# Discrete Mathematics: Module 6
# Graphs and subgraphs

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# Graphs and Subgraphs

## Graphs and Subgraphs Total Lectures (05 )

**6.1** Definitions, Paths and circuits, Types of Graphs , Eulerian and Hamiltonian
**6.2** Planer graphs
**6.3** Isomorphism of graphs
**6.4** Subgraph

# Topics

- Definitions
- Paths and circuits
- Types of Graphs
- Representation
- Sub-graph
- Connectivity
- Eulerian and Hamiltonian definitions
- Isomorphism of graphs
- Planar Graphs

# Graph Theory

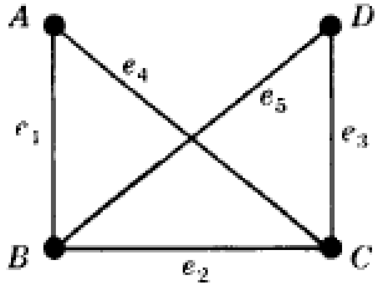
- Applications:
  - Computer networks
  - Distinguish between two chemical compounds with the same molecular formula but different structures
  - Solve shortest path problems between cities
  - Scheduling exams and assign channels to television stations

# Graph

- Definition:
- A generalization of the simple concept of a set of dots, links, edges or arcs.
- Representation: A Graph G =(V, E) consists of two things:
- (i) A set $V = V(G)$ whose elements are called *vertices, points,* or *nodes* of *G*.
- (ii) A set $E = E(G)$ of unordered pairs of distinct vertices called *edges* of *G*.

# Graphs cont….

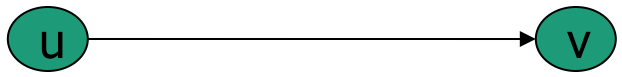
- Vertices *u* and *v* are said to be *adjacent* or *neighbors* if there is an edge *e* = {*u, v*}.
- In such a case, *u* and *v* are called the *endpoints* of *e*, and *e* is said to *connect u* and *v*. Also, the edge *e* is said to be *incident* on each of its endpoints *u* and *v*.
- Graphs are pictured by diagrams in the plane in a natural way.



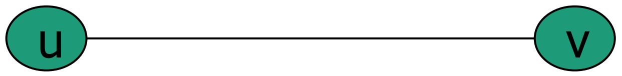

(a) Graph

# Definitions – Edge Type

**Directed:** Ordered pair of vertices. Represented as (u, v) directed from vertex u to v.



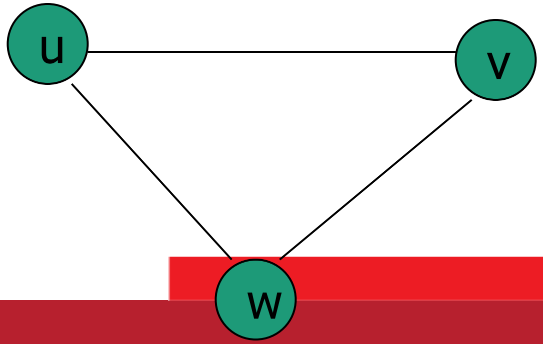

**Undirected:** Unordered pair of vertices. Represented as {u, v}. Disregards any sense of direction and treats both end vertices interchangeably.



# Definitions – Graph Type

**Simple (Undirected) Graph:** consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges (undirected)

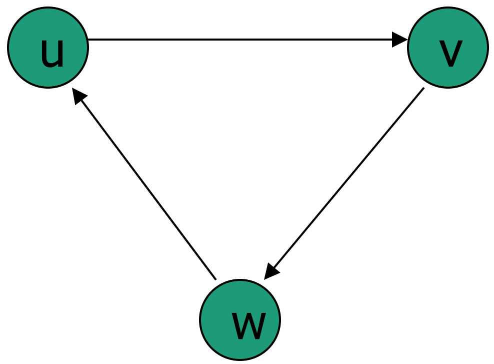
Representation Example: G(V, E), V = {u, v, w}, E = {{u, v}, {v, w}, {u, w}}



# Definitions – Graph Type

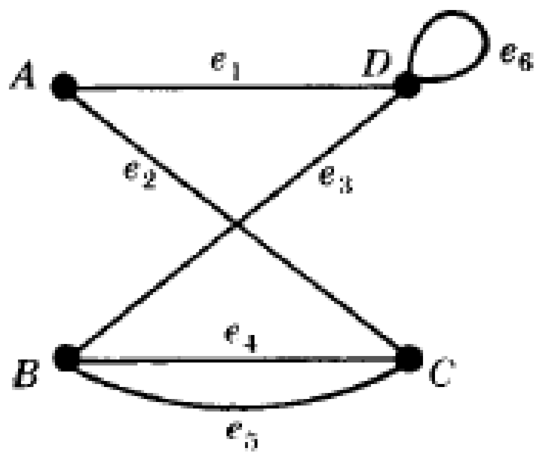
**Directed Graph:** G(V, E), set of vertices V, and set of Edges E, that are ordered pair of elements of V (directed edges)

Representation Example: G(V, E), V = {u, v, w}, E = {(u, v), (v, w), (w, u)}



# Multigraph

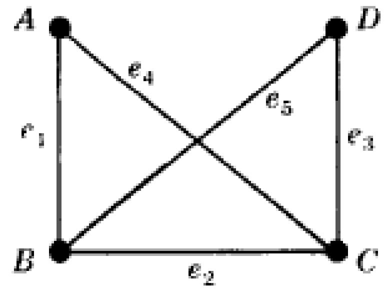
- The edges *e4* and *e5* are called *multiple edges* since they connect the same endpoints, and
- The edge *e6* is called a *loop* since its endpoints are the same vertex.
- Such a diagram is called a *multigraph*;
- the formal definition of a graph permits neither multiple edges nor loops.
- Thus a graph may be defined to be a multigraph without multiple edges or loops.




(b) Multigraph

# Degree of Vertex

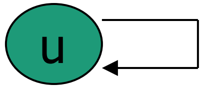
- The *degree of a vertex v* in a graph *G*, written deg *(v)*, is equal to the number of edges in *G* which contain *v*, that is, which are incident on *v*.
- Since each edge is counted twice in counting the degrees of the vertices of *G*, we have the following simple but important result.
- The sum of the degrees of the vertices of a graph *G* is equal to twice the number of edges in *G*.
- deg*(A)* = 2, deg*(B)* = 3, deg*(C)* = 3, deg*(D)* = 2.



(a) Graph

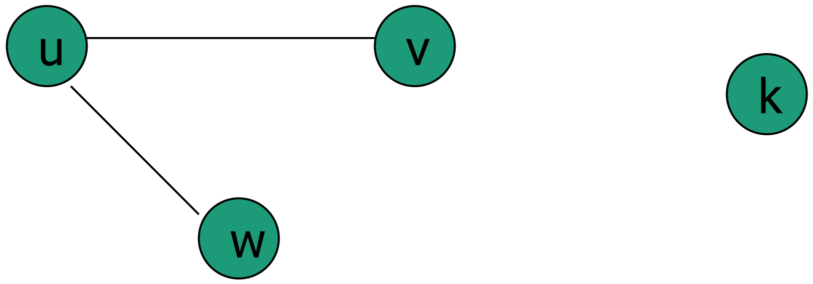
- The sum of the degrees equals 10 which, as expected, is twice the number of edges.
- A vertex is said to be *even* or *odd* according as its degree is an even or an odd number.
- Thus *A* and *D* are even vertices whereas *B* and *C* are odd vertices.

- **Isolated Vertex**: A vertex of degree zero is called an *isolated* vertex.
- **Loop**: A graph may contain an edge from a vertex to itself referred to as a loop



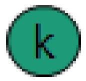

- **Adjacent vertices** : A pair of vertices that determine an edge.

- For V = {u, v, w} ,
  E = { {u, w}, {u, w}, (u, v) },
  deg (u) = 2, deg (v) = 1, deg (w) = 1, deg (k) = 0,
- k is isolated vertex



# Finite graph and trivial Graph

- A multigraph is said to be *finite* if it has a finite number of vertices and a finite number of edges.
- The finite graph with one vertex and no edges, i.e., a single point, is called the *trivial graph*.
- Example of Trivial Graph

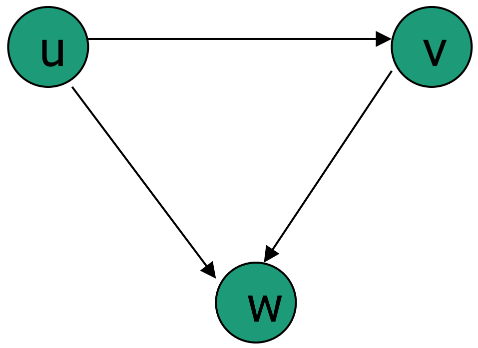


# Directed graphs

- **In-degree (u):** number of in coming edges

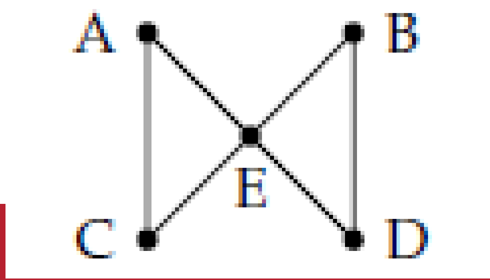
- **Out-degree (u):** number of outgoing edges

**Representation Example:** For V = {u, v, w} , E = { (u, w), ( v, w), (u, v) },
indeg(u) = 0, outdeg (u) = 2,
indeg(v) = 1,outdeg(v)=1
indeg(w) = 2, outdeg (w) = 0



# Problems

- Al, Bob, Cam, Dan, and Euler are all members of the social networking website *Facebook*. The site allows members to be "friends" with each other. It turns out that Al and Cam are friends, as are Bob and Dan. Euler is friends with everyone. Represent this situation with a graph.
- Solution:
  - Each person will be represented by a vertex and
  - each friendship will be represented by an edge.



