Fourier Integral and Transform

Fourier Integral Theorem

Consider a function $f(x)$ which satisfies Dirichlet's conditions in any interval (-c,c) , then Fourier Integral Theorem states that

$$f(x) = \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\ \cos u(t-x)\, dudt$$

Fourier Sine and Cosine Integral

We know that

$$f(x) = \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\ \cos u(t-x)\, dudt$$

$$= \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)(\cos ut\ \cos ux + \ \sin ut\ \sin ux)\, dudt$$

$$= \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{\infty} f(t) \cos ut\ \cos ux\, dudt + \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{\infty} f(t) \sin ut\ \sin ux\, dudt$$

Formula for odd even function

For odd function $\int_{-a}^{a} f(x)dx = 0$

For even function $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If $f(x)$ is odd,

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty}\int_{t=0}^{\infty} f(t)(\sin ut \sin ux))dudt = \frac{2}{\pi}\int_{u=0}^{\infty} \sin ux \int_{t=0}^{\infty} f(t) \sin ut\ dudt$$

If $f(x)$ is even,

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty}\int_{t=0}^{\infty} f(t)(\cos ut\ \cos ux)\, dudt = \frac{2}{\pi}\int_{u=0}^{\infty} \cos ux \int_{t=0}^{\infty} f(t) \cos ut\, dudt$$

List of Formulae for Fourier Integrals

1. Fourier Integral for $f(x)$

$$f(x) = \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\ \cos u(t-x)\, dudt$$

2. Fourier Sine Integral for $f(x)$ OR if $f(x)$ is odd function of (x)

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty} \sin ux \int_{t=0}^{\infty} f(t) \sin ut\ \ dudt$$

3. Fourier Cosine Integral for $f(x)$ OR if $f(x)$ is even function of (x)

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty} \cos ux \int_{t=0}^{\infty} f(t) \cos ut\, dudt$$

Ex1-Express the function $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ as a Fourier Integral and hence evaluate $\int_{\lambda=0}^{\infty} \left[\frac{\sin\lambda\cos\lambda x}{\lambda}\right] d\lambda$

Solution We know that Fourier Integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos u(t-x)\,du\,dt \quad ; f(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$

$$= \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-1}^{1} \cos u(t-x)\,du\,dt = \frac{1}{\pi}\int_{u=0}^{\infty}\left[\frac{\sin u(t-x)}{u}\right]_{-1}^{1} du$$

$$= \frac{1}{\pi}\int_{u=0}^{\infty}\left[\frac{\sin u(1-x)-\sin u(-1-x)}{u}\right] du = \frac{1}{\pi}\int_{u=0}^{\infty}\left[\frac{\sin u(1-x)+\sin u(1+x)}{u}\right] du$$

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty}\left[\frac{\sin u\cos ux}{u}\right] du = \frac{2}{\pi}\int_{\lambda=0}^{\infty}\left[\frac{\sin\lambda\cos\lambda x}{\lambda}\right] d\lambda \qquad \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\int_{\lambda=0}^{\infty}\left[\frac{\sin\lambda\cos\lambda x}{\lambda}\right] d\lambda = \frac{\pi}{2} f(x) = f(x) = \begin{cases} \pi/2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

For $|x|$=1 (point of discontinuity), $f(x) = \frac{\pi/2+0}{2} = \frac{\pi}{4}$. hence $f(x) = \begin{cases} \pi/2, & |x| < 1 \\ 0, & |x| > 1 \\ \pi/4, & |x| = 1 \end{cases}$

OR

In the given function, replace x by $-x$, we get $f(-x) = \begin{cases} 1 \quad , & |x| < 1 \\ 0, & |x| > 1 \end{cases} = f(x)$

So the function is even function and we can apply . Fourier Cosine Integral formula instead of Fourier Integral formula.

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty} \cos ux \int_{t=0}^{\infty} f(t) \cos ut \, dudt \;= \frac{2}{\pi}\int_{u=0}^{\infty} \cos ux \int_{t=0}^{1} \cos ut \, dudt$$

$$= \frac{2}{\pi}\int_{u=0}^{\infty} \cos ux \left[\frac{\sin u(t)}{u}\right]_0^1 du = \frac{2}{\pi}\int_{u=0}^{\infty} \cos ux \frac{\sin u}{u} du = \frac{2}{\pi}\int_{\lambda=0}^{\infty} \left[\frac{\sin \lambda \cos \lambda x}{\lambda}\right] d\lambda$$

$$\int_{\lambda=0}^{\infty} \left[\frac{\sin \lambda \cos \lambda x}{\lambda}\right] d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

For $|x|$=1 (point of discontinuity), $f(x) = \frac{\pi/2 + 0}{2} = \frac{\pi}{4}$. hence $f(x) = \begin{cases} \pi/2, & |x| < 1 \\ 0, & |x| > 1 \\ \pi/4, & |x| = 1 \end{cases}$

Ex 2- Using Fourier Cosine Integral representation of an appropriate function , show that $\int_0^{\infty} \frac{\cos wx}{k^2+w^2} dw = \frac{\pi e^{-kx}}{2k}$

Solution We know that . Fourier Cosine Integral for $f(x)$

$$f(x) = \frac{2}{\pi} \int_{u=0}^{\infty} \cos ux \int_{t=0}^{\infty} f(t) \cos ut \, dudt$$

Let $f(x) = e^{-kx}$.

