CHAPTER

8 Vector Differentiation

1. Introduction

Having studied algebra of vectors in the previous chapter, we shall go to the next logical step *i.e.* calculus of vectors. Before considering how to differentiate a vector we shall first learn how a curve in space is represented in vector form. After defining the derivative of a vector we shall learn how it represents a tangent to a curve in space.

We shall then prove the rules of differentiation of vectors. In article 10 we shall discuss point functions. After defining a new operator ∇ we shall learn to find gradient, divergence and curl of a vector. Lastly we shall consider second order differential operators and an important operator ∇^2 called Laplacian. We shall then learn the technique of operating by these operators on different functions.

2. Vector Function of a Scalar Quantity

If t is a scalar quantity and if to each value of t in an interval there corresponds a vector $\bar{r}$ then we say that $\bar{r}$ is a vector function of a scalar variable t and we denote it as

$$\bar{r} = \bar{f}(t)$$

If c is a particular value of the scalar t, then we denote the corresponding vector $\bar{c}$ under $\bar{f}$ by

$$\bar{c} = \bar{f}(c)$$

Illustration : Suppose a particle P moves along a curve and t denotes time and $\bar{r}$ denotes its position vector. Then clearly $\bar{r}$ is a function of t. Similarly, the velocity and acceleration of the moving particle P are also vector functions of time t.

3. Decomposition of a Vector Function

If we can write $\bar{r} = \bar{f}(t)$ as $\boxed{\bar{f}(t) = f_1(t)\,i + f_2(t)\,j + f_3(t)\,k}$

where $f_1(t)$, $f_2(t)$, $f_3(t)$, are scalar functions of t then it is called the **decomposition** of the vector function $\bar{f}(t)$.

Illustrations : If t is a parameter then we know that the equations of circle, ellipse, hyperbola and parabola in parametric form are $x = a\cos t$, $y = a\sin t$; $x = a\cos t$, $y = b\sin t$; $x = a\sec t$, $y = b\tan t$; $x = at^2$, $y = 2at$.

If $\bar{r}$ is the position vector of a point P on a curve then clearly,

$$\bar{r} = xi + yj + zk$$

if (x, y, z) are the coordinates of P.

Hence, using the above parametric forms of the curves the position vector of a point on circle, ellipse, hyperbola and parabola can be given by

$$\bar{r} = a\cos t\, i + a\sin t\, j + ok$$

$$\bar{r} = a\cos t\, i + b\sin t\, j + ok$$

$$\bar{r} = a\sec t\, i + b\tan t\, j + ok$$

$$\bar{r} = at^2 i + 2\,at\, j + ok.$$

Thus, above equations give us vector decomposition of a position vector of a point on these curves.

Since the above equations give us the position vector of a point on the curves they can also be called as the equations of these curves in vector form.

4. Curves in Space

If $\bar{r}$ is a position vector of a point $P(x, y, z)$ on a curve $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$, then

$$\bar{r} = xi + yj + zk$$

where $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ is the vector equation of the curve in space.

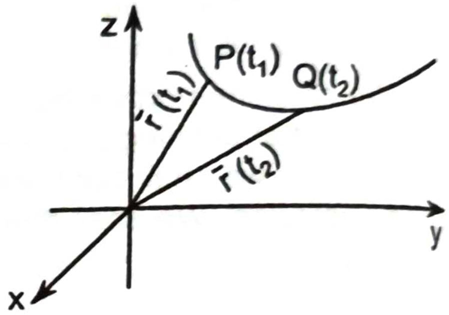

Fig. 8.1

For a fixed value of t, say t_1, $x_1 = f_1(t_1)$, $y_1 = f_2(t_1)$, $z_1 = f_3(t_1)$ are constants and $\bar{r} = x_1 i + y_1 j + z_1 k$ is a fixed point P on the curve. As t changes from t_1 to t_2, we move on from a point P to another point Q on the curve.

The following are the equations of some more curves in space,

$$\bar{r} = 3t\, i + 3t^2 j + 2t^3 k,$$

$$\bar{r} = e^t \cos t\, i + e^t \sin t\, j + e^t k$$

$$\bar{r} = a\cos t\, i + b\sin t\, j + tk$$

$$\bar{r} = t\cos t\, i + t\sin t\, j + atk \quad \text{etc.}$$

5. Point Functions

(a) Scalar Valued Point Functions

Consider any region R of space and suppose that to each point P of R there corresponds by some law a scalar quantity denoted by $\Phi\,(P)$. Then Φ is called a scalar point function defined for the region R.

Illustrations : Consider a material body occupying some region R. If $\Phi(P)$ denotes the density of the material at P, temperature at P or charge at P then Φ is a scalar point function over R.

(b) Vector Valued Point Functions

Consider any region R of space and suppose that to each point P of R there corresponds by some law a vector quantity $\bar{f}(P)$. Then $\bar{f}$ is called a vector point function defined for the region R.

Illustrations : Consider a fluid in motion. At any time t if $\bar{f}(P)$ denotes velocity at a point P which varies from point to point or acceleration at a point P which varies from point to point then $\bar{f}$ is a vector point function.

6. Vector Operator Del ∇

We define a new operation ∇ (read as 'del' or 'nabla') by

$$\boxed{\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}}$$

where i, j, k have usual meanings.

7. Gradient

If Φ is a scalar point function then the vector function $\nabla\Phi$ is called the gradient of Φ.

Thus,
$$\boxed{\text{grad } \Phi = \nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}}$$

Remark ....

We also denote grad Φ as,

$$\boxed{\text{grad } \Phi = \nabla\Phi = \sum i\frac{\partial \Phi}{\partial x} = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}}$$

8. Standard Results

The following results can be proved very easily by using the definition of ∇.

(i) $\nabla(\Phi \pm \Psi) = \nabla\Phi \pm \nabla\Psi$

(ii) $\nabla(\Phi\Psi) = \Phi(\nabla\Psi) + (\nabla\Phi)\Psi$

(iii) $\nabla f(u) = i\frac{\partial}{\partial x}f(u) + j\frac{\partial}{\partial y}f(u) + k\frac{\partial}{\partial z}f(u) = f'(u)\nabla u$

Example 1 : If $\Phi = xz^2 - 5yz + xz$, find $\nabla\Phi$ at $(1, -1, 2)$.

Sol. : By definition $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$$\begin{aligned}\therefore \quad \nabla\Phi &= i(z^2 + z) + j(-5z) + k(2xz - 5y + x)\\ &= i(4+2) + j(-10) + k(4+5+1)\\ &= 6i - 10j + 10k.\end{aligned}$$

Example 2 : If $\Phi = \log(x^2 + y^2 + z^2)$, find $\nabla\Phi$.

Sol. : By definition $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$$\begin{aligned}\therefore \quad \nabla\Phi &= i\frac{1}{x^2+y^2+z^2}\cdot 2x + j\frac{1}{x^2+y^2+z^2}\cdot 2y + k\frac{1}{x^2+y^2+z^2}\cdot 2z\\ &= \frac{2}{x^2+y^2+z^2}(xi + yj + zk)\\ &= \frac{2\bar{r}}{r^2} \quad \text{where } \bar{r} = xi + yj + zk.\end{aligned}$$

Example 3 : If $\Phi = (x^2 + y^2 + z^2) \cdot e^{-\sqrt{x^2+y^2+z^2}}$, find $\nabla\Phi$.

Sol. : $$\frac{\partial\Phi}{\partial x} = (x^2 + y^2 + z^2) \cdot e^{-\sqrt{x^2+y^2+z^2}} \times \frac{-1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x + e^{-\sqrt{x^2+y^2+z^2}} \cdot (2x)$$

where, $r = \sqrt{x^2 + y^2 + z^2}$.

$$\therefore \quad \frac{\partial\Phi}{\partial x} = -r \cdot e^{-r}\, x + e^{-r} \cdot 2x = e^{-r}(2x - xr)$$

Similarly, $\dfrac{\partial\Phi}{\partial y} = e^{-r}(2y - yr)$, $\quad \dfrac{\partial\Phi}{\partial z} = e^{-r}(2z - zr)$

Hence, $$\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$$
$$= e^{-r}(2-r)[xi + yj + zk] = e^{-r}(2-r)\bar{r}.$$

Example 4 : If $\Phi = x^2 + y^2 + z^2$, $\Psi = x^2y^2 + y^2z^2 + z^2x^2$, find $\nabla[\nabla\Phi \cdot \nabla\Psi]$.

Sol. : $$\nabla\Phi = 2x\,i + 2y\,j + 2z\,k$$
$$\nabla\Psi = (2xy^2 + 2xz^2)\,i + (2yx^2 + 2yz^2)\,j + (2zx^2 + 2zy^2)\,k$$
$$\therefore \quad \nabla\Phi \cdot \nabla\Psi = 4x^2(y^2 + z^2) + 4y^2(x^2 + z^2) + 4z^2(x^2 + y^2)$$
$$= 8(x^2y^2 + y^2z^2 + z^2x^2)$$
$$\therefore \quad \nabla(\nabla\Phi \cdot \nabla\Psi) = 16x(y^2 + z^2)\,i + 16y(z^2 + x^2)\,j + 16z(x^2 + y^2)\,k$$

Example 5 : If $u = x + y + z$, $v = x + y$, $w = -2xz - 2yz - z^2$, show that $\nabla u \cdot [\nabla v \times \nabla w] = 0$.

Sol. : $$\nabla u = i + j + k, \quad \nabla v = i + j$$
$$\nabla w = -2z\,i - 2z\,j - (2x + 2y + 2z)\,k$$
$$\nabla u \cdot [\nabla v \times \nabla w] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -2z & -2z & -2x - 2y - 2z \end{vmatrix} = 0 \qquad \text{[By (B), page 7-4]}$$

(Since first two columns are identical.)

Example 6 : Prove that $\nabla f(r) = f'(r)\dfrac{\bar{r}}{r}$ and hence, find f if $\nabla f = 2r^4\,\bar{r}$. **(M.U. 1996, 2008)**

Sol. : We have, $\nabla\Phi = i\dfrac{\partial\Phi}{\partial x} + j\dfrac{\partial\Phi}{\partial y} + k\dfrac{\partial\Phi}{\partial z}$

Here, $\Phi = f(r)$ and f is a function of r and r is a function of (x, y, z).

$$\therefore \quad \nabla f(r) = i\frac{df}{dr}\frac{\partial r}{\partial x} + j\frac{df}{dr}\frac{\partial r}{\partial y} + k\frac{df}{dr}\frac{\partial r}{\partial z}$$

But $r^2 = x^2 + y^2 + z^2 \qquad \therefore \quad 2r\dfrac{\partial r}{\partial x} = 2x$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla f(r) = \frac{f'(r)}{r}[xi + yj + zk] = \frac{f'(r)}{r}\bar{r}$$

Comparing this with the given expression. *i.e.*, comparing

$$\nabla f(r) = f'(r)\frac{\bar{r}}{r} \text{ with } \nabla f(r) = 2r^4\bar{r} = 2r^5\frac{\bar{r}}{r} \qquad \text{we see that } f'(r) = 2r^5.$$

Here, by integration $\quad f(r) = \dfrac{2r^6}{6} + C = \dfrac{r^6}{3} + C.$

Example 7 : Prove that $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\bar{r}}{r^3}$.

Sol. : Here, $f(r) = \dfrac{1}{r} \quad \therefore \; f'(r) = -\dfrac{1}{r^2}$

But $\quad \boxed{\nabla f(r) = f'(r)\cdot\dfrac{\bar{r}}{r}} \qquad$ [By Ex. 6 above]

(This can be used as a standard result *e.g.* $\nabla r^2 = 2r\cdot\dfrac{\bar{r}}{r}$, $\nabla \log r = \dfrac{1}{r}\cdot\dfrac{\bar{r}}{r}$)

$$\therefore \quad \nabla\left(\frac{1}{r}\right) = f'(r)\frac{\bar{r}}{r} = -\frac{1}{r^2}\frac{\bar{r}}{r} = -\frac{\bar{r}}{r^3}.$$

Alternatively, we have $\quad \nabla\Phi = \dfrac{\partial\Phi}{\partial x}i + \dfrac{\partial\Phi}{\partial y}j + \dfrac{\partial\Phi}{\partial z}k.$

Here, $\Phi = \dfrac{1}{r}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{d\Phi}{dr} = -\frac{1}{r^2} \quad \text{and} \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla\Phi = \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial x}i + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial y}j + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial z}k$$

$$= -\frac{1}{r^2}\cdot\frac{x}{r}i - \frac{1}{r^2}\cdot\frac{y}{r}j - \frac{1}{r^2}\cdot\frac{z}{r}k$$

$$= -\frac{1}{r^3}(xi + yj + zk) = -\frac{\bar{r}}{r^3}.$$

Example 8 : Prove that $\nabla r^n = n r^{n-2}\bar{r}$. **(M.U. 1994, 2006)**

Sol. : As proved in Ex. 6 above $\nabla f(r) = f'(r)\dfrac{\bar{r}}{r}$

Here, $f(r) = r^n \quad \therefore \; f'(r) = n r^{n-1} \qquad \therefore \; \nabla f(r) = n r^{n-1}\dfrac{\bar{r}}{r} = n r^{n-2}\bar{r}$

Alternatively we have $\quad \nabla\Phi = \dfrac{\partial\Phi}{\partial x}i + \dfrac{\partial\Phi}{\partial y}j + \dfrac{\partial\Phi}{\partial z}k$

Here, $\Phi = r^n$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{d\Phi}{dr} = n r^{n-1} \quad \text{and} \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla\Phi = \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial x}i + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial y}j + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial z}k$$

$$= n r^{n-1}\left(\frac{x}{r}\right)i + n r^{n-1}\left(\frac{y}{r}\right)j + n r^{n-1}\left(\frac{z}{r}\right)k$$

$$= n\,r^{n-2}(xi + yj + zk)$$

$$\therefore \quad \boxed{\nabla\Phi = \nabla r^n = n\,r^{n-2}\,\bar{r}}$$

Example 9 : Find $\nabla(e^{r^2})$. **(M.U. 2004)**

Sol. : Here $f(r) = e^{r^2}$ $\quad\therefore\ f'(r) = e^{r^2}\cdot 2r$

$$\therefore \quad \nabla f(r) = f'(r)\cdot\frac{\bar{r}}{r} = e^{r^2}2r\cdot\frac{\bar{r}}{r} = 2e^{r^2}\bar{r}$$

Alternatively we have $\quad \nabla\Phi = \dfrac{\partial\Phi}{\partial x}i + \dfrac{\partial\Phi}{\partial y}j + \dfrac{\partial\Phi}{\partial z}k$

Here, $\Phi = e^{r^2}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{d\Phi}{dr} = e^{r^2}2r \quad\text{and}\quad \frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla\Phi = \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial x}i + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial y}j + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial z}k$$

$$= e^{r^2}2r\left(\frac{x}{r}\right)i + e^{r^2}2r\left(\frac{y}{r}\right)j + e^{r^2}\left(\frac{z}{r}\right)k$$

$$= 2e^{r^2}r\cdot\frac{(xi + yj + zk)}{r} = 2e^{r^2}\cdot\bar{r}.$$

Example 10 : Show that $\nabla\left[\dfrac{(\bar{a}\cdot\bar{r})}{r^n}\right] = \dfrac{\bar{a}}{r^n} - \dfrac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}}$. **(M.U. 1995, 98, 2005)**

