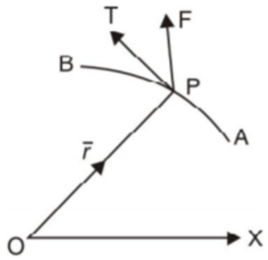
Line Integral

Let $\vec{F}(x,y,z)$ be a vector function and a curve AB.

Line Integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB .

Line Integral= $\int_c \vec{F}.\overrightarrow{dr}$



Remark

(1) Work. If $\overline{F}$ represents the variable force acting on a particle along arc AB, then the total work done = $\int_A^B \vec{F}\cdot\vec{dr}$

(2) Circulation. If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V}\cdot\vec{dr}$ is called the circulation of V round the closed curve c.

If the circulation of V round every closed curve is zero then V is said to be irrotational there.

(3) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

Ex-1 If a force $\vec{F} = 2x^2y\hat{\imath} + 3xy\hat{\jmath}$ displaces a particle in the xyplane from (0,0) to (1,4)along a curve $y = 4x^2$. Find the work done.

Solution

$$\text{Work done } = \int_c \vec{F}.\vec{dr} \qquad \left[\begin{array}{l} \vec{r} = x\hat{i} + y\hat{j} \\ \vec{dr} = dx\hat{i} + dy\,\hat{j} \end{array}\right]$$

$$= \int_c (2\,x^2 y\hat{i} + 3xy\,\hat{j}).(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_c (2\,x^2 y\; dx + 3xy\; dy)$$

Putting the values of y and dy, we get $\begin{pmatrix} y = 4x^2 \\ dy = 8x\,dx \end{pmatrix}$

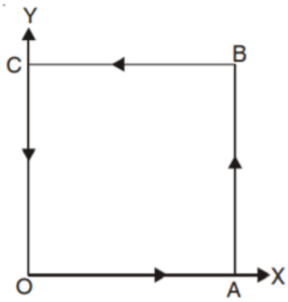
$$= \int_0^1 [2x^2\,(4x^2)\,dx + 3x\,(4x^2)\,8x\,dx]$$

$$= 104\int_0^1 x^4\,dx = 104\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5} \qquad \textbf{Ans.}$$

Ex2-Evaluate $\int_c \vec{F}.\overrightarrow{dr}$ where $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath}$ and c is the boundary of the square in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a$ & $y = a$

Solution

$$\int_C \vec{F}.\vec{d}\,r = \int_{OA}\vec{F}.\vec{dr} + \int_{AB}\vec{F}.\vec{dr} + \int_{BC}\vec{F}.\vec{dr} + \int_{CO}\vec{F}.\vec{dr}$$



Here $\quad \vec{r} = x\hat{i} + y\hat{j}, \quad \vec{d}\,r = dx\hat{i} + dy\hat{j}, \quad \vec{F} = x^2\hat{\imath} + xy\,\hat{\jmath}$

$$\vec{F}.\vec{dr} = x^2dx + xydy \qquad \ldots(1)$$

On $OA, y = 0$

$$\therefore\ \vec{F}.\vec{dr} = x^2dx$$

$$\int_{OA}\vec{F}.\vec{dr} = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad \ldots(2)$$

On $AB, x = a$
(1) becomes

$$\therefore\ dx = 0$$

$$\therefore\ \vec{F}.\vec{dr} = aydy$$

$$\int_{Ab}\vec{F}.\vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2} \qquad \ldots(3)$$

On $BC, y = a$
$\Rightarrow$ (1) becomes

$$\therefore\ dy = 0$$

$$\vec{F}.\vec{dr} = x^2dx$$

$$\int_{BC}\vec{F}.\vec{dr} = \int_a^0 x^2dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3} \qquad \ldots(4)$$

On $CO, x = 0$,
(1) becomes

$$\therefore\ \vec{F}.\vec{dr} = 0$$

$$\int_{CO}\vec{F}.\vec{dr} = 0 \qquad \ldots(5)$$

$$\int_C \vec{F}.\vec{d}\,r = \frac{a^3}{2}$$

Ex3- If $\vec{A} = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}$, evaluate $\oint \vec{A}.\overrightarrow{dr}$ from (0,0,0) to (1,1,1) along the curve $x = t, y = t^2, z = t^3$

Solution We have, $\int_C \vec{A}.\vec{dr} = \int_C [(3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}].[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$

$$= \int_C [(3x^2 + 6y)\,dx - 14yz dy + 20xz^2 dz]$$

If $x = t$, $y = t^2$, $z = t^3$, then points (0, 0, 0) and (1, 1, 1) correspond to $t = 0$ and $t = 1$ respectively.

