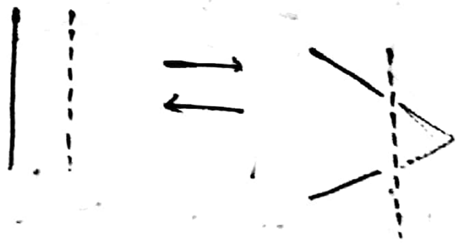
Suppose D is a diagram of K which is tricolourable, we will show that under 3 R-moves, tricolourability is preserved.

Ω_1



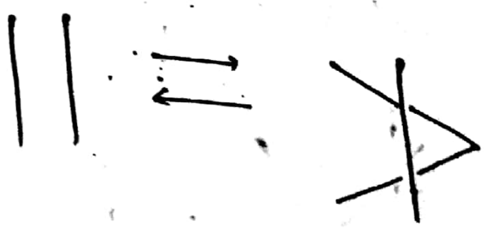
Tricolourable because of only one colour

Ω_2 (case 2)



3 different colours Hence, tricolourable

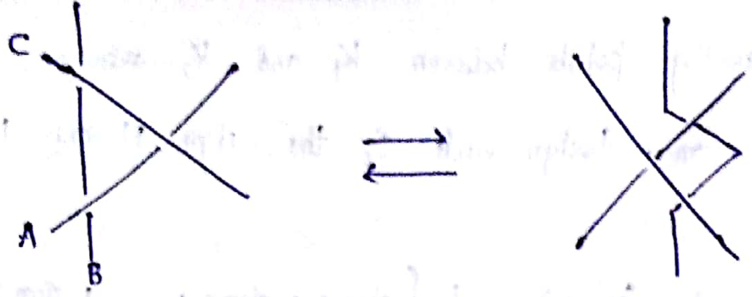
Ω_2 (case 1)



Tricolourable. Same colour.

Ω_3

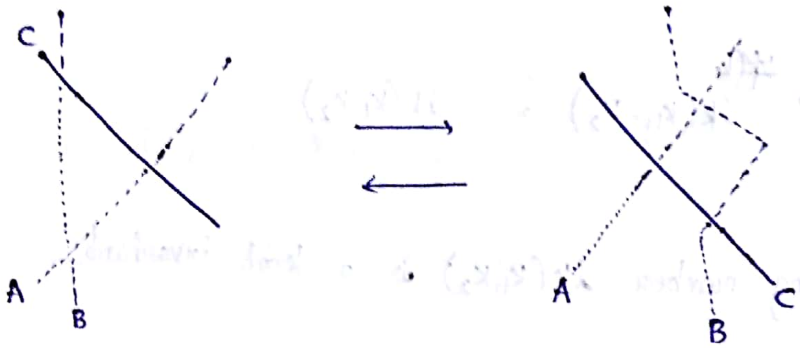
(case 1)



A, B, C are of same colour

Tricolourable

(case 2)



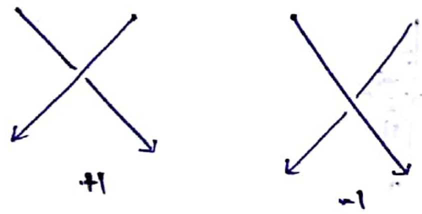
A & B have same colour

C has different colour

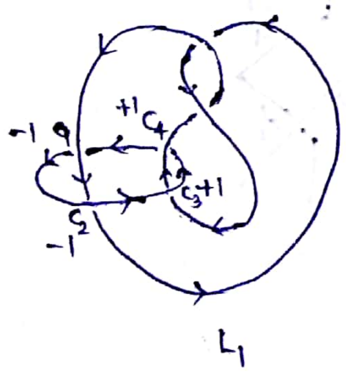
(case 3)

A & B & C have different colours.

* <u>Linking Number</u> :



Let D be an oriented 2-component link $L = \{K_1, K_2\}$



L_1

$\frac{1}{2}(-1-1+1+1) = 0$

L_2

$\frac{1}{2}(1+1+1+1)$

$= 2$

Suppose crossing points between K_1 and K_2 are $c_1, c_2, \ldots, c_m$. Assign each c_i the sign $+1$ or -1.

Then,

$$\text{Linking number}(K_1, K_2) = \frac{1}{2}\{\text{sign } c_1 + \text{sign } c_2 + \cdots + \text{sign } c_m\}.$$

$$\rightarrow \quad lk(K_1, -K_2) = -\,lk(K_1, K_2)$$

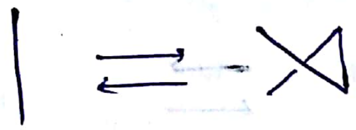
* **Theorem:**

Linking number $lk(K_1, K_2)$ is a knot invariant.

Proof:

We will show invariance under 3 R-moves

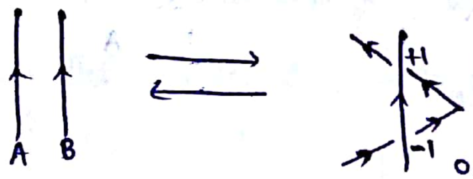
Ω_1



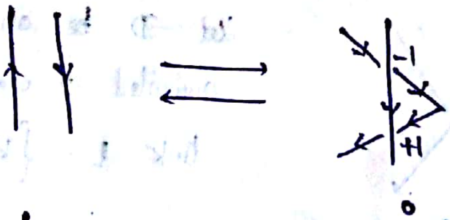
Unchanged since only one component

Ω_2

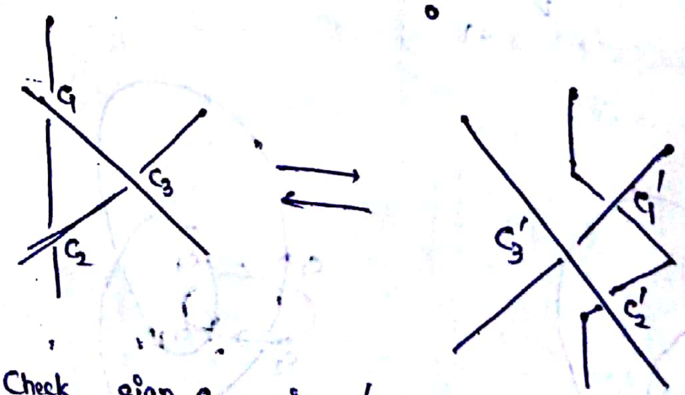
(case 1)



(case 2)



Ω_3



Check sign c_1 = sign c_1'

sign c_2 = sign c_2'

sign c_3 = sign c_3'

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* $K = \{K_1, K_2, \dots, K_n\}$