Sol. : We have $\quad \dfrac{\bar{a}\cdot\bar{r}}{r^n} = \dfrac{(a_1 i + a_2 j + a_3 k)\cdot(xi + yj + zk)}{r^n} = \dfrac{a_1 x + a_2 y + a_3 z}{r^n}$

Let $\quad \Phi = \dfrac{\bar{a}\cdot\bar{r}}{r^n} = \dfrac{a_1 x + a_2 y + a_3 z}{r^n}$

$$\therefore \quad \frac{\partial\Phi}{\partial x} = \frac{r^n\cdot a_1 - (a_1 x + a_2 y + a_3 z)\,n\,r^{n-1}(\partial r/\partial x)}{r^{2n}}$$

But $r^2 = x^2 + y^2 + z^2 \quad\therefore\ 2r\dfrac{\partial r}{\partial x} = 2x \quad\therefore\ \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$$\therefore \quad \frac{\partial\Phi}{\partial x} = \frac{a_1 r^n - (a_1 x + a_2 y + a_3 z)\cdot n\,r^{n-2}\cdot x}{r^{2n}} = \frac{a_1}{r^n} - \frac{n(a_1 x + a_2 y + a_3 z)\,x}{r^{n+2}}$$

Similarly, $\quad \dfrac{\partial\Phi}{\partial y} = \dfrac{a_2}{r^n} - \dfrac{n(a_1 x + a_2 y + a_3 z)\,y}{r^{n+2}}$

and $\quad \dfrac{\partial\Phi}{\partial z} = \dfrac{a_3}{r^n} - \dfrac{n(a_1 x + a_2 y + a_3 z)\,z}{r^{n+2}}$

$$\therefore \quad \nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$$

$$= \frac{1}{r^n}(a_1 i + a_2 j + a_3 k) - \frac{n}{r^{n+2}}\left[(a_1 x + a_2 y + a_3 z)(xi + yj + zk)\right]$$

But $\bar{a}\cdot\bar{r} = (a_1 i + a_2 j + a_3 k)\cdot(xi + yj + zk) = a_1 x + a_2 y + a_3 z$

$$\therefore \quad \nabla\Phi = \frac{\bar{a}}{r^n} - \frac{n}{r^{n+2}}(\bar{a}\cdot\bar{r})\,\bar{r}.$$

Example 11 : Prove that $\nabla\left[\dfrac{\bar{a}\cdot\bar{r}}{r^3}\right] = \dfrac{\bar{a}}{r^3} - \dfrac{3(\bar{a}\cdot\bar{r})\,\bar{r}}{r^5}$.

Sol. : Putting $n = 3$ in the above Ex. 10, we get the required result.

Example 12 : If $\bar{r} = xi + yj + zk$ and $\bar{a}, \bar{b}$ are constant vectors, prove that

$$\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5} - \frac{\bar{a}\cdot\bar{b}}{r^3}.$$

(M.U. 1999, 2007)

Sol. : As in the Ex. 7, we have

$$\nabla\frac{1}{r} = -\frac{1}{r^2}\cdot\frac{\bar{r}}{r} = -\frac{\bar{r}}{r^3} = -\frac{1}{r^3}(xi + yj + zk)$$

$$\therefore \quad \bar{b}\cdot\nabla\left(\frac{1}{r}\right) = (b_1 i + b_2 j + b_3 k)\cdot\left(-\frac{1}{r^3}(xi + yj + zk)\right)$$

$$= -\frac{1}{r^3}(b_1 x + b_2 y + b_3 z) = \Phi \text{ say}$$

$$\therefore \quad \nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} \qquad \text{.................... (1)}$$

Now, $$\frac{\partial\Phi}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{b_1 x + b_2 y + b_3 z}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= -\frac{(x^2 + y^2 + z^2)^{3/2}\cdot b_1 - (b_1 x + b_2 y + b_3 z)\cdot(3/2)(x^2 + y^2 + z^2)^{1/2}\,2x}{(x^2 + y^2 + z^2)^3}$$

$$= -\frac{b_1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3(b_1 x + b_2 y + b_3 z)\,x}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= -\frac{b_1}{r^3} + \frac{3(b_1 x + b_2 y + b_3 z)\,x}{r^5}$$

Similarly, $\dfrac{\partial\Phi}{\partial y} = -\dfrac{b_2}{r^3} + \dfrac{3(b_1 x + b_2 y + b_3 z)\,y}{r^5}$; $\quad \dfrac{\partial\Phi}{\partial z} = -\dfrac{b_3}{r^3} + \dfrac{3(b_1 x + b_2 y + b_3 z)\,z}{r^5}$

Hence, from (1)

$$\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = -\frac{1}{r^3}(b_1 i + b_2 j + b_3 k) + \frac{3}{r^5}(b_1 x + b_2 y + b_3 z)(xi + yj + zk)$$

(But $\bar{b}\cdot\bar{r} = (b_1 i + b_2 j + b_3 k)\cdot(xi + yj + zk) = b_1 x + b_2 y + b_3 z$)

$$\therefore \quad \nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = -\frac{\bar{b}}{r^3} + \frac{3(\bar{b}\cdot\bar{r})\,\bar{r}}{r^5}$$

$$\therefore \quad \bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5} - \frac{\bar{a}\cdot\bar{b}}{r^3}.$$

Example 13 : Find $\Phi(r)$ such that $\nabla\Phi = -\dfrac{\bar{r}}{r^5}$ and $\Phi(2) = 3$. **(M.U. 1988, 95)**

Sol. : We have

$$\nabla\Phi = -\bar{r}(x^2+y^2+z^2)^{-5/2} \qquad [\because\ r = \sqrt{x^2+y^2+z^2}\,]$$

$$= -(x^2+y^2+z^2)^{-5/2}(xi+yj+zk) \qquad \ldots\ldots (1)$$

But $$\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} \qquad \ldots\ldots (2)$$

Comparing (1) and (2), we get,

$$\frac{\partial\Phi}{\partial x} = -x(x^2+y^2+z^2)^{-5/2}, \qquad \frac{\partial\Phi}{\partial y} = -y(x^2+y^2+z^2)^{-5/2},$$

$$\frac{\partial\Phi}{\partial z} = -z(x^2+y^2+z^2)^{-5/2}.$$

But $$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = -(x^2+y^2+z^2)^{-5/2}(xdx+ydy+zdz)$$

Now let $x^2+y^2+z^2 = t$

$$\therefore\ 2(x\,dx+y\,dy+z\,dz) = dt \qquad \therefore\ d\Phi = -t^{-5/2}\cdot\frac{dt}{2}$$

Integrating, $$\Phi = -\frac{1}{2}\frac{t^{-3/2}}{-3/2} + C = \frac{t^{-3/2}}{3} + C$$

Now resubstituting $t = x^2+y^2+z^2$,

$$\therefore\ \Phi = \frac{1}{3}(x^2+y^2+z^2)^{-3/2} + C = \frac{1}{3}\cdot\frac{1}{r^3} + C$$

But by data $\Phi(r) = 3$ when $r = 2$

$$\therefore\ 3 = \frac{1}{3}\cdot\frac{1}{8} + C \quad \therefore\ C = \frac{71}{24} \qquad \therefore\ \Phi = \frac{1}{3}\cdot\frac{1}{r^3} + \frac{71}{24} = \frac{1}{3}\left(\frac{1}{r^3} + \frac{71}{8}\right).$$

EXERCISE - I

1. If $\Phi = 2xz^2 - 3xy - 4x$, find $\nabla\Phi$ at (1, –1, 2). [**Ans. :** $7i - 3j + 8k$]
2. If $\Phi = 5x^2y - 3y^2z^2$, find $\nabla\Phi$ at (1, –2, 1). [**Ans. :** $-20i + 17j - 24k$]
3. If $\Phi = 2xz^4 - x^2y$, find $\nabla\Phi$ and $|\nabla\Phi|$ at (2, –2, 1). [**Ans. :** $10i - 4j + 16k$, $2\sqrt{93}$]
4. If $\Phi = x^2+y^2+z^2$ and $\Psi = x^2y^2 + y^2z^2 + z^2x^2$, find $\nabla[\nabla\Phi\cdot\nabla\Psi]$.
   [**Ans. :** $16x(y^2+z^2) + 16y(z^2+x^2) + 16z(x^2+y^2)$]
5. If $u = x+y+z$, $v = x+y$, $w = -2xz - 2yz - z^2$, show that $\nabla u[\nabla v\times\nabla w] = 0$.
6. If $\bar{F} = 2x^2i - 3yzj + xz^2k$ and $\Phi = 2z - x^3y$, find $\bar{F}\cdot\nabla\Phi$, at (1, –1, 1). [**Ans. :** 5]
7. Find $\nabla\Phi$ if (i) $\Phi = \sqrt{x^2+y^2+z^2}$ (ii) $\Phi = \log(x^2+y^2+z^2)$ **(M.U. 1991)**
   (iii) $\Phi = e^{r^2}$ where $r^2 = x^2+y^2+z^2$. (iv) $\Phi = 3x^2y - y^3z^2$ at (1, –2, 1). **(M.U. 1992)**
   [**Ans. :** (i) $(x^2+y^2+z^2)^{-1/2}(xi+yj+zk)$ (ii) $2(x^2+y^2+z^2)^{-1}(xi+yj+zk)$
   (iii) $2e^{r^2}(xi+yj+zk)$ (iv) $-12i - 9j + 16k$.]

8. (a) Find $\Phi(r)$ such that $\nabla\Phi = -\frac{\bar{r}}{r^5}$ and $\Phi(1) = 0$. [**Ans. :** $\Phi(r) = \frac{1}{3}\left(\frac{1}{r^3} - 1\right)$]

(M.U. 1994, 96)

(b) If $\nabla u = 2r^4\bar{r}$, find u. **(M.U. 1996, 2008)** [**Ans. :** $u = \frac{(x^2+y^2+z^2)^3}{3} + c$]

9. If $\bar{a}$ is a constant vector and r and $\bar{r}$ have usual meanings, prove that

(i) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$ **(M.U. 2000)** (ii) $\nabla\left(\bar{a}\cdot\nabla\frac{1}{r}\right) = -\frac{\bar{a}}{r^3} + \frac{3(\bar{a}\cdot\bar{r})\bar{r}}{r^5}$ **(M.U. 2000, 01)**

10. If $\Phi_1 = x+y+z$, $\Phi_2 = x+y+z^2$, $\Phi_3 = 2xz+2yz+z^3$, prove that

$$\nabla\Phi_1\cdot[\nabla\Phi_2\times\nabla\Phi_3] = 0.$$

11. If $\Phi = 3x^2y$, $\Psi = xz^2 - 2y$, show that $\nabla(\nabla\Phi\cdot\nabla\Psi) = (6yz^2 - 12x)\,i + 6xz^2\,j + 12xyzk$.

12. If $\Phi(x,y,z) = x^2yz$, $\bar{u} = 3x^2yi + yz^2j - xy^2k$, find $\frac{\partial^2}{\partial y\,\partial z}(\Phi\,\bar{u})$ at $(1,-2,1)$.

[**Ans. :** $-12\,(i+j+k)$]

9. Geometrical Meaning of Grad Φ

Consider a scalar point function and let

$$\bar{r} = xi + yi + zk$$

be the position vector of a point P on the surface $\Phi(x,y,z) = c$.

Such a surface for which the value of the function is constant is called a **level surface**.

Now, $\overline{dr} = dx\,i + dy\,j + dz\,k$ and it lies in the plane tangential to the surface $\Phi(x,y,z) = c$.

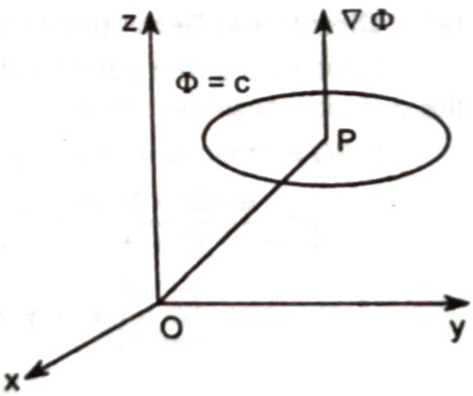

Fig. 8.2

Also $\quad d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz.$

Since $\quad \Phi(x,y,z) = c,\ d\Phi = 0$

$$\therefore \quad \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = 0$$

Hence, $\quad \nabla\Phi\cdot\overline{dr} = \left(i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}\right)\cdot(dx\,i + dy\,j + dz\,k)$

$$= \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = 0$$

$\nabla\Phi$ is a vector perpendicular to $\overline{dr}$. But since $\overline{dr}$ lies in the tangent plane, $\nabla\Phi$ *is a* **vector perpendicular to the tangent plane to the surface.** $\Phi(x,y,z) = c$. (See Ex. of (A) on page 8-12)

10. Directional Derivative

Let Φ be a scalar point function and let $\Phi(P)$ and $\Phi(Q)$ be the values of Φ at two neighbouring points P and Q in the field. Then,

$$\lim_{Q\to P}\frac{\Phi(Q)-\Phi(P)}{Q-P}$$

if it exists is called the **directional derivative** of Φ in the direction of PQ.

If $\bar{f}$ is a vector point function then $\lim_{Q \to P} \dfrac{\bar{f}(Q) - \bar{f}(P)}{Q - P}$

Fig. 8.3

if it exists is called the **directional derivative of** $\bar{f}$ in the direction of PQ.

If we take the direction of PQ along the coordinate axes then we get the directional derivatives along the coordinate axes.

Taking PQ along the x-axis

$$\lim_{Q \to P} \frac{\Phi(Q) - \Phi(P)}{Q - P} = \lim_{\delta x \to 0} \frac{\Phi(x + \delta x, y, z) - \Phi(x, y, z)}{\partial x} = \frac{\partial \Phi}{\partial x}$$

$\therefore$ **$\dfrac{\partial \Phi}{\partial x}, \dfrac{\partial \Phi}{\partial y}, \dfrac{\partial \Phi}{\partial z}$ are the directional derivatives of Φ in the direction of the coordinate axes at P.**

Similarly, it can be seen that

$\dfrac{\partial \bar{f}}{\partial x}, \dfrac{\partial \bar{f}}{\partial y}, \dfrac{\partial \bar{f}}{\partial z}$ are the directional derivatives of $\bar{f}$ in the direction of the coordinate axes.

(a) Directional Derivative in the given direction

Let Φ be a scalar point function in a scalar field. Let P be (x, y, z). Let a line segment PQ have the direction cosines l, m, n.