# Problems

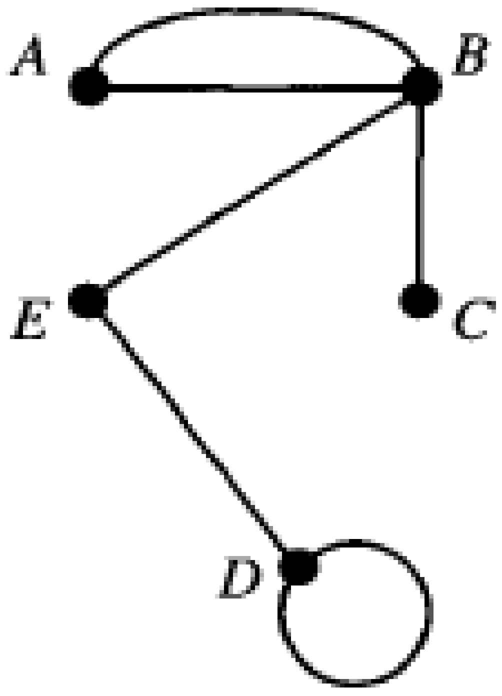



Figure 6.4

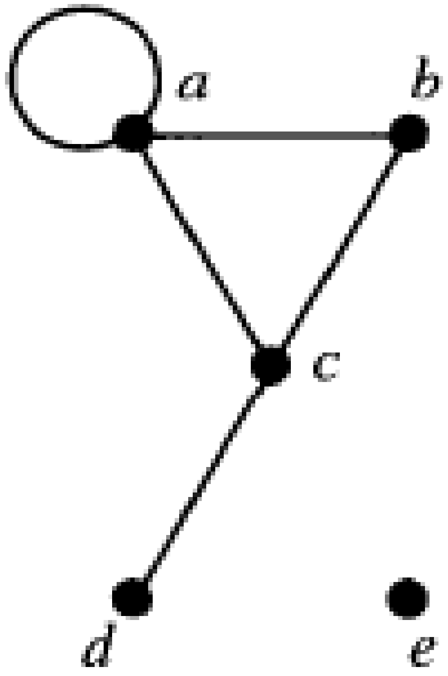



Figure 6.5

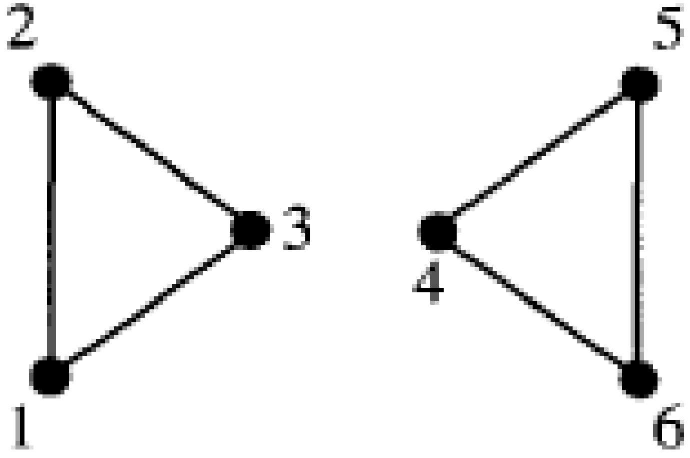



Figure 6.6

Back

# Path and Circuits

- **PATHS:**
- A path π in a graph G is a sequence π: $v_l, v_2, \ldots, v_k$ of vertices, each adjacent to the next, and a choice of an edge between each $v_i$ and $v_{i+1}$ so that no edge is chosen more than once.
- Pictorially, this means that it is possible to begin at $v_i$ and travel along edges to $v_k$ and never use the same edge twice.
- **CIRCUITS:**
- A circuit is a path that begins and ends with the same vertex. we call such paths cycles;
- A path $v_l, v_2, \ldots, v_k$ is called **simple path** if no vertex appears more than once.
- Similarly, a circuit $v_l, v_2, \ldots, v_{k-1}, v_l$ is **simple** if the vertices $v_l, v_2, \ldots, v_{k-1}$ are all distinct.

# Paths

- Figure 6.2 is π1: 1, 3, 4, 2
- Figure 6.4 π2: D, E, B, C, π3: A, B, E, D, D
- π4: 1,2,1 in fig 6.3
- Figure 6.5 are π5: a, b, c, a and π6: d,c,a,a
- Figure 6.6 the sequence 1, 2, 3, 2
- π7: c, a, b, c, d in Figure 6.5

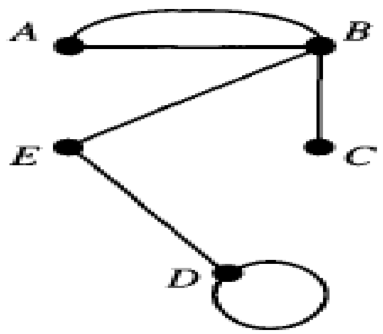



Figure 6.4

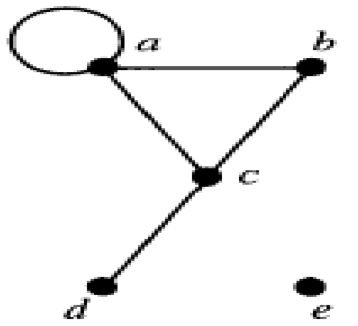



Figure 6.5

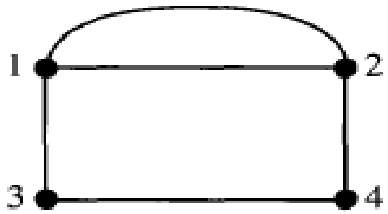



Figure 6.2

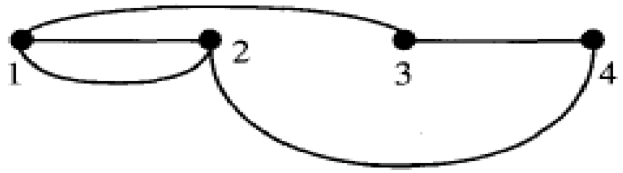



Figure 6.3

(a) One path in the graph represented by Figure 6.2 is π1: 1, 3, 4, 2.

(b) Paths in the graph of Figure 6.4 π2: D, E, B, C, π3: A, B, E, D, D, and

π4: 1,2,1 in fig 6.3. Note that in π4 we do not specify which edge between A and B is used first.

(c) Examples of paths in the graph of Figure 6.5 are π5: a, b, c, a and π6: d, c, a,a. The path π5 is a circuit.

(d) In Figure 6.6 the sequence 1, 2, 3, 2 is not a path, since the single edge between 2 and 3 would be traveled twice.

(e) The path π7: c, a, b, c, d in Figure 6.5 is not simple.

# Types of Graphs

- **Discrete Graph**:
  - For each integer $n \geq 1$, we let $U_n$ denote the graph with *n* vertices and no edges.

U1 U2 U3

- **Linear Graph:**

- For each integer $n \geq 1$, we let $L_n$ denote the graph with *n* vertices $\{v_1, v_2, \ldots\ldots\ldots, v_n\}$ and with edges $\{v_i, v_{i+1}\}$ for $1 \leq i < n$. We call $L_n$ the linear graph.

$L_2$ $L_3$

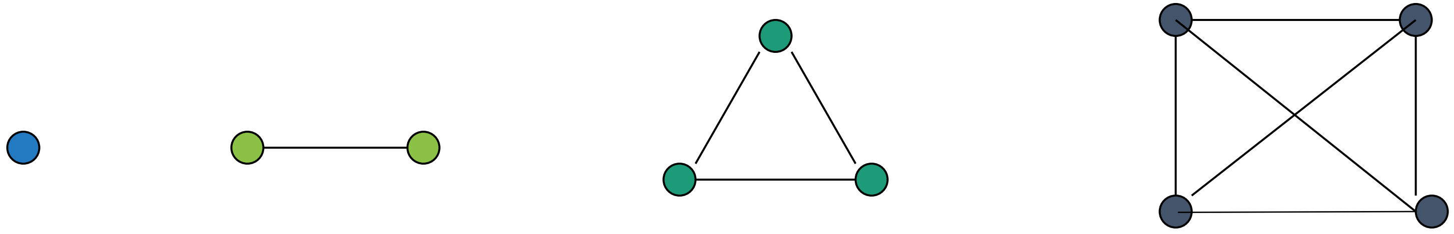
# Types of Graphs

- **Connected Graph**:
  - A graph is called as **connected** of there is a path from any vertex to any other vertex in the graph. Otherwise, the graph is **disconnected**.
  - If a graph is **disconnected**, the various connected pieces are called the **components** of the graph.
  - Identify the types of Graphs

# Complete Graph

- **Complete graph:** $K_n$, where every vertex is connected to every other vertex.
- **If $K_n$ is complete graph for n vertices then number of edges are**

  **$n(n-1)/2$**

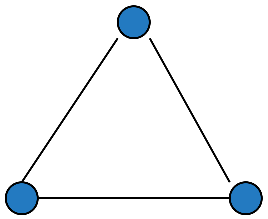


$K_1$ $K_2$ $K_3$ $K_4$

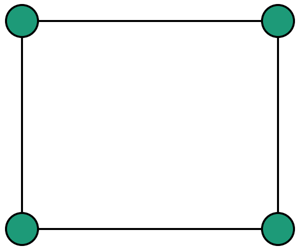
# Simple Graph

- Cycle: $C_n$, $n \geq 3$ consists of n vertices $v_1, v_2, v_3 \ldots v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\} \ldots \{v_{n-1}, v_n\}$, $\{v_n, v_1\}$