Linking number $lk(K_i, K_j)$

$\frac{n(n-1)}{2}$ linking numbers

$$\sum_{1 \le i, j \le n} lk(K_i, K_j) = lk(L)$$

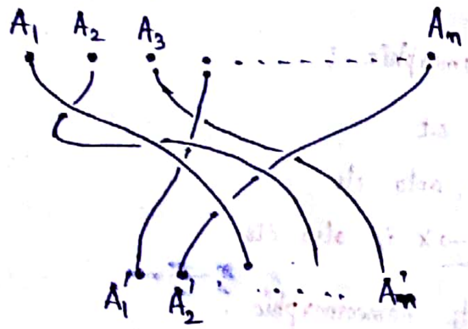
↘ invariant of L

* <u>Braid Group and its relation to Knot Theory</u>:

An n-braid is defined as follows

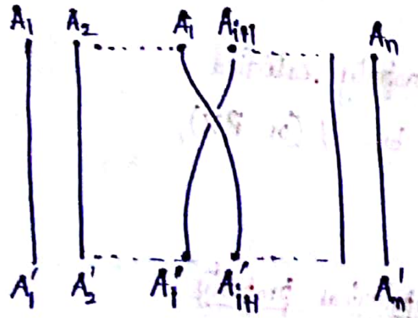
$A_1 \cdot \; A_2 \cdot \; \dots\dots\dots \cdot A_n$

We connect A_i to A_j' by linearly moving monotone downwards.

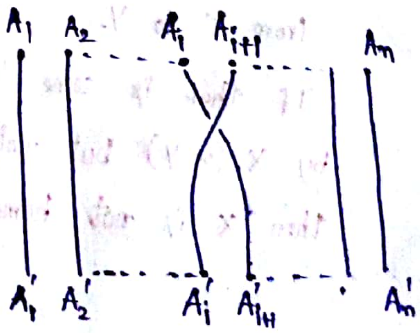


• Generators

σ_i

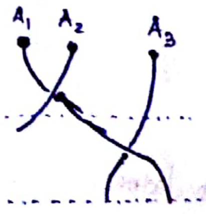


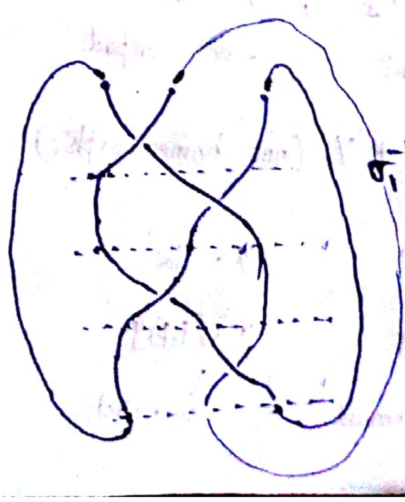
σ_i^{-1}



• Product of braids

$\sigma_1^{-1}\sigma_2$



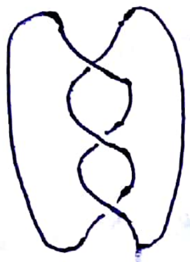


$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

$\approx 4_1$

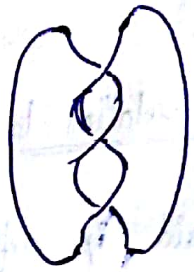
From a braid we form the braid closure by connecting A_1 to A_1', A_2 to A_2', ..., A_n to A_n' by arcs from outside.
This will give us a knot (or link)

σ_1^3 closure



$\approx 3_1$

σ_1^{-3} closure



$\approx 3_1^*$

30/01/201[illegible]
Tuesday

* X, Y – topological spaces

Q) Is X and Y are homeomorphic topological space ?

Ans: • X and Y are homeomorphic :

$\exists\ f : X \longrightarrow Y$ s.t
f is one-one, onto, cts
and $f^{-1} : Y \longrightarrow X$ is also cts.

• X and Y are not homeomorphic :

There doesn't exist a homeomorphism from X to Y.

If there is some topological property satisfied by X (or Y) but not satisfied by Y (or X), then X is not homeomorphic.

$X \underset{f^{-1}\ \text{continuous}}{\overset{f\ \text{continuous}}{\rightleftarrows}} Y$

$X \sim Y$ homeomorphic
$X \xrightarrow{f} Y$
f cts, one-one, onto
f^{-1} cts

Eg: ① $X = [0,1]$, $Y = \mathbb{R}$
– compact – not compact

Topological property
Compactness

$X \not\approx Y$ (not homeomorphic)

② $X = \mathbb{R}$, $Y = \mathbb{R}^2$

Connectedness

$\mathbb{R} \setminus \{p\}$ → not connected
$\mathbb{R}^2 \setminus \{f(p)\}$ → connected

$X \xrightarrow{f} Y$
$p \longmapsto f(p)$

③ $X = \mathbb{R}^2$ is not homeomorphic to $Y = \mathbb{R}^3$ Local connectedness; Metrizability and so on.

Simply connected.

$\mathbb{R}^2$ $\mathbb{R}^3$

simply connected

$\mathbb{R}^2 \setminus \{0\}$ $\mathbb{R}^3 \setminus \{0\}$

not simply connected

simply connected.

* Homotopy :

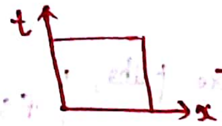
Let X, Y be two topological spaces.

Two continuous maps $f, g : X \to Y$ are called homotopic if there is a continuous map $H : X \times I \to Y$ such that

$$H\big|_{X \times \{0\}} = f \quad \text{and} \quad H\big|_{X \times \{1\}} = g$$

$X \times I \longrightarrow Y$

$X \times [0,1]$

 $\longrightarrow Y$

Eg: $X = S^1$ — unit circle.

$X \times I$ = 

$(x, t) \in X \times I$

$H(x,t) \in Y$

$H(x,0) = f(x)$

$H(x,1) = g(x)$

Notation :

$f \sim g$

and H is the homotopy between f and g.

- In particular, if $g : X \to Y$ is a constant map and $f \sim g$, then f is said to be nulhomotopic.

* Defn :

Two topological spaces X and Y are called homotopically equivalent if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

$X \xrightarrow{f} Y \xrightarrow{g} X$

$g \circ f \sim id_X$

$g \circ f \sim id_Y$

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$$H = H(x,t) \quad x \in X,\ t \in I$$

$$H = \{H_t\}$$

$$H_t : X \to Y$$

$$H_t(x) = H(x,t)$$

* **Def^n:**

Let A be a subset of the topological space X and $f, g : X \to Y$ be two continuous maps such that $f = g$ on A. We say that f and g are <u>homotopic relative to A</u> if there is a continuous map $H : X \times [0,1] \to Y$ such that

$$H\big|_{X \times \{0\}} = f, \quad H\big|_{X \times \{1\}} = g \quad \text{and} \quad H\big|_{A \times [0,1]} = f\big|_A = g\big|_A$$

Notation:

$f \sim_A g$ - f is homotopic to g relative to A

Case:

If $X = [0,1]$, that is if f, g are paths, we denote the homotopic equivalence of f and g relative to the end points by $f \sim_p g$

$f : X \xrightarrow{cts} Y$

$f : [0,1] \xrightarrow[\text{path}]{cts} Y$

$A \subset X = [0,1]$

$A = \{0,1\}$

Eg:

X, Y - top spaces

$$X \xrightarrow[cts]{f,g,h} Y$$

Given $f \sim g$

Q) Is $f \sim f$? Yes.