If $PQ = r$ then Q is $(x + lr, y + mr, z + nr)$.

$$\therefore \lim_{Q \to P} \frac{\Phi(Q) - \Phi(P)}{Q - P} = \lim_{r \to 0} \frac{\Phi(x + lr, y + mr, z + nr) - \Phi(x, y, z)}{r}$$

$$= \lim_{r \to 0} \frac{1}{r}\left[\Phi(x, y, z) + lr\frac{\partial \Phi}{\partial x}(x, y, z) + mr\frac{\partial \Phi}{\partial y}(x, y, z) + nr\frac{\partial \Phi}{\partial z}(x, y, z) - \Phi(x, y, z)\right]$$

[By Taylor's Theorem]

$$= l\frac{\partial \Phi}{\partial x} + m\frac{\partial \Phi}{\partial y} + n\frac{\partial \Phi}{\partial z}$$

This is the directional derivative of a scalar function Φ in the direction of a line whose directions are l, m, n.

$$\boxed{\left.\begin{array}{l}\textbf{The directional derivative of } \Phi \\ \textbf{in the direction } l, m, n,\end{array}\right\} = l\frac{\partial \Phi}{\partial x} + m\frac{\partial \Phi}{\partial y} + n\frac{\partial \Phi}{\partial z}}$$

[See Ex. 3, page 8-11]

If $\bar{f}$ is a vector point function then the directional derivative of $\bar{f}$ in the direction of the line whose direction cosines are l, m, n is

$$\boxed{l\frac{\partial \bar{f}}{\partial x} + m\frac{\partial \bar{f}}{\partial y} + n\frac{\partial \bar{f}}{\partial z}}$$

(b) Directional Derivative in the direction of a vector $\bar{a}$

Since $\nabla\Phi$ is a vector quantity its **component** (or resolved part) in the direction of a vector $\bar{a}$ is $\dfrac{\nabla\Phi \cdot \bar{a}}{|\bar{a}|}$. This component is called the **directional derivative of Φ in the direction of $\bar{a}$**.

$$\boxed{\text{Thus, the directional derivative of } \Phi \text{ in the direction of } \bar{a} = \frac{\nabla\Phi \cdot \bar{a}}{|\bar{a}|}}$$

[See Ex. 4, page 8-17]

(c) Maximum Directional Derivative

Since the resolved part of a vector is maximum in its own direction, the directional derivative is maximum in the direction $\nabla\Phi$. Since $\nabla\Phi$ is normal to the surface, we can also say that $\nabla\Phi$ is maximum in the direction of the normal to the surface and the maximum directional derivative is $|\Delta\Phi|$.

Remark ....

Though Φ is a scalar point function, grad Φ is a vector point function whose components are $\partial\Phi/\partial x$, $\partial\Phi/\partial y$ and $\partial\Phi/\partial z$. [See Ex. 7, page 8-13]

(d) Angle between Two Surfaces

We know that $\nabla\Phi$ is perpendicular to the tangent plane to the surface $\Phi(x, y, z) = c$. Hence, if $\Phi(x, y, z) = c_1$ and $\Psi(x, y, z) = c_2$ are two surfaces the angle between the two surfaces is equal to the angle between the normals *i.e.*, the angle between $\nabla\Phi$ and $\nabla\Psi$. [See Ex. 1, 2, page 8-14]

(e) Gradient of a constant

If Φ is a constant then $\dfrac{\partial\Phi}{\partial x} = \dfrac{\partial\Phi}{\partial y} = \dfrac{\partial\Phi}{\partial z} = 0$. $\therefore$ grad $\Phi = \bar{0}$.

(f) Differential $d\Phi$ where Φ is a scalar point function

We have
$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz \quad \text{.................... (1)}$$

Also $d\bar{r} = i\,dx + j\,dy + k\,dz$ and $\nabla\Phi = \dfrac{\partial\Phi}{\partial x}i + \dfrac{\partial\Phi}{\partial y}j + \dfrac{\partial\Phi}{\partial z}k$

$$\therefore\quad \nabla\Phi \cdot d\bar{r} = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz \quad \text{.................... (2)}$$

Hence, from (1) and (2), we get $\boxed{d\Phi = \nabla\Phi \cdot d\bar{r}}$

(g) Differential $d\bar{f}$ where $\bar{f}$ is a vector point function

We have,
$$d\bar{f} = \frac{\partial\bar{f}}{\partial x}dx + \frac{\partial\bar{f}}{\partial y}dy + \frac{\partial\bar{f}}{\partial z}dz = dx\frac{\partial\bar{f}}{\partial x} + dy\frac{\partial\bar{f}}{\partial y} + dz\frac{\partial\bar{f}}{\partial z} \quad \text{.................... (1)}$$

Also $d\bar{r} = i\,dx + j\,dy + k\,dz$ and $\nabla = i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}$

$$\therefore\quad d\bar{r} \cdot \nabla = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} \quad \text{.......... (2)}$$

From (1) and (2) we get $\boxed{d\bar{f} = (d\bar{r} \cdot \nabla)\bar{f}}$

(A) To Find Unit Normal

Example : Find the unit vector normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$. **(M.U. 2008)**

Sol. : We know that $\nabla\Phi$ is the vector normal to the surface $\Phi(x, y, z) = c$ at P.

Now, $$\nabla\Phi = i\frac{\partial}{\partial x}(xy^3z^2) + j\frac{\partial}{\partial y}(xy^3z^2) + k\frac{\partial}{\partial z}(xy^3z^2)$$

$$= y^3z^2 i + 3xy^2z^2 j + 2xy^3 zk$$

$$= -4i - 12j + 4k \text{ at } (-1, -1, 2)$$

unit vector normal to the surface at $(-1, -1, 2)$.

$$= \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{-4i - 12j + 4k}{\sqrt{16 + 144 + 16}} = -\frac{1}{\sqrt{11}}(i + 3j - k)$$

(B) To Find The Directional Derivative

Example 1 : Find the directional derivative of $\Phi = x^4 + y^4 + z^4$ at point $A(1, -2, 1)$ in the direction of AB where B is $(2, 6, -1)$. **(M.U. 1993, 2005, 09, 14)**

Sol. : $$\nabla\Phi = i\frac{\partial}{\partial x}(x^4 + y^4 + z^4) + j\frac{\partial}{\partial y}(x^4 + y^4 + z^4) + k\frac{\partial}{\partial z}(x^4 + y^4 + z^4)$$

$$\therefore \quad \nabla\Phi = 4(x^3 i + y^3 j + z^3 k) = 4(i - 8j + k) \text{ at } (1, -2, 1)$$

$$\overline{AB} = \overline{0B} - \overline{0A} = (2-1)i + (6+2)j + (-1-1)k = i + 8j - 2k$$

$\therefore$ Directional derivative at A in the direction of $\overline{AB}$

$$= 4(i - 8j + k) \cdot \frac{(i + 8j - 2k)}{\sqrt{1 + 64 + 4}} = \frac{4(1 - 64 - 2)}{\sqrt{69}} = -\frac{260}{\sqrt{69}}$$

Example 2 : Find the directional derivative of $\Phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at $(1, 2, 3)$. **(M.U. 1996, 2007)**

Sol. : $$\nabla\Phi = i\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + j\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + k\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= 2(xi + yi + zk) = 2(i + 2j + 3k) \text{ at } (1, 2, 3)$$

Given direction $= 3i + 4j + 5k$.

Directional derivative in the given direction

$$= \nabla\Phi \cdot \frac{\bar{a}}{|\bar{a}|} = 2(i + 2j + 3k) \cdot \frac{(3i + 4j + 5k)}{\sqrt{9 + 16 + 25}} = \frac{2(3 + 8 + 15)}{5\sqrt{2}} = \frac{26}{5}\sqrt{2}.$$

Example 3 : Find the directional derivative of $\Phi(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $i + 2j + 2k$. **(M.U. 2005)**

Sol. : $$\nabla\Phi = i\frac{\partial}{\partial x}(xy^2 + yz^3) + j\frac{\partial}{\partial y}(xy^2 + yz^3) + k\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= y^2 i + (2xy + z^3) j + 3yz^2 k$$

$$= i - 3j - 3k \text{ at } (2, -1, 1).$$

Directional derivative in the direction of $(i + 2j + 2k)$

$$= (i - 3j - 3k) \cdot \frac{(i + 2j + 2k)}{\sqrt{1+4+4}} = \frac{1}{3}(1 - 6 - 6) = -\frac{11}{3}.$$

Example 4 : Find the directional derivative of $\Phi = x^2 y \cos z$ at $(1, 2, \pi/2)$ in the direction of $\bar{a} = 2i + 3j + 2k$. **(M.U. 2013)**

Sol. :

$$\nabla\Phi = i\frac{\partial}{\partial x}(x^2 y \cos z) + j\frac{\partial}{\partial y}(x^2 y \cos z) + k\frac{\partial}{\partial z}(x^2 y \cos z)$$

$$= 2xy \cos z\, i + x^2 \cos z\, j - x^2 y \sin z\, k$$

At $(1, 2, \pi/2)$, $\nabla\Phi = 0\,i + 0\,j - 2\,k$.

Directional derivative in the direction of $2i + 3j + 2k$

$$= (0i + 0j - 2k) \cdot \frac{2i + 3j + 2k}{\sqrt{4+9+4}} = -\frac{4}{\sqrt{17}}.$$

Example 5 : Find the directional derivative of $\Phi = \dfrac{y}{x^2 + y^2}$ at $(0, 1)$ in the direction making are angle of 30° with positive x-axis.

Sol. : Directional derivative

$$\nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} = \left[-\frac{y}{(x^2+y^2)^2} \cdot 2x\right] i + \left[\frac{(x^2+y^2) \cdot 1 - y \cdot 2y}{(x^2+y^2)^2}\right] j$$

$$= \frac{-2xy}{(x^2+y^2)^2}\, i + \frac{(x^2 - y^2)}{(x^2+y^2)^2}\, j = 0i - j \text{ at } (0, 1)$$

Unit vector making an angle of 30° with the x-axis

$$= \cos 30^0\, i + \sin 30^0\, j = \frac{\sqrt{3}}{2} i + \frac{1}{2} j$$

$\therefore$ Required directional derivative $= (0i - j) \cdot \left(\dfrac{\sqrt{3}}{2} i + \dfrac{1}{2} j\right) = -\dfrac{1}{2}.$

Example 6 : In what direction from the point $(2, 1, -1)$ is the directional derivative of $\Phi = x^2 y z^3$ maximum ? What is its magnitude ?

Sol. :

$$\nabla\Phi = \nabla x^2 y z^3 = i\frac{\partial}{\partial x}(x^2 y z^3) + j\frac{\partial}{\partial y}(x^2 y z^3) + k\frac{\partial}{\partial z}(x^2 y z^3)$$

$$= 2xyz^3 i + x^2 z^3 j + 3x^2 y z^2 k$$

$$= -4i - 4j + 12k \text{ at } (2, 1, -1)$$

Directional derivative is maximum in the direction of $\nabla\Phi$. Hence, directional derivative is maximum in the direction of $-4i - 4j + 12k$.

Its magnitude $= \sqrt{16 + 16 + 144} = 4\sqrt{11}.$

Example 7 : Find the maximum directional derivative of $\Phi = (4x - y + 2z)^2$ at $(1, 2, 1)$. **(M.U. 2001)**

Sol. : Directional derivative of Φ is given by

$$\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$$

$$= 2\,(4x - y + 2z)\,4i + 2\,(4x - y + 2z)\,(-1)\,j + 2\,(4x - y + 2z)\cdot 2k$$

$$= 2\,(4x - y + 2z)\,(4i - j + 2k)$$

At $(1, 2, 1)$, $\quad \nabla\Phi = 8\,(4i - j + 2k)$

Maximum directional derivative $= |\nabla\Phi| = 8\sqrt{16 + 1 + 4} = 8\sqrt{21}.$

(C) To Find The Angle Between The Normals

Example 1 : Find the angle between the normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ and $(-3, -3, 3)$. **(M.U. 1998, 2005)**

Sol. : Let $\Phi = xy - z^2$

$$\therefore \quad \nabla\Phi = i\frac{\partial}{\partial x}(xy - z^2) + j\frac{\partial}{\partial y}(xy - z^2) + k\frac{\partial}{\partial z}(xy - z^2)$$

$$= yi + xj - 2zk$$

$$\nabla\Phi = 4i + j - 4k \text{ at } (1, 4, 2)$$

and $\quad \nabla\Phi = -3i - 3j - 6k$ at $(-3, -3, 3)$

But these are the normals to the surface at the given points.

Angle between two vectors $\bar{a}, \bar{b}$ is given by $\bar{a}\cdot\bar{b} = |a|\,|b|\cos\theta$.

If θ is the angle between them,

$$(4i + j - 4k)\cdot(-3i - 3j - 6k) = |4i + j - 4k|\cdot|-3i - 3j - 6k|\cos\theta$$

$$\therefore \quad -12 - 3 + 24 = 9 = \sqrt{33}\,\sqrt{54}\cos\theta$$

$$\therefore \quad \cos\theta = \frac{9}{\sqrt{3}\,\sqrt{11}\,\sqrt{27}\,\sqrt{2}} = \frac{9}{9\sqrt{22}} = \frac{1}{\sqrt{22}}.$$

Example 2 : Find the angle between the surfaces $x\log z + 1 - y^2 = 0$, $x^2y + z = 2$ at $(1, 1, 1)$. **(M.U. 2003, 05)**

Unit Normal $\rightarrow \cos\alpha$

Sol. : Let $\quad \Phi = x\log z + 1 - y^2.$

$$\therefore \quad \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} = \log z\, i - 2yj + \frac{x}{z}k$$

$$= 0i - 2j + k \text{ at } (1, 1, 1).$$

Unit normal vector at $(1, 1, 1) = \dfrac{0i - 2j + k}{\sqrt{5}}$

Let $\quad \Psi = x^2y + z - 2$

$$\nabla\Psi = i\frac{\partial\Psi}{\partial x} + j\frac{\partial\Psi}{\partial y} + k\frac{\partial\Psi}{\partial z} = 2xy\,i + x^2 j + k = 2i + j + k$$

Unit normal vector at $(1, 1, 1) = \dfrac{2i + j + k}{\sqrt{6}}$

$$\cos\theta = \frac{(0 - 2j + k)}{\sqrt{5}}\cdot\frac{(2i + j + k)}{\sqrt{6}} = -\frac{1}{30}.$$

(D) To Find The Constants

Example 1 : Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1, -1, 2)$.

(M.U. 2000, 06)

Sol. : Let $u = ax^2 - byz - (a+2)x$ and $v = 4x^2y + z^3 - 4$.

$$\therefore \quad \nabla u = (2ax - a - 2)\,i + (-bz)\,j + (-by)\,k$$
$$= (a-2)\,i - 2bj + bk \quad \text{at } (1, -1, 2)$$

The direction ratios of the normal to this surface at $(1, -1, 2)$ are $a-2, -2b, b$.

And $\nabla v = 8xyi + 4x^2 j + 3z^2 k = -8i + 4j + 12k$ at $(1, -1, 2)$

The direction ratios of the normal to this surface at $(1, -1, 2)$ are $-8, 4, 12$ *i.e.* $-2, 1, 3$.

Since, the surfaces are orthogonal, normals are perpendicular to each other.

$$\therefore \quad (-2)(a-2) + (1)(-2b) + (3)(b) = 0 \quad i.e. \quad -2a + b = -4 \qquad \text{.....................} (1)$$

Since $(1, -1, 2)$ lies on the surface

$ax^2 - byz - (a+2)\,x = 0$, we have $a + 2b - a - 2 = 0$.

i.e. $\quad b = 1. \qquad$ .... (2)

Then from (1) we get $a = 5/2$. Hence, $a = 5/2$ and $b = 1$.

