CHAPTER

9 Vector Integration

1. The Line Integral

Let $\overline{F}$ be a vector function defined throughout some region of space and let C be any curve in that region. If $\bar{r}$ is the position vector of a point $P(x, y, z)$ on C then the integral $\int_C \overline{F} \cdot d\bar{r}$ is called **the line integral** of $\overline{F}$ taken over C.

Now, since $\bar{r} = xi + yj + zk$ $\quad \therefore \quad d\bar{r} = dxi + dyj + dzk$

and if $\overline{F} = F_1 i + F_2 j + F_3 k$ then $\boxed{\int_C \overline{F} \cdot d\bar{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)}$

2. Line Integral in Parametric Form

The line integral is usually computed by expressing $F_1 dx + F_2 dy + F_3 dz$ in terms of a single variable. It is clear that if the curve C can be expressed in the parametric form $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ then the line integral will be reduced to $\int \Phi(t)\, dt$, integrated between the proper limits t_1 and t_2 where t is a parameter. **The line integral** defined as above in general **depends upon the path** C.

3. Condition for Independence of the Path in the Line Integral

If $\overline{F}$ is the gradient of some scalar point function Φ *i.e.* if $\overline{F} = \nabla \Phi$ then the line integral is independent of the path from A to B.

Proof : Let $\overline{F} = \nabla \Phi = i\dfrac{\partial \Phi}{\partial x} + j\dfrac{\partial \Phi}{\partial y} + k\dfrac{\partial \Phi}{\partial z}$

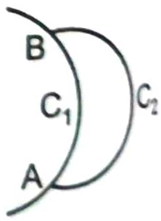


Fig. 9.1

$$\therefore \quad \int_C \overline{F} \cdot d\bar{r} = \int_A^B \left(i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}\right) \cdot (i\,dx + j\,dy + k\,dz)$$

$$\therefore \quad \int_C \overline{F} \cdot d\bar{r} = \int_A^B \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz = \int_A^B d\Phi = \Phi_B - \Phi_A$$

Thus, the line integral does not depend upon the path but depends upon the end points A and B.

Conversely it can be proved that : *If the line integral is independent of the path then $\overline{F}$ is the gradient of some scalar function Φ.*

Definition : If $\overline{F} = \nabla \Phi$ such a field is called **conservative**. In other words if the line integral $\int_A^B \overline{F} \cdot d\bar{r}$ is independent of the path joining A and B then the field is **conservative**.

Remarks ....

1. If $\bar{F} = \nabla\Phi$ then $F_1 i + F_2 j + F_3 k = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k$

   $\therefore\ F_1 = \frac{\partial \Phi}{\partial x},\ F_2 = \frac{\partial \Phi}{\partial y},\ F_3 = \frac{\partial \Phi}{\partial z}$

2. If the curve is closed and the field is conservative then since $\Phi_A = \Phi_B$ ($\because\ A = B$),

   $\int_C \bar{F} \cdot d\bar{r} = \Phi_A - \Phi_A = 0$.

4. Circulation of a Vector

Definition : If the curve C is closed, the line integral $\int_C \bar{F} \cdot d\bar{r}$ is called the **circulation** of $\bar{F}$ along C.

If the circulation of $\bar{F}$ along every closed curve in the region is zero than $\bar{F}$ is called **irrotational.**

Note ....

If the field is conservative then $\int_A^A \bar{F} \cdot d\bar{r}$ along every closed curve will be zero *i.e.* the circulation will be zero. This means the field is irrotational. Thus, **irrotational field is conservative.**

(A) To Evaluate The Line Integral

Example 1 : Integrate $\bar{F} = x^2 i + xy j$

(i) from O to P along OP, P being (0, 1), (ii) along the x-axis from $x = 0$ to $x = 1$,

(iii) along the line $x = 1$ from $y = 0$ to $y = 1$.

Also integrate $\bar{F}$ from (0, 0) to (1, 1) along the parabola $y^2 = x$.

Sol : We have $\bar{F} \cdot d\bar{r} = x^2 dx + xy\, dy$ .... (1)

(i) **From *O* to *P* along *OP* :** The equation of the line OP is $y = x$ $\therefore$ $dy = dx$.

Putting $y = x$ and $dy = dx$, in (1), we get

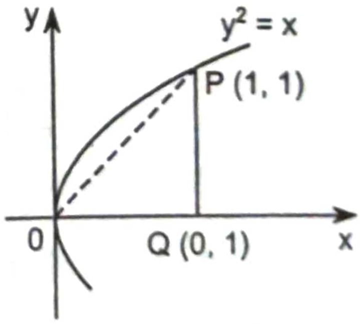


Fig. 9.2

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^1 x^2 dx + x^2\, dx$$

$$= \int_0^1 2x^2 dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

(ii) **From $x = 0$ to $x = 1$ along the x-axis *i.e.* from *O* to *Q* along the x-axis**

The equation of the x-axis is $y = 0$ $\therefore$ $dy = 0$.

Putting $y = 0$ and $dy = 0$ in (1), we get

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^1 (x^2 dx + 0) = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right] = \frac{1}{3}$$

(iii) From $y = 0$ to $y = 1$ along the line $x = 1$ *i.e.* from Q to P along the line QP

The equation of the line QP is $x = 1$ $\quad \therefore \quad dx = 0$

Putting $x = 1$ and $dx = 0$ in (1), we get

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^1 (x^2 \cdot 0 + y\, dy) = \int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

(iv) Along the parabola $y^2 = x$: The equation of the parabola is $y^2 = x$.

$\therefore \quad 2y\, dy = dx$ and y varies from 0 to 1.

Putting $x = y^2$ and $dx = 2y\, dy$ in (1), we get

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^1 (y^4 \cdot 2y\, dy) + (y^2 \cdot y\, dy)$$

$$= \int_C (2y^5 + y^3)\, dy = \left[\frac{y^6}{3} + \frac{y^4}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Example 2 : Prove that $\displaystyle\int_A^B (2xy^3 - y^2 \cos x)\, dx + (1 - 2y \sin x + 3x^2y^2)\, dy = \frac{\pi^2}{4}$

along arc $2x = \pi y^2$ from $A(0, 0)$ to $B(\pi/2, 1)$.

(M.U. 1999, 2000)

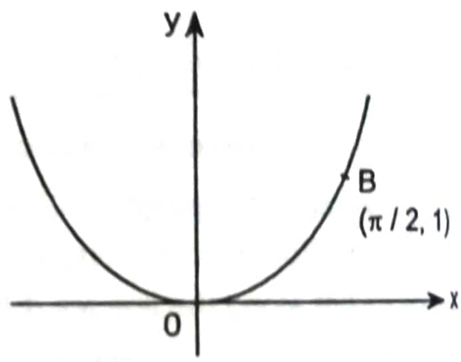


Fig. 9.3

Sol. : Along the arc $\quad y = \sqrt{\frac{2}{\pi}} \cdot \sqrt{x}$

$$dy = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2 \cdot \sqrt{x}}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{x}}\, dx$$

$$\therefore \int_A^B \bar{F} \cdot d\bar{r} = \int_0^{\pi/2} \left(2x \cdot \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \cdot x\sqrt{x} - \frac{2}{\pi} \cdot x \cos x\right) dx$$

$$+ \left(1 - 2 \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sqrt{x} \sin x + 3x^2 \cdot \frac{2x}{\pi}\right) \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{x}}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{4\sqrt{2}}{\pi\sqrt{\pi}} \cdot x^{5/2} - \frac{2x}{\pi} \cos x + \frac{1}{\sqrt{2\pi}} \cdot x^{-1/2} - \frac{2}{\pi} \sin x + \frac{6}{\sqrt{2} \cdot \pi\sqrt{\pi}} \cdot x^{5/2}\right] dx$$

$$= \int_0^{\pi/2} \left[\left(\frac{4\sqrt{2} + 3\sqrt{2}}{\pi\sqrt{\pi}}\right) x^{5/2} - \frac{2}{\pi} x \cos x + \frac{1}{\sqrt{2\pi}} x^{-1/2} - \frac{2}{\pi} \sin x\right] dx$$

But $\int x \cos x\, dx = x \sin x + \cos x$

$$\therefore \int_A^B \bar{F} \cdot d\bar{r} = \left[\frac{4\sqrt{2} + 3\sqrt{2}}{\pi\sqrt{\pi}} \cdot \frac{x^{7/2}}{7/2} - \frac{2}{\pi}\{x \sin x + \cos x\} + \frac{2}{\sqrt{2\pi}} \cdot x^{1/2} + \frac{2}{\pi} \cos x\right]_0^{\pi/2}$$

$$= \left[\left\{\frac{7\sqrt{2}}{\pi\sqrt{\pi}} \cdot \frac{2}{7}\left(\frac{\pi}{2}\right)^{7/2} - \frac{2}{\pi}\left\{\frac{\pi}{2}\right\} + \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}}\right\} - \left\{-\frac{2}{\pi} + \frac{2}{\pi}\right\}\right]$$

$$\therefore \int_A^B \bar{F} \cdot d\bar{r} = \pi^{(7/2)-(3/2)} \cdot 2^{(3/2)-(7/2)} = \frac{\pi^2}{4}$$

Aliter : Let $\bar{F} = (2xy^3 - y^2 \cos x)\, i + (1 - 2y \sin x + 3x^2 y^2)\, j$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3 - y^2 \cos x & 1 - 2y \sin x + 3x^2y^2 & 0 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(-2y \cos x + 6xy^2 - 6xy^2 + 2y \cos x)$$

$$= 0$$

$\therefore$ $\bar{F}$ is conservative.

$\therefore$ The integral depends upon the end-points A and B and not on the curve.

$$\therefore \int_A^B \bar{F} \cdot dr = \int_A^B (2xy^3 - y^2 \cos x)\, dx + (1 - 2y \sin x + 3x^2y^2)\, dy$$

$$= \int_A^B (2xy^3 dx + 3x^2y^2 dy) - (y^2 \cos x\, dx + 2y \sin x\, dy) + dy$$

$$= \int_A^B d(x^2y^3) - d(y^2 \sin x) + d(y) = \left[x^2y^3 - y^2 \sin x + y\right]_{(0,0)}^{(\pi/2,\, 1)}$$

$$= \left(\frac{\pi^2}{4} \cdot 1 - 1 + 1\right) - (0 - 0 + 0) = \frac{\pi^2}{4}.$$

Example 3 : Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = yzi + zxj + xyk$ and C is the portion of the curve $\bar{r} = (a \cos t)\, i + (b \sin t)\, j + ctk$ from $t = 0$ to $t = \pi/4$.

Sol. : Since $\bar{r} = a \cos t\, i + b \sin t\, j + ct\, k$ the parametric equations of the curve are $x = a \cos t$, $y = b \sin t$, $z = ct$. .... (1)

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_C (yzi + zxj + xyk) \cdot (dxi + dyj + dzk)$$

$$= \int_C yz\, dx + zx\, dy + xy\, dz$$

Putting the values of x, y, z from (i) and $dx = -a \sin t\, dt$, $dy = b \cos t\, dt$, $dz = c\, dt$.

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^{\pi/4} (bct \sin t)(-a \sin t\, dt) + (act \cos t)(b \cos t\, dt) + (ab \cos t \sin t)\, c\, dt$$

$$= abc \int_0^{\pi/4} \left[t(\cos^2 t - \sin^2 t) + \sin t \cos t\right] dt$$

$$= abc \int_0^{\pi/4} \left[t \cos 2t + \frac{\sin 2t}{2}\right] dt \qquad \text{[Integrate the first term by parts]}$$

$$= abc \left[t \frac{\sin 2t}{2} + \frac{\cos 2t}{4} - \frac{\cos 2t}{4}\right]_0^{\pi/4} = abc \left[t \frac{\sin 2t}{2}\right]_0^{\pi/4} = \frac{\pi abc}{8}.$$

Doubt ?

Example 4 : Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = \cos y \cdot i - x \sin y \ j$ and c is the curve $y = \sqrt{1 - x^2}$ in the xy-plane from (1, 0) to (0, 1). **(M.U. 2004)**

Sol. : The curve $y = \sqrt{1 - x^2}$ *i.e.*, $x^2 + y^2 = 1$ is a circle with centre at the origin and radius unity. We have

$$\therefore \quad \int_C \bar{F} \cdot d\bar{r} = \int_C (\cos y i - x \sin y \ j) \cdot (dx i + dy \ j)$$

$$= \int_C (\cos y \ dx - x \sin y \ dy)$$

$$= \int_{(1,0)}^{(0,1)} d(x \cos y) = \left[x \cos y\right]_{(1,0)}^{(0,1)} = 0 - 1 = -1.$$

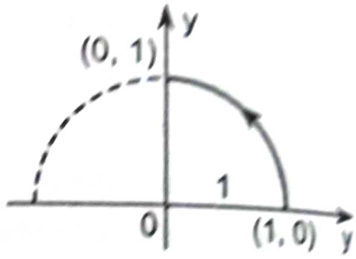


Fig. 9.4

Example 5 : Evaluate $\int \underline{F} \cdot d\underline{r}$ along the arc of the curve from the point (1, 0) to $(e^{2\pi}, 0)$ where $\bar{F} = \dfrac{xi + yj}{(x^2 + y^2)^{3/2}}$ and the vector equation of the curve is $\bar{r} = (e^t \cos t) \, i + (e^t \sin t) \, j$.

(M.U. 2001)

Sol. : We have $\quad \bar{r} = (e^t \cos t) \, i + (e^t \sin t) \, j \quad$ *i.e.*, $\quad x = e^t \cos t, \quad y = e^t \sin t$

and $\quad dx = (-e^t \sin t + e^t \cos t) \, dt; \quad dy = (e^t \cos t + e^t \sin t) \, dt$

$$\therefore \quad \bar{F} = \frac{xi + yj}{(x^2 + y^2)^{3/2}} = \frac{e^t \cos t \ i + e^t \sin t \ j}{[e^{2t} \cos^2 t + e^{2t} \sin^2 t]^{3/2}} = \frac{e^t \cos t i + e^t \sin t j}{e^{3t}}$$

$$\therefore \quad \int_C \bar{F} \cdot d\bar{r} = \int_C \frac{(e^t \cos t \ i + e^t \sin t \ j) \cdot (dx \ i + dy \ j)}{e^{3t}} = \int_C \frac{e^t \cos t \cdot dx + e^t \sin t \cdot dy}{e^{3t}}$$

$$= \int_C [(e^t \cos t)(-e^t \sin t + e^t \cos t) + (e^t \sin t)(e^t \cos t + e^t \sin t)] \frac{dt}{e^{3t}}$$

$$= \int_C e^{2t} (-\cos t \sin t + \cos^2 t + \sin t \cos t + \sin^2 t) \frac{dt}{e^{3t}}$$

$$= \int_C e^{-t} \cdot dt$$

Now, for the point (1, 0) when $x = 1$, $y = 0$, $t = 0$ and for the point $(e^{2\pi}, 0)$when $x = e^{2\pi}$, $y = 0$, $t = 2\pi$.

$$\therefore \quad \int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} e^{-t} \, dt = -\left[e^{-t}\right]_0^{2\pi} = 1 - e^{-2\pi}$$

Example 6 : Evaluate $\int_C \bar{F} \times d\bar{r}$ where $\bar{F} = (2xy + z^2) \, i + x^2 j + 3xz^2 k$ along the curve $x = t$, $y = t^2$, $z = t^3$ from (0, 0, 0) to (1, 1, 1). **(M.U. 2003, 09)**

Sol. : We have
$$\bar{F} \times d\bar{r} = \begin{vmatrix} i & j & k \\ 2xy + z^2 & x^2 & 3xz^2 \\ dx & dy & dz \end{vmatrix}$$

But $x = t,\ y = t^2,\ z = t^3$. $\quad\therefore\ dx = dt,\ \ dy = 2t\,dt,\ \ dz = 3t^2\,dt.$

$$\therefore\ \bar{F} \times d\bar{r} = \begin{vmatrix} i & j & k \\ 2t^3 + t^6 & t^2 & 3t^7 \\ dt & 2t\,dt & 3t^2\,dt \end{vmatrix}$$

$$= (3t^4 - 6t^8)\,dt\,i - (6t^5 + 3t^8 - 3t^7)\,dt\,j + (4t^4 + 2t^7 - t^2)\,dt\,k$$

$$\therefore\ \int_C \bar{F} \times d\bar{r} = \int_0^1 (3t^4 - 6t^8)\,dt\,i - (6t^5 + 3t^8 - 3t^7)\,dt\,j + (4t^4 + 2t^7 - t^2)\,dt\,k$$

$$= \left[\left(\frac{3}{5}t^5 - \frac{6}{9}t^9\right)i - \left(6\frac{t^6}{6} + \frac{3t^9}{9} - 3\frac{t^8}{8}\right)j + \left(4\frac{t^5}{5} + 2\frac{t^8}{8} - \frac{t^3}{3}\right)k\right]_0^1$$

$$= \left(\frac{3}{5} - \frac{2}{3}\right)i + \left(1 + \frac{1}{3} - \frac{3}{8}\right)j + \left(\frac{4}{5} + \frac{1}{4} - \frac{1}{3}\right)k$$

$$= -\frac{1}{15}i + \frac{23}{24}j + \frac{43}{60}k$$

Example 7 : Evaluate $\int_C \bar{F} \cdot \bar{dr}$ where $\bar{F} = 2xi + (xz - y)\,j + 2zk$ from $O(0, 0, 0)$ to $P(3, 1, 2)$ along the line OP.

Sol. : We have $\int_C \bar{F} \cdot d\bar{r} = \int_C 2x\,dx + (xz - y)\,dy + 2z\,dz$

The equation of the line OP is $\dfrac{x}{3} = \dfrac{y}{1} = \dfrac{z}{2} = t$ say.

$\therefore\ x = 3t,\ y = t,\ z = 2t \qquad \therefore\ dx = 3dt,\ dy = dt,\ dz = 2dt$

When $x = 0,\ y = 0,\ z = 0,\ t = 0$ and when $x = 3,\ y = 1,\ z = 2,\ t = 1$.

$$\therefore\ \int_C \bar{F} \cdot d\bar{r} = \int_0^1 (6t)\,3\,dt + (6t^2 - t)\,dt + (4t)\,2\,dt = \left[9t^2 + 2t^3 - \frac{t^2}{2} + 4t^2\right]_0^1 = \frac{29}{2}.$$

Example 8 : Find the circulation of $\bar{F}$ round the curve C where

$\bar{F} = y\,i + z\,j + x\,k$ and C is the circle $x^2 + y^2 = a^2,\ z = 0$. **(M.U. 2005)**

Sol. : The parametric equations of the circle are $x = a\cos\theta,\ y = a\sin\theta,\ z = 0$.

Also $\bar{r} = x\,i + y\,j + z\,k = a\cos\theta\,i + b\sin\theta\,j + ok$

$d\bar{r} = (-a\sin\theta\,i + a\cos\theta\,j)\,d\theta$

Now, circulation $= \int_C \bar{F} \cdot d\bar{r} = \int_C (y\,i + z\,j + x\,k) \cdot d\bar{r}$

$$= \int_C (a\sin\theta\,i + 0\,j + a\cos\theta\,k) \cdot (-a\sin\theta\,i + a\cos\theta\,j)\,d\theta$$

$$= \int_C -a^2\sin^2\theta\,d\theta = -a^2\int_0^{2\pi}\sin^2\theta\,d\theta = -a^2\int_0^{2\pi}\left(\frac{1 - \cos 2\theta}{2}\right)d\theta$$

$$= -\frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -\frac{a^2}{2} \cdot 2\pi = -\pi a^2$$

(B) To Find The Work Done

Example 1 : Find the work done in moving a particle once round the circle $x^2 + y^2 = a^2$, $z = 0$ in the force field given by $\bar{F} = \sin y\, i + (x + x\cos y)\, j$. **(M.U. 2005)**

Sol. : Work done $= \int_C \bar{F} \cdot d\bar{r} = \int_C \sin y\, dx + (x + x\cos y)\, dy$

$$= \int_C (\sin y\, dx + x\cos y\, dy) + x\, dy$$

$$= \int_C [d(x\sin y) + x\, dy] = \int_C d(x\sin y) + \int_C x\, dy$$

Now, $\int_C d(x\sin y) = 0$ since C is a closed curve by remark 2, page 9-2.

Further putting $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$ as parametric equations of the circle.

$$\int_C x\, dy = \int_0^{2\pi} (a\cos\theta)(a\cos\theta)\, d\theta$$

$$= a^2 \int_0^{2\pi} \frac{(1+\cos 2\theta)}{2}\, d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi a^2$$

$\therefore$ Work done $= \pi a^2$.

Example 2 : Find the work done when a force $\bar{F} = (x^2 - y^2 + x)\, i - (2xy + y)\, j$ moves a particle in the xy plane from $(0, 0)$ to $(1,1)$ along the parabola $y^2 = x$. Is the work done different when the path is the straight line $y = x$?

Sol. : The work done $w = \int_C \bar{F} \cdot d\bar{r}$

$$\therefore \quad w = \int_C (x^2 - y^2 + x)\, dx - (2xy + y)\, dy$$

(a) Now, for the parabola $y^2 = x$, we put $y = t$ and $x = t^2$; $dy = dt$ and $dx = 2t\, dt$; $t = 0$ and $t = 1$.

$$\therefore \quad w = \int_0^1 (t^4 - t^2 + t^2) \cdot 2t\, dt - (2t^3 + t)\, dt = \int_0^1 (2t^5 - 2t^3 - t)\, dt$$

$$= \left[\frac{t^6}{3} - \frac{t^4}{2} - \frac{t^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} - \frac{1}{2} = -\frac{2}{3}$$

(b) For the straight line $y = x$, we put $x = t$, $y = t$; $dx = dt$; $dy = dt$; $t = 0$ and $t = 1$.

$$\therefore \quad w = \int_0^1 (t^2 - t^2 + t)\, dt - (2t^2 + t)\, dt$$

$$= \int_0^1 -2t^2\, dt = -2\left[\frac{t^3}{3}\right]_0^1 = -\frac{2}{3}$$

$\therefore$ The work done along both paths is the same.

Example 3 : Find the total work done in moving a particle in the force field $\bar{F} = 3xy\,i - 5z\,j + 10x\,k$ along $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ and $t = 2$.

(M.U. 1991 , 93, 99, 2014)

Sol. : We have

$$\bar{F} \cdot d\bar{r} = (3xy\,i - 5z\,j + 10x\,k) \cdot (dx\,i + dy\,j + dz\,k)$$
$$= 3xy\,dx - 5z\,dy + 10x\,dz$$
$$= 3(t^2+1)(2t^2)(2t\,dt) - 5(t^3)(4t\,dt) + 10(t^2+1)(3t^2\,dt)$$
$$= (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt$$
$$= (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

$$\text{Work done} = \int_C \bar{F} \cdot d\bar{r} = \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$
$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$$
$$= (128 + 64 + 48 + 80) - (2 + 2 + 3 + 10)$$
$$= 303$$

Example 4 : Prove that $\bar{F} = (y^2 \cos x + z^3)\,i + (2y \sin x - 4)\,j + (3xz^2 + 2)\,k$ is a conservative field. Find (i) scalar potential for $\bar{F}$ (ii) the work done in moving an object in this field from $(0, 1, -1)$ to $(\pi/2, -1, 2)$. **(M.U. 1994, 2002, 03, 06, 10, 11)**

Sol. : (a) The field is conservative if curl $\bar{F} = 0$.

$$\text{Now,} \quad \text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$\therefore \quad \text{curl}\,\bar{F} = (0-0)\,i - (3z^2 - 3z^2)\,j + (2y \cos x - 2y \cos x)\,k = \bar{0}$$

$\therefore$ $\bar{F}$ is conservative.