Q) Is $g \sim f$? Yes

Q) $f \sim g$, $g \sim h$. is $f \sim h$? Yes.

* **Proposition :**

The relation $\sim$ and $\sim_p$ are equivalence relations.

Proof : Let us show that $\sim$ is an equivalence relation

- $f \sim f$ since $H(x,s) = f(x)$ is a homotopy between f and f
- Also, if $H(x,t)$ is a homotopy between f and g,

  then $H'(x,t) = H(x,1-t)$ is a homotopy between g and f

  Hence $f \sim g \Rightarrow g \sim f$

- Finally, if $f \sim g$ and $g \sim h$ with homotopies $F(x,t)$, $G(x,t)$

  then

$$H(x,t) = \begin{cases} F(x,2t) & , t \in [0,\frac{1}{2}] \\ G(x,2t-1) & , t \in [\frac{1}{2},1] \end{cases}$$

$F_0 = f$, $F_1 = g$, $G_0 = g$, $G_1 = h$

Pasting Lemma

$H : X \times I \to Y$

$H_0 = f$

$H_1 = h$

is a homotopy between f and h.

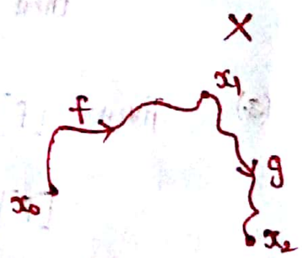
So, $f \sim h$

* **Notation :**

- $[f]$ is the equivalence class of f. (f ↗ continuous func / or ↘ path)

* **Def^n :**

Let f be a path from x_0 to x_1 and g be a path from x_1 to x_2 then the product of f and g denoted by $f * g$.



$f : I \longrightarrow X$

$f(0) = x_0$

$f(1) = x_1$

$$f*g : [0,1] \longrightarrow X$$

$$f*g(0) = x_0 \quad , \quad f*g(1) = x_2$$

$$f*g(t) = \begin{cases} f(2t) & , t \in [0,\frac{1}{2}] \\ g(2t-1), & t \in [\frac{1}{2},1] \end{cases}$$

also path

Def^n: $[f] * [g] = [f*g]$

* Proposition :

The product of paths factors to a product of homotopy equivalence classes of paths relative to the end point.

Proof :

Note that if $f \sim_p f'$ and $g \sim_p g'$ with homotopies relative to end points $F(t,s)$ and $G(t,s)$ respectively.

Then $$H(t,s) = \begin{cases} \cancel{F(t,s)}\ F(2t,s) & , t \in [0,\frac{1}{2}] \\ G(2t-1,s) & , t \in [\frac{1}{2},1] \end{cases}$$

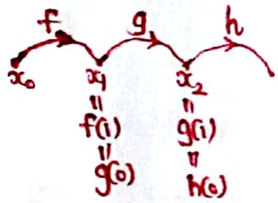
is a homotopy between $f*g$ and $f'*g'$

* Theorem :

The operation $*$ on equivalence classes of paths relative to the end points has the following properties

(1) if $f(1) = g(0)$ and $g(1) = h(0)$, then

(Associativity) $[f]*([g]*[h]) = ([f]*[g])*[h]$



(2) (Left and right identities)

Given $x \in X$, define $e_x : I \longrightarrow X$
$t \longmapsto x$

then $[f] * [e_{f(1)}] = [f] = [e_{f(0)}] * [f]$

(3) Given $[f]$, there is a path $[\bar{f}]$ such that

$[f] * [\bar{f}] = [e_{f(0)}]$ and

$[\bar{f}] * [f] = [e_{f(1)}]$

$\bar{f}(t) = f(1-t)$

Q) $(*, [f])$ is a group ? No

Since To define $f*g$, final point of f must be the same as initial point of g.

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01/02/2018
Thursday

Recall :

* X, Y – topological spaces

$f, g : X \longrightarrow Y$ continuous

- $f \sim g$ homotopy relation between f and g given by the homotopy map $H : X \times I \longrightarrow Y$

- $\sim \rightarrow$ equivalence relation.

- $\sim_p \rightarrow$ path homotopy (i.e. homotopy related to $A = \{0,1\}$ - end points)

- $* \rightarrow$ product of paths

$f * g : [0,1] \longrightarrow X$.

$$f * g = \begin{cases} f(2t) & , t \in [0, \frac{1}{2}] \\ g(2t-1) & , t \in [\frac{1}{2}, 1] \end{cases}$$

$[f] * ([g] * [h])$
$= [f] * ([g * h])$
$= [f * (g * h)]$

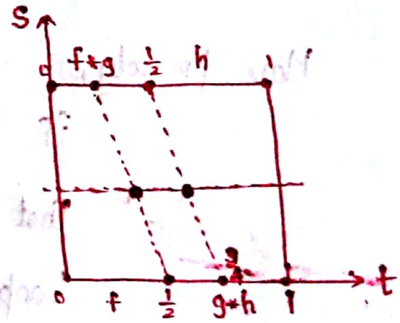
- The product operation $*$ satisfies

(i) Associativity condition

$[f], [g], [h]$

$[f] * ([g] * [h]) = ([f] * [g]) * [h]$

i.e. $[f * (g * h)] = [(f * g) * h]$



(ii) Left and Right identity

$[f] * [e_{f(1)}] = [f] = [e_{f(0)}] * [f]$

(iii) Inverse

$[f] * [\bar{f}] = [e_{f(0)}]$

and $[\bar{f}] * [f] = [e_{f(1)}]$

$e_{f(1)} : I \longrightarrow X$
$t \longmapsto f(1)$ (constant)

$[f] * [\bar{f}] = [f * \bar{f}] = [e_{f(0)}]$
$\Rightarrow$ There is a homotopy between $f * \bar{f}$ and $e_{f(0)}$

Proof : (of Theorem)

Hint of the required homotopies

(i)

A homotopy between $f*(g*h)$ and $(f*g)*h$ is given by

$$H(t,s) = \begin{cases} f\left(\frac{4}{s+1}t\right) & , t \in \left[0, \frac{s+1}{4}\right] \\ g(4t-s-1) & , t \in \left[\frac{s+1}{4}, \frac{s+2}{4}\right] \\ h\left(\frac{4}{2-s}t - \frac{2+s}{2-s}\right) & , t \in \left[\frac{s+2}{4}, 1\right] \end{cases}$$

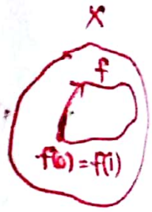
(ii)

A homotopy between f and $e_{f(0)} * f$ is given by

$$H(t,s) = \begin{cases} f(0) & , t \in \left[0, \frac{s}{2}\right] \\ f\left(\frac{2t}{2-s} - \frac{s}{2-s}\right) & \text{for } t \in \left[\frac{s}{2}, 1\right] \end{cases}$$

* **Def^n:**

A path whose end points coincide is called a Loop.