Example 2 : Find the values of a, b, c if the directional derivative of $\Phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has maximum magnitude 64 in the direction parallel to the z axis. **(M.U. 2003, 04)**

Sol. : We have $\Phi = axy^2 + byz + cz^2x^3$

$$\therefore \quad \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} = (ay^2 + 3cx^2z^2)\,i + (2axy + bz)\,j + (by + 2czx^3)\,k$$
$$= (4a + 3c)\,i + (4a - b)\,j + (2b - 2c)\,k \quad \text{at } (1, 2, -1) \qquad \text{.....................} (1)$$

The directional derivative is maximum in the direction of $\nabla\Phi$ *i.e.* in the direction of $(4a + 3c)\,i + (4a - b)\,j + (2b - 2c)\,k$.

But by data the directional derivative is maximum in the direction of the z-axis *i.e.* in the direction of $0i + 0j + k$.

$$\therefore \quad \frac{4a + 3c}{0} = \frac{4a - b}{0} = \frac{2b - 2c}{1} \qquad \therefore \quad 4a + 3c = 0 \text{ and } 4a - b = 0$$

Hence, from (1), $\quad \nabla\Phi = (2b - 2c)\,k \qquad \therefore \quad |\nabla\Phi| = |2b - 2c|$.

But directional derivative is maximum in the direction of $\nabla\Phi$ and is given to be 64,

$$\therefore \quad 2b - 2c = 64 \qquad \therefore \quad b - c = 32.$$

Solving $4a + 3c = 0$, $4a - b = 0$ and $b - c = 32$, we get $a = 6$, $b = 24$, $c = -8$.

EXERCISE - II

(A) To Find The Unit Normal

1. Find the unit normal to the surface $2x^2 + 4yz - 5z^2 = -10$ at $(3, -1, 2)$.

[**Ans. :** $\dfrac{3i + 2j + 6k}{7}$]

2. Find the unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$.

[**Ans. :** $(i + j + k)/\pm\sqrt{3}$]

Hence

3. Find unit vector normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2).

[**Ans.** : $(8i + j + 8k)/\sqrt{129}$]

(B) To Find The Directional Derivative

1. Find the directional derivative of $\Phi = xy + yz + zx$ at (1, 2, 3) in the direction of $3i + 4j + 5k$.

[**Ans.** : $46/5\sqrt{2}$]

2. Find the directional derivative of $\Phi = x^2y + y^2z + z^2x^2$ at $P(1, 2, 1)$ in the direction of the normal to the surface $x^2 + y^2 - z^2x = 1$ at $Q(1, 1, 1)$. **(M.U. 2000)** [**Ans.** : $4/3$]

3. Find the directional derivative of $4xz^2 + x^2yz$ at (1, −2, −1) in the direction of $2i - j - 2k$.

(M.U. 1999) [**Ans.** : $37/3$]

4. Find the directional derivative of $\Phi = x/(x^2 + y^2)$ at (0, 1) in the direction making an angle of 30° with positive x-axis. [**Ans.** : $\sqrt{3}/2$]

5. Find the directional derivative of $\Phi = 2x^3y - 3y^2z$ at $P(1, 2, -1)$ in the direction towards Q (3, −1, 5). In what direction from P is the directional derivative maximum ? Find the magnitude of maximum directional derivative. **(M.U. 1993)** [**Ans.** : $-90/7$; $12i + 14j - 12k$; 22]

6. Find the directional derivative of the function $\Phi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point (5, 0, 4). In what direction will it be maximum ? Find also the magnitude of the maximum. **(M.U. 1998)** [**Ans.** : $4\sqrt{7}/\sqrt{3}$; $\nabla\Phi = 2i - 4j + k$; $|\nabla\Phi| = 2\sqrt{41}$]

7. Find the maximum directional derivative of $xy(x - y + 2z)$ at (1, 1, 0). **(M.U. 2000)**

[**Ans.** : $\sqrt{6}$]

8. Find the maximum directional derivative of $\Phi = x(x - y) + y(y + z)$ at $P(1, 2, 1)$.

(M.U. 1997) [**Ans.** : $2\sqrt{5}$]

9. Find the directional derivative of $\Phi = 4xz^3 - 3x^2y^2z$ at (2, −1, 2) in the direction from this point towards the point (4, − 4, 8). [**Ans.** : $376/7$]

10. Find the directional derivative of $\Phi = xy^2 + yz^2$ at (2, −1, 1) in the direction of the vector $i + 2j + 2k$. [**Ans.** : -3]

11. In what direction is the directional derivative of $\Phi = x^2y^2z^4$ at (3, −1, − 2) maximum ? Find its magnitude. [**Ans.** : 418·45]

12. In what direction is the directional derivative of $\Phi = 2xz - y^2$ at (1, 3, 2) maximum ? Find its magnitude. [**Ans.** : $2\sqrt{14}$]

13. Find the directional derivative of $\Phi = x^2y + y^2z + z^2x$ at (2, 2, 2) in the direction of the normal to the surface $4x^2y + 2z^2 = 2$ at the point (2, −1, 3). [**Ans.** : $36/\sqrt{41}$]

14. Find the directional derivative of $\Phi = xy^2 + yz^3$ at (2, −1, 1) in the direction of the normal to $x \log z - y^2 + 4 = 0$ at (−1, 2, 1). **(M.U. 1999, 2003, 13)** [**Ans.** : $-15/\sqrt{17}$]

15. Find the directional derivative of $\Phi = xy^2 + yz^2$ at (2, −1, 1) in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at (1, 1, 1). [**Ans.** : $-13/\sqrt{29}$]

16. Find the rate of change of $\Phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1). **(M.U. 1998, 2005)** [**Ans.** : $9/\sqrt{29}$]

17. Find the rate of change of $\Phi = xy + yz + zx$ in the direction of the normal to the surface $x^2 + y^2 = z + 4$ at (1, 1, − 2) . [**Ans.** : -2]

(C) To Find the Angle Between The Normals

1. Find the angle between the normals to the surfaces $x^2y + 2xz = 4$ at $(2, -2, 3)$ and to $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$.
[**Ans.** : $\cos\theta = 11/3\sqrt{14}$]

2. Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 9$, $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.
(M.U. 2003, 14) [**Ans.** : $\cos\theta = 8/3\sqrt{21}$]

3. Find the angle between the normals to the surface $xy = z^2$ at $P(1, 1, 1)$ and $Q(4, 1, 2)$.
(M.U. 2003) [**Ans.** : $\cos\theta = 13/\sqrt{198}$]

4. Find the angle between the normals to the surfaces $x^2y + z = 3$ and $x \log z - y^2 + 4 = 0$ at $(-1, 2, 1)$.
[**Ans.** : $\cos\theta = -5/3\sqrt{34}$]

5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 12$ and $x^2 + y^2 - z = 6$ at $(2, -2, 2)$.
[**Ans.** : $\cos\theta = 7/3\sqrt{11}$]

6. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 7$ at the point $(2, 1, -2)$.
(M.U. 1995) [**Ans.** : $\cos\theta = 4/\sqrt{21}$]

• 7. Find the angle between the two surfaces $x^2 + y^2 + az^2 = 6$ and $z = 4 - y^2 + bxy$ at $P(1, 1, 2)$. [**Ans.** : $(1, 1, 2)$ lies on both surfaces $\therefore a = 1, b = -1$, $\cos\theta = \sqrt{6/11}$]

• 8. Find the angle between the surfaces $ax^2 + y^2 + z^2 - xy = 1$ and $bx^2y + y^2z + z = 1$ at $(1, 1, 0)$.
(M.U. 2000, 06) [**Ans.** : $(1, 1, 0)$ lies on both the surfaces $\therefore a = 1$, $b = 1$, $\cos\theta = 1/\sqrt{2}$, $\theta = 45^\circ$]

(D) To Find The Constants

1. Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ cut orthogonally at $(1, -1, 2)$.
(M.U. 2004) [**Ans.** : $a = 4$, $b = 1$]

2. Find the constants a, b such that the surface $ax^2 - bxy + xz = 10$ is orthogonal to the surface $x^2 + y^2 = 4 + xz$ at $(1, 2, 1)$.
(M.U. 2002) [**Ans.** : $a = 27$, $b = 9$]

3. Find the constants a, b, c if the normal to the surface $ax^2 + bxz + z^2y = c$ at $P(-1, 1, 2)$ is parallel to the normal to the surface $x^2 - y^2 + 2z = 2$ at $Q(1, 1, 1)$.
(M.U. 2001)
[**Ans.** : $a = 10$, $b = 8$, $c = -2$]

4. Find the constants a and b such that the surface $ax^2 - 2byz = (a + 4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1, -1, 2)$.
(M.U. 1996) [**Ans.** : $a = 5$, $b = 1$]

5. Find the constants a, b, c if the normal to the surface $ax^2 + yz + bxz^3 = c$ at $P(1, 2, 1)$ is parallel to the normal to the surface $y^2 + xz = 61$ at $(10, 1, 6)$.
(M.U. 1997, 2010)
[**Ans.** : $a = 1$, $b = 1$, $c = 4$]

6. Find the constants a, b if the angle between the surfaces

$x^2 + axz + byz = 2$ and $x^2z + xy + y + 1 = z$ at $(0, 1, 2)$ is $\cos^{-1}(1/\sqrt{3})$.
(M.U. 2000)
[**Ans.** : $a = 1$, $b = 1$]

7. If the directional derivative of $\Phi = ax^2 + by + 2z$ at $(1, 1, 1)$ is maximum in the direction of $i + j + k$, find a and b.
(M.U. 2001) [**Ans.** : 1, 2]

11. Divergence and Curl

We have seen in the previous article that if we operate upon a scalar point function by the operator ∇ we get a vector point function called grad denoted by $\nabla\Phi$. In this article we shall see that by operating upon a vector point function $\bar{f}$ by the operator ∇ *scalarly* (*i.e.* by taking dot product) we get *divergence* $\bar{f}$ denoted by *div* $\bar{f} = (\nabla \cdot \bar{f})$ and by operating on $\bar{f}$ *vectorially* (*i.e.* by taking cross product) we get *curl* $\bar{f}$ denoted by *curl* $\bar{f} = (\nabla \times \bar{f})$.

From the results (i) and (ii) of § 8, page 8-3, it is clear that ∇ is a differential operator just as d/dx is in differential calculus. We can apply ordinary rules of differential calculus to the differential operator ∇ keeping in mind its vector character.

(a) Definition : Let $\bar{f} = f_1 i + f_2 j + f_3 k$ then

$$\nabla \cdot \bar{f} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (f_1 i + f_2 j + f_3 k) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

is called the **divergence** of $\bar{f}$. Thus,

$$\boxed{\text{div } \bar{f} = \nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}} \qquad \text{.................... (1)}$$

Note that divergence $\bar{f}$ is a scalar point function. **(M.U. 2002)**

Remark ....

If $\bar{f}$ is a vector point function such that $\nabla \cdot \bar{f} = 0$ then $\bar{f}$ is called **solenoidal.**

Note ....

Writing $\nabla \cdot \bar{f}$ as $\bar{f} \cdot \nabla$ is wrong. $\nabla \cdot \bar{f}$ is a scalar while $\bar{f} \cdot \nabla$ is an operator

$$\left(f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z}\right).$$

We can also write $\nabla \cdot \bar{f}$ as,

$$\boxed{\text{div}\, \bar{f} = \nabla \cdot \bar{f} = i \cdot \frac{\partial \bar{f}}{\partial x} + j \cdot \frac{\partial \bar{f}}{\partial y} + k \cdot \frac{\partial \bar{f}}{\partial z}} \qquad \text{.................... (2)}$$

$$\boxed{\nabla \cdot \bar{f} = \sum i \cdot \frac{\partial \bar{f}}{\partial x}} \qquad \text{.................... (2A)}$$

The equivalence between (1) and (2) can be proved as follows.

$$\nabla \cdot \bar{f} = i \cdot \frac{\partial \bar{f}}{\partial x} + j \cdot \frac{\partial \bar{f}}{\partial y} + k \cdot \frac{\partial \bar{f}}{\partial z} \qquad \text{[By (2)]}$$

$$= i \cdot \left(i \frac{\partial f_1}{\partial x} + j \frac{\partial f_2}{\partial x} + k \frac{\partial f_3}{\partial x}\right) + j \cdot \left(i \frac{\partial f_1}{\partial y} + j \frac{\partial f_2}{\partial y} + k \frac{\partial f_3}{\partial y}\right) + k \cdot \left(i \frac{\partial f_1}{\partial z} + j \frac{\partial f_2}{\partial z} + k \frac{\partial f_3}{\partial z}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Hence, the equality between (1) and (2).

(b) Definition: If $\bar{f} = f_1 i + f_2 j + f_3 k$ then

$$\nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$ is called the **curl of** $\bar{f}$. Thus,

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} \qquad \text{.................... (3)}$$

Note that curl $\bar{f}$ is a vector point function.

Remark ....

If $\bar{f}$ is a vector point function such that curl $\bar{f} = 0$ then $\bar{f}$ is called **irrotational** or **conservative**. (The reason for this nomenclature will be made clear later.)

We can also write curl $\bar{f}$ as

$$\boxed{\nabla \times \bar{f} = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + k \times \frac{\partial \bar{f}}{\partial z}} \qquad \text{.................... (4)}$$

$$\boxed{\nabla \times \bar{f} = \sum i \times \frac{\partial \bar{f}}{\partial x}}$$

The equivalence between (1) and (2) can be proved as follows.

$$\nabla \times \bar{f} = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + k \times \frac{\partial \bar{f}}{\partial z} \qquad \text{[By (4)]}$$

$$= i \times \left(i \frac{\partial f_1}{\partial x} + j \frac{\partial f_2}{\partial x} + k \frac{\partial f_3}{\partial x} \right) + j \times \left(i \frac{\partial f_1}{\partial y} + j \frac{\partial f_2}{\partial y} + k \frac{\partial f_3}{\partial y} \right) + k \times \left(i \frac{\partial f_1}{\partial z} + j \frac{\partial f_2}{\partial z} + k \frac{\partial f_3}{\partial z} \right)$$

$$= k \frac{\partial f_2}{\partial x} - j \frac{\partial f_3}{\partial x} - k \frac{\partial f_1}{\partial y} + i \frac{\partial f_3}{\partial y} + j \frac{\partial f_1}{\partial z} - i \frac{\partial f_2}{\partial z}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Hence, the equality between (3) and (4).

We denote cut $\bar{f}$ also as curl $\bar{f} = \nabla \times \bar{f} = \sum i \times \dfrac{\partial \bar{f}}{\partial x}$.

(c) Standard Result : The following results can be proved very easily by using the above definition.

1. $\text{div}(\bar{f} \pm \bar{g}) = \text{div } \bar{f} \pm \text{div } \bar{g}$ or $\nabla \cdot (\bar{f} \pm \bar{g}) = \nabla \cdot \bar{f} \pm \nabla \cdot \bar{g}$
2. curl $(\bar{f} \pm \bar{g}) = \text{curl } \bar{f} \pm \text{curl } \bar{g}$ or $\nabla \times (\bar{f} \pm \bar{g}) = \nabla \times \bar{f} \pm \nabla \times \bar{g}$
3. $\nabla \cdot (\bar{f} \times \bar{g}) = \bar{g} \cdot (\nabla \times \bar{f}) - \bar{f} \cdot (\nabla \times \bar{g})$ **(M.U. 2003)**
4. $\nabla \cdot (\Phi \bar{f}) = \Phi (\nabla \cdot \bar{f}) + \nabla \Phi \cdot \bar{f}$

Type I : To Find div and curl of $\bar{F}$

Example 1 : If $\bar{F} = x^2 z\, i - 2y^3 z^3\, j + xy^2 z^2\, k$ find div $\bar{f}$ and curl $\bar{f}$ at (1, −1, 1).

