FOURIER SERIES

Find the Fourier series for the following functions.

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(0, 2\pi)$

1. $f(x)=x^2 \quad in \;\; (0, 2\pi)$ Hence deduce that $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\cdots\cdots$

[Ans: $f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx-4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$]

2. $f(x)=e^{-x}, 0<x<2\pi$ & $f(x+2\pi)=f(x)$ Hence deduce the value of $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$

[Ans: $f(x)=\frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$]

3. $f(x)=x\sin x$ in the interval $0\le x\le 2\pi$. Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2-1}=\frac{3}{4}$

[Ans: $f(x)=-1-\frac{1}{2}\cos x+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx+\pi\sin x$]

4. $f(x)=\sqrt{1-\cos x} \;\; in \;\; (0, 2\pi)$ Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}$

5. $f(x)=x, \quad 0<x\le\pi$
$\quad =2\pi-x, \quad \pi\le x<2\pi$ Hence deduce that $\frac{\pi^2}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\cdots\cdots$

[Ans: $f(x)=\frac{\pi}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos nx$]

6. $f(x)=x \quad in \;\; (0, 2\pi)$ [Ans: $f(x)=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$]

7. $f(x)=\frac{3x^2-6x\pi+2\pi^2}{12} \quad in \;\; (0, 2\pi)$ Hence deduce that $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\cdots\cdots$

[Ans: $f(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$]

8. $f(x)=\left(\frac{\pi-x}{2}\right)$ in the interval $0\le x\le 2\pi$ Also deduce that $\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots\cdots$

9. $f(x)=1, \quad 0<x\le\pi$
$\quad =2-\frac{x}{\pi}, \quad \pi\le x<2\pi$

[Ans: $f(x)=\frac{3}{4}-\frac{2}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right]+\frac{1}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\cdots\right]$]

10. $f(x)=2x \quad in \;\; (0, 2\pi)$ Also find a_4 & b_{10}. [Ans: $f(x)=2\pi-4\sum_{n=1}^{\infty}\frac{\sin nx}{n}, 0, -0.4$]

11. $f(x)=\cos px, \quad in \quad (0, 2\pi)$ where p is not an integer.

12. $f(x) = kx, 0 \le x \le 2\pi$ .Also find a_4&b_{10}.

13. $f(x) = e^{2x}$in $(0,2\pi)$

14. $f(x) = e^{-2x}$in $(0,2\pi)$

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(-\pi, \pi)$

15.state the value of $f(x)\, at\ x=0$ if $f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

[Ans: $f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{[1-2(-1)^n]}{n}\sin nx$]

16. $\begin{aligned} f(x) &= 1/2, \quad -\pi < x < 0 \\ &= x/\pi, \quad 0 < x < \pi \end{aligned}$ Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

[Ans: $f(x) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin 2nx}{n}$]

17. $\begin{aligned} f(x) &= -x-\pi, \quad -\pi \le x \le 0 \\ &= x+\pi, \quad 0 \le x \le \pi \end{aligned}$ [Ans: $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + 4\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)}$]

18. $\begin{aligned} f(x) &= 0, \quad -\pi \le x \le 0 \\ &= x, \quad 0 \le x \le \pi \end{aligned}$,

Hence, deduce that i) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \cdots\cdots$ ii) $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\cdots$

19.Obtain Fourier Series for $f(x) = e^{-|x|}, -\pi \le x \le \pi$

20. . $\begin{aligned} f(x) &= 0, \quad -\pi \le x \le 0 \\ &= \sin x, \quad 0 \le x \le \pi \end{aligned}$, Hence, deduce that i) $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\cdots$

ii) $\frac{1}{4}(\pi - 2) = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\cdots$[Ans: $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\left[\frac{\cos 2x}{4\cdot 1^2 - 1} + \frac{\cos 4x}{4\cdot 2^2 - 1} + \cdots\right]$]

21. It is given that for $-\pi < x < \pi$, $x^2 = \frac{\pi^2}{3} + 4\sum(-1)^n \frac{\cos nx}{n^2}$

Using Parsvel's identity prove that $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$.