More precisely,

if the path $f : [0,1] \longrightarrow X$ has the property that $f(0) = f(1) = x_0$, then f is called a Loop based at x_0.

Remark :

(1) Alternatively, a loop based at x_0 can be identified with a map $f : S^1 \longrightarrow X$, with $f(0) = x_0$

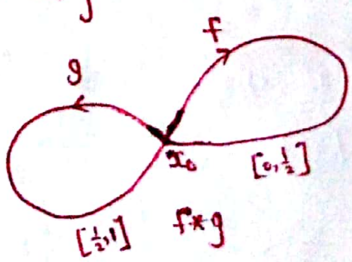
(2) The set of homotopy equivalence classes relative to end points of loops based at x_0, denoted by $\pi_1(X, x_0)$ is a group.

$$\pi_1(X, x_0) = \{ [f] \mid f \text{ is loop based at } x_0 \}$$

$(\pi_1(X, x_0), *)$ is a group

Fundamental group of space X based at x_0

$$[f] * [g] = [f*g]$$



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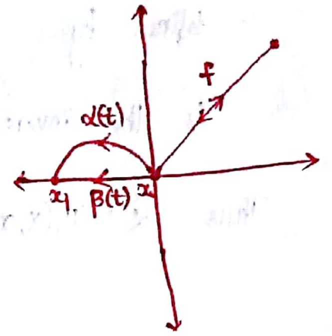
*** Def^n:**

The group $\pi_1(X, x_0)$ is called the <u>Fundamental group</u> (or the first homotopy group) of X at x_0.

Eg: $X = \mathbb{R}^2$, $x_0 = (0,0)$



$\pi_1(\mathbb{R}^2, x_0) = ?$

$= \{[e_{x_0}]\}$

trivial group with identity element only.

$\alpha \sim_p \beta$

$H_s(t) = (1-s)\alpha(t) + s\beta(t)$

In $\mathbb{R}^2$ for any function f

$[f] = [e_{x_0}]$

In particular, $\pi_1(\mathbb{R}^n, x_0) = \{[e_{x_0}]\}$ is trivial.

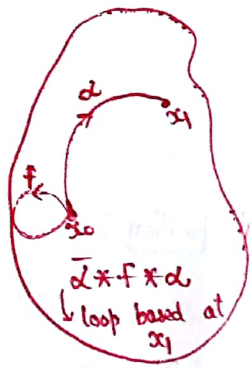
($\mathbb{R}^n$ is simply connected)

*** Theorem:**

Let X be a path connected topological space and x_0, x_1 are two points in X. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$

Proof:

Suppose α be a path joining x_0 and x_1



Let $[f] \in \pi_1(X, x_0)$

$\bar{\alpha}(t) = \alpha(1-t)$

Define the map $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

given by $\hat{\alpha}([f]) = [\bar{\alpha} * f * \alpha]$

for $[f] \in \pi_1(X, x_0)$

$[f] \longmapsto [\bar{\alpha} * f * \alpha] \in \pi_1(X, x_1)$

This is a well-defined map

$\hat{\alpha}[f] * \hat{\alpha}[g] = [\bar{\alpha} * f * \alpha] * [\bar{\alpha} * g * \alpha]$

$\alpha * \bar{\alpha} \sim_p e_{x_0}$

$= [\bar{\alpha} * f * \alpha * \bar{\alpha} * g * \alpha]$

$\bar{\alpha} * f * \alpha * \bar{\alpha} * g * \alpha \sim_p \bar{\alpha} * f * e_{x_0} * g * \alpha$

$\sim_p \bar{\alpha} * f * g * \alpha$

$= [\bar{\alpha} * f * g * \alpha] = \hat{\alpha}([f * g])$

$\Rightarrow \hat{\alpha}$ is a group homomorphism.

Consider $\hat{\beta} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$

defined by $\hat{\beta}[h] = [\alpha * h * \bar{\alpha}]$

is the inverse group homomorphism of $\hat{\alpha}$.

Thus $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ is a group isomorphism.

06/02/2011
Tuesday

* **Recall :**

X – topological space

$\pi_1(X, x_0)$ – Fundamental group of X based at point x_0

– First homotopy group of X based at x_0.

* **The behaviour of the fundamental group under continuous transformation**

Def^n :

Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map.
$(f(x_0) = y_0)$

Define $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

$[\alpha] \longmapsto [f \circ \alpha]$

by $f_*([\alpha]) = [f \circ \alpha]$

$\alpha : [0,1] \longrightarrow X$

$\alpha(0) = \alpha(1) = x_0$

$f \circ \alpha : [0,1] \longrightarrow Y$

$f \circ \alpha(0) = f(\alpha(0))$
$= f(x_0)$
$= f(\alpha(1))$
$= f \circ \alpha(1)$

* **Proposition :**

The map $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is well-defined.

Moreover, if $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$ are continuous, then $(g \circ f)_* = g_* \circ f_*$

$$\begin{tikzcd}
(X, x_0) \arrow[r, "f"] \arrow[dr, dashed, "g \circ f"'] & (Y, y_0) \arrow[d, "g"] \\
& (Z, z_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, dashed, "(g \circ f)_*"'] & \pi_1(Y, y_0) \arrow[d, "g_*"] \\
& \pi_1(Z, z_0)
\end{tikzcd}$$

$g_* \circ f_* = (g \circ f)_*$

To Show f_* is Group homomorphism

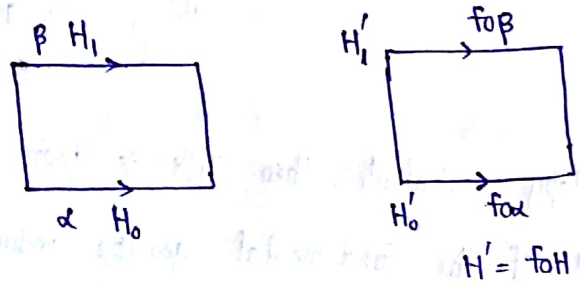
Show :

$f_*([\alpha] * [\beta])$

$= [f \circ \alpha] * [f \circ \beta]$

Proof :

1. The fact that $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is well-defined follows from the fact that if H is a homotopy relative to the base point between the loops α and β based at x_0, then $f\circ H$ is a homotopy relative to the base point between the loops $f\circ\alpha$ and $f\circ\beta$ based at y_0.