  Representation Example: $C_3$, $C_4$



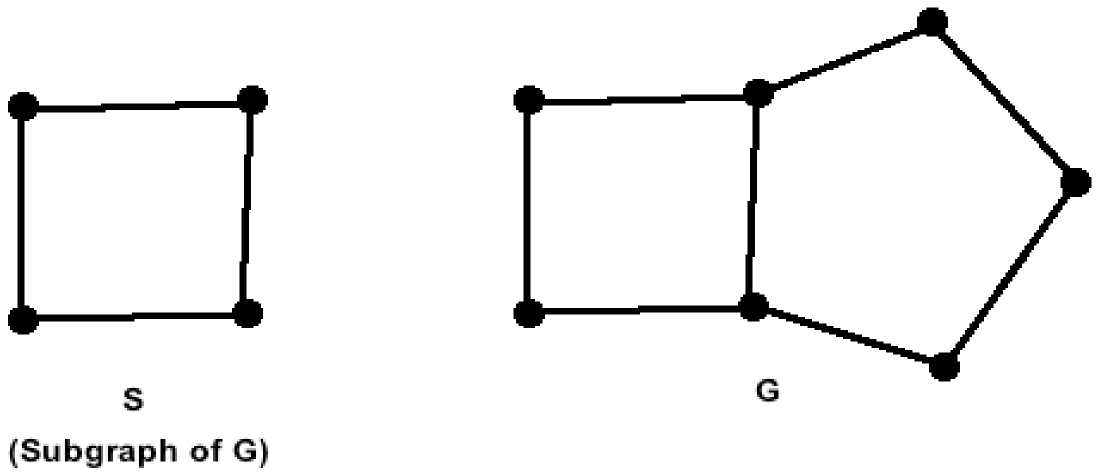
C3



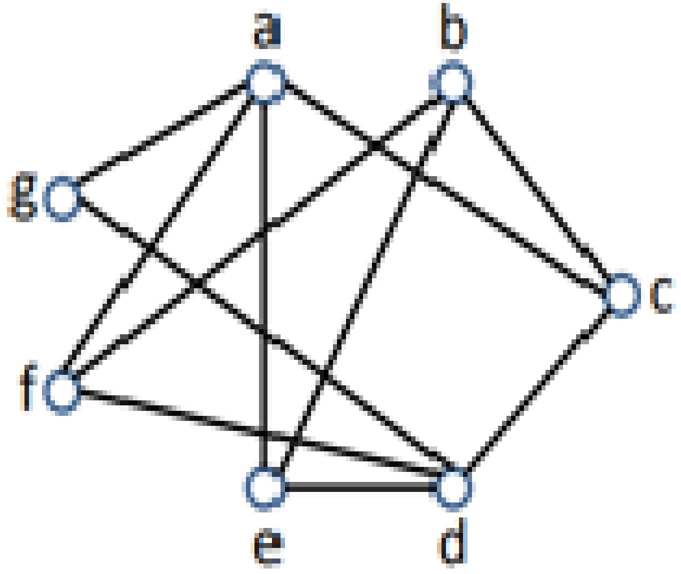
C4

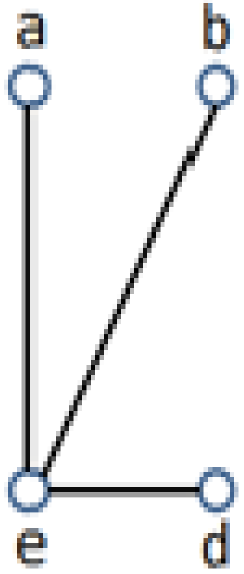
# Subgraph

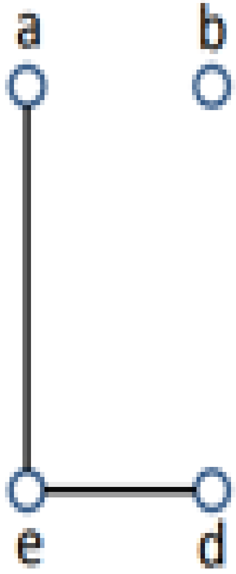
- **Definition:**
- A **Subgraph** $S$ of a graph $G$ is a graph whose vertex set $V(S)$ is a subset of the vertex set $V(G)$, that is $V(S) \subseteq V(G)$, and whose edge set $E(S)$ is a subset of the edge set $E(G)$, that is $E(S) \subseteq E(G)$.

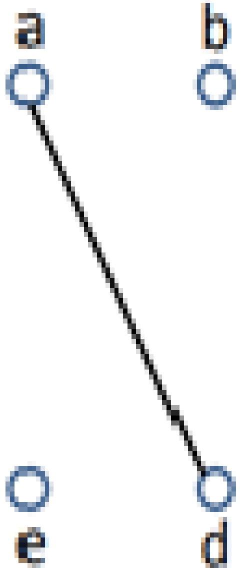


# Which one is a subgraph of the leftmost graph G?

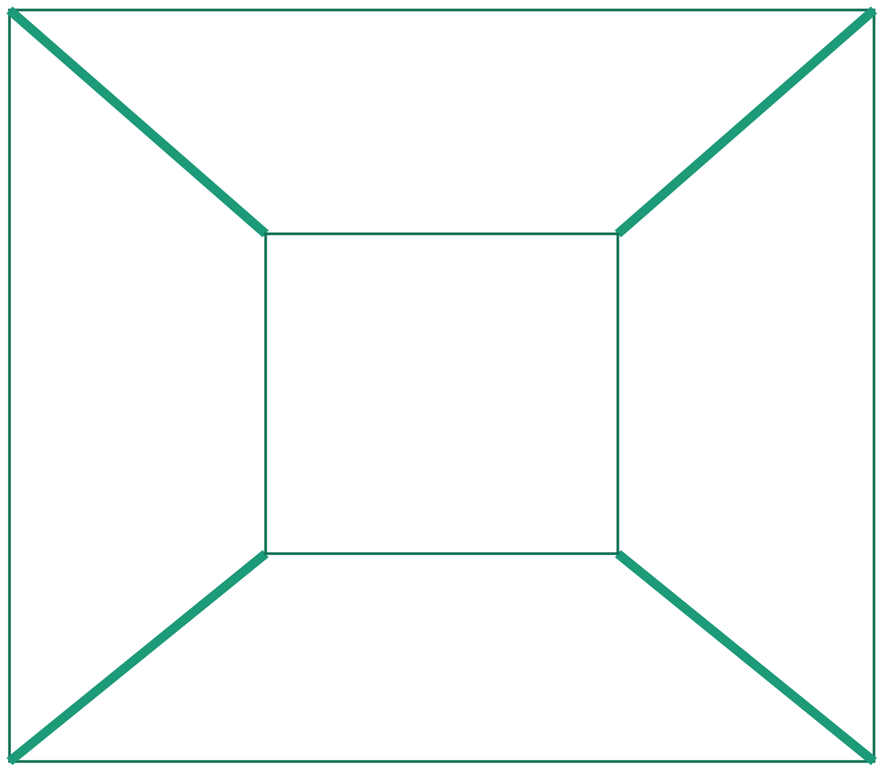


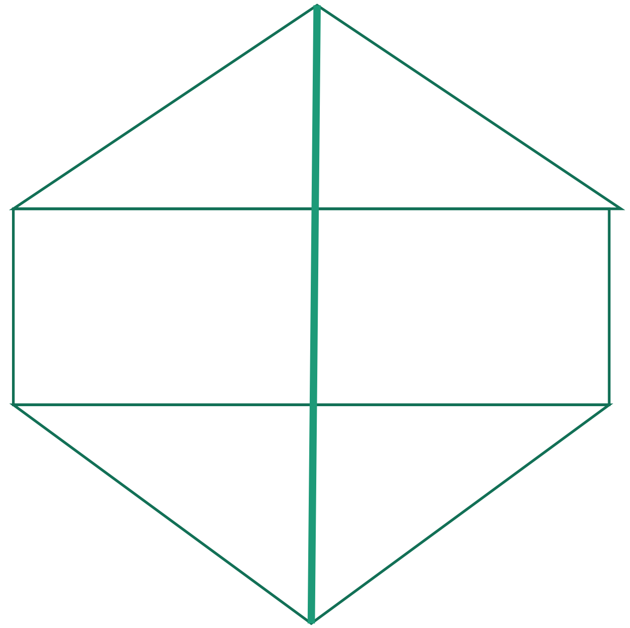



# Find Sub graphs of G

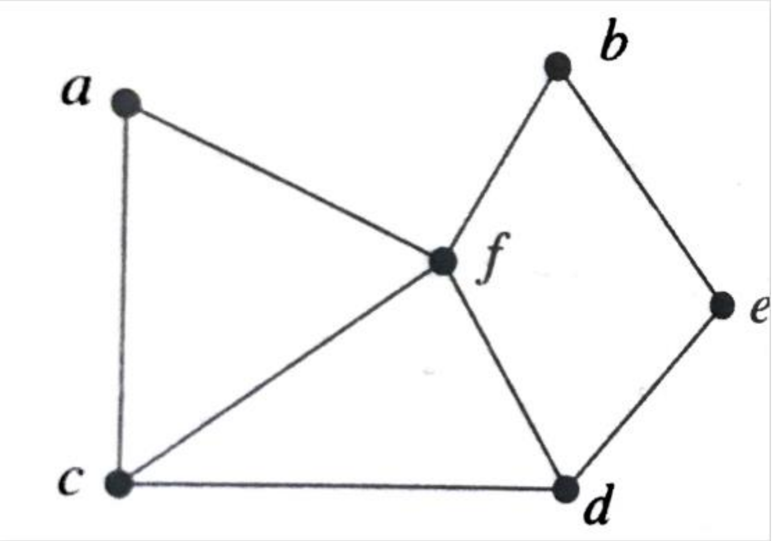


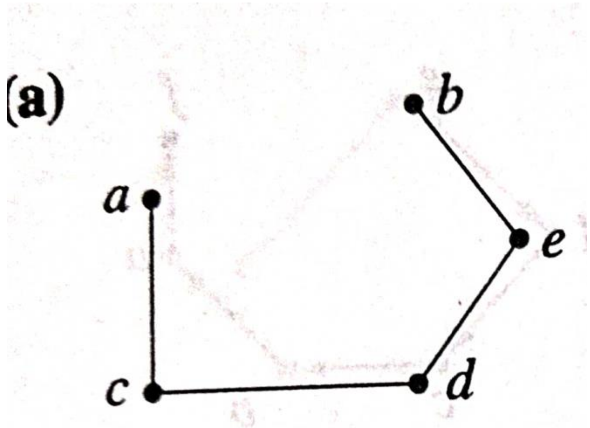
- Identify subgraphs of G.

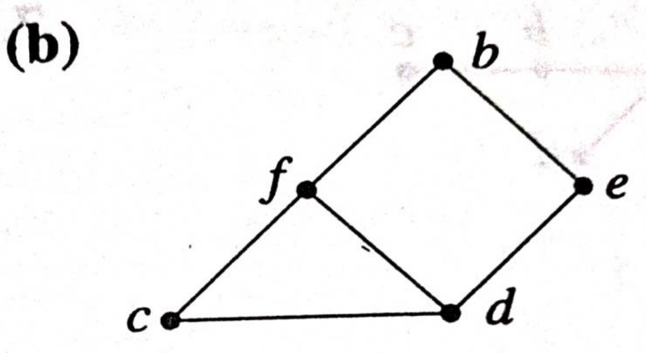


a) Give the largest subgraph of G that does not contain f.

b) Give the largest subgraph of G that does not contain a.