Now, $\int_C \vec{A}.\vec{dr} = \int_{t=0}^{t=1} [(3t^2 + 6t^2)\,d\,(t) - 14t^2\,t^3\,d\,(t^2) + 20t\,(t^3)^2\,d\,(t^3)]$

$$= \int_{t=0}^{t=1} [9t^2\,dt - 14t^5\,.2t\,d\,t + 20\,t^7\,.3\,t^2\,dt] = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[9\left(\frac{t^3}{3}\right) - 28\left(\frac{t^7}{7}\right) + 60\left(\frac{t^{10}}{10}\right)\right]_0^1 \qquad = 3 - 4 + 6 = 5$$

Ex4- Evaluate $\int_c \vec{F}.\overrightarrow{dr}$ where $\vec{F} = \frac{\hat{\imath}y - \hat{\jmath}x}{x^2 + y^2}$ and c is the circle $x^2 + y^2 = 1$ traversed counter clockwise

Solution

$$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z,\ d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$\int_c \vec{F}.d\vec{r} = \int_c \frac{\hat{i}y - \hat{j}x}{x^2 + y^2} \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c \frac{ydx - xdy}{x^2 + y^2} = \int_c (ydx - xdy) \qquad \ldots(1)\ [\because\ x^2 + y^2 = 1]$$

Putting $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\,\theta$, $dy = \cos\theta\,d\,\theta$ in (1), we get

$$\int_C \vec{F}\,d\,\vec{r} = \int_0^{2\pi} \sin\theta\,(-\sin\theta\,d\,\theta) - \cos\theta\,(\cos\theta\,d\,\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\,\theta = -\int_0^{2\pi} d\,\theta = -(\theta)_0^{2\pi} = -2\pi$$

Theorem on Conservative Field

Definition of Conservative Field

Let $\vec{F}(x,y,z)$ be a vector function such that curl $\vec{F}=0$, then $\vec{F}$ is irrotational and $\vec{F}$ is said to be Conservative.

Theorem

If $\vec{F}$ is Conservative, there exists a scalar function (or scalar potential)$\emptyset$ such that $\vec{F}=\overrightarrow{\nabla\emptyset}$

To find the scalar potential function ϕ

$$\vec{F}=\vec{\nabla}\phi$$

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{i}dx+\hat{j}dy+\hat{k}dz\right)$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi.\left(d\vec{r}\right)=\nabla\phi.d\vec{r}=\vec{F}.d\vec{r}$$

Ex1- Show that the vector field $\vec{F}=2x(y^2+z^3)\hat{\imath}+2x^2y\hat{\jmath}+3x^2z^2\hat{k}$ is conservative. Find the scalar potential and work done in moving a particle from (-1,2,1) to (2,3,4)

Solution

Step 1– To show that $\vec{F}$ is conservative $\Rightarrow$TST $\vec{F}$ is irrotational

$\Rightarrow$ TST curl $\vec{F}=0$

$$\vec{F}=2x(y^2+z^3)\hat{i}+2x^2y\,\hat{j}+3x^2z^2\hat{k}$$

$$\text{Curl }\vec{F}=\nabla\times\vec{F}$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2\end{vmatrix}=(0-0)i-(6xz^2-6xz^2)\hat{j}+(4xy-4xy)\hat{k}=0$$

Hence, vector field $\vec{F}$ is irrotational.

Since F is Conservative, there exists a scalar function (or scalar potential)$\emptyset$ such that $\vec{F}=\overrightarrow{\nabla\emptyset}$

Step2- To find scalar potential $\emptyset$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.\left(d\vec{r}\right) = \nabla\phi.d\vec{r} = \vec{F}.d\vec{r}$$

$$= \left[2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}\right](\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2x(y^2+z^3)dx + 2x^2y\,dy + 3x^2z^2dz$$

Method 1

$$\phi = \int\left[2x(y^2+z^3)dx + 2x^2ydy + 3x^2z^2dz\right] + C$$

$$\int(2xy^2dx + 2x^2ydy) + (2xz^3dx + 3x^2z^2dz) + C = x^2y^2 + x^2z^3 + C$$

Hence, the scalar potential is $x^2y^2 + x^2z^3 + C$

Method 2

Compare the expressions of $d\emptyset$ and $\vec{F}.\overrightarrow{dr}$

$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$ and $\vec{F}.d\vec{r} = 2x(y^2+z^3)dx + 2x^2y\,dy + 3x^2z^2dz$

