POSET and Lattice

Module 4: POSET and Lattice (Total Lectures: **09)**

4.1 Partial ordered relations (Posets) ,Hasse diagram

4.2 Lattice, sublattice

4.3 Types of Lattice ,Boolean Algebra

POSETS(Partially Ordered Sets)

A relation R on a set A is called **partial order if R** is

REFLEXIVE,

ANTISYMMETRIC AND

TRANSITIVE

The set A together with the partial order R is called a POSET (A, R) or A

HASSE DIAGRAM

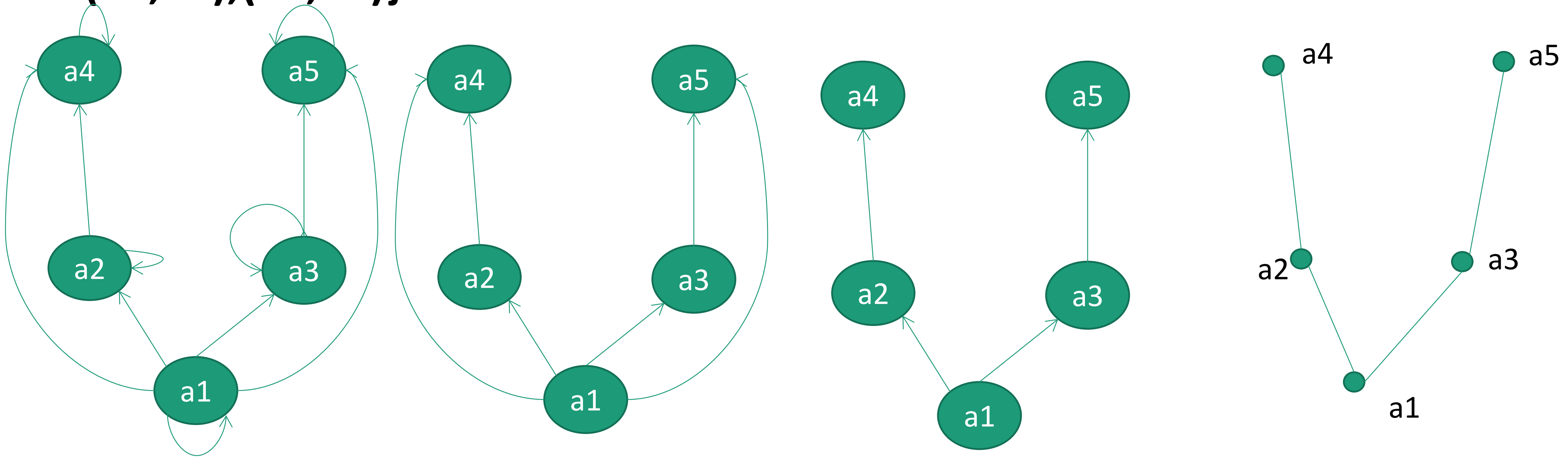
Mathematical diagram used to represent a finite partially ordered set

STEPS:

1. Draw the digraph of the given relation
2. Delete all cycles from the graph
3. Eliminate all edges that are implied by transitive relations
4. Draw the digraph of a partial order with all edges pointing upward, so that arrows may be omitted from the edges.
5. Replace the circles representing the vertices by dots. The resulting diagram of a partial order is called the Hasse diagram of the partial order of the poset.

Hasse Diagram for POSET

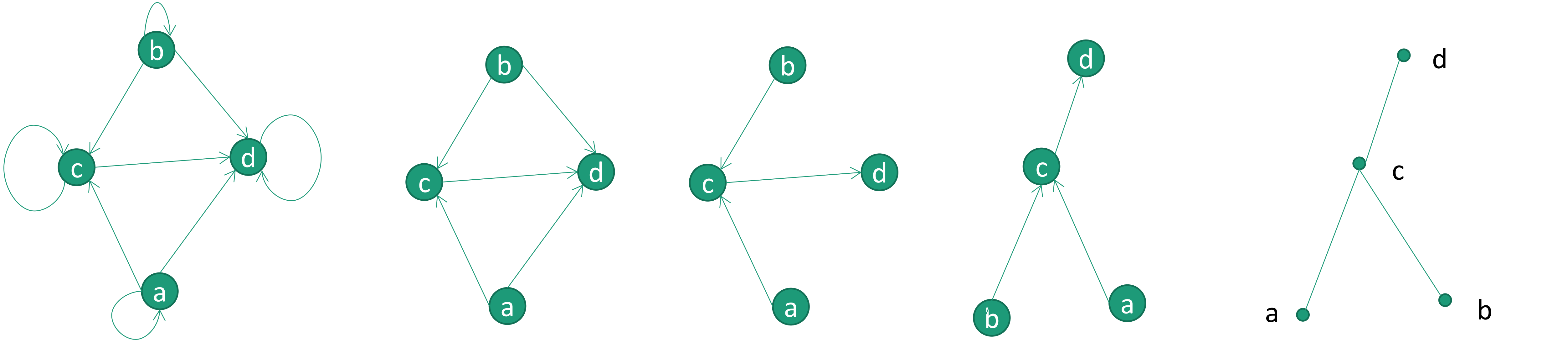
- **R={(a1,a1),(a1,a2),(a1,a3),(a1,a4),(a1,a5),(a2,a2),a2,a4),(a3,a3),(a3,a5),(a4,a4),(a5,a5)}**