(b) Since, $\bar{F}$ is conservative there exists a scalar potential Φ such that $\bar{F} = \nabla\Phi$.

$$\therefore \quad (y^2 \cos x + z^3)\,i + (2y \sin x - 4)\,j + (3xz^2 + 2)\,k = \frac{\partial \Phi}{\partial x}\,i + \frac{\partial \Phi}{\partial y}\,j + \frac{\partial \Phi}{\partial z}\,k$$

$$\therefore \quad \frac{\partial \Phi}{\partial x} = y^2 \cos x + z^3, \quad \frac{\partial \Phi}{\partial y} = 2y \sin x - 4, \quad \frac{\partial \Phi}{\partial z} = 3xz^2 + 2$$

$$\text{Now,} \quad d\Phi = \frac{\partial \Phi}{\partial x}\,dx + \frac{\partial \Phi}{\partial y}\,dy + \frac{\partial \Phi}{\partial z}\,dz$$
$$= (y^2 \cos x + z^3)\,dx + (2y \sin x - 4)\,dy + (3xz^2 + 2)\,dz$$
$$= (y^2 \cos x\,dx + 2y \sin x\,dy) + (z^3 dx + 3xz^2\,dz) + (-4\,dy) + (2\,dz)$$
$$= d(y^2 \sin x + z^3 x - 4y - 2z)$$

$$\therefore \quad \Phi = y^2 \sin x + 2^3 x - 4y - 2z$$

[**Alternatively**, integrating partially

$$\frac{\partial \Phi}{\partial x} = y^2 \cos x + z^3, \quad \therefore \quad \Phi = y^2 \sin x + z^3 x + \Psi_1(y, z) \qquad \text{......(1)}$$

$$\frac{\partial \Phi}{\partial y} = 2y \sin x - 4, \quad \therefore \quad \Phi = y^2 \sin x - 4y + \Psi_2(x, z) \qquad \text{......(2)}$$

$$\frac{\partial \Phi}{\partial z} = 3xz^2 + 2, \quad \therefore \quad \Phi = z^3 x + 2z + \Psi_3(x, y) \qquad \text{......(3)}$$

Comparing (1), (2) and (3)

$$\Psi_1(y, z) = -4y + 2z, \quad \Psi_2(x, z) = z^3 x + 2z, \quad \Psi_3(x, y) = y^2 \sin x - 4y$$

$$\therefore \quad \Phi = y^2 \sin x + z^3 x - 4y + 2z \text{]}$$

(c) Now, work done $= \int_C \bar{F} \cdot d\bar{r}$

$$= \int_C (y^2 \cos x + z^3)\, dx + (2y \sin x - 4)\, dy + (3xz^2 + 2)\, dz$$

$$= \int_C d(y^2 \sin x + z^3 x - 4y + 2z) \text{ (as shown above)}$$

$$= \left[y^2 \sin x + z^3 x - 4y + 2z \right]_{(0,\,1,\,-1)}^{(\pi/2,\,-1,\,2)}$$

$$= \left[1 + 8\frac{\pi}{2} + 4 + 4 \right] - [-4 - 2] = 4\pi + 15$$

Example 5 : If the vector field $\bar{F}$ is irrotational find the constants a, b, c where $\bar{F}$ is given by

$$\bar{F} = (x + 2y + az)\, i + (bx - 3y - z)\, j + (4x + cy + 2z)\, k$$

Show that $\bar{F}$ can be expressed as the gradient of a scalar function. Then find the work done in moving a particle in this field from $(1, 2, -4)$ to $(3, 3, 2)$ along the straight line joining these points.

(M.U. 2004, 05, 06)

Sol. : (a) The field $\bar{F}$ is irrotational if curl $\bar{F} = 0$.

Now, curl $\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$

$$= (c + 1)\, i - (4 - a)\, j + (b - 2)\, k = \bar{0}$$

$$\therefore \quad a = 4, \; b = 2, \; c = -1$$

$$\therefore \quad \bar{F} = (x + 2y + 4z)\, i + (2x - 3y - z)\, j + (4x - y + 2z)\, k$$

(b) Since, now $\bar{F}$ is irrotational there exists a scalar function Φ such that $\bar{F} = \nabla \Phi$.

$$\therefore \quad (x + 2y + 4z)\, i + (2x - 3y - z)\, j + (4x - y + 2z)\, k = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k$$

$$\therefore \quad \frac{\partial \Phi}{\partial x} = x + 2y + 4z, \; \frac{\partial \Phi}{\partial y} = 2x - 3y - z, \; \frac{\partial \Phi}{\partial z} = 4x - y + 2z$$

$$\therefore \quad d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$$

$$\therefore \quad d\Phi = (x+2y+4z)\,dx + (2x-3y-z)\,dy + (4x-y+2z)\,dz$$

$$= x\,dx - 3y\,dy + 2z\,dz + 2(y\,dx + x\,dy) + 4(z\,dx + x\,dz) - (z\,dy + y\,dz)$$

$$= d\left[\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - zy\right]$$

$$\therefore \quad \Phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - zy$$

(c) Now, work done $= \int_C \bar{F} \cdot d\bar{r}$

$$= \int_C (x+2y+4z)\,dx + (2x-3y-z)\,dy + (4x-y+2z)\,dz$$

$$= \int_C d\left[\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - zy\right] \text{ (as above)}$$

$$= \left[\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - zy\right]_{(1,\,2,\,-4)}^{(3,\,3,\,2)}$$

$$= \left(\frac{9}{2} - \frac{27}{2} + 4 + 18 + 24 - 6\right) - \left(\frac{1}{2} - 6 + 16 + 4 - 16 + 8\right)$$

$$= \frac{49}{2}$$

Example 6 : Show that $\bar{F} = (ye^{xy}\cos z)\,i + (xe^{xy}\cos z)\,j - (e^{xy}\sin z)\,k$ is irrotational and find the scalar potential for $\bar{F}$ and evaluate $\int \bar{F} \cdot d\bar{r}$ along the curve joining the points (0, 0, 0) and $(-1, 2, \pi)$. **(M.U. 1996, 2001, 09)**

Sol. : (a) The field $\bar{F}$ is irrotational if curl $\bar{F} = 0$.

Now, $\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{xy}\cos z & xe^{xy}\cos z & -e^{xy}\sin z \end{vmatrix}$

$$= (-xe^{xy}\sin z + xe^{xy}\sin z)\,i + (-ye^{xy}\sin z + ye^{xy}\sin z)\,j$$
$$+ (xy\,e^{xy}\cos z + e^{xy}\cos z - xy\,e^{xy}\cos z - e^{xy}\cos z)\,k$$

$$= \bar{0}$$

$\therefore$ $\bar{F}$ is irrotational.

(b) Since, $\bar{F}$ is irrotational there exists a scalar function Φ such that $\bar{F} = \nabla\Phi$.

$$\therefore \quad (ye^{xy}\cos z)\,i + (xe^{xy}\cos z)\,j - e^{xy}\sin zk = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$$

$$\therefore \quad \frac{\partial\Phi}{\partial x} = ye^{xy}\cos z,\ \frac{\partial\Phi}{\partial y} = xe^{xy}\cos z,\ \frac{\partial\Phi}{\partial z} = -e^{xy}\sin z$$

$$\therefore \quad d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

$$\therefore \quad d\Phi = ye^{xy}\cos z\,dx + xe^{xy}\cos z\,dy - e^{xy}\sin z\,dz$$
$$= d(e^{xy}\cos z)$$
$$\therefore \quad \Phi = e^{xy}\cos z$$

[**Alternatively :** Integrating partially

$$\because \quad \frac{\partial \Phi}{\partial x} = ye^{xy}\cos z, \quad \Phi = e^{xy}\cos z + \Psi_1(y,z)$$

$$\because \quad \frac{\partial \Phi}{\partial y} = xe^{xy}\cos z, \quad \Phi = e^{xy}\cos z + \Psi_2(x,z)$$

$$\because \quad \frac{\partial \Phi}{\partial z} = -e^{xy}\sin z, \quad \Phi = e^{xy}\cos z + \Psi_3(x,y)$$

Comparing all these results, we should have

$$\Psi_1(y,z) = \Psi_2(x,z) = \Psi_3(x,y) = 0 \qquad \therefore \quad \Phi = e^{xy}\cos z \text{]}$$

(c) Now, work done $= \int_C \bar{F}\cdot d\bar{r} = \int_C d(e^{xy}\cos z) = \left[e^{xy}\cos z\right]_{(0,\,0,\,0)}^{(-1,\,2,\,\pi)}$

$$= e^{-2}\cos\pi - 1 = -\frac{1}{e^2} - 1 = -\frac{(e^2+1)}{e^2}.$$

Example 7 : Show that $\bar{F} = (2xyz^2)\,i + (x^2z^2 + z\cos yz)\,j + (2x^2yz + y\cos yz)\,k$ is conservative. Find scalar potential Φ such that $\bar{F} = \nabla\Phi$ and hence, find the work done by $\bar{F}$ in displacing a particle from $A\,(0, 0, 1)$ to $B\,(1, \pi/4, 2)$ along the straight line AB. **(M.U. 1997, 2006)**

Sol. : (a) The field $\bar{F}$ is irrotational if curl $\bar{F} = 0$.

$$\text{Now, curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz)\,i$$
$$+ (4xyz - 4xyz)\,j + (2xz^2 - 2xz^2)\,k$$
$$= \bar{0}$$

$\therefore$ $\bar{F}$ is irrotational.

(b) Since $\bar{F}$ is irrotational there exists a scalar function Φ, such that $\bar{F} = \nabla\Phi$.

$$\therefore \quad (2xyz^2)\,i + (x^2z^2 + z\cos yz)\,j + (2x^2yz + y\cos yz)\,k = \frac{\partial \Phi}{\partial x}i + \frac{\partial \Phi}{\partial y}j + \frac{\partial \Phi}{\partial z}k$$

$$\therefore \quad \frac{\partial \Phi}{\partial x} = 2xyz^2, \quad \frac{\partial \Phi}{\partial y} = x^2z^2 + z\cos yz, \quad \frac{\partial \Phi}{\partial z} = 2x^2yz + y\cos yz$$

$$\therefore \quad d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz$$
$$= 2xyz^2\,dx + x^2z^2\,dy + z\cos y z\,dy + 2x^2yz\,dz + y\cos z\,dz$$
$$= (2xyz^2dx + x^2z^2dy + 2x^2yz\,dz) + (z\cos yz\,dy + y\cos yz\,dz)$$
$$= d[x^2yz^2 + \sin yz]$$

$x^2yz^2 + x^1yz^2 + x^1yz^2$

$$\therefore \quad \Phi = x^2yz^2 + \sin yz$$

+

(c) Now, work done $= \int_C \bar{F} \cdot d\bar{r} = \int_C d(x^2yz^2 + \sin yz)$

$$= \left[x^2yz^2 + \sin yz\right]_{(0,\,0,\,1)}^{(1,\,\pi/4,\,2)} = \pi + 1.$$

EXERCISE - I

(A) **To Evaluate The Line Integral.**

1. Evaluate $\int_A^B (y\,dx + x\,dy)$ along $y = x^2$ form $A(0, 0)$ to $B(1, 1)$. **(M.U. 2001) [Ans. : 1]**

2. Evaluate $\int_C (x^2 - xy^2)\,dx + (y^2 - 2xy)\,dy$ where C is the square with the vertices (0, 0,), (2, 0), (2, 2), (0, 2). **[Ans. : – 16 / 3]**

3. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2 + y^2)\,i - 2xy\,j$ and C is the rectangle in the xy-plane bounded by $y = 0$, $x = a$, $y = b$, $x = 0$. **[Ans. : $-2ab^2$]**

4. Evaluate $\int_A^B (y^2 dx + xy\,dy)$ along $x = t^2$, $y = 2t$ from $A(1, -2)$ to $B(0, 0)$. **(M.U. 2002)** **[Ans. : 3]**

5. Evaluate $\int_A^B (3xy\,dx - y^2 dy)$ along the parabola $y = 2x^2$ from $A(0, 0)$ to $B(1, 2)$. What is the integral if the path is a straight line joining A to B? **(M.U. 2003) [Ans. : $-\frac{7}{6}, -\frac{2}{3}$]**

6. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (3x^2 + 6y)\,i - 14yzj + 20xz^2 k$ and C is the straight line joining (0, 0, 0) and (1, 1, 1). **(M.U. 2004) [Ans. : Put $x = t$, $y = t$, $z = t$ from $t = 0$ to $t = 1$; $\frac{13}{3}$]**

7. Prove that $\oint_C [(x^2 + 4)\,i + (y^2 - 4)\,j] \cdot d\bar{r} = 0$ where C is $x^2 + y^2 = 4$. **(M.U. 2001)**

8. Evaluate $\int_A^B (3x^2y - 2xy)\,dx + (x^3 - x^2)\,dy$ along $y^2 = 2x^3$ from $A(0, 0)$ and $B(2, 4)$. **(M.U. 2002) [Ans. : 16]**

9. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (3x^2 + 6y)\,i - 14yz\,j + 20xz^2\,k$ from (0, 0, 0) to (1, 1, 1) along the curve C, $x = t$, $y = t^2$, $z = t^3$. **(M.U. 1992) [Ans. : 5]**

10. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = x^2 i + xyj$ (i) from $O(0, 0)$ to $P(2, 2)$, (ii) along the x-axis from $x = 0$ to $x = 2$, (iii) along the line $x = 2$ from $y = 0$ to $y = 2$, (iv) along the parabola $y^2 = 2x$ from O to P. **[Ans. : (i) 16 / 3, (ii) 8 / 3, (iii) 4, (iv) 44 / 15]**

11. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2 - y)\,i + (y^2 + x)\,j$ (i) from $O(0, 1)$ to $P(1, 2)$ along the line OP, (ii) along the straight line from (0, 1) to (1, 1) and then along the line from (1, 1) to (1, 2), (iii) along the parabola $x = t,\ y = t^2 + 1$. [Ans. : (i) 5 / 3, (ii) 8 / 3, (iii) 2]

12. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = 3xi + (2xz - y)\,j + zk$ from (0, 0, 0) to (2, 1, 3) along the line joining the two points. [Ans. : 14]

13. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (3x^2 - 6yz)\,i + (2y + 3xz)\,j + (1 - 4xyz^2)\,k$ from (0, 0, 0) to (1, 1, 1). (i) along the straight line joining these points.
(ii) along $x = t,\ y = t^2,\ z = t^3$. [Ans. : (i) 6 / 5, (ii) 2]

14. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = yz\,i + (zx + 1)\,j + xy\,k$ along the line joining $A(1, 0, 0)$ to $B(2, 1, 4)$. (M.U. 2005) [Ans. : 9]

15. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = xy\,i + yz\,j + zx\,k$ and C is the portion of the curve $\bar{r} = t\,i + t^2 j + t^3 k$ from $t = -1$ to $t = 1$. [Ans. : 10/7]

16. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = z\,i + x\,j + y\,k$ and C is the arc of the curve $\bar{r} = \cos t\,i + \sin t\,j + t\,k$ from $t = 0$ to $t = 2\pi$. (M.U. 2003) [Ans. : 3π]

17. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ along the arc of the curve $\bar{r} = (e^t \cos t)\,i + (e^t \sin t)\,j$ from (1, 0) to $(e^{2\pi}, 0)$ where $\bar{F} = \dfrac{xi + yj}{(x^2 + y^2)^{3/2}}$. (M.U. 2001) [Ans. : $1 - e^{-2\pi}$]

18. A vector field is given by $\bar{F} = \sin y\,i + x(1 + \cos y)\,j$, evaluate the line integral over the circular path $x^2 + y^2 = a^2,\ z = 0$. (M.U. 2006) [Ans. : πa^2]

(B) To Find The Work Done

1. Find the work done in moving a particle once round the ellipse
$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ in the plane $z = 0$ in the force field given by
$\bar{F} = (3x - 4y + 2z)\,i + (4x + 2y - 3z^2)\,j + (2xz - 4y^2 + z^3)\,k$. [Ans. : 96π]

2. If $\bar{F}_1 = (3x + y)\,i - x\,j + (y - z)\,k,\ \bar{F}_2 = 2i - 3j + k$ evaluate
$\int_C (\bar{F}_1 \times \bar{F}_2) \cdot d\bar{r}$ around the circle $x^2 + y^2 = 4,\ z = 0$. [Ans. : -24π]

3. Find the work done in moving a particle once round the circle $x^2 + y^2 = 9,\ z = 0$ in the force field, $\bar{F} = (2x - y + z)\,i + (x + y - z^2)\,j + (3x - 2y + 4z)\,k$. [Ans. : 18π]

4. Find the work done in moving a particle from $A(1, 0, 1)$ to $B(2, 1, 2)$ along the straight line AB in the force field $\bar{F} = x^2 i + (x - y)\,j + (y + z)\,k$. (M.U. 2000) [Ans. : 13 / 3]

5. A particle situated at the origin moves along the parabola $y^2 = x$ to the point (1, 1) under the action of force $\overline{F} = (x^2 + y^2)\,i + (x^2 - y^2)\,j$. Find the work done. [**Ans. :** 7 / 10]

6. Calculate the work done when a force $\overline{F} = 3xy\,i - y^2\,j$ moves a particle in xy-plane from (0, 0) to (1, 2) along the parabola $y = 2x^2$. [**Ans. :** – 7 / 6]

7. Find the work done in moving a particle in a force field given by $\overline{F} = 3xz\,i - 4y\,j + zk$ along the curve $x = t^2 + 1$, $y = t^3$, $z = 2t + 3$ from A (1, 0, 3) to B (2, 1, 5).

(M.U. 2001) [**Ans. :** 25·9]

8. Prove that $\overline{F} = (4xy + 3x^2 z)\,i + (2x^2 - 2z)\,j + (x^3 - 2y)\,k$ is conser-vative. Find the work done in moving a particle from A (1, 0, 1) to B (2, 1, 1). **(M.U. 2002)** [**Ans. :** 13]

9. Show that $\overline{F} = (2xy + z^3)\,i + x^2\,j + 3z^2 x\,k$ is a conservative field. Find its scalar potential and also work done in moving a particle from (1, –2, 1) to (3, 1, 4). [**Ans. :** $\Phi = x^2 y + z^3 x$; 202]

10. A vector field is given by $\overline{F} = (x^2 - y^2 + x)\,i - (2xy + y)\,j$. Show that the field is irrotational and find its scalar potential. Also evaluate the line integral from (1, 2) to (2, 1).

[**Ans. :** $\Phi = \dfrac{x^3}{3} - xy + \dfrac{x^2}{2} - \dfrac{y^2}{2}$; $-\dfrac{22}{3}$]

11. Prove that the vector field given by $\overline{F} = 3x^2 y\,i + (x^3 - 2yz^2)\,j + (3z^2 - 2y^2 z)\,k$ is irrotational. Also find Φ such that $\overline{F} = \nabla\Phi$. Also evaluate the line integral from (2, 1, 1), (2, 0, 1).

(M.U. 2012) [**Ans. :** $\Phi = x^3 y - y^2 z^2 + z^3$; – 7]

12. Prove that the vector field given by $\overline{F} = (x^2 - yz)\,i + (y^2 - zx)\,j + (z^2 - xy)\,k$ is irrotational. Find its scalar potential. Also evaluate the line integral from (2, 0, 1), (1, 1, 0).

(M.U. 2002) [**Ans. :** $\Phi = \dfrac{1}{3}(x^3 + y^3 + z^3 - 3xyz)$; $-\dfrac{7}{3}$]

13. Find the values of a, b, c such that $\overline{F} = (axy + bz^3)\,i + (3x^2 - cz)\,j + (3xz^2 - y)\,k$ is irrotational. For these values of a, b, c, find the scalar potential Φ such that $\overline{F} = \nabla\Phi$ and Φ (1, 1, 1) = 1. **(M.U. 2004)** [**Ans. :** $\Phi = 3x^2 y + z^3 x - zy - 2$]

14. For the force field $\overline{F} = (3x^2 yz - 3y)\,i + (x^3 z - 3x)\,j + (x^3 y + 2z)\,k$, show that a scalar potential Φ exists such that $\overline{F} = \text{grad}\,\Phi$. Hence, find the work done in moving a partical from A (0, 0, 0) to B (1, 1, 1) along the line AB. [**Ans. :** $\Phi = x^3 yz - 3xy + z^2$; 1]

15. Prove that $\overline{F} = (2xy + z)\,i + (x^2 + 2yz^3)\,j + (3y^2 z^2 + x)\,k$ is irrotational. Find the scalar point function Φ such that $\overline{F} = \text{grad}\,\Phi$ and evaluate $\int_A^B \overline{F} \cdot d\overline{r}$ along the straight line joining A (1, 2, 0) to B (2, 2, 1). **(M.U. 1997)** [**Ans. :** $\Phi = x^2 y + zy + y^2 z^3$; 12]

16. Prove that $\overline{F} = (6xy^2 - 2z^3)\,i + (6x^2 y + 2yz)\,j + (y^2 - 6z^2 x)\,k$ is a conservative field. Find the scalar potential Φ such that $\nabla\Phi = \overline{F}$. Hence or otherwise find the work done by $\overline{F}$ in displacing a particle from A (1, 0, 2) to B (0, 1, 1) along the straight line AB.

[**Ans. :** Yes ; $\Phi = 3x^2 y^2 - 2z^3 x + y^2 z$; – 17]

17. A fluid motion is given by $\bar{v} = (y+z)\,i + (z+x)\,j + (x+y)\,k$. Is this motion irrotational ? If so find the velocity potential. [**Ans. :** Yes ; $\Phi = xy + yz + zx$]

18. Find the constant a so that $\bar{F} = (axy - z^3)\,i + (a-2)\,x^2\,j + (1-a)\,xz^2 k$ is a conservative field. Find its scalar potential and work done in moving a particle from $(1, 2, -3)$ to $(1, -4, 2)$ in the field. **(M.U. 2006)** [**Ans. :** $a = 4$, $\Phi = 2x^2 y - xz^3$; -47]

19. Prove that $\int_{(1,2)}^{(3,4)} (6xy^2 - y^3)\,dx + (6x^2 y - 3xy^2)\,dy$ is independent of the path joining the points $(1, 2)$, $(3, 4)$. [**Ans. :** $\Phi = 3x^2y^2 - y^3 x$]

20. Prove that $\bar{F} = 2xye^z\,i + x^2 e^z\,j + x^2 ye^z\,k$ is irrotational vector and find the corresponding scalar Φ such that $\bar{F} = \nabla\Phi$. Also find the work done in moving a partical in this field from $(0, 0, 0)$ to $(1, 1, 1)$. [**Ans. :** $\Phi = x^2 ye^z$; e]

21. Prove that $\bar{F} = (2xy + z)\,i + (x^2 + 2yz^3)\,j + (3y^2 z^2 + x)\,k$ is irrotational. Find the scalar point function Φ such that $\bar{F} = \text{grad}\,\Phi$ and evaluate $\int_A^B \bar{F} \cdot d\bar{r}$ along the straight line joining $A\,(1, 2, 0)$ to $B\,(2, 2, 1)$. **(M.U. 1997)** [**Ans. :** $\Phi = x^2 y + zy + y^2 z^3$; 12]

5. Green's Theorem

If P and Q are two functions of x, y and their partial derivatives $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous single valued functions over the closed region bounded by a curve C then

$$\boxed{\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.}$$ **(M.U. 1998)**

Proof : Let the region R be bounded by a simple closed curve C which is such that it can be cut by any line parallel to the axes in at the most two points. Consider the lines $x = d$, $x = b$, $y = a$, $y = c$ which touch the curve as shown in the figure.

Let the equations of the arcs DAB and BCD be $y = f_1(x)$ and $y = f_2(x)$. Let the equations of the arcs ABC and CDA be $x = f_3(y)$ and $x = f_4(y)$.

