More Problems on Gradient

EX5: Find $\nabla(e^{r^2})$

Solution: We know that $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$

$$\therefore \nabla(e^{r^2}) = e^{r^2}\, 2r\frac{\bar{r}}{r} = 2\, e^{r^2}\, \bar{r}$$

EX 6: Prove that $\nabla(r^n) = nr^{n-2}\bar{r}$

Solution: We know that $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$

$$\therefore \nabla(r^n) = nr^{n-1}\frac{\bar{r}}{r}$$
$$= nr^{n-2}\bar{r}$$

Ex 7: Show that $\nabla\left[\frac{(\bar{a}\cdot\bar{r})}{r^n}\right] = \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$

Solution:

: We have $\frac{\bar{a}\cdot\bar{r}}{r^n} = \frac{(a_1 i + a_2 j + a_3 k)\cdot(xi + yj + zk)}{r^n}$

$$= \frac{a_1 x + a_2 y + a_3 z}{r^n}$$

Let $\Phi = \frac{\bar{a}\cdot\bar{r}}{r^n} = \frac{a_1 x + a_2 y + a_3 z}{r^n}$

$$\therefore \frac{\partial \Phi}{\partial x} = \frac{r^n \cdot a_1 - (a_1 x + a_2 y + a_3 z)\, n\, r^{n-1}(\partial r/\partial x)}{r^{2n}}$$

But $r^2 = x^2 + y^2 + z^2 \quad \therefore 2r\frac{\partial r}{\partial x} = 2x \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\therefore \frac{\partial \Phi}{\partial x} = \frac{a_1 r^n - (a_1 x + a_2 y + a_3 z)\cdot n\, r^{n-2}\cdot x}{r^{2n}}$$

$$= \frac{a_1}{r^n} - \frac{n(a_1x + a_2y + a_3z)\,x}{r^{n+2}}$$

Similarly, $$\frac{\partial \Phi}{\partial y} = \frac{a_2}{r^n} - \frac{n(a_1 x + a_2 y + a_3 z)\,y}{r^{n+2}}$$

and $$\frac{\partial \Phi}{\partial z} = \frac{a_3}{r^n} - \frac{n(a_1 x + a_2 y + a_3 z)\,z}{r^{n+2}}$$

$$\therefore \quad \nabla \Phi = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k$$

$$= \frac{1}{r^n}(a_1 i + a_2 j + a_3 k) - \frac{n}{r^{n+2}}\left[(a_1 x + a_2 y + a_3 z)(xi + yj + zk)\right]$$

But $\bar{a} \cdot \bar{r} = (a_1 i + a_2 j + a_3 k) \cdot (xi + yj + zk) = a_1 x + a_2 y + a_3 z$

$$\therefore \quad \nabla \Phi = \frac{\bar{a}}{r^n} - \frac{n}{r^{n+2}}(\bar{a} \cdot \bar{r})\,\bar{r}.$$

EXERCISE

If $\bar{r} = xi + yj + zk$ and $\bar{a}, \bar{b}$ are constant vectors, prove that $\bar{a} \cdot \nabla\left(\bar{b} \cdot \nabla \frac{1}{r}\right) = \frac{3(\bar{a} \cdot \bar{r})(\bar{b} \cdot \bar{r})}{r^5} - \frac{\bar{a} \cdot \bar{b}}{r^3}$.

EX8:Find $\phi(r) s.t\ \nabla\phi = -\frac{\bar{r}}{r^5}$ and $\phi(2)$=3

Solution: We have

$$\nabla\Phi = -\bar{r}(x^2+y^2+z^2)^{-5/2} \qquad [\because r = \sqrt{x^2+y^2+z^2}\,]$$

$$= -(x^2+y^2+z^2)^{-5/2}(xi+yj+zk) \qquad \text{..........}\ (1)$$

But $$\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} \qquad \text{..........}\ (2)$$

Comparing (1) and (2), we get,

$$\frac{\partial\Phi}{\partial x} = -x(x^2+y^2+z^2)^{-5/2},$$

$$\frac{\partial\Phi}{\partial y} = -y(x^2+y^2+z^2)^{-5/2},$$

$$\frac{\partial\Phi}{\partial z} = -z(x^2+y^2+z^2)^{-5/2}.$$

But $$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

$$= -(x^2+y^2+z^2)^{-5/2}(xdx+ydy+zdz)$$

Now let $x^2+y^2+z^2 = t$

$$\therefore\ 2(x\,dx + y\,dy + z\,dz) = dt$$

$$\therefore\ d\Phi = -t^{-5/2}\cdot\frac{dt}{2}$$

Integrating, $$\Phi = -\frac{1}{2}\frac{t^{-3/2}}{-3/2} + C = \frac{t^{-3/2}}{3} + C$$

Now resubstituting $t = x^2+y^2+z^2$,

$$\therefore\ \Phi = \frac{1}{3}(x^2+y^2+z^2)^{-3/2} + C$$

$$= \frac{1}{3}\cdot\frac{1}{r^3} + C$$

But by data $\Phi(r) = 3$ when $r = 2$

$$\therefore\ 3 = \frac{1}{3}\cdot\frac{1}{8} + C \qquad \therefore\ C = \frac{71}{24}$$

$$\therefore\ \Phi = \frac{1}{3}\cdot\frac{1}{r^3} + \frac{71}{24} = \frac{1}{3}\left(\frac{1}{r^3} + \frac{71}{8}\right).$$

