Prerequisite:

Product of Two Vectors

The product of two vectors results in two different ways, the one is a number and the other is vector. So, there are two types of product of two vectors, namely scalar product and vector product. They are written as $\vec{a} \cdot \vec{b}$ and $\vec{a} \times \vec{b}$.

Scalar or Dot Product

The scalar, or dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined to be $|\vec{a}|\,|\vec{b}| \cos\theta$ *i.e.*,

scalar where θ is the angle between $\vec{a}$ and $\vec{b}$.

Symbolically, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

USEFUL RESULTS

$\hat{i} \cdot \hat{i} = (1)(1)\cos 0^\circ = 1$ Similarly, $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = (1)(1)\cos 90^\circ = 0$ Similarly, $\hat{j} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{i} = 0$

Note. If the dot product of two vectors is zero then vectors are prependicular to each other.

Vector or Cross Product

Let $\hat{\eta}$ be a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \,.\, \hat{\eta}$$

2. Useful results

Since $\hat{i}, \hat{j}, \hat{k}$ are three mutually perpendicular unit vectors, then

$$\begin{aligned}
\hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\
\hat{i} \times \hat{j} &= -\hat{j} \times \hat{i} = \hat{k} \\
\hat{j} \times \hat{k} &= -\hat{k} \times \hat{j} = \hat{i} \\
\hat{k} \times \hat{i} &= -\hat{i} \times \hat{k} = \hat{j}
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
\hat{j} \times \hat{i} &= -\hat{i} \times \hat{j} \\
\hat{k} \times \hat{j} &= -\hat{j} \times \hat{k} \\
\hat{i} \times \hat{k} &= -\hat{k} \times \hat{i}
\end{aligned}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar Triple Product

If $\bar{a}, \bar{b}, \bar{c}$ are three vectors then the **scalar** product $\bar{a} \cdot (\bar{b} \times \bar{c})$ is called **scalar triple product** of the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$. It is denoted by $[\bar{a}, \bar{b}, \bar{c}]$.

Thus, $$\boxed{[\bar{a}, \bar{b}, \bar{c}] = \bar{a} \cdot (\bar{b} \times \bar{c})} \quad \text{.......... (A)}$$

If $\bar{a} = a_1 i + a_2 j + a_3 k,\ \ \bar{b} = b_1 i + b_2 j + b_3 k,\ \ \bar{c} = c_1 i + c_2 j + c_3 k$

then $$\bar{a} \cdot (\bar{b} \times \bar{c}) = (a_1 i + a_2 j + a_3 k) \cdot \left\{(b_2 c_3 - b_3 c_2) i + (b_3 c_1 - b_1 c_3) j + (b_1 c_2 - b_2 c_1) k\right\}$$

$$\therefore \quad \boxed{\bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \quad \text{.......... (B)}$$

Deductions : (These are based on the laws of determinants.)

(i) $[\bar{a}, \bar{b}, \bar{c}] = [\bar{c}, \bar{a}, \bar{b}] = [\bar{b}, \bar{c}, \bar{a}]$ changing the order of vectors cyclically does not change the value of the product.

(ii) $[\bar{a}, \bar{b}, \bar{c}] = -[\bar{a}, \bar{c}, \bar{b}] = -[\bar{b}, \bar{a}, \bar{c}] = -[\bar{c}, \bar{b}, \bar{a}]$ interchanging the positions of two vectors changes the sign of the product.

(iii) $[\bar{a}, \bar{a}, \bar{b}] = [\bar{a}, \bar{b}, \bar{a}] = [\bar{b}, \bar{a}, \bar{a}] = 0$ if two vectors are same the value of the product is zero.

(iv) $[k\bar{a}, \bar{b}, \bar{c}] = [\bar{a}, k\bar{b}, \bar{c}] = [\bar{a}, \bar{b}, k\bar{c}]$ multiplying any vector by k multiplies the product by k.

(v) $\bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$. Dot and cross can be interchanged.

