Functions and Pigeon Hole Principle (04)

5.1 Definition and types of functions: Injective, Surjective and Bijective

5.2 Composition, Identity and Inverse

5.3 Pigeon-hole principle , Extended Pigeon-hole principle

Functions

Rosen 6^{th} ed.

Definition of Functions

- Given any sets A, B, a function f from *(or "mapping") A to B* (f:$A \rightarrow B$) is an assignment of **exactly one** element $\boldsymbol{f(x) \in B}$

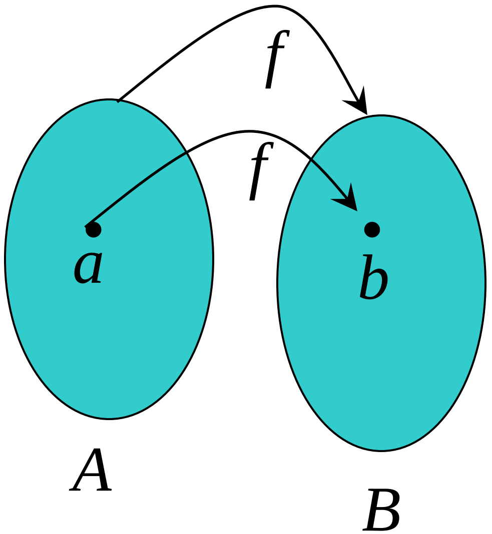
  to each element $\boldsymbol{x \in A}$.

Generic Functions

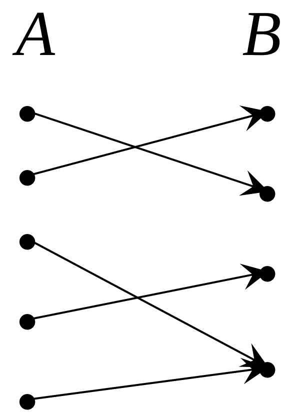
- A function $f: X \rightarrow Y$ is a relationship between elements of X to elements of Y, when each element from X is related to a unique element from Y
- X is called domain of f, range of f is a subset of Y so that for each element y of this subset there exists an element x from X such that $y = f(x)$
- Sample functions:
  - $f : R \rightarrow R, f(x) = x^2$
  - $f : Z \rightarrow Z, f(x) = x + 1$
  - $f : Q \rightarrow Z, f(x) = 2$

Graphical Representations

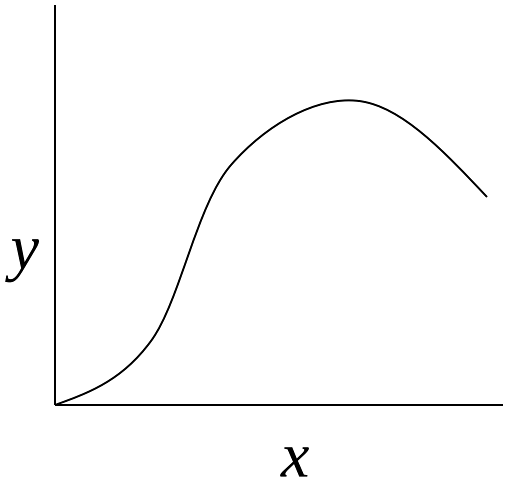
- Functions can be represented graphically in several ways:



Like Venn diagrams



Graph



Plot

Some Function Terminology

- If f:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then:
  - – A is the *domain* of f.
  - – B is the *codomain* of f.
  - – b is the *image* of a under f.
  - – a is a *pre-image* of b under f.
    - In general, b may have more than one pre-image.
  - – The *range* $R \subseteq B$ of f is $\{b \mid \exists a\, f(a)=b\}$.