As $d\emptyset = \vec{F}.\vec{dr}$ So we have

$$\frac{\partial\emptyset}{\partial x} = 2x(y^2+z^3) \quad \frac{\partial\emptyset}{\partial y} = 2x^2y \quad \frac{\partial\emptyset}{\partial z} = 3x^2z^2$$

Integrating above eqns w.r.t. x, y, z respectively partially, we get

$$\emptyset = \int 2x(y^2+z^3)dx = x^2(y^2+z^3) + k(y,z)$$

$$\emptyset = \int 2x^2ydy = x^2y^2 + k(x,z)$$

$$\emptyset = \int 3x^2z^2dz = x^2z^3 + k(x,y)$$

Excluding the repeated terms, we get

$$\emptyset = x^2y^2 + x^2z^3 + c$$

Step 3 To find work done

Since the field is conservative, work done does not depend on the path. It depends only on the end points.

$$\text{Work done} = \int_{(-1,2,1)}^{(2,3,4)}\vec{F}.d\vec{r} = \int_{(-1,2,1)}^{(2,3,4)}d\phi = [\phi]_{(-1,2,1)}^{(2,3,4)} = \left[x^2y^2 + x^2z^3 + c\right]_{(-1,2,1)}^{(2,3,4)}$$

$$= (36 + 256) - (2 - 1) = 291$$

Ex 2- Determine whether the integral $\int (2xyz^2)dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$ is independent of path of integration? If so, then evaluate it from (1,0,1) to $(0,\frac{\pi}{2},1)$.

Solution $\int_c (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$

$$= \int_c [(2xyz^2\hat{i}) + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].(\hat{i}dx + \hat{j}dy + \hat{k}dz) = \int_c \vec{F}\cdot\vec{dr}$$

This integral is independent of path if curl $\vec{F} = 0$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz) = \hat{i} - (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k} = 0$$

Hence the integral is independent of path of integration and therefore there exists a scalar function (or scalar potential)$\emptyset$ such that $\vec{F} = \overrightarrow{\nabla\emptyset}$

Evaluation of integral from (1,0,1) to $(0,\frac{\pi}{2},1)$.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz) = \nabla\phi\cdot dr = \vec{F}\cdot\vec{d}\,r$$

$$= [(2xyz^2)\,\hat{i} + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].\,(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$$

$$= [(2x\,dx)\,yz^2 + x^2\,(dy)\,z^2 + x^2y\,(2z\,dz)] + [(\cos yz\,dy)\,z + (\cos yz\,dz)\,y]$$

$$= d\,(x^2yz^2) + d\,(\sin yz)$$

$$\phi = \int d\,(x^2yz^2) + \int d\,(\sin yz) = x^2yz^2 + \sin yz$$

The value of integral is

$$[\phi]_A^B = \phi\,(B) - \phi\,(A)$$

$$= [x^2yz^2 + \sin yz]_{(0,\frac{\pi}{2},1)} - [x^2yz^2 + \sin yz]_{(1,0,1)} = \left[0 + \sin\left(\frac{\pi}{2}\times 1\right)\right] - [0 + 0]$$

=1

EX3- A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{\imath} + (y^2 + x^2 y)\hat{\jmath}$. Show that the vector field $\vec{A}$ is irrotational and find the scalar potential.

Solution $\vec{A}$ is irrotational if curl $\vec{A} = 0$

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy - 2xy) = 0$$

Hence, $\vec{A}$ is irrotational. If ϕ is the scalar potential, then

$$\vec{A} = \text{grad } \phi$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \qquad \text{[Total differential coefficient]}$$

$$= \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) . (\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz) = \text{grad } \phi \, . \, dr$$

$$= \vec{A}.dr = [(x^2 + xy^2)\hat{i} + (y^2 + x^2 y)\hat{j}].(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz) = (x^2 + xy^2)\, dx + (y^2 + x^2 y)\, dy$$

To find Ø

So we have, $\frac{\partial Ø}{\partial x} = (x^2 + xy^2) \qquad \frac{\partial Ø}{\partial y} = (y^2 + x^2 y)$