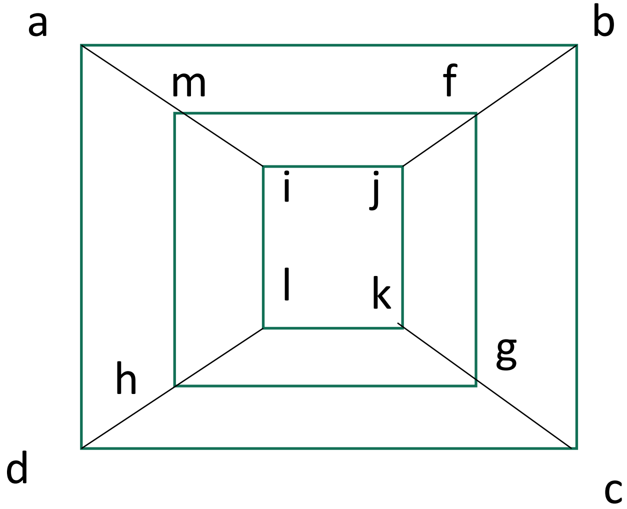



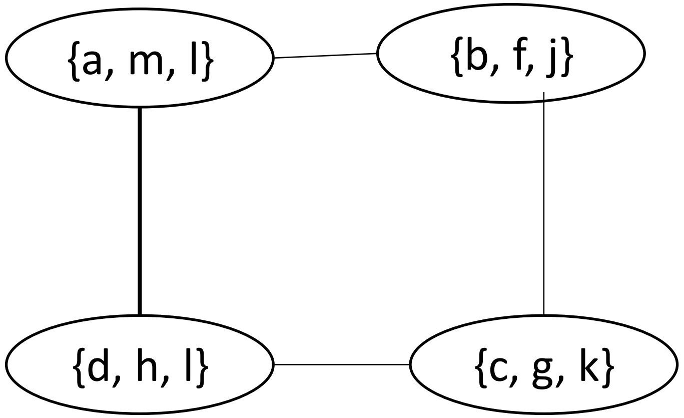
# Quotient Graph

- Construction is Defined for graphs without multiple edges between the same vertices.
- Suppose, G=(V,E,$\gamma$) is graph without multiple edges between the same vertices and that R is an equivalence relation on the set V. Then we construct **Quotient graph** $G^R$ in the following way.
- The Vertices of $G^R$ are the equivalence classes of V produced by R. if [v] and [w] are the equivalence classes of vertices *v* and *w* of G, then there is an edge in $G^R$ from [v] to [w] iff **some vertex in [v] is connected to some vertex in [w]** In the graph G.
- We get $G^R$ by merging all the vertices in each equivalence class into a single vertex and combining any edge that are superimposed by such a process.

- R is a equivalence relation on V defined by partition

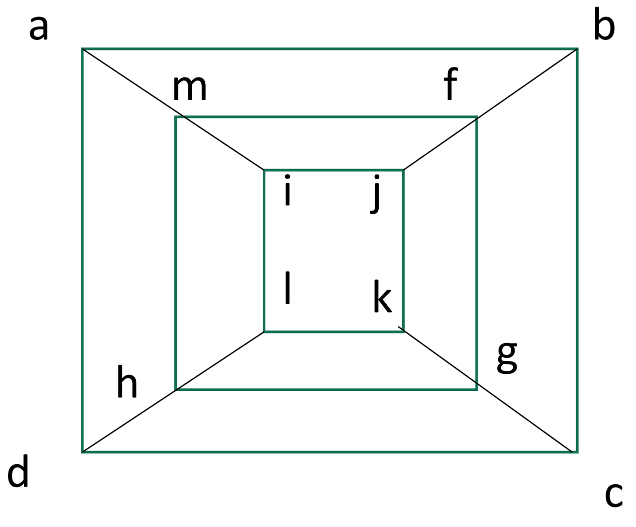
{{ a, m, i }, { b, f, j }, { c, g, k }, { d, h, l } }

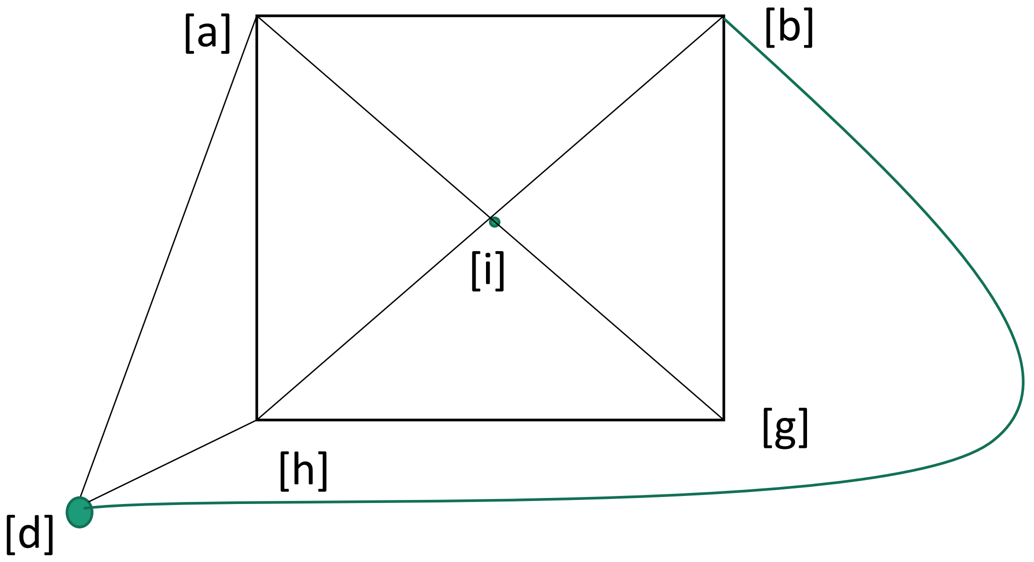


- R is a equivalence relation on V defined by partition

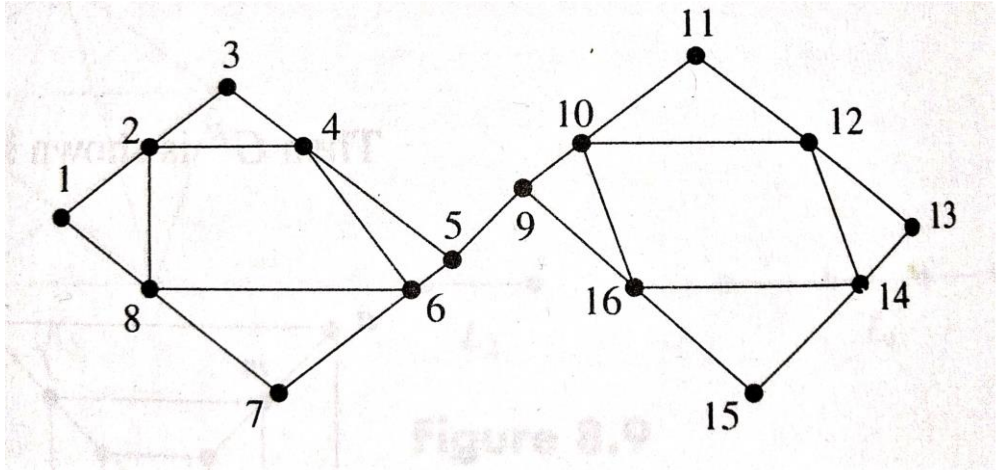
{{ i, j, k, l }, { a, m }, { f, b, c}, { d}, { g } , {h } }

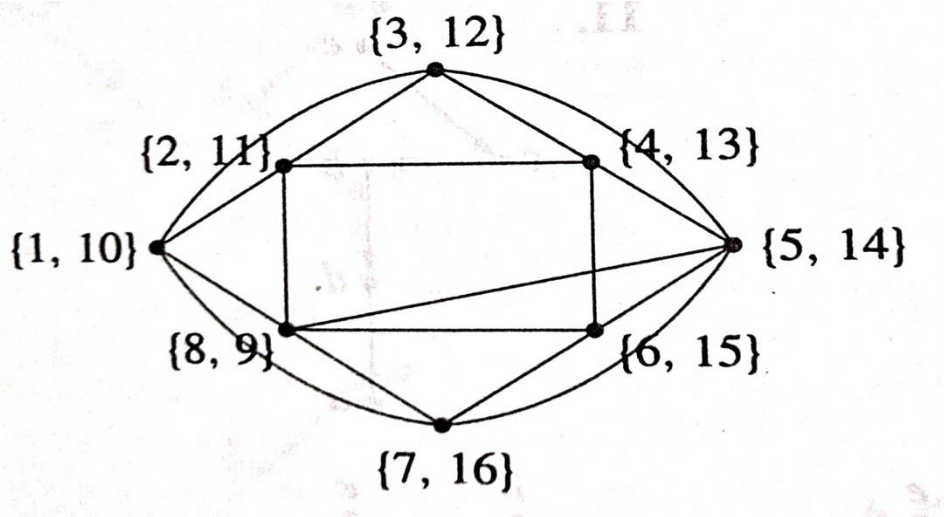


# Problems to solve

- Use graph G, Let R = {(1,1),(2,2),(3,3),(4,4,),(5,5), (6,6), (7,7), (8,8), (9,9), (10,10), (11,11), (12,12), (13,13), (14,14), (15,15), (16,16), (1,10), (10,1), (3,12), (12,3), (5,14), (14,5), (2,11), (11,2), (4,13), (13,4), (6,15), (15,6), (7,16), (16,7), (8,9), (9,8)}. Draw the quotient graph $G^R$