STEP 1 STEP 2 STEP 3,4 STEP 5

Draw the Hasse diagram of R

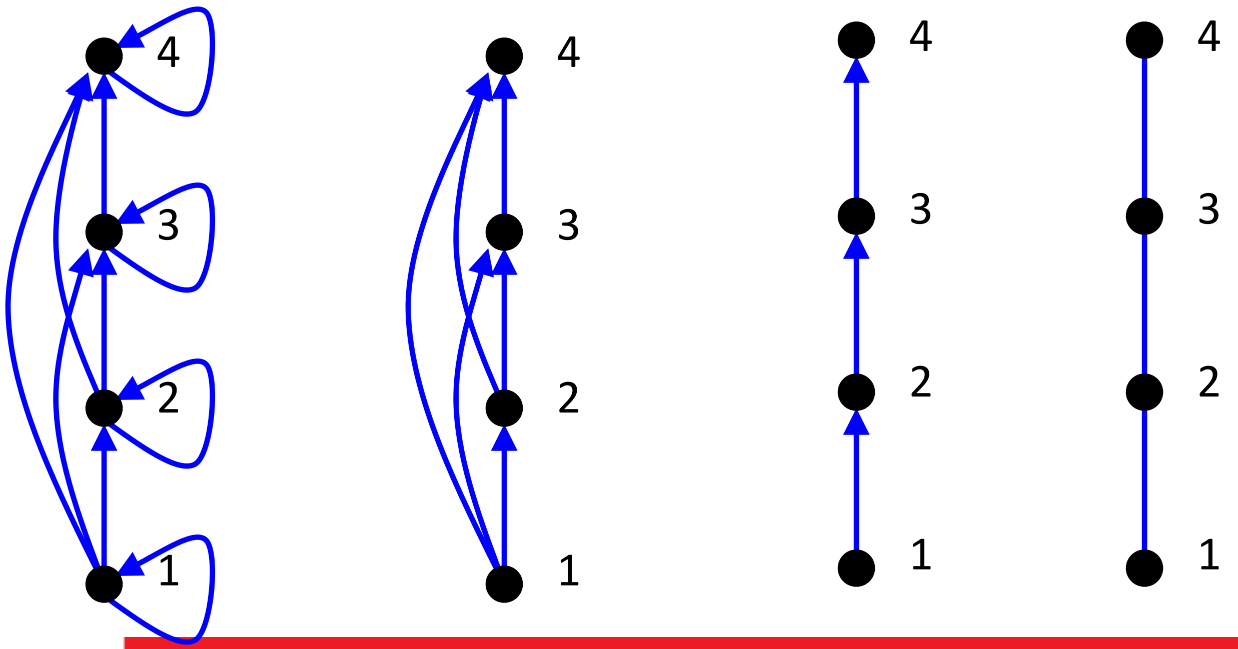
R= {a,a),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(c,d),(d,d)}



Step 1 Step 2 Step 3 Step 4 Step 5

Problems

- Draw the Hasse Diagram for the **POSET({1,2,3,4},<=)**
- Consider the POSET({1,2,3,4,6,8,12},|)



- Let A = { 1 , 2 , 3 , 4 } and R be a relation on A whose matrix is

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Prove that R is a partial order and draw the Hasse diagram.

Problems

Draw a Hasse diagram for A, (divisibility relation) where

(i) A = {1, 2, 3, 4, 5, 6, 7, 8};

(ii) A = {1, 2, 3, 5, 11, 13};

(iii) A = {2, 3, 4, 5, 6, 30, 60};

(iv) A = {1, 2, 3, 6, 12, 24};

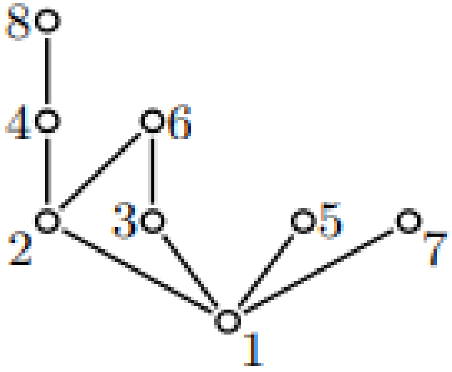
(v) A = {1, 2, 4, 8, 16};

(vi) A = {2, 4, 6, 12, 24, 36}

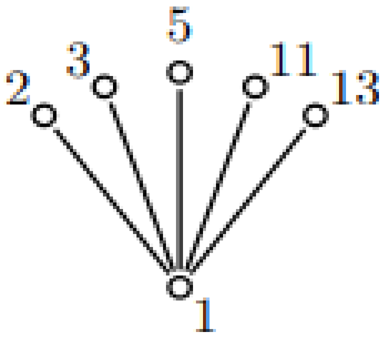
Solution

Solution:

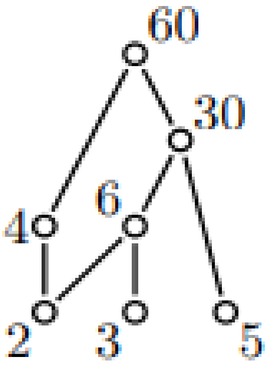
(i):



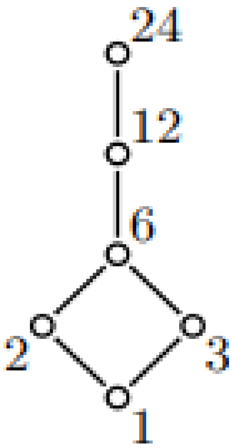
(ii):



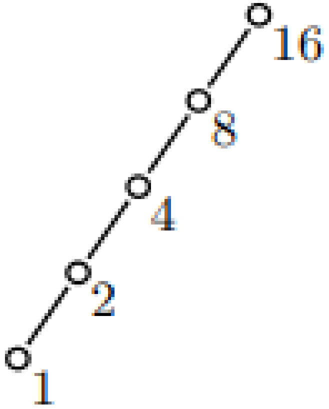
(iii):



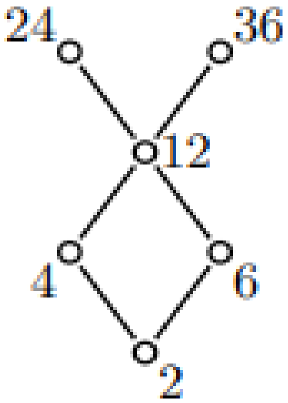
(iv):



(v):



(vi):



Problems

- **Draw Hasse diagram for D_6, D_{16}, D_{30}, D_{80}**

- If R^{-1} is reflexive, antisymmetric, and transitive. Thus R^{-1} is also partial order. The POSET(A, R^{-1}) is called the **dual** of the POSET(A,R), and the partial order R^{-1} is called the **dual** of the partial order R.
- If (A,≤) is poset, the elements a and b of a are said to be **comparable** if

$$a \leq b \quad \text{or} \quad b \leq a.$$

Example – In the poset $(Z^+, |)$ (where Z^+ is the set of all positive integers and $|$ is the divides relation) are the integers 3 and 9 comparable? Are 7 and 10 comparable?

The word partial in partial order set means some elements may not be comparable.