Example 1 $\{(-3,1),(0,2),(2,4)\}$

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Domain Range

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Domain

Range

-3

0

<u>Example 1</u> $\{(-3,1),(0,2),(2,4)\}$

Domain Range

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Domain

-3

0

2

Range

Example 1 $\{(-3,1),(0,2),(2,4)\}$

Domain Range

Example 1 {(-3,1),(0,2),(2,4)}

Domain

-3

0

2

Range

1

Example 1 $\{(-3,1),(0,2),(2,4)\}$

Domain

-3
0
2

Range

1
2

Example 1 $\{(-3,1),(0,2),(2,4)\}$

Domain

-3

0

2

Range

1

2

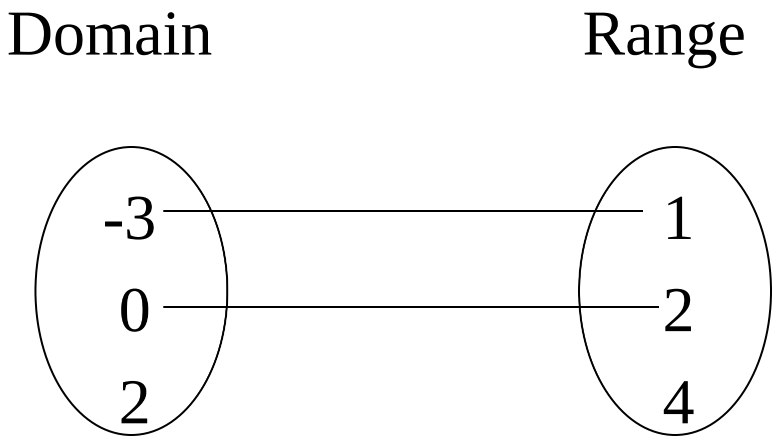
Example 1 $\{(-3,1),(0,2),(2,4)\}$

Domain	Range
-3	1
0	2
2	4

Example 1 {(-3,1),(0,2),(2,4)}

Domain Range

Example 1 {(-3,1),(0,2),(2,4)}



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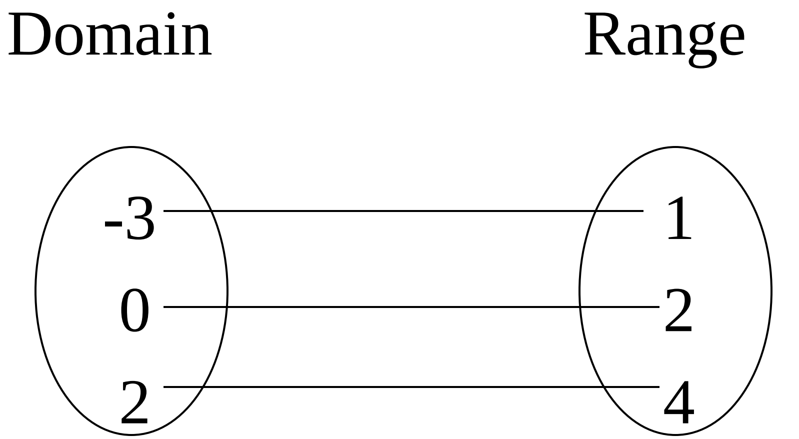


Example 1 {(-3,1),(0,2),(2,4)}

Domain Range

{(-1,5),(1,3),(4,5)}

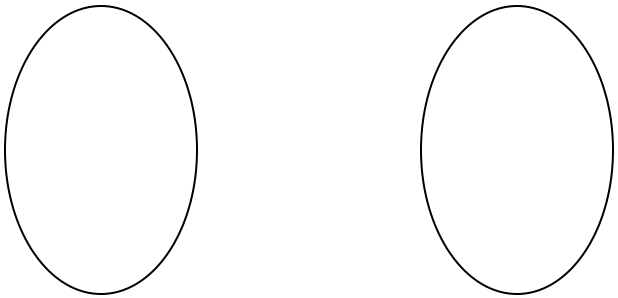
{(5,6),(-3,0),(1,1),(-3,6)}

Example 1 {(-3,1),(0,2),(2,4)}

Domain Range

{(-1,5),(1,3),(4,5)}

Domain Range



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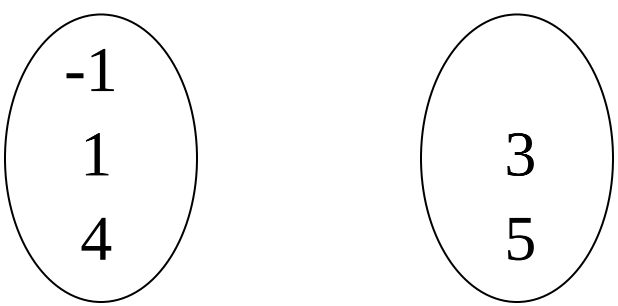