# SPANNING SUBGRAPH

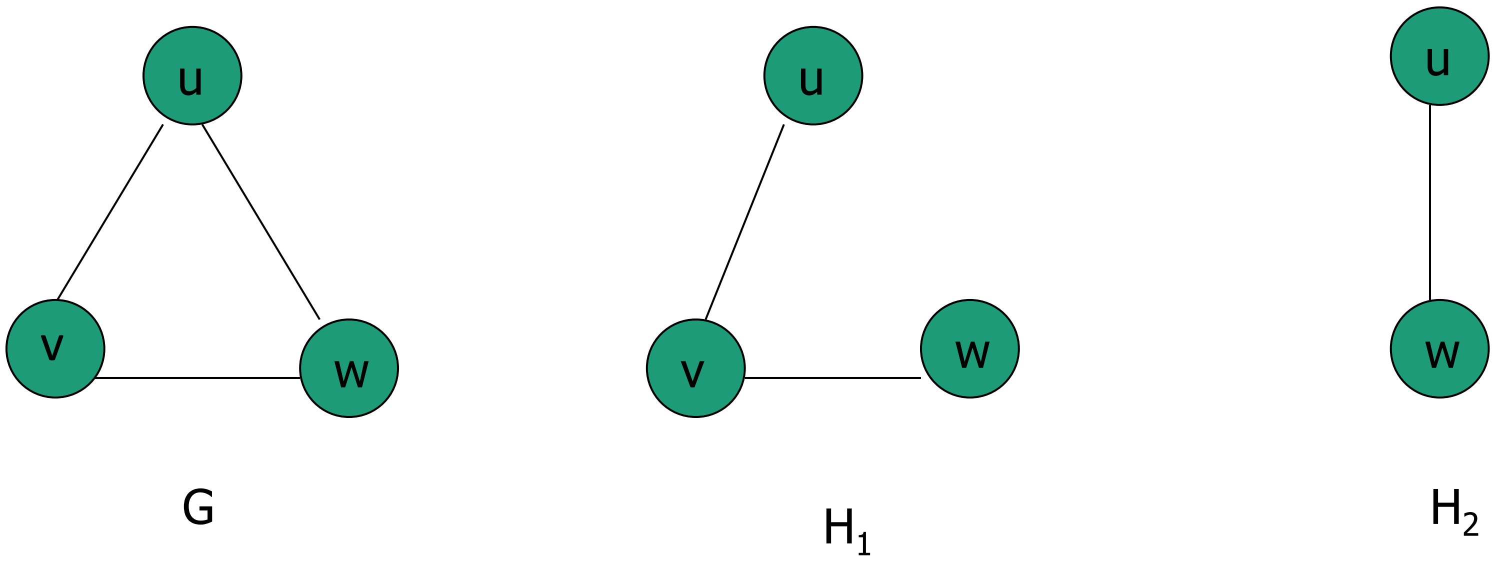
- A sub graph that contains all the vertices of G is called a **SPANNING SUBGRAPH**

## Subgraphs and Complements

If $G = (V, E)$ is a graph, then the complement of G, denoted by $\overline{G}$, is a graph with the same vertex set, such that

an edge e exists in $\overline{G}$ $\Leftrightarrow$ e does not exist in G

- H1 and H2 are complement of the graph



# Handshaking Lemma

Consider a Graph G with e number of edges and n number of vertices , the sum of the degrees of all vertices in G is twice the number of edges in G

$$\sum_{i=1}^{n} d(v_i) = 2e$$

# Problems

- Determine the number of edges in a graph with 6 nodes in which 2 of degree 4 and 4 of degree 2. Draw two such graphs
- Is it possible to construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining nodes have degree 4
- Is it possible to draw a simple graph with 4 vertices and 7 edges . Justify ?

- **Path** : A path is a sequence of vertices where <u>no edge is chosen more than once</u>
  - A path is called <u>*simple*</u> if no vertex repeats more than once
- **Length of Path** : Number of edges in a path is called as length of path
- **Circuit**: A circuit is a path that begins and ends with the same vertex

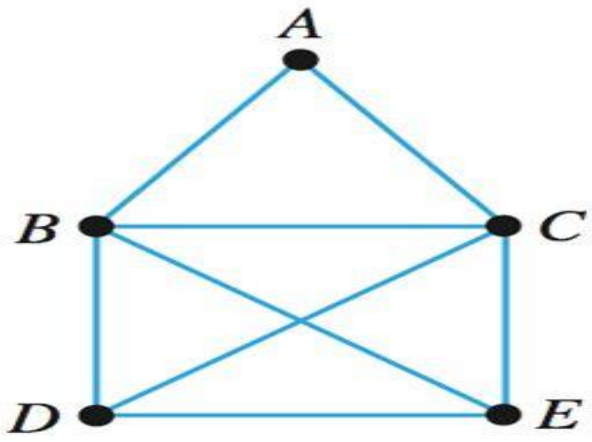
# EULER PATH AND EULER CIRCUIT

- EULER PATH
  - A path in a graph G is called an Euler path if it includes every edge exactly once
- EULER CIRCUIT
  - A Euler path that is a circuit

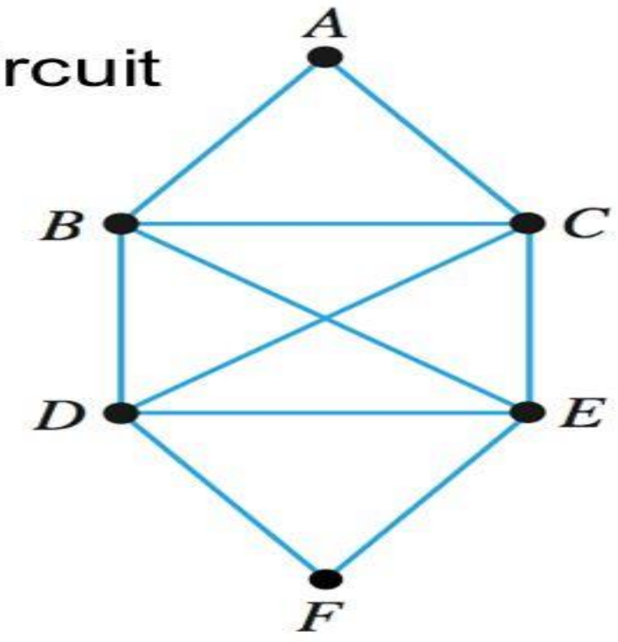
# Examples

- Euler path

D, E, B, C, A, B, D, C, E



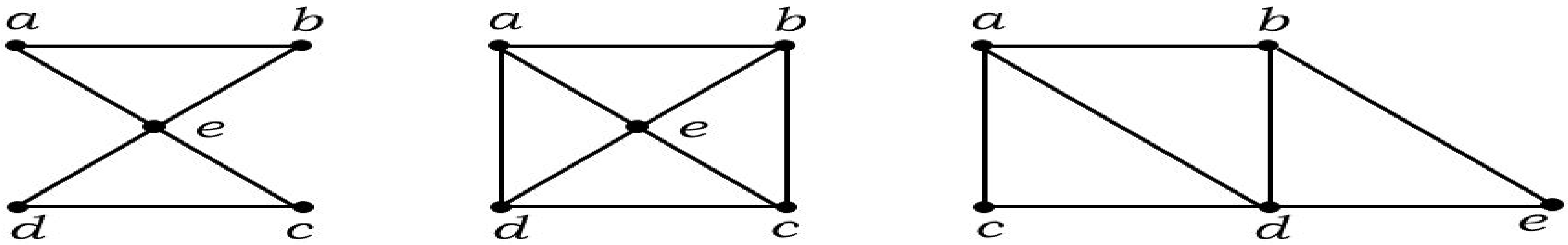

- Euler circuit



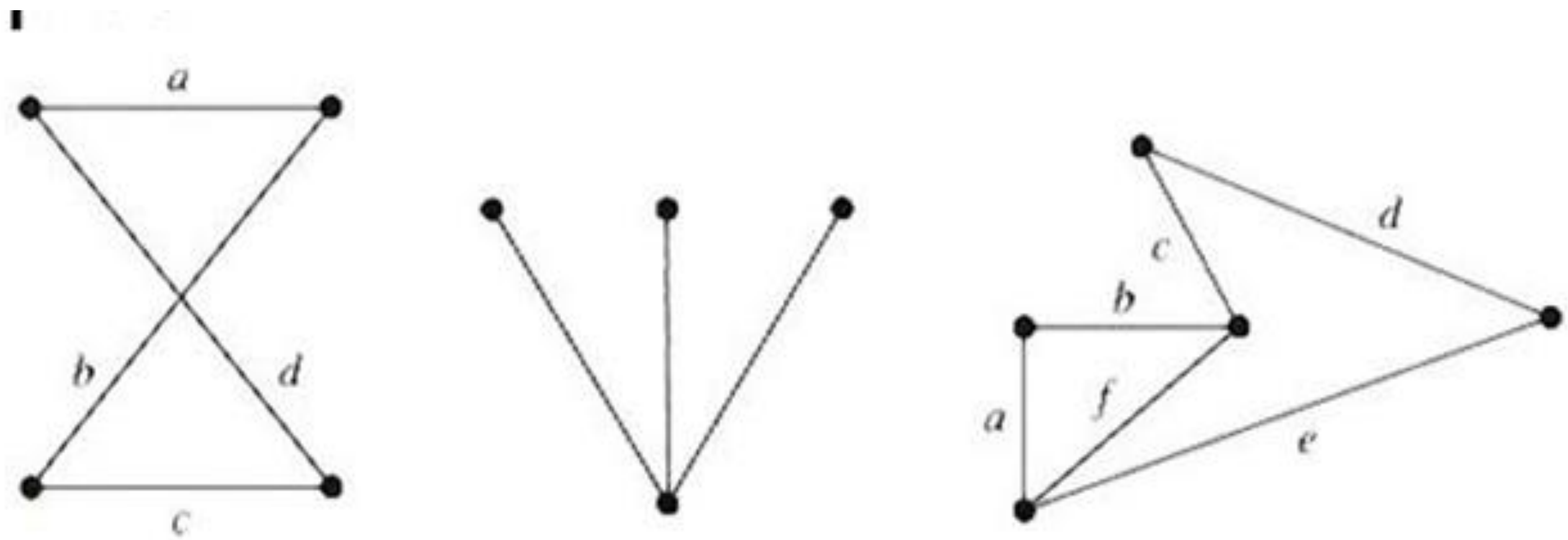

D, E, B, C, A, B, D, C, E, F, D

# Example

- Which of the following graphs has an Euler *circuit*?




(a, e, c, d, e, b, a)




(a) (b) (c)

- The path a, b, c, d in (a) is an Euler circuit since all edges are included exactly once.
- The graph (b) has neither an Euler path nor circuit.
- The graph (c) has an Euler path a, b, c, d, e, f but not an Euler circuit.