Sol. : By definition

$$\text{div } \bar{F} = \nabla \cdot \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^3) + \frac{\partial}{\partial z}(xy^2 z^2)$$

$$= 2xz - 6y^2 z^3 + 2xy^2 z$$

$$\therefore \quad \text{div } \bar{F} = (2 - 6 + 2) = -2 \text{ at } (1, -1, 1)$$

$$\therefore \quad \text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & -2y^3 z^3 & xy^2 z^2 \end{vmatrix} = i(2xyz^2 + 6y^3 z^2) - j(y^2 z^2 - x^2) + k(0 - 0)$$

$$= i(-2 - 6) - j(1 - 1) + k(0) = -8\, i \text{ at } (1, -1, 1).$$

Example 2 : If $\bar{F} = xy\, e^{2z}\, i + xy^2 \cos z\, j + x^2 \cos xy\, k$ find div $\bar{F}$ and curl $\bar{F}$. **(M.U. 2013)**

Sol. :

$$\text{div } \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = y\, e^{2z} + 2xy \cos z + 0\, k$$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy\, e^{2z} & xy^2 \cos z & x^2 \cos xy \end{vmatrix}$$

$$= i(-x^3 \sin xy + xy^2 \sin z) - j(-x^2 y \sin xy + 2x \cos xy - 2xy\, e^{2z})$$
$$+ k(y^2 \cos z - x\, e^{2z})$$

Example 3 : If $\bar{a}$ is a constant vector find div $\bar{a}$ and curl $\bar{a}$.

Sol. : Let $\bar{a} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are constants

$$\therefore \quad \nabla \cdot \bar{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = 0$$

$$\nabla \times \bar{a} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \end{vmatrix} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) + j\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial z}\right)$$

$$\therefore \quad \nabla \times \bar{a} = 0\, i + 0\, j + 0\, k = \bar{0}$$

$$\therefore \quad \boxed{\textbf{div } \bar{a} = 0 \quad \textbf{and} \quad \textbf{curl } \bar{a} = \bar{0}.}$$

Example 4 : If $\bar{r} = xi + yj + zk$ find (i) grad r, (ii) div $\bar{r}$ and (iii) curl $\bar{r}$.

Sol. : (i) $\text{grad } r = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\sqrt{x^2 + y^2 + z^2}$

$$\text{Now, } \frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{r} \qquad \left[\because r = \sqrt{x^2 + y^2 + z^2}\right]$$

$\therefore$ grad $r = \dfrac{x}{r}i + \dfrac{y}{r}j + \dfrac{z}{r}k = \dfrac{1}{r}(xi + yj + zk) = \dfrac{1}{r}\bar{r}$

(ii) $\text{div}\,\bar{r} = \left(\dfrac{\partial}{\partial x}x\right) + \left(\dfrac{\partial}{\partial y}y\right) + \left(\dfrac{\partial}{\partial z}z\right) = 1 + 1 + 1 = 3.$

(iii) $\text{curl}\,\bar{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0\,i + 0\,j + 0\,k = \bar{0}$

Thus, $\boxed{\textbf{grad } r = \dfrac{\bar{r}}{r}, \quad \textbf{div } \bar{r} = 3, \quad \textbf{crul } \bar{r} = \bar{0}}$

Example 5 : If $\bar{a}$ is a constant vector and $\bar{r} = xi + yj + zk$, prove that

(1) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$ (2) div $(\bar{a}\times\bar{r}) = 0$ (3) div $(\bar{a}\cdot\bar{r})\,\bar{a} = a^2$

(4) curl $(\bar{a}\times\bar{r}) = 2\bar{a}$ (5) div $[\bar{a}\times(\bar{r}\times\bar{a})] = 2a^2$ **(M.U. 1999, 2005, 08)**

Sol. : (1) We have

$$\bar{a}\cdot\bar{r} = (a_1 i + a_2 j + a_3 k)\cdot(xi + yj + zk) = a_1 x + a_2 y + a_3 z$$

$$\therefore \quad \nabla(\bar{a}\cdot\bar{r}) = i\frac{\partial}{\partial x}(a_1 x + a_2 y + a_3 z) + j\frac{\partial}{\partial y}(\ldots\ldots) + k\frac{\partial}{\partial z}(\ldots\ldots)$$

$$= a_1 i + a_2 j + a_3 k = \bar{a}$$

(2) We have

$$\bar{a}\times\bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\,i - (a_1 z - a_3 x)\,j + (a_1 y - a_2 x)\,k$$

$$\text{div}\,(\bar{a}\times\bar{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) = 0$$

(3) We have, as above

$$\bar{a}\cdot\bar{r} = a_1 x + a_2 y + a_3 z$$

$$(\bar{a}\cdot\bar{r})\,\bar{a} = (a_1 x + a_2 y + a_3 z)\,a_1 i + (\ldots.)\,a_2 j + (\ldots.)\,a_3 k$$

$$\text{div}\,(\bar{a}\cdot\bar{r})\,\bar{a} = \frac{\partial}{\partial x}[(a_1 x + a_2 y + a_3 z)\,a_1] + \frac{\partial}{\partial y}[(\ldots.)\,a_2] + \frac{\partial}{\partial z}[(\ldots.)\,a_3]$$

$$= a_1^2 + a_2^2 + a_3^2 = a^2$$

(4) We have as in (2)

$$\bar{a}\times\bar{r} = (a_2 z - a_3 y)\,i + (a_3 x - a_1 z)\,j + (a_1 y - a_2 x)\,k$$

$$\therefore \quad \text{curl}\,(\bar{a}\times\bar{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$\therefore \quad \text{curl}\,(\bar{a}\times\bar{r}) = i(a_1 + a_1) + j(a_2 + a_2) + k(a_3 + a_3)$$

$$= 2(a_1 i + a_2 j + a_3 k) = 2\bar{a}.$$

(5) We have as in (2)

$$\bar{a}\times\bar{r} = (a_2 z - a_3 y)\,i + (a_3 x - a_1 z)\,j + (a_1 y - a_2 x)\,k$$

$\therefore \quad \bar{r} \times \bar{a} = -(\bar{a} \times \bar{r}) = (a_3 y - a_2 z) i + (a_1 z - a_3 x) j + (a_2 x - a_1 y) k$

$$\therefore \quad \bar{a} \times (\bar{r} \times \bar{a}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_3 y - a_2 z & a_1 z - a_3 x & a_2 x - a_1 y \end{vmatrix}$$

$$= [a_2 (a_2 x - a_1 y) - a_3 (a_1 z - a_3 x)] i + [.....] j + [.....] k$$

$$\text{div } [\bar{a} \times (\bar{r} \times \bar{a})] = \frac{\partial}{\partial x} [a_2 (a_2 x - a_1 y) - a_3 (a_1 z - a_3 x)] + \frac{\partial}{\partial y} [.....] + \frac{\partial}{\partial z} [.....]$$

$$= (a_2^2 + a_3^2) + (a_3^2 + a_1^2) + (a_1^2 + a_2^2)$$

$$= 2 (a_1^2 + a_2^2 + a_3^2) = 2 \bar{a}.$$

List of Formulae

1. If $\bar{a}$ is constant, then $\boxed{\nabla \cdot \bar{a} = 0 \quad \text{and} \quad \nabla \times \bar{a} = \bar{0}}$
2. If $\bar{r} = xi + yj + zk$, then $\boxed{\nabla \cdot \bar{r} = 3 \quad \text{and} \quad \nabla \times \bar{r} = \bar{0}}$
3. If $\bar{a}$ is constant and $\bar{r}$ is a position vector then

$$\boxed{\nabla (\bar{a} \cdot \bar{r}) = \bar{a} \quad \text{and} \quad \nabla \cdot (\bar{a} \times \bar{r}) = 0 \quad \text{and} \quad \nabla \times (\bar{a} \times \bar{r}) = 2 \bar{a}}$$

4. $\boxed{\nabla r^n = n r^{n-2} \bar{r}}$ [Ex. 8, page 8-5]

Example 6 : If $\Phi = x^3 + y^3 + z^3 - 3xyz$, find (i) $\bar{r} \cdot \nabla \Phi$, (ii) div $\bar{F}$ and curl $\bar{F}$ where $\bar{F} = \nabla \Phi$. **(M.U. 2003)**

Sol. : (a) $\nabla \Phi = i \frac{\partial \Phi}{\partial x} + j \frac{\partial \Phi}{\partial y} + k \frac{\partial \Phi}{\partial z}$

$\therefore \quad \bar{F} = \nabla \Phi = i (3x^2 - 3yz) + j (3y^2 - 3xz) + k (3z^2 - 3xy)$

But $\quad \bar{r} = xi + yj + 2k$

$\therefore \quad \bar{r} \cdot \nabla \Phi = \bar{r} \cdot \bar{F} = x (3x^2 - 3yz) + y (3y^2 - 3xz) + z (3z^2 - 3xy)$

$$= 3 (x^3 + y^3 + z^3 - 3xyz) = 3\Phi$$

(b) div $\bar{F} = \nabla \cdot \bar{F}$

$$\therefore \quad \text{div } \bar{F} = \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3xz) + \frac{\partial}{\partial z} (3z^2 - 3xy)$$

$$= 6x + 6y + 6z = 6 (x + y + z)$$

(c) $$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ 3 (x^2 - yz) & 3 (y^2 - xz) & 3 (z^2 - xy) \end{vmatrix}$$

$$= i (-3x + 3x) + j (3y - 3y) + k (-3z + 3z)$$

$$= 0 i + 0 j + 0 k = \bar{0}$$

Example 7 : If $\bar{f} = (x + y + 1) i + j - (x + y) k$, prove that $\bar{f} \cdot \text{curl } \bar{f} = 0$. **(M.U. 2004)**

Sol. : $$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ x + y + 1 & 1 & -x - y \end{vmatrix} = i (-1) - j (-1) + k (-1)$$

$$= -i + j - k$$

$$\therefore \quad \bar{f}\cdot \text{curl}\,\bar{f} = [(x+y+1)\,i + j - (x+y)\,k]\cdot[-i+j-k]$$
$$= -(x+y+1)+1+(x+y) = 0$$

Example 8 : Prove that $\nabla\cdot(\nabla\times\bar{F}) = 0$ where $\bar{F}$ is a vector point function. **(M.U. 2000)**

Sol. : Let $\bar{F} = F_1 i + F_2 j + F_3 k$

Then $$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + j\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\therefore \quad \nabla\cdot(\nabla\times\bar{F}) = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k\right]$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_1}{\partial y\partial z} = 0.$$

Example 9 : If $u\bar{F} = \nabla v$ where u and v are scalar fields and $\bar{F}$ is a vector field, prove that $\bar{F}\cdot\text{curl}\,\bar{F} = 0$.

Sol. : Let $\bar{F} = F_1 i + F_2 j + F_3 k$. By data $u\bar{F} = \nabla v$

$$\therefore \quad uF_1\, i + uF_2\, j + uF_3\, k = i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}$$

$$\therefore \quad uF_1 = \frac{\partial v}{\partial x},\quad uF_2 = \frac{\partial v}{\partial y},\quad uF_3 = \frac{\partial v}{\partial z}$$

$$\therefore \quad F_1 = \frac{1}{u}\frac{\partial v}{\partial x},\quad F_2 = \frac{1}{u}\frac{\partial v}{\partial y},\quad F_3 = \frac{1}{u}\frac{\partial v}{\partial z} \qquad \text{.................... (1)}$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= i\left[\left\{-\frac{1}{u^2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{1}{u}\frac{\partial^2 v}{\partial y\partial z}\right\} - \left\{-\frac{1}{u^2}\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} + \frac{1}{u}\frac{\partial^2 v}{\partial y\partial z}\right\}\right] + j[\ldots\ldots] + k[\ldots\ldots]$$

$$= i\frac{1}{u^2}\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + j(\ldots\ldots) + k(\ldots\ldots)$$

Now, using (1), we get

$$\therefore \quad \bar{F}\cdot\text{curl}\,\bar{F} = \frac{1}{u^3}\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + (\ldots\ldots) + (\ldots\ldots) = 0$$

(The terms are cancelled because of symmetry.)

Type II : Div and Curl of Functions of $\bar{r}$

Example 1 : Prove that $\nabla\left\{\nabla\cdot\frac{\bar{r}}{r}\right\} = -\frac{2}{r^3}\bar{r}$. **(M.U. 2005)**

Sol. : We have $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2+y^2+z^2}$

Now, let $\bar{f} = \dfrac{\bar{r}}{r} = \dfrac{xi}{\sqrt{x^2+y^2+z^2}} + \dfrac{yj}{\sqrt{x^2+y^2+z^2}} + \dfrac{zk}{\sqrt{x^2+y^2+z^2}}$

$$= f_1 i + f_2 j + f_3 k$$

$$\therefore \quad \nabla \cdot \frac{\bar{r}}{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Now, $\dfrac{\partial f_1}{\partial x} = \dfrac{\sqrt{x^2+y^2+z^2} - x \cdot x \big/ \sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)} = \dfrac{r^2 - x^2}{(r^2)^{3/2}} = \dfrac{r^2-x^2}{r^3}$

Similarly, we get two more results.

$$\therefore \quad \nabla \cdot \frac{\bar{r}}{r} = \frac{(r^2-x^2)+(r^2-y^2)+(r^2-z^2)}{r^3} = \frac{3r^2 - (x^2+y^2+z^2)}{r^3}$$

$$= \frac{2r^2}{r^3} = \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \nabla\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right) = 2\left[i \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + j(\ldots\ldots) + k(\ldots\ldots)\right]$$

$$= 2\left[i \cdot \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot 2x + j(\ldots\ldots) + k(\ldots\ldots)\right]$$

$$= -2\left[\frac{xi}{r^3} + \frac{yj}{r^3} + \frac{zk}{r^3}\right] = \frac{-2}{r^3}\bar{r}$$

Example 2 : Prove that $\nabla \cdot \left(r \nabla \dfrac{1}{r^n}\right) = \dfrac{n(n-2)}{r^{n+1}}$. **(M.U. 2006, 09)**

Sol. : We have $r^{-n} = (x^2+y^2+z^2)^{-n/2}$

$$\therefore \quad \nabla\left(\frac{1}{r^n}\right) = \left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-\frac{n}{2}-1} \cdot (2x)\, i + (\ldots\ldots)\, j + (\ldots\ldots)\, k$$

$$= -nr^{-n-2}[xi + yj + zk]$$

$$\therefore \quad r\nabla\left(\frac{1}{r^n}\right) = -nr^{-n-1}xi - nr^{-n-1}yj - nr^{-n-1}zk$$

$$\therefore \quad \nabla \cdot \left[r\nabla\left(\frac{1}{r^n}\right)\right] = -n\frac{\partial}{\partial x}\left[r^{-n-1}x\right] - [\ \ldots\ldots\ ] - [\ \ldots\ldots\ ]$$

$$= -n\left[(-n-1)x \cdot r^{-n-2}\frac{\partial r}{\partial x} + r^{-n-1} \cdot 1\right] - \left[\ldots\ldots\right] - \left[\ldots\ldots\right]$$

$$= -n\left[(-n-1)\frac{x^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right] - \left[\ldots\ldots\right] - \left[\ldots\ldots\right]$$

$$= -n\left[\left\{(-n-1)\frac{x^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right\} + \left\{(-n-1)\frac{y^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right\} + \left\{(-n-1)\frac{z^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right\}\right]$$

$$\therefore \quad \nabla\cdot\left[r\,\nabla\left(\frac{1}{r^n}\right)\right] = -n\left[-\frac{(n+1)}{r^{n+3}}(x^2+y^2+z^2)+\frac{3}{r^{n+1}}\right]$$

$$= -n\left[-\frac{n+1}{r^{n+1}}+\frac{3}{r^{n+1}}\right] = \frac{n(n-2)}{r^{n+1}}$$

Example 3 : Prove that $\nabla\cdot\left(r\,\nabla\frac{1}{r^3}\right) = \frac{3}{r^4}$. **(M.U. 1999, 2006)**

Sol. : Putting $r = 3$ in the above example, we get the result.