22. $\begin{aligned} f(x) &= 1 + \frac{2x}{\pi}, \quad -\pi \le x \le 0 \\ &= 1 - \frac{2x}{\pi}, \quad 0 \le x \le \pi \end{aligned}$, Deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \cdots\cdots$

[Ans: $f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}[1-(-1)^n]\cos nx$]

23. $$f(x) = x + \frac{\pi}{2}, \quad -\pi < x < 0$$
$$= \frac{\pi}{2} - x, \quad 0 < x < \pi$$
, Hence, Deduce that
$$i)\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots$$
$$ii)\ \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \cdots\cdots$$

[Ans: $f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi\, n^2} [1 - (-1)^n] \cos nx$]

24. Prove that $\sinh ax = \frac{2}{\pi} \sinh a\pi \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{n^2 + a^2} \sin nx \right]$

25. $f(x) = x \cos x, \quad in \quad (-\pi, \pi)$ [Ans: $f(x) = -\frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} (-1)^n \cdot \frac{n}{n^2 - 1} \sin nx$]

26. $f(x) = x + x^2, \quad in \quad (-\pi, \pi)$. Hence deduce that i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\cdots$ ii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots$

27. $f(x) = \cos px, \quad in \quad (-\pi, \pi)$. Where p is not an integer. Hence, prove that

$\cot p\pi = \frac{2p}{\pi} \left[\frac{1}{2p^2} - \frac{1}{p^2 - 1^2} + \frac{1}{p^2 - 2^2} - \frac{1}{p^2 - 3^2} + \cdots \right]$ And deduce that $\cos\theta = \frac{1}{\theta} - \sum_{n=1}^{\infty} \frac{2\theta}{n^2\pi^2 - \theta^2}$

Also deduce that $\frac{1}{2} - \frac{\pi\sqrt{3}}{18} = \frac{1}{9 \cdot 1^2 - 1} + \frac{1}{9 \cdot 2^2 - 1} + \frac{1}{9 \cdot 3^2 - 1} + \cdots$

28. $f(x) = |\sin x|, \quad in \quad (-\pi, \pi)$ [Ans: $f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots \right]$]

29. $f(x) = |x|, \quad in \quad (-\pi, \pi)$ Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots$

[Ans: $f(x) = \frac{\pi}{2} - \frac{\pi}{2} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right]$]

30. $f(x) = x \sin x, \quad in \quad (-\pi, \pi)$. Hence deduce that $\frac{1}{4}(\pi - 2) = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots\cdots$

[Ans: $f(x) = 1 - \frac{1}{2} \cos x - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - 1} \cos nx$]

31. $f(x) = \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}}, \quad in \quad (-\pi, \pi)$ [Ans: $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{n^2 + a^2} \cdot \sin nx$]

32. $f(x) = \frac{x(\pi^2 - x^2)}{12}, \quad in \quad (-\pi, \pi)$ [Ans: $f(x) = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots$]

33. $$f(x) = 0, \quad -\pi \le x \le 0$$
$$= x^2, \quad 0 \le x \le \pi$$

34 .If $x^2 = \frac{\pi}{3} + 4 \sum (-1)^n \cdot \frac{\cos nx}{n^2} \quad for \quad -\pi < x < \pi, \ prove\ that \quad \sum \frac{1}{n^4} = \frac{\pi^4}{90}$

35. $f(x) = \sin x, \quad in \quad (-\pi, \pi)$

36. $f(x)=\frac{\pi^2}{12}-\frac{x^2}{4}, \quad in \quad (-\pi,\pi)$ [Ans: $f(x)=\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\cdots$]

37. $$\begin{aligned} f(x) &= x, & -\pi<x<0 \\ &= 0, & 0<x<\pi/2 \\ &= x-\pi/2, & \pi/2<x<\pi \end{aligned}$$

38. $f(x)=x^2, \quad in \quad (-\pi,\pi)$

39. $\boldsymbol{f(x)}=\begin{cases} 0, & -\pi<x<0 \\ x^2, & \mathbf{0}<\mathrm{x}>\pi \end{cases}$

40. $f(x)=x\cos x \;\; in\ (-\pi,\pi)$

41. $f(x)=\cosh \mathrm{p}\, x \;\; in\ (-\pi,\pi)$, p is not an integer

42. $f(x)=\frac{x(\pi-x)(\pi+x)}{12} \;\; in\ (-\pi,\pi)$

43.$f(x)=x|\mathrm{x}|, -\pi\le x\le\pi$

44.$f(x)=e^{-|\mathrm{x}|}, -\pi\le x\le\pi$

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(0, 2l)$

45. $f(x)=x^2, in \;\; (0, a)$ Hence deduce that $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots\cdots$