# EULER CIRCUIT and EULER PATH

**Theorem: EULER CIRCUIT**

A) If graph G has a vertex of odd degree , then there can be no Euler circuit in G

B) If G is a connected graph and every vertex has an even degree then there is a Euler circuit in G

**Theorem: EULER PATH**

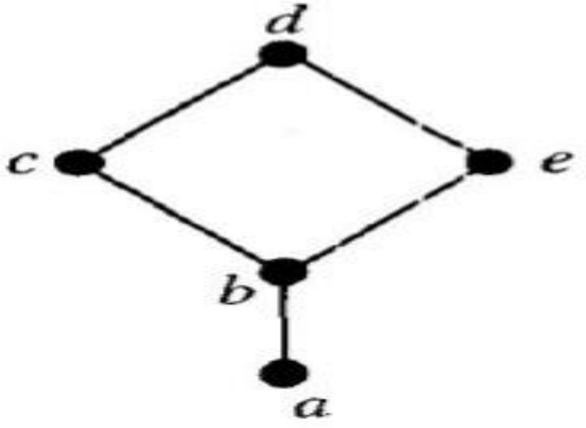
A) If a graph G has more than two vertices of odd degree then there can be no Euler path in G

B) If G is connected and has exactly two vertices of odd degree then there is a Euler path in G

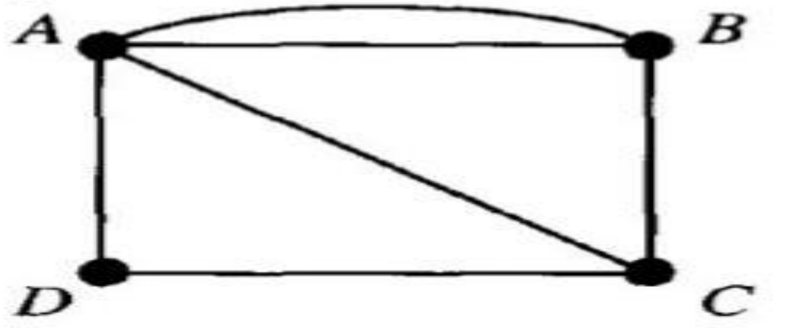
# HAMILTONIAN PATH AND CIRCUIT

- A Hamiltonian path contains each **vertex exactly once.**
- A Hamiltonian circuit is a circuit that contains each vertex exactly once except for the first vertex which is also the last

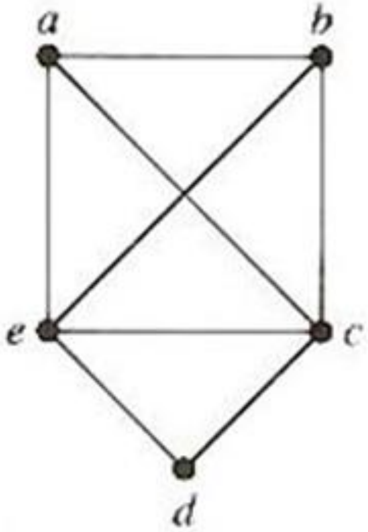
# Examples



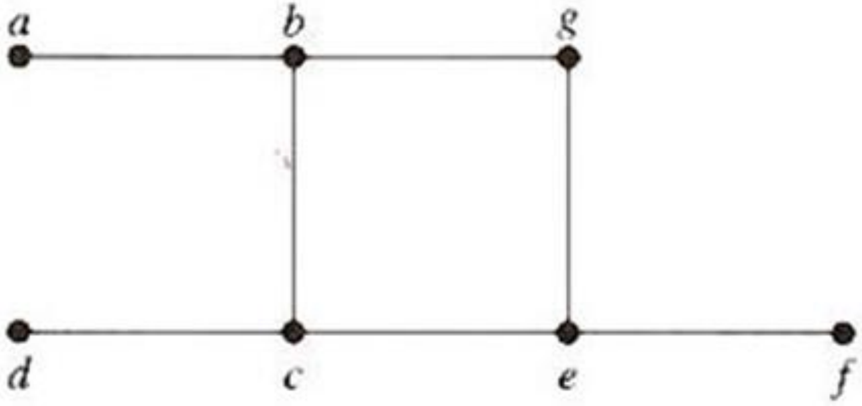

**Hamiltonian path: a, b, c, d, e**



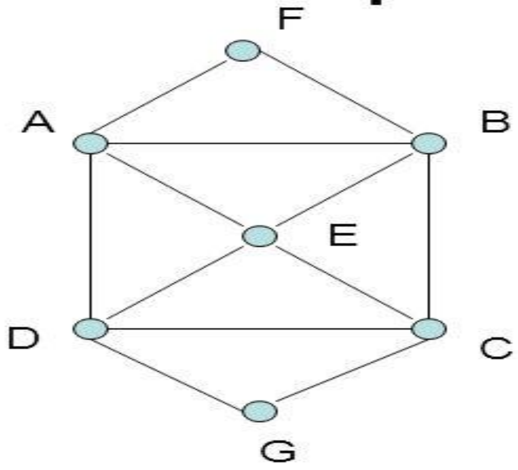

**Hamiltonian circuit: A, D, C, B, A**


a
b
e
c
d


a
b
d
c


a
b
g
d
c
e
f

# Examples of Hamilton circuits




Graph 3

Has many **Hamilton circuits**:
1) A, F, B, E, C, G, D, A
2) A, F, B, C, G, D, E, A

Has many **Hamilton paths**:
1) A, F, B, E, C, G, D
2) A, F, B, C, G, D, E

Has **Euler circuit** => Every vertex has even degree

**Theorem: HAMILTONIAN CIRCUIT**

A) G has a Hamiltonian circuit if for any two vertices u and v of G that are not adjacent ,degree(u)+degree(v) ≥ number of vertices

B) G has a Hamiltonian circuit if each vertex has degree greater than or equal to n/2

# Graph Isomorphism

Graphs $G = (V, E)$ and $H = (U, F)$ are isomorphic if we can set up a bijection $f : V \rightarrow U$ such that

$$x \text{ and } y \text{ are adjacent in } G$$

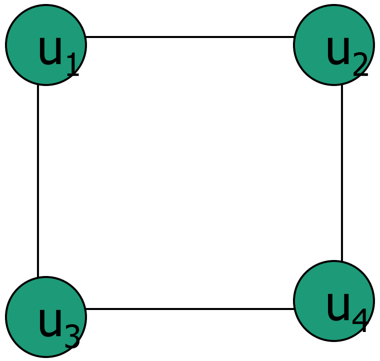
$$\Leftrightarrow f(x) \text{ and } f(y) \text{ are adjacent in } H$$

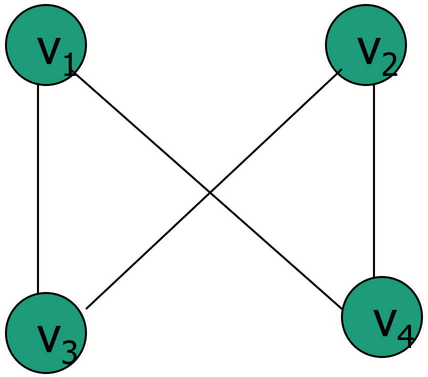
- Function f is called isomorphism
  - Same nos of vertices
  - Same nos of edges
  - Equal nos of vertices with a given degree
  - Adjacency of vertices

# Graph - Isomorphism

Representation example: G1 = (V1, E1) , G2 = (V2, E2)

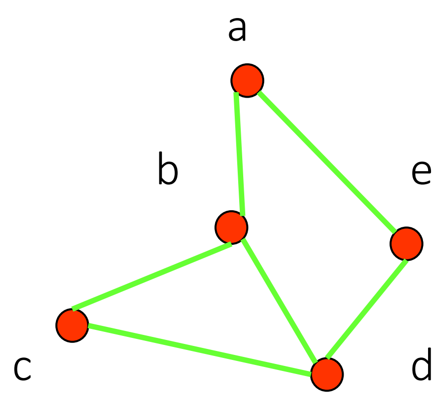
$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$,

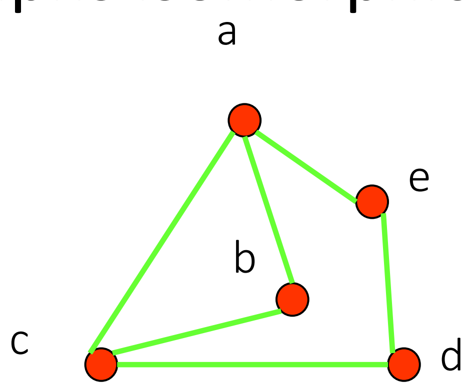


# Isomorphism of Graphs

- **Example I:** Are the following two graphs isomorphic?

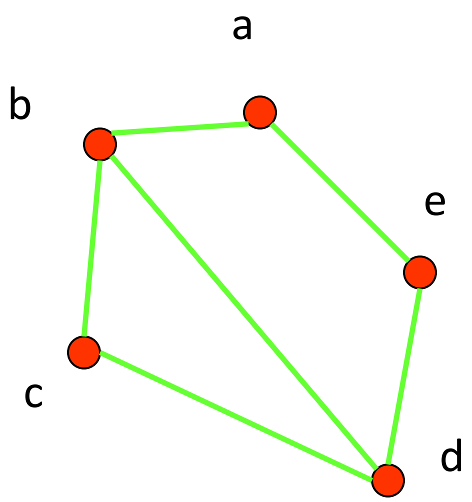


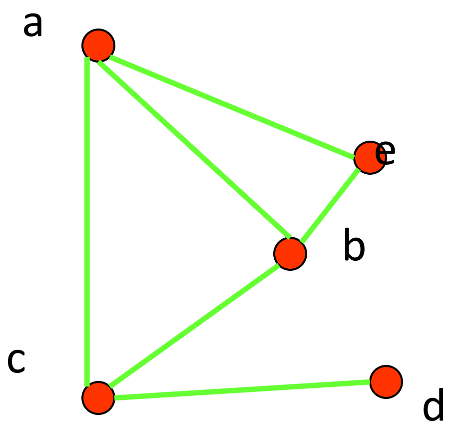




**Solution:** Yes, they are isomorphic, because they can be arranged to look identical. You can see this if in the right graph you move vertex b to the left of the edge {a, c}. Then the isomorphism f from the left to the right graph is: f(a) = e, f(b) = a, f(c) = b, f(d) = c, f(e) = d.