Vector triple product:

- If $\overline{a}$, $\overline{b}$, $\overline{c}$ are 3 vectors then vector triple product of these 3 vectors is given by
- $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\, \overline{b} - (\overline{a} \cdot \overline{b})\, \overline{c}$
- Or $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c})\, \overline{b} - (\overline{b} \cdot \overline{c})\, \overline{a}$

Scalar product of 4 vectors:

- If $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$ are 4 vectors then scalar product of these 4 vectors is given by
- $(\overline{a} \times \overline{b}) \cdot (\overline{c} \times \overline{d}) = \begin{vmatrix} \overline{a} \cdot \overline{c} & \overline{b} \cdot \overline{c} \\ \overline{a} \cdot \overline{d} & \overline{b} \cdot \overline{d} \end{vmatrix}$
- This result is known as Lagrange's identity.

Vector product of 4 vectors:

- If $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$ are 4 vectors then vector product of these 4 vectors is given by
- $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = [\overline{a}\ \ \overline{c}\ \ \overline{d}]\, \overline{b} - [\overline{b}\ \ \overline{c}\ \ \overline{d}]\, \overline{a}$
- Or $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = [\overline{a}\ \ \overline{b}\ \ \overline{d}]\, \overline{c} - [\overline{a}\ \ \overline{b}\ \ \overline{c}]\, \overline{d}$

Vector differentiation

Point Functions

(a) Scalar Valued Point Functions

Consider any region R of space and suppose that to each point P of R there corresponds by some law a scalar quantity denoted by $\Phi(P)$. Then Φ is called a scalar point function defined for the region R.

Illustrations : Consider a material body occupying some region R. If $\Phi(P)$ denotes the density of the material at P, temperature at P or charge at P then Φ is a scalar point function over R.

(b) Vector Valued Point Functions

Consider any region R of space and suppose that to each point P of R there corresponds by some law a vector quantity $\bar{f}(P)$. Then $\bar{f}$ is called a vector point function defined for the region R.

Illustrations : Consider a fluid in motion. At any time t if $\bar{f}(P)$ denotes velocity at a point P which varies from point to point or acceleration at a point P which varies from point to point then $\bar{f}$ is a vector point function.

Vector operator Del (∇)

- Define an operator del (or nabla) as follows
- $\nabla = i\ \frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$
- Gradient
- If ϕ is a scalar point function then a vector point function $\nabla\phi$ is given by
- $\nabla\phi = = i\ \frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$
- Standard result:
- (1) $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$
- (2)) $\nabla(\phi\ \psi) = \phi\nabla\psi + \psi\nabla\phi$
- (3) $\nabla\ f(u) = \ i\ \frac{\partial f(u)}{\partial x} + j\frac{\partial f(u)}{\partial y} + k\frac{\partial f(u)}{\partial z} = \ f'(u)\ \nabla u$

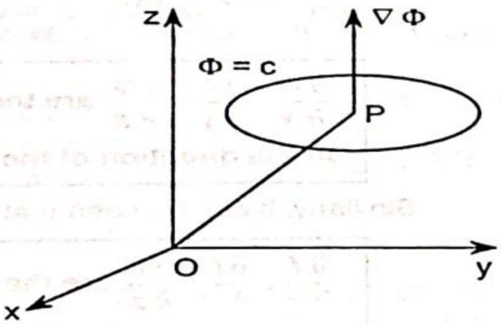
Geometrical Meaning of Grad Φ

Consider a scalar point function and let

$$\bar{r} = xi + yi + zk$$

be the position vector of a point P on the surface $\Phi(x, y, z) = c$.

Such a surface for which the value of the function is constant is called a **level surface.**



Now, $\overline{dr} = dx\, i + dy\, j + dz\, k$ and it lies in the plane tangential to the surface $\Phi(x, y, z) = c$.

Also $d\Phi = \dfrac{\partial \Phi}{\partial x} dx + \dfrac{\partial \Phi}{\partial y} dy + \dfrac{\partial \Phi}{\partial z} dz.$

Since $\Phi(x, y, z) = c$, $d\Phi = 0$

$$\therefore \quad \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz = 0$$

Hence, $\nabla\Phi \cdot \overline{dr} = \left(i \dfrac{\partial \Phi}{\partial x} + j \dfrac{\partial \Phi}{\partial y} + k \dfrac{\partial \Phi}{\partial z} \right) \cdot (dx\, i + dy\, j + dz\, k)$

$$= \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz = 0$$

$\nabla\Phi$ is a vector perpendicular to $\overline{dr}$. But since $\overline{dr}$ lies in the tangent plane, $\nabla\Phi$ *is a* **vector perpendicular to the tangent plane to the surface.** $\Phi(x, y, z) = c$.

