Complex Form of Fourier Series (C.F.F.S)

Let $f(x)$ be defined in the interval $(C, C + 2l)$. The complex form of Fourier Series for $f(x)$ in this interval is given by

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}}$$

where, $$\boxed{C_n = \frac{1}{2l}\int_C^{C+2l} f(x)\, e^{-in\pi x/l}\, dx} \quad n = 0, \pm 1, \pm 2, \ldots..$$

Proof

$$f(x) = a_0 + \sum_{1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{1}^{\infty} a_n \left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + \sum_{1}^{\infty} b_n \left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)$$

$$= a_0 + \sum_{1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{in\pi x/l} + \sum_{1}^{\infty} \left(\frac{a_n + ib_n}{2}\right) e^{-in\pi x/l}$$

$$= C_0 + \sum_{1}^{\infty} C_n e^{in\pi x/l} + \sum_{1}^{\infty} C_{-n} e^{-in\pi x/l}$$

where, $C_n = \dfrac{a_n - ib_n}{2}$, $C_{-n} = \dfrac{a_n + ib_n}{2}$

$$\therefore \quad f(x) = \sum^{\infty} C_n e^{in\pi x/l}$$

RESULTS

Cor. 1 : If the interval is C to $C+2\pi$ replacing l by π in the above result

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where,
$$C_n = \frac{1}{2\pi}\int_C^{C+2\pi} f(x)\, e^{-inx}\,dx$$

Cor. 2 : If the interval is $(0, 2l)$, putting $C = 0$ in the above result

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$$

where,
$$C_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-in\pi x/l}\,dx$$

Cor. 3 : If the interval is $(0, 2\pi)$, putting $l = \pi$ in the above corollary 2,

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where,
$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx$$

RESULTS

Cor. 4 : If the interval is $(-l, l)$, putting $C = -l$ in the above result

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \\ C_n &= \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\, dx \end{aligned}}$$

where,

Cor. 5 : If the interval is $(-\pi, \pi)$, putting $l = \pi$ in corollary 4,

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{inx} \\ C_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \end{aligned}}$$

where,

EX.1 Obtain C.F.F.S. for $f(x) = e^{ax}$ in $(-\pi, \pi)$

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where, $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} \cdot e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi} \cdot e^{-in\pi} - e^{-a\pi} \cdot e^{in\pi}\right]$$

But $\quad e^{\pm in\pi} = \cos(\pm n\pi) + i\sin(\pm n\pi)$

$$= (-1)^n + i(0) = (-1)^n$$

$$\therefore \quad C_n = \frac{1}{2\pi(a-in)}\left[(-1)^n e^{a\pi} - (-1)^n e^{-a\pi}\right]$$

$$= \frac{(-1)^n}{\pi(a-in)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right) = \frac{(-1)^n}{\pi(a-in)} \sin h\, a\pi$$

$$= \frac{(-1)^n \sin h\, a\pi}{\pi(a-in)} \cdot \frac{(a+in)}{(a+in)} = \frac{(-1)^n \sin h\, a\pi\,(a+in)}{\pi(a^2+n^2)}$$

Hence, $e^{ax} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin h\, a\pi \cdot (a+in)}{\pi(a^2+n^2)} e^{inx}$ .... (i)

EX.2 Obtain C.F.F.S. for $\cos(\alpha x)$ in $(-\pi, \pi)$ using EX.1

Hence, $$e^{ax} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh a\pi \cdot (a+in)}{\pi(a^2+n^2)} e^{inx} \qquad \text{..........(i)}$$

For deductions, replace a by $i\alpha$ in (i)

$$\therefore \quad e^{i\alpha x} = \sum \frac{(-1)^n \sinh i\alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n (i) \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha - n) \cdot e^{inx}$$

