Algebraic Structures (11)

7.1 Algebraic structures with one binary operation: semigroup, monoids and groups

7.2 Cyclic groups, Normal subgroups

7.3 Hamming Code ,Minimum Distance

7.4 Group codes ,encoding-decoding techniques

7.5 Parity check Matrix ,Maximum Likelihood

7.6 Mathematics of Cryptography - Modular Arithmetic, Matrices, Linear Congruence, GF Fields, Primes and Related Congruence Equations- Primes, Primality Testing, Factorization, Quadratics Congruence, Chinese reminder theorem, Exponentiation and Logarithm.

Algebraic systems

- **$N = \{1,2,3,4,\ldots..\propto\}$ = Set of all natural numbers.**

  $Z = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots.. \propto\}$ = Set of all integers.

  Q = Set of all rational numbers, R = Set of all real numbers.

- **Binary Operation:** The binary operator * is said to be a binary operation (closed operation) on a non empty set A, if

  $a * b \in A$ for all $a, b \in A$ (Closure property).

  Ex: The set N is closed with respect to addition and multiplication

  but not w.r.t subtraction and division.

- **Algebraic System:** A set 'A' with one or more binary(closed) operations defined on it is called an algebraic system.

  Ex: $(N, +)$, $(Z, +, -)$, $(R, +, \cdot, -)$ are algebraic systems.

Properties

- **Commutative:** Let $*$ be a binary operation on a set A.

  The operation $*$ is said to be commutative in A if

  $a * b = b * a$ for all a, b in A

- **Associativity:** Let $*$ be a binary operation on a set A.

  The operation $*$ is said to be associative in A if

  $(a * b) * c = a *(b * c)$ for all a, b, c in A

(Addition , Subtraction)

- **Idempotent :** Let $*$ be a binary operation on a set A.

  The operation $*$ is said to be idempotent in A if

  $a * a = a$

- **Identity:** For an algebraic system (A, *), an element 'e' in A is said to be an identity element of A if

  $a * e = e * a = a$ for all $a \in A$.

- **Inverse:** Let (A, *) be an algebraic system with identity 'e'. Let a be an element in A. An element b is said to be inverse of A if

  $a * b = b * a = e$

Semi group

- **Semi Group:** An algebraic system (A, *) is said to be a semi group if

  1. * is closed operation on A.

  2. * is an associative operation, for all a, b, c in A.

- Ex. (N, +) is a semi group.
- Ex. (N, .) is a semi group.
- Ex. (N, –) is not a semi group.

- **Monoid:** An algebraic system (A, *) is said to be a **monoid** if the following conditions are satisfied.

  1) * is a closed operation in A.

  2) * is an associative operation in A.

  3) There is an identity in A.

Monoid

- Ex. Show that the set 'N' is a monoid with respect to multiplication.
- Solution: Here, N = {1,2,3,4,……}

1. Closure property : We know that product of two natural numbers is again a natural number.

i.e., $a.b = b.a$ for all $a,b \in N$

$\therefore$ Multiplication is a closed operation.

2. Associativity : Multiplication of natural numbers is associative.

i.e., $(a.b).c = a.(b.c)$ for all $a,b,c \in N$

3. Identity : We have, $1 \in N$ such that

$a.1 = 1.a = a$ for all $a \in N$.

$\therefore$ Identity element exists, and 1 is the identity element.

Hence, N is a monoid with respect to multiplication.

Subsemigroup & submonoid

Subsemigroup : Let (S, *) be a semigroup and let **T be a subset of S.**

If T is closed under operation * , then (T, *) is called a subsemigroup of (S, *).

Ex: (N, .) is semigroup and T is set of multiples of positive integer m then (T,.) is a sub semigroup.

Submonoid : Let (S, *) be a monoid with identity e, and let T be a non-empty subset of S. If T is closed under the operation * and e $\in$ T, then (T, *) is called a submonoid of (S, *).

Group

- **Group:** An algebraic system (G, *) is said to be a **group** if the following conditions are satisfied.

  1) * is a closed operation.

  2) * is an associative operation.