Integrating above eqns w.r.t. x, y, z respectively partially, we get

$$Ø = \int (x^2 + xy^2) dx = \frac{x^3}{3} + \frac{x^2 y^2}{2} + k(y)$$

$$Ø = \int (y^2 + x^2 y) dy = \frac{y^3}{3} + \frac{x^2 y^2}{2} + k(x)$$

Hence $\qquad Ø = \frac{x^3}{3} + \frac{x^2 y^2}{2} + \frac{y^3}{3} + c$

EX4- A fluid motion is given by $\vec{v} = (y \sin z - \sin x)\hat{\imath} + (x \sin z + 2yz)\hat{\jmath} + (xy \cos z + y^2)\hat{k}$. Is the motion is irrotational and find the velocity potential.

Solution

$$\text{Curl } \vec{v} = \vec{\nabla} \times \vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (y \sin z - \sin x)\hat{i} + (x \sin z + 2yz)\hat{j} + (xy \cos z + y^2)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= (x \cos z + 2y - x \cos z - 2y)\hat{i} - [y \cos z - y \cos z]\hat{j} + (\sin z - \sin z)\hat{k} = 0$$

Hence, the motion is irrotational.

To find the velocity potential

So, $\bar{v} = \bar{\nabla}\phi$ where ϕ is called velocity potential.

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \qquad \text{[Total differential coefficient]}$$

$$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \vec{\nabla}\phi.d\vec{r} = \vec{v}.d\vec{r}$$

$$= [(y \sin z - \sin x)\hat{i} + (x \sin z + 2yz)\hat{j} + (xy \cos z + y^2)\hat{k}].[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$$

$$= (y \sin z - \sin x)\,dx + (x \sin z + 2\,y\,z)\,dy + (x\,y \cos z + y^2)\,dz$$

$$= (y \sin z\,dx + x\,dy \sin z + x\,y \cos z\,dz) - \sin x\,dx + (2\,y\,z\,dy + y^2\,dz)$$

$$= d\,(x\,y \sin z) + d\,(\cos x) + d\,(y^2\,z)$$

$$\phi = \int d\,(xy \sin z) + \int d\,(\cos x) + \int d(y^2 z)$$

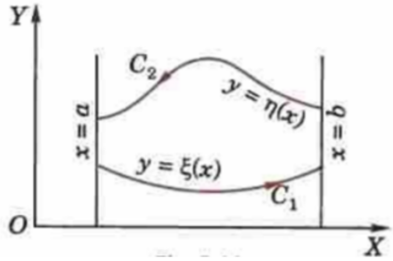
$$\phi = xy \sin z + \cos x + y^2 z + c$$

Hence, Velocity potential $= xy \sin z + \cos x + y^2 z + c$. **Ans.**

Green's Theorem

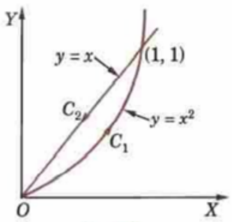
If P and Q are two functions of x and y such that their partial derivatives $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous single valued functions over the closed region R bounded by a curve c, then

$$\int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$



Type 1- Verification of Green's Theorem

Ex-1 Verify Green's Theorem for $\int_c [(xy + y^2)dx + x^2 dy]$ where c is bounded by $y = x$ and $y = x^2$



Solution Here $P = xy + y^2$ and $Q = x^2$

$$\int_c Pdx + Qdy = \int_{c1} + \int_{c2}$$

Along C_1, $y = x^2$ and x varies from 0 to 1

$$\therefore \quad \int_{C_1} = \int_0^1 [\{x(x)^2 + (x^2)^2\}]\, dx + x^2 d(x^2)]$$

$$= \int_0^1 (3x^3 + x^4)\, dx = \frac{19}{20}$$

Along C_2, $y = x$ and x varies from 1 to 0.

$$\therefore \quad \int_{C_2} = \int_1^0 [\{x(x) + (x)^2\}\, dx + x^2\, d(x)] = \int_1^0 3x^2\, dx = -1.$$

$$\int_c Pdx + Qdy = \int_{c1} + \int_{c2} = \frac{19}{20} - 1 = -\frac{1}{20} \qquad \text{(i)}$$

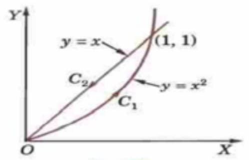
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xy + y^2) = x + 2y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x$$

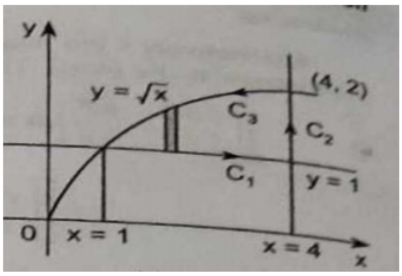
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dydx = \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx = \int_0^1 (x^4 - x^3)\, dx = -\frac{1}{20} \qquad \text{(ii)}$$

Hence, Green theorem is verified from the equality of (i) and (ii).