Example 4 : Prove that div grad $r^n = n(n+1)\,r^{n-2}$. **(M.U. 2004)**

Sol. : As proved in Ex. 8, page 8-5

$$\text{grad } r^n = n\,r^{n-2}\,\bar{r}.$$

$$\therefore \quad \text{div grad } r^n = \nabla\cdot\left[n\,r^{n-2}(xi+yj+zk)\right]$$

$$= n\left[\frac{\partial}{\partial x}\left(r^{n-2}\cdot x\right)+\frac{\partial}{\partial y}\left(r^{n-2}\cdot y\right)+\frac{\partial}{\partial z}\left(r^{n-2}\cdot z\right)\right]$$

$$= n\left[r^{n-2}+x\cdot(n-2)\cdot r^{n-3}\frac{\partial r}{\partial x}+\ldots\ldots+\ldots\ldots\right]$$

$$= n\left[\left\{r^{n-2}+x\cdot(n-2)\cdot r^{n-3}\frac{x}{r}\right\}+\{\ \ldots\ldots\ \}+\{\ \ldots\ldots\ \}\right]$$

$$= n\left[\left\{r^{n-2}+(n-2)\,r^{n-4}\cdot x^2\right\}+\left\{r^{n-2}+(n-2)\,r^{n-4}\cdot y^2\right\}\right.$$
$$\left.+\left\{r^{n-2}+(n-2)\,r^{n-4}\cdot z^2\right\}\right]$$

$$= n\left[3r^{n-2}+(n-2)\,r^{n-4}\cdot(x^2+y^2+z^2)\right]$$

$$= n\left[3r^{n-2}+(n-2)\,r^{n-2}\right]$$

$$= n(n+1)\,r^{n-2}$$

(The term r^{n-2} comes from three terms.)

Example 5 : Prove that **(i)** $\nabla\cdot(\bar{a}\times\bar{r}) = 0$, **(ii)** $\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r}\right) = 0$. **(M.U. 2004)**

Sol. : Assuming the result that $\nabla\cdot(\bar{a}\times\bar{r})$ can be looked upon as a scalar triple product treating ∇ as a vector we have [See (B), page 7-4]

$$\text{(i)}\quad \nabla\cdot(\bar{a}\times\bar{r}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}(a_2z-a_3y)+\frac{\partial}{\partial y}(a_3x-a_1z)+\frac{\partial}{\partial z}(a_1y-a_2x) = 0$$

(ii) In the same way

$$\nabla \cdot \left(\frac{\bar{a} \times \bar{r}}{r}\right) = \nabla \cdot \left(\bar{a} \times \frac{\bar{r}}{r}\right) = \begin{vmatrix} \partial / \partial x & \partial / \partial y & \partial / \partial z \\ a_1 & a_2 & a_3 \\ x / r & y / r & z / r \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{a_2 z - a_3 y}{r}\right) + \frac{\partial}{\partial y}\left(\frac{a_3 x - a_1 z}{r}\right) + \frac{\partial}{\partial z}\left(\frac{a_1 y - a_2 x}{r}\right)$$

$$= -\frac{1}{r^2}(a_2 z - a_3 y)\frac{\partial r}{\partial x} + \ldots\ldots + \ldots\ldots$$

$$= -(a_2 z - a_3 y)\frac{x}{r^3} + \ldots\ldots + \ldots\ldots$$

$$= -\frac{1}{r^3}\left[(a_2 z - a_3 y)\, x + (a_3 x - a_1 z)\, y + (a_1 y - a_2 x)\, z\right]$$

$$= 0$$

Example 6 : Prove that $\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \frac{(2-n)\bar{a}}{r^n} + \frac{n(\bar{a} \cdot \bar{r})\bar{r}}{r^{n+2}}$. **(M.U. 1996, 2002, 03, 05, 09)**

Sol. : We have $\frac{\bar{a} \times \bar{r}}{r^n} = \frac{1}{r^n}(\bar{a} \times \bar{r}) = \frac{1}{r^n}\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$$= \frac{1}{r^n}(a_2 z - a_3 y)\, i + \frac{1}{r^n}(a_3 x - a_1 z)\, j + \frac{1}{r^n}(a_1 y - a_2 x)\, k$$

$$\therefore \quad \nabla \times \frac{(\bar{a} \times \bar{r})}{r^n} = \begin{vmatrix} i & j & k \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ \dfrac{a_2 z - a_3 y}{r^n} & \dfrac{a_3 x - a_1 z}{r^n} & \dfrac{a_1 y - a_2 x}{r^n} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\left(\frac{a_1 y - a_2 x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_3 x - a_1 z}{r^n}\right)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$\because \quad r^2 = x^2 + y^2 + z^2 \qquad \therefore \quad 2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= i\left[\left\{-n\, r^{-n-1} \cdot \left(\frac{y}{r}\right)(a_1 y - a_2 x) + \frac{1}{r^n}a_1\right\}\right.$$

$$\left. - \left\{-n\, r^{-n-1}\left(\frac{z}{r}\right)(a_3 x - a_1 z) + \frac{1}{r^n}(-a_1)\right\}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[-\frac{n}{r^{n+2}}(a_1 y^2 - a_2 xy) + \frac{a_1}{r^n} + \frac{n}{r^{n+2}}(a_3 xz - a_1 z^2) + \frac{a_1}{r^n}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[\frac{2a_1}{r^n} - \frac{n}{r^{n+2}}a_1(y^2 + z^2)\frac{n}{r^{n+2}}(a_2 xy + a_3 xz)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

Adding $\frac{n}{r^{n+2}} a_1 x^2$ to the third term and subtracting it from the second term

$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}(x^2+y^2+z^2) + \frac{n}{r^{n+2}}(a_1x^2 + a_2xy + a_3xz)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}r^2 + \frac{n}{r^{n+2}}x(a_1x + a_2y + a_3z)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}r^2 + \frac{n}{r^{n+2}}x(a_1x + a_2y + a_3z)\right]$$
$$+ j\left[\frac{2a_2}{r^n} - \frac{na_2}{r^{n+2}}r^2 + \frac{n}{r^{n+2}}y(a_1x + a_2y + a_3z)\right]$$
$$+ k\left[\frac{2a_3}{r^n} - \frac{na_3}{r^{n+2}}r^2 + \frac{n}{r^{n+2}}z(a_1x + a_2y + a_3z)\right]$$

$$= \frac{(2-n)}{r^n}(a_1i + a_2j + a_3k) + \frac{n}{r^{n+2}}(a_1x + a_2y + a_3z)(xi + yj + zk)$$

$$= \frac{(2-n)}{r^n}\bar{a} + \frac{n}{r^{n+2}}(\bar{a}\cdot\bar{r})\bar{r}$$

Example 7 : Prove that $\nabla \times (\bar{a} \times \bar{r})\, r^n = (n+2)\, r^n\, \bar{a} - n\, r^{n-2}(\bar{a}\cdot\bar{r})\,\bar{r}$. **(M.U. 1993)**

Sol. : Change the sign of n in the above Ex. 6 or try independently on the same lines.

Example 8 : Prove that $\nabla \times \left(\frac{\bar{a}\times\bar{r}}{r}\right) = \frac{\bar{a}}{r} + \frac{\bar{a}\cdot\bar{r}}{r^3}\bar{r}$.

Sol. : Put $n = 1$ in the above Ex. 6 or try independently.

Example 9 : Prove that $\nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{a}{r^3} + \frac{3(\bar{a}\cdot\bar{r})\,r}{r^5}$. **(M.U. 1998, 2006)**

Sol. : Put $n = 3$ in Ex. 6 or try independently.

Example 10 : Prove that $\nabla \log r = \frac{\bar{r}}{r^2}$ and hence, show that

$$\nabla \times (\bar{a} \times \nabla \log r) = 2\frac{(\bar{a}\cdot\bar{r})\bar{r}}{r^4}$$ where $\bar{a}$ is a constant vector. **(M.U. 2000)**

Sol. : $\log r = \frac{1}{2}\log(x^2+y^2+z^2)$ $\quad \therefore \quad \frac{\partial}{\partial x}(\log r) = \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{x}{r^2}$

Similarly, $\frac{\partial}{\partial y}(\log r) = \frac{y}{r^2}$, $\frac{\partial}{\partial z}(\log r) = \frac{z}{r^2}$.

$$\therefore \quad \nabla \log r = i\frac{x}{r^2} + j\cdot\frac{y}{r^2} + k\cdot\frac{z}{r^2} = \frac{1}{r^2}(xi + yj + zk) = \frac{\bar{r}}{r^2}$$

Now, $\bar{a} \times \nabla \log r = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix} = \frac{(a_2z - a_3y)}{r^2}\,i + \frac{(a_3x - a_1z)}{r^2}\,j + \frac{(a_1y - a_2x)}{r^2}\,k$

$$\therefore \quad \nabla \times (\bar{a} \times \nabla \log r) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{(a_2 z - a_3 y)}{r^2} & \dfrac{(a_3 x - a_1 z)}{r^2} & \dfrac{(a_1 y - a_2 x)}{r^2} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\left\{\left(\frac{a_1 y - a_2 x}{r^2}\right)\right\} - \frac{\partial}{\partial z}\left\{\left(\frac{a_3 x - a_1 z}{r^2}\right)\right\}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$\because \quad r^2 = x^2 + y^2 + z^2,\ 2r\frac{\partial r}{\partial x} = 2x,$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= i\left[\left\{\frac{r^2(a_1) - (a_1 y - a_2 x)\, 2r\,(y/r)}{r^4}\right\}\right.$$

$$\left. - \left\{\frac{r^2(-a_1) - (a_3 x - a_1 z)\, 2r\,(z/r)}{r^4}\right\}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= \frac{2i}{r^4}\left[a_1 r^2 - a_1(y^2 + z^2) + (a_2 xy + a_3 xz)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

[By adding $a_1 x^2$ to the second term and sebtracting $a_1 x^2$ from the third term.]

$$= \frac{2i}{r^4}\left[a_1 r^2 - a_1(x^2 + y^2 + z^2) + (a_1 x^2 + a_2 xy + a_3 xz)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= \frac{2i}{r^4}\left[a_1 r^2 - a_1 r^2 + x(a_1 x + a_2 y + a_3 z)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= \frac{2}{r^4}(a_1 x + a_2 y + a_3 z)(xi + yj + zk) = \frac{2}{r^4}(\bar{a} \cdot \bar{r})\,\bar{r}$$

Example 11 : Prove that $\nabla \cdot \left\{\dfrac{f(r)}{r}\bar{r}\right\} = \dfrac{1}{r^2}\dfrac{d}{dr}\left[r^2 f(r)\right]$. **(M.U. 2002, 08)**

Hence, or otherwise prove that $\operatorname{div}(r^n\,\bar{r}) = (n+3)\,r^n$.

Sol. : $\dfrac{f(r)}{r}\bar{r} = \dfrac{f(r)}{r}(xi + yj + zk)$

$$\therefore \quad \nabla \cdot \left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right] + \frac{\partial}{\partial y}\left[\frac{f(r)}{r}y\right] + \frac{\partial}{\partial z}\left[\frac{f(r)}{r}z\right]$$

$$= \frac{f'(r)}{r}x\frac{\partial r}{\partial x} - \frac{f(r)}{r^2}x\frac{\partial r}{\partial x} + \frac{f(r)}{r} + \ldots\ldots + \ldots\ldots$$

But $r^2 = x^2 + y^2 + z^2 \quad \therefore \dfrac{\partial r}{\partial x} = \dfrac{x}{r},\ \dfrac{\partial r}{\partial y} = \dfrac{y}{r},\ \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \quad \nabla \cdot \left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{f'(r)}{r}\frac{x^2}{r} - \frac{f(r)}{r^2}\frac{x^2}{r} + \frac{f(r)}{r} + [\ldots\ldots] + [\ldots\ldots]$$

$$\therefore \quad \nabla \cdot \left\{ \frac{f(r)}{r} \bar{r} \right\} = f'(r) \frac{x^2}{r^2} - \frac{f(r)}{r} \frac{x^2}{r^2} + \frac{f(r)}{r} + [.....] + [.....]$$

$$= f'(r) \frac{(x^2 + y^2 + z^2)}{r^2} - \frac{f(r)}{r} \frac{(x^2 + y^2 + z^2)}{r^2} + 3 \frac{f(r)}{r}$$

$$= f'(r) + 2 \frac{f(r)}{r} \qquad \text{.................... (1)}$$

[$\because$ The term $\frac{f(r)}{r}$ will come from each bracket.]

Now, $\frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right] = \frac{1}{r^2} \left[r^2 f'(r) + 2r f(r) \right] = f'(r) + \frac{2f(r)}{r}$ ....(2)

From (1) and (2) we get the required result.

$$\left\{ e.g. \quad \nabla \cdot \left[\frac{e^r}{r} \bar{r} \right] = e^r + \frac{2e^r}{r} \right\}$$

Now put $\frac{f(r)}{r} = r^n$ i.e. $f(r) = r^{n+1}$

$$\therefore \quad \nabla \cdot \left\{ r^n \bar{r} \right\} = \frac{1}{r^2} \frac{d}{dr} \left\{ (r^{n+1}) r^2 \right\} = \frac{1}{r^2} \frac{d}{dr} \left\{ r^{n+3} \right\}$$

$$= \frac{1}{r^2} (n+3) r^{n+2} = (n+3) r^n.$$

(Also see the next example.)

Example 12 : Prove that $\nabla \cdot \left[\frac{\log r}{r} \bar{r} \right] = \frac{1}{r} [1 + 2 \log r]$.

Sol. : Putting $f(r) = \log r$ in the result (1) of the above Ex. 11, we get

$$\nabla \cdot \left[\frac{\log r}{r} \bar{r} \right] = \frac{1}{r^2} + \frac{2 \log r}{r} \qquad [\text{ Because } f(r) = \log r]$$

$$= \frac{1}{r} [1 + 2 \log r]$$

EXERCISE - III

State true or false with proper justification.

(i) Maximum value of the directional derivative of $\Phi = x^3 + 2xy + 3z$ cannot be less than 3. **(M.U. 1997)**

(ii) Grad r^3 is an irrotational vector. **(M.U. 1997)**

(iii) Div $\bar{r} = 3$, curl $\bar{r} = 0$.

(iv) If $\bar{a}$ is a constant vector then $\bar{a} \times \bar{r}$ is irrotational.

