More Problems on Directional Derivatives

Ex 4: Find the directional derivative of $\phi = \frac{y}{x^2+y^2}$ at (0,1) in the direction making an angle of 30^0 with positive x-axis.

- Solution:

$$\nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y}$$

$$= \left[-\frac{y}{(x^2+y^2)^2}\cdot 2x\right]i + \left[\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}\right]j$$

$$= \frac{-2xy}{(x^2+y^2)^2}i + \frac{(x^2-y^2)}{(x^2+y^2)^2}j$$

$$= 0i - j \text{ at } (0, 1)$$

Unit vector making an angle of 30^o with the x-axis

$$= \cos 30^0 i + \sin 30^0 j = \frac{\sqrt{3}}{2}i + \frac{1}{2}j$$

$\therefore$ Required directional derivative $= (0i - j)\cdot\left(\frac{\sqrt{3}}{2}i + \frac{1}{2}j\right) = -\frac{1}{2}$.

EX 5: Find the angle between the surfaces $x\log z + 1 - y^2 = 0$, $x^2y + z = 2 \ at \ (1,1,1)$

- Solution:

Let $\Phi = x\log z + 1 - y^2$.

$$\therefore \quad \nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z} = \log z i - 2yj + \frac{x}{z}k$$

$$= 0i - 2j + k \text{ at } (1, 1, 1).$$

Unit normal vector at (1, 1, 1) $= \frac{0i - 2j + k}{\sqrt{5}}$

Let $\Psi = x^2y + z - 2$

$$\nabla\Psi = i\frac{\partial \Psi}{\partial x} + j\frac{\partial \Psi}{\partial y} + k\frac{\partial \Psi}{\partial z} = 2xy\, i + x^2 j + k = 2i + j + k$$

Unit normal vector at (1, 1, 1) $= \frac{2i + j + k}{\sqrt{6}}$

$$\cos\theta = \frac{(0 - 2j + k)}{\sqrt{5}}\cdot\frac{(2i + j + k)}{\sqrt{6}} = -\frac{1}{30}.$$

EX 6: Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at (1,-1,2)

- Solution:

Let $u = ax^2 - byz - (a+2)x$ and $v = 4x^2y + z^3 - 4$.

$$\therefore \quad \nabla u = (2ax - a - 2)i + (-bz)j + (-by)k$$

$$= (a-2)i - 2bj + bk \text{ at } (1, -1, 2)$$

The direction ratios of the normal to this surface at (1, –1, 2) are $a - 2, -2b, b$.

And $\nabla v = 8xyi + 4x^2 j + 3z^2 k = -8i + 4j + 12k$ at $(1, -1, 2)$

The direction ratios of the normal to this surface at (1, –1, 2) are – 8, 4, 12 *i.e.* – 2, 1, 3.

Since, the surfaces are orthogonal, normals are perpendicular to each other.

$$\therefore \quad (-2)(a-2) + (1)(-2b) + (3)(b) = 0 \quad i.e. \quad -2a + b = -4 \qquad \text{.................... (1)}$$

Since (1, –1, 2) lies on the surface

$ax^2 - byz - (a+2)x = 0$, we have $a + 2b - a - 2 = 0$.

i.e. $b = 1$. .... (2)

Then from (1) we get $a = 5/2$. Hence, $a = 5/2$ and $b = 1$.

EX 7: Find the values of a, b, c if the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1,2,-1) has magnitude 64 in the direction parallel to the z –axis.

Solution:

We have $\Phi = axy^2 + byz + cz^2x^3$

$$\therefore \quad \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} = (ay^2 + 3cx^2z^2)i + (2axy + bz)j + (by + 2czx^3)k$$

$$= (4a + 3c)i + (4a - b)j + (2b - 2c)k \text{ at } (1, 2, -1) \qquad \text{.................... (1)}$$

The directional derivative is maximum in the direction of $\nabla\Phi$ *i.e.* in the direction of $(4a + 3c)i + (4a - b)j + (2b - 2c)k$.