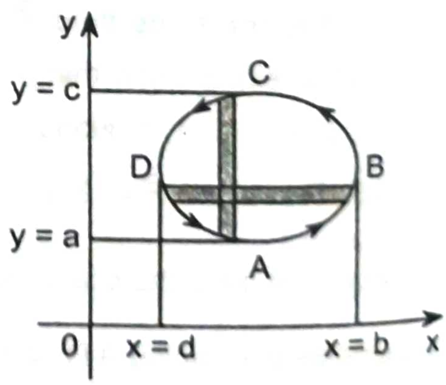


Fig. 9.5

Considering a strip parallel to the x-axis, we get,

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_a^c dy \int_{x=f_4(y)}^{x=f_3(y)} \frac{\partial Q}{\partial x}\,dx = \int_a^c [Q(x,y)]_{f_4(y)}^{f_3(y)}\,dy$$

$$= \int_a^c Q(f_3, y)\,dy - \int_a^c Q(f_4, y)\,dy$$

$$= \int_a^c Q(f_3, y)\,dy + \int_c^a Q(f_4, y)\,dy$$

$$= \int_{ABC} Q\,dy + \int_{CDA} Q\,dy = \int_C Q\,dy \qquad \text{.................... (1)}$$

Similarly, considering a strip parallel to the y-axis, we get

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_d^b dx \int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial P}{\partial y}\cdot dy = \int_d^b \left[P(x,y)\right]_{f_1(x)}^{f_2(x)} dx$$

$$= \int_d^b P(x,f_2)\,dx - \int_d^b P(x,f_1)\,dx = -\left[\int_b^d P(x,f_2)\,dx + \int_d^b P(x,f_1)\,dx\right]$$

$$= -\left[\int_{BCD} P\,dx + \int_{DAB} P\,dx\right] = -\int_C P\,dx \qquad \text{.................... (2)}$$

Subtracting (2) from (1), we get

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$ **(* Proof is not expected)**

George Green (1793 - 1841)

He was a British mathematical physicist. His father was a baker and owned a brick-windmill used to grind grain. George Green had only one year of formal education; he was almost entirely self-taught. He joined Cambridge at the age of fourty and became 4th Wrangler in 1838. His work was not well known to mathematical community during his lifetime. It was rediscovered and popularised by Lord Kelvin after death of Green. Green's theorem and functions were important tools in classical mechanics and were revised by Schwinger in 1948 which led to his 1965 Nobel prize. Albert Eienstein commented that Green had been 20 years ahead of his time. He is known for Green's Theorem, Green's Functions, Green's Identities, Green's measure, Green's matrix.

Corollary : Vector form of Green's Theorem

If we put $\bar{F} = Pi + Qj$ and $\bar{r} = xi + yj$ then Green's Theorem can be written as

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \iint_R \bar{N}\cdot(\nabla\times\bar{F})\,ds}$$

where $\bar{N}$ is the unit vector along the z-axis.

Proof : By Green's Theorem, we have

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad \text{.................... (1)}$$

Since, $\bar{r} = xi + yj \quad \therefore\ d\bar{r} = dx\,i + dy\,j$ .... (2)

$$\therefore\ P\,dx + Q\,dy = (Pi + Qj)\cdot(dx\,i + dy\,j) = \bar{F}\cdot d\bar{r}$$

Further, $$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = -i\frac{\partial Q}{\partial z} + j\frac{\partial P}{\partial z} + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\therefore \quad \overline{N} \cdot \nabla \times \overline{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \qquad \text{.......... (3)}$$

where for the sake of uniformity, we have used $\overline{N}$ as the unit vector along the z-axis.

$$\therefore \quad \overline{N} \cdot \overline{k} = 1$$

Using (2) and (3) in (1), we get

$$\int_C \overline{F} \cdot d\overline{r} = \iint_R \overline{N} \cdot (\nabla \times \overline{F})\, ds$$

(✱ Proof is not expected)

(A) To Evaluate The Integral By Using Green's Theorem

Example 1 : By using Green's Theorem, show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int (x\,dy - y\,dx)$. **(M.U. 1998)**

Sol. : By Green's Theorem $\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Now, put $P = -y$ and $Q = x$. $\quad \therefore \frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = -1$

$$\therefore \quad \int_C (x\,dy - y\,dx) = \iint_R (1+1)\,dx\,dy = \iint_R 2dx\,dy = 2A$$

$$\therefore \quad A = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

Example 2 : Find the area of (i) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ by applying Green's Theorem for a closed curve C in the xy-plane such that $\frac{1}{2}\int_C (x\,dy - y\,dx)$ is the area enclosed by C. **(M.U. 2003, 05)**

Sol. : (I) Consider the parametric equations of the ellipse as $x = a\cos\theta$, $y = b\sin\theta$. Hence, the area of the ellipse is

$$A = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_C [a\cos\theta\, b\cos\theta\, d\theta - (b\sin\theta)(-a\sin\theta\, d\theta)]$$

$$= \frac{1}{2}\int_C ab(\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_C ab\, d\theta = \frac{1}{2} ab \cdot [\theta]_0^{2\pi} = \frac{1}{2} \cdot ab \cdot 2\pi = \pi ab.$$

Cor. : If $a = b$ the curve is a circle and its area $= \pi a^2$.

(ii) Consider the parametric equations of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

$$\therefore \quad dx = -3a\cos^2\theta \sin\theta\, d\theta\,; \qquad dy = 3a\sin^2\theta\cos\theta\, d\theta$$

$$\therefore \quad A = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_C (a\cos^3\theta \cdot 3a\sin^2\theta\cos\theta\,d\theta)$$

$$- a\sin^3\theta\,(-3a\cos^2\theta\sin\theta)\,d\theta$$

$$= \frac{1}{2}\cdot 4 \cdot \int_0^{\pi/2} 3a^2(\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)\,d\theta$$

$$= 6a^2\left[\frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} + \frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right]$$

$$= 6a^2 \cdot \frac{\pi}{16} = \frac{3\pi a^2}{8}.$$

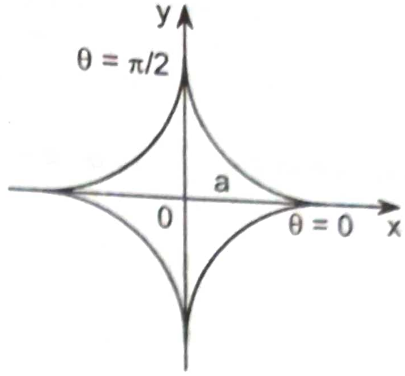


Fig. 9.6

[See Applied Mathematics - II, Chapter 2, § 7.]

Example 3 : Evaluate by Green's Theorem $\int_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$ where C is the rectangle whose vertices are $(0, 0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$.

Sol. : By Green's Theorem, $\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Now, $P = e^{-x}\sin y$, $Q = e^{-x}\cos y$

$$\therefore \quad \frac{\partial Q}{\partial x} = -e^{-x}\cos y, \quad \frac{\partial P}{\partial y} = e^{-x}\cos y$$

$$\therefore \quad \int_C (P\,dx + Q\,dy) = \iint_R (-2e^{-x}\cos y)\,dx\,dy = -2\int_{x=0}^{\pi}\int_{y=0}^{\pi/2} e^{-x}\cos y\,dx\,dy$$

$$= -2\int_0^{\pi} e^{-x}dx \int_0^{\pi/2}\cos y\,dy = -2\left[-e^{-x}\right]_0^{\pi}\left[\sin y\right]_0^{\pi/2}$$

$$= +2\left[e^{-\pi} - 1\right](1-0) = 2(e^{-\pi} - 1).$$

Example 4 : Evaluate by Green's Theorem $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = -xy\,(xi - yj)$ and C is $r = a(1 + \cos\theta)$. **(M.U. 1992, 2003, 06, 12)**

Sol. : By Green's Theorem, $\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Now, $\int_C \bar{F}\cdot d\bar{r} = \int_C (-x^2y\,i + xy^2\,j)\cdot(dx\,i + dy\,j) = \int_C (-x^2y\,dx + xy^2\,dy)$

By Comparison $P = -x^2y$, $Q = xy^2$

$$\therefore \quad \frac{\partial Q}{\partial x} = y^2, \quad \frac{\partial P}{\partial y} = -x^2 \qquad \therefore \quad \int_C \bar{F}\cdot d\bar{r} = \iint_R (y^2 + x^2)\,dx\,dy$$

To evaluate the integral, we put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$. For the Cardioid $r = a(1+\cos\theta)$, we take the integral from $\theta = 0$ to $\theta = \pi$, $r = 0$ to $r = a(1+\cos\theta)$ twice.

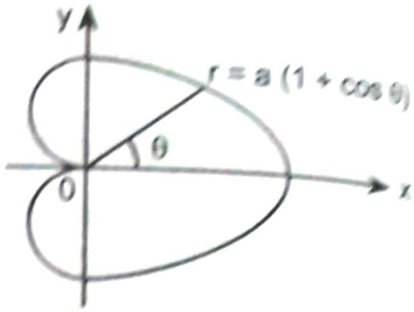


Fig. 9.7

$$\int_C \bar{F}\cdot d\bar{r} = \iint_R r^2\cdot r\,dr\,d\theta = 2\int_{\theta=0}^{\pi}\int_{r=0}^{r=a(1+\cos\theta)} r^3\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_0^{\pi} a^4(1+\cos\theta)^4\,d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi}\left[2\cos^2(\theta/2)\right]^4 d\theta = 8a^4\int_0^{\pi}\cos^8\left(\frac{\theta}{2}\right)d\theta$$

Put $\dfrac{\theta}{2} = t$, $d\theta = 2dt$ and use reduction formula.

[See Applied Mathematics - II, Chapter 2, § 7.]

$$\therefore\quad I = 8a^4\int_0^{\pi/2} 2\cos^8 t\,dt = 16a^4\left[\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{35\pi}{16}a^4$$

Example 5 : Using Green's theorem evaluate $\int_C \bar{F}\cdot d\bar{r}$ where C is the curve enclosing the region bounded by $y^2 = 4ax$, $x = a$ in the plane $z = 0$ and

$$\bar{F} = (2x^2y + 3z^2)\,i + (x^2 + 4yz)\,j + (2y^2 + 6xz)\,k. \qquad \textbf{(M.U. 2003)}$$

Sol. : In the plane $z = 0$, we have

$$\bar{F} = 2x^2y\,i + x^2\,j + 2y^2k \quad\text{and}\quad d\bar{r} = dx\,i + dy\,j$$

$$\therefore\quad \bar{F}\cdot d\bar{r} = 2x^2y\,dx + x^2\,dy$$

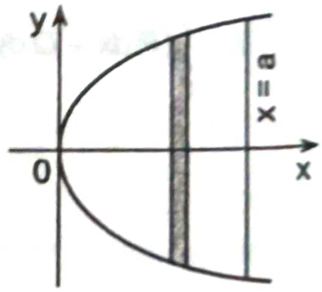


Fig. 9.8

By Greens Theorem $\displaystyle\int_C (P\,dx + Q\,dy) = \iint_R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

Here, $P = 2x^2y$, $Q = x^2$. $\quad\therefore\ \dfrac{\partial Q}{\partial x} = 2x$, $\dfrac{\partial P}{\partial y} = 2x^2$

$$\therefore\quad \int_C (P\,dx + Q\,dy) = \int_0^a\int_{-2\sqrt{ax}}^{2\sqrt{ax}} (2x - 2x^2)\,dx\,dy$$

$$= 2\int_0^a\int_0^{2\sqrt{ax}} 2\cdot(x - x^2)\,dx\,dy = 4\int_0^a (x - x^2)[y]_0^{2\sqrt{ax}}\,dx$$

$$= 4\int_0^a (x - x^2)\,2\sqrt{ax}\,dx = 8\int_0^a (a^{1/2}x^{3/2} - a^{1/2}x^{5/2})\,dx$$

$$= 8\left[a^{1/2}\frac{x^{5/2}}{5/2} - a^{1/2}\frac{x^{7/2}}{7/2}\right]_0^a = 8\left[\frac{2}{5}\cdot a^3 - \frac{2}{7}a^4\right]$$

$$= \frac{16a^3}{35}(7 - 5a).$$

Example 6 : Using Green's Theorem evaluate

$$\oint_C (e^{x^2} - xy)\,dx - (y^2 - ax)\,dy \quad \text{where } C \text{ is the circle } x^2 + y^2 = a^2. \qquad \textbf{(M.U. 2003)}$$

Sol. : By Green's Theorem $\displaystyle\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Here $P = e^{x^2} - xy,\ Q = -(y^2 - ax) \quad \therefore \dfrac{\partial Q}{\partial x} = a,\ \dfrac{\partial P}{\partial y} = -x.$

$$\therefore \oint_C (e^{x^2} - xy)\,dx - (y^2 - ax)\,dy = \iint_R (a + x)\,dx\,dy$$

where, R is the area of the circle $x^2 + y^2 = a^2$.

To change to polar coordinates, we put $x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$.

$$\therefore\ I = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (a + r\cos\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{ar^2}{2} + \frac{r^3}{3}\cos\theta\right]_0^a d\theta$$

$$= \int_0^{2\pi} \left(\frac{a^3}{2} + \frac{a^3}{3}\cos\theta\right) d\theta = \left[\frac{a^3}{2}\theta + \frac{a^3}{3}\sin\theta\right]_0^{2\pi} = a^3 \cdot \pi.$$

(B) To Verify Green's Theorem

Example 1 : Verify Green's theorem for $\overline{F} = x^2 i - xy\,j$ and C is the triangle having vertices A (0, 2), B (2, 0), C (4, 2). **(M.U. 2001, 10)**

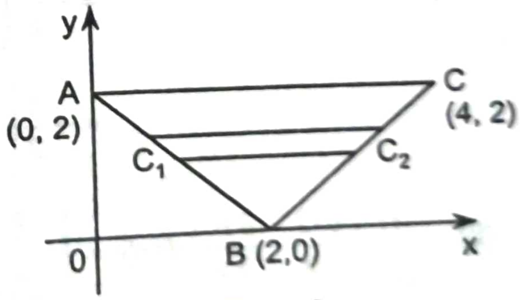


Fig. 9.9

Sol. : By Green theorem.

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here, $P = x^2,\ Q = -xy \quad \therefore \dfrac{\partial Q}{\partial x} = -y,\ \dfrac{\partial P}{\partial y} = 0.$

(a) Along AB, since the equation of AB is

$$\frac{y-2}{2-0} = \frac{x-0}{0-2},\quad y = 2 - x.$$

Putting $P = x^2$ and $Q = -xy = -x(2-x)$, $dy = -dx$.

$$\int_{C_1} P\,dx + Q\,dy = \int_0^2 [x^2 + x(2-x)]\,dx = \int_0^2 2x\,dx = \left[x^2\right]_0^2 = 4$$

Along BC, since the equation of BC, $\dfrac{y-0}{0-2} = \dfrac{x-2}{2-4}$ i.e., is $y = x - 2$.

$$\iint_{C_2} (P\,dx + Q\,dy) = \int_2^4 (x^2 - x^2 + 2x)\,dx$$

$$\iint_{C_2} (P\,dx + Q\,dy) = \int_2^4 2x\,dx = \left[x^2\right]_2^4 = 12$$

Along CA, since the equation of CA is $y = 2$, $dy = 0$.

$$\iint_{C_3} (P\,dx + Q\,dy) = \int_4^0 x^2\,dx = \left[\frac{x^3}{3}\right]_4^0 = -\frac{64}{3}$$

$$\therefore \int_C (P\,dx + Q\,dy) = 16 - \frac{64}{3} = -\frac{16}{3}. \qquad \text{.......... (1)}$$

(b) $$\iint_{ABC} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{ABC} -y\,dx\,dy = \int_0^2 \int_{2-y}^{2+y} -y\,dx\,dy = \int_0^2 -y[x]_{2-y}^{2+y}\,dy$$

$$= -\int_0^2 y[2 + y - 2 + y]\,dy = -\int_0^2 2y^2 dy$$

$$= -2\left[\frac{y^3}{3}\right]_0^2 = -\frac{16}{3}. \qquad \text{.......... (2)}$$

From (1) and (2), the theorem is verified.

Alternatively : We may calculate the above double integral by taking strips parallel to the y-axis. Then,

(i) $$\iint_{ABD} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^2 \int_{2-x}^{2} -y\,dx\,dy = -\int_0^2 \left[\frac{y^2}{2}\right]_{2-x}^{2} dx$$

$$= -\frac{1}{2}\int_0^2 \left[4 - (2-x)^2\right] dx = -\frac{1}{2}\int_0^2 (4x - x^2)\,dx$$

$$= -\frac{1}{2}\left[2x^2 - \frac{x^3}{3}\right]_0^2 = -\frac{1}{2}\left[8 - \frac{8}{3}\right] = -\frac{8}{3}$$

(ii) $$\iint_{BCD} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_2^4 \int_{x-2}^{2} -y\,dx\,dy = -\int_2^4 \left[\frac{y^2}{2}\right]_{x-2}^{2} dx$$

$$= -\frac{1}{2}\int_2^4 \left[4 - (x-2)^2\right] dx = -\frac{1}{2}\int_2^4 (4x - x^2)\,dx$$

$$= -\frac{1}{2}\left[2x^2 - \frac{x^3}{3}\right]_2^4 = -\frac{1}{2}\left[\left(32 - \frac{64}{3}\right) - \left(8 - \frac{8}{3}\right)\right]$$

$$= -\frac{1}{2}\left[24 - \frac{56}{3}\right] = -\frac{1}{2}\cdot\frac{16}{3} = -\frac{8}{3}.$$

$$\therefore \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -\frac{16}{3} \text{ as before.}$$

Example 2 : Verify Green's Theorem for $\int_C \left(\frac{1}{y}dx + \frac{1}{x}dy\right)$ where C is the boundary of the region defined by $x = 1$, $x = 4$, $y = 1$ and $y = \sqrt{x}$.

Sol. : By Green's Theorem (M.U. 1994, 99, 2005)

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here, $P = \frac{1}{y}$, $Q = \frac{1}{x}$, $\frac{\partial Q}{\partial x} = -\frac{1}{x^2}$, $\frac{\partial P}{\partial y} = -\frac{1}{y^2}$. The region of integration is ABC.

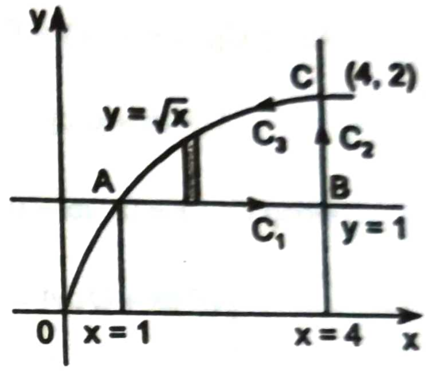


Fig. 9.10

(a) Along C_1, $y = 1$, $dy = 0$

$$\therefore \int_{C_1} (P\,dx + Q\,dy) = \int_1^4 \left(\frac{1}{y}dx + \frac{1}{x}0\right) = \int_1^4 dx = [x]_1^4 = 3$$

Along C_2, $x = 4$, $dx = 0$

$$\therefore \int_{C_2} (P\,dx + Q\,dy) = \int_1^2 \left(\frac{1}{y}0 + \frac{1}{x}dy\right) = \int_1^2 \frac{dy}{4} = \frac{1}{4}[y]_1^2 = \frac{1}{4}$$

Along C_3, $y = \sqrt{x}$ $\therefore$ $dy = \frac{1}{2\sqrt{x}}dx$

$$\int_{C_3} (P\,dx + Q\,dy) = \int_4^1 \left(\frac{1}{y}dx + \frac{1}{x}dy\right) = \int_4^1 \left(\frac{1}{\sqrt{x}} + \frac{1}{2x^{3/2}}\right) dx$$

$$= \left[2\sqrt{x} - x^{-x/2}\right]_4^1 = \left[(2-1) - \left(4 - \frac{1}{2}\right)\right] = -\frac{5}{2}$$

$$\therefore \int_C (P\,dx + Q\,dy) = 3 + \frac{1}{4} - \frac{5}{2} = \frac{3}{4} \qquad \text{.................... (1)}$$

(b) $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_1^4 \int_1^{\sqrt{x}} \left(-\frac{1}{x^2} + \frac{1}{y^2}\right) dx\,dy = \int_1^4 \left[-\frac{y}{x^2} - \frac{1}{y}\right]_1^{\sqrt{x}} dx$$

$$= \int_1^4 \left[-\frac{1}{x^{3/2}} - \frac{1}{\sqrt{x}} + \frac{1}{x^2} + 1\right] dx = \left[2x^{-1/2} - 2\sqrt{x} - \frac{1}{x} + x\right]_1^4$$

$$= \left(1 - 4 - \frac{1}{4} + 4\right) - (2 - 2 - 1 + 1) = \frac{3}{4} \qquad \text{.................... (2)}$$

From (1) and (2) the theorem is verified.

Example 3 : Verify Green's Theorem in the plane for $\oint (x^2 - y)\,dx + (2y^2 + x)\,dy$ around the boundary of the region defined by $y = x^2$ and $y = 4$. (M.U. 1996, 2006)

Sol. : By Green's Theorem $\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Here, $P = x^2 - y$, $Q = 2y^2 + x$, $\therefore$ $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial P}{\partial y} = -1$.

(a) Along C_1, $y = x^2$ $\therefore$ $dy = 2x\,dx$

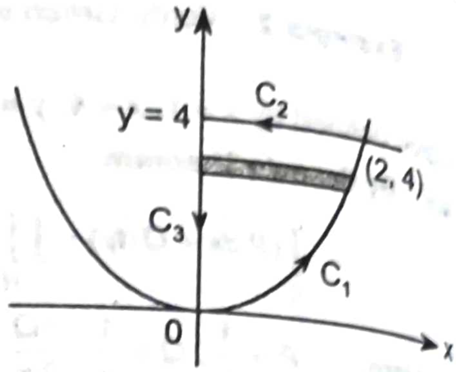


Fig. 9.11

$$\therefore \int_{C_1} (P\,dx + Q\,dy) = \int_0^2 \left[(x^2 - x^2) + (2x^4 + x)\,2x\right] dx$$

$$= \int_0^2 (4x^5 + 2x^2)\,dx = \left[\frac{2x^6}{3} + \frac{2x^3}{3}\right]_0^2$$

$$\therefore \int_{C_1} (P\,dx + Q\,dy) = \frac{128}{3} + \frac{16}{3} = \frac{144}{3}.$$

Along C_2, $y = 4$, $dy = 0$.

$$\int_{C_2} (P\,dx + Q\,dy) = \int_2^0 (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_2^0 = \left[0 - \left(\frac{8}{3} - 8\right)\right] = \frac{16}{3}$$

Along C_3, $x = 0$, $dx = 0$.

$$\int_C (P\,dx + Q\,dy) = \int_4^0 (2y^2)\,dy = \left[\frac{2y^3}{3}\right]_4^0 = 0 - \frac{2}{3} \cdot 64 = -\frac{128}{3}$$

$$\therefore \int_C (P\,dx + Q\,dy) = \frac{144}{3} + \frac{16}{3} - \frac{128}{3} = \frac{32}{3} \qquad \text{.................... (1)}$$

(b) $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint (1+1)\,dx\,dy = \int_0^4 \int_0^{\sqrt{y}} 2\,dx\,dy = 2\int_0^4 [x]_0^{\sqrt{y}}\,dy$$

$$= 2\int_0^4 \sqrt{y}\,dy = 2 \cdot \frac{2}{3}\left[y^{3/2}\right]_0^4 = \frac{4}{3} \cdot [8] = \frac{32}{3} \qquad \text{.................... (2)}$$

From (1) and (2), the theorem is proved.

Example 4 : Verify Green's Theorem in the plane for $\int_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. **(M.U. 1994, 98, 2001, 03, 05, 09)**

Sol. : By Green's Theorem

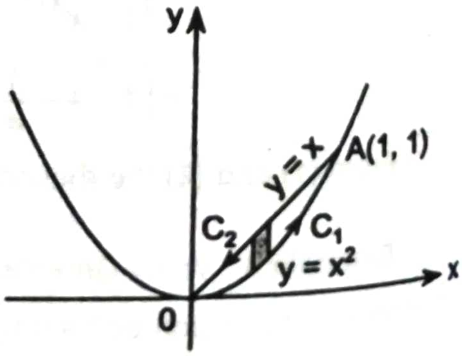


Fig. 9.12

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here, $P = xy + y^2$, $Q = x^2$

$$\therefore \frac{\partial Q}{\partial x} = 2x, \frac{\partial P}{\partial y} = x + 2y$$

(a) Along C_1, $y = x^2$ and $dy = 2x\,dx$ and x varies from 0 to 1,

$$\therefore \int_{C_1} (P\,dx + Q\,dy) = \int_0^1 (x \cdot x^2 + x^4)\,dx + x^2 \cdot 2x\,dx$$

$$= \int_0^1 (x \cdot x^2 + x^4)\,dx + x^2 \cdot 2x\,dx$$

$$\therefore \int_{C_1} (P\,dx + Q\,dy) = \int_0^1 (3x^3 + x^4)\,dx = \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}$$

Along C_2, $y = x$ and $dy = dx$ and x varies from 1 to 0

$$\therefore \int_{C_2} (P\,dx + Q\,dy) = \int_1^0 (x \cdot x + x^2)\,dx + x^2\,dx = \int_1^0 3x^2\,dx$$

$$\therefore \int_C (P\,dx + Q\,dy) = \left[x^3\right]_1^0 = -1 = \frac{19}{20} - 1 = -\frac{1}{20} \qquad \text{.................... (1)}$$

(b) $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dx\,dy$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx = \int_0^1 \left[(x^2 - x^2) - (x^3 - x^4)\right] dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right] = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20} \qquad \text{.................... (2)}$$

From (1) and (2), the theorem is verified.