$[\alpha] = [\beta]$
We have to show
$[f\circ\alpha] = [f\circ\beta]$
(For well-definedness)

2.
$$(g\circ f)_* : \pi_1(X, x_0) \longrightarrow \pi_1(Z, z_0)$$
$$[\alpha] \longrightarrow [(g\circ f)\circ\alpha]$$
$$(g\circ f_*)[\alpha] = [(g\circ f)\circ\alpha]$$

$$g_*\circ f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Z, z_0)$$
$$\begin{aligned}(g_*\circ f_*)[\alpha] &= g_*(f_*[\alpha]) \\ &= g_*([f\circ\alpha]) \\ &= [g\circ(f\circ\alpha)]\end{aligned}$$

We have
$$(g\circ f)\circ\alpha = g\circ(f\circ\alpha) \quad \text{for any three continuous functions.}$$

So,
$$\begin{aligned}(g\circ f)_*[\alpha] &= [(g\circ f)\circ\alpha] \\ &= [g\circ(f\circ\alpha)] \\ &= g_*([f\circ\alpha]) \\ &= g_*(f_*[\alpha]) \\ &= (g_*\circ f_*)[\alpha]\end{aligned}$$

$$(X, x_0) \xrightarrow{1_X} (X, x_0)$$
$$\pi_1(X, x_0) \xrightarrow{1_{X*}} \pi_1(X, x_0)$$
$$[\alpha] \longmapsto [1_X\circ\alpha] = [\alpha]$$

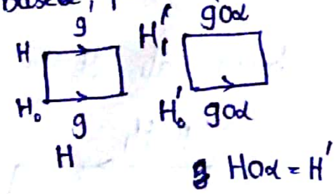
* **Proposition :**

If $g, g' : (X, x_0) \longrightarrow (Y, y_0)$ are homotopic (relative to x_0)

then $g_* = g'_*$

Proof :

This follows from the fact that for every loop α based at x_0, $g \circ \alpha$ and $g' \circ \alpha$ are homotopic relative to the based point



* **Theorem :**

If X and Y are homotopy equivalent, then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$, with the isomorphism of the fundamental groups induced by the homotopy equivalence between the two space.

Proof :

Let $g : X \longrightarrow Y$ and $h : Y \longrightarrow X$ be such that

$h \circ g \sim Id_X$ and $g \circ h \sim Id_Y$ $\quad (Id_X = 1_X)$

We claim that g_* is an isomorphism (or h^* is an isomorphism)

Lemma :

Let $g : (X, x_0) \longrightarrow (X, x_1)$ be a map homotopic to the identity. Then $g_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ is a group isomorphism (Prove it)

By this Lemma,

$(h \circ g)_*$ is an isomorphism

$\parallel$

$h_* \circ g_*$

g_* is a group homomorphism

h_* " " " "

g_* is bijective.

$(\because h_* \circ g_*$ is isomorphism$)$

$$\begin{tikzcd} \pi_1(X) \arrow[r, "g_*"] & \pi_1(Y) \arrow[l, bend left, "h_*"] \end{tikzcd}$$

$\Rightarrow h^*$ is isomorphism

$\Rightarrow g^*$ is also isomorphism

(Check this once again)

Q) $\pi_1(\mathbb{R}^n, x_0) \to$ Trivial

$\alpha o \beta$ surj $\Rightarrow$ β surj

$\alpha o \beta$ inj $\Rightarrow$ α inj

$\mathbb{R}^n$ is simply connected

Q) $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$

S^n is simply connected for $n > 1$

* **Corollary :**

The space $\mathbb{R}^n \setminus \{0\}$, $n \geq 3$ are Simply connected.

Proof :

Let $X = \mathbb{R}^n \setminus \{0\}$, $n \geq 3$

$Y = S^{n-1}$

We will show that X is homotopy equivalent to S^{n-1} for all $n \geq 1$

Consider, $S^{n-1} \xhookrightarrow{i} \mathbb{R}^n \setminus \{0\}$ and the map $i \to$ inclusion map

$$r : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$$

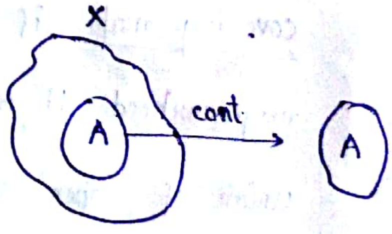
$$x \longrightarrow \frac{x}{\|x\|}$$

Then $r \circ i = 1_{S^{n-1}}$ and $i \circ r \sim 1_{\mathbb{R}^n \setminus \{0\}}$

with homotopy $H(x,t) = \dfrac{x}{\|x\|^t}$

* **Def^n :**

If $A \subset X$, a retraction of X onto A is a continuous map $r : X \longrightarrow A$ s.t $r|_A = 1_A$. In this case, A is called a Retract of X



* **Proposition :**

If A is a retract of X and $i : A \hookrightarrow X$ is the inclusion, then $i_* : \pi_1(A) \to \pi_1(X)$ is a monomorphism and $r_* : \pi_1(X) \longrightarrow \pi_1(A)$ is an epimorphism.

Proof : We have $r \circ i = 1_A$ then $r_* \circ i_* = 1_{\pi_1(A)}$

$\Rightarrow i_*$ is one-to-one

and r_* is onto map

*** Def^n :**

Let A be a subspace of X. We say that A is a deformation retract of X if 1_X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy. The homotopy is called deformation retraction.

Eg: $S^1 \subset \mathbb{C} \setminus \{0\}$

The circle is a deformation retract of $\mathbb{C} \setminus \{0\}$.

The deformation retraction is $H(z,t) = \dfrac{z}{|z|^t}$

*** Proposition :**

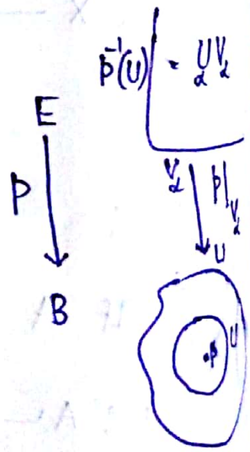
Let A be a deformation retract of X. Then the inclusion $i : (A, x_0) \hookrightarrow (X, x_0)$ induces an isomorphism at the level of fundamental groups.