Problems on Gradient :

Ex 1.If $\phi = x^2 + y^2 + z^2, \psi = x^2y^2 + y^2z^2 + z^2x^2$ Find $\nabla[\nabla\phi \cdot \nabla\psi]$

Solution:

$$\nabla\Phi = 2x\, i + 2y\, j + 2z\, k$$

$$\nabla\Psi = (2xy^2 + 2xz^2)\, i + (2yx^2 + 2yz^2)\, j + (2zx^2 + 2zy^2)\, k$$

$$\therefore \quad \nabla\Phi \cdot \nabla\Psi = 4x^2(y^2 + z^2) + 4y^2(x^2 + z^2) + 4z^2(x^2 + y^2)$$

$$= 8(x^2y^2 + y^2z^2 + z^2x^2)$$

$$\therefore \quad \nabla(\nabla\Phi \cdot \nabla\Psi) = 16x(y^2 + z^2)\, i + 16y(z^2 + x^2)\, j + 16z(x^2 + y^2)\, k$$

Ex 2. If $\phi = (x^2+y^2+z^2)e^{-\sqrt{x^2+y^2+z^2}}$ Find $\nabla\phi$

Solution:

$$\frac{\partial \Phi}{\partial x} = (x^2+y^2+z^2)\cdot e^{-\sqrt{x^2+y^2+z^2}}$$

$$\times \frac{-1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x + e^{-\sqrt{x^2+y^2+z^2}} \cdot (2x)$$

$$= -r \cdot e^{-r}\, x + e^{-r} \cdot 2x$$

$$= e^{-r}(2x - x\,r)$$

Similarly, $$\frac{\partial \Phi}{\partial y} = e^{-r}(2y - y\,r)$$

$$\frac{\partial \Phi}{\partial z} = e^{-r}(2z - z\,r)$$

Hence, $$\nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}$$

$$= e^{-r}(2-r)[\,xi + y\,j + zk\,]$$

$$= e^{-r}(2-r)\,\bar{r}.$$

EX 3: If $u = x + y + z, v = x + y, w = -2xz - 2yz - z^2$

Show that $\nabla u \cdot [\nabla v \times \nabla w] = 0$

Solution:

$$\nabla u = i + j + k,\ \ \nabla v = i + j$$

$$\nabla w = -2z\,i - 2z\,j - (2x + 2y + 2z)\,k$$

$$\nabla u \cdot [\nabla v \times \nabla w] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -2z & -2z & -2x - 2y - 2z \end{vmatrix} = 0$$

(Since first two columns are identical.)

EX4: Prove that $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$. Hence find f if $\nabla f = 2r^4\bar{r}$

- Solution:

We have, $\nabla \Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}$

Here, $\Phi = f(r)$ and f is a function of r and r is a function of (x, y, z).

$$\therefore \quad \nabla f(r) = i\frac{df}{dr}\frac{\partial r}{\partial x} + j\frac{df}{dr}\frac{\partial r}{\partial y} + k\frac{df}{dr}\frac{\partial r}{\partial z}$$

But $r^2 = x^2 + y^2 + z^2 \qquad \therefore \quad 2r\frac{\partial r}{\partial x} = 2x$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla f(r) = \frac{f'(r)}{r}[xi + yj + zk] = \frac{f'(r)}{r}\bar{r}$$

Comparing this with the given expression. *i.e.*, comparing

$$\nabla f(r) = f'(r)\frac{\bar{r}}{r} \quad \text{with} \quad \nabla f(r) = 2r^4\bar{r} = 2r^5\frac{\bar{r}}{r}$$

we see that $f'(r) = 2r^5$.

Here, by integration

$$\therefore \quad f(r) = \frac{2r^6}{6} + C = \frac{r^6}{3} + C$$