Vector IntegrationNotes-2

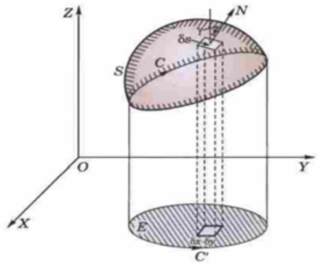
Stoke's Theorem
(Relation between Line Integral and Surface Integral)

Surface Integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the Line Integral of the vector point function $\vec{F}$ taken along the closed curve C

$$\oint \vec{F}.\overrightarrow{dr} = \int\int_S \text{curl } \vec{F}.\hat{n}ds$$

Where $\hat{n}$ is a unit external normal to any surface ds.

$dx\,dy$ = projection of ds on the xy-plane



Ex 1 Verify Stoke's Theorem for the vector field $\vec{F} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy −plane.

Solution

The projection of the upper half of given sphere on the xy-plane ($z = 0$) is the circle $c[x^2 + y^2 = 1]$

$$\oint_c \mathbf{F}.d\mathbf{R} = \oint_c [(2x - y)dx - yz^2\,dy - y^2\,zdz] = \oint_c (2x - y)dx \qquad [z = 0 \text{ in the } xy\text{-plane}]$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta) \qquad [\text{Putting } x = \cos\theta,\ y = \sin\theta]$$

$$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\,d\theta = \times 0 + 4\int_0^{\pi/2} \sin^2\theta\,d\theta = \pi. \qquad ...(i)$$

Now
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{I} + 0\,\mathbf{J} + \mathbf{K} = \mathbf{K}$$

$$\therefore \quad \int \text{curl } \mathbf{F}.\,\mathbf{N}ds = \int_S K.\mathbf{N}\,ds = \int_A \mathbf{K}.\mathbf{N}\frac{dxdy}{|\mathbf{N}.\mathbf{K}|}$$

where A is the projection of S on xy-plane and $ds = dxdy/\mathbf{N}.\mathbf{K}$

$$= \int_A dx\,dy = \text{area of circle } C = \pi$$

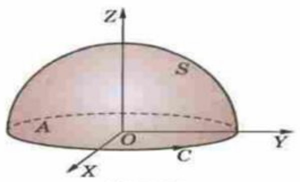
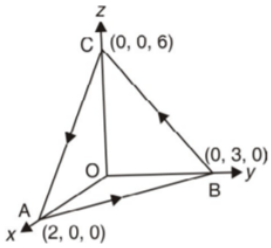


Fig. 8.17

Hence, the Stokes theorem is verified from the equality of (i) and (ii).

Ex 2- Use Stoke's Theorem to evaluate $\int_c (x+y)dx + (2x - z)dy + (y+z)dz$ where c is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,6)



Solution Here $\vec{F} = (x+y)\hat{\imath} + (2x - z)\hat{\jmath} + (y+z)\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\,\hat{\imath} + \hat{k}$$

Eqn of the plane through A,B,C is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

Or $3x + 2y + z = 6$

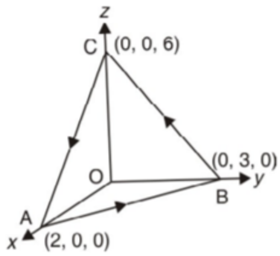
Vector normal to the plane is $grad(3x + 2y + z - 6) = 3\,\hat{\imath}$ -

$$\hat{n} = \frac{3\,\hat{\imath} + 2\hat{\jmath} + \hat{k}}{\sqrt{(9+4+1)}} = \frac{1}{\sqrt{14}}(3\,\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

$$ds = \frac{dxdy}{\hat{n}.\hat{k}}$$

By Stoke's Theorem $\oint \vec{F}.\overrightarrow{dr} = \int\int_S \text{curl } \vec{F}.\hat{n}ds$

$$\int_c (x+y)dx + (2x-z)dy + (y+z)dz$$



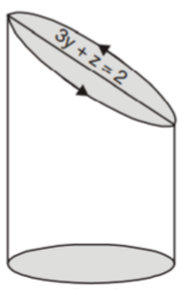
$$= \iint_s \text{curl } \vec{F}\cdot\hat{n}\,ds = \iint_s (2\hat{i} + \hat{k})\cdot\frac{1}{\sqrt{4}}(3\hat{i} + 2\hat{j} + \hat{k})\frac{dx\,dy}{\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}).\hat{k}}$$

$$= \iint_s \frac{(6+1)}{\sqrt{14}}\frac{dx\,dy}{\frac{1}{\sqrt{14}}} = 7\iint dx\,dy = 7 \text{ Area of } \Delta \text{ OAB}$$

$$= 7\left(\frac{1}{2}\times 2\times 3\right) = 21$$

(projection on xy plane is triangle OAB. Area of a triangle is $\frac{1}{2}ab\sin\theta$ where θ is the angle between the sides a & b.)