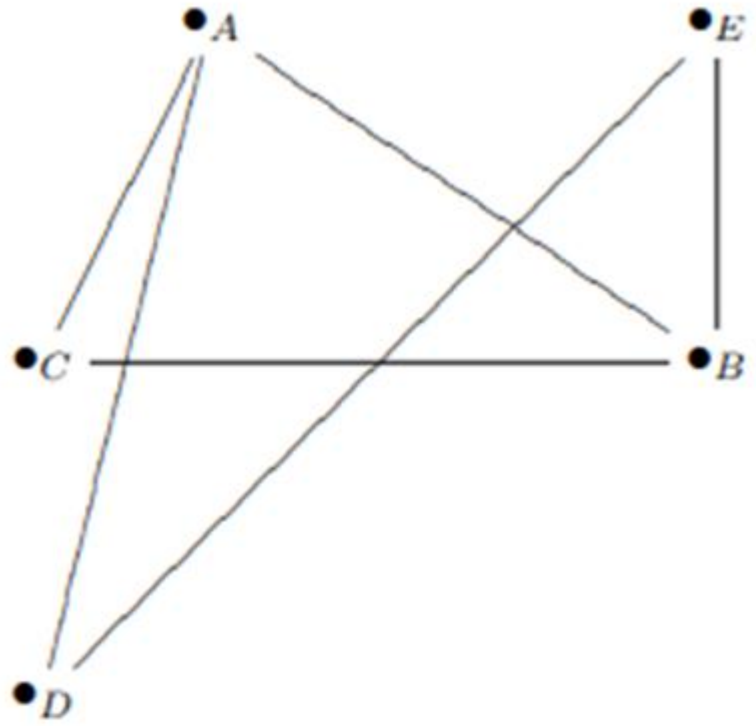
# Isomorphism of Graphs

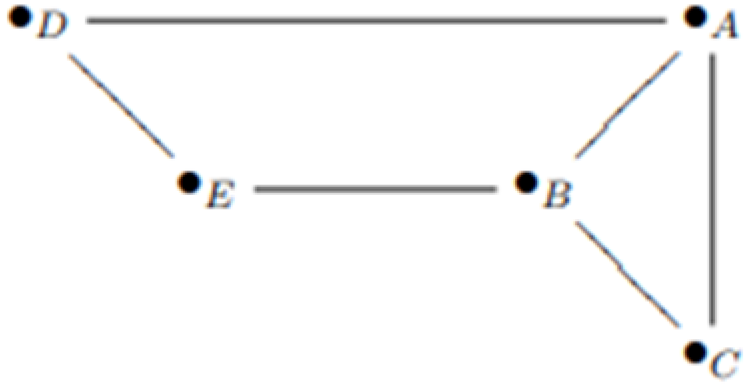
- **Example II:** How about these two graphs?



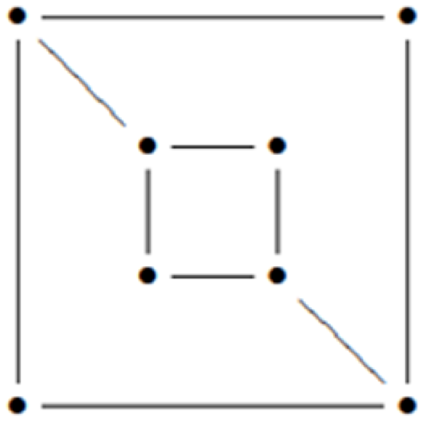

**Solution:** No, they are not isomorphic, because they differ in the degrees of their vertices. Vertex d in right graph is of degree one, but there is no such vertex in the left graph.



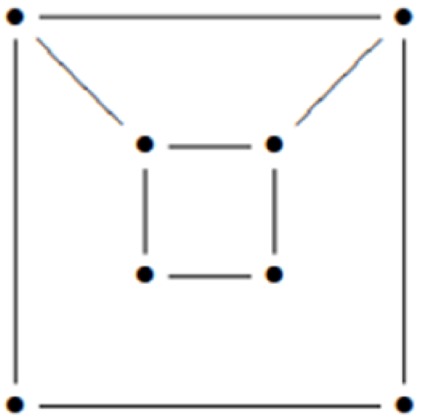




- A is adjacent to: B, C, D
- B is adjacent to: A, C, E
- C is adjacent to: A, B
- D is adjacent to: A, E
- E is adjacent to: B, D

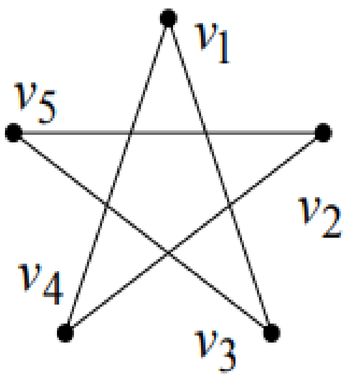
$G$ :



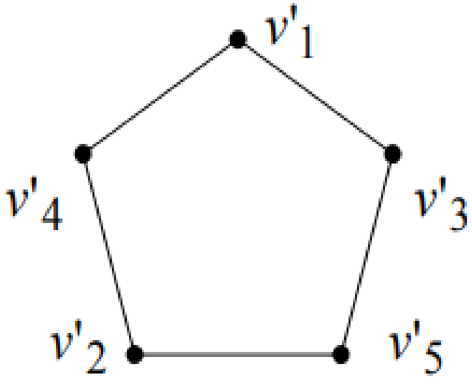
$H$ :



- Both graphs contain
- 8 vertices and 10 edges
- Nos of vertices of degree 2 = 4
- Nos of vertices of degree 3 = 4
- Adjacency : There exists no vertex of degree 3 whose adjacent vertices have same degree in both graphs
- So its not ISOMORPHIC




$G$




$G'$

$G = \{V, E\}$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and

$$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)\}$$
$$= \{e_1, e_2, e_3, e_4, e_5\}$$

$G' = \{V', E'\}$ where $V' = \{v_1', v_2', v_3', v_4', v_5'\}$ and

$$E' = \{(v_1', v_2'), (v_2', v_3'), (v_3', v_4'), (v_4', v_5'), (v_5', v_1')\}$$
$$= \{e_1', e_2', e_3', e_4', e_5'\}$$

Construct 2 functions: $f : V \to V'$ and $g : E \to E'$

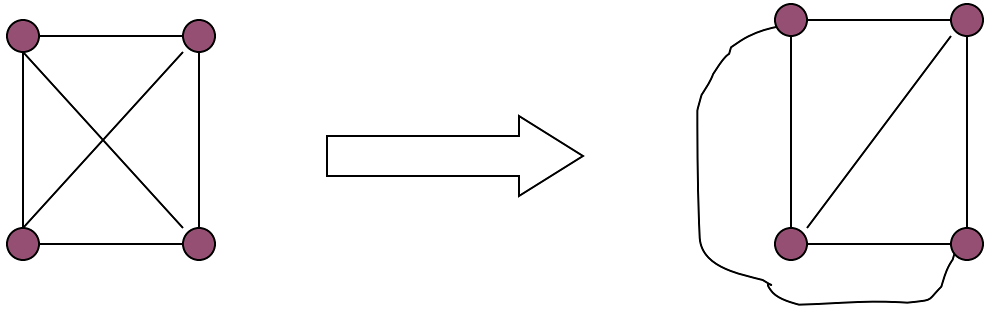
| $f : V \to V'$ | | $g : E \to E'$ | |
|---|---|---|---|
| $V$ | $V'$ | $E$ | $E'$ |
| $v_1$ | $v_1'$ | $e_1$ | $e_1'$ |
| $v_2$ | $v_2'$ | $e_2$ | $e_2'$ |
| $v_3$ | $v_3'$ | $e_3$ | $e_3'$ |
| $v_4$ | $v_4'$ | $e_4$ | $e_4'$ |
| $v_5$ | $v_5'$ | $e_5$ | $e_5'$ |