Ex 2- Verify Green's Theorem for $\int_{c} \left[\frac{1}{y} dx + \frac{1}{x} dy\right]$ where c is bounded by $\mathrm{x} = 1$, $\mathrm{x} = 4, y = 1$ and $y = \sqrt{x}$



Solution Here $\mathrm{P} = \frac{1}{y}$ and $\mathrm{Q} = \frac{1}{x}$,

$$\int_{c} Pdx + Qdy = \int_{c1} + \int_{c2} + \int_{c3}$$

Along c1, $y = 1, dy = 0$

$$\int_{c1} = \int_{c1} \left(\frac{1}{y} dx + \frac{1}{x}.0\right) = \int_{1}^{4} dx = [x]_1^4 = 3$$

Along c2, $\mathrm{x} = 4, dx = 0,$ $\int_{c2} = \int_{c2} \left(\frac{1}{y}.0 + \frac{1}{x} dy\right) = \int_1^2 \frac{dy}{4} = \frac{1}{4}[y]_1^2 = \frac{1}{4}$

Along c3, $y = \sqrt{x}, dy = \frac{1}{2\sqrt{x}} dx$

$$\int_{c3} = \int_4^1 \left(\frac{1}{y} dx + \frac{1}{x} dy\right) = \int_4^1 \left(\frac{1}{\sqrt{x}} dx + \frac{1}{2x^{3/2}} dx\right) = [2\sqrt{x} - x^{-1/2}]_4^1 = -\frac{5}{2}$$

$$\int_{c} Pdx + Qdy = 3 + \frac{1}{4} - \frac{5}{2} = \frac{3}{4} \qquad \text{(i)}$$

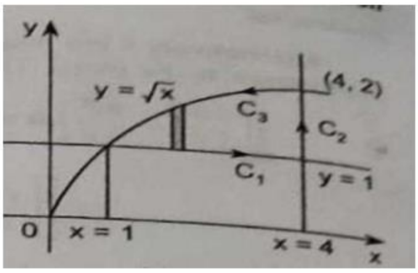
.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_1^4 \int_1^{\sqrt{x}} \left(-\frac{1}{x^2} + \frac{1}{y^2}\right) dxdy = \int_1^4 [-\frac{y}{x^2} - \frac{1}{y}]_1^{\sqrt{x}}$$

$$= \int_{1}^{4} \left[-\frac{1}{x^{\frac{3}{2}}} - \frac{1}{\sqrt{x}} + \frac{1}{x^2} + 1\right] dx$$

$$= \left[2x^{-1/2} - 2\sqrt{x} - \frac{1}{x} + x\right]_1^4 = \frac{3}{4}$$



(ii)

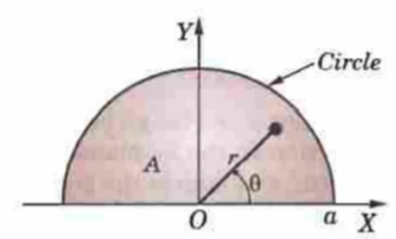
Hence, Green theorem is verified from the equality of (*i*) and (*ii*).

Type 2- Evaluation

Ex 1-Apply Greens Theorem to Evaluate $\int_c [(2x^2 - y^2)dx + (x^2+y^2)dy]$ where c is boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$

Solution By Greens Theorem $\int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

Here $\mathrm{P} = 2x^2 - y^2$ and $Q = x^2 + y^2$



$\frac{\partial P}{\partial y} = -2y$ and $\frac{\partial Q}{\partial x} = 2x$

$$\int_c [(2x^2 - y^2)dx + (x^2+y^2)dy] =$$

$= 2 \iint_A (x + y)\, dxdy$, where A is the region

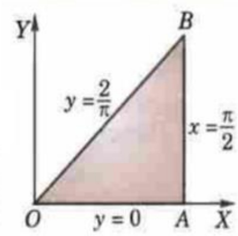
$$= 2\int_0^a \int_0^\pi r\ (\cos\theta + \sin\theta) \cdot r d\theta\, dr \quad = 2\int_0^a r^2 dr \int_0^\pi (\cos\theta + \sin\theta)\, d\theta = 2 \cdot \frac{a^3}{3} \cdot (1+1) = \frac{4a^3}{3}.$$