08/02/2018
Thursday

*** Covering space :**

Def^n:

A continuous surjective map $p : \underset{\uparrow \text{ Ambient space}}{E} \longrightarrow \underset{\uparrow \text{ Base space}}{B}$ is called a covering map if for every point $b \in B$ has an open neighbourhood U in B such that $p^{-1}(U)$ is a disjoint union of open sets in E and the restriction of p to each of these open sets is a homeomorphism onto U.

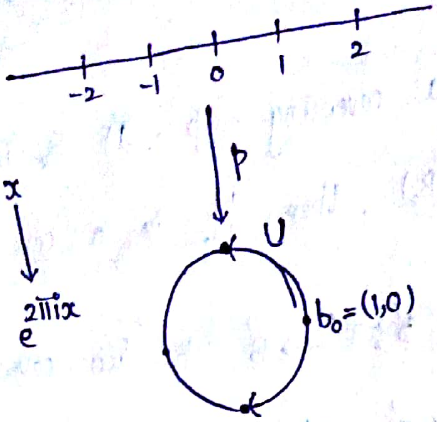


Here, E is said to be the covering space of B.

Also, the open neighbourhood U is said to be evenly covered by p and the disjoint open sets in $p^{-1}(U)$ are called Slices

Eg: $E = \mathbb{R}$

$B = S^1$ $\quad x \mapsto e^{2\pi i x}$



$\mathbb{Z} = p^{-1}(b_0) \subseteq \mathbb{R}$

$p^{-1}(U) = \underbrace{\bigsqcup_{i \in \mathbb{Z}} V_i}_{\text{Disjoint union}} \quad i \in \mathbb{Z}$

$V_0 = \left(\frac{-1}{4}, \frac{1}{4}\right)$

$V_1 = \left(1 - \frac{1}{4}, 1 + \frac{1}{4}\right)$

$$V_0 = \left(\frac{-1}{4}, \frac{1}{4}\right) \xrightarrow{p|_{V_0}} U \qquad p(x) = e^{2\pi i x}$$

continuous
injective
inverse also exists
$\Rightarrow$ bijective
$\Rightarrow$ Homeomorphism

$p^{-1}(b_0) = 0$

The map $p : \mathbb{R} \longrightarrow S^1$
defined by $p(x) = e^{2\pi i x}$ is a Covering map.

Eg 2:

$p : \mathbb{R}_+ \longrightarrow S^1$

$p(x) = e^{2\pi i x}$

p is not a covering map

$\ast \rightarrow$ $p : E \rightarrow B$ covering map

$p^{-1}(b) \longrightarrow$ is called ~~Fibler~~ Fiber

- $p^{-1}(b) \subset E$

$p^{-1}(b) \cap V_\alpha$ is Singleton set

$\therefore$ $p^{-1}(b)$ is discrete topology.

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* **Proposition :**

(1.) Let $p : E \longrightarrow B$ is a covering map. If B_0 is a subspace of B and $E_0 = p^{-1}(B_0)$, then $p|_{E_0}$ is a covering map.

(2.) Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ be two covering maps then $p'' : E \times E' \longrightarrow B \times B'$ defined by $p''(e, e') = (p(e), p'(e'))$ is a covering map.

Proof :

(2.) Suppose $p^{-1}(U) = \bigsqcup_{\alpha} U_\alpha$ with U_α disjoint and each of them covers U and is homeomorphic to U via the map p.

and similarly $p'^{-1}(U') = \bigsqcup_{\beta} U_\beta$

then $p''^{-1}(U \times U') = \bigsqcup_{\alpha, \beta} U_\alpha \times U_\beta$

and the product $U_\alpha \times U_\beta$ are disjoint and each covers $U \times U'$ and is homeomorphic to $U \times U'$ via the map p''.

$$\begin{tikzcd} E \times E' \arrow[d, "p''"] \\ B \times B' \end{tikzcd}$$

$b'' = (b, b')$

U'' $\quad U \times U' = U''$

$p''^{-1}(U'')$

Eg :

The map $p : \mathbb{R}^2 \longrightarrow S^1 \times S^1$

$(x, y) \longmapsto \left(e^{2\pi i x}, e^{2\pi i y}\right)$

is a covering map.

$S^1 \times S^1$ ↳ torus.

Def^n : Lifting

Let $p : E \longrightarrow B$ be a continuous surjective map and let $f : X \longrightarrow B$ be a continuous map where X is a topological space, then a lifting of f is a map $\tilde{f} : X \longrightarrow E$ such that $p \circ \tilde{f} = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

$f = p \circ \tilde{f}$

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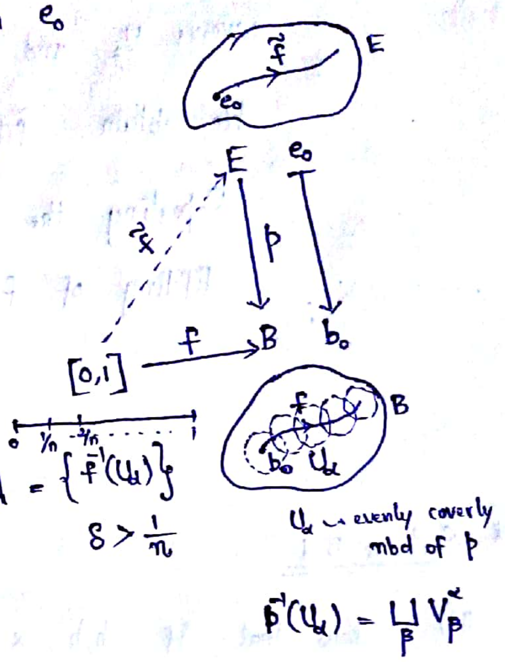
* **Theorem:** (The path lifting Lemma)

Let $p: E \to B$, $(E,e_0) \to (B,b_0)$ be a covering map such that $p(e_0) = b_0$, then any path $f:[0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}:[0,1] \to E$ beginning at e_0

Proof:



Lebesgue Number Lemma:

Let (X,d) be a compact metric space and $\mathcal{A}$ be an open cover of X, then $\exists$ a δ (lebesgue number) > 0 s.t for each subset U of X with diameter less than δ is contained in an element of $\mathcal{A}$.

Cover the path $f([0,1])$ by open sets U that are evenly covered by p. Because $[0,1]$ is Compact, so, by Lebesgue Number Lemma, there is $n \geq 1$ such that the partition of $[0,1]$ into n equal subintervals has the image of each of this subinterval lie inside an open set evenly covered by p.

Consider the first interval $[0,\frac{1}{n}]$ and let U_1 be an open set that contains $f[0,\frac{1}{n}]$ and is evenly covered by p. then the lift of f restricted to this interval must lie in the slice V_1 that contains e_0 because $p: V_1 \to U_1$ is a homeomorphism and $\tilde{f}\big|_{[0,\frac{1}{n}]} = p^{-1} \circ f[0,\frac{1}{n}]$.