Linearly ordered set

- Definition: If $(S,\prec)$ is a poset and every two elements of S are comparable, S is called a **totally ordered set or linearly ordered set**.
- If every pair of elements in poset S is comparable, then S is **chain**.
- The relation $\prec$ is said to be a **total order or linear order**.
- Example
  - The relation "less than or equal to" over the set of integers $(\mathbb{Z}, \leq)$ since for every $a,b \in \mathbb{Z}$, it must be the case that $a \leq b$ or $b \leq a$
- What happens if we replace $\leq$ with $<$?

Lexicographic Orderings

- Lexicographic ordering is the same as any dictionary or phone-book ordering:
  - We use alphabetic ordering
    - Starting with the first character in the string
    - Then the next character, if the first was equal, etc.
  - If a word is shorter than the other, than we consider that the ‘no character’ of the shorter word to be less than ‘a’

Lexicographic order

- Formally, lexicographic ordering is defined by two other orderings
- **Definition**: Let $(A_1,\prec_1)$ and $(A_2,\prec_2)$ be two posets. The lexicographic ordering $\prec$ on the Cartesian product $A_1 \times A_2$ is defined by

$$(a_1,a_2) \prec (a'_1,a'_2) \text{ if } (a_1 \prec_1 a'_1) \text{ or } (a_1 = a'_1 \text{ and } a_2 \prec_2 a'_2)$$

- If we add equality to the lexicographic ordering $\prec$ on $A_1 \times A_2$, we obtain a partial ordering
- lexicographical order: $(a,b) \leq (c,d)$ if $a < c$ or ($a = c$ and $b \leq d$);
- product order: $(a,b) \leq (c,d)$ if $a \leq c$ and $b \leq d$;

Extremal Elements: Summary

We will define the following terms:

- A **maximal/minimal element** in a poset (S, $\prec$)
- The maximum (**greatest**)/minimum (**Least**) element of a poset (S, $\prec$)
- An **upper/lower bound element** of a subset A of a poset (S, $\prec$)
- The **greatest lower (glb)/least upper bound (lub)** element of a subset A of a poset (S, $\prec$)

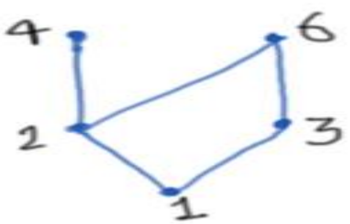
Maximal /Minimal element

- **Maximal element**: If in a poset, **an element in not related to any other element**, then it is called maximal element.

Minimal element: If in a poset, **if no element is related to an element**, then it is called minimal element.

Considering poset(A,≤), An element a∈A is called maximal element of a if there is no element c in A such that a<c. An element b∈A is called minimal element of A if there is no element c in A such that c<b.

Example:
[{1,2,3,4,6};A(|)]



Maximal Element = 4, 6
Minimal Element = 1

- **Definition**: An element a in a poset (S, $\prec$) is called maximal if it is not less than any other element in S.
- If there is one unique maximal element **a**, we call it the maximum element (or the greatest element (I))

Minimum

- **Definition**: An element a in a poset (S, $\prec$) is called minimal if it is not greater than any other element in S.
- If there is one unique minimal element **a**, we call it the minimum element (or the least element or Zero element (O))

- Greatest Element **I (unit element)**
- Least Element **O (Zero Element)**

A POSET HAS ATMOST 1 GREATEST ELEMENT AND ATMOST 1 LEAST ELEMENT

Greatest Element = Universal Upper bound(UB)

Least Element = Universal Lower bound(LB)

LUB → Least Upper Bound (LUB)

GLB → Greatest Lower bound (GLB)

Extremal Elements :**Upper bound**

- **Definition**: Let $(S,\leq)$ be a poset and let $A\subseteq S$.

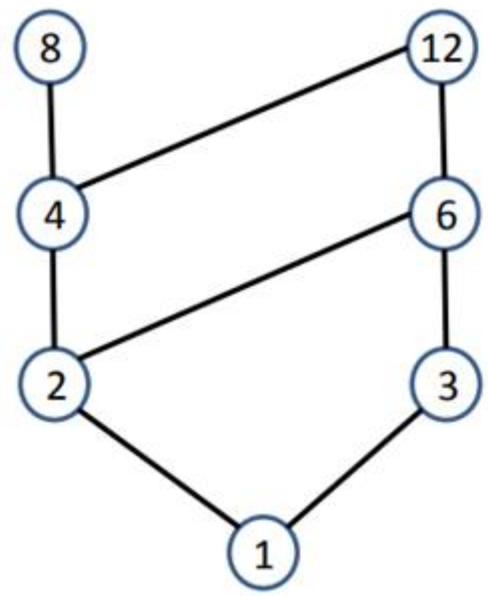
  If u is an element of S such that $a \leq u$ for all $a\in A$ then u is an <u>upper bound of A</u>

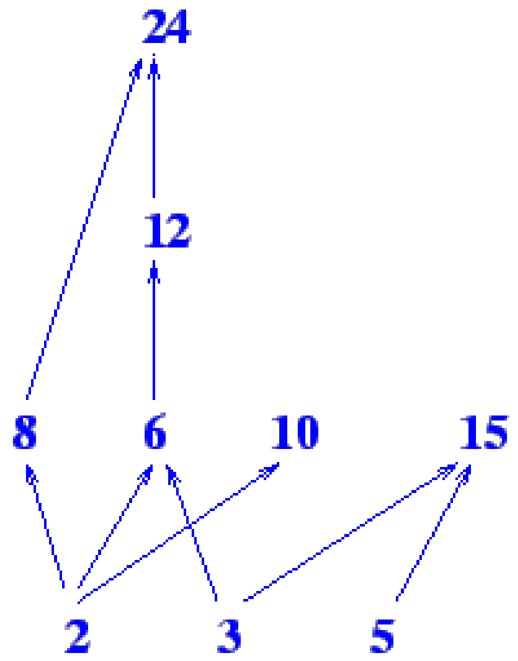
- An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the <u>least upper bound on A</u>. We abbreviate it as LUB

Extremal Elements: Lower Bound

- **Definition**: Let $(S, \leq)$ be a poset and let $A \subseteq S$. If l is an element of S such that $l \leq a$ for all $a \in A$ then l is an lower bound of A
- An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the greatest lower bound on A. We abbreviate it GLB.