[Ans: $f(x)=\frac{a^2}{3}+\sum_{n=1}^{\infty}\frac{a^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{a}\right)-\sum_{n=1}^{\infty}\frac{a^2}{n\,\pi}\sin\left(\frac{n\pi x}{a}\right)$]

46. $f(x)=2x-x^2, 0\le x\le 3$ [Ans: $f(x)=\sum_{n=1}^{\infty}\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right)+\sum_{n=1}^{\infty}\frac{3}{n\,\pi}\sin\left(\frac{2n\pi x}{3}\right)$]

47. $$\begin{aligned} f(x) &= \pi x, & 0<x<1 \\ &= 0, & 1<x<2 \end{aligned},$$ [Ans: $f(x)=\frac{\pi}{4}-\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos n\pi x-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$]

48. $$\begin{aligned} f(x) &= \pi x, & 0\le x\le 1 \\ &= \pi(2-x), & 1\le x\le 2 \end{aligned}$$, with period 2, show that $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$

49. $$\begin{aligned} f(x) &= \pi x, & 0<x<1 \\ &= 0, & x=1 \\ &= \pi(2-x), & 1<x<2 \end{aligned}$$, Hence show that $\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots\cdots$ [Ans: $f(x)=\frac{\pi}{4}+2\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x$]

50. $f(x) = 3kx/l, \quad 0 < x < (l/3)$
$= 3k(l-2x)/l, \ (l/3) < x < (2l/3),$
$= \pi(2-x), \quad (2l/3) < x < l$

[Ans: $\dfrac{9k}{\pi^2}\sum \dfrac{1}{n^2}\sin\dfrac{2n\pi}{3}\cdot\sin\dfrac{2n\pi x}{l}$]

51.If $x^2 = \dfrac{4l^2}{3} + \dfrac{4l^2}{\pi^2}\sum \dfrac{1}{n^2}\cos\left(\dfrac{n\pi x}{l}\right) - \dfrac{4l^2}{\pi}\sum \dfrac{1}{n}\sin\left(\dfrac{n\pi x}{l}\right)$ *in* $0 < x < 2l,$ find the sum of the series $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} \cdots\cdots$ [Ans: $\dfrac{\pi^2}{6}$]

52. $f(x) = k\,x$ in the interval $0 \le x \le 2$. Hence deduce that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$

53. Find Fourier series to represent $f(x) = 2x - x^2$ in (0,3) and prove that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$

54. $f(x) = 2 - \frac{x^2}{2}$ in $0 \le x \le 2$

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(-l, l)$

55. $f(x) = 0, \quad -c < x < 0$
$= a, \quad 0 < x < c$,

[Ans: $f(x) = \dfrac{a}{2} + \dfrac{2a}{\pi}\left[\dfrac{1}{1}\sin\dfrac{\pi x}{c} + \dfrac{1}{3}\sin\dfrac{3\pi x}{c} + \cdots\right]$]

56. $f(x) = -x, \quad -1 < x < 0$
$= x, \quad 0 < x < 1$,

[Ans: $f(x) = 1 - \dfrac{4}{\pi^2}\sum \dfrac{1}{(2n-1)^2}\cos n\pi x$]

57.
$f(x) = x, \quad -1 < x < 0$
$= x+2, \quad 0 < x < 1$,

[Ans: $f(x) = 1 + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[1 - 2\,(-1)^n\right]\sin n\pi x$

58. $f(x) = |\ x\ |, \quad -2 < x < 2$, Hence deduce that $\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^4} = \dfrac{\pi^4}{96}$

[Ans: $f(x) = 1 - \dfrac{8}{\pi^2}\sum \dfrac{1}{(2n-1)^2}\cos\left[\dfrac{(2n-1)\pi x}{2}\right]$]

59. $f(x) = 1 - x^2, \quad -1 < x < 1,$ [Ans: $f(x) = \dfrac{2}{3} - \dfrac{4}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos n\pi x$

60. $f(x) = \sin ax, \quad -l < x < l$, [Ans: $f(x) = 2\pi\sin al\sum \dfrac{(-n)(-1)^n}{n^2\pi^2 - a^2l^2}\sin\dfrac{n\pi x}{l}$]