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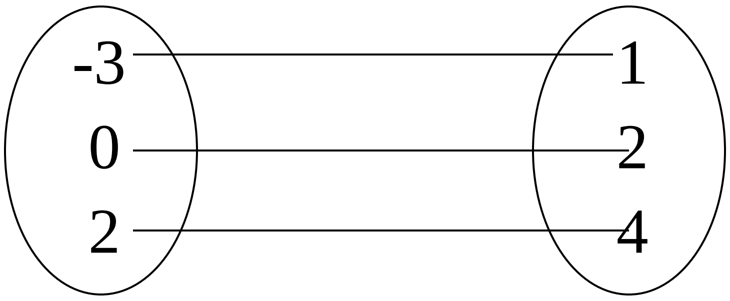
Domain Range



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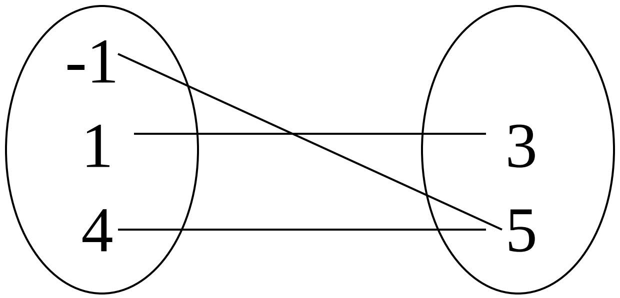
Domain Range



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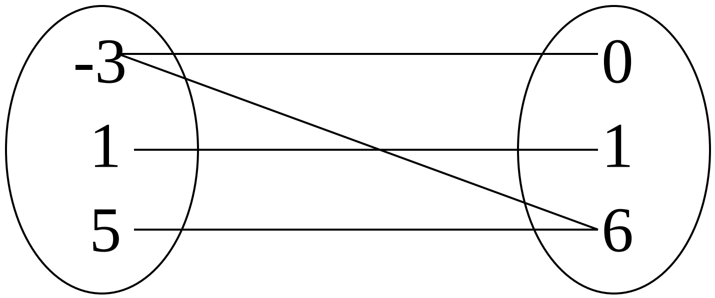

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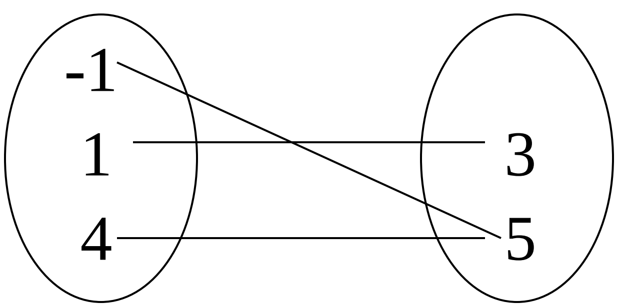
Domain Range

FUNCTION

1-1

{(-1,5),(1,3),(4,5)}

Domain Range



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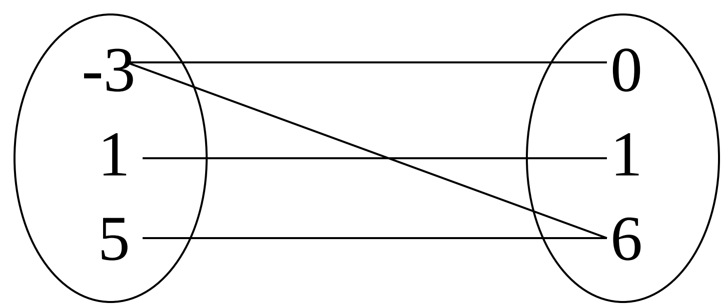