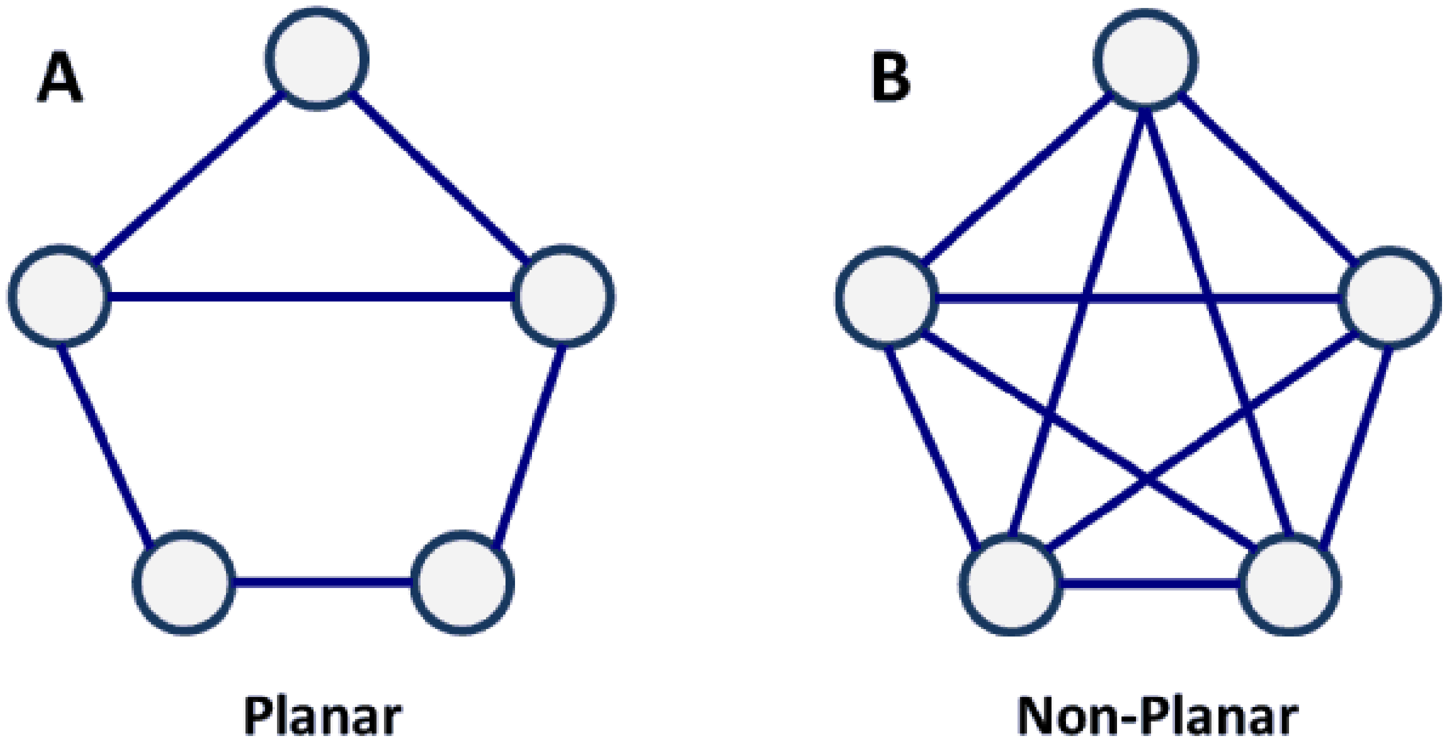
# Planar Graphs

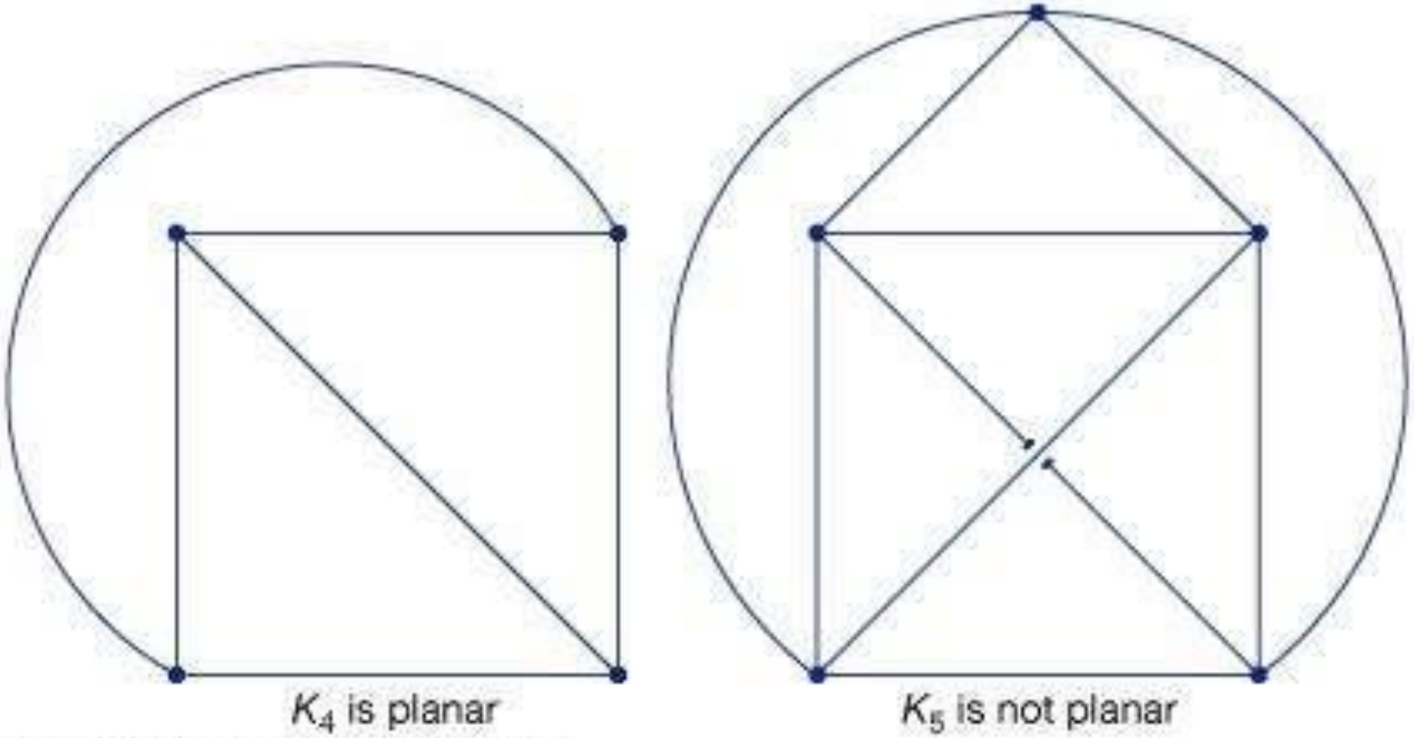
- A graph (or multigraph) $G$ is called *planar* if $G$ can be drawn in the plane with its edges intersecting only at vertices of $G$, such a drawing of $G$ is called an *embedding* of $G$ in the plane.

  Application Example: VLSI design (overlapping edges requires extra layers), Circuit design (cannot overlap wires on board)

  Representation examples: $K1$,$K2$,$K3$,$K4$ are planar, $Kn$ for $n$>4 are non-planar

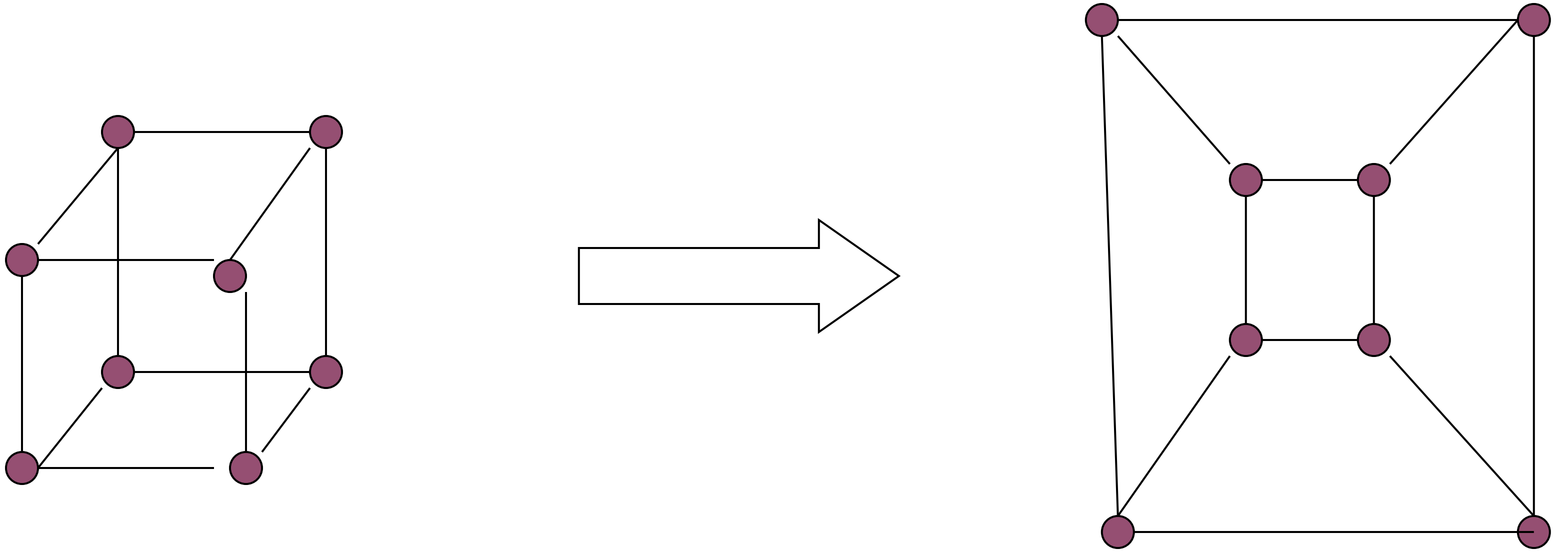



A
B
Planar
Non-Planar


$K_4$ is planar
$K_5$ is not planar

# Planer Graph

- Representation examples: $Q_3$

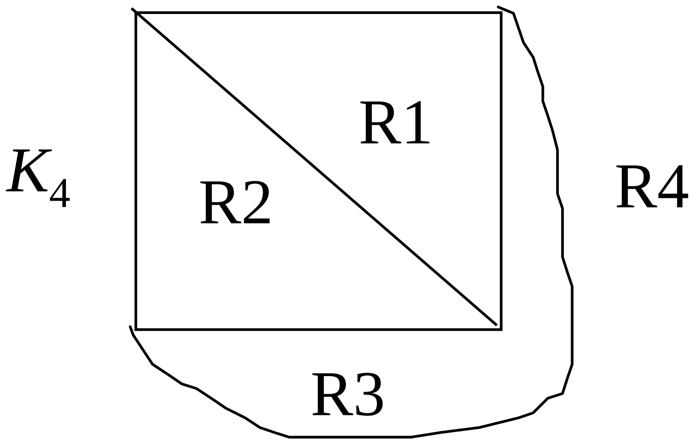


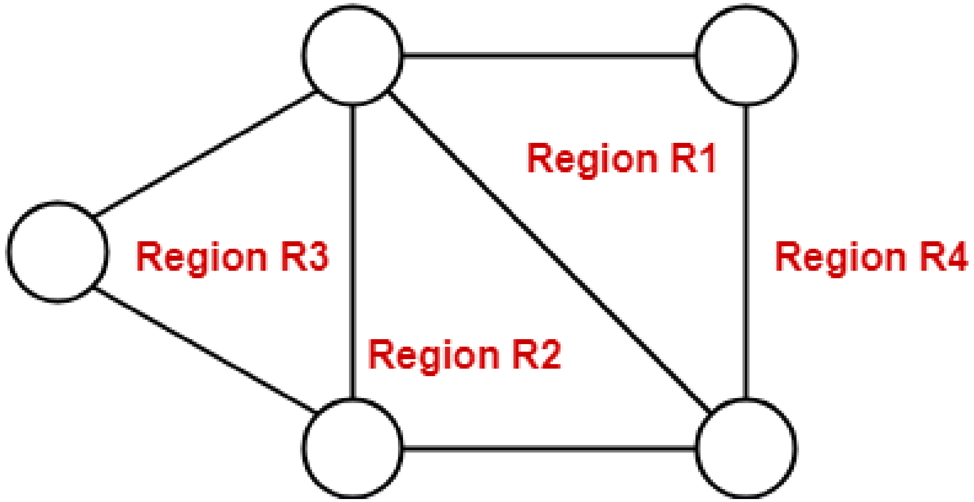
# **Theorem :** *Euler's planar graph theorem*

- For a **connected** planar graph or multigraph

$$\underbrace{v}_{\text{number of vertices}} - \underbrace{e}_{\text{number of Edges}} + \underbrace{r}_{\text{number of Regions}} = 2$$

- A planar graph divides the plane into several regions (faces), one of them is the infinite region.




$v$=4,$e$=6,$r$=4, $v$-$e$+$r$=2

# References

"Discrete Mathematics and its Applications" Kenneth Rosen, 5th Edition, McGraw Hill.