Find Maximal and Minimal





Problems

Consider the poset {3, 5, 9, 15, 24, 45}, for divisibility relation.

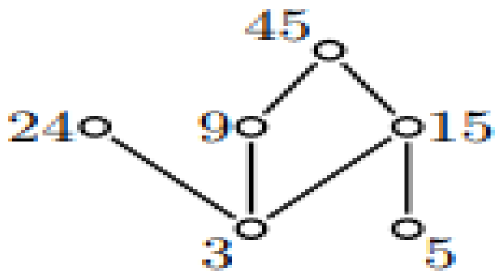
(i) Draw its Hasse diagram.

(ii) Find its maxima, minima, greatest and least elements when they exist.

(iii) Find maxima, minima, greatest and least elements of the set M = {3, 9, 15}, when they exist.

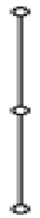
Solution:

(ii): Max 24,45, greatest DNE, min 3,5, least DNE.
(iii): Max 9,15, greatest DNE, min 3, least 3.



Chain and Anti chain

- Chain-A subset of A is called a chain if every two elements in the subset are related
- Anti chain- If no two distinct elements in the subset are related



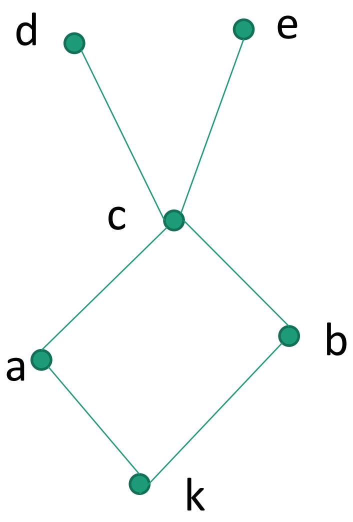
(a) chain

(b) antichain

SOMAIYA VIDYAVIHAR UNIVERSITY
K J Somaiya College of Engineering

Examples

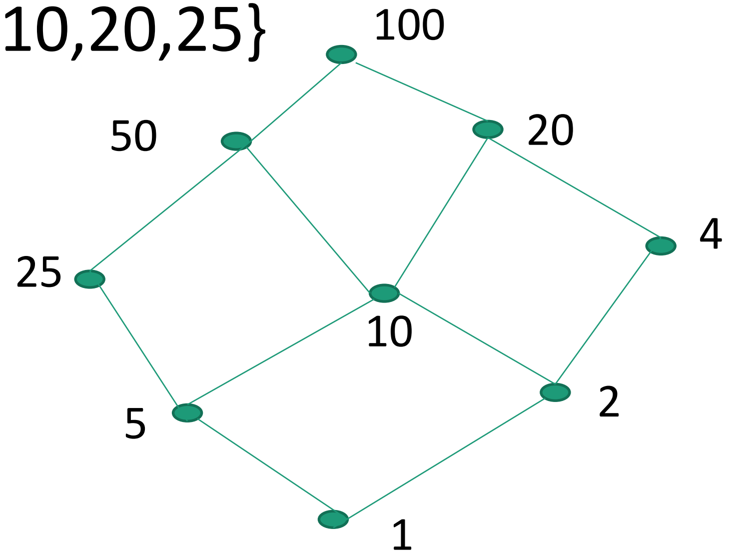
- Determine LUB, GLB of B={a,b,c}



- LUB is c GLB is k

- Determine LUB, GLB of a. B={10,20}

b. B={5,10,20,25}



a. GLB is 10 LUB is 20
b. GLB is 5 LUB is 100

Lattice

A LATTICE IS A POSET (L, $\leq$) IN WHICH EVERY SUBSET { a , b } consisting of 2 elements has a **LUB and a GLB**

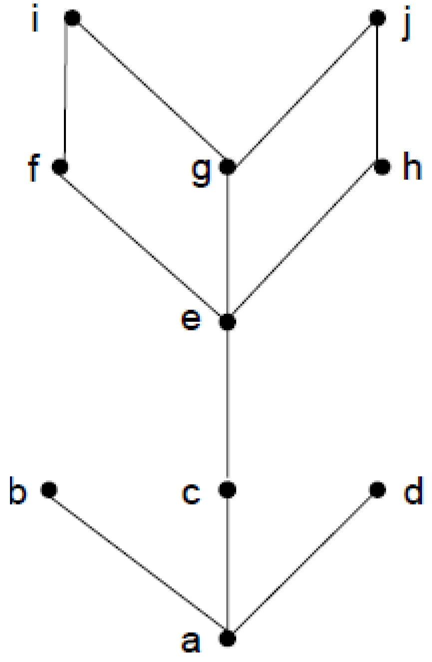
LUB ({ a , b }) by **a V b (join of a and b)**

GLB ({ a , b }) by **a ∧ b (meet of a and b)**

Example

- Is the example shown in fig a Lattice?

No, because the pair {b,c} does Not have a least upper bound



- Is this example a Lattice?