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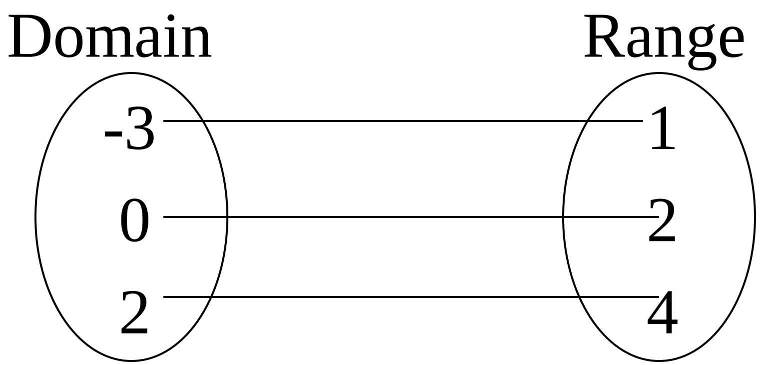

FUNCTION

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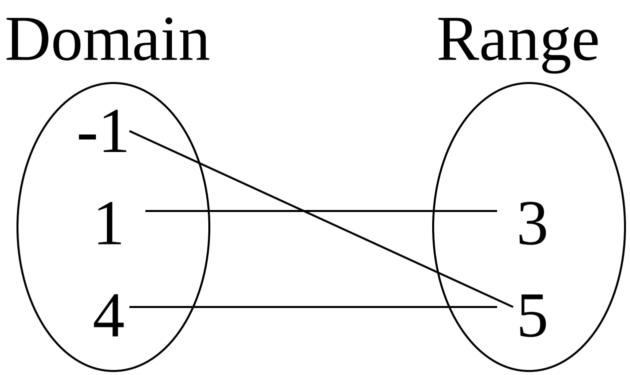

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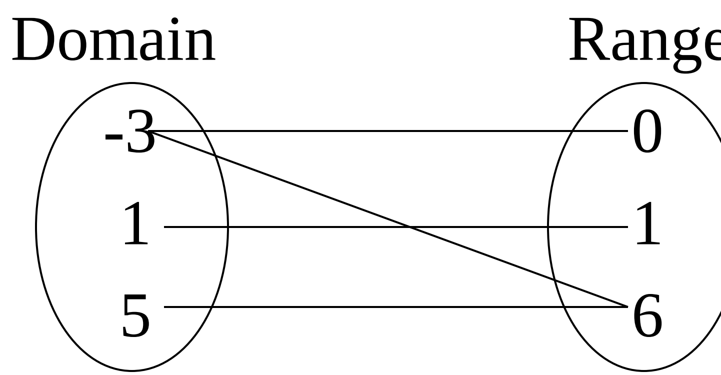

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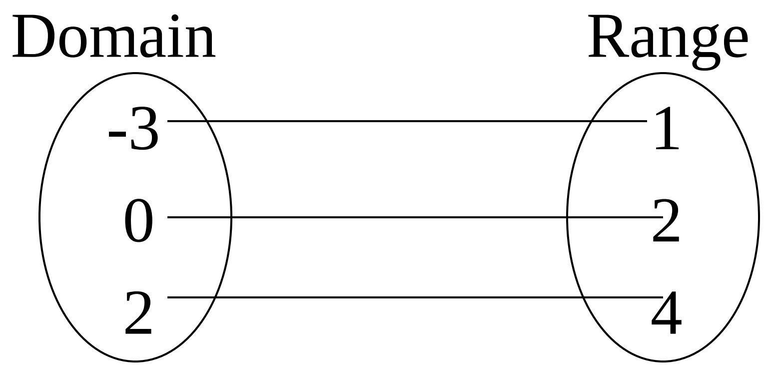

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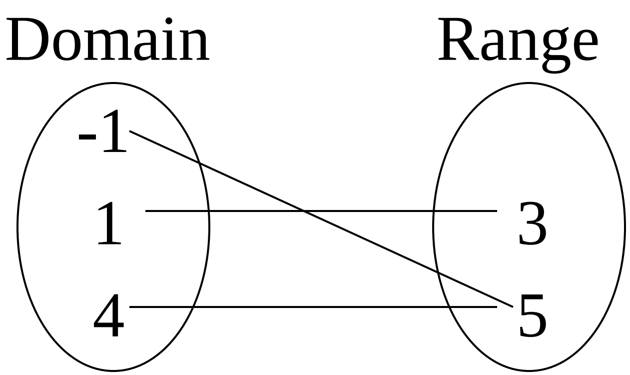