(v) If $\bar{a}$ is a constant vector then $\bar{a}$ is solenoidal.

(vi) Grad r^n is an irrotational vector.

(vii) If $\bar{a}$ is a constant vector then curl $(\bar{a} \times \bar{r}) = 0$.

(viii) If $\bar{a}$ is a constant vector then curl $(\bar{a} \cdot \bar{r}) = 0$.

(ix) If $\bar{a}$ is a constant vector then div $(\bar{a} \cdot \bar{r})\bar{a} = a^2$.

[**Ans. :** **(i)** True, **(ii)** True, **(iii)** True, **(iv)** False, **(iv)** True, **(v)** True, **(vi)** True, **(vii)** False, **(viii)** False, **(ix)** True.]

EXERCISE - IV

1. If $\bar{f} = x^2zi - 2y^3z^2j + xy^2zk$, find $\nabla \cdot \bar{f}$ at $(1, -1, 1)$. **(M.U. 1993)** [**Ans. :** -3]

2. If $\bar{f} = 3x^2i + 5xyj + xyz^3k$, find div $\bar{f}$ and curl $\bar{f}$ at (1, 2, 3).

   [**Ans. : (i)** 65, **(ii)** $27i - 18j + 10k$]

3. Find div $\bar{F}$ and curl $\bar{F}$ where

   (a) $\bar{F} = (x^2 + yz)\, i + (y^2 + zx)\, j + (z^2 + xy)\, k$.

   (b) $\bar{F} = (x^2 - y^2)\, i + 2xyj + (y^2 - xy)\, k$.

   (c) $\dfrac{xi - yj}{x^2 + y^2}$ **(M.U. 2000, 10)**

   [**Ans. :** **(a) (i)** $2(x + y + z)$, **(ii)** $\bar{0}$; **(b) (i)** $4x$, **(ii)** $(2y - x)\, i + yj + 4yk$

   (c) (i) $-\dfrac{2(x^2 - y^2)}{(x^2 + y^2)^2}$, **(ii)** $\dfrac{4xy}{(x^2 + y^2)^2}$.]

4. If $\bar{A} = \nabla(xy + yz + zx)$, find $\nabla \cdot \bar{A}$ and $\nabla \times \bar{A}$. **(M.U. 2004, 14)** [**Ans. :** $0, \bar{0}$]

5. If $\bar{F} = x^2i + xzj + yzk$ and $\bar{r} = xi + yj + zk$, find div $(\bar{F} \times \bar{r})$ and curl $(\bar{F} \times \bar{r})$. **(M.U. 2001)**

   [**Ans. : (i)** $z^2 + xz - x^2$, **(ii)** $(2x^2 - xy)\, i + (4xz - 2xy - y^2)\, j + (3yz - 2xz)\, k$]

6. If $\bar{u} = x^2yi + y^2x^3j - 3x^2z^2k$ and $\bar{v} = 2xz^2i - yzj + x^2y^3k$, find $\nabla \cdot (\bar{u} \times \bar{v})$ at (1, 2, 1).

   [**Ans. :** 96]

7. If $\bar{u} = x^2yi + y^2x^3j - 3x^2z^2k$ and $\Phi = x^2yz$, find $\nabla \cdot (\Phi\, \bar{u})$ at (1, 2, 1). [**Ans. :** 10]

8. If $\bar{A} = \dfrac{x}{r}i + \dfrac{y}{r}j + \dfrac{z}{r}k$ and $r = \sqrt{x^2 + y^2 + z^2}$, find div $\bar{A}$. [**Ans. :** $\dfrac{2}{r}$]

9. If $\bar{F} = (\bar{a} \cdot \bar{r})\bar{r}$ where $\bar{a}$ is a constant vector, find curl $\bar{F}$ and prove that it is perpendicular to $\bar{a}$. **(M.U. 2002)**

10. Prove that $\nabla \cdot \left(\dfrac{\bar{r}}{r^3}\right) = 0$. **(M.U. 1992)**

11. If $u = x^2 + y^2 + z^2$ prove that curl grad $u = \bar{0}$.

12. If $\bar{A} = yz^2i + zx^2j + xy^2k$, prove that $\bar{A} \cdot$ curl $\bar{A} = xyz(xy + yz + zx)$.

12. Physical Interpretation of Divergence

Consider a region of space filled with a fluid which moves so that its velocity vector at any point $P(x, y, z)$ is $\bar{f}(x, y, z)$. With usual notation let, $\bar{f} = f_1i + f_2j + f_3k$, so that f_1, f_2, f_3 are scalar functions of x, y, z and are the components of velocity parallel to the axes.

Now construct a parallelepiped having centre at $P(x, y, z)$ and edges parallel to the coordinate axes and having magnitudes δx, δy, δz respectively.

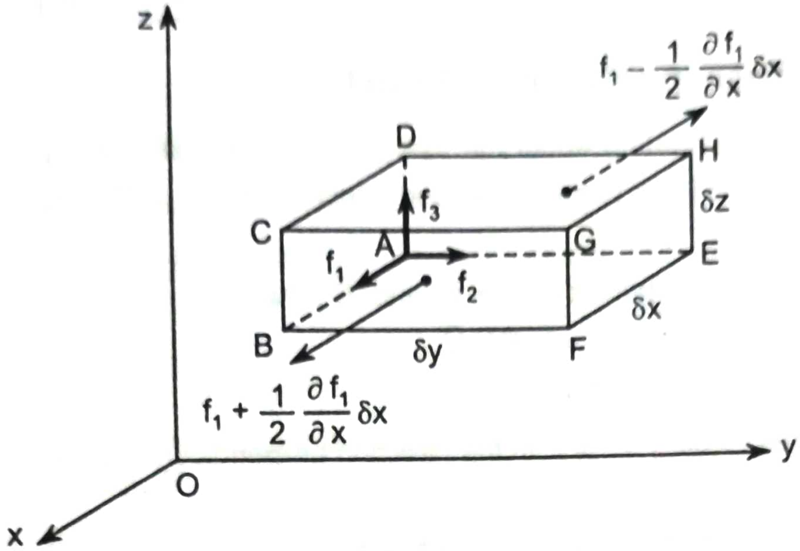

Fig. 8.4

Since the component of $\bar{f}$ normal to any face is responsible for the flow through that face,

The amount of fluid, leaving the face $CBFG$ in time δt

$$= \text{velo. comp. normal to } CBFG \times \text{area} \times \text{time}$$

$$= \left(f_1 + \frac{1}{2}\frac{\partial f_1}{\partial x}\delta x\right)\delta y\,\delta z\,\delta t.$$

Similarly, amount of fluid entering the face $DAEH$

$$= \text{velo. como. normal to } DAEH \times \text{area} \times \text{time}$$

$$= \left(f_1 - \frac{1}{2}\frac{\partial f_1}{\partial x}\delta x\right)\delta y\,\delta z\,\delta t.$$

$\therefore$ Gain of fluid in the parallelepiped in the direction of the x-axis.

$$= \frac{\partial f_1}{\partial x}\delta x\,\delta y\,\delta z\,\delta t.$$

Similarly, we can calculate the gain of fluid in the parallelepiped in the directions of the y and z-axis.

$\therefore$ Total gain of fluid in the parallelepiped

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)\delta x\,\delta y\,\delta z\,\delta t.$$

But $\delta x\,\delta y\,\delta z$ is the volume of the parallelepiped.

$\therefore$ Total gain of fluid in the parallelepiped per unit volume per unit time

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)\left(\frac{\delta x\,\delta y\,\delta z}{\delta x\,\delta y\,\delta z}\right)\left(\frac{\delta t}{\delta t}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \text{div}\,\bar{f} = \nabla \cdot \bar{f}$$

Definition : A vector $\bar{F}$ whose divergence $\bar{F}$ is zero is called **solenoidal**. For such a vector there is no loss or gain of fluid.

Thus, we have another definition.

Definition : A vector $\overline{F}$ is said to be **solenoidal** if $\boxed{\nabla \cdot \overline{F} = 0}$

13. Physical Interpretation of curl

Let a rigid body be rotating about the axis OA with angular velocity w radians per sec. Let P be a point of the body such that $\overline{OP} = \overline{R}$ and $\angle AOP = \theta$. Let $PA \perp OA$.

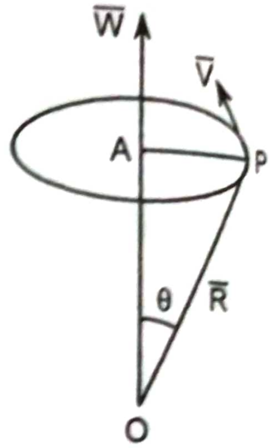

Fig. 8.5

If $\overline{N}$ is a unit vector perpendicular to $\overline{w}$ and $\overline{R}$, then

$$\overline{w} \times \overline{R} = wR\sin\theta \cdot \overline{N} = (wPA)\,\overline{N}$$
$$= (\text{speed of } P)\,\overline{N}$$
$= $ velocity $\overline{V}$ of P in the direction perpendicular to the plane AOP.

If $\overline{w} = w_1 i + w_2 j + w_3 k$ and $\overline{R} = xi + yj + zk$

Then

$$\overline{V} = \overline{w} \times \overline{R} = \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = (w_2 z - w_3 y)\,i + (w_3 x - w_1 z)\,j + (w_1 y - w_2 x)\,k$$

$$\therefore \quad \text{curl } \overline{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix} = (w_1 + w_1)\,i + (w_2 + w_2)\,j + (w_3 + w_3)\,k$$
$$= 2w_1 i + 2w_2 j + 2w_3 k = 2\overline{w}$$

$$\therefore \qquad \overline{w} = \frac{1}{2}\,\text{curl } \overline{V}$$

Thus, the angular velocity of rotation at any point is equal to half the curl of the velocity vector.

Definition : Any motion in which the curl of the velocity vector is zero *i.e.* if curl $\overline{v} = 0$ then $\overline{w} = 0$ *i.e.* angular velocity is zero, the motion is said to be **irrotational**.

Note ....

In view of this interpretation of curl, curl $\overline{F}$ is also called the **rotation** of $\overline{F}$ and is **sometimes denoted by rot** $\overline{F}$.

Definition : A vector $\overline{F}$ is said to be **irrotational** if $\boxed{\text{curl } \overline{F} = \overline{0}}$

(A) To Show That A Vector Is Solenoidal or Irrotational

Example 1 : If $\overline{F} = (x + 3y)\,i + (y - 2z)\,j + (az + x)\,k$ is solenoidal, find the value of a.

(M.U. 1998)

Sol. : We know that $\overline{F}$ is solenoidal if div $\overline{F} = \nabla \cdot \overline{F} = 0$.

Now, $$\nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(az + x)$$
$$= 1 + 1 + a = 2 + a$$

$\overline{F}$ is solenoidal if divergence $\overline{F} = 0$. $\quad \therefore \; 2 + a = 0 \quad \therefore \; a = -2.$

Example 2 : Find a, b, c if $\overline{F} = (axy + bz^3)i + (3x^2 - cz)j + (3xz^2 - y)k$ is irrotational. **(M.U. 1999, 2005)**

Sol. : $\overline{F}$ is irrotational if curl $\overline{F} = 0$.

$$\therefore \text{ curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy + bz^3 & 3x^2 - cz & 3xz^2 - y \end{vmatrix} = i(-1+c) - j(3z^2 - 3bz^2) + k(6x - ax)$$

$$\therefore \ c - 1 = 0, \qquad 3z^2 - 3bz^2 = 0, \qquad 6x - ax = 0$$

$$\therefore \ c = 1, \qquad 3z^2(1-b) = 0, \qquad x(6-a) = 0$$

$$\therefore \ c = 1, \ b = 1, \ a = 6.$$

Example 3 : Prove that $\overline{F} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is solenoidal and determine the constants a, b, c if $\overline{F}$ is irrotational. **(M.U. 1995, 2000, 04, 14)**

Sol. : $\overline{F}$ is solenoidal if $\nabla \cdot \overline{F} = 0$.

Now, $\nabla \cdot \overline{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z} = 1 - 3 + 2 = 0$

Hence, for all values of a, b, c, $\overline{F}$ is solenoidal. $\overline{F}$ is irrotational if curl $\overline{F} = \overline{0}$.

$$\text{Now, curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

and $\quad F_1 = x + 2y + az, \ F_2 = bx - 3y - z, \ F_3 = 4x + cy + 2z$

$$\therefore \text{ curl } \overline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k$$

$$= (c+1)\,i + (a-4)\,j + (b-2)\,k = 0i + 0j + 0k$$

$$\therefore \ c + 1 = 0, \qquad a - 4 = 0, \qquad b - 2 = 0$$

$$\therefore \ a = 4, \ b = 2, \ c = -1.$$

Example 4 : Is $\overline{F} = \dfrac{\overline{a} \times \overline{r}}{r^n} a$, solenoidal vector ? ($\overline{a}$ is a constant vector). **(M.U. 1997)**

Sol. : By data $\quad \overline{F} = \dfrac{\overline{a} \times \overline{r}}{r^n} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x/r^n & y/r^n & z/r^n \end{vmatrix}$

$$\therefore \ \overline{F} = i\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) + j\left(\frac{a_3 x}{r^n} - \frac{a_1 z}{r^n}\right) + k\left(\frac{a_1 y}{r^n} - \frac{a_2 x}{r^n}\right)$$

$$\text{Now, } \nabla \cdot \overline{F} = \frac{\partial}{\partial x}\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) + \frac{\partial}{\partial y}\left(\frac{a_3 x}{r^n} - \frac{a_1 z}{r^n}\right) + \frac{\partial}{\partial z}\left(\frac{a_1 y}{r^n} - \frac{a_2 x}{r^n}\right)$$

$$\text{Now, } \frac{\partial}{\partial x}\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) = (a_2 z - a_3 y)\frac{\partial}{\partial x} r^{-n} = (a_2 z - a_3 y)\left(-n\, r^{-n-1}\frac{\partial r}{\partial x}\right)$$

$$= (a_2 z - a_3 y)\left(-n\, r^{-n-1}\frac{x}{r}\right) = (a_2 z - a_3 y)\left(\frac{-nx}{r^{n+2}}\right)$$

$\left(\because \ r^2 = x^2 + y^2 + z^2 \quad \therefore \ 2r\dfrac{\partial r}{\partial x} = 2x\right)$

By symmetry, we get two more expressions.

$$\therefore \quad \nabla \cdot \bar{F} = \frac{n}{r^{n+2}}(a_3xy - a_2xz + a_1zy - a_3xy + a_2xz - a_1yz) = 0$$

Hence, $\bar{F}$ is solenoidal.

Example 5 : If $\bar{r}$ is the position vector of a point (x, y, z) and r is the modulus of $\bar{r}$ then prove that $r^n\,\bar{r}$ is an irrotational vector for any value of n but is solenoidal only if $n = -3$.

(M.U. 2002, 03, 06)

Sol. : (a) By definition

$$\text{curl } r^n\,\bar{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n\,x & r^n\,y & r^n\,z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(r^n\,z) - \frac{\partial}{\partial z}(r^n\,y)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[z\,n\,r^{n-1}\frac{\partial r}{\partial y} - y\,n\,r^{n-1}\frac{\partial r}{\partial z}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

Now, $r^2 = x^2 + y^2 + z^2 \quad \therefore \quad 2r\dfrac{\partial r}{\partial x} = 2x \quad \therefore \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \quad \text{curl } r^n\,\bar{r} = i[\,n\,r^{n-2}\,zy - n\,r^{n-2}\,zy\,] + j[\ldots] + k[\ldots] = \bar{0}$$

Hence, $r^n\,\bar{r}$ is irrotational for any value of n.