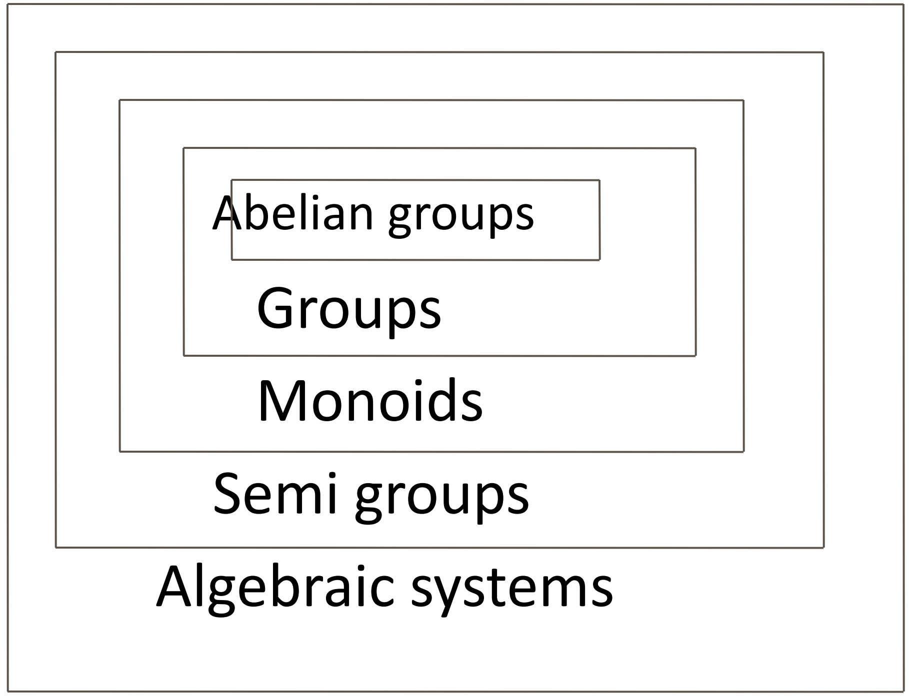
  3) There is an identity in G.

  4) Every element in G has inverse in G.

- **Abelian group (Commutative group):** A group (G, *) is said to be ***abelian*** (or ***commutative)*** if

  $a * b = b * a$ for all a, b belongs to G.

Algebraic systems



Theorems –Self Study

- **In a Group (G, *) the following properties hold good**

1. Identity element is unique.
2. Inverse of an element is unique.
3. Cancellation laws hold good

   $a * b = a * c \Rightarrow b = c$ (left cancellation law)

   $a * c = b * c \Rightarrow a = b$ (Right cancellation law)

4. $(a * b)^{-1} = b^{-1} * a^{-1}$

- In a group, the identity element is its own inverse.

- ***<u>Order of a group</u>*** : The number of elements in a group is called order of the group.

- <u>Finite group</u>: If the order of a group G is finite, then G is called a finite group.

Ex. Show that, the set of all integers is a group with respect to **addition**.

- Solution: Let Z = set of all integers.

  Let a, b, c are any three elements of Z.

1. **Closure property** : We know that, Sum of two integers is again an integer.

   i.e., $a + b \in Z$ for all $a,b \in Z$

2. **Associativity**: We know that addition of integers is associative.

   i.e., $(a+b)+c = a+(b+c)$ for all $a,b,c \in Z$.

3. **Identity** : We have $0 \in Z$ and $a + 0 = a$ for all $a \in Z$.

   $\therefore$ Identity element exists, and '0' is the identity element.

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Contd.,

4. **Inverse**: To each $a \in Z$, we have $-a \in Z$ such that

$$a + (-a) = 0$$

Each element in Z has an inverse

- 5. **Commutativity:** We know that addition of integers is commutative.

  i.e., $a + b = b + a$ for all $a, b \in Z$.

Hence, (Z , +) is an abelian group.

Ex. Show that set of all non zero real numbers is a group with respect to multiplication .

- Solution: Let R^* = set of all non zero real numbers.

  Let a, b, c are any three elements of R^* .

1. Closure property : We know that, product of two nonzero real numbers is again a nonzero real number .

   i.e., $a . b \in R^*$ for all $a,b \in R^*$.