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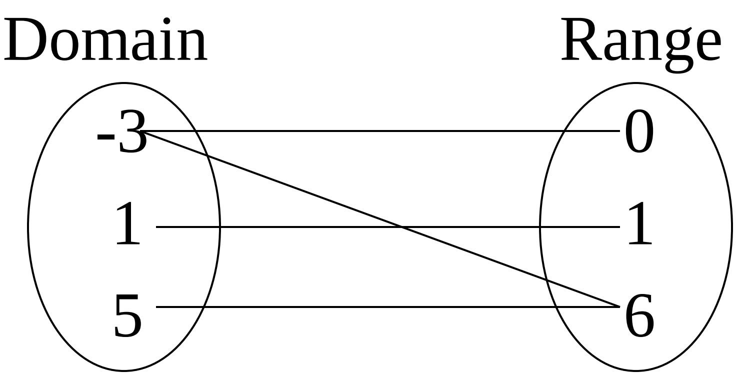

Not a Function

Example 3

Given $f(x) = 3x - 5$ and $g(x) = x^2 + 2$, find:

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(a) $f(-3)$

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$f(-3) =$

Example 3

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(a) $f(-3)$

$f(x) = 3x - 5$

$f(-3) = 3$

Example 3

Given $f(x) = 3x - 5$ and $g(x) = x^2 + 2$, find:

(a) $f(-3)$

$f(x) = \mathbf{3}x - 5$

$f(-3) = \mathbf{3}$

Example 3

Given $f(x) = 3x - 5$ and $g(x) = x^2 + 2$, find:

(a) $f(-3)$

$$f(x) = 3x - 5$$

$$f(-3) = 3(-3)$$

Example 3

Given $f(x) = 3x - 5$ and $g(x) = x^2 + 2$, find:

(a) $f(-3)$

$$f(\boldsymbol{x}) = 3x - 5$$

$$f(\mathbf{-3}) = 3(-3)$$

Example 3

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$$f(\mathbf{-3}) = 3\mathbf{(-3)}$$

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(a) $f(-3)$

$$f(x) = 3x - 5$$

$$f(-3) = 3(-3) - 5$$

$$= -9 - 5$$

Example 3

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(a) $f(-3)$

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$$f(-3) = 3(-3) - 5$$

$$= -9 - 5 = -14$$

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$g(2z)$

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$$g(2z) = (2z)$$

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(b) $g(2z)$

$$g(\boldsymbol{x}) = x^2 + 2$$

$$g(\mathbf{2}\boldsymbol{z}) = (2z)$$

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(b) $g(2z)$

$g(\boldsymbol{x}) = \boldsymbol{x}^2 + 2$

$g(\mathbf{2}\boldsymbol{z}) = (\mathbf{2}\boldsymbol{z})$

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(b) $g(2z)$

$$g(x) = x^2 \mathbf{+ 2}$$

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(b) $g(2z)$

$g(x) = x^2 + 2$

$g(2z) = (2z)^2 + 2$

$= (2)^2$

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$= -9 - 5 = -14$

(b) $g(2z)$

$g(x) = x^2 + 2$

$g(2z) = (2z)^2 + 2$

$= (2)^2(\text{z})^2 + 2$

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(a) $f(-3)$

$$f(x) = 3x - 5$$

$$f(-3) = 3(-3) - 5$$

$$= -9 - 5 = -14$$

(b) $g(2z)$

$$g(x) = x^2 + 2$$

$$g(2z) = (2z)^2 + 2$$

$$= (2)^2(z)^2 + 2$$

$$= 4z^2 + 2$$

Problems

- Let X={x,y,z,k} and Y={a,b,c}.Determine whether the relation S from X to Y is a function, If it is function, give its domain and range, co-domain of a function.
  – S={(x,a),(y,a),(x,b),(y,c)}
  – S={(x,c),(y,a),(z,b)}
  – S={(x,a),(y,b),(z,a),(k,b)}