(b) $\quad \text{div}\,(r^n\,\bar{r}) = \nabla \cdot (r^n\,xi + r^n\,yj + r^n\,zk)$

$$= \frac{\partial}{\partial x}(r^n\,x) + \frac{\partial}{\partial y}(r^n\,y) + \frac{\partial}{\partial z}(r^n\,z)$$

$$= \left[r^n + xnr^{n-1}\frac{\partial r}{\partial x}\right] + \left[\ldots\ldots\right] + \left[\ldots\ldots\right]$$

$$= \left[r^n + nxr^{n-1}\frac{x}{r}\right] + \left[r^n + nyr^{n-1}\frac{y}{r}\right] + \left[r^n + nzr^{n-1}\frac{z}{r}\right]$$

$$= 3\,r^n + nr^{n-2}(x^2 + y^2 + z^2)$$

$$= 3\,r^n + nr^{n-2} \cdot r^2 + n\,r^n = (n+3)\,r^r.$$

Hence, $\text{div}\,(r^n\,\bar{r}) = 0$ if $n = -3$.

Example 6 : Find $f(r)$, so that the vector $f(r)\,\bar{r}$ is both solenoidal and irrotational.

(M.U. 1996, 2003, 05, 09)

Sol. : (a) We have $f(r)\,\bar{r} = f(r)\,xi + f(r)\,yj + f(r)\,zk$

$$\text{div}\,[f(r)\,\bar{r}] = \nabla \cdot f(r)\,\bar{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot [f(r)\,xi + f(r)\,yj + f(r)\,zk]$$

$$= \frac{\partial}{\partial x}[f(r)\,x] + \frac{\partial}{\partial y}[f(r)\,y] + \frac{\partial}{\partial z}[f(r)\,z]$$

Now, $\dfrac{\partial f(r)}{\partial x} = f'(r)\dfrac{\partial r}{\partial x} = f'(r)\dfrac{x}{r}$

Similarly, $\dfrac{\partial}{\partial y} f(r) = f'(r)\dfrac{y}{r}$, $\dfrac{\partial f(r)}{\partial z} = f'(r)\dfrac{z}{r}$

$\therefore \quad \dfrac{\partial}{\partial x}[f(r)\,x] = x\dfrac{\partial}{\partial x} f(r) + f(r) = x\dfrac{f'(r)}{r}\cdot x + f(r)$

$$\therefore \text{ div } [f(r)\,\bar{r}] = f'(r)\frac{x}{r}\cdot x + f(r) + f'(r)\frac{y}{r}\cdot y + f(r) + f'(r)\frac{z}{r}\cdot z + f(r)$$

$$= 3f(r) + f'(r)\cdot\frac{1}{r}[x^2 + y^2 + z^2] = 3f(r) + f'(r)\,r$$

If $f(r)\bar{r}$ is solenoidal

$$\text{div } [f(r)\,\bar{r}] = 3f(r) + f'(r)\,r = 0 \qquad \therefore \quad \frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating $\log f(r) = -3\log r + \log c$

$$\therefore \quad \log f(r) = \log\frac{c}{r^3} \qquad \therefore \quad f(r) = \frac{c}{r^3}$$

Thus, $f(r)\,\bar{r}$ is solenoidal if $f(r) = \dfrac{c}{r^3}$.

(b) Now, $\text{curl}\,[f(r)\,\bar{r}] = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y} z f(r) - \frac{\partial}{\partial z} y f(r)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[z f'(r)\cdot\frac{y}{r} - y\cdot f'(r)\frac{z}{r}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= f'(r)\,i\left[\frac{zy}{r} - \frac{zy}{r}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right] = 0$$

Thus, $f(r)\,\bar{r}$ is irrotational if for any $f(r)$.

Hence, $f(r)\,\bar{r} = \dfrac{c}{r^3}\bar{r}$ is both solenoidal and irrotational.

Note ....

Note the similarity and difference between the above two examples 5 and 6. Also compare them with the following Example 7.

Example 7 : Prove that $\bar{F} = \dfrac{\bar{r}}{r^3}$ is both irrotational and solenoidal. **(M.U. 2002, 04, 14)**

Sol. : As above.

Example 8 : Show that $\bar{F} = \dfrac{\bar{r}}{r^2}$ is irrotational.

Find F such that $\bar{F} = -\nabla\Phi$ where $\bar{r} = xi + yj + zk$. **(M.U. 2004)**

Sol. : We have $\quad \text{curl}\ \frac{\bar{r}}{r^2} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix}$

$$\therefore \quad \text{curl}\ \frac{\bar{r}}{r^2} = \left[\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right)\right] i + \left[\cdots\cdots\right] j + \left[\cdots\cdots\right] k$$

$$= \left(-\frac{2}{r^3}\frac{yz}{r} + \frac{2}{r^3}\frac{z}{r} \cdot y\right) i + \left(\cdots\cdots\right) j + \left(\cdots\cdots\right) k$$

$$= 0i + 0j + 0k = \bar{0} \quad \therefore \ \bar{F} \text{ is irrotational.}$$

Now, $\bar{F} = -\nabla\Phi$ gives $\frac{\bar{r}}{r^2} = \frac{xi + yj + zk}{r^2} = -\left[\frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k\right]$.

$$\therefore \quad \frac{\partial \Phi}{\partial x} = -\frac{x}{r^2} = -\frac{x}{x^2 + y^2 + z^2}; \qquad \frac{\partial \Phi}{\partial y} = -\frac{y}{r^2} = -\frac{y}{x^2 + y^2 + z^2};$$

$$\frac{\partial \Phi}{\partial z} = -\frac{z}{r^2} = -\frac{z}{x^2 + y^2 + z^2}.$$

But $\quad d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz = -\frac{xdx + ydy + zdz}{x^2 + y^2 + z^2}$

By integration, $\Phi = -\frac{1}{2}\log(x^2 + y^2 + z^2)$.

(B) To Find The Scalar Potential

Example 1 : A vector field is given by $\bar{F} = (x^2 + xy^2) i + (y^2 + x^2 y) j$. Show that $\bar{F}$ is irrotational and find its scalar potential. **(M.U. 2004)**

Sol. :- We have $\quad \text{curl}\ \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2 + x^2 y)\right] i - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 + xy^2)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(y^2 + x^2 y) - \frac{\partial}{\partial y}(x^2 + xy^2)\right] k$$

$$= 0\,i + 0\,j + (2xy - 2xy)\,k = 0\,i + 0\,j + 0\,k$$

Hence, $\bar{F}$ is irrotational.

If Φ is the scalar potential then $\bar{F} = \nabla\Phi$.

$$\therefore \quad (x^2 + xy^2)\,i + (y^2 + x^2 y)\,j + 0\,k = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial y} k$$

$$\therefore \quad \frac{\partial \Phi}{\partial x} = x^2 + xy^2 \quad \text{.......... (1)} \qquad \frac{\partial \Phi}{\partial y} = y^2 + x^2 y \quad \text{.......... (2)}$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{.......... (3)}$$

But $d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$

$$= \left[x^2 + xy^2\right] dx + \left[y^2 + x^2 y\right] dy + 0\, dz$$

$$= x^2\, dx + y^2\, dy + (xy^2\, dx + x^2 y\, dy)$$

By integration $\Phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{1}{2} x^2 y^2$

Aliter : Integrating (1), (2), (3) w.r.t. x, y, z respectively treating the other variables constant, we get,

$$\Phi = \frac{x^3}{3} + \frac{x^2 y^2}{2} + \Psi_1(y, z); \quad \Phi = \frac{y^3}{3} + \frac{x^2 y^2}{2} + \Psi_2(x, z); \quad \Phi = \Psi_3(x, y).$$

Comparing these equations, we find that

$$\Psi_1(y, z) = \frac{y^3}{3}, \quad \Psi_2(x, z) = \frac{x^3}{3}, \quad \Psi_3(x, y) = 0.$$

$$\therefore \quad \Phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2 y^2}{2}.$$

Example 2 : A vector field $\overline{F}$ is given by

$$\overline{F} = (y \sin z - \sin x)\, i + (x \sin z + 2yz)\, j + (xy \cos z + y^2)\, k$$

Prove that it is irrotational and hence, find its scalar potential. **(M.U. 2013)**

Sol. : We have

$$\text{curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy \cos z + y^2) - \frac{\partial}{\partial z}(x \sin z + 2yz)\right] i$$

$$+ \left[\frac{\partial}{\partial z}(y \sin z - \sin x) - \frac{\partial}{\partial x}(xy \cos z + y^2)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(x \sin z + 2yz) - \frac{\partial}{\partial y}(y \sin z - \sin x)\right] k$$

$$= [x \cos z + 2y - x \cos z - 2y]\, i + [y \cos z - y \cos z]\, j + [\sin z - \sin z]\, k$$

$$= 0\, i + 0\, j + 0k = \overline{0}$$

Hence, $\overline{F}$ is irrotational.

If Φ is the scalar potential then $\overline{F} = \nabla \Phi$.

$$\therefore \quad (y \sin z - \sin x)\, i + (x \sin z + 2yz)\, j + (xy \cos z + y^2)\, k = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k$$

$$\therefore \quad \frac{\partial \Phi}{\partial x} = y \sin z - \sin x \quad \text{.......... (1)} \qquad \frac{\partial \Phi}{\partial y} = x \sin z + 2yz \quad \text{.......... (2)}$$

$$\frac{\partial \Phi}{\partial z} = xy \cos z + y^2 \quad \text{.......... (3)}$$

But $\quad d\Phi = \dfrac{\partial \Phi}{\partial x}dx + \dfrac{\partial \Phi}{\partial y}dy + \dfrac{\partial \Phi}{\partial z}dz$

$$= (y \sin z - \sin x)\,dx + (x \sin z + 2yz)\,dy + (xy \cos z + y^2)\,dz$$

$$= [y \sin z\,dx + x \sin z dy + xy \cos z dz] + (-\sin x)\,dx + (2yzdy + y^2 dz)$$

By integration, $\Phi = xy \sin z + \cos x + y^2 z$

Aliter : Integrating (1), (2) and (3) partially w.r.t. x, y, z treating other variables constant, we get,

$$\Phi = (xy \sin z + \cos x) + \Psi_1(y, z) \qquad \text{.....................} (4)$$

$$\Phi = (xy \sin z + y^2 z) + \Psi_2(x, z) \qquad \text{.....................} (5)$$

$$\Phi = (xy \sin z + y^2 z) + \Psi_3(x, y) \qquad \text{.....................} (6)$$

Comparing (4), (5) and (6), we find that

$$\Psi_1(y, z) = y^2 z,\ \Psi_2(x, z) = \cos x,\ \Psi_3(x, y) = \cos x$$

$$\therefore \quad \Phi = xy \sin z + \cos x + y^2 z$$

EXERCISE - V

(A) To Show That A Vector is Solenoidal or Irrotational

1. Show that the vector $\bar{F} = (z + \sin y)\,i + (x \cos y - z)\,j + (x - y)\,k$ is irrotational.
2. Show that the vector $\bar{F} = (y + z)\,i + (z + x)\,j + (x + y)\,k$ is solenoidal.
3. Show that the vector $\bar{F} = \dfrac{xi + yj}{x^2 + y^2}$ is solenoidal.
4. Determine the constant a, so that the vector
   $\bar{F} = (x + 3y^2)\,i + (2y + 2z^2)\,j + (x^2 + az)\,k$ is solenoidal. [**Ans. :** $a = -3$]
5. Show that the vector $\bar{F} = yzi + zxj + xyk$ is solenoidal.
6. Show that the vector $\bar{F} = \dfrac{-yi + xj}{x^2 + y^2}$ is irrotational.
7. If the vector $(ax + 3y + 4z)\,i + (x - 2y + 3z)\,j + (3x + 2y - z)\,k$
   is solenoidal, determine the constant a. [**Ans. :** $a = 3$]
8. Show that if $(xyz)^b (x^a i + y^a j + z^a k)$ is an irrotational vector then either $b = 0$ or $a = -1$.
   (M.U. 1995)

(B) To Find The Scalar Potential

1. Show that the velocity given by $\bar{V} = (y + z)\,i + (z + x)\,j + (x + y)\,k$ is irrotational and find its scalar potential. [**Ans. :** $\Phi = yz + zx + xy$]
2. Is the above motion possible for an incompressible fluid ?
   [**Ans. :** Fluid motion is possible because $\nabla \cdot \bar{V} = 0$.]
3. Show that the vector, $\bar{F} = (x^2 - yz)\,i + (y^2 - zx)\,j + (z^2 - xy)\,k$
   is irrotational and hence, find Φ such that $\bar{F} = \nabla\Phi$. [**Ans. :** $\Phi = \dfrac{x^3}{3} + \dfrac{y^3}{3} + \dfrac{z^3}{3} - xyz$]

4. Show that the vector $\overline{F} = (6xy + z^3)\, i + (3x^2 - z)\, j + (3xz^2 - y)\, k$
is irrotational. Find the function Φ such that $\overline{F} = \nabla\Phi$. **[Ans. :** $\Phi = 3x^2y + z^3x - yz$ **]**

5. Show that the vector $\overline{F} = (y^2 \cos x + z^3)\, i + (2y \sin x - 4)\, j + (3xz^2 + 2)\, k$
is irrotational and find its scalar potential. **[Ans. :** $\Phi = y^2 \sin x + z^3x - 4y + 2z$ **]**

6. Show that the vector field given by $\overline{F} = 2xyz^3\, i + x^2z^3\, j + 3x^2yz\, k$
is irrotational and find its scalar potential. **[Ans. :** $\Phi = x^2yz^3$ **]**

7. Prove that $\overline{F} = (z^2 + 2x + 3y)\, i + (3x + 2y + z)\, j + (y + 2zx)\, k$ is irrotational and find scalar potential function Φ such that $\overline{F} = \nabla\Phi$ and $\Phi\,(1, 1, 0) = 4$.
Hence, find the workdone by $\overline{F}$ in moving a particle from $A\,(0, 1, 1)$ to $B\,(3, 0, 2)$.
(M.U. 2002, 03) [Ans. : $\Phi = x^2 + y^2 + z^2x + 3xy + zy - 1$; 19 **]**

8. Show that $\overline{F} = (y^2 - z^2 + 3yz - 2x)\, i + (3xz + 2xy)\, j + (3xy - 2xz + 2z)\, k$
is both solenoidal and irrotational. **(M.U. 2005)**

9. If $\nabla\Phi = (y^2 - 2xyz^3)\, i + (3 + 2xy - x^2z^3)\, j + (6z^3 - 3x^2yz^2)\, k$,
find Φ where $\Phi\,(1, 0, 1) = 8$. **(M.U. 2005) [Ans. :** $xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 - \frac{13}{2}$ **]**

EXERCISE - VI

Theory

1. Define div. and curl of a vector point function. **(M.U. 2002)**
2. Define the gradient of a scalar point function Φ and prove that the directional derivative of Φ is maximum in the direction of grad Φ. **(M.U. 2001)**

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