  No

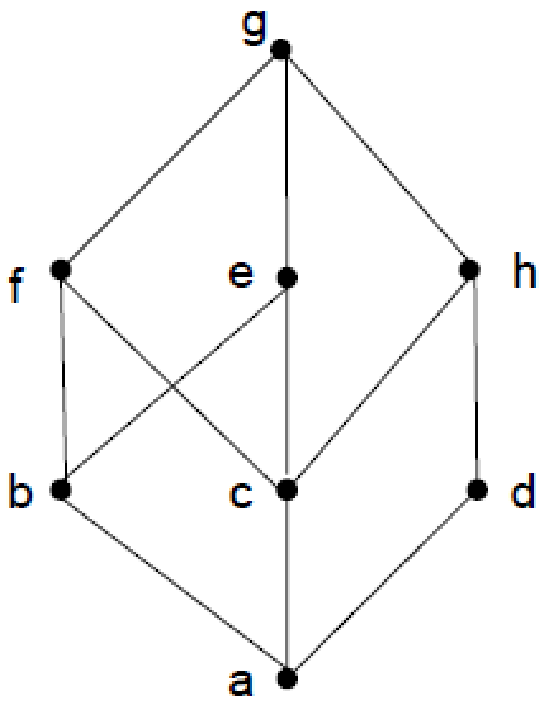
  The lower bound of A={e,f} is {a,b,c}

  No glb for A

  Simillarly ,

  The upper bound for B= {b,c} is {f,e,g}

  no ulb for B



Identify lattice

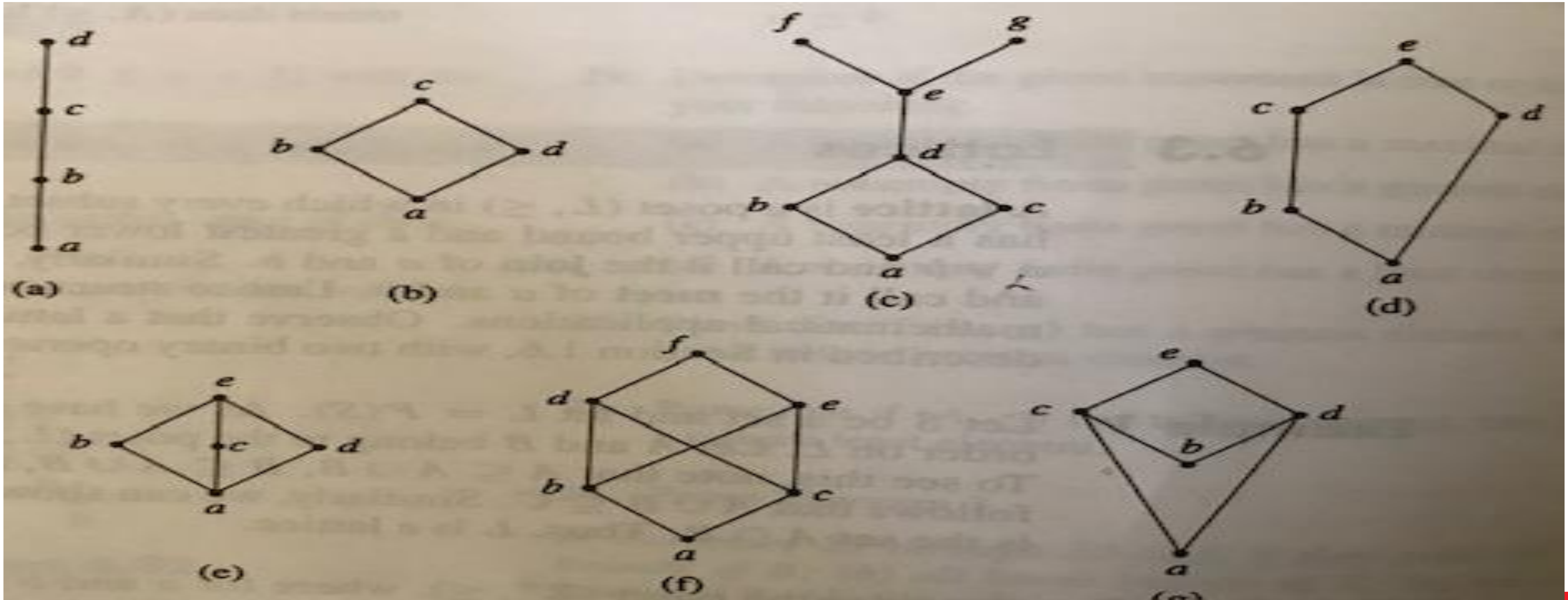
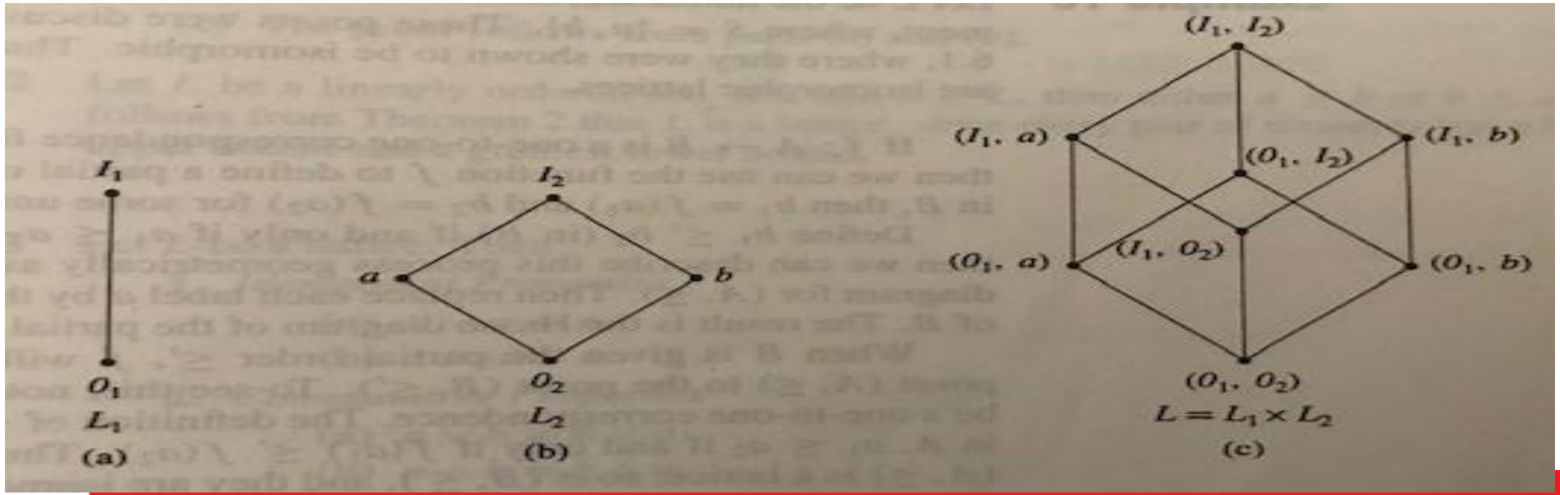