TYPES OF FUNCTIONS

- One to One (Injective)
- Onto (Surjective)
- One to One Onto Function (Bijective)

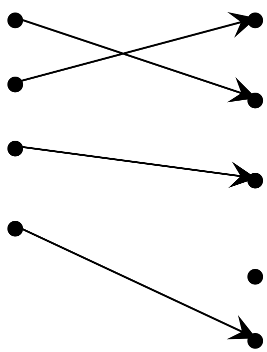
One-to-One Functions

- Function $f : X \rightarrow Y$ is called one-to-one (injective) when for all elements x_1 and x_2 from X if $f(x_1) = f(x_2)$, then $x_1 = x_2$
- Iff every element of its range has **only one** pre-image.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
- A={1,2,3,4} and B={a,b,c,d,e} and

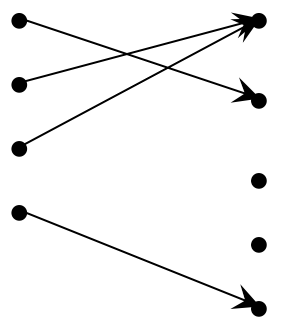
  f={(1,a),(2,e),(3,c),(4,d)}

One-to-One Illustration

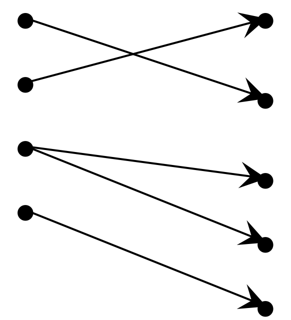
- Graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

- Let A = {1, 2, 3} and B = {a, b, c, d}. Which of the following is a one-to-one function?

Choices:

A. {1,a 2,c, 3,a}
B. {1,b 2,d 3,a}
C. {1,a, 2,a, 3,a}
D. {1,c 2,b 1,a, 3,d}

Correct Answer: B

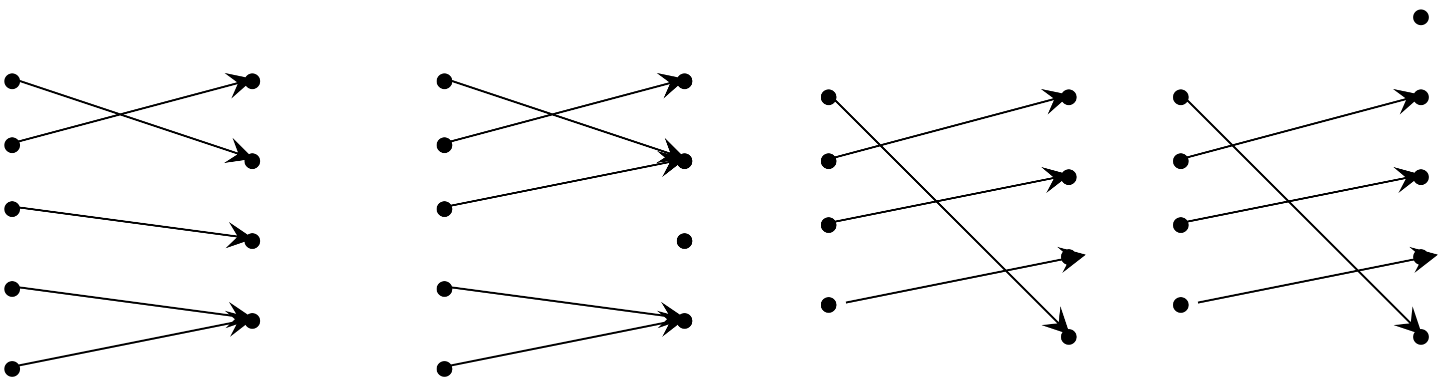
Onto Functions

- Function $f : X \rightarrow Y$ is called onto (surjective) when given any element y from Y, there exists x in X so that $f(x) = y$
- Determine whether the following functions is onto:
- $A=\{1,2,3,4\}$ and $B=\{a,b,c\}$ and