(C) To Find The Work Done

Example : Find the work done in moving a particle once round the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ in the plane $z = 0$ in the force field given by $\bar{F} = (3x - 2y)\,i + (2x + 3y)\,j + y^2 k$ by using Green's Theorem.

(M.U. 2005)

Sol. : We have

$$\text{Work done} = \int_C \bar{F} \cdot d\bar{r} = \int_C (3x - 2y)\,dx + (2x + 3y)\,dy + y^2\,dz.$$

In the plane $z = 0$, $dz = 0$.

$$\therefore \text{Work done} = \int_C (3x - 2y)\,dx + (2x + 3y)\,dy = \int_C P\,dx + Q\,dy$$

By Green's Theorem,

$$\text{Work done} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here $P = 3x - 2y$ and $Q = 2x + 3y$.

$$\therefore \frac{\partial Q}{\partial x} = 2 \text{ and } \frac{\partial P}{\partial y} = -2.$$

$$\therefore \text{Work done} = \iint_R (2 + 2)\,dx\,dy = 4 \iint_R dx\,dy$$

$$= 4 \cdot \text{Area of the ellipse}$$

$$= 4 \cdot \pi \cdot 4 \cdot 3 = 48\,\pi.$$

EXERCISE - II

(A) To Evaluate The Integral By Using Green's Theorem

1. Evaluate $\int_C \left[(x^2 y \cos x + 2xy \sin x - y^2 e^x)\, dx + (x^2 \sin x - 2y\, e^x)\, dy\right]$ where C is the closed curve $x^{2/3} + y^{2/3} = a^{2/3}$. **(M.U. 2001)** [**Ans. :** 0]

2. If C is the triangle formed by $y = 2x$, $2x + y = 8$ and $x = 0$, find

$$\int_C (x\, dy - y\, dx)$$ **(M.U. 2002)** [**Ans. :** 16]

3. Evaluate $\oint_C \left(\frac{y}{x^2 + y^2}\right) dx - \left(\frac{x}{x^2 + y^2}\right) dy$ by Green's Theorem. **(M.U. 2002, 05)** [**Ans. :** 0]

4. Using Green's Theorem in plane evaluate $\oint_C x^2 (1 - y)\, dx + (2y + 1)\, dy$ where C is the boundary of the region defined by $x^2 + y^2 = a^2$. **(M.U. 2002)** [**Ans. :** $\pi a^2 / 4$]

5. Using Green's Theorem evaluate $\int_C (x^2 - y^3)\, dx + (x^2 + y^2)\, dy$ where C is $x^2 + 4y^2 = 64$.
(M.U. 2005)

6. Evaluate by Green's Theorem $\int_C \left[(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy\right]$ where C is the boundary of the region bounded by (i) $y = \sqrt{x}$ and $y = x$. (ii) $y = \sqrt{x}$, $y = x^2$.
(M.U. 2003, 05) [**Ans. :** (i) 5 / 6, (ii) 3 / 2]

7. Evaluate by Green's Theorem $\int_C (2x^2 - y^2)\, dx + (x^2 + y^2)\, dy$ where C is the boundary of the surface in the xy-plane enclosed by the x-axis and the semi-circle $y = \sqrt{1 - x^2}$. [**Ans. :** 4 / 3]

8. Using Green's Theorem evaluate $\int_C x^2 y\, dx + y^3 dy$ where C is the closed path formed by $y = x$, $y = x^2$ from (0, 0) to (1, 1) to (0, 0) traversed in the positive sense. **(M.U. 1994)**
[**Ans. :** $-1/20$]

9. Using Green's Theorem evaluate $\int_C \left[(x^2 - \cosh y)\, dx + (y + \sin x)\, dy\right]$ where C is the plane rectangle with vertices (0, 0), $(\pi, 0)$, $(\pi, 1)$, (0, 1). **(M.U. 2006)**
[**Ans. :** $\pi(\cosh 1 - 1)$]

10. Using Green's Theorem evaluate $\int_C (2x^2 - y^2)\, dx + (x^2 + y^2)\, dy$ where C is the boundary of the surface enclosed by the lines $x = 0$, $y = 0$, $x = 2$, $y = 3$.
(M.U. 1999, 2006, 14) [**Ans. :** 13]

11. Using Green's Theorem evaluate $\int_C (2x - 5y)\, dx - (2y - 3x)\, dy$ when C is the parallelogram having vertices (0, 0), (2, 0), (4, 2), (2, 2). **(M.U. 2000)** [**Ans. :** 32]

13. Use Green's Theorem to evaluate $\int_C (x^2 + xy)\,dx + (x^2 + y^2)\,dy$ where C is the square formed by the line $x = \pm 1$, $y = \pm 1$. **(M.U. 2004) [Ans. : 6]**

13. Evaluate by Green's Theorem $\oint_C (e^{2x} - xy^2)\,dx + (ye^x + y^2)\,dy$ where C is the closed curve bounded by $y^2 = x$ and $x^2 = y$. **(M.U. 2002) [Ans. : 1 / 6]**

(B) To Verify Green's Theorem

1. Verify Green's Theorem for $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2 - y^2)\,i + (x + y)\,j$ and C is the triangle with vertices (0, 0), (1, 1), (2, 1). **(M.U. 2000, 02, 04, 09) [Ans. : 7 / 6]**

2. Verify Green's Theorem for $\int_C (2xy - x^2)\,dx + (x + y^2)\,dy$ where C is the closed curve in the xy-plane bounded by $y = x^2$ and $x = y^2$. **[Ans. : 1 / 30]**

3. Verify Green's Theorem for $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2 - xy)\,i + (x^2 - y^2)\,j$ and C is the triangle whose vertices are (0, 0), (1, 1) and (1, −1). **(M.U. 2002) [Ans. : 2]**

4. Verify Green's Theorem for $\oint_C [(y - \sin x)\,dx + \cos x\,dy]$ where C is the boundary of the $\Delta\,OAB$

(a) whose vertices are $O\,(0, 0)$, $A\,(\pi/2, 0)$, $B\,(\pi/2, 1)$. **(M.U. 2003, 06)**

(b) whose sides are $y = 0$, $x = \dfrac{\pi}{2}$, $y = \dfrac{2x}{\pi}$. **(M.U. 1994, 95, 2007)**

[Ans. : (a) & (b) $-\dfrac{2}{\pi} - \dfrac{\pi}{4}$]

5. Verify Green's Theorem for $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = x^2 i + xy\,j$ and C is the triangle whose vertices are (0, 2), (2, 0) and (4, 2). **(M.U. 2003) [Ans. : 16 / 3]**

6. Verify Green's Theorem for $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2 - xy)\,i + (x^2 - y^2)\,j$ and C is the closed curve bounded by $x^2 = 2y$ and $x = y$. **(M.U. 2000) [Ans. : 5 / 8]**

7. Verify Green's Theorem in plane for $\oint_C x^2(1 - y)\,dx + (2y + 1)\,dy$ where C is the boundary of the region defined by $x^2 + y^2 = a^2$. **(M.U. 2002) [Ans. : $\pi a^2 / 4$]**

8. Verify Green's Theorem for $\int_C (x^2 - y^3)\,dx + (x^2 + y^2)\,dy$ where C is $x^2 + 4y^2 = 64$. **(M.U. 2005)**

9. Verify Green's Theorem for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region defined by $x = 0$, $y = 0$, $x + y = 1$. **(M.U. 2014)**

(C) To Find The Work Done

1. Find the work done by $\bar{F} = (4x - 2y)\,i + (2x - 4y)\,j$ in moving a particle once counter clockwise around the circle $(x - 2)^2 + (y - 2)^2 = 4$ by using Green's Theorem. **[Ans. : 16 π]**

6. Stoke's Theorem

The integral of the normal component of the curl of a vector $\overline{F}$ over a surface S is equal to the line integral of the tangential component of $\overline{F}$ around the curve bounding S *i.e.*

$$\boxed{\iint_S \overline{N} \cdot (\nabla \times \overline{F})\, ds = \int_C \overline{F} \cdot d\bar{r}}$$

where $\overline{N}$ is the unit <u>outward normal</u> vector to the element *dS*.

Let *S* be the surface and the closed curve *C* be its boundary. If $\overline{N}$ is the unit normal to *dS* at point *P* then surface integral of the normal component of curl $\overline{F}$ is equal to line integral of $\overline{F} \cdot d\bar{r}$ over the curve *C*.

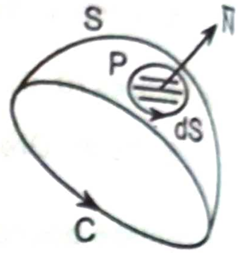


Fig. 9.13

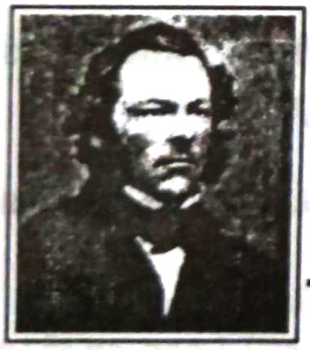
We shall accept this theorem without proof.

However, we shall see the validity of this important theorem by varyfing it in examples 5 to 11.

Procedure

1. First find the direction ratios and then the direction cosines of the outward normal to the given surface and then write the unit normal vector $\overline{N}$.
2. Then find the curl of the given vector $\overline{F}$ *i.e.* find $\nabla \times \overline{F}$.
3. Then find the dot product of the above two vectors *i.e.* find $\overline{N} \cdot (\nabla \times \overline{F})$.
4. Then find the required integral. In examples 4, 5, 6 and 7, $ds = dx\, dy$ and the integral is easy to calculate. In example 10, we use $ds = dx\, dy = r\, d\theta\, dr$ and polar coordinates because the boundaries are circles.

George Gabriel Stokes (1819 - 1903)



He studied at Cambridge and was a Senior Wrangler (the top student in the class) and first Smith's prizeman. Immediately after this he became the Lucasian Professor at Cambridge in 1849, a post previously held by Newton and the one which Stokes held until his death in 1903. He wrote various papers and developed the ideas proposed by many contemporary mathematicians such as Lagrange, Laplace, Fourier, Poisson and Cauchy. He was secretary of the Royal Society of London for thirty year and president for five years, was president of the Victoria Institute created in 1865 to explore the relationship between religion and science. His measure publications include *Mathematical and Physical Papers, On Light (Buruett Lectures), Natural Theology, Röntgen Rays*, etc.

Example 1 : Deduce Green's Theorem from Stokes Theorem.

Sol. : By Stokes Theorem, we have $\iint_S \overline{N} \cdot \nabla \times \overline{F}\, ds = \int_C \overline{F} \cdot d\bar{r}$ .... (1)

If $\overline{F} = Pi + Qj$ and $\bar{r} = xi + yj$

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$\therefore \quad \nabla \times \overline{F} = -i\frac{\partial Q}{\partial z} + j\frac{\partial P}{\partial z} + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

If $\overline{N}$ is a unit normal vector to the x-y plane (*i.e.*, $\overline{N} = k$)

$$\overline{N} \cdot \nabla \times \overline{F} = \overline{N} \cdot \left[-i\frac{\partial Q}{\partial z} + j\frac{\partial P}{\partial z} + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right]$$

$$= \frac{\partial \Phi}{\partial x} - \frac{\partial P}{\partial y} \qquad \left[\because \ \overline{N} \cdot k = 1\right]$$

And $\quad \overline{F} \cdot d\overline{r} = (Pi + Qj) \cdot (dx\, i + dy\, j) = P\,dx + Q\,dy$

Hence, from (1), we get,

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C (P\,dx + Q\,dy) \text{ which is Green's Theorem.}$$

Example 2 : From Stoke's Theorem deduce that curl grad $\Phi = 0$.

Sol. : Suppose S be an open surface bounded by a simple closed curve C. We have by Stoke's Theorem

$$\iint_S \overline{N} \cdot \nabla \times \overline{F}\, ds = \int \overline{F} \cdot d\overline{r}$$

Now, let $\overline{F} = \text{grad}\,\Phi \qquad \therefore \quad \iint_S \overline{N} \cdot \nabla \times \text{grad}\,\Phi\, ds = \int_C \text{grad}\,\Phi \cdot d\overline{r}$ .... (1)

Now, $\quad \text{grad}\,\Phi \cdot d\overline{r} = \left(i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}\right) \cdot (i\,dx + j\,dy + k\,dz)$

$$= \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz = d\Phi$$

$\therefore$ From (1), $\quad \iint_S N \cdot \text{curl grad}\,\Phi\, ds = \int_C d\Phi = 0$

Since, S is arbitrary curl grad $\Phi = 0$.

Example 3 : Prove that $\iint_S \overline{N} \cdot \nabla\Phi\, ds = \int_C \Phi \cdot d\overline{r}$.

Sol. : We have by Stoke's Theorem,

$$\iint_S \overline{N} \cdot \nabla \times \overline{F}\, ds = \int_C \overline{F} \cdot d\overline{r} \qquad \text{............ (1)}$$

Let $\overline{F} = \Phi\overline{a}$ where Φ is a continuously differentiable scalar point function and $\overline{a}$ is an arbitrary constant vecotor *i.e.* $\Phi\overline{a} = \Phi a_1 i + \Phi a_2 j + \Phi a_3 k$.

Now, $\quad \nabla \times \overline{F} = \nabla \times \Phi\overline{a} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \Phi a_1 & \Phi a_2 & \Phi a_3 \end{vmatrix}$

$$= \left(a_3\frac{\partial \Phi}{\partial y} - a_2\frac{\partial \Phi}{\partial z}\right)i + \left(a_1\frac{\partial \Phi}{\partial z} - a_3\frac{\partial \Phi}{\partial y}\right)j + \left(a_2\frac{\partial \Phi}{\partial x} - a_1\frac{\partial \Phi}{\partial y}\right)k$$

$$\therefore \quad \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \nabla\Phi \times \bar{a}$$

$$\therefore \quad \bar{N} \cdot \nabla \times \bar{F} = \bar{N} \cdot \nabla\Phi \times \bar{a} = \bar{N} \times \nabla\Phi \cdot \bar{a} = \bar{a} \cdot \bar{N} \times \nabla\Phi$$

Hence, from (1), we get,

$$\iint \bar{a} \cdot \bar{N} \times \nabla\Phi \, ds = \int_C \Phi \bar{a} \cdot d\bar{r} \quad \therefore \quad \bar{a} \cdot \left[\iint_R \bar{N} \times \nabla\Phi \, ds - \int_C \Phi \, d\bar{r} \right] = 0$$

But $\bar{a} \neq 0 \quad \therefore \quad \iint_R \bar{N} \times \nabla\Phi \, ds = \int_C \Phi \, d\bar{r}$

Example 4 : Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = x^2 i + xy\, j$ and C is the boundary of the rectangle $x = 0$, $y = 0$, $x = a$, $y = b$. **(M.U. 2006, 09)**

Sol. : We have by Stoke's Theorem

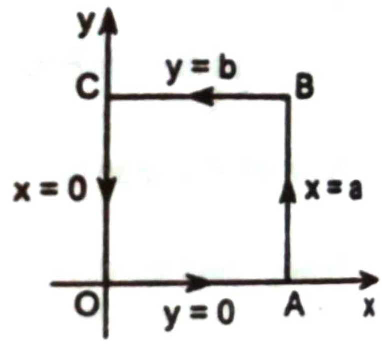


Fig. 9.14

$$\int_C \bar{F} \cdot d\bar{r} = \iint_C \bar{N} \cdot \nabla \times \bar{F} \, ds$$

$$\text{Now,} \quad \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = yk.$$

In the xy plane, $ds = dx\,dy$, $\bar{N} = k$

$$\therefore \quad \iint_S \bar{N} \cdot \nabla \times \bar{F} \, ds = \iint_S k \cdot yk \, dx\, dy = \int_0^a \int_0^b y \, dx\, dy$$

$$= \int_0^a \left[\frac{y^2}{2} \right]_0^b dx = \int_0^a \frac{b^2}{2} dx = \frac{b^2}{2} [x]_0^a = \frac{ab^2}{2}$$

$$\therefore \quad \int_C \bar{F} \cdot d\bar{r} = \frac{ab^2}{2}.$$

Example 5 : Evaluate by Stoke's Theorem $\int_C (xy\,dx + xy^2\,dy)$ where C is the square in the xy-plane with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. **(M.U. 1987, 93, 2014)**

Sol. : By Stoke's Theorem, $\int_C \bar{F} \cdot d\bar{r} = \iint_C \bar{N} \cdot \nabla \times \bar{F} \, ds$

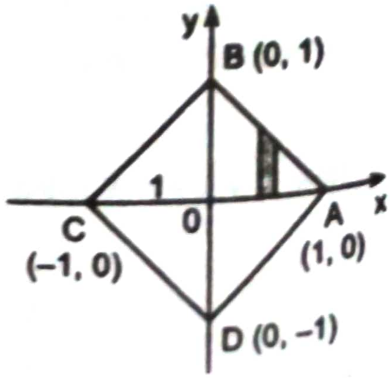


Fig. 9.15

In the xy-plane $\bar{r} = x i + y j + 0k$

$\therefore \quad d\bar{r} = dx\, i + dy\, j$

Hence, from $\bar{F} \cdot d\bar{r} = xy\,dx + xy^2\,dy$, we get

$\bar{F} = xy\, i + xy^2\, j + 0k$

$$\therefore \quad \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = (0-0)\,i + (0-0)\,j + (y^2 - x)\,k$$

Further, $\bar{N} = k$ and $ds = dx\,dy$

$$\therefore \iint_S \bar{N} \cdot \nabla \times \bar{F}\,ds = \iint_S k \cdot (y^2 - x)\,k\,dx\,dy$$

$$= \iint_S (y^2 - x)\,dx\,dy \text{ where } S \text{ is the square } ABCD.$$

$$\therefore \iint_S \bar{N} \cdot \nabla \times \bar{F}\,ds = 4 \iint_{\Delta 0AB} (y^2 - x)\,dx\,dy$$

The equation of the line AB is $\frac{y-1}{1-0} = \frac{x-0}{0-1} \quad \therefore \; y = 1 - x$

$$\therefore \iint_{\Delta 0AB} (y^2 - x)\,dx\,dy = \int_{x=0}^{1} \int_{y=0}^{y=1-x} (y^2 - x)\,dy\,dx = \int_0^1 \left[\frac{y^3}{3} - xy\right]_0^{1-x} dx$$

$$= \int_0^1 \left[\frac{(1-x)^3}{3} - x \cdot (1-x)\right] dx = \frac{1}{3}\int_0^1 (1 - 6x + 6x^2 - x^3)\,dx$$

$$= \frac{1}{3}\left[x - 3x^2 + 2x^3 - \frac{x^4}{4}\right]_0^1 = \frac{1}{3}\left[1 - 3 + 2 - \frac{1}{4}\right] = -\frac{1}{12}$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \iint_S \bar{N} \cdot \nabla \times \bar{F}\,ds = 4 \cdot \left(-\frac{1}{12}\right) = -\frac{1}{3}$$

Example 6 : Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = yz\,i + zx\,j + xy\,k$ and C is the boundary of the circle $x^2 + y^2 + z^2 = 1$, $z = 0$. **(M.U. 2005)**

Sol. : We have by Stoke's Theorem $\int_C \bar{F} \cdot d\bar{r} = \iint_C \bar{N} \cdot \nabla \times \bar{F}\,ds$

$$\text{Now, } \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x - x)\,i + (y - y)\,j + (z - z)\,k = \bar{O}$$

$$\therefore \iint_S \bar{N} \times \nabla \bar{F}\,ds = \iint_S 0\,ds = 0 \qquad \therefore \int_C \bar{F} \cdot d\bar{r} = 0.$$

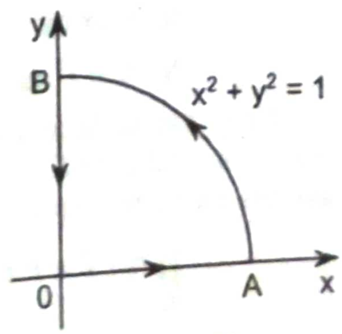


Fig. 9.16

Example 7 : Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = 4xz\,i - y^2\,j + yz\,k$ and C is the area in the plane $z = 0$ bounded by $x = 0$, $y = 0$ and $x^2 + y^2 = 1$. **(M.U. 1994, 98)**

Sol. : We have by Stoke's Theorem

$$\int_C \bar{F} \cdot d\bar{r} = \iint_C \bar{N} \cdot \nabla \times \bar{F}\,ds$$

Now, $\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xz & -y^2 & yz \end{vmatrix} = z\,i + 4x\,j$

In the xy-plane $\overline{N} = k$, $ds = dx\,dy$

$$\therefore \quad \iint_S \overline{N} \cdot \nabla \times \overline{F}\, ds = \iint_S k \cdot (zi + 4xj)\, dx\, dy$$

$$= \int\int_S 0\, dx\, dy = 0 \qquad [\because\ k \cdot i = k \cdot j = 0]$$

$$\therefore \quad \int_C \overline{F} \cdot d\overline{r} = 0.$$

Example 8 : Use Stoke's Theorem to evaluate $\int_C \overline{F} \cdot d\overline{r}$ where $\overline{F} = y\,i + z\,j + x\,k$ and C is the boundary of the surface $x^2 + y^2 = 1 - z$, $z > 0$. **(M.U. 1993, 94, 95, 2003, 06)**

Sol. : We have by Stoke's Theorem $\int_C \overline{F} \cdot d\overline{r} = \iint_C \overline{N} \cdot \nabla \times \overline{F}\, ds$

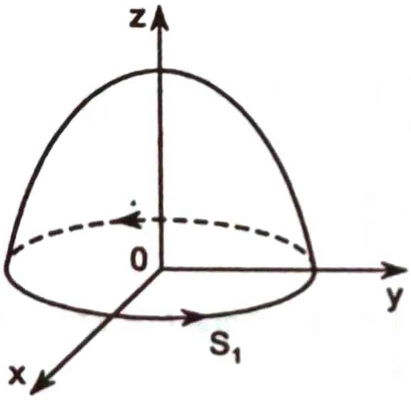


Fig. 9.17 (a)

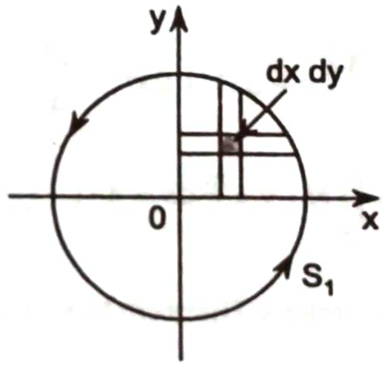


Fig. 9.17 (b)

Now, $\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \begin{matrix} = (0-1)\,i + (0-1)\,j + (0-1)\,k \\ = -i - j - k \end{matrix}$

Since, the base circle S_1 is traversed in anticlockwise direction as shown in the figure, the **outward normal** to the surface is along the z-axis *i.e.* $\overline{N} = k$

$$ds = dx\,dy$$

$$\therefore \quad \overline{N} \cdot \nabla \times \overline{F}\, ds = k \cdot (-i - j - k)\, dx\, dy = -\, dx\, dy$$

Now, the surface integral over the curved surface S of the paraboloid is the same as the surface integral over S_1, the circle of intersection of the paraboloid and the plane $z = 0$ because they have the same boundary.

$$\therefore \quad \iint_{S_1} \overline{N} \cdot \nabla \times \overline{F}\, ds = -\iint_{S_1} dx\, dy = -\pi \qquad \text{(Area of the circle is } \pi \cdot 1^2 = \pi)$$

$$\therefore \quad \int_C \overline{F} \cdot d\overline{r} = \pi \qquad \text{(Numerically)}$$

Example 9 : Use Stoke's theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (2x - y)\, i - yz^2\, j - y^2 zk$ and S is the surface of hemisphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane. **(M.U. 2001, 14)**

Sol. : We have by Stoke's Theorem, $\int_C \bar{F} \cdot d\bar{r} = \iint_S \bar{N} \cdot \nabla \times \bar{F}\, ds$

The hemi-sphere lying above the xy-plane has a boundary on the xy-plane given by $x^2 + y^2 + z^2 = a^2$.