2. Associativity: We know that multiplication of real numbers is associative.

   i.e., $(a.b).c = a.(b.c)$ for all $a,b,c \in R^*$.

3. Identity : We have $1 \in R^*$ and $a .1 = a$ for all $a \in R^*$.

   $\therefore$ Identity element exists, and '1' is the identity element.

4. Inverse: To each $a \in R^*$, we have $1/a \in R^*$ such that

   $a .(1/a) = 1$ i.e., Each element in R^* has an inverse.

Contd.,

- 5.Commutativity: We know that multiplication of real numbers is commutative.

  i.e., a . b = b . a for all a,b $\in R^*$.

  Hence, (R^* , .) is an abelian group.

- **Ex: Show that set of all real numbers 'R' is not a group with respect to multiplication.**
- Solution: We have $0 \in R$.

  The multiplicative inverse of 0 does not exist.

  Hence. R is not a group.

Modulo systems

Addition modulo m ($+_m$)

let m be a positive integer. For any two positive integers a and b

$a +_m b = a + b$ if $a + b < m$

$a +_m b = r$ if $a + b \geq m$ where r is the remainder obtained by dividing (a+b) with m.

Ex $14 +_6 8 = 22 \% 6 = 4$; Ex $9 +_{12} 3 = 12 \% 12 = 0$

Multiplication modulo p ($\times_p$)

let p be a positive integer. For any two positive integers a and b

$a \times_p b = a\,b$ if $a\,b < p$

$a \times_p b = r$ if $a\,b \geq p$ where r is the remainder obtained by dividing (ab) with p.

Ex. $3 \times_5 4 = 2$, $5 \times_5 4 = 0$, $2 \times_5 2 = 4$

Ex.The set G = {0,1,2,3,4,5} is a group with respect to addition modulo 6.

Solution: The composition table of G is

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

1. Closure property: Since all the entries of the composition table are the elements of the given set, the set G is closed under $+_6$.

Contd.,

2. <u>Associativity</u>: The binary operation $+_6$ is associative in G.

for ex. $(2 +_6 3) +_6 4 = 5 +_6 4 = 3$ and

$2 +_6 (3 +_6 4) = 2 +_6 1 = 3$

3. <u>Identity</u> : Here, The first row of the table coincides with the top row.

The element heading that row , i.e., 0 is the identity element.

4. . <u>Inverse</u>: From the composition table, we see that the inverse

elements of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively.

5. Commutativity: The corresponding rows and columns of the table are

identical. Therefore the binary operation $+_6$ is commutative.

Hence, (G, $+_6$) is an abelian group.

Ex.The set G = {1,2,3,4,5,6} is a group with respect to multiplication modulo 7.

Solution: The composition table of G is

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

1. <u>Closure property:</u> Since all the entries of the composition table are the elements of the given set, the set G is closed under $\times_7$.

Contd.,

2. Associativity: The binary operation $\times_7$ is associative in G.

for ex. $(2 \times_7 3) \times_7 4 = 6 \times_7 4 = 3$ and

$2 \times_7 (3 \times_7 4) = 2 \times_7 5 = 3$

3. Identity : Here, The first row of the table coincides with the top row.

The element heading that row , i.e., 1 is the identity element.

4. . Inverse: From the composition table, we see that the inverse elements

of 1, 2, 3, 4. 5 ,6 are 1, 4, 5, 2, 5, 6 respectively.

5. Commutativity: The corresponding rows and columns of the table are

identical. Therefore the binary operation $\times_7$ is commutative.

Hence, $(G, \times_7)$ is an abelian group.

Cyclic group

- A **cyclic group** is a group that can be generated by a single element.
- Every element of a cyclic group is a power of some specific element which is called a **generator**.
- A cyclic group can be generated by a generator ‘g’, such that every other element of the group can be written as a power of the generator ‘g’.

CYCLIC GROUPS:

A group (G , *) is said to cyclic group if there exists an

element , such that every element of G , can be $a \in G$

expressed as a^n , some integral power of a.

Examples:
(Z,.+) is generated by 1 or -1. Zn, the integers mod n
under modular addition, is generated by 1 or by any element k in Zn
which is relatively prime to n.