  $f=\{(1,a),(2,a),(4,c),(3,b)\}$

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Onto (but not 1-1)

Not Onto (or 1-1)

Both 1-1 and onto

1-1 but not onto

One to One Onto(Bijective) Functions

- Function from A to B is said to be one to one onto (bijective) if it is both one to one and onto function
- Determine whether the following functions is bijective:
- $A=\{1,2,3,4\}$ and $B=\{a,b,c,d\}$ and

  $f=\{(1,b),(2,c),(3,d),(4,a)\}$

PROBLEMS

1. Consider set X and Y and function $f: X \rightarrow Y$. Determine whether following functions are

   i. one to one function
   ii. Onto function
   iii. neither one-to-one nor onto
   iv. one-to-one onto function

a. $X=\{x,y,z\}$ $Y=\{1,2,3,4\}$, $f=\{(x,1),(y,1),(z,3)\}$
b. $X=\{a,b,c,d\}=Y$, $f=\{(a,a),(b,c),(c,d),(d,b)\}$
c. $X=\{a,b,c\}$, $Y=\{k,l\}$, $f=\{(a,k),(b,k),(c,l)\}$

PROBLEMS

- Let A=B={1,2,3,4}. Define functions

f:A$\longrightarrow$ B (if possible) such that

i. F is one to one
ii. F is neither one-to-one nor onto
iii. F is onto but not one-to-one
iv. F is one-to-one but not onto

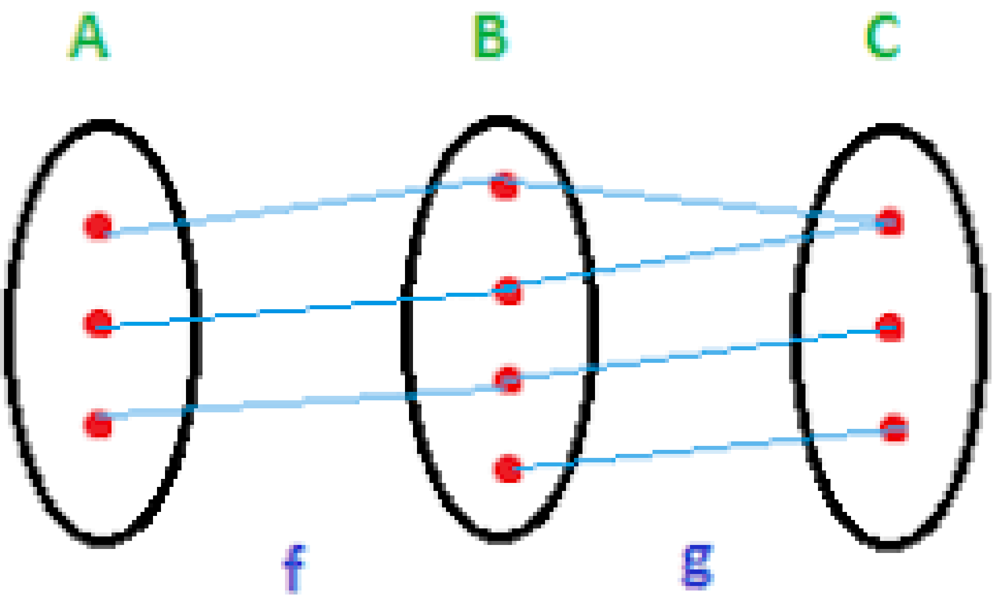
Everywhere defined function

- A function f is said to be everywhere defined if Dom(f) = A
- Ex A= {1,2,3}, B={a,b,c}

f ={ (1,c), (2,b) , (3,a)}

Composition of Functions

- Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, let range of f be a subset of the domain of g. Then we can define a composition of $g \circ f : X \rightarrow Z$
- $(g \circ f)(x)=g[f(x)]$
- Find the image of x under f and then find the image of f(x) under g.
- Composition of two one-to-one functions is one-to-one
- Composition of two onto functions is onto