Figure 6.40

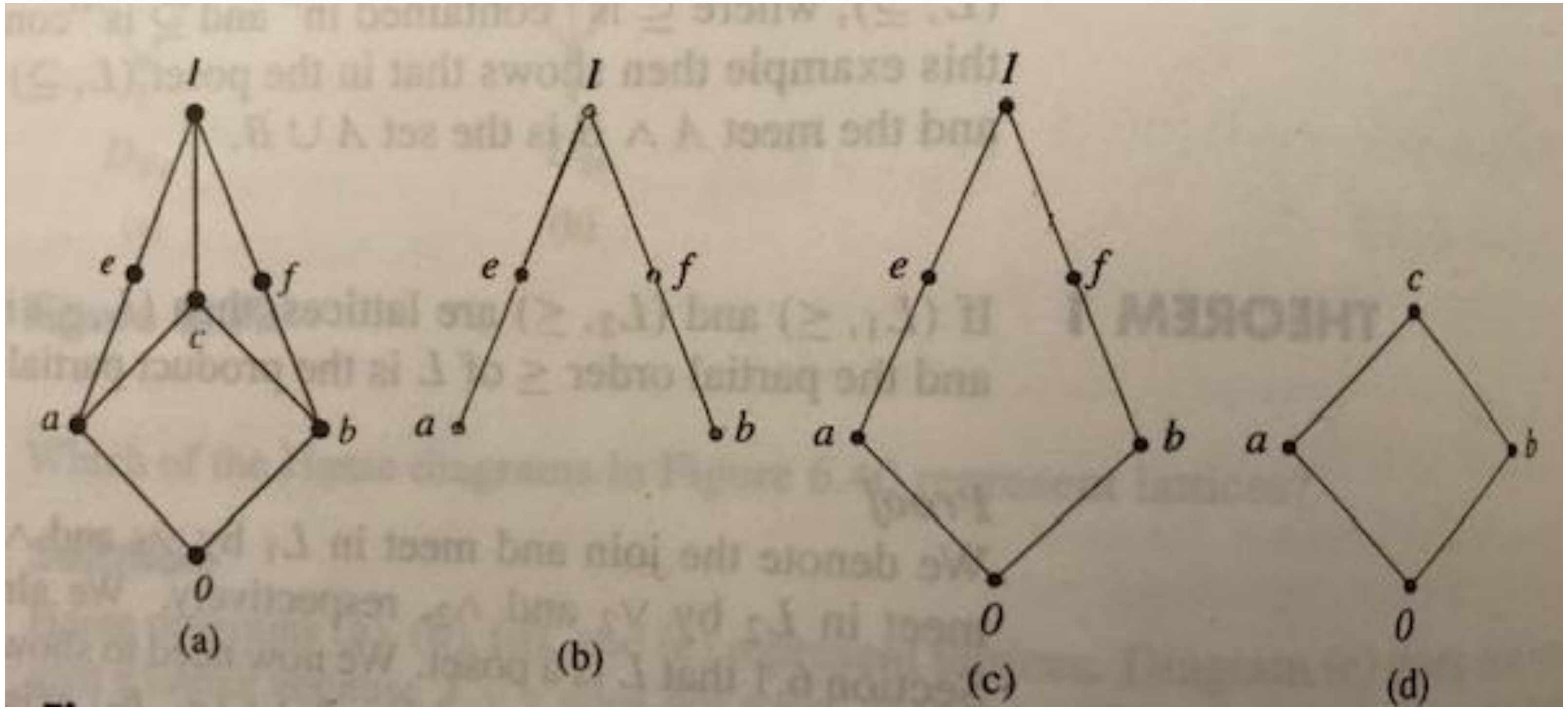
Sublattice

- Let L1 and L2 be the lattices shown in fig (a) and (b). Then L= L1 x L2 is the lattice.
- Let $(L, \leq)$ be a lattice. A non empty subset S of L is called a sublattice of L if $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$.



Identify the sublattice of (a)

- State whether the (b),(c) and (d) are sublattice of (a)?



Properties of Lattices

1. Idempotent Properties

a) $a \vee a = a$

b) $a \wedge a = a$

2. Commutative Properties

a) $a \vee b = b \vee a$

b) $a \wedge b = b \wedge a$

3. Associative Properties

a) $a \vee (b \vee c) = (a \vee b) \vee c$

b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Properties

a) $a \vee (a \wedge b) = a$

b) $a \wedge (a \vee b) = a$

THEOREMS-Self Study-Kolman

a) $a \vee b = b$ iff $a \leq b$
b) $a \wedge b = a$ iff $a \leq b$
c) $a \wedge b = a$ iff $a \vee b = b$
d) If $a \leq b$ then
   a) $a \vee c \leq b \vee c$
   b) $a \wedge c \leq b \wedge c$
e) $a \leq c$ and $b \leq c$ iff $a \vee b \leq c$
f) $c \leq a$ and $c \leq b$ iff $c \leq a \wedge b$
g) if $a \leq b$ and $c \leq d$ then
   a) $a \vee c \leq b \vee d$
   b) $a \wedge c \leq b \wedge d$

ISOMORPHIC LATTICES

If $f: L1 \to L2$ is an isomorphism from the poset $(L1, \leq 1)$ to the poset $(L2, \leq 2)$ then L1 is a lattice iff L2 is a lattice.

If a and b are elements of L1 then

$f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$

If two lattices are isomorphic as posets we say they are isomorphic lattices.

- TYPES OF LATTICE
  - BOUNDED
  - COMPLEMENTED
  - DISTRIBUTIVE

Bounded Lattice

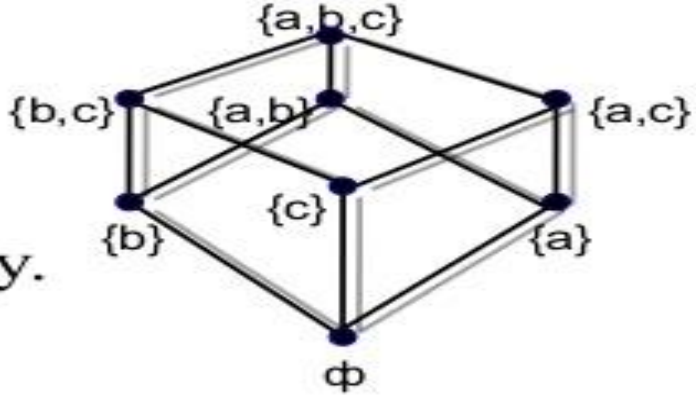
- Let 'L' be a lattice w.r.t R if there exists an element I ∈ L such that (a R I) ∀ x ∈ L , then I is called Upper Bound of a Lattice L.
- Similarly if there exists an element O ∈ L such that (O R a) ∀ a ∈ L , then O is called Lower Bound of Lattice L.
- **In a Lattice if Upper Bound and Lower exists then it is called Bounded Lattice.**