Divergence and Curl of a vector point function

- Definition 1:Let $\overline{F} = f_1 i + f_2 j + f_3 k$ then Divergence of $\overline{F}$ is given by
- div $\overline{F}$ = $\nabla \cdot \overline{F}$ = $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$
- Definition 2: Curl of $\overline{F}$ is given by
- Curl $\overline{F}$= $\nabla \times \overline{F}$ = $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$
- Definition 3: A vector function $\overline{F}$ is said to be solenoidal if $\nabla \cdot \overline{F} = 0$
- Definition 4: A vector function $\overline{F}$ is said to be irrotational if $\nabla \times \overline{F} = \overline{0}$
- Remember :
- If $\overline{r} = xi + yj + zk$ then div $\overline{r}$=3 and Curl $\overline{r}$ = $\overline{0}$

Exercise

1 If $\overline{F} = x^2 z\, i - 2y^3 z^3 j + xy^2 z^2 k$ find div $\overline{F}$ and Curl $\overline{F}$ at (1,-1,1)

2 Show that $\overline{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal & irrotational.

3 If $\overline{r}$ is the position vector of point (x, y, z) and r is the modulus of $\overline{r}$, then prove that $r^n \overline{r}$ is an irrotational vector for any value of n but solenoidal only if $n = -3$.

Directional derivative

- Definition: Let ϕ be a scalar point function then directional derivative of ϕ at the point p in the direction of a unit vector $\hat{u}$ is given by

  $D_{\hat{u}}\phi$(p)= $\nabla\phi$(p)$\cdot$ $\hat{u}$

- Properties of Directional derivatives:

1) $D_{\hat{u}}\phi$(p)= $\nabla\phi$(p)$\cdot$ $\hat{u} = |\nabla\phi(\text{p})|\ |\hat{u}|\ \cos\theta$

$= |\nabla\phi(\text{p})| \cos\theta$ ($|\hat{u}|$=1)

$D_{\hat{u}}\phi$(p) is maximum if $\cos\theta$=1 $i.e\ \theta = 0$

$i.e$ $\nabla\phi$ and $\hat{u}$ are in the same direction .

But $D_{\hat{u}}\phi$(p) is in the direction of $\hat{u}$,therefore $D_{\hat{u}}\phi$(p) is maximum in the direction of $\nabla\phi$

and the magnitude of its maximum direction is $|\nabla\phi(\text{p})|$

2) $Similarly$ $D_{\hat{u}}\phi$(p) is minimum if $\cos\theta$= -1

$i.e\ \theta = \pi\ i.e$ $\nabla\phi$ and $\hat{u}$ are in the opposite direction .

But $D_{\hat{u}}\phi$(p) is in the direction of $\hat{u}$,therefore $D_{\hat{u}}\phi$(p) is minimum in the direction of - $\nabla\phi$

and the magnitude of its minimum direction is -$|\nabla\phi(\text{p})|$.

Problems on Directional derivatives

EX1: Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at point A = (1,-2,1) in the direction of AB where B = (2,6,-1)

Solution:

$$\nabla\Phi = i\frac{\partial}{\partial x}(x^4 + y^4 + z^4) + j\frac{\partial}{\partial y}(x^4 + y^4 + z^4) + k\frac{\partial}{\partial z}(x^4 + y^4 + z^4)$$

$$\therefore \quad \nabla\Phi = 4(x^3 i + y^3 j + z^3 k)$$

$$= 4(i - 8j + k) \text{ at } (1, -2, 1)$$

$$\overline{AB} = \overline{OB} - \overline{OA} = (2-1)\,i + (6+2)\,j + (-1-1)\,k$$

$$= i + 8j - 2k$$

$\therefore$ Directional derivative at A in the direction of $\overline{AB}$

$$= 4(i - 8j + k) \cdot \frac{(i + 8j - 2k)}{\sqrt{1 + 64 + 4}}$$

$$= \frac{4(1 - 64 - 2)}{\sqrt{69}} = -\frac{260}{\sqrt{69}}$$

EX 2: Find the directional derivative of $\phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at (1,2,3)

- Solution: $\nabla\Phi = i\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + j\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + k\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$

$$= 2(xi + yi + zk)$$

$$= 2(i + 2j + 3k) \text{ at } (1, 2, 3)$$

Given direction $= 3i + 4j + 5k$.

Directional derivative in the given direction

$$= \nabla\Phi \cdot \frac{\bar{a}}{|\bar{a}|} = 2(i + 2j + 3k)\frac{(3i + 4j + 5k)}{\sqrt{9 + 16 + 25}}$$

$$= \frac{2(3 + 8 + 15)}{5\sqrt{2}} = \frac{26}{5}\sqrt{2}.$$

EX 3 :In what direction from the point (2,1,-1) is the directional derivative of $\phi = x^2yz^3$ a maximum? What is its magnitude?

- Solution:

$$\nabla\Phi = \nabla x^2yz^3$$

$$= i\frac{\partial}{\partial x}(x^2yz^3) + j\frac{\partial}{\partial y}(x^2yz^3) + k\frac{\partial}{\partial z}(x^2yz^3)$$

$$= 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$$

$$= -4i - 4j + 12k \text{ at } (2, 1, -1)$$

Directional derivative is maximum in the direction of $\nabla\Phi$. Hence, directional derivative is maximum in the direction of $-4i - 4j + 12k$.

Its magnitude $= \sqrt{16 + 16 + 144} = 4\sqrt{11}$.