A
B
C
f
g

PROBLEMS

- Let X={1,2,3} , Y={a,b}, Z={5,6,7}
- f={(1,a),(2,a),(3,b)} anf g={(a,5},(b,7)}
- Find composite of gof.
- gof(1)=f(1)=g(a)=5
- gof(2)=f(2)=g(a)=5
- gof(3)=f(3)=g(b)=7

Problem

1. $f(x)=x^2+3x+1$, $g(x)=2x-3$

- Find composition functions
  - fof
  - fog
  - gof

2. $f(n) = n+1$, $g(n)=2n$, $h(n)=n-1$ determine :
ho(fog) , fogoh, gof, f^2 .(fof)

Problems

3. $f(x)=x+9$ and $g(y)=y^2+3$. Find the following composition function:
   - (fof)(a)
   - gog(a)
   - gof(4)
   - fog(-4)

IDENTITY FUNCTION

- Let A be a non empty set, then we can define f: A → A(i.e B = A) as f(a)=a for all a belongs to A
- Eg A ={1,2,3} and f: A→A is an identity function since f(1)=1

  f(2)=2

  f(3)=3

Inverse Functions

- If $f : X \rightarrow Y$ is a bijective function, then it is possible to define an inverse function $f^{-1}: Y \rightarrow X$ so that $f^{-1}(y) = x$ whenever $f(x) = y$
- Find an inverse for the following functions:
  - $f(1)=2, f(2)=3, f(3)=1$
    - $f^{-1}(1) = \{3\}, \ldots$

A FUNCTION f FOR WHICH f^{-1} EXISTS IS CALLED INVERTIBLE

- Steps

Set y = f (x)

Interchange x and y

Solve for y which is $f^{-1}(x)$

1) $f(x) = 1 / (x - 2)$

2) $f(x) = x^3 + 2$

3) $f(x) = 5x - 7$

4) $f(x) = 8 / (9 - 3x)$

1. Sol: $y=1/(x-2)$

$X=1/(y-2)$

$Y-2= 1/x$

$Y=1/x +2$

Problems

- Find inverse of function
- $f(x) = (x + 1) / x$
- $f(x) = (4x + 3) / (5x - 2)$
- $f(x) = (7 + 4x) / (6 - 5x)$

Pigeonhole Principle

- **If n pigeons fly into m pigeonholes and n > m, then at least one hole must contain two or more pigeons**
- In a group of 8 people chosen anyway there must be at least two who will be born on the same day of the week?
  - 7 days in a week and 8 people
- Show that if any 5 numbers are chosen from 1 to 8, two of them will add up to 9

A1 ={1,8}, A2={2,7},A3={3,6},A4={4,5}

1) Golf: Let us suppose that there are 8 balls and 7 holes. If balls are to be put in different holes, then at least one hole must have more than one ball.

2) Handshake: If a number of people does handshake with one another, then according to pigeonhole principle, there must exist two people who shake hands with same people.

3) Birthday: Let us consider that n people are chosen at random from a group of people. Then, in order to find the probability of having same birthday, pigeonhole principle is applied. It says that at least two people will have same birthday.

4) Marble picking: Consider that we have a mixture of different color marbles in a jar. In order to find at least how many marbles will be picked before two same color marbles are guaranteed. It can be calculated using pigeonhole principle assuming one pigeonhole per color will be assumed.

Extended Pigeonhole Principle:

- It states that if n pigeons are assigned to m pigeonholes (The number of pigeons is very large than the number of pigeonholes), then one of the pigeonholes must contain at least **[(n-1)/m]+1** pigeons.

Example 1

Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have atleast have 2045 pages

Solution

- Let pages be pigeons and dictionaries the pigeonholes
- Using extended hole pigeonhole principle
  - **[(n-1)/m]+1**
  - **[(61327-1)/30]+1**
  - **2045**

Example 2

- Six friends discover that they have a total of 2161 Rs with them. Show that one or more of them must be have at least 361 Rs

- **[(n-1)/m]+1**
- **[(2160-1)/6]+1**
- **361**

Problems

- Show that if seven colors are used to paint 50 bicycles, at least eight bicycles will be of same color.
- Let T be an equilateral triangle whose sides are of length 1 unit. Show that if any five points are chosen lying on or inside the triangle, then two of them must be no more than ½ unit apart.