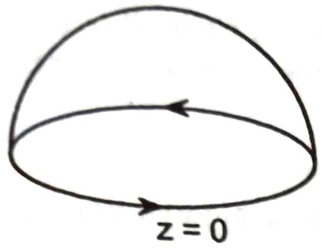


Fig. 9.18 (a)

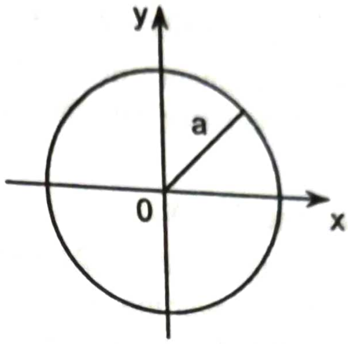


Fig. 9.18 (b)

Now, $\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)\, i - (0 - 0)\, j + (0 + 1)\, k = k$

$$\therefore \iint_{S_1} \bar{N} \cdot (\nabla \times \bar{F})\, ds = \iint_{S_1} k \cdot k\, ds = \iint_{S_1} dx\, dy$$

where S_1 is the area of the circle $x^2 + y^2 = a^2$ *i.e.* πa^2.

$$\therefore \iint_{S_1} \bar{N} \cdot (\nabla \times \bar{F})\, ds = \iint_{S_1} dx\, dy = \pi a^2$$

Example 10 : Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where

$$\bar{F} = (y^2 + z^2 - x^2)\, i + (z^2 + x^2 - y^2)\, j + (x^2 + y^2 - z^2)\, k$$

over the boundary of the surface $x^2 + y^2 - 2ax + az = 0$ above the plane $z = 0$. **(M.U. 2003)**

Sol. : We have by Stoke's Theorem $\int_C \bar{F} \cdot d\bar{r} = \iint_C \bar{N} \cdot \nabla \times \bar{F}\, ds$

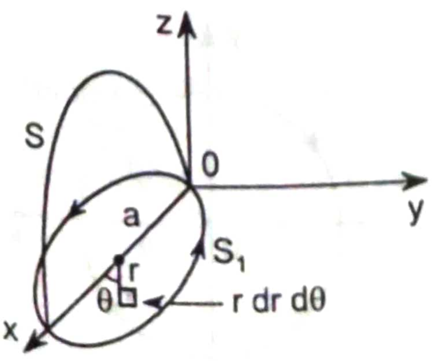


Fig. 9.19 (a)

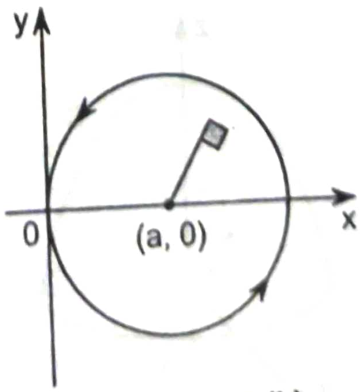


Fig. 9.19 (b)

The equation of the given surface can be written as $(x-a)^2 + y^2 = -a(z-a)$.

This is a paraboloid with its vertex at $(a, 0, a)$ opening downwards. It meets z-plane, $z = 0$ in the circle $(x-a)^2 + y^2 = a^2$. The parametric equations of the circle are $x - a = a\cos\theta$, $y = a\sin\theta$, $z = 0$.

$$\text{Now,} \quad \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y^2+z^2-x^2) & (z^2+x^2-y^2) & (x^2+y^2-z^2) \end{vmatrix}$$

$$\therefore \quad \nabla \times \bar{F} = (2y-2z)\,i + (2z-2x)\,j + (2x-2y)\,k$$

Now, we take a small element $r\,dr\,d\theta$ at (r, θ) in the circle. If (r, θ) are the coordinate of a point P w.r.t. the centre of the circle then the coordinates of the point w.r.t. O are $x = a + r\cos\theta$, $y = r\sin\theta$.

Further, in the xy-plane $\bar{N} = k$.

$$\therefore \quad \bar{N} \cdot \nabla \times \bar{F} = k \cdot [(2y-2z)\,i + (2z-2x)\,j + (2x-2y)\,k] = 2(x-y)$$

Now, the surface integral over the curved surface S is the same as the surface integral over S_1, the circle of intersection of the paraboloid and the plane $z = 0$ because they have the same boundary.

$$\therefore \quad \iint_{S_1} \bar{N} \cdot \nabla \times \bar{F}\,ds = \iint 2(x-y)\,ds = \int_0^a \int_0^{2\pi} 2(a + a\cos\theta - a\sin\theta)\,r\,dr\,d\theta$$

$$= 2\int_0^a [a\theta + a\sin\theta + a\cos\theta]_0^{2\pi}\,r\,dr = 2\int_0^a [2\pi a + a - a]\,r\,dr$$

$$= 2\int_0^a 2\pi a\,r\,dr = 4\pi a\left[\frac{r^2}{2}\right]_0^a = 2\pi a^3.$$

$$\therefore \quad \int_C \bar{F} \cdot d\bar{r} = 2\pi a^3.$$

Example 11 : Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where

$\bar{F} = (x-y-z)\,i + (y-z-x)\,j + (z-x-y)\,k$ over the paraboloid $x^2 + y^2 = 4 - z$, $z \geq 0$.

(M.U. 1998)

Sol. : We have by Stoke's Theorem $\int_C \bar{F} \cdot d\bar{r} = \iint_S \bar{N} \cdot \nabla \times \bar{F}\,ds$

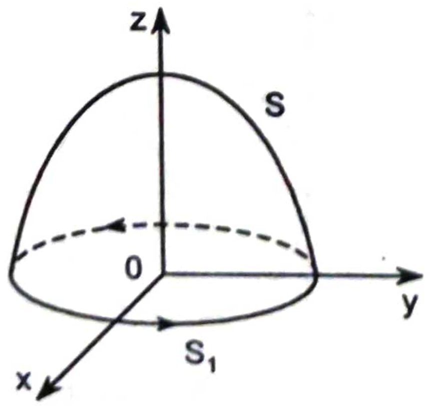


Fig. 9.20 (a)

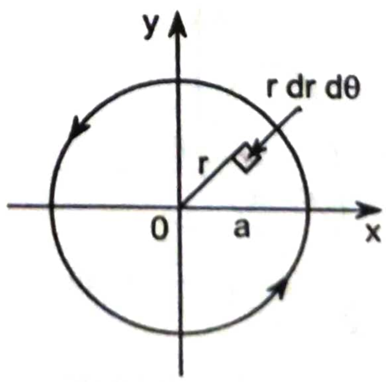


Fig. 9.20 (b)

The given surface $x^2 + y^2 = -(z-4)$ is a paraboloid with its vertex at (0, 0, 4) opening downwards. It meets the x-y plane, $z = 0$ in the circle $x^2 + y^2 = 4$.

$$\text{Now,} \quad \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y-z & y-z-x & z-x-y \end{vmatrix}$$

$$= (-1+1)\, i + (-1+1)\, j + (-1+1)\, k = \bar{0}$$

$$\therefore \iint_{S_1} \bar{N} \cdot \nabla \bar{F}\, ds = 0 \qquad \therefore \int_C \bar{F} \cdot d\bar{r} = 0$$

Example 12 : Evaluate $\iint_S (\nabla \times \bar{F}) \cdot d\bar{s}$ where

$$\bar{F} = (2y^2 + 3z^2 - x^2)\, i + (2z^2 + 3x^2 - y^2)\, j + (2x^2 + 3y^2 - z^2)\, k$$

over the part of the sphere $x^2 + y^2 + z^2 - 2ax + az = 0$ cut off by the plane $z = 0$. **(M.U. 2003)**

Sol. : The section of the given sphere in the *xy*-plane *i.e.* $z = 0$ is given by $x^2 + y^2 - 2ax = 0$ *i.e.* $(x-a)^2 + y^2 = a^2$ which is a circle with centre at $(a, 0)$ and radius a.

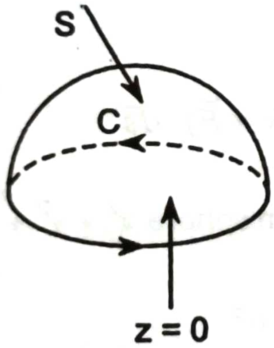


Fig. 9.21 (a)

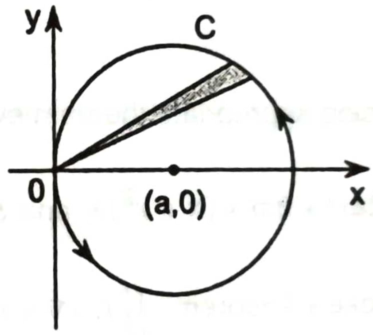


Fig. 9.21 (b)

$$\text{Now,} \quad \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 + 3z^2 - x^2 & 2z^2 + 3x^2 - y^2 & 2x^2 + 3y^2 - z^2 \end{vmatrix}$$

$$= (6y - 4z)\, i + (6z - 4x)\, j + (6x - 4y)\, k$$

and $\quad ds = dx\, dy, \ \bar{N} = k$

$$\therefore \iint_S N \cdot (\nabla \times \bar{F})\, ds = \iint_S k \cdot [(6y - 4z)\, i + (6z - 4x)\, j + (6x - 4y)\, k]\, ds$$

$$= \iint_S (6x - 4y)\, dx\, dy$$

To evaluate the integral, we change it to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$. The equation of the circle $x^2 + y^2 - 2ax = 0$, then changes to $r^2 = 2ar\cos\theta$ *i.e.* $r = 2a\cos\theta$. On this circle r varies from 0 to $2a\cos\theta$ and θ varies from $-\pi/2$ to $\pi/2$.

$$\therefore \iint_S (6x - 4y)\, dx\, dy = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} (6r\cos\theta - 4r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_{-\pi/2}^{\pi/2} (6\cos\theta - 4\sin\theta)\, d\theta \int_0^{2a\cos\theta} r^2\, dr$$

$$\therefore \quad \iint_S (6x - 4y)\,dx\,dy = \int_{-\pi/2}^{\pi/2} (6\cos\theta - 4\sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{8a^3}{3}\int_{-\pi/2}^{\pi/2} (6\cos\theta - 4\sin\theta)\cos^3\theta\,d\theta$$

$$= \frac{8a^3}{3}\int_{-\pi/2}^{\pi/2} (6\cos^4\theta - 4\cos^3\theta\sin\theta)\,d\theta$$

$$\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 2\int_0^{\pi/2}\cos^4\theta\,d\theta = 2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{8} \qquad \text{[By reduction formula]}$$

and $$\int_{-\pi/2}^{\pi/2}\cos^3\theta\sin\theta\,d\theta = -\left[\frac{\cos^4\theta}{4}\right]_{-\pi/2}^{\pi/2} = 0$$

$$\therefore \quad \iint_S (6x - 4y)\,dx\,dy = \frac{8a^3}{3}\cdot 6\cdot\frac{3\pi}{8} = 6\pi a^3$$

Example 13 : Using appropriate theorem evaluate $\int_S (\nabla \times \bar{F})\cdot d\bar{s}$

where $\bar{F} = (x^2 + y - 2a)\,i + 3xj + (x + z^2)\,k$ and S is the hemisphere $x^2 + y^2 + z^2 = a^2$ lying above the x-axis.

(M.U. 2002)

Sol. : We shall use Stoke's Theorem $\iint_S \bar{N}\cdot\nabla\times\bar{F}\,ds = \oint_C \bar{F}\cdot d\bar{r}$

Let $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta \le 2\pi$ be the parametric equations of C.

$$\therefore \quad \oint_C \bar{F}\cdot d\bar{r} = \int_C [(x^2 + y - 2a)]\,dx + 3x\,dy + 0\cdot dz$$

$$= \int_0^{2\pi} [(a^2\cos^2\theta + a\sin\theta - 2a)(-a\sin\theta\,d\theta) + 3a\cos\theta\cdot(a\cos\theta)\,d\theta]$$

$$= \int_0^{2\pi} -a^3\cos^2\theta\sin\theta\,d\theta - \int_0^{2\pi} a^2\sin^2\theta\,d\theta + 2a^2\int_0^{2\pi}\sin\theta\,d\theta + 3a^2\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi} -a^3\cos^2\theta\sin\theta\,d\theta - a^2\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta + 2a^2\int_0^{2\pi}\sin\theta\,d\theta + 3a^2\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^3}{3}\left[\cos^3\theta\right]_0^{2\pi} - \frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} + 2a^2\left[\cos\theta\right]_0^{2\pi} + \frac{3a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$\therefore \quad \oint_C \bar{F}\cdot d\bar{r} = 0 - \frac{a^2}{2}\cdot 2\pi + \frac{3a^2}{2}\cdot 2\pi = a^2\cdot 2\pi = 2a^2\pi.$$

Example 14 : Evaluate $\iint_S (\nabla \times \bar{F}) \cdot d\bar{s}$ where

$$\bar{F} = (2x - y + z)\, i + (x + y - z^2)\, j + (3x - 2y + 4z)\, k$$

and S is the surface of the cylinder $x^2 + y^2 = 4$ bounded by the plane $z = 9$ and open at the other end. **(M.U. 1993, 2002)**

Sol. : By Stoke's Theorem $\iint_S (\nabla \times \bar{F})\, d\bar{s} = \int_C \bar{F} \cdot d\bar{r}$

where C is the circle $x^2 + y^2 = 4$ in the plane $z = 9$.

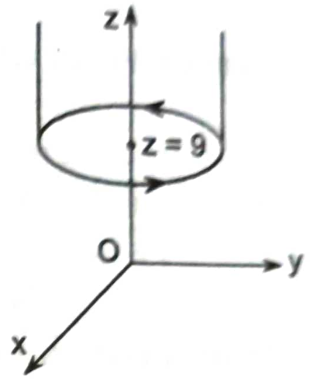


Fig. 9.22

$$\therefore\ \bar{F} \cdot d\bar{r} = [(2x - y + z)\, i + (x + y - z^2)\, j + (3x - 2y + 4z)\, k] \cdot (dx\, i + dy\, j + dz\, k)$$

$$= (2x - y + z)\, dx + (x + y - z^2)\, dy + (3x - 2y + 4z)\, dz$$

Since, $z = 9$, $dz = 0$ and since $x^2 + y^2 = 4$,

we put $x = 2\cos\theta$, $y = 2\sin\theta$.

$$\therefore\ dx = -2\sin\theta\, d\theta,\ dy = 2\cos\theta\, d\theta$$

$$\therefore\ \int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} [(4\cos\theta - 2\sin\theta + 9)(-2\sin\theta\, d\theta) + (2\cos\theta + 2\sin\theta - 81)(2\cos\theta\, d\theta)]$$

$$= \int_0^{2\pi} (-8\sin\theta\cos\theta + 4\sin^2\theta - 18\sin\theta + 4\cos^2\theta + 4\sin\theta\cos\theta - 162\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} (4 - 4\sin\theta\cos\theta - 18\sin\theta - 162\cos\theta)\, d\theta$$

$$= \left[4\theta - \frac{4\sin^2\theta}{2} + 18\cos\theta - 162\sin\theta\right]_0^{2\pi}$$

$$= (8\pi + 18) - (18) = 8\pi.$$

Example 15 : If $\bar{F} = a(x + y)\, i + a(y - x)\, j + z^2 k$, evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is the boundary of the plane surface of the hemi-sphere $x^2 + y^2 + z^2 = a^2$, $z = 0$. **(M.U. 1998)**

Sol. : By Stoke's Theorem $\int_C \bar{F} \cdot d\bar{r} = \iint_S \bar{N} \cdot (\nabla \times \bar{F})\, ds$

Now, $\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a(x+y) & a(y-x) & z^2 \end{vmatrix}$

$$= (0 - 0)\, i + (0 - 0)\, j + (-a - a)\, k = -2ak$$

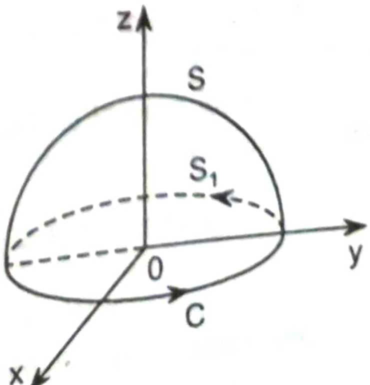


Fig. 9.23

Now, the surface integral over the surface S of the hemi-sphere is the same as the surface integral over S_1, the plane face of the hemi-sphere as both have the same boundary C.

Further, in xy-plane $\bar{N} = k$, $dS = dx\, dy$

$$\therefore \iint_S \bar{N}\cdot(\nabla\times\bar{F})\,ds = \iint_{S_1} k\cdot(-2ak)\,dx\,dy = -2a\iint_{S_1} dx\,dy = -2aS_1$$

where, S_1 is the area of the circle $x^2+y^2=a^2$ which is πa^2.

$$\therefore \iint_S \bar{N}\cdot\nabla\times\bar{F}\,ds = -2a(\pi a^2) = -2\pi a^3.$$

Example 16 : By using Stoke's Theorem evaluate $\int_C [(x^2+y^2)\,i+(x^2-y^2)\,j]\cdot d\bar{r}$ where C is the boundary of the region enclosed by circles $x^2+y^2=4,\ x^2+y^2=16.$ **(M.U. 2004, 05)**

Sol. : By Stoke's Theorem the given line integral

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{N}\cdot(\nabla\times\bar{F})\,ds$$ where S is the region bounded by the two circles.

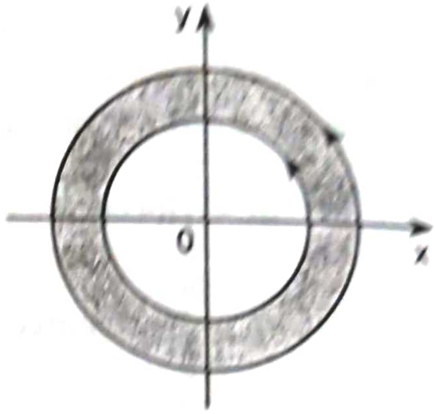


Fig. 9.24

In the xy-plane $\bar{N}=k,\ ds=dx\,dy$

And $$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & x^2-y^2 & 0 \end{vmatrix} = (2x-2y)\,k.$$

$$\therefore \bar{N}\cdot\nabla\times\bar{F}\,ds = k\cdot(2x-2y)\,k\,dx\,dy$$
$$= (2x-2y)\,dx\,dy$$

If we use polar coordinates the equations of the circles are $r=2,\ r=4$; $2x-2y=2r(\cos\theta-\sin\theta)$; $dx\,dy=r\,d\theta\,dr$.

$$\therefore I = \int_0^{2\pi}\int_2^4 2r(\cos\theta-\sin\theta)\,r\,d\theta\,dr$$

$$= 2\left[\frac{r^3}{3}\right]_2^4\left[\sin\theta+\cos\theta\right]_0^{2\pi} = \frac{2}{3}(64-8)(1-1) = 0$$

Example 17 : Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F}=(x+y)\,i+(y+z)\,j-xk$ and C is the boundary of the plane $2x+y+z=2$ cut-off by coordinate planes. **(M.U. 2000, 15)**

Sol. : By Stoke's Theorem $$\int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{N}\cdot\nabla\times\bar{F}\,ds$$

The equation of the plane is $\Phi = 2x+y+z=2$. We know that grad Φ is a vector perpendicular to the surface $\Phi(x,y,z)=C$.

$\therefore$ Vector normal to the surface *i.e.* normal to the plane ABC is given by

$$\text{grad}\,\Phi = \nabla\Phi = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(2x+y+z)$$
$$= 2i+j+k$$

Unit normal vector in the direction of $\nabla\Phi$.

$$\bar{N} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{2i+j+k}{\sqrt{4+1+1}} = \frac{2i+j+k}{\sqrt{6}}$$

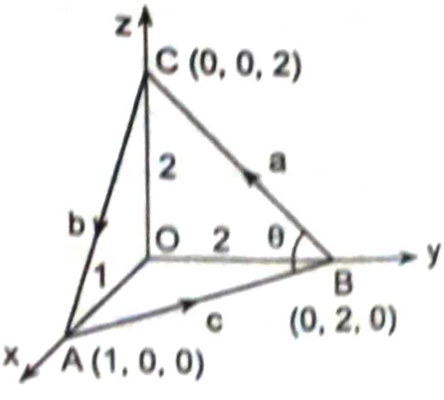


Fig. 9.25

Now, $\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & y+z & -x \end{vmatrix} = (0-1)\,i - (-1+0)\,j + (-1)\,k$
$= -i + j - k$

$$\therefore \iint_S \bar{N} \cdot \nabla \times \bar{F}\, ds = \iint_S \left(\frac{2i+j+k}{\sqrt{6}}\right) \cdot (-i+j-k)\, ds$$

$$= \iint_S \left(-\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\right) ds = -\frac{2}{\sqrt{6}} \iint_S ds$$

The equation of the plane is $\frac{x}{1} + \frac{y}{2} + \frac{z}{2} = 1$. $\therefore$ $OA = 1$, $OB = 2$, $OC = 2$.

[OR by putting $y = 0$, $z = 0$ in $2x + y + z = 2$, we have $x = 1$. Similarly, putting $x = 0$, $z = 0$, we get $y = 2$ and putting $x = 0$, $y = 0$, we get $z = 2$. Hence the coordinates of A, B, C are $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.]

But $\iint_S ds$ over the triangle ABC is the area of the triangle ABC. If $AB = c$, $BC = a$ and θ is the acute angle between AB and BC then the area of $\Delta\, ABC = \frac{1}{2} ac \sin\theta$.

We know by cosine rule, that $b^2 = a^2 + c^2 - 2ac\cos B$.

Now, $a^2 = 8$, $b^2 = 5$, $c^2 = 5$. $\therefore$ $5 = 8 + 5 - 2\sqrt{40}\cos B$

$$\therefore \cos B = \frac{8}{2\sqrt{40}} = \frac{4}{\sqrt{40}} = \frac{2}{\sqrt{10}} \qquad \therefore \sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\frac{4}{10}}$$

$$\therefore \sin\theta = \frac{\sqrt{6}}{\sqrt{10}} = \sqrt{\frac{3}{5}}.$$

$$\therefore \text{Area of the } \Delta\, ABC = \frac{1}{2} ac \sin\theta = \frac{1}{2} \cdot 2\sqrt{2} \cdot \sqrt{5} \cdot \frac{\sqrt{3}}{\sqrt{5}} = \sqrt{6}$$

$$\therefore \iint ds = \sqrt{6} \qquad \therefore \iint_S \bar{N} \cdot \nabla \times \bar{F}\, ds = -\frac{2}{\sqrt{6}} \cdot \sqrt{6} = -2.$$

Example 18 : Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = 2y(1-x)\,i + (x - x^2 - y^2)\,j + (x^2 + y^2 + z^2)\,k$ and C is the boundary of the plane $x + y + z = 2$ cut-off by coordinate planes.

