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14) Prove $\int_{(1,2)}^{(3,4)} (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$ is independent of path joining points $(1,2)$ & $(3,4)$ & hence evaluate

$$\int F\,dr = \int_{(1,2)}^{(3,4)} (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

$$\vec{F} = (6xy^2 - y^3)\hat{i} + (6x^2y - 3xy^2)\hat{j}$$

$$\text{curl } \vec{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy^2 - y^3 & 6x^2y - 3xy^2 & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} + (0-0)\hat{j} + (12xy - 3y^2 - 12xy + 3y^2)\hat{k}$$

$$= 0$$

Since curl $\vec{F} = 0$, $\vec{F}$ is irrotational & hence the field is conservative

Scalar Potential $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x} = 6xy^2 - y^3 \qquad \frac{\partial\phi}{\partial y} = 6x^2y - 3xy^2$$

Partial Integrations

$$\phi = 3x^2y^2 - xy^3 + c(y,z) \qquad \phi = 3x^2y^2 - xy^3 + c(x,z)$$

Hence $\phi = 3x^2y^2 - xy^3 + C$

Thus $\int_{(1,2)}^{(3,4)} F\,dr = [\phi]_{(1,2)}^{(3,4)}$

$$\Rightarrow [3[3]^2[4]^2 - [3][4]^3 + c] - [3[1]^2[2]^2 - [1][2]^3 + c]$$

$$\Rightarrow 3[144 - 4] - [192 - 8] + c - c$$

$\Rightarrow$ 236 units

28) $\int_c (xy\,dx + xy^2\,dy)$ where C = square in x-y plane with vertices $(0,1),(1,0),(-1,0)$ & $(0,-1)$

Here $P = xy \quad Q = xy^2$

$\frac{\partial P}{\partial y} = x \qquad \frac{\partial Q}{\partial x} = y^2$

By Greens' Theorem

$$\int_c [P\,dx + Q\,dy] = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] \partial x \partial y$$

$$I = \iint_R [y^2 - x]\,dx\,dy$$

Limits for one quadront

$x \in (0,1) \qquad y \in (0, 1-x)$

$$\therefore I = \int_0^1 \int_0^{1-x} (y^2 - x)\,dx\,dy$$

$$= \int_0^1 \left[\frac{y^3}{3} - xy\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^3}{3} - (x - x^2)\,dx$$

$$= \frac{1}{3}\int_0^1 (1 - 3x + 3x^2 - x^3 - 3x + 3x^2)\,dx$$

$$= \frac{1}{3}\int_0^1 [1 - 6x + 6x^2 - x^3]\,dx$$

$$= \frac{1}{3}\left[x - 3x + 2x^3 - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3}[1 - 3 + 2 - 1/4] = \frac{1}{3}[-1/4] = \frac{-1}{12}$$

For complete square $\quad 4 * \frac{-1}{12} = \boxed{\frac{-1}{3}}$

38) $\int_c \bar{F}\, d\bar{r}$ where $\bar{F} = z\hat{i} + x\hat{j} + y\hat{k}$ & C is boundary of hemisphere $z = \sqrt{1-x^2-y^2}$ in xy plane

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$z = \sqrt{1-x^2-y^2}$

$z^2 + x^2 + y^2 = 1 \qquad z = 0$ (x-y plane)

$\oint_c [x\,dy + y\,\cancel{dz}^{\,0}]$

$x = \cos\theta \qquad y = \sin\theta$

$\oint [\cos\theta\, dy] \qquad dy = \cos\theta\, d\theta$

$\oint_c [\cos^2\theta\, d\theta]$

$\oint_c \frac{1+\cos 2\theta}{2}\, d\theta$

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$

$= \frac{1}{2}\left[2\pi + 0 - (0-0)\right] \Rightarrow \pi$