Normal Subgroup

A subgroup is called a ***normal subgroup*** *if for any $a \in G$, $aH = Ha$.*

Note 1:

aH = Ha does not necessarily mean that $a * h = h * a$ for every $h \in H$.

It only means that $a * h_i = hj * a$ for some h_i, $hj \in H$.

Note2:

Every subgroup of an abelian group is normal.

Proposition 2.3.1. *$Hg = gH$, for all $g \in G$, if and only if H is a normal subgroup* of *G.*

Let H={$[0]_6$, $[3]_6$},Find left and right cosets in group Z_6
Is it a normal subgroup

- It is abelian group ,$a +_6 b = b +_6 a$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Left coset of H, **a H** = { a * h | h ∈ H }

$$0\,H = \{\, 0 +_6 0 ,\ 0 +_6 3 \,\} = \{\, 0 , 3 \,\}$$

$$1\,H = \{\, 1 +_6 0 ,\ 1 +_6 3 \,\} = \{\, 1 , 4 \,\}$$

$$2\,H = \{\, 2 +_6 0 ,\ 2 +_6 3 \,\} = \{\, 2 , 5 \,\}$$

$$3\,H = \{\, 3 +_6 0 ,\ 3 +_6 3 \,\} = \{\, 3 , 0 \,\}$$

$$4\,H = \{\, 4 +_6 0 ,\ 4 +_6 3 \,\} = \{\, 4 , 1 \,\}$$

$$5\,H = \{\, 5 +_6 0 ,\ 5 +_6 3 \,\} = \{\, 5 , 2 \,\}$$

Right coset of H, **H a**={ h * a | h $\in$ H }

$$\mathbf{H\,0 = \{ 0 +_6 0 , 3 +_6 0 \} = \{ 0 , 3 \}}$$

H1,H2,H3,H4,H5

H is a normal sub group of Z_6

Hamming distance

The Hamming distance $d(x, y)$ between two words x, y is the weight $|x \oplus y|$ of $x \oplus y$, (bits in which they differ)

Eg. $d(00111, 11001) = 4$

Find the distance between x and y

$x = 110110$; $y = 000101$

$x = 001100$; $y = 010110$

$x = 0100100$; $y = 0011010$

Theorems

- The minimum weight of all non zero words in a group code is equal to its minimum distance
- A code can **detect** all combinations of k or fewer iff the minimum distance between any two code words is at least **k + 1**
- A code can **correct** all combinations of k or fewer errors iff the minimum distance between any two code words is at least **2 k + 1**

- Consider the (2,4) encoding function , how many errors will ‘e’ detect ?

e (00)=0000 e (10)=0111

e (01)=1011 e (11)=1100

Soln: since 2>=k+1

k<=1,Will detect 1 or fewer errors

- Consider the encoding function $B^2 \rightarrow B^6$ defined as follows

e (00)=0010000 e (10)=100010

e (01)=010100 e (11)=110001

How many errors can it correct and detect?

Error detection 3>=k+1; k <=2 or fewer errors

Error correction 3>=2k+1;k<=1 or fewer errors

Group Codes

An (m,n) encoding function $e:B^m \rightarrow B^n$ is called

a group code if $e(B^m) = \{e(b) | b \in B^m\}$=Ran

(e) is a subgroup of B^n

Subgroup if:

Identity element of B^n is in N

If x and y belong to N , then $x \oplus y \in N$

If x is in N, then its inverse in N

Consider the encoding function $B^2 \rightarrow B^5$ defined as follows

e (00)=00000 e (10)=10101

e (01)=01110 e (11)=11011

is a group code

Soln: Let N={ 00000 , 10101, 01110, 11011 } be set of code words

$\oplus$	00000	01110	10101	11011
00000	00000	01110	10101	11011
01110	01110	00000	11011	10101
10101	10101	11011	00000	01110
11011	11011	10101	01110	00000

$a \oplus b \in N$ which is closed operation,associative,identity,inverse

1. Closed operation : For any a,b $\in$ N, a $\oplus$ b $\in$ N , So N is closed under $\oplus$ operation
2. Identity element of B^5 i.e 00000 also belongs to N