But by data the directional derivative is maximum in the direction of the *z*-axis *i.e.* in the direction of $0i + 0j + k$.

$$\therefore \quad \frac{4a + 3c}{0} = \frac{4a - b}{0} = \frac{2b - 2c}{1} \qquad \therefore \quad 4a + 3c = 0 \text{ and } 4a - b = 0$$

Hence, from (1), $\nabla\Phi = (2b - 2c)k \qquad \therefore \quad |\nabla\Phi| = |2b - 2c|$.

But directional derivative is maximum in the direction of $\nabla\Phi$ and is given to be 64,

$$\therefore \quad 2b - 2c = 64 \qquad \therefore \quad b - c = 32.$$

Solving $4a + 3c = 0$, $4a - b = 0$ and $b - c = 32$, we get $a = 6$, $b = 24$, $c = -8$.

Problems on Divergence and Curl of a vector point function

EX 1: If $\overline{F} = x^2z\,i - 2y^3z^3j + xy^2z^2k$ find div $\overline{F}$ and Curl $\overline{F}$at (1,-1,1)

- Solution:

$$\text{div } \overline{F} = \nabla \cdot \overline{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3z^3) + \frac{\partial}{\partial z}(xy^2z^2)$$

$$= 2xz - 6y^2z^3 + 2xy^2z$$

$$\therefore \quad \text{div } \overline{F} = (2 - 6 + 2) = -2 \text{ at } (1, -1, 1)$$

$$\text{curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & -2y^3z^3 & xy^2z^2 \end{vmatrix}$$

$$= i(2xyz^2 + 6y^3z^2) - j(y^2z^2 - x^2) + k(0 - 0)$$

$$\therefore \quad \text{curl } \overline{F} = i(-2 - 6) - j(1 - 1) + k(0)$$

$$= -8\,i \quad \text{at } (1, -1, 1).$$

EX 2: If $\overline{a}$ is a constant vector and If $\overline{r} = xi + yj + zk$ then prove that div$[\overline{a} \times (\overline{r} \times \overline{a})]$ = 2 $[{a_1}^2 + {a_2}^2 + {a_3}^2]$

- Solution: We have

$$\bar{a} \times \bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)\,i - (a_1 z - a_3 x)\,j + (a_1 y - a_2 x)\,k$$

$$\therefore \quad \bar{r} \times \bar{a} = -(\bar{a} \times \bar{r})$$

$$= (a_3 y - a_2 z)\,i + (a_1 z - a_3 x)\,j + (a_2 x - a_1 y)\,k$$

$$\therefore \quad \bar{a} \times (\bar{r} \times \bar{a}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_3 y - a_2 z & a_1 z - a_3 x & a_2 x - a_1 y \end{vmatrix}$$

$$= [a_2(a_2 x - a_1 y) - a_3(a_1 z - a_3 x)]\,i + [\ldots\ldots]\,j + [\ldots\ldots]\,k$$

$$\text{div}\,[\bar{a}\times(\bar{r}\times\bar{a})] = \frac{\partial}{\partial x}[a_2(a_2x - a_1y) - a_3(a_1z - a_3x)] + \frac{\partial}{\partial y}[\ldots\ldots] + \frac{\partial}{\partial z}[\ldots\ldots]$$

$$= (a_2^2 + a_3^2) + (a_3^2 + a_1^2) + (a_1^2 + a_2^2)$$

$$= 2(a_1^2 + a_2^2 + a_3^2)$$

EX 3: Prove that $\nabla\cdot\left\{\frac{f(r)}{r}\,\bar{r}\right\} = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$. Hence or otherwise prove that $\nabla\cdot(r^n\bar{r})$=(n+3) r^n

Solution: We know that $\nabla\cdot(\phi\,\bar{f}) = \phi(\nabla\cdot\bar{f}) + \nabla\phi\cdot\bar{f}$