Ex 2 Apply Greens Theorem to Evaluate $\int_c [(y - \sin x)dx + \cos x\, dy]$ where c is the plane of the triangle enclosed by the lines $y = 0\, , x = \pi/2$ and $y = \frac{2}{\pi} x$

Solution By Greens Theorem $\int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

Here $\mathrm{P} = y - \sin x$ and $\mathrm{Q} = \cos x$ $\frac{\partial P}{\partial y} = 1 - \cos x$ and $\frac{\partial Q}{\partial x} = 2x$

$$\int_c [(y - \sin x)dx + \cos x\, dy]$$



$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{y=2x/\pi} (-\sin x - 1)\, dy\, dx = -\int_0^{\pi/2} (\sin x + 1) \Big| y \Big|_0^{2x/\pi} dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} x(\sin x + 1)\, dx = -\frac{2}{\pi}\left\{ \Big| x(-\cos x + x) \Big|_0^{\pi/2} - \int_0^{\pi/2} 1.(-\cos x + x)\, dx \right\}$$

$$= -\frac{2}{\pi}\left\{ \frac{\pi^2}{4} - \Big| -\sin x + \frac{x^2}{2} \Big|_0^{\pi/2} \right\} = -\frac{\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right) = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Type 3- Work done

Ex 1- Find the work done in moving a particle once round the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ in the plane $z=0$ in the force field given by $\vec{F}=(3x-2y)\hat{\imath}+(2x+3y)\hat{\jmath}+y^2\hat{k}$ by using Greens Theorem.

Solution

Work done $=\int_c \vec{F}.\overrightarrow{dr}=\int_c (3x-2y)dx+(2x+3y)dy\ +y^2dz$

$=\int_c (3x-2y)dx+(2x+3y)dy$ as $z=0$, $dz=0$

Here $P=(3x-2y)$ and $Q=(2x+3y)$ $\quad \frac{\partial P}{\partial y}=-2$ and $\frac{\partial Q}{\partial x}=2$

By Greens Theorem $\int_c Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

Work done $=\int_c \vec{F}.\overrightarrow{dr}$

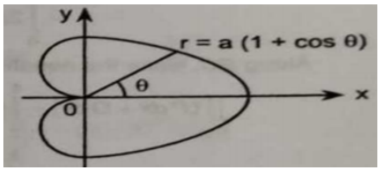
$$=\iint_R (2+2)dxd=4\iint_R dxdy=4\,Area\ of\ ellipse=4\pi ab=4\pi 4.3=48\pi$$

Ex 2-Evaluate by Greens Theorem $\int_c \vec{F}.\overrightarrow{dr}$ where $\vec{F}=-xy(x\hat{\imath}-y\hat{\jmath})$ and c is $r=a(1+\cos\theta)$

Solution By Greens Theorem $\int_c Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

And $\int_c \vec{F}.\overrightarrow{dr}=\int_c(-xy(x\hat{\imath}-y\hat{\jmath})).(dx\hat{\imath}+dy\hat{\jmath})$

$$=\int_c -x^2ydx+xy^2dy$$



By comparison, $P=-x^2y$ and $Q=xy^2$ $\quad \frac{\partial P}{\partial y}=-x^2$ and $\frac{\partial Q}{\partial x}=y^2$

$\int_c \vec{F}.\overrightarrow{dr}=\iint_R (y^2+x^2)dxdy$

Change to Polar Coordinates, $x=r\cos\theta$, $y=r\sin\theta$, $dxdy=rd\theta dr$

$\int_c \vec{F}.\overrightarrow{dr}=\iint_R r^3d\theta dr=2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3d\theta dr=2\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)}d\theta$

$\frac{1}{2}\int_0^{\pi}a^4(1+\cos\theta)^4d\theta=8\,a^4\int_0^{\pi}cos^8\left(\frac{\theta}{2}\right)d\theta=16a^4\int_0^{\pi/2}cos^8(t)dt$ (put $\frac{\theta}{2}=t$)

$=\frac{35\pi}{16}a^4$ (using formula of Beta function)