Sol. : By Stoke's Theorem $\int_C \bar{F} \cdot d\bar{r} = \iint_S \bar{N} \cdot \nabla \times \bar{F}\, ds$

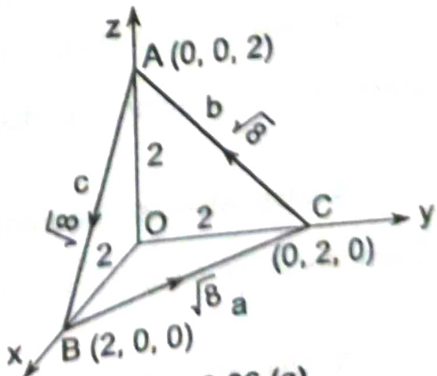


Fig. 9.26 (a)

Now, $\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y(1-x) & x - x^2 - y^2 & x^2 + y^2 + z^2 \end{vmatrix}$

$= i(2y - 0) - j(2x - 0) + k(1 - 2x - 2 + 2x)$

$= 2y\, i - 2x\, j - k$

Further, $\Phi \equiv x + y + z - 2 = 0$

$\therefore$ Normal to the plane ABC,

$$\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} = i + j + k$$

Unit normal to the plane of the ΔABC,

$$\bar{N} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{i+j+k}{\sqrt{3}}$$

Let ds be the element in the plane ABC. Then its projection in the xy-plane will be $dx\,dy$. If θ is the angle between plane ABC and the xy-plane, then

$$dx\,dy = \cos\theta\,ds \quad \text{and} \quad \cos\theta = \bar{N}\cdot k = \left(\frac{i+j+k}{\sqrt{3}}\right)\cdot k = \frac{1}{\sqrt{3}}.$$

$$\therefore \quad dx\,dy = \frac{1}{\sqrt{3}}\,ds \qquad \therefore \quad ds = \sqrt{3}\,dxy$$

$$\therefore \quad \bar{N}\cdot\nabla\times\bar{F}\,ds = \frac{(i+j+k)}{\sqrt{3}}\cdot(2y\,i - 2x\,j - k)\sqrt{3}\,dx\,dy$$

$$= (2y - 2x - 1)\,dx\,dy$$

$$\therefore \quad \iint_S \bar{N}\cdot\nabla\times\bar{F}\,ds = \iint_{\Delta OAB}(2y - 2x - 1)\,dx\,dy$$

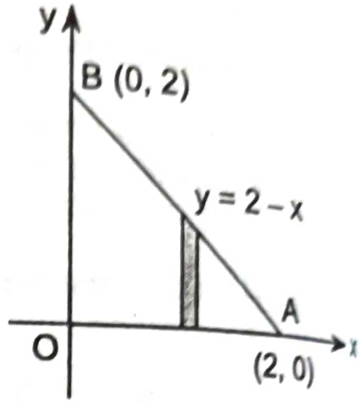


Fig. 9.26 (b)

$$= \int_{x=0}^{2}\int_{y=0}^{y=2-x}(2y - 2x - 1)\,dx\,dy = \int_0^2\left[y^2 - 2xy - y\right]_0^{2-x}dx$$

$$= \int_0^2[(2-x)^2 - 2x(2-x) - (2-x)]\,dx = \int_0^2[(2-x)^2 - 2(2x - x^2) - (2-x)]\,dx$$

$$= \left[-\frac{(2-x)^3}{3} - 2\left(x^2 - \frac{x^3}{3}\right) + \frac{(2-x)^2}{2}\right]_0^2 = \left[-2\left(4 - \frac{8}{3}\right)\right] - \left[-\frac{8}{3} + 2\right] = -2$$

EXERCISE - III

1. Evaluate $\int_C (e^x\,dx + 2y\,dy - dz)$ where C is $x^2 + y^2 = 4$, $z = 2$ by Stokes Theorem. **(M.U. 2004)** [**Ans. :** $\nabla\times\bar{F} = 0, 0$]

2. Use Stoke's Theorem to evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (x^2 - y^2)\,i + 2xyj$ and C is the rectangle in the plane $z = 0$, bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$. [**Ans. :** $2ab^2$]

3. Use Stoke's Theorem to evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (x^2 + y^2)\,i + 4xy\,j$ and C is the boundary of the region bounded by the parabola $y^2 = 4x$ and the line $x = 4$. **(M.U. 1992, 94)** [**Ans. :** 64]

4. Use Stoke's Theorem to evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = xy\,i + yz\,j + z^2k$ and C is the boundary of the square in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = a$. [**Ans. :** $-a^3/2$]

5. Use Stoke's Theorem to evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (x^2 + y^2)\,i - 2\,xyj$ and C is the boundary of the rectangle bounded by the lines $x = \pm\,a$, $y = 0$, $y = b$. **(M.U. 2007)** [**Ans. :** $-4ab^2$]

6. Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = 4xy\,i - y^2 j + yzk$ and C is the area bounded by $x = 0$, $y = 0$, $x^2 + y^2 = 1$, $z = 0$. (M.U. 1998) [Ans. : $-4/3$]

7. Use Stoke's Theorem to evaluate $\int_C (x^3 - 3y)\,dx + (x + \sin y)\,dy$ where C is the boundary of the triangle whose vertices are (0, 0), (1, 0), (0, 2). [Ans. : 4]

8. Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2 - y^2)\,i + 2xy\,j$ and C is the region bounded by $y = 0$, $x = 2$, $y = x$ in the xy-plane. [Ans. : $16/3$]

9. Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = z^2\,i + x^2\,j + y^2\,k$ and C is the curved surface of the hemi-sphere $x^2 + y^2 + z^2 = 100$, $z \geq 0$. (M.U. 1999) [Ans. : $8a^3/3$]

10. Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = y^2\,i + xy\,j - xz\,k$ and C is the boudnary of the sphere $x^2 + y^2 + z^2 = a^2$, $z = 0$. (M.U. 1999)

11. By using Stoke's Theorem evaluate $\int_C \left\{[2x^2 - y^2]\,i + (x^2 + y^2)\,j\right\} \cdot d\bar{r}$ where C is the boundary of the surface in the xy-plane enclosed by the x-axis, circle $x^2 + y^2 = 1$ and $y > 0$.

12. By using Stoke's Theorem evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (2x + y)\,i - 4z^2 - y^2 z\,k$ and C is the boundary of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z = 0$. [Ans. : $-\pi a^2$]

13. Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = -xy\,i + 2yz\,j + y^2 k$ and C is the boundary of half of the sphere $x^2 + y^2 + z^2 = a^2$, $z = 0$. (M.U. 2004) [Ans. : $\frac{4}{3}a^3$]

14. Use Stoke's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = zi + xj + yk$ and C is the boundary of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ in the xy-plane. (M.U. 2004, 06) [Ans. : πa^2]

7. Gauss Divergence Theorem

The surface integral of the normal component of a vector over a closed surface S is equal to the volume integral of the divergence of $\bar{F}$ throughout the volume bounded by S.

$$\boxed{\iint_S \bar{N} \cdot \bar{F}\,ds = \iiint_V \nabla \cdot \bar{F}\,dv}$$

where $\bar{N}$ is the unit outward normal.

We shall accept this important theorem without proof.

Karl Friedrich Gauss (1777 - 1855)

Karl Friedrich Gauss was a great German mathematician and scientist. He is called the "**prince of mathematicians**". He is ranked with Isaac Newton and Anchimedes. It is said that Gauss at the age of three had pointed out an error in calculation made by his father while preparing a payroll.

In his doctoral thesis he gave the first complete proof of the fundamental theorem of algebra that a polynomial of *n*th degree has *n* roots. He is considered to have laid the founation of number theory. We are all familiar with Gaussian probability distribution (Normal Distribution). He gave first the geometric interpretation of complex numbers, developed the theory of conformal mapping. He did fundamental work in electromagnetism.

He knew many languages and read extensively. It is said that if Gauss had published all of his discoveries the state of mathematics would have advanced by 50 years.

Procedure

1. From given $\overline{F}$ first find $\nabla \cdot \overline{F} = \Phi$ say.
2. For rectangular coordinates $dV = dx\,dy\,dz$.
3. Then find the triple integral $\iiint \Phi \, dx\,dy\,dz$ with proper limits.

Note ....

1. We first note that the Gauss Divergence Theorem expresses a surface integral as a volume integral.

2. If $\overline{F} = Pi + Qj + Rk$ then div $\overline{F} = \nabla \cdot \overline{F}$

$$\therefore \quad \text{div } \overline{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (Pi + Qj + Rk)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\overline{N} \cdot \overline{F}\,ds = Pi \cdot \overline{N}\,ds + Qj \cdot \overline{N}\,ds + Rk \cdot \overline{N}\,ds$$

$$= P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy$$

Gauss's Theorem now can be stated as

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz = \iint_S (P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy)$$

List of Formulae

(A) If *n* is an even positive integer

$$\int_0^{\pi/2} \sin^n \theta\, d\theta = \int_0^{\pi/2} \cos^n \theta\, d\theta = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdots\cdots \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

(B) If *n* is an odd positive integer

$$\int_0^{\pi/2} \sin^n \theta\, d\theta = \int_0^{\pi/2} \cos^n \theta\, d\theta = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdots\cdots \frac{2}{3} \cdot 1$$

(C) $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots\ldots} P$

where, $P = \begin{cases} \pi/2, & \text{if } m, n \text{ both are even} \\ 1, & \text{otherwise} \end{cases}$

Type I : Theoretical Examples

Example 1 : If $\overline{F} = \text{curl } \overline{G}$ and S is any closed curve then prove that $\iint_S \overline{N} \cdot \overline{F} \, ds = 0$.

Sol. : By Gauss's Divergence Theorem $\iint_S \overline{N} \cdot \overline{F} \, ds = \iiint_V \nabla \cdot \overline{F} \, dv$

$$\therefore \quad \nabla \cdot \overline{F} = \nabla \cdot \text{curl } \overline{G} = \nabla \cdot \nabla \times G$$

Now, if $\overline{G} = g_1 i + g_2 j + g_3 k$, then

$$\nabla \times \overline{G} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ g_1 & g_2 & g_3 \end{vmatrix} = i\left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right) + j\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right) + k\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)$$

$$\therefore \quad \nabla \cdot \nabla \times \overline{G} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \nabla \times \overline{G}$$

$$= \frac{\partial^2 g_3}{\partial x \partial y} - \frac{\partial^2 g_2}{\partial x \partial z} + \frac{\partial^2 g_1}{\partial y \partial z} - \frac{\partial^2 g_3}{\partial x \partial y} + \frac{\partial^2 g_2}{\partial x \partial z} - \frac{\partial^2 g_1}{\partial y \partial z} = 0$$

$$\therefore \quad \nabla \cdot \overline{F} = 0 \qquad \therefore \quad \iint_S \overline{N} \cdot \overline{F} \, ds = 0$$

Example 2 : From Gauss's Theorem deduce that div curl $\overline{G} = 0$.

Sol. : Let S be a closed surface enclosing a volume V. Then by Gauss's Divergence Theorem

$$\iiint_V \nabla \cdot \overline{F} \, dv = \iint_S \overline{N} \cdot \overline{F} \, ds$$

Now, let $\overline{F} = \text{curl } \overline{G}$

$$\therefore \quad \iiint_V \nabla \cdot \text{curl } \overline{G} \, dv = \iint_S \overline{N} \cdot \text{curl } \overline{G} \, ds \qquad \ldots\ldots (1)$$

Now, divide the surface S into two parts S_1 and S_2, by a closed curve C. Now, C is the common boundary to S_1 and S_2.

$\therefore$ By Stoke's Theorem

$$\iint_{S_1} \overline{N} \cdot \text{curl } \overline{G} \, ds = \int_C \overline{G} \cdot dr \qquad \ldots\ldots (2)$$

and

$$\iint_{S_2} \overline{N} \cdot \text{curl } \overline{G} \, ds = -\int_C \overline{G} \cdot dr \qquad \ldots\ldots (3)$$

(This is so because the positive direction of C on S_1 is opposite to the positive direction of C on S_2.)

Adding (2) and (3), we get $\iint_{S_2} \overline{N} \cdot \text{curl } \overline{G}\, ds = 0$

Hence, from (1), we get, $\iiint_V \nabla \cdot \text{Curl } \overline{G}\, dv = 0$

Since, V is arbitrary $\quad \text{div curl } \overline{G} = 0$.

Example 3 : Show that $\oint_S r^n\, \bar{r}\, ds = (n+3)\int_V r^n\, dv$ where S any closed surface enclosing volume V. **(M.U. 2003)**

Sol. : We have by Gauss Divergence Theorem

$$\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V (\nabla \cdot \overline{F})\, dv \qquad \text{i.e.} \qquad \oint_S \overline{F} \cdot d\bar{s} = \int_V (\nabla \cdot \overline{F})\, dv$$

By data $\overline{F} = r^n\, \bar{r}$, $\quad \therefore\ \nabla \cdot \overline{F} = \nabla \cdot (r^n\, \bar{r})$.

$$\text{Now,}\quad \nabla \cdot r^n\, \bar{r} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (r^n\, xi + r^n\, yj + r^n zk)$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= n\, r^{n-1} \frac{\partial r}{\partial x} x + r^n \cdot 1 + n\, r^{n-1} \frac{\partial r}{\partial y} y + r^n \cdot 1 + n\, r^{n-1} \frac{\partial r}{\partial z} z + r^n \cdot 1$$

$$= n\, r^{n-1} \cdot \frac{x^2}{r} + n\, r^{n-1} \cdot \frac{y^2}{r} + n\, r^{n-1} \cdot \frac{z^2}{r} + 3\, r^n$$

$$= n\, r^{n-2} (n^2 + y^2 + z^2) + 3\, r^n = n\, r^{n-2}\, r^2 + 3\, r^n$$

$$\therefore\quad \nabla \cdot r^n\, \bar{r} = n\, r^n + 3\, r^n = (n+3)\, r^n.$$

$$\therefore\quad \oint \overline{F} \cdot ds = \int_V (n+3)\, r^n\, dv.$$

Example 4 : Show that $\iint_S (\nabla \Phi \times \nabla \Psi) \cdot d\bar{s} = 0$ for any closed surface S. **(M.U. 2003)**

Sol. : Let $\overline{F} = \nabla \Phi \times \nabla \Psi \quad \therefore\ \nabla \cdot \overline{F} = \nabla \cdot [\nabla \Phi \times \nabla \Psi]$

$$\text{Now,}\quad \overline{F} = \nabla \Phi \times \nabla \Psi = \begin{vmatrix} i & j & k \\ \dfrac{\partial \Phi}{\partial x} & \dfrac{\partial \Phi}{\partial y} & \dfrac{\partial \Phi}{\partial z} \\ \dfrac{\partial \Psi}{\partial x} & \dfrac{\partial \Psi}{\partial y} & \dfrac{\partial \Psi}{\partial z} \end{vmatrix}$$

$$\therefore\quad \overline{F} = i\left(\frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial z} - \frac{\partial \Phi}{\partial z}\frac{\partial \Psi}{\partial y}\right) - j\left(\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial z} - \frac{\partial \Phi}{\partial z}\frac{\partial \Psi}{\partial x}\right) + k\left(\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial y} - \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial x}\right)$$

$$\therefore\quad \nabla \cdot \overline{F} = \nabla \cdot [\nabla \Phi \times \nabla \Psi]$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left[i\left(\frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial z} - \frac{\partial \Phi}{\partial z}\frac{\partial \Psi}{\partial y}\right)\right.$$

$$\left. - j\left(\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial z} - \frac{\partial \Phi}{\partial z}\frac{\partial \Psi}{\partial x}\right) + k\left(\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial y} - \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial x}\right)\right]$$

$$\therefore\ \nabla\cdot\bar{F}=\frac{\partial^2\Phi}{\partial x\partial y}\cdot\frac{\partial\Psi}{\partial z}+\frac{\partial\Phi}{\partial y}\cdot\frac{\partial^2\Psi}{\partial x\partial z}-\frac{\partial^2\Phi}{\partial x\partial z}\cdot\frac{\partial\Psi}{\partial y}$$
$$-\frac{\partial\Phi}{\partial z}\cdot\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial^2\Phi}{\partial x\partial y}\cdot\frac{\partial\Psi}{\partial z}-\frac{\partial\Phi}{\partial x}\cdot\frac{\partial^2\Psi}{\partial y\partial z}$$
$$+\frac{\partial^2\Phi}{\partial y\partial z}\cdot\frac{\partial\Psi}{\partial x}+\frac{\partial\Phi}{\partial z}\cdot\frac{\partial^2\Psi}{\partial x\partial y}+\frac{\partial^2\Phi}{\partial z\partial x}\cdot\frac{\partial\Psi}{\partial y}$$
$$+\frac{\partial\Phi}{\partial x}\cdot\frac{\partial^2\Psi}{\partial y\partial z}-\frac{\partial^2\Phi}{\partial z\partial y}\cdot\frac{\partial\Psi}{\partial x}-\frac{\partial\Phi}{\partial y}\cdot\frac{\partial^2\Psi}{\partial x\partial y}=0$$

Now, by Gauss's Divergence Theorem

$$\iint_S \bar{N}\cdot\bar{F}\,ds=\iiint_V \nabla\cdot\bar{F}\,dv$$
$$=\iiint_V \nabla\cdot(\nabla\Phi\times\nabla\Psi)\,dv=\iiint_V 0\,dv=0$$

Example 5 : Prove that $\iint_S \frac{\bar{r}\cdot\bar{N}}{r^2}\,ds=\iiint_V \frac{1}{r^2}\,dV.$

Sol. : By Gauss's Divergence Theorem $\iint_S \bar{N}\cdot\bar{F}\,ds=\iiint_V \nabla\cdot\bar{F}\,dV$

Now, let $\bar{F}=\frac{\bar{r}}{r^2}=\frac{xi}{r^2}+\frac{yj}{r^2}+\frac{zk}{r^2}$

$$\therefore\ \nabla\cdot\bar{F}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(\frac{xi}{r^2}+\frac{yj}{r^2}+\frac{zk}{r^2}\right)$$

But $\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)=\frac{r^2\cdot 1-x\cdot 2r(\partial/\partial x)}{r^4}=\frac{r^2-2\cdot xr(x/r)}{r^4}$ $\quad[\because\ r^2=x^2+y^2+z^2]$

$$=(r^2-2x^2)/r^4$$

Similarly, $\frac{\partial}{\partial y}\left(\frac{y}{r^2}\right)=\frac{r^2-2y^2}{r^4},\quad \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)=\frac{r^2-2z^2}{r^4}$

$$\therefore\ \nabla\cdot\bar{F}=\frac{3r^2-2(x^2+y^2+z^2)}{r^4}=\frac{1}{r^2}\qquad\therefore\ \iint_S\frac{\bar{r}\cdot\bar{N}}{r^2}\,ds=\iiint_V\frac{1}{r^2}\,dv$$

Example 6 : Use divergence theorem to show that $\iint_S \nabla r^2\,\overline{ds}=6V$ where S is any closed surface enclosing a volume V. **(M.U. 1998, 2005, 08)**

Sol. : By Divergence Theorem $\iint_S \bar{N}\cdot\bar{F}\,ds=\iiint_V \nabla\cdot\bar{F}\,dv$

Here, $\bar{F}=\nabla r^2=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$

$$=2xi+2yj+2zk$$

$$\therefore \quad \nabla \cdot \bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (2x\,i + 2y\,j + 2z\,k) = 2 + 2 + 2.$$

$$\therefore \quad \iiint_V \nabla \cdot \bar{F}\, dv = \iiint_V 6\, dv = 6V$$

Example 7 : Evaluate $\iint_S \nabla r^2 \cdot \bar{N}\, ds$ where S is the surface of the sphere

$x^2 + y^2 + z^2 = 25$ and $r^2 = x^2 + y^2 + z^2$. (M.U. 1985, 9

Sol. : Following as above

$$\iint_S \nabla r^2 \cdot \bar{N}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv = 6V = 6\left(\frac{4}{3}\pi \cdot 5^3 \right) = 1000\,\pi$$

Example 8 : Show that $\iiint_V \nabla\Phi\, dv = \iint_S \Phi\, \bar{N}\, ds.$

Sol. : By Divergence Theorem, $\iint_S \bar{N} \cdot \bar{F}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$ .... (1

Comparing this with the given problem let $\bar{F} = \Phi\,\bar{a}$ where $\bar{a}$ is an arbitrary constant vecto
Hence, by (1)

$$\iint_S \bar{N} \cdot \Phi\,\bar{a}\, ds = \iiint_V \nabla \cdot \Phi\,\bar{a}\, dv \quad \text{..................... (2}$$

Now, $\nabla \cdot \Phi\,\bar{a} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (\Phi\, a_1\, i + \Phi\, a_2\, j + \Phi\, a_3\, k)$

$$= \frac{\partial}{\partial x}(\Phi\, a_1) + \frac{\partial}{\partial y}(\Phi\, a_2) + \frac{\partial}{\partial z}(\Phi\, a_3) = a_1\frac{\partial \Phi}{\partial x} + a_2\frac{\partial \Phi}{\partial y} + a_3\frac{\partial \Phi}{\partial z}$$

$$= (a_1\, i + a_2\, j + a_3\, k) \cdot \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\Phi = \bar{a} \cdot \nabla\Phi$$

Hence, from (2), we get,

$$\iint_S \bar{N} \cdot \Phi\,\bar{a}\, ds = \iiint_V \bar{a} \cdot \nabla\Phi\, dv$$

$$\therefore \quad \iint_S \bar{a} \cdot \Phi\,\bar{N}\, ds = \iiint_V \bar{a} \cdot \nabla\Phi\, dv \qquad \therefore \quad \iint_S \Phi\,\bar{N}\, ds = \iiint_V \nabla\Phi\, dv$$

Example 9 : If $\bar{G} = \nabla\Phi$ and $\nabla^2\Phi = 0$, prove that $\iiint_V G^2\, dv = \iint_S \Phi\,\bar{G} \cdot \bar{N}\, ds.$

Sol. : By Gauss's Divergence Theorem $\iiint_V \nabla \cdot \bar{F}\, dv = \iint_S \bar{N} \cdot \bar{F}\, ds$

Comparing r.h.s. of this with r.h.s. of the required result, we put $\bar{F} = \Phi\,\bar{G}$

Now, $\nabla \cdot \bar{F} = \nabla \cdot (\Phi\,\bar{G})$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (\Phi\, G_1\, i + \Phi\, G_2\, j + \Phi\, G_3\, k)$$

$$= \Phi\frac{\partial G_1}{\partial x} + G_1\frac{\partial \Phi}{\partial x} + \Phi\frac{\partial G_2}{\partial y} + G_2\frac{\partial \Phi}{\partial y} + \Phi\frac{\partial G_3}{\partial z} + G_3\frac{\partial \Phi}{\partial z}$$

$$\therefore \quad \nabla \cdot \overline{F} = \Phi\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\right) + G_1\frac{\partial \Phi}{\partial x} + G_2\frac{\partial \Phi}{\partial y} + G_3\frac{\partial \Phi}{\partial z} \qquad \text{.................... (1)}$$

But, by data $\overline{G} = \nabla\Phi$ *i.e.* $G_1 i + G_2 j + G_3 j = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k$

$$\therefore \quad G_1 = \frac{\partial \Phi}{\partial x},\ G_2 = \frac{\partial \Phi}{\partial y},\ G_3 = \frac{\partial \Phi}{\partial z}$$

$$\therefore \quad \frac{\partial G_1}{\partial x} = \frac{\partial^2 \Phi}{\partial x^2},\ \frac{\partial G_2}{\partial y} = \frac{\partial^2 \Phi}{\partial y^2},\ \frac{\partial G_3}{\partial z} = \frac{\partial^2 \Phi}{\partial z^2}$$

Hence, from (1), we get, $\nabla \cdot \overline{F} = \Phi\,\nabla^2\Phi + (G_1^{\,2} + G_2^{\,2} + G_3^{\,2})$ .... (2)

But by data, $\nabla^2\Phi = 0$ and $G_1^{\,2} + G_2^{\,2} + G_3^{\,2} = G^2$

$\therefore$ From (2), we get $\nabla \cdot \overline{F} = G^2$

Hence, $\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V G^2\, dv$ gives $\iint_S \Phi\, \overline{N} \cdot \overline{G}\, ds = \iiint_V G^2 dv$.