Let $\phi = \frac{f(r)}{r}$ and $\bar{f} = \bar{r}$ then by above result

$$\text{LHS} = \nabla\cdot\left\{\frac{f(r)}{r}\,\bar{r}\right\} = \frac{f(r)}{r}(\nabla\cdot\bar{r}) + \nabla\frac{f(r)}{r}\cdot\bar{r}$$

$$= \frac{f(r)}{r}\,3 + \left\{\frac{\partial}{\partial x}\left[\frac{f(r)}{r}\right]i + \frac{\partial}{\partial y}\left[\frac{f(r)}{r}\right]j + \frac{\partial}{\partial z}\left[\frac{f(r)}{r}\right]k\right\}\cdot\bar{r}$$

$$= \frac{f(r)}{r}\,3 + \left[\left[\frac{rf'(r)-f(r)}{r^2}\right]\frac{\partial r}{\partial x}\,i + \left[\frac{rf'(r)-f(r)}{r^2}\right]\frac{\partial r}{\partial y}\,j + \left[\frac{rf'(r)-f(r)}{r^2}\right]\frac{\partial r}{\partial z}\,k\right]\cdot\bar{r}$$

But $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore\ \nabla\cdot\left\{\frac{f(r)}{r}\,\bar{r}\right\} = \frac{f(r)}{r}\,3 + \left[\frac{rf'(r)-f(r)}{r^3}\right](xi + yj + zk)\cdot\bar{r}$$

$$= \frac{f(r)}{r}\,3 + \left[\frac{rf'(r)-f(r)}{r^3}\right](\bar{r}\cdot\bar{r})$$

$= \frac{f(r)}{r} 3 + [\frac{rf'(r) - f(r)}{r^3}] r^2$

$= f'(r) + 2\frac{f(r)}{r}$.........(1)

RHS = $\frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$.

$= \frac{1}{r^2}[r^2 f'(r) + 2r f(r)]$

$= f'(r) + 2\frac{f(r)}{r}$.........(2)

from (1) and (2) $\nabla \cdot \left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$.

From the above result $\nabla \cdot (r^n\bar{r}) = \frac{1}{r^2}\frac{d}{dr}[r^2 r^{(n+1)}]$ as $(\frac{f(r)}{r} = r^n)$

$= \frac{1}{r^2}(n+3)\, r^{(n+2)}$

$= (n+3)\, r^n$

Problems on Solenoidal and irrotational vector function:

EX 1:Prove that $\bar{F} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is Solenoidal and determine the constants a, b, c if $\bar{F}$ is irrotational.

- Solution:

Sol. : $\bar{F}$ is solenoidal if $\nabla \cdot \bar{F} = 0$.

Now, $\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= 1 - 3 + 2 = 0$

Hence, for all values of a, b, c, $\bar{F}$ is solenoidal. $\bar{F}$ is irrotational if curl $\bar{F} = 0$.

$$\text{Now,}\quad \text{curl}\ \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

and $F_1 = x + 2a + az,\ F_2 = bx - 3y - z,\ F_3 = 4x + 6y + 2z$

$$\therefore \quad \text{curl}\ \overline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k$$

$$= (c+1)\, i + (a-4)\, j + (b-2)\, k = 0i + 0j + 0k$$

$$\therefore \quad c + 1 = 0,\ a - 4 = 0,\ b - 2 = 0$$

$$\therefore \quad a = 4,\ b = 2,\ c = -1.$$

EX2: If $\overline{r}$ is the position vector of a point (x,y,z) and r is the modulus of $\overline{r}$ then prove that $r^n\overline{r}$ is an irrotational vector for any value of n but solenoidal only if n = - 3

- Solution:

: **(a)** By definition

$$\text{curl}\ r^n\, \overline{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] + j[\ldots.] + k[\ldots.]$$

$$= i\left[z\, n\, r^{n-1}\frac{\partial r}{\partial y} - y\, n\, r^{n-1}\frac{\partial r}{\partial z}\right] + j[\ldots.] + k[\ldots.]$$

$$\text{Now,}\quad r^2 = x^2 + y^2 + z^2 \quad \therefore\ 2r\frac{\partial r}{\partial x} = 2x \quad \therefore\ \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\text{Similarly,}\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \text{curl}\ r^n\, \overline{r} = i\,[n\, r^{n-2}\, zy - n\, r^{n-2}\, zy] + j[\ldots.] + k[\ldots.]$$

$$= \overline{0}$$

Hence, $r^n\, \overline{r}$ is irrotational.