61. $f(x) = x - x^2, \quad -1 < x < 1,$ [Ans: $f(x) = -\dfrac{1}{3} - \dfrac{4}{\pi^2}\sum \dfrac{(-1)^n}{n^2}\cos n\pi x - \dfrac{2}{\pi}\sum \dfrac{(-1)^n}{n}\sin n\pi x$]

62. $f(x) = a^2 - x^2, \quad -a < x < a$, [Ans: $f(x) = \dfrac{2a^2}{3} + \dfrac{4a^2}{\pi^2}\left[\dfrac{1}{1^2}\cos\dfrac{\pi x}{a} - \dfrac{1}{2^2}\cos\dfrac{2\pi x}{a} + \dfrac{1}{3^2}\cos\dfrac{3\pi x}{a} - \cdots\cdots\right]$]

63. $f(x)=x^2, \quad -1<x<1,$ [Ans: $f(x)=\frac{1}{3}-\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x-\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x-\cdots\right]$]

64. $f(x)=9-x^2, \quad -3<x<3,$ [Ans: $f(x)=6+\frac{36}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3}-\frac{1}{2^2}\cos\frac{2\pi x}{3}+\frac{1}{3^2}\cos\frac{3\pi x}{3}-\cdots\cdots\right]$]

65. $f(x)=x-x^3 \ in \ (-1,1)$ [Ans: $f(x)=-\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{n^3}$]

66. $f(x)=x-x^2 \ in \ (-1,1)$ [Ans: $f(x)=-\frac{1}{3}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi x}{n^2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{n}$]

67. $f(x)=x^2-2, \ -2\le x\le 2$ [Ans: $f(x)=-\frac{2}{3}-\frac{16}{\pi^2}\left[\cos\frac{\pi x}{2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}\cdots\right]$]

68 $f(x)=\begin{cases}0, & -2<x<-1\\ 1+x, & -1<x<0\\ 1-x, & 0<x<1\\ 0, & 1<x<2\end{cases}$ [Ans: $f(x)=\frac{1}{4}+\sum\frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi x}{2}\right)$]

69. $f(x)=e^{-x}, \quad (-a,a)$

[Ans: $f(x)=\frac{\sinh a}{a}+2a\sinh a\sum\frac{(-1)^n}{a^2+n^2\pi^2}\cos\frac{n\pi x}{a}+2\pi\sinh a\sum\frac{(-1)^{n+1}\cdot n}{a^2+n^2\pi^2}\sin\frac{n\pi x}{a}$]

70. $f(x)=|\ x\ |, \quad -1<x<1$

71. $f(x)=\begin{cases}0, & -2<x<-1\\ k, & -1<x<1\\ 0, & 1<x<2\end{cases}$

72. $f(x)=\begin{cases}0, & -5<x<0\\ 7, & 0<x<5\end{cases}$ period of the function is 10.

73. $f(x)=\begin{cases}0, & -2<x<0\\ x+5, & 0<x<2\end{cases}$

74. $f(x)=1-x^2 \ in \ (-1,1)$ hence find $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots$

75. $f(x)=\begin{cases}-\sin\frac{\pi x}{k}, & -k<x<0\\ \sin\frac{\pi x}{k}, & 0<x<k\end{cases}$

76. $f(x)=\begin{cases}2(x-4), & -4<x<0\\ 2(x+4), & 0<x<4\end{cases}$

77. $f(x)=x^2-2 \ on \ (-2,2)$

HALF RANGE SERIES

78.Obtain half range sine series for $\begin{aligned} f(x) &= x, \quad 0 < x < \pi/2 \\ &= \pi - x, \quad \pi/2 < x < \pi \end{aligned}$ Hence find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$. [Ans: $f(x) = \sum \frac{4}{\pi} \frac{\sin(n\pi/2)}{n^2} \cdot \sin nx$]

79.Find half range cosine series for $f(x) = x, \quad (0, 2)$. Using Parsvel's identity, deduce that

i) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \cdots\cdots$ ii) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$.

80.Obtain the expression of $f(x) = x(\pi - x), 0 < x < \pi$ as a half-range cosine series. Hence, show that i) $\frac{\pi^2}{6} = \sum_1^{\infty} \frac{1}{n^2}$

ii) $\frac{\pi^2}{12} = \sum_1^{\infty} \frac{(-1)^{n+1}}{n^2}$ iii) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$.