Distributive lattice

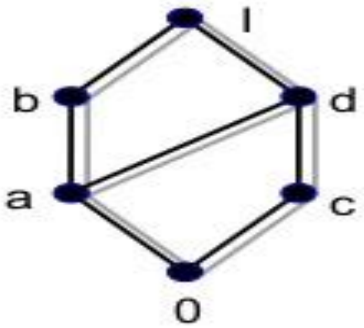
- If a lattice satisfies the following two distribute properties, it is called a distributive lattice.
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Distributive Lattices

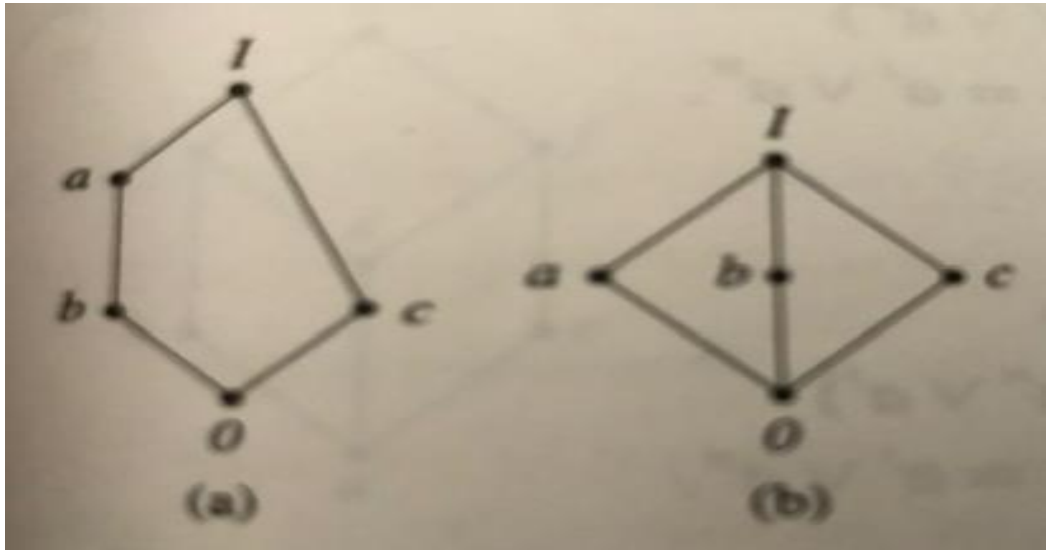
- Example
  For a set S, the lattice P(S) is distributive, since join and meet each satisfy the distributive property.



- Example
  The lattice whose Hasse diagram shown in adjacent diagram is distributive.



- Identify the shown lattice is distributive lattice or not?



(a) $a \wedge (b \vee c) =$

$a \wedge I = a$

$(a \wedge b) \vee (a \wedge c) =$

$b \vee o = b$

(b) $a \vee (b \wedge c) =$

$a \vee O = a$

$(a \vee b) \wedge (a \vee c) =$

$I \wedge I = I$

Complemented Lattice

- A lattice L is said to be complemented if it is bounded and if every element in L has a complement
- **Theorem**: Let L be a bounded lattice with greatest element I and least element 0 and let a belong to L.

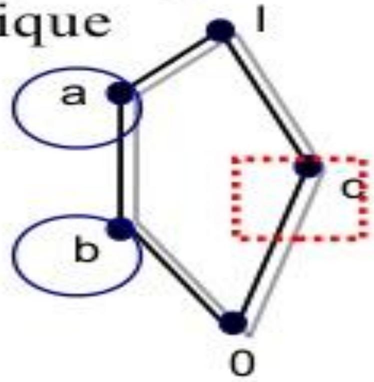
  An element a’ belong to L is a complement of a if $a \vee a' = I$ and $a \wedge a' = 0$

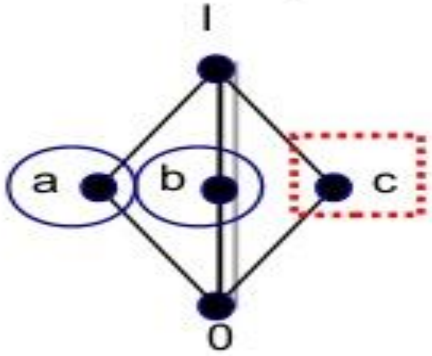
Complemented Lattice

- Example

  The lattice L=P(S) is such that every element has a complement, since if A in L, then its set complement $\bar{A}$ has the properties $A \vee \bar{A} = S$ and $A \wedge \bar{A} = \phi$. That is, the set complement is also the complement in L.

- Example : complemented lattices where complement of element is not unique





Boolean Algebra

- Lattice which satisfies Compliment and Distributive property

Quasi ordered set

Suppose $\prec$ is a relation on a set S satisfying the following two properties:

[Q_1] (Irreflexive) For any $a \in A$, we have $a \nprec a$.

[Q_2] (Transitive) If $a \prec b$, and $b \prec c$, then $a \prec c$.

Then $\prec$ is called a *quasi-order* on S.

- LUB-> Supremum : sup(A) -> LCM
- GLB-> Infimum : Inf(A) ->GCD

Well order Set

Definition 14.1: An ordered set S is said to be *well-ordered* if every subset of S has a first element.

The classical example of a well-ordered set is the set **N** of positive integers with the usual order $\leq$. The following facts follow from the definition.

(1) A well-ordered set is linearly ordered. For if $a, b, \in S$, then $\{a, b\}$ has a first element; hence a and b are comparable.

(2) Every subset of a well-ordered set is well-ordered.

(3) If X is well-ordered and Y is isomorphic to X, then Y is well-ordered.

(4) All finite linearly ordered sets with the same number n of elements are well-ordered and are all isomorphic to each other. In fact, they are all isomorphic to $\{1, 2, \ldots, n\}$ with the usual order $\leq$.

(5) Every element $a \in S$, other than a last element, has an immediate successor. For, let $M(a)$ denote the set of elements which strictly succeed a. Then the first element of $M(a)$ is the immediate successor of a.