Thus the restriction $f\big|_{[0,\frac{1}{n}]}$ has a unique lifting that starts at e_0.

Let us pass now to the next subinterval $[\frac{1}{n}, \frac{2}{n}]$

The lifting of $f|_{[\frac{1}{n},\frac{2}{n}]}$ must start at $\tilde{f}(\frac{1}{n})$.

Let $f([\frac{1}{n},\frac{2}{n}]) \subset U_2$ with U_2 evenly covered by V_2.

The lift of $f|_{[\frac{1}{n},\frac{2}{n}]}$ must lie in the slice V_2 that covers U_2 and contains $\tilde{f}(\frac{1}{n})$.

We obtain $\tilde{f}([\frac{1}{n},\frac{2}{n}]) = p^{-1}\circ f([\frac{1}{n},\frac{2}{n}])$

Repeating the procedure construct $\tilde{f}$ as the required lifting of f.

13/02/2018
Tuesday

Munkres Chapter 9

* Section 5.1

#1) Show that if $h,h' : X \longrightarrow Y$ are homotopic and $k,k' : Y \longrightarrow Z$ are homotopic, $k\circ h$ and $k'\circ h'$ are homotopic

Sol^n:

h and h' are Homotopic

$\Rightarrow \exists$ a continuous map $H : X \times I \longrightarrow Y$

$H(x,0) = h(x)$

$H(x,1) = h'(x)$

k and k' are Homotopic

$\Rightarrow \exists$ cts map $K : Y \times I \longrightarrow Z$

$K(x,0) = k(x)$

$K(x,1) = k'(x)$

Define $F : X \times I \longrightarrow Z$

$F(x,t) = K(H(x,t), t)$

H, K cts $\Rightarrow F$ cts

$F(x,0) = K(H(x,0), 0) = K(h(x), 0)$

$= k(h(x))$

$= k\circ h(x)$

$F(x,1) = k'\circ h'(x)$

#2) Given spaces X and Y, let [X, Y] denote the set of homotopy classes of maps on X into Y (a) $I = [0,1]$. Show that the set [X I] has a single element

Sol:

Let $\varphi : X \to I$ be a cts map

Let $\tilde{\varphi} : X \to I$ s.t

$\tilde{\varphi}(x) = 0 \quad \forall x \in X$

Define

$H : X \times I \longrightarrow Y$

$H(x,t) = (1-t)\varphi(x) + t\,\underset{0}{\tilde{\varphi}(x)}$

$= (1-t)\varphi(x)$

$H(x,0) = \varphi(x)$

$H(x,1) = 0$

$\Rightarrow \varphi$ and 0 are homotopic to each other

5/02/20..
..esday

(b) Show that if Y is path-connected, the sets [I Y] has a single element.

Proof:

$f : I \longrightarrow Y$

$g : I \longrightarrow Y \qquad g(x) = f(0) \ \forall x \in I$

$F : I \times I \longrightarrow Y$

$F(x,t) = f((1-t)x)$

$F(x,0) = f(x)$

$F(x,1) = f(0)$

$h : I \longrightarrow Y$

$k : I \longrightarrow Y$

$k(x) = h(0)$

$H(x,t) = h((1-t)x)$

$H(x,0) = h(x)$

$H(x,1) = h(0)$

$p : [0,1] \longrightarrow Y$

$p(0) = f(0)$

$p(1) = h(0)$

$$L : I \times I \longrightarrow Y$$

$$L(x,t) = p(t)$$

$$L(x,0) = p(0) = f(0)$$

$$L(x,1) = p(1) = h(0).$$

#3) X is contractible if $Id : X \longrightarrow X$ is homotopic constant

a) Show that I & R are

Solⁿ: Claim $Id_X \simeq$ constant

Let $f(x) = 0$

$$H : X \times I \longrightarrow X$$

$$H(x,t) = (1-t)x$$

$$H(x,0) = x$$

$$H(x,1) = 0$$

b) Show that any contractible is path connected

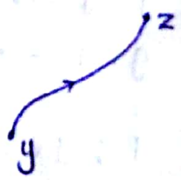
Solⁿ: $X \longrightarrow$ contractible

$Id_X \simeq$ constant map $f(x) = p \in X$

$\exists\ H : X \times I \longrightarrow X$ s.t

$$H(x,0) = x$$

$$H(x,1) = f(x) = p$$



Let $y, z \in X$

Let us define

$$\gamma(t) = \begin{cases} H(y, 2t) & 0 \le t \le \frac{1}{2} \\ H(z, 2-2t) & \frac{1}{2} \le t \le 1 \end{cases}$$

(c) Show that if Y is contractible, the set $[X\ Y]$ has a single element

Pf:

Y is contractible

$\Rightarrow \exists\ y_0 \in Y$ st

$Id : \cancel{X} Y \longrightarrow Y \simeq y_0$

$\exists\ F : \cancel{X} \times I \longrightarrow Y$ cts

s.t $\quad F(y,0) = id(y)$

$F(y,1) = y_0$

$\alpha : X \longrightarrow Y$ cts

Define $H : X \times I \longrightarrow Y$

$H(x,t) = F(\alpha(x),t)$ is cts

$H(x,0) = F(\alpha(x),0)$

$= \alpha(x)$

$H(x,1) = F(\alpha(x),1)$

$= y_0$

(d) If X is contractible, Y is path connected, then $[X,Y]$ has a single element.

Pf:

$H : X \times I \longrightarrow X$

$H(x,0) = Id_X(x) = x$

$H(x,1) = f(x) = p\ \forall\ x \in X$

Let $\phi : X \to Y$

and $\psi : X \longrightarrow Y$,

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52

(4) $A \subset X$

$r: X \to A$ a ~~retraction~~.

i.e. $r: X \to A$ cont s.t $r(a) = a$ $\forall$

51, 52, 53 Sections

53.

$p: S^1 \to S^1$

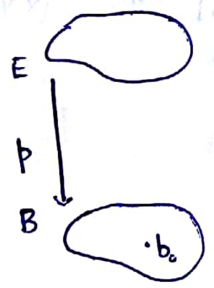
Show $p(z) = z^2$ is a covering map.

15/02/20
Thursday

Section 53

#3) Let $p: E \to B$ be a covering map. Let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a k-fold covering of B.