Ex 3- Use Stoke's Theorem to evaluate $\int_c \vec{F}.\overrightarrow{dr}$ where $\vec{F} = -y^2\hat{\imath} + x\hat{\jmath} + z^2\hat{k}$ and c is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$



Solution $\oint_C \vec{F}.\vec{dr} = \iint_S \text{curl } \vec{F}.\hat{n}\,ds = \iint_S \text{curl}(-y^2\hat{\imath} + x\hat{\jmath} + z^2\hat{k})\,\hat{n}\,ds$

$F(x,y,z) = -y^2\hat{\imath} + x\hat{\jmath} + z^2\hat{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}(1+2y) = (1+2y)\hat{k}$$

Normal vector $= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y+z-2) = \hat{\jmath} + \hat{k}$ = grad(y+z-2)

Unit normal vector $\hat{n} = \frac{\hat{\jmath} + \hat{k}}{\sqrt{2}}$ $\qquad ds = \frac{dx\,dy}{\hat{n}.\hat{k}}$

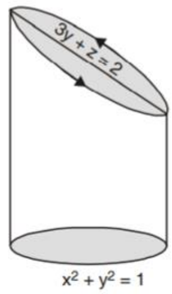
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (1+2y)\,\hat{k}\cdot\frac{\hat{j}+\hat{k}}{\sqrt{2}}\,\frac{dx\,dy}{\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right)\cdot\hat{k}}$$

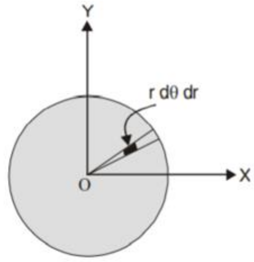
$$= \iint \frac{1+2y}{\sqrt{2}}\,\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi}\left[\frac{1}{2}+\frac{2}{3}\sin\theta\right]d\theta$$

$$= \left[\frac{\theta}{2}-\frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi-\frac{2}{3}-0+\frac{2}{3}\right) = \pi \qquad \textbf{Ans.}$$



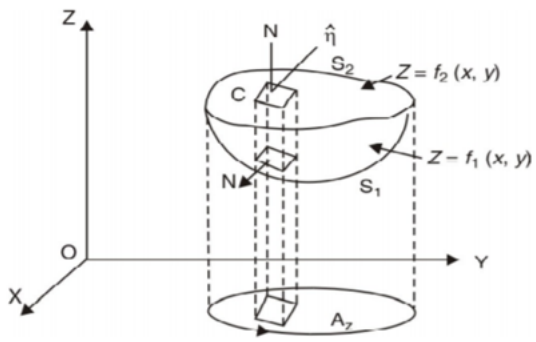


Gauss Divergence Theorem
(Relation between Surface Integral and Volume Integral)

Statement. The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically

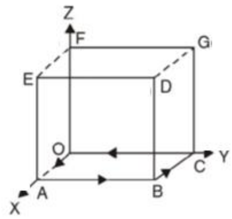
$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\,\vec{F}dw$$



Ex 1 Evaluate $\int\int_{s} \vec{F}.\hat{n}ds$ where $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$ and s is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

Solution By Divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,ds = \iint_v\int(\nabla.\vec{F})\,dv$$
$$= \iiint_v\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(4xz\hat{i}-y^2\hat{j}+yz\hat{k})\,dv$$
$$= \iint_v\int\left[\frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}(-y^2)+\frac{\partial}{\partial z}(yz)\right]dx\,dy\,dz$$
$$= \iint_v\int(4z-2y+y)\,dx\,dy\,dz$$
$$= \iint_v\int(4z-y)\,dx\,dy\,dz = \int_0^1\int_0^1\left(\frac{4z^2}{2}-yz\right)_0^1 dx\,dy$$
$$= \int_0^1\int_0^1(2z^2-yz)_0^1\,dx\,dy = \int_0^1\int_0^1(2-y)\,dx\,dy$$
$$= \int_0^1\left(2y-\frac{y^2}{2}\right)_0^1 dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}[x]_0^1 = \frac{3}{2}(1) = \frac{3}{2}.$$