Problems to solve

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ be ordered by divisibility. State whether each of the following subsets of $\mathbf{N}$ are linearly (totally) ordered.

(*a*) $\{24, 2, 6\}$; (*c*) $\mathbf{N} = \{1, 2, 3 \ldots\}$; (*e*) $\{7\}$;
(*b*) $\{3, 15, 5\}$; (*d*) $\{2, 8, 32, 4\}$; (*f*) $\{15, 5, 30\}$.

(*a*) Since 2 divides 6 which divides 24, the set is linearly ordered.

(*b*) Since 3 and 5 are not comparable, the set is not linearly ordered.

(*c*) Since 2 and 3 are not comparable, the set is not linearly ordered.

(*d*) This set is linearly ordered since $2 \prec 4 \prec 8 \prec 32$.

(*e*) Any set consisting of one element is linearly ordered.

(*f*) Since 5 divides 15 which divides 30, the set is linearly ordered.

b)3 $\parallel$ 5 incomparable elements

Prerequisites in college is a familiar partial ordering of available classes. We write $A \prec B$ if course A is a prerequisite for course B. Let C be the ordered set consisting of the mathematics courses and their prerequisites appearing in Fig. (a).

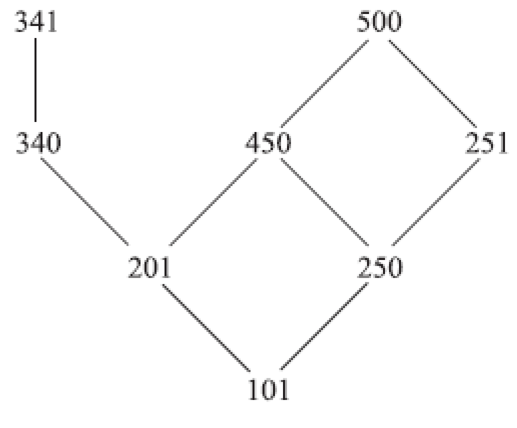
(a) Draw the Hasse diagram for the partial ordering C of these classes.

(b) Find all minimal and maximal elements of C.

(c) Does C have a first element or a last element?

Class	Prerequisites
Math 101	None
Math 201	Math 101
Math 250	Math 101
Math 251	Math 250
Math 340	Math 201
Math 341	Math 340
Math 450	Math 201, Math 250
Math 500	Math 450, Math 251

(a)



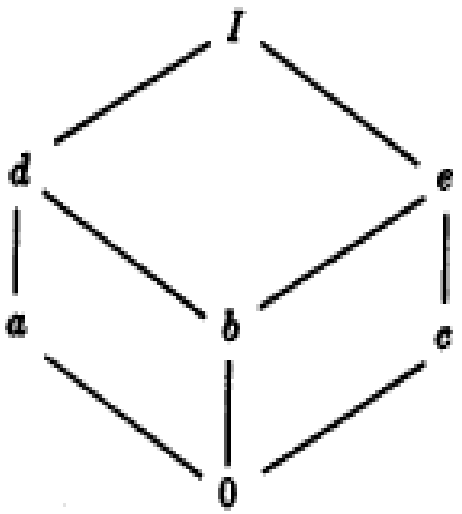
(b)

Join irreducible element and Atom

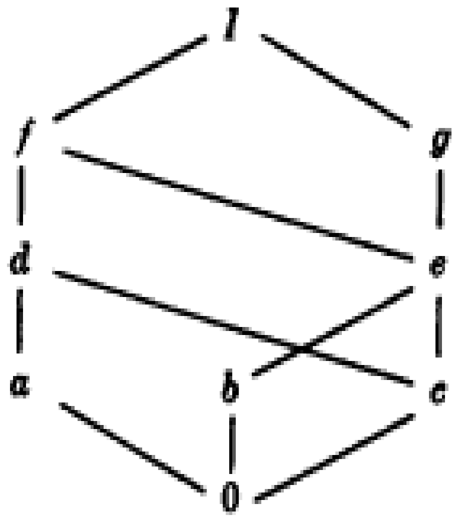
Definition: Let L be a lattice. An element $x \in L$ is join-irreducible if it cannot be written as the join of two other elements. That is, if $x = y \vee z$ then either $x = y$ or $x = z$. The subposet (not sublattice!) of L consisting of all join-irreducible elements is denoted $\mathrm{Irr}(L)$.

- In other word, $a \neq O$, is join irreducible if and only if ***a*** has a unique immediate predecessor. Those elements which immediately succeed O, called *atoms*, are join irreducible.

- Find join irreducible elements and atoms for fig.(a) and (b)
- Fig(a): a,b,c,0 (a,b,c)
- Fig(b): a,b,c,g,0 (a,b,c)
- Find complements of a and b
- c,e : complement of a
- (b has no complement)
- Fig.b : a has g; b has none



(a)



(b)

Problem to solve

- Let A={a, b, c, d, e, f, g, h} and R be the relation defined by M_R.show that (A,R) is poset. Is the poset (A,R) is complimented (if yes give all pair of complements) and Prove or disprove that (A,R) is a Boolean algebra.

$$M_R = \begin{matrix} 1&1&1&1&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&1&1&1&1&1&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1 \\ 0&0&0&1&0&0&0&1 \end{matrix}$$

Problem to solve

- Let A={a, b, c, d, e, f, g, h} and R be the relation defined by M_R. Prove or disprove that (A,R) is a Boolean algebra.

$$
MR= \begin{matrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix}
$$

c) Mention all the elements of set D_{36} also specify R on D_{36} as aRb if a | b. Mention Domain and Range of R. Explain if the relation is Equivalence Relation or a Partially Ordered Relation. If it is a Partially Ordered Relation, draw its Hasse Diagram. (08 M)

(c) Determine the matrix of the partial order of divisibility on the set A. Draw the Hasse diagram of the Poset. Indicate those which are chains [8M]

(1) A={1,2,3,5,6,10,15,30}

(2) A={3,6,12,36,72}

Q.2 (a) Find the complement of each element in D_{42}. [6M]

- X={2,3,6,12,24,36} a relation ≤ is defined as x≤y if x divides y. Draw the hasse diagram for (X,≤). Find maximal, minimal elements. Give example of chain and antichain. Is the poset a lattice.

ANY ????