Now, replace a by $-i\alpha$ in (i)

$$\therefore \quad e^{-i\alpha x} = \sum \frac{(-1)^n \sinh(-i\alpha\pi)}{\pi(-\alpha^2+n^2)} \cdot (-i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n (-i) \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha + n) \cdot e^{inx}$$

$$\therefore \quad \cos\alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2} = \sum \frac{(-1)^n \sin\alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha) \cdot e^{inx}$$

$$= \frac{\sin\alpha\pi}{\pi} \sum \frac{(-1)^n \cdot \alpha}{(\alpha^2-n^2)} \cdot e^{inx}$$

EX.3 Obtain C.F.F.S. for $f(x) = e^{ax}$ in $(0, a)$

$$f(x) = \sum_{-\infty}^{\infty} C_n \cdot e^{2in\pi x/a}$$

where $$C_n = \frac{1}{a}\int_0^a e^{ax} \cdot e^{-2in\pi x/a}\,dx$$

$$\therefore \quad C_n = \frac{1}{a}\int_0^a e^{(a-2in\pi/a)x}dx$$

$$= \frac{1}{a}\left[\frac{e^{(a-2in\pi/a)x}}{(a-2in\pi/a)}\right]_0^a$$

$$= \frac{1}{a} \cdot \frac{a}{(a^2-2in\pi)} \cdot \left[e^{(a^2-2in\pi)} - 1\right]$$

$$= \frac{1}{(a^2-2in\pi)}\left[e^{a^2} \cdot e^{-2in\pi} - 1\right]$$

$$= \frac{1}{(a^2-2in\pi)}(e^{a^2} - 1) \quad \left[\because e^{-2in\pi} = \cos 2n\pi - i\sin 2n\pi = 1\right]$$

$$\therefore \quad e^{ax} = (e^{a^2} - 1)\sum_{-\infty}^{\infty} \frac{e^{2in\pi x/a}}{(a^2-2in\pi)}.$$

EX.4 Obtain C.F.F.S. for $f(x)=\begin{cases}0, 0<x<l\\ a, l<x<2l\end{cases}$ $f(x+2)=f(x)$

$$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $C_n=\frac{1}{2l}\int_0^{2l} f(x)\, e^{-in\pi x/l}\, dx$

$$\therefore \quad C_n=\frac{1}{2l}\left[\int_0^l 0\cdot dx+\int_l^{2l} a\, e^{-in\pi x/l}\, dx\right]$$

$$=\frac{a}{2l}\int_l^{2l} e^{-in\pi x/l}\, dx=\frac{a}{2l}\left[\frac{e^{-in\pi x/l}}{-in\pi/l}\right]_l^{2l}$$

$$=\frac{-a}{2in\pi}\left[e^{-2in\pi}-e^{-in\pi}\right] \quad \text{except when } n=0.$$

Case 1 : Where $n=0$ from (1)

$$C_0=\frac{1}{2l}\left[\int_0^l 0\, dx+\int_l^{2l} a\, e^0\, dx\right]$$

$$\therefore \quad C_0=\frac{1}{2l}\int_l^{2l} a\, dx=\frac{a}{2l}[2l-l]$$

$$=\frac{al}{2l}=\frac{a}{2}$$

Case 2 : When $n=\pm 1, \pm 3, \ldots\ldots$ from (2)

$$C_1=\frac{-a}{2i\pi}\left[e^{-2i\pi}-e^{-i\pi}\right]$$

$$=\frac{-a}{2i\pi}[\cos(-2\pi)+i\sin(-2\pi)-\cos(-\pi)-i\sin(-\pi)]$$

$$=\frac{-a}{2i\pi}[1+i(0)-(-1)+i(0)]=\frac{-a}{2i\pi}\cdot 2=\frac{ai}{\pi}$$

$$C_{-1} = \frac{a}{2i\pi}\left[\cos 2\pi + i\sin 2\pi - \cos\pi - i\sin\pi\right]$$

$$= \frac{a}{2i\pi}\left[1 + i(0) - (1) + i\,0\right] = \frac{a}{2i\pi}\cdot 2 = -\frac{ai}{\pi}$$