Example 10 : Show that $\iint_S (\nabla r^n) \cdot d\,\overline{s} = n(n+1)\iiint_V r^{n-2} dv.$

Sol. : We have by Gauss's Divergence Theorem,

$$\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V \nabla \cdot \overline{F}\, dv$$

Comparing l.h.s. of this with l.h.s. of the given result, we see that

$$\overline{F} = \nabla r^n = i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z}$$

But $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{\partial}{\partial x}(r^n) = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{n/2} = \frac{n}{2}(x^2 + y^2 + z^2)^{(n/2)-1} \cdot 2x$$

$$= nx(x^2 + y^2 + z^2)^{(n/2)-1} = nx\, r^{n-2}$$

$$\overline{F} = \nabla r^n = nx\, r^{n-2} i + ny\, r^{n-2} j + nz\, r^{n-2} k$$

$$\therefore \quad \nabla \cdot \overline{F} = \frac{\partial}{\partial x}(nx\, r^{n-2}) + \frac{\partial}{\partial y}(ny\, r^{n-2}) + \frac{\partial}{\partial z}(n\, r^{n-2} z)$$

Now, $\frac{\partial}{\partial x}(nx\, r^{n-2}) = n \cdot 1 \cdot r^{n-2} + nx(n-2)\, r^{n-3}\frac{\partial r}{\partial x}$

$$= n\, r^{n-2} + nx(n-2)\, r^{n-3}\frac{x}{r}$$

$$= n\, r^{n-2} + nx^2(n-2)\, r^{n-4}$$

$$\therefore \quad \nabla \cdot \overline{F} = 3n\, r^{n-2} + n(n-2)\, r^{n-4}(x^2 + y^2 + z^2)$$

$$= 3n\, r^{n-2} + n(n-2)\, r^{n-2} = n(n+1)\, r^{n-2}$$

$$\therefore \quad \iint \overline{N} \cdot \nabla r^n ds = \iiint_V n(n+1)\, r^{n-2}\, dv.$$

Example 11 : Prove that $\iint_S \frac{\bar{r}\cdot\bar{N}}{r^3}\,ds = 0$ if the origin lies outside the closed surface S and $\iint_S \frac{\bar{r}\cdot\bar{N}}{r^3}\,ds = 4\pi$ if the origin lies inside the surface S where $\bar{r} = x\,i + y\,j + z\,k$ and $r = |\bar{r}|$.

(M.U. 1993, 99, 2001)

Sol. : ***Case I :*** When the origin lies outside the closed surface S, $\bar{F} = \frac{\bar{r}}{r^3}$ is well defined in the closed region and is differentiable everywhere within S.

Now, by divergence theorem $\iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$

$$\therefore \quad \bar{F} = \frac{\bar{r}}{r^3} = \frac{x\,i}{r^3} + \frac{y\,j}{r^3} + \frac{z\,k}{r^3}$$

$$\therefore \quad \nabla\cdot\bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(\frac{x\,i}{r^3} + \frac{y\,j}{r^3} + \frac{z\,k}{r^3}\right)$$

$$= \left\{\frac{(x^2+y^2+z^2)^{3/2} - x(3/2)(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3}\right\} + \{\ldots\ldots\} + \{\ldots\ldots\}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}(x^2+y^2+z^2-3x^2) + \{\ldots\ldots\} + \{\ldots\ldots\}$$

$$= \frac{1}{(x^2+y^2+z^2)^{5/2}}\{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)\}$$

$$\therefore \quad \nabla\cdot\bar{F} = 0$$

$$\therefore \quad \iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V 0\cdot dv = 0$$

Case II : When the origin lies inside the closed surface S, $\bar{F} = \frac{\bar{r}}{r^3}$ is not defined at the origin and hence, is not differentiable and divergence theorem is not applicable. Now, consider a sphere S_1 with centre at the origin and a small radius a. Now, $\bar{F}$ is well defined and differentiable everywhere within the region enclosed between two surfaces S_1 and S.

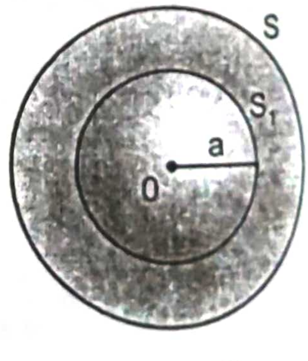


Fig. 9.27

Hence, by Divergence theorem

$$\iint_S \frac{\bar{r}\cdot\bar{N}}{r^3}\,ds + \iint_{S_1} \frac{\bar{r}\cdot\bar{N}}{r^3}\,ds = \iiint_{V_1} \nabla\cdot\bar{F}\,dv$$

where V_1 is the volume of the region between the sphere S_1 and the given surface S. But since, the region does not enclose the origin by case I,

$$\iiint_{V_1} \nabla\cdot\bar{F}\,dv = 0 \qquad \therefore \quad \iint_S \frac{\bar{r}\cdot\bar{N}}{r^3}\,ds = -\iint_{S_1} \frac{\bar{r}\cdot\bar{N}}{r^3}\,ds$$

But for the inner sphere of radius a with centre at the origin outward normal at any point is negative of its radius vector at that point.

$$\therefore \quad \bar{N} = -\frac{\bar{r}}{r}$$

And for the surface of the sphere $r = a$.

$$\therefore \iint_S \frac{\bar{r} \cdot \bar{N}}{r^3} ds = \iint_{S_1} \frac{\bar{r} \cdot \bar{r}}{a^3 \times a} ds = \frac{1}{a^2} \iint_S ds = \frac{1}{a^2} 4\pi a^2 = 4\pi.$$

Example 12 : Prove that

$$\iint_S (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot \bar{N} \, ds = \iiint (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) \, dv.$$

(M.U. 1994)

Sol. : By Gauss's Divergence Theorem $\iint_S \bar{N} \cdot \bar{F} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$ .... (1)

Let $\bar{F} = \Phi \nabla \Psi$. By (1), we have,

$$\therefore \iint_S \Phi \nabla \Psi \cdot \bar{N} \, ds = \iiint_V \nabla \cdot (\Phi \nabla \Psi) \, dv$$

$$\text{Now, } \nabla \cdot \Phi \nabla \Psi = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot \left(\Phi \frac{\partial \Psi}{\partial x} i + \Phi \frac{\partial \Psi}{\partial y} j + \Phi \frac{\partial \Psi}{\partial z} k \right)$$

$$= \Phi \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial \Phi}{\partial x} \frac{\partial \Psi}{\partial x} + \Phi \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial \Phi}{\partial y} \frac{\partial \Psi}{\partial y} + \Phi \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial \Phi}{\partial z} \frac{\partial \Psi}{\partial z}$$

$$= \Phi \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + \frac{\partial \Phi}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \Psi}{\partial y} + \frac{\partial \Phi}{\partial z} \frac{\partial \Psi}{\partial z}$$

$$= \Phi \nabla^2 \Psi + \nabla \Phi \cdot \nabla \Psi$$

$$\therefore \iint_S \Phi \nabla \Psi \cdot \bar{N} \, ds = \iiint_V (\Phi \nabla^2 \Psi + \nabla \Phi \cdot \nabla \Psi) \, dv \quad \text{.......................... (2)}$$

Interchanging Φ and Ψ in (2), we get,

$$\iint_S \Psi \nabla \Phi \cdot \bar{N} \, ds = \iiint_V (\Psi \nabla^2 \Phi + \nabla \Psi \cdot \nabla \Phi) \, dv \quad \text{.......................... (3)}$$

Subtracting (3) from (2), we get,

$$\iint_S (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot \bar{N} \, ds = \iiint_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) \, dv$$

Type II : When $\bar{F}$ is given

Example 1 : If S is any closed surface enclosing a volume V and if $\bar{F} = ax\, i + by\, j + cz\, k$ then prove that $\iint_S \bar{N} \cdot \bar{F} \, ds = (a + b + c) V$. **(M.U. 2005)**

Sol. : By Gauss's Divergence Theorem $\iint_S \bar{N} \cdot \bar{F} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$

$$\text{Now, } \nabla \cdot \bar{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) = a + b + c$$

$$\therefore \iint_F \bar{N} \cdot \bar{F} \, ds = \iiint_V (a + b + c) \, dv$$

$$= (a + b + c) \iiint_V dV = (a + b + c) V$$

Example 2 : Evaluate $\iint_S (ax\,i + by\,j + cz\,k) \cdot \overline{N}\,ds$ where S is $ax^2 + by^2 + cz^2 = abc$ (a, b, c are positive constants). (M.U. 2000)

Sol. : Following as in Ex. 1 above $\iint_S (ax\,i + by\,j + cz\,k) = (a + b + c)V$

where V is the volume of the ellipsoid $\frac{x^2}{bc} + \frac{y^2}{ca} + \frac{z^2}{ab} = 1$.

i.e. $V = \frac{4\pi}{3}\sqrt{bc}\,\sqrt{ca}\,\sqrt{ab} = \frac{4\pi}{3}abc \qquad \therefore \; I = \frac{4\pi}{3}abc(a + b + c)$

Example 3 : Evaluate $\iint_S (9x\,i + 6y\,j - 10z\,k) \cdot d\overline{s}$ where S is the surface of the sphere with radius 2. (M.U. 2014)

Sol. : By Ex. 1 above $\iint_S (9x\,i + 6y\,j - 10z\,k) \cdot d\overline{s} = (9 + 6 - 10) \cdot \frac{4}{3}\pi \cdot 2^3 = \frac{400}{3}\pi$.

Example 4 : If S is a closed surface enclosing volume V, prove that

$$\iint_S \overline{r} \cdot d\overline{s} = 3V.$$

(M.U. 2006)

Sol. : Let $\overline{F} = \overline{r}$ then

$$\nabla \cdot \overline{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (x\,i + y\,j + z\,k) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

Hence, by Gauss's Divergence Theorem

$$\iint_S \overline{N} \cdot \overline{F}\,ds = \iiint_V 3dv = 3V.$$

Cor. : If S is a closded surface enclosing a unit volume then find

$$\iint_S \overline{r} \cdot d\overline{s}.$$

(M.U. 2001) [Ans. : 3]

Example 5 : Use Gauss's Divergence Theorem to evaluate $\iint_S \overline{N} \cdot \overline{F}\,ds$ where $\overline{F} = x^2 i + zj + yzk$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Sol. : By the Gauss's Divergence Theorem $\iint_S \overline{N} \cdot \overline{F}\,ds = \iiint_V \nabla \cdot \overline{F}\,dV$

But $\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz) = 2x + y$

$$\therefore \; \iiint_V \nabla \cdot \overline{F}\,dV = \int_0^1\int_0^1\int_0^1 (2x + y)\,dx\,dy\,dz = \int_0^1\int_0^1 [2xz + yz]_0^1\,dx\,dy$$

$$= \int_0^1\int_0^1 (2x + y)\,dx\,dy = \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \left(2x + \frac{1}{2}\right)dx$$

$$\therefore \; \iiint_V \nabla \cdot \overline{F}\,dV = \left[x^2 + \frac{1}{2}x\right]_0^1 = \frac{3}{2} \qquad \therefore \; \iint_S \overline{N} \cdot \overline{F}\,ds = \frac{3}{2}.$$

Example 6 : Use Gauss's Divergence Theorem to evaluate $\iint\limits_S \overline{N} \cdot \overline{F}\, ds$ where $\overline{F} = 4x\,i + 3y\,j - 2z\,k$ and S is the surface bounded by $x = 0,\ y = 0,\ z = 0$ and $2x + 2y + z = 4$.

Sol. : By Divergence Theorem $\iint\limits_S \overline{N} \cdot \overline{F}\, ds = \iiint\limits_V \nabla \cdot \overline{F}\, dv$

Now, $\overline{F} = 4x\,i + 3y\,j - 2z\,k$ $\quad\therefore\ \nabla \cdot \overline{F} = 4 + 3 - 2 = 5$

Now, $$\iiint\limits_V \nabla \cdot \overline{F}\, dv = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{z=4-2x-2y} 5\, dx\, dy\, dz$$

$$= \int_0^2 \int_0^{2-x} 5(4 - 2x - 2y)\, dx\, dy$$

$$= 5\int_0^2 \left[4y - 2xy - y^2\right]_0^{2-x} dx$$

$$= 5\int_0^2 \left[4(2 - x) - 2x(2 - x) - (2 - x)^2\right] dx$$

$$= 5\int_0^2 \left[4 - 4x + x^2\right] dx = 5\left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2$$

$$= 5\left[8 - 8 + \frac{8}{3}\right] = \frac{40}{3}$$

$$\therefore \iint\limits_S \overline{N} \cdot \overline{F}\, ds = \frac{40}{3}.$$

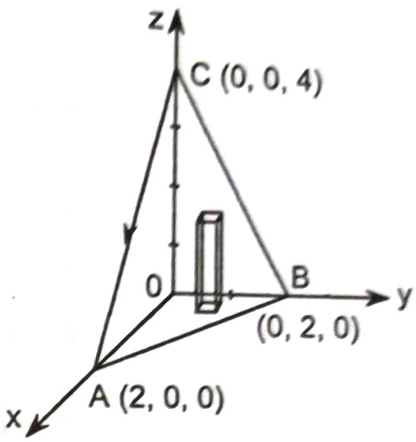


Fig. 9.28

Example 7 : Use Gauss's Divergence Theorem to evaluate $\iint\limits_S \overline{N} \cdot \overline{F}\, ds$ where $\overline{F} = 4x\,i - 2y^2\,j + z^2\,k$ and S is the region bounded by $x^2 + y^2 = 4,\ z = 0,\ z = 3$.

(M.U. 1992, 94, 98, 2001, 06)

Sol. : By Divergence Theorem $\iint\limits_S \overline{N} \cdot \overline{F}\, ds = \iiint\limits_V \nabla \cdot \overline{F}\, dv$

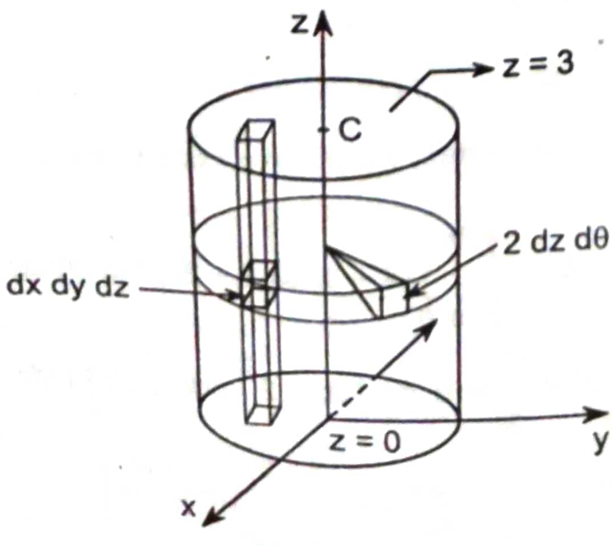


Fig. 9.29 (a)

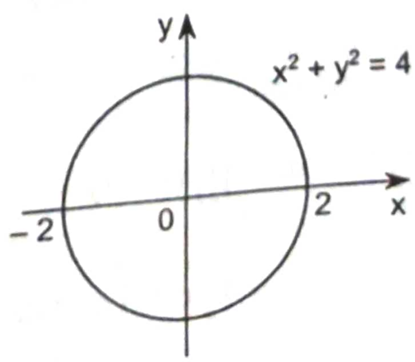


Fig. 9.29 (b)

Now, $\overline{F} = 4x\,i - 2y^2\,j + z^2\,k \qquad \therefore \quad \nabla \cdot \overline{F} = 4 - 4y + 2z$

$$\therefore \quad \iiint\limits_V \nabla \cdot \overline{F}\, dv = \iiint\limits_V (4 - 4y + 2z)\, dx\, dy\, dz$$

For the whole volume z varies from 0 to 3, y varies from $-\sqrt{4-x^2}$ to $+\sqrt{4-x^2}$ and x varies from -2 to 2.

$$\therefore \quad \iiint\limits_V \nabla \cdot \overline{F}\, dv = \int\limits_{x=-2}^{2} \int\limits_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int\limits_{z=0}^{3} (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3 dx\, dy$$

$$= \int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dxdy$$

$$= \int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dx\, dy$$

[$\because \int\limits_{-a}^{a} 12y\, dy = 0$ as $12y$ is an odd function and $\int\limits_{-a}^{a} 21 dy = 2 \times 21 \int\limits_{0}^{a} dy$ as 21 dy is an even function.]

$$= \int\limits_{-2}^{2} \int\limits_{0}^{\sqrt{4-x^2}} 2 \times 21\, dx\, dy = \int\limits_{-2}^{2} \left[42\, y\right]_0^{\sqrt{4-x^2}} dx$$

$$= 42 \int\limits_{-2}^{2} \sqrt{4-x^2}\, dx = 42 \times 2 \int\limits_{0}^{2} \sqrt{4-x^2}\, dx$$

$$= 84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84 \cdot 2 \cdot \frac{\pi}{2} = 84\pi \qquad \text{.......................... (1)}$$

Alternatively, changing to cylindrical polar coordinates, we put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $dx\, dy\, dz = r\, dr\, d\theta\, dz$.

$$\therefore \quad \iiint\limits_V \nabla \cdot \overline{F}\, dv = \iiint (4 - 4y + 2z)\, dx\, dy\, dz = \int\limits_0^a \int\limits_0^{2\pi} \int\limits_0^b (4 - 4r\sin\theta + 2z)\, r\, dr\, d\theta\, dz$$

$$= 4\int\limits_0^a r\, dr \int\limits_0^{2\pi} d\theta \int\limits_0^b dz - 4\int\limits_0^a r^2\, dr \int\limits_0^{2\pi} \sin\theta\, d\theta \int\limits_0^b dz + 2\int\limits_0^a r\, dr \int\limits_0^{2\pi} d\theta \int\limits_0^b z\, dz$$

$$= 4 \cdot \frac{(a^2)}{2}(2\pi)(b) - 4 \cdot \frac{(a^3)}{2}\left[-\cos\theta\right]_0^{2\pi} b + 2 \cdot \frac{(a^2)}{2}(2\pi)\left(\frac{b^2}{2}\right)$$

$$= 4\pi a^2 b + 0 + \pi a^2 b^2$$

Putting $a = 2,\ b = 3$

$$\iiint_V \nabla \cdot \overline{F}\, dv = 4\pi \cdot 4 \cdot 3 + \pi \cdot 4 \cdot 9 = 84\pi.$$

$$\therefore \iint_S \overline{N} \cdot \overline{F}\, ds = 84\pi.$$

Example 8 : Use Gauss's Divergence Theorem to evaluate $\iint_S \overline{N} \cdot \overline{F}\, ds$ where $\overline{F} = 2xi + xyj + zk$ over the region bounded by the cylinder $x^2 + y^2 = 4,\ z = 0,\ z = 6.$

(M.U. 1996)

Sol. : By Divergence Theorem $\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V \nabla \cdot \overline{F}\, dv$

Here, $\overline{F} = 2xi + xyj + zk$

$$\therefore \quad \nabla \cdot \overline{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(z) = 2 + x + 1 = 3 + x$$

$$\therefore \quad \iiint_V \nabla \cdot \overline{F} = \iiint_V (3 + x)\, dv = \iiint_V (3 + x)\, dx\, dy\, dz \qquad \text{.......... (A)}$$

Now, to cover the whole volume bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$ and $z = 6$, z varies from 0 to 6, y varies from $-\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$, and x varies from -2 to 2 (as in the previous example).

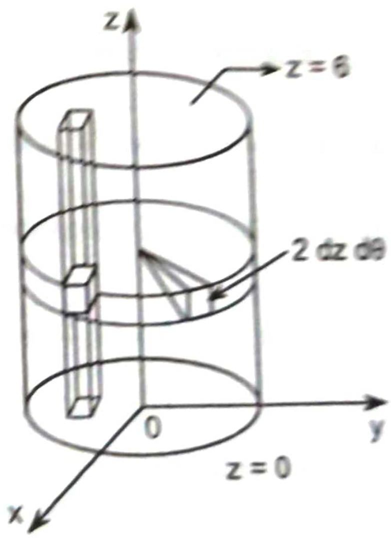


Fig. 9.30

$$\therefore \iiint_V (3 + x)\, dx\, dy\, dz = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{6} (3 + x)\, dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [3z + xz]_0^6\, dx\, dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (18 + 6x)\, dx\, dy$$

$$= \int_{-2}^{2} [18y + 6xy]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\, dx$$

$$= \int_{-2}^{2} \left[\left(18\sqrt{4 - x^2} + 6x\sqrt{4 - x^2}\right) - \left(-18\sqrt{4 - x^2} - 6x\sqrt{4 - x^2}\right)\right] dx$$

$$= \int_{-2}^{2} \left(36\sqrt{4 - x^2} + 12x\sqrt{4 - x^2}\right) dx$$

$$= \left[36\left\{\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right\} - 4(4 - x^2)^{3/2}\right]_{-2}^{2}$$

$$= 36\left\{2 \cdot \frac{\pi}{2} + 2\frac{\pi}{2}\right\} = 72\pi$$

Aliter : By using cylindrical coordinates, we put

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \text{ and } dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

$$0 \le x \le 2,\quad 0 \le \theta \le 2\pi,\ \ 0 \le z \le 6$$

$\therefore$ From (A), we get

$$\iiint_V \nabla\cdot\bar{F} = \iiint_V (3+x)\,dx\,dy\,dz = \int_0^2\int_0^{2\pi}\int_0^6 (3+r\cos\theta)\,r\,dr\,d\theta\,dz$$

$$= \int_0^2\int_0^{2\pi}\int_0^6 3r\,dr\,d\theta\,dz + \int_0^2\int_0^{2\pi}\int_0^6 r^2\cos\theta\,dr\,d\theta\,dz$$

$$= 3\int_0^2 r\,dr\cdot\int_0^{2\pi} d\theta\cdot\int_0^6 dz + \int_0^2 r^2\,dr\cdot\int_0^{2\pi}\cos\theta\,d\theta\cdot\int_0^6 dz$$

$$= 3\left[\frac{r^2}{2}\right]_0^2\cdot[\theta]_0^{2\pi}\cdot[z]_0^6 + \left[\frac{r^3}{3}\right]_0^2\cdot[\sin\theta]_0^{2\pi}\cdot[z]_0^6$$

$$\therefore\quad \iiint_V \nabla\cdot\bar{F} = 3\cdot\frac{4}{2}\cdot 2\pi\cdot 6 + 0 = 72\pi.$$

$$\therefore\quad \iint_S \bar{N}\cdot\bar{F}\,ds = 72\pi.$$

Example 9 : Use Gauss's Divergence Theorem to evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = xi + yj + z^2k$ and S is the closed surface bounded by the cone $x^2+y^2=z^2$ and the plane $z=1$.

Sol. : By Gauss's Divergence Theorem, we have $\iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$.

Now, $$\nabla\cdot\bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(xi+yj+z^2k)$$

$$= 1+1+2z = 2(1+z)$$

We shall obtain the volume integral by using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $dx\,dy\,dz = r\,dr\,d\theta\,dz$.

$\because\quad r^2 = x^2+y^2$ and by data $x^2+y^2 = z^2$.

$\therefore\quad z = r$. Hence, z varies from r to 1.

$$\therefore\quad I = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r}^{z=1} 2(1+z)\,r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^1 2\left[z+\frac{z^2}{2}\right]_r^1 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2\left[1+\frac{1}{2}-r-\frac{r^2}{2}\right]r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2\left(\frac{3}{2}r - r^2 - \frac{r^3}{2}\right)dr\,d\theta$$

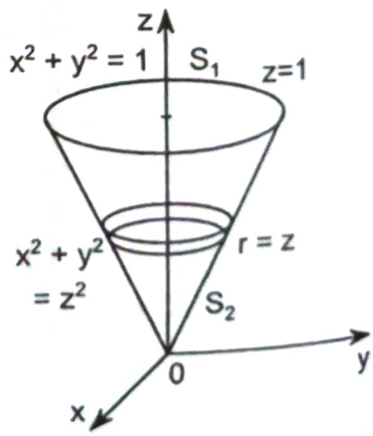


Fig. 9.31

$$\therefore \quad I = \int_0^{2\pi} 2\left[\frac{3}{4}r^2 - \frac{r^3}{3} - \frac{r^4}{8}\right]_0^1 d\theta = \int_0^{2\pi} 2\left[\frac{3}{4} - \frac{1}{3} - \frac{1}{8}\right] d\theta = \int_0^{2\pi} \frac{7}{12}\, d\theta$$

$$= \frac{7}{12}\left[\theta\right]_0^{2\pi} = \frac{7}{12} \cdot 2\pi = \frac{7}{6}\pi$$

$$\therefore \quad \iint_S \overline{N} \cdot \overline{F}\, ds = \frac{7}{6}\pi.$$

Example 10 : Using Gauss's Divergence Theorem, prove that

$$\iint_S (y^2z^2 i + z^2x^2 j + z^2y^2 k) \cdot \overline{N}\, ds = \frac{\pi}{12}$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane. **(M.U. 2000)**

Sol. : By Divergence Theorem $\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V \nabla \cdot \overline{F}\, dv$

Now, we have $\overline{F} = y^2z^2 i + z^2x^2 j + z^2y^2 k$

$$\therefore \quad \nabla \cdot \overline{F} = \frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2) = 2zy^2$$

$$\therefore \quad \iint_S (y^2z^2 i + z^2x^2 j + z^2y^2 k) \cdot \overline{N}\, ds = \iiint_V 2zy^2\, dv \qquad \text{.................... (A)}$$

where V is the volume of the given sphere above the xy-plane.