(b) $$\text{div}\,(r^n \bar{r}) = \nabla \cdot (r^n xi + r^n yj + r^n zk)$$
$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$
$$= \left[r^n + xnr^{n-1}\frac{\partial r}{\partial x}\right] + \left[\ldots\ldots\right] + \left[\ldots\ldots\right]$$
$$= \left[r^n + nxr^{n-1}\frac{x}{r}\right] + \left[r^n + nyr^{n-1}\frac{y}{r}\right] + \left[r^n + nzr^{n-1}\frac{z}{r}\right]$$
$$= 3r^n + nr^{n-1}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right)$$
$$= 3r^n + nr^{n-1}\frac{(x^2 + y^2 + z^2)}{r}$$
$$= 3r^n + nr^{n-1} \cdot r = (n+3)\,r^n$$

Hence, $\text{div}\,(r^n \bar{r}) = 0$ if $n = -3$.

Ex 3: Find $f(r)$ so that the vector $f(r)\bar{r}$ is both solenoidal and irrotational.

- Solution:

: We have $f(r)\,\bar{r} = f(r)\,xi + f(r)\,yj + f(r)\,zk$

$$\text{div}\,[f(r)\,\bar{r}] = \nabla \cdot f(r)\,\bar{r}$$
$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot [f(r)\,xi + f(r)\,yj + f(r)\,zk]$$
$$= \frac{\partial}{\partial x}[f(r)\,x] + \frac{\partial}{\partial y}[f(r)\,y] + \frac{\partial}{\partial z}[f(r)\,z]$$

Now, $\dfrac{\partial f(r)}{\partial x} = f'(r)\dfrac{\partial r}{\partial x} = f'(r)\dfrac{x}{r}$

Similarly, $\dfrac{\partial}{\partial y}f(r) = f'(r)\dfrac{y}{r}$, $\dfrac{\partial f(r)}{\partial z} = f'(r)\dfrac{z}{r}$

$\therefore$ $\dfrac{\partial}{\partial x}[f(r)\,x] = x\dfrac{\partial}{\partial x}f(r) + f(r)$

$= x\dfrac{f'(r)}{r} \cdot x + f(r)$ and so on.

$\therefore$ $\text{div}\,[f(r)\,\bar{r}] = f'(r)\dfrac{x}{r} \cdot x + f(r) + f'(r)\dfrac{y}{r} \cdot y + f(r) + f'(r)\dfrac{z}{r} \cdot z + f(r)$

$\therefore$ $\text{div}\,[f(r)\,\bar{r}] = 3f(r) + f'(r) \cdot \dfrac{1}{r}[x^2 + y^2 + z^2]$

$= 3f(r) + f'(r)\,r$

If $f(r)\bar{r}$ is solenoidal

$$\text{div}\ [f(r)\bar{r}] = 3f(r) + f'(r)\,r = 0$$

$$\therefore \quad \frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating $\log f(r) = -3\log r + \log c$

$$\therefore \quad \log f(r) = \log\frac{c}{r^3} \qquad \therefore \quad f(r) = \frac{c}{r^3}$$

Now, $\text{curl}\left[f(r)\bar{r}\right] = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x\,f(r) & y\,f(r) & z\,f(r) \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y} z\,f(r) - \frac{\partial}{\partial z} y\,f(r)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[z\,f'(r)\cdot\frac{y}{r} - y\cdot f'(r)\frac{z}{r}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= f'(r)\,i\left[\frac{zy}{r} - \frac{zy}{r}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right] = 0$$

Hence, $f(r)\bar{r} = \dfrac{c}{r^3}\bar{r}$ is both solenoidal and irrotational.