Put value of $f(t)$ and replace u by w.

$$e^{-kx} = \frac{2}{\pi} \int_{u=0}^{\infty} \cos wx \int_{t=0}^{\infty} e^{-kt} \cos wt \, dwdt$$

$$= \frac{2}{\pi} \int_{u=0}^{\infty} \cos wx \left[\frac{e^{-kt}}{k^2+w^2} \{-k \cos wt + w \sin wt\}_0^{\infty} \right] \mathrm{dw}$$

$$= \frac{2}{\pi} \int_{u=0}^{\infty} \cos wx \left[0 + \frac{k}{k^2+w^2} \right] \mathrm{dw} \quad = \frac{2k}{\pi} \int_{u=0}^{\infty} \frac{\cos wx}{k^2+w^2} dw$$

Hence $\int_0^{\infty} \frac{\cos wx}{k^2+w^2} dw = \frac{\pi e^{-kx}}{2k}$

Ex 3-Find Fourier Sine Integral for $f(x) = e^{-\beta x}$. Hence show that $\frac{\pi e^{-\beta x}}{2} = \int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2+\lambda^2} dw$

Solution Fourier Sine Integral for $f(x)$

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty} \sin ux \int_{t=0}^{\infty} f(t) \sin ut \; dudt$$

Put $f(t) = e^{-\beta t}$ and replace u *by* λ

$$f(t) = \frac{2}{\pi}\int_{\lambda=0}^{\infty} \sin \lambda x \int_{t=0}^{\infty} e^{-\beta t} \sin \lambda t \; d\lambda dt$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty} \sin \lambda x \left[\frac{e^{-\beta t}}{\beta^2+\lambda^2}\{-\beta \sin \lambda t - \lambda \cos \lambda t\}_0^\infty\right] \mathrm{d}\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty} \sin \lambda x \left[0 + \frac{\lambda}{\beta^2+\lambda^2}\right] \mathrm{d}\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty} \frac{\lambda \sin \lambda x}{\beta^2+\lambda^2} d\lambda$$

Hence $\frac{\pi e^{-\beta x}}{2} = \int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2+\lambda^2} dw$

Fourier Transform of $f(x)$ or Complex Fourier Transform of $f(x)$

If a function $f(x)$ is defined on $(-\infty, \infty)$, is piecewise continuous in each finite interval and is absolutely integrable in $(-\infty, \infty)$ then Fourier Transform of $f(x)$ is defined as $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$

Formulae: Fourier Transform of $f(x)$ and Invers Fourier Transform of $F(s)$

1. Fourier Transform of $f(x) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$
2. Inverse Fourier Transform of $F(s) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$
3. Fourier Sine Transform of $f(x) = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin st \; dt$
4. Invers Fourier Transform of $F_s(s) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx \, ds$
5. Fourier Cosine Transform of $f(x) = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos st \; dt$
6. Invers Fourier Transform of$F_c(s) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx \, ds$

Note: If $f(x)$ is odd, Fourier Sine formula is used and If $f(x)$ is even, Fourier Cosine is used.

Ex 1- Find the Fourier Transform of $f(x) = \begin{cases} 1+\frac{x}{a} & -a<x<0 \\ 1-\frac{x}{a} & 0<x<a \\ 0 & otherwise \end{cases}$

Solution Fourier Transform of $f(x) = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$

$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{0}(1+\frac{t}{a})e^{ist}dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{a}(1-\frac{t}{a})e^{ist}dt$ Integrating by parts, we get $\left[\int [uv]_1 = uv_1 - u'v_2 + u''u_3 \ldots\right]$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(1+\frac{t}{a}\right)\left(\frac{e^{ist}}{is}\right) - \left(\frac{1}{a}\right)\left(\frac{e^{ist}}{-s^2}\right)\right]_{-a}^{0} + \frac{1}{\sqrt{2\pi}}\left[\left(1-\frac{t}{a}\right)\left(\frac{e^{ist}}{is}\right) - \left(-\frac{1}{a}\right)\left(\frac{e^{ist}}{-s^2}\right)\right]_{0}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{is} + \frac{1}{as^2} - 0 - \frac{e^{-isa}}{as^2}\right] + \frac{1}{\sqrt{2\pi}}\left[0 - \frac{e^{isa}}{as^2} - \frac{1}{is} + \frac{1}{as^2}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} - \frac{1}{as^2}(e^{-isa} + e^{isa})\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} - \frac{1}{as^2}(2\cos as)\right] = \frac{1}{\sqrt{2\pi}}\frac{2}{as^2}(1-\cos as)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2}{as^2}2\sin^2\frac{as}{2} = \frac{2\sqrt{2}}{\sqrt{\pi}as^2}\sin^2\frac{as}{2}$$