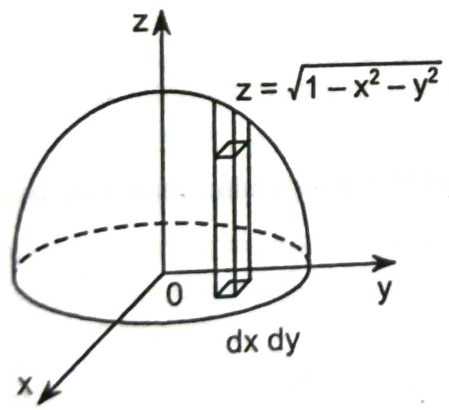


Fig. 9.32 (a)

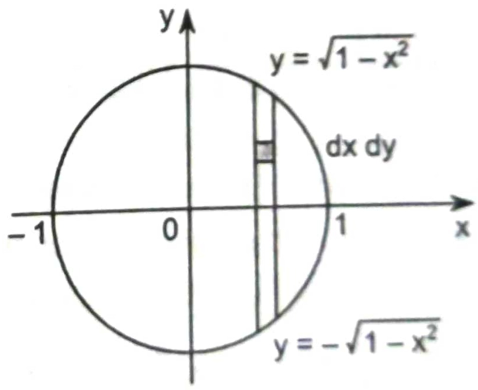


Fig. 9.32 (b)

From the above figure it is clear that z varies from 0 to $\sqrt{1 - x^2 - y^2}$, y varies from $-\sqrt{1 - x^2}$ to $\sqrt{1 - x^2}$ and x varies from -1 to 1.

$$\therefore \quad I = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} 2zy^2\, dx\, dy\, dz = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[y^2z^2\right]_0^{\sqrt{1-x^2-y^2}} dx\, dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2(1 - x^2 - y^2)\, dx\, dy = \int_{-1}^{1} \left[(1 - x^2)\frac{y^3}{3} - \frac{y^5}{5}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ds$$

$$= 2\int_{-1}^{1} \left[\frac{(1 - x^2)^{5/2}}{3} - \frac{(1 - x^2)^{5/2}}{5}\right] dx$$

$$\therefore \quad I = \frac{4}{15}\int_{-1}^{1}(1-x^2)^{5/2}\,dx = \frac{8}{15}\int_{0}^{1}(1-x^2)^{5/2}\,dx$$

Now, put $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$\therefore \quad I = \frac{8}{15}\int_{0}^{\pi/2}\cos^5\theta\cos\theta\,d\theta = \frac{8}{15}\int_{0}^{\pi/2}\cos^6\theta\,d\theta$$

$$= \frac{8}{15}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{12} \quad \text{[By reduction formula]}$$

Aliter : Changing to spherical coordinates, we put

$x = r\sin\theta\cos\Phi$, $y = r\sin\theta\sin\Phi$, $z = r\cos\theta$, $dv = r^2\sin\theta\, dr\, d\theta\, d\Phi$.

Since, the region of integration is the hemi-sphere, we put

$0 \le r \le a = 1$, $0 \le \theta \le \pi/2$, $0 \le \Phi \le 2\pi$.

$$\therefore \quad I = \iiint_{2\pi} 2zy^2\,dv = \int_0^1\int_0^{\pi/2}\int_0^{2\pi} 2\cdot r\cos\theta\cdot r^2\sin^2\theta\sin^2\Phi\cdot r^2\sin\theta\,dr\,d\theta\,d\Phi$$

$$= 2\int_0^1 r^5\,dr\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\int_0^{2\pi}\sin^2\Phi\,d\Phi$$

$$= 2\int_0^1 r^5\,dr\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\cdot 4\cdot\int_0^{\pi/2}\sin^2\Phi\,d\Phi$$

$$\therefore \quad I = 2\left[\frac{r^6}{6}\right]_0^1\cdot\left[\frac{2\cdot1\cdot1}{4\cdot2\cdot1}\right]\cdot4\cdot\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{\pi}{12}.$$

Example 11 : Evaluate $\iiint \nabla\cdot\bar{F}\,dV$ taken over the rectangular parallelepiped

$0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ and where

$\bar{F} = (x^2 - yz)\,i + (y^2 - zx)\,j + (z^2 - xy)\,k.$ **(M.U. 2004)**

Sol. : Here $\nabla\cdot\bar{F} = \dfrac{\partial}{\partial x}(x^2 - yz) + \dfrac{\partial}{\partial y}(y^2 - zx) + \dfrac{\partial}{\partial z}(z^2 - xy)$

$= 2x + 2y + 2z$

$$\therefore \quad \iiint \nabla\cdot\bar{F}\,dV = 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xy + yz + \frac{z^2}{2}\right]_0^c dx\,dy = 2\int_{x=0}^{a}\int_{y=0}^{b}\left[cx + cy + \frac{c^2}{2}\right]dx\,dy$$

$$= 2\int_{x=0}^{a}\left[cxy + \frac{cy^2}{2} + \frac{c^2}{2}\cdot y\right]_0^b dx = 2\int_{x=0}^{a}\left[bcx + \frac{b^2c}{2} + \frac{bc^2}{2}\right]dx$$

$$= 2\left[bc\frac{x^2}{2} + \frac{b^2cx}{2} + \frac{bc^2}{2}\cdot x\right]_0^a$$

$$= (a^2bc + ab^2c + abc^2) = abc\,(a+b+c)$$

Example 12 : Use Divergence Theorem to evaluate $\iint \bar{F} \cdot d\bar{s}$ where $\bar{F} = x^3 i + y^3 j + z^3 k$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

(M.U. 1995, 2005, 06)

Sol. : By Divergence Theorem $\iint_S \bar{N} \cdot \bar{F}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$

Now, $\nabla \cdot \bar{F} = \nabla \cdot (x^3 i + y^3 j + z^3 k) = \dfrac{\partial}{\partial x}(x^3) + \dfrac{\partial}{\partial y}(y^3) + \dfrac{\partial}{\partial z}(z^3)$

$$= 3(x^2 + y^2 + z^2) = 3r^2$$

Now, V is the volume of the sphere of radius a and centre at the origin. For integration we use spherical polar coordinates by putting $x = r \sin\theta \cos\Phi$, $y = r\sin\theta \sin\Phi$, $z = r\cos\theta$ where $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \Phi \le 2\pi$ and $x^2 + y^2 + z^2 = r^2$ and $dV = r^2 \sin\theta\, dr\, d\theta\, d\Phi$.

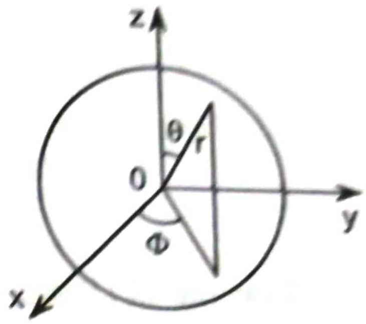


Fig. 9.33

$$\therefore \iiint_V \nabla \cdot \bar{F}\, dv = 3 \int_{\Phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} (r^2)\, r^2 \sin\theta\, dr\, d\theta\, d\Phi$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \left[\frac{r^5}{5}\right]_0^a \sin\theta\, d\theta\, d\Phi = \frac{3a^5}{5} \int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\Phi$$

$$= \frac{6a^5}{5} \int_0^{2\pi} d\Phi = \frac{6a^5}{5} [\Phi]_0^{2\pi} = \frac{12a^5}{5}.$$

Note ....

If the region of integration is a sphere or an ellipsoid it is convenient to use spherical polar coordinates $x = r\sin\theta\cos\Phi$, $y = r\sin\theta\sin\Phi$, $z = r\cos\theta$, $dV = dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\Phi$.

(i) Limits for complete sphere are $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \Phi \le 2\pi$

(ii) Limits for hemisphere are $0 \le r \le a$, $0 \le \theta \le \pi/2$, $0 \le \Phi \le 2\pi$

(iii) Limits for an octant are $0 \le r \le a$, $0 \le \theta \le \pi/2$, $0 \le \Phi \le \pi/2$

Example 13 : Using Gauss's Divergence Theorem evaluate $\iint_S \bar{F} \cdot d\bar{s}$

where $\bar{F} = 4xi - 2y^2 j + 3z^2 k$ and S is the surface $x^2 + y^2 + z^2 = a^2$, $z = 0$, $z = b$.

Sol. : By Gauss's Divergence Theorem $\iint_S \bar{N} \cdot \bar{F}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$

Now, $\nabla \cdot \bar{F} = \dfrac{\partial}{\partial x}(4x) - \dfrac{\partial}{\partial y}(2y^2) + \dfrac{\partial}{\partial z}(3z^2) = 4 - 4y + 6z$

$$\therefore \iiint_V \nabla \cdot \bar{F}\, dv = \iiint_V (4 - 4y + 6z)\, dx\, dy\, dz$$

We now change to cylindrical coordinates and put

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dx\, dy\, dz = r\, dr\, d\theta\, dz$.

$$\therefore \iiint_V \nabla \cdot \bar{F}\, dv = \int_0^{2\pi} \int_0^a \int_0^b (4 - 4r\sin\theta + 6z)\, r\, dr\, d\theta\, dz$$

$$\therefore \iiint_V \nabla \cdot \bar{F}\, dv = \int_0^{2\pi}\int_0^a \left[4rz - 4r^2 \sin\theta z + 3z^2 r\right]_0^b dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^a (4rb - 4r^2 b\sin\theta + 3b^2 r)\, dr\, d\theta = \int_0^{2\pi}\left[2r^2 b - 4\frac{r^3}{3}\sin\theta + 3b^2\frac{r^2}{2}\right]_0^a d\theta$$

$$= \int_0^{2\pi}\left[2a^2 b - \frac{4a^3}{3}\sin\theta + \frac{3a^2b^2}{2}\right] d\theta = \left[2a^2 b\theta + \frac{4a^3}{3}\cos\theta + \frac{3a^2b^2}{2}\theta\right]_0^{2\pi}$$

$$= \left(2a^2 b + \frac{3a^2b^2}{2}\right)2\pi = (4 + 3b)\pi a^2 b$$

Example 14 : Using Gauss's Divergence Theorem evaluate $\iint_S \bar{F} \cdot d\bar{s}$ where $\bar{F} = 2x^2 yi - y^2 j + 4xz^2 k$ and s is the region bounded by $y^2 + z^2 = 9$ and $x = 2$ in the first octant. **(M.U. 2003)**

Sol. : By Divergence Theorem $\iint_S \bar{N} \cdot \bar{F}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$

Now, $\bar{F} = 2x^2 y\, i - y^2 j + 4xz^2 k$

$\therefore \quad \nabla \cdot \bar{F} = 4xy - 2y + 8xz$

$$\therefore \iiint_V \nabla \cdot \bar{F}\, dv = \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}}\int_{x=0}^{2}(4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}}\left[2x^2 y - 2xy + 4x^2 z\right]_{x=0}^{2} dy\, dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}}(4y + 16z)\, dy\, dz$$

$$= \int_{z=0}^{3}\left[2y^2 + 16yz\right]_{y=0}^{\sqrt{9-z^2}}$$

$$= \int_0^3\left[2(9 - z^2) + 16z\sqrt{9 - z^2}\right] dz$$

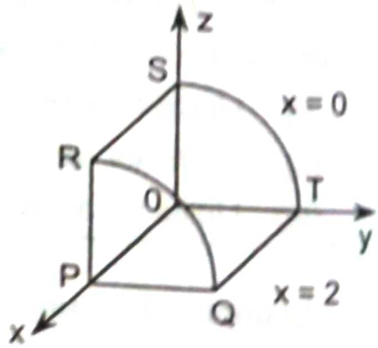


Fig. 9.34

$$= \left[2\left(9z - \frac{z^3}{3}\right) + 16(9 - z^2)^{3/2}\left(\frac{2}{3}\right)\left(-\frac{1}{2}\right)\right]_0^3$$

$$= 2(18) + 16(9) = 180$$

$$\therefore \iint_S \bar{N} \cdot \bar{F}\, ds = 180$$

Type III : $\overline{F}$ is to be found

Example 1 : Using Gauss's Divergence Theorem evaluate $\iint_S (ax^2 + by^2 + cz^2)\, ds$ over the sphere $x^2 + y^2 + z^2 = 1$. **(M.U. 1993, 2000, 06)**

Sol. : By Divergence Theorem $\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V \nabla \cdot \overline{F}\, dv$ .... (1)

We first note that we are given $\overline{N} \cdot \overline{F} = ax^2 + by^2 + cz^2$. We have to find $\overline{F}$.

Now, unit normal to the surface Φ is $\dfrac{\nabla\Phi}{|\nabla\Phi|}$. [See Example (A), page 8-12]

If $\Phi \equiv x^2 + y^2 + z^2 = 1$, then $\overline{N} = \dfrac{\nabla\Phi}{|\nabla\Phi|}$.

Now, $\nabla\Phi = i\dfrac{\partial \Phi}{\partial x} + j\dfrac{\partial \Phi}{\partial y} + k\dfrac{\partial \Phi}{\partial z} = 2xi + 2yj + 2zk$

$$\therefore \quad \overline{N} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 4z^2}} = xi + yj + zk \qquad [\because\ x^2 + y^2 + z^2 = 1, \text{ by data}]$$

$$\therefore \quad \overline{N} \cdot \overline{F} = (ax^2 + by^2 + cz^2) = (xi + yj + zk) \cdot \overline{F}$$

Hence, by comparing the two sides, we see that $\overline{F} = axi + byj + czk$.

Now, $\nabla \cdot \overline{F} = \dfrac{\partial}{\partial x}(ax) + \dfrac{\partial}{\partial y}(by) + \dfrac{\partial}{\partial z}(cz) = a + b + c$

Hence, from (1) $\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V (a + b + c)\, dv$

But $\iiint_V dv = (\text{Volume of the sphere } x^2 + y^2 + z^2 = 1) = \dfrac{4}{3}\pi$

$$\therefore \quad \iint_S (ax^2 + by^2 + cz^2)\, ds = \frac{4}{3}\pi(a + b + c)$$

(This is a typical example where we have to find $\overline{F}$ first.)

Example 2 : Evaluate $\iint_S \dfrac{ds}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$ where S is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$. **(M.U. 2003)**

Sol. : As in the above example, we have to find $\overline{F}$.

Unit normal to the surface $ax^2 + by^2 + cz^2 - 1 = 0$ is $\dfrac{\nabla\Phi}{|\nabla\Phi|}$. [See Example (A), page 8-12]

$$\therefore \quad \overline{N} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{2axi + 2byj + 2czk}{2\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \frac{axi + byj + czk}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

$$\therefore \quad \overline{N} \cdot \overline{F} = \frac{1}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \frac{axi + byj + czk}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \cdot \overline{F}$$

Comparing the two sides we see that

$$\overline{F} = xi + yj + zk$$

and $\quad \bar{N} \cdot \bar{F} = \dfrac{axi + byj + czk}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \cdot (xi + yj + zk) = \dfrac{ax^2 + by^2 + cz^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$

$$= \frac{1}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \qquad [\because \ ax^2 + by^2 + cz^2 = 1, \text{ by data }]$$

Now, $\quad \nabla \cdot \bar{F} = \dfrac{\partial(x)}{\partial x} + \dfrac{\partial(y)}{\partial y} + \dfrac{\partial(z)}{\partial z} = 3.$

$\therefore$ By divergence theorem $\quad \iint_S \bar{N} \cdot \bar{F}\, ds = \iiint_V (\nabla \cdot \bar{F})\, dv$

$$\therefore \iint_S \frac{ds}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \iiint_V 3\, dv = 3 \iiint_V dv = 3v = 3 \text{ (Volume of the ellipsoid)}$$

$$= 3\left[\frac{4\pi}{3} \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}} \cdot \frac{1}{\sqrt{c}}\right] = \frac{4\pi}{\sqrt{abc}}.$$

(The ellipsoid is $\dfrac{x^2}{(1/\sqrt{a})^2} + \dfrac{y^2}{(1/\sqrt{b})^2} + \dfrac{z^2}{(1/\sqrt{c})^2} = 1$.)

Example 3 : Evaluate $\iint_S \bar{F} \cdot d\bar{s}$ where $\bar{F} = 4xi - 2y^2 j + z^2 k$ and S is the region bounded by $y^2 = 4x$, $x = 1$, $z = 0$, $z = 3$. **(M.U. 1994, 2000, 07)**

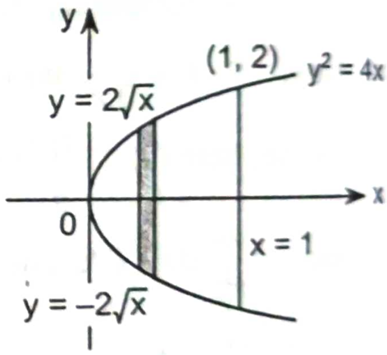


Fig. 9.35

Sol. : By Divergence Theorem $\iint_S \bar{N} \cdot \bar{F}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$

But $\quad \nabla \cdot \bar{F} = \dfrac{\partial}{\partial x}(4x) + \dfrac{\partial}{\partial y}(-2y^2) + \dfrac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

and $\quad dv = dx\, dy\, dz$

$$\therefore \iint_S \bar{N} \cdot \bar{F}\, ds = \int_0^1 \int_{-2\sqrt{x}}^{2\sqrt{x}} \int_0^3 (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_{-2\sqrt{x}}^{2\sqrt{x}} \left[4z - 4yz + z^2\right]_0^3 dx\, dy = \int_0^1 \int_{-2\sqrt{x}}^{2\sqrt{x}} [12 - 12y + 9]\, dx\, dy$$

$$= \int_0^1 \left[21y - 6y^2\right]_{-2\sqrt{x}}^{2\sqrt{x}} dx = \int_0^1 84\sqrt{x}\, dx = 84\left[\frac{x^{3/2}}{3/2}\right]_0^1 = 56$$

Example 4 : Evaluate $\iint_S x^2 dy\, dz + y^2 dz\, dx + 2z(xy - x - y)\, dx\, dy$ where S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Sol. : We have to first find $\bar{F}$. If $\bar{N}$ is the outward normal to the surfaces then

We have ($\because \bar{N} \cdot i\, ds = dy\, dz$, $\bar{N} \cdot j\, ds = dz\, dx$, $\bar{N} \cdot k\, ds = dx\, dy$)

$$\iint_S x^2 dy\, dz + y^2 dz\, dx + 2z(xy - x - y)\, dx\, dy = \iint_S \bar{N} \cdot \bar{F}\, ds$$

where $\bar{F} = x^2 i + y^2 j + 2z(xy - x - y)\, k$.

By Divergence Theorem $$\iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

Now, $$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}[2z(xy - x - y)]$$

$$= 2x + 2y + 2(xy - x - y) = 2xy$$

$$\therefore \iint_S \bar{N}\cdot\bar{F}\,ds = \int_0^1\int_0^1\int_0^1 (2xy)\,dx\,dy\,dz = \int_0^1\int_0^1 [2x\,yz]_0^1\,dx\,dy = \int_0^1\int_0^1 2xy\,dx\,dy$$

$$= \int_0^1 \left[xy^2\right]_0^1 dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

EXERCISE - IV

1. Evaluate the surface integral $\iint_S (yzi + zxj + xyk)\,ds$ where S is the surface of the sphere in the first octant. **(M.U. 1994) [Ans : 0, $\nabla\cdot\bar{F} = 0$]**

2. Use divergence theorem to show $\iint_S \bar{N}\cdot\bar{F}\,ds = 6V$ where $\bar{F} = 3xi + 2yj + zk$ and S is any closed surface enclosing a volume V.

3. Evaluate $\iint_S \frac{\bar{r}\cdot\bar{N}}{r^3}\,ds$ where S is $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$. **(M.U. 2000) [Ans. : 0]**

4. Evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = xi - yj + 2zk$ over the sphere $x^2 + y^2 + (z-a)^2 = a^2$.

[Ans. : $8\pi a^3/3$]

5. Evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = (2x + 3z)\,i - (xz + y)\,j + (y^2 + 2z)\,k$ and S is the surface of the sphere with centre (3, –1, 2) and radius 3. **(M.U. 1987) [Ans. : 108π]**

6. Evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = (2x + 3z^2)\,i - (xz^2 + y)\,j + (y^2 + 2z)\,k$ and S is the surface of the sphere with centre (3, –14, –17) and radius 3. **(M.U. 1993) [Ans. : 108π]**

7. Evaluate $\iint \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = 4xz\,i - y^2\,j + yz\,k$ and S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$. **(M.U. 2001, 02) [Ans. : 3/2]**

8. Using appropriate theorem evaluate $\oint_C \left\{[a^2(x - yz)]\,i + [b^2(y - zx)]\,j + [c^2(z^2 - xy)]\,k\right\}\cdot d\bar{s}$ where, $ax^2 + by^2 + cz^2 = abc$. **[Ans. : $\oint \bar{F}\cdot d\bar{s} = (a^2 + b^2 + c^2)\int_V dv = (a^2 + b^2 + c^2)\cdot\frac{4}{3}\pi abc$]**

9. Using Gauss's Divergence Theorem evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = xi - yj + zk$ and S is the surface of the cylinder $x^2 + y^2 = a^2$ bounded by the planes $z = 0$ and $z = b$. **[Ans. : $\pi a^2 b$]**

10. Evaluate $\iint_S (x^3 dy\,dz + y^3 dz\,dx + z^3 dx\,dy)$ over the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$. [**Ans.** : $3a^5$]

11. Evaluate $\iint_S \overline{N} \cdot \overline{F}\, ds$ where $\overline{F} = 2x^2yi - y^2j + 4xz^2k$ and S is the region in the first octant bounded by $y^2 + z^2 = 9$, $x = 2$. **(M.U. 2003, 05, 06)**

(**Hint** : $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^2 (4xy - 2y + 8xz)\, dy\, dz\, dx = 180$) [**Ans.** : 180]

12. Evaluate $\iint_S (yi + xj + z^2k) \cdot d\overline{s}$ where S denotes the cylindrical surface bounded by $x^2 + y^2 = a^2$, $z = 0$, $z = h$. [**Ans.** : $\pi a^2 h^2$]

13. Evaluate $\iint_S \overline{N} \cdot \overline{F}\, ds$ where $\overline{F} = 4xzi - y^2j + yzk$ and S is the surface of the cube $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$. **(M.U. 1991, 2003)** [**Ans.** : $3a^4/2$]

14. Evaluate $\iint_S (x^3 dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$ where S is the surface of the closed cylinder $x^2 + y^2 = a^2$, $z = a$, $z = b$. **(M.U. 1999)** [**Ans.** : $5a^4b/4$]

15. Evaluate $\iiint_V \nabla \cdot \overline{F}\, dv$ where $\overline{F} = 4xi - 2y^2j + z^2k$ over the cylindrical region $x^2 + y^2 = a^2$, $z = 0$, $z = b$. [**Ans.** : $\pi a^2 b(b+4)$]

16. Evaluate $\iiint_V \nabla \cdot \overline{F}\, dv$ where $\overline{F} = yi + xj + z^2k$ over the cylindrical region $x^2 + y^2 = a^2$, $z = 0$, $z = b$. **(M.U. 2003)** [**Ans.** : $\pi a^2 b^2$]

17. Use Gauss Theorem to evaluate $\iint_S \overline{F} \cdot d\overline{s}$ where $\overline{F} = x\,i - 3y^2j + zk$ over the surface of the cylinder $x^2 + y^2 = 16$ between $z = 0$ and $z = 5$. **(M.U. 1998)** [**Ans.** : 160π]

18. Use Divergence Theorem to evaluate $\iint_S \overline{F} \cdot d\overline{s}$ where $\overline{F} = (x^2 - yz)\,i + (y^2 - 2z)\,j + (z^2 - xy)\,k$ and S is the surface of the rectangular parallelepiped bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, $z = c$. **(M.U. 1999)** [**Ans.** : $abc\,(a + b + c)$]

EXERCISE - V

Theory

1. Express Green's Theorem (for plane regions) in vector notation. **(M.U. 1993, 2003)**
2. State and explain Stoke's Theorem.
3. State and explain Gauss's Divergence Theorem.