[Ans: $f(x) = \frac{\pi^2}{6} - \left[\frac{1}{1^2}\cos 2x + \frac{1}{2^2}\cos 4x + \frac{1}{3^2}\cos 6x + \cdots\right]$

81.Show that if $0 < x < \pi$, $\cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin 2mx$

82.Expand $f(x) = lx - x^2, 0 < x < l$ in a half range i) cosine series, ii) sine series.

Hence from sine series deduce that $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$

[Ans: i) $f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \cdots\right]$

ii) $f(x) = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots\right]$]

83.Find half range cosine series for $f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$

[Ans: $f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \cdots\right]$]

84.Prove that in the interval $0 < x < \pi$, $\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} - \cdots\right]$

85.Obtain half-range sine series for $f(x) = x(2-x)$ *in* $0 < x < 2$ and hence find $\sum \frac{1}{n^6} = \frac{\pi^6}{945}$

86.Obtain half range sine series for $f(x) = \begin{cases} (1/4) - x, & 0 < x < (1/2) \\ x - (3/4), & (1/2) < x < 1 \end{cases}$

[Ans: $f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$]

87.Obtain half-range cosine series for $f(x) = x$ *in* $0 < x < l$. Hence deduce that $\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots = \frac{\pi^4}{1440}$

88.Obtain half-range cosine series for $f(x)=\begin{cases} kx, & 0<x<(l/2) \\ l-x, & (l/2)<x<l \end{cases}$

Hence, deduce that i) $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\cdots$ ii) $\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\cdots\cdots$

[Ans: $f(x)=\frac{kl}{4}-\frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{6^2}\cos\frac{6\pi x}{l}+\frac{1}{10^2}\cos\frac{10\pi x}{l}+\cdots\right]$]

89.Find half range sine series of period $2l$ for $f(x)=\begin{cases} \frac{2x}{l}, & 0<x<(l/2) \\ \frac{2}{l}(l-x), & (l/2)<x<l \end{cases}$

[Ans: $f(x)=\frac{8}{\pi^2}\sum\frac{1}{n^2}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{l}$]

90.Obtain sine series for $f(x)=\begin{cases} mx, & 0<x\le(\pi/2) \\ m(\pi-x), & (\pi/2)\le x<\pi \end{cases}$ [Ans: $f(x)=\frac{4m}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\cdots\right]$]

91.Obtain half range cosine series for $f(x)=\sin\left(\frac{\pi x}{l}\right)$ $in\ \ 0<x<l$.

[Ans: $f(x)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{1\cdot 3}\cos\frac{2\pi x}{l}+\frac{1}{3\cdot 5}\cos\frac{4\pi x}{l}+\cdots\right]$]

92.Obtain half-range cosine series for $f(x)=(x-1)^2$ $in\ \ 0<x<1$.Hence, find $\sum_{n=1}^{\infty}\frac{1}{n^2}$ & $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

[Ans: $f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$]

93.Find HRSSfor $f(x)=\begin{cases} \frac{2x}{3}, & 0\le x\le\frac{\pi}{3} \\ \frac{\pi-x}{3}, & \frac{\pi}{3}\le x\le\pi \end{cases}$

[Ans: $f(x)=\frac{\sqrt{3}}{\pi}\left[\frac{1}{1^2}\sin x+\frac{1}{2^2}\sin 2x-\frac{1}{4^2}\sin 4x-\frac{1}{5^2}\sin 5x+\cdots\right]$]

94.Obtain the half range sine series for $f(x)=x(\pi-x), 0<x<\pi$ Hence, find $\sum_{n=1}^{\infty}\frac{(-1)^3}{(2n-1)^3}$

95.Show that in the interval $0<x<\pi$, $\sin x=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\cdots\right]$

96.Obtain half-range sine series for $f(x)=x^2$ $in\ \ 0<x<3$.

97.Obtain HRCS for $f(x)=x(2-x)$ $in\ \ 0<x<2$ [Ans: $f(x)=\frac{2}{3}-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{[1+(-1)^n]}{n^2}\cos\left(\frac{n\pi x}{2}\right)$]

98.Find half range cosine series for f(x)=$\begin{cases} kx & 0<x<l/2 \\ 0 & l/2<x<l \end{cases}$ Hence deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$

99. Find half range cosine series for f(x) = x on (0,2) hence deduce that $\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$