00000 $\oplus$ 00000 = 00000 $\oplus$ 00000

01110 $\oplus$ 00000= 00000 $\oplus$ 01110

10101 $\oplus$ 00000 = 00000 $\oplus$ 10101

11011 $\oplus$ 00000= 00000 $\oplus$ 11011

3. $\oplus$ Associative operation

01110 $\oplus$ (00000 $\oplus$ 10101) = (01110 $\oplus$ 00000) $\oplus$ 10101

01110 $\oplus$ 10101 = 01110 $\oplus$ 10101

11011 = 11011

4 . Inverse

Ex: 01110 $\oplus$ 01110 = 01110 $\oplus$ 01110 = 00000

Show that (3,5)encoding function $e: B^3 \rightarrow B^6$ defined as follows

e (000)=000000

e (001)=000110

e (100)=100101

e (110)=110111

e (010)=010010

e (011)=010100

e (101)=100011

e (111)=110001

PARITY CHECK MATRIX

Consider the parity check matrix given by H;

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine the group code $e_H : B^2 \rightarrow B^5$

Soln: $B^2 = \{00,01,10,11\}$

Then $e(00) = 00\, x_1\, x_2\, x_3 = B^5$

$x_1 = 0 .1 + 0. 0 = 0$

$x_2 = 0 .1 + 0. 1 = 0$

$X_3 = 0 .0 + 0. 1 = 0$

e (00) = 00000

Next $e(01) = 01\, x_1\, x_2\, x_3 = B^5$

$x_1 = 0 .1 + 1. 0 = 0$

$x_2 = 0 .1 + 1. 1 = 1$

$X_3 = 0 .0 + 1. 1 = 1$

e (01) = 01011

Next $e(10) = 10\ x_1\ x_2\ x_3\ = B^5$

$x_1\ = 1 .1\ + 0. 0\ = 1$

$x_2\ = 1 .1\ + 0. 1\ = 1$

$X_3\ = 1 .0\ + 0. 1 = 0$

e (10) = 10110

Next $e(11) = 11\ x_1\ x_2\ x_3\ = B^5$

$x_1\ = 1 .1\ + 1. 0\ = 1$

$x_2\ = 1 .1\ + 1. 1\ = 0$

$X_3\ = 1 .0\ + 1. 1 = 1$

e (11) = 11101

$e_H : B^2 \rightarrow B^5$ is as above for e (00) , e (01), e (10) ,e (11)

Problem 1

Consider the parity check matrix given by H;

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine the group code $e_H : B^2 \rightarrow B^5$

$e(00) = 00000$

$e(01) = 01011$

$e(00) = 10011$

$e(00) = 11000$

Problem 2

Consider the parity check matrix given by H;

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine the group code $e_H : B^3 \rightarrow B^6$

$e(000) = 000000$

$e(001) = 001111$

$e(010) = 010011$

$e(011) = 011100$

$e(100) = 100100$

$e(101) = 101011$

$e(110) = 110111$

$e(111) = 111000$

MAXIMUM LIKELIHOOD DECODING TECHNIQUE

Consider the encoding function $B^2 \rightarrow B^4$ defined as follows

e (00)=**0000**

e (01)=**0110**

e (10)=**1011**

e (11)=**1101**

Decode the foll words relative to MLD function,

(i) **0101** (ii) **1010** (iii) **1101**

Step 1: Construct Decoding Table

	0000	**0110**	**1011**	**1101**
0000	0000	0110	1011	**1101**
0001	0001	0111	**1010**	1100
0010	0010	0100	1001	1111
1000	1000	1110	0011	**0101**

Consider the encoding function $B^2 \rightarrow B^5$ defined as follows

e (00)=**00000** e (10)=**10101**

e (01)=**01110** e (11)=**11011**

Decode the foll words relative to MLD function,

(i) 11110 (ii)10011(iii)10100

	e (00)	e (01)	e (10)	e (11)
	00000	**01110**	**10101**	**11011**
00000	00000	01110	10101	11011
00001	00001	01111	**10100**	11010
00010	00010	01100	10111	11001
00100	00100	01010	10001	11111
01000	01000	00110	11101	**10011**
10000	10000	**11110**	00101	01011