Ex2- Find the Fourier Transform of $f(x) = e^{-|x|}$

Solution We know that $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$, $f(t) = e^{-|t|}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} (\cos st + i \sin st)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} \cos st\, dt + i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} \sin st\, dt$$

Since the first integral is even and second is odd, by Formula for odd even function

$$F(s) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-|t|} \cos st\, dt = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} \cos st\, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-t}}{1+s^2} \{-\cos st + s \sin st\} \right]_0^{\infty} = \sqrt{\frac{2}{\pi}} \left[0 + \frac{1}{1+s^2} \right] = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$$

Note- Since the given function is even function ie. $f(x) = f(-x)$, we can also apply formula of Fourier Cosine Transform.

Ex 3- Find the Fourier Sine Transform of $f(x) = \frac{e^{-ax}}{x}$

Solution Fourier Sine Transform of $f(x) = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \; dt$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \sin st \; dt$$

To apply DUIS Rule, differentiate both sides w.r.t.s

$$\frac{d}{dx}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \; t \cos st \, dt = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-at}}{a^2+s^2}\{-a\cos st + t \sin st\}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2}(-a)\right] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right]$$

Integrating w.r.t. s, we get $\quad F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{s}{a} + c, put\ s = 0\ to\ get$ c=0.

Hence $F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{s}{a}$

Ex 4- Find the Fourier Cosine Transform of $f(x) = e^{-2x} + 4e^{-3x}$

Solution Fourier Cosine Transform of $f(x) = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \ dt$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty (e^{-2t} + 4e^{-3t}) \cos st \ dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2t} \cos st \ dt + \sqrt{\frac{2}{\pi}} \int_0^\infty 4e^{-3t} \cos st \ dt \qquad \left(\because \int e^{-ax} \cdot \cos bx \, dx = \frac{e^{-ax}}{a^2 + b^2} [b \sin bx - a \cos bx] \right)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-2t}}{2^2+s^2} \{s \sin st - 2 \cos st\} \right]_0^\infty + \sqrt{\frac{2}{\pi}} \left[\frac{4e^{-3t}}{3^2+s^2} \{s \sin st - 3 \cos st\} \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 + \frac{2}{2^2+s^2} \right] + \sqrt{\frac{2}{\pi}} \left[0 + \frac{12}{3^2+s^2} \right]$$

$$= 2 \sqrt{\frac{2}{\pi}} \left[\frac{1}{4+s^2} + \frac{6}{9+s^2} \right]$$

Ex 5- Find the Fourier Sine Transform of $e^{-x}, x \geq 0$ and hence deduce that $\int_0^\infty \frac{x \sin mx}{1+x^2} = \frac{\pi}{2} e^{-m}, (m > 0)$

Solution Fourier Sine Transform of $f(x) = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \; dt$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \sin st \; dt = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-t}}{1^2+s^2}\{-\sin st\} - s\cos st\}\right]_0^\infty = \sqrt{\frac{2}{\pi}} \left[0 + \frac{s}{1^2+s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{1^2+s^2}$$

For deduction, we use Inverse Fourier Sine Transform .

By definition, Invers Fourier Transform of $F_s(s) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{1^2 + s^2} \sin sx \, ds = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{1^2 + s^2} ds$$

Replace the dummy variable s by m to get $\int_0^\infty \frac{x \sin mx}{1+x^2} = \frac{\pi}{2} e^{-m}$

Ex 6- Find the Fourier Cosine Transform of $f(x) = \begin{cases} 1 \ \ , & 0 < x < k \\ 0, & x \geq k \end{cases}$ and also find $f(x)$ if $F_c(s) = \sqrt{\frac{2}{\pi}} \frac{\sin ks}{s}$

Solution Fourier Cosine Transform of $f(x) = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \ dt$

$= \sqrt{\frac{2}{\pi}} \int_0^k 1. \cos st \ dt = \sqrt{\frac{2}{\pi}} \left[\frac{\sin st}{s}\right]_0^k = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sk}{s}\right]$

For second part, we use Inverse Fourier Cosine Transform .

By definition, Invers Fourier Transform of $F_c(s) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{\sin sk}{s}\right] \cos sx \, ds = \frac{2}{\pi} \int_0^\infty \frac{1}{2s} \{\sin(k+x)s + \sin(k-x)\} \, ds$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\{\sin(k+x)s + \sin(k-x)\}}{s} ds = \frac{1}{\pi} \int_0^\infty \frac{\sin(k+x)s}{s} ds + \frac{1}{\pi} \int_0^\infty \frac{\sin(k-x)}{s} ds$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}\right) + \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1 \ \text{ if } 0 < x < k$$

and $= \frac{1}{\pi}\left(\frac{\pi}{2}\right) + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = 0$ if $x \geq k$ Thus $f(x) = \begin{cases} 1 \ \ , & 0 < x < k \\ 0, & x \geq k \end{cases}$