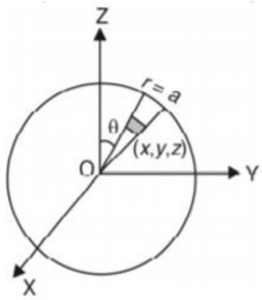
Ex 2- Use Divergence theorem to evaluate $\int\int_{s} \vec{A}.\overrightarrow{ds}$ where $\vec{A} = x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}$ and s is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution

$$\iint_S \vec{A}\cdot\vec{ds} = \iiint_v \operatorname{div}\vec{A}\,dV$$
$$= \iiint_v\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^3\hat{i}+y^3\hat{j}+z^3\hat{k})\,dV$$
$$= \iiint_v(3x^2+3y^2+3z^2)\,dV = 3\iiint_v(x^2+y^2+z^2)\,dV$$

On putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we get

$$= 3\iiint_v r^2(r^2\sin\theta\,dr\,d\theta\,d\phi) = 3\times 8\int_0^{\frac{\pi}{2}}d\phi\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^a r^4\,dr$$
$$= 24\,(\phi)_0^{\frac{\pi}{2}}\,(-\cos\theta)_0^{\frac{\pi}{2}}\left(\frac{r^5}{5}\right)_0^a = 24\left(\frac{\pi}{2}\right)(-0+1)\left(\frac{a^5}{5}\right) = \frac{12\pi a^5}{5}$$



Few important limits for sphere

If the region of integration is a sphere or an ellipsoid, it is convenient to use spherical polar co-ordinates

$$x = r\sin\theta\cos\emptyset, y = r\sin\theta\sin\emptyset, z = r\cos\theta,\ dV = dxdydz = r^2\sin\theta drd\theta d\emptyset$$

(i) Limits for complete sphere are

$0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \emptyset \le 2\pi$

(ii) Limits for hemisphere are

$0 \le r \le a,\ 0 \le \theta \le \pi/2,\ 0 \le \emptyset \le 2\pi$

(iii) Limits for octant are

$0 \le r \le a,\ 0 \le \theta \le \pi/2,\ 0 \le \emptyset \le \pi/2$

Ex 3- Use Divergence theorem to $\int\int_S \nabla(x^2+y^2+z^2)\overrightarrow{ds} = 6V$ where s is any closed surface enclosing volume V.

Solution

Here $\nabla\ (x^2+y^2+z^2) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2)$

$$= 2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k} = 2\,(x\,\hat{i}+y\,\hat{j}+z\,\hat{k})$$

$$\therefore \quad \iint_S \nabla(x^2+y^2+z^2)\cdot ds = \iint_S \nabla(x^2+y^2+z^2)\cdot\hat{n}\,ds$$

$\hat{n}$ being outward drawn unit normal vector to S

$$= \iint_S 2\,(x\hat{i}+y\,\hat{j}+z\hat{k})\cdot\hat{n}\,ds$$

$$= 2\iiint_V div\,(x\hat{i}+y\,\hat{j}+z\,\hat{k})\,dv \qquad ...(1)$$

(By Divergence Theorem)
(V being volume enclosed by S)

Now, div. $(x\,\hat{i}+y\,\hat{j}+z\hat{k}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i}+y\,\hat{j}+z\,\hat{k})$

$$= \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3 \qquad ...(2)$$

From (1) & (2), we have

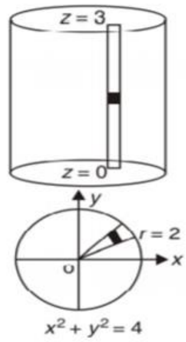
$$\iint \nabla(x^2+y^2+z^2)\cdot dS = 2\iiint_V 3\,dv = 6\iiint_V dv = 6\,V$$

Proved.