Proof:



$|p^{-1}(b_0)| = k$ evenly covered

Let U be the neighbourhood of b_0

$p^{-1}(U) = \bigsqcup V_i$, V_i open in E

$p|_{V_i} : V_i \to U$ homeomorphism

Suppose $\exists$ $b' \in B$ s.t $|p^{-1}(b')| \neq k$

$C = \{b \in B : |p^{-1}(b)| = k\}$ $b_0 \in C$

$D = \{b \in B : |p^{-1}(b)| \neq k\}$ $b' \in D$ $\Rightarrow C \neq D$

$B = C \cup D$ ($\because$ Connected)

Aim: C and D form separation for B (C and D both open)

Let $c \in C \subseteq B$

$c \in U_c \subseteq B$

$c \in U_c$. $p^{-1}(U_c) = \sqcup V_{i_\alpha}$

$p\big|_{V_{i_\alpha}} : V_{i_\alpha} \longrightarrow U_c$ homeo

Aim :

$U_c \subseteq C$

$x \in U_c \Rightarrow |p^{-1}(x)| = k$

$\Rightarrow x \in C$

$\Rightarrow U_c \subseteq C$

$\Rightarrow$ C is open

In the same way D is also open .

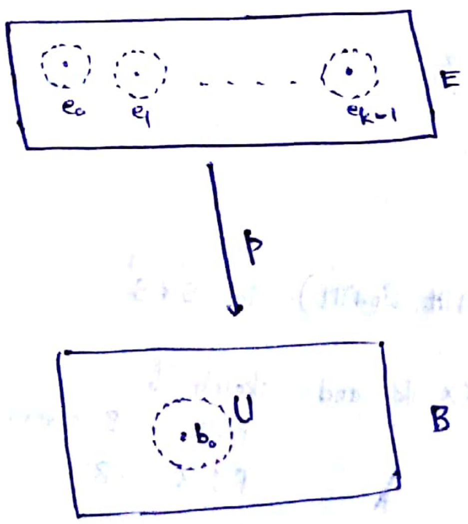
$\Rightarrow C \cap D = \phi$

$\Rightarrow$ C and D form a separation for B

$\Rightarrow$ B is disconnected.

$\Rightarrow\Leftarrow$ Contradiction.

$\therefore$ $|p^{-1}(b)| = k \quad \forall\ b \in B.$



$|p^{-1}(b_0)| = k$

$p^{-1}(U) = \sqcup V_i$

$p\big|_{V_i} : V_i \longrightarrow U$ homeo

#5)

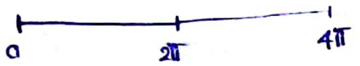
$p : S^1 \longrightarrow S^1$, $p(z) = z^2$.

Show that $p: S^1 \longrightarrow S^1$ is a covering map.
↳ 2-fold cover

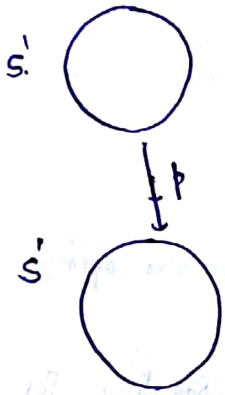
Sol^n:

Let $z = e^{it}$

$p(z) = z^2$
$= e^{2it}$



$|p^{-1}(e^{it})| = 2$

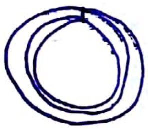


$p^{-1}(e^{it}) = \left\{ e^{it/2}, e^{i(\pi + t/2)} \right\}$

$p^{-1}\left(e^{i(t-\Delta t)}, e^{i(t+\Delta t)} \right)$

$= \left\{ \left(e^{i\left(\frac{t-\Delta t}{2}\right)}, e^{i\left(\frac{t+\Delta t}{2}\right)} \right), \left(e^{i\left(\left(\frac{t-\Delta t}{2}\right)+\pi\right)}, e^{i\left(\left(\frac{t+\Delta t}{2}\right)+\pi\right)} \right) \right\}$

$p(z) = z^3$ ↳ 3-fold covering



$p^{-1}(e^{it}) = \left\{ e^{it/3}, e^{i(2\pi/3 + t/3)}, e^{i(4\pi/3 + t/3)} \right\}$

$|p^{-1}(e^{it})| = 3$

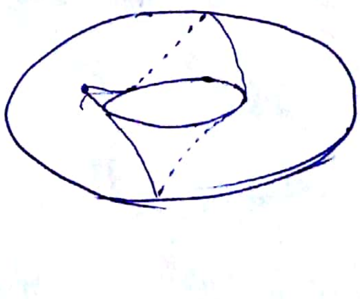
Section 54:

#5) $p \times p : \mathbb{R} \times \mathbb{R} \longrightarrow S^1 \times S^1$

Consider path $f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$ in $S^1 \times S^1$

Find lifting of f to $\mathbb{R} \times \mathbb{R}$ and sketch it.

Sol^n:



$p : E \longrightarrow B$ covering

$f : X \longrightarrow B$

$\tilde{f} : X \longrightarrow E$

$p \circ \tilde{f} = f$

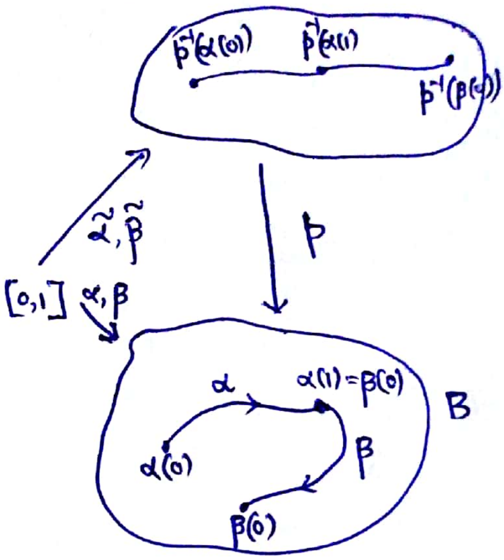
Scanned by CamScanner

#3) $p : E \longrightarrow B$ covering map.

Let α, β are paths in B with $\alpha(1) = \beta(0)$.

Let $\tilde{\alpha}, \tilde{\beta}$ be liftings of them s.t $\tilde{\alpha}(1) = \tilde{\beta}(0)$

Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.



Sol^n:

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) &, 0 \le t \le \frac{1}{2} \\ \beta(2t-1) &, \frac{1}{2} \le t \le 1 \end{cases}$$

$$p \circ (\tilde{\alpha} * \tilde{\beta}) = \begin{cases} p \circ \tilde{\alpha}(2t) &, 0 \le t \le \frac{1}{2} \\ p \circ \tilde{\beta}(2t-1) &, \frac{1}{2} \le t \le 1 \end{cases}$$

$$= \begin{cases} \alpha(2t) &, 0 \le t \le \frac{1}{2} \\ \beta(2t-1) &, \frac{1}{2} \le t \le 1 \end{cases}$$

$$= \alpha * \beta$$

Knot part → Murasugi

Homotopy → Munkres