Ex 4- Use Divergence theorem to evaluate $\int\int_S \vec{F}.\overrightarrow{ds}$ where $\vec{F} = 4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}$ and s is the surface of the region $x^2+y^2 = 4, z = 3$

Solution

By Divergence Theorem,



$$\iint_S \vec{F}\cdot dS = \iiint_V div\,\vec{F}\,dV$$

$$= \iiint_V\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(4x\hat{i}-2y^2\hat{j}+z^2\hat{k})\,dV$$

$$= \iiint_V (4-4y+2z)\,dx\,dy\,dz$$

$$= \iint dx\,dy\int_0^3 (4-4y+2z)dz = \iint dx\,dy\,[4z-4yz+z^2]_0^3$$

$$= \iint (12-12y+9)\,dx\,dy = \iint (21-12y)\,dx\,dy$$

Let us put $x = r\cos\theta$, $y = r\sin\theta$

$$= \iint (21-12r\sin\theta)\,r\,d\theta\,dr = \int_0^{2\pi} d\theta\int_0^2 (21r-12r^2\sin\theta)\,dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{21r^2}{2}-4r^3\sin\theta\right]_0^2 = \int_0^{2\pi} d\theta\,(42-32\sin\theta) = (42\,\theta+32\cos\theta)_0^{2\pi}$$

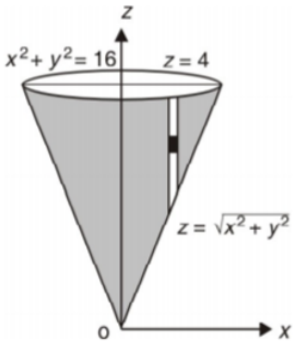
$$= 84\,\pi+32-32 = 84\,\pi$$

Ex 5 Evaluate $\int\int_S \vec{F}.\hat{n}ds$ over the entire surface of the region above the $xy-$ plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$ if $\vec{F} = 4xz\hat{\imath} + xyz^2\hat{j} + 3z\hat{k}$.

Solution

If V is the volume enclosed by S, then V is bounded by the surfaces $z = 0$, $z = 4$, $z^2 = x^2 + y^2$.

By divergence theorem, we have



$$\iint_S \overline{F}\,.\,\hat{n}\,ds = \iiint_V div\,\vec{F}\,dx\,dy\,dz$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)\right] dx\,dy\,dz$$

$$= \iiint_V (4z + xz^2 + 3)\,dx\,dy\,dz$$

Limits of z are $\sqrt{x^2+y^2}$ and 4.

$$\iiint_{\sqrt{x^2+y^2}}^{4} (4z + xz^2 + 3)\,dz\,dy\,dx = \iint\left[2z^2 + \frac{xz^3}{3} + 3z\right]_{\sqrt{x^2+y^2}}^{4} dy\,dx$$

$$= \iint\left[\left(32 + \frac{64x}{3} + 12\right) - \{2(x^2+y^2) + x(x^2+y^2)^{3/2} + 3\sqrt{x^2+y^2}\}\right] dy\,dx$$

$$= \iint\left(44 + \frac{64x}{3} - 2(x^2+y^2) - x(x^2+y^2)^{3/2} - 3\sqrt{x^2+y^2}\right) dy\,dx$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$= \iint\left(44 + \frac{64r\cos\theta}{3} - 2r^2 - r\cos\theta\, r^3 - 3r\right) r\,d\theta\,dr$$

Limits of r are 0 to 4.
and limits of θ are 0 to 2π.

$$= \int_0^{2\pi}\int_0^4 \left(44r + \frac{64r^2\cos\theta}{3} - 2r^3 - r^5\cos\theta - 3r^2\right) d\theta\,dr$$

$$= \int_0^{2\pi}\left[22r^2 + \frac{64\times r^3\cos\theta}{9} - \frac{r^4}{2} - \frac{r^6}{6}\cos\theta - r^3\right]_0^4 d\theta$$

$$= \int_0^{2\pi}\left[22(4)^2 + \frac{64\times(4)^3\cos\theta}{9} - \frac{(4)^4}{2} - \frac{(4)^6}{6}\cos\theta - (4)^3\right] d\theta$$

$$= \int_0^{2\pi}\left[352 + \frac{64\times 64}{9}\cos\theta - 128 - \frac{(4)^6}{6}\cos\theta - 64\right] d\theta$$

$$= \int_0^{2\pi}\left[160 + \left(\frac{64\times 64}{9} - \frac{(4)^6}{6}\right)\cos\theta\right] d\theta$$

$$= \left[160\,\theta + \left(\frac{64\times 64}{9} - \frac{(4)^6}{6}\right)\sin\theta\right]_0^{2\pi} = 160\,(2\pi) + \left(\frac{64\times 64}{9} - \frac{(4)^6}{6}\right)\sin 2\pi$$

$= 320\,\pi$ **Ans.**