Similarly, $C_3 = \dfrac{ia}{3\pi}$, $C_{-3} = -\dfrac{ai}{3\pi}$

$C_5 = \dfrac{ia}{5\pi}$, $C_{-5} = -\dfrac{ai}{5\pi}$

Case 3 : When $n = \pm 2, \pm 4, \ldots..$

$$C_2 = \frac{-a}{4i\pi}\left[e^{-4i\pi} - e^{-2i\pi}\right]$$

$$= \frac{-a}{4i\pi}\left[\cos(-4\pi) + i\sin(-4\pi) - \cos(-2\pi) + i\sin(-2\pi)\right]$$

$$= \frac{-a}{4i\pi}\left[1 + i(0) - (1) - i(0)\right] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots..$

$\therefore \quad f(x) = \dfrac{a}{2} + \dfrac{ai}{\pi}\left[(e^{u} - e^{-u}) + \dfrac{1}{3}(e^{3u} - e^{-3u}) + \ldots.\right]$ where $u = \dfrac{i\pi x}{l}$

EX.5Obtain C.F.F.S. for $f(x) = \begin{cases} 1, 0 < x < 1 \\ 0, 1 < x < 2 \end{cases}$ $f(x+2) = f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $C_n = \dfrac{1}{2l}\int_0^{2l} f(x)\, e^{-in\pi x/l}\, dx$

Since, in this case $l = 1$,

$$\therefore \quad C_n = \frac{1}{2}\int_0^2 f(x)\, e^{-in\pi x} dx$$

$$= \frac{1}{2}\left[\int_0^1 1 \cdot e^{-in\pi x} dx + \int_1^2 0 \cdot e^{-in\pi x} dx\right]$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi x}}{-in\pi}\right]_0^1 = \frac{1}{-2in\pi}\left[e^{-in\pi} - 1\right]$$

$$= \frac{1}{2in\pi}\left[1 - e^{-in\pi}\right] \text{ except when } n = 0.$$

Case 1 : When $n = 0$, from (1)

$$C_0 = \frac{1}{2}\left[\int_0^1 1 \cdot dx + \int_1^2 0 \cdot dx\right] = \frac{1}{2}[\,x\,]_0^1 = \frac{1}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \pm 5, \ldots\ldots$ from (2)

$$C_1 = \frac{1}{2i\pi}\left[1 - e^{-i\pi}\right] = \frac{1}{2i\pi}[1 - \{\cos\pi - i\sin\pi\}]$$

$$= \frac{1}{2i\pi}[1 - (-1)] = \frac{2}{2i\pi} = \frac{1}{i\pi}$$

$$C_{-1} = \frac{1}{-2i\pi}\left[1 - e^{i\pi}\right] = -\frac{1}{2i\pi}[1 - \{\cos\pi + i\sin\pi\}]$$

$$= -\frac{1}{2i\pi}[1 - (-1)] = -\frac{1}{i\pi}$$

Similarly, $C_3 = \dfrac{1}{3i\pi}$, $\quad C_{-3} = -\dfrac{1}{3i\pi}$, $\quad C_5 = \dfrac{1}{5i\pi}$, $\quad C_{-5} = -\dfrac{1}{5i\pi}$, ...

Case 3 : When $n = \pm 2, \pm 4, \ldots\ldots$ from (2)

$$C_2 = \frac{1}{4i\pi}\left[1 - e^{-2i\pi}\right] = \frac{1}{4i\pi}[1 - \{\cos 2\pi + i\sin 2\pi\}] = \frac{1}{4i\pi}[1 - 1] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots\ldots = 0$

Hence,
$$f(x) = \frac{1}{2} + \frac{1}{i\pi}(e^{i\pi x} - e^{-i\pi x}) + \frac{1}{3i\pi}(e^{3i\pi x} - e^{-3i\pi x}) + \ldots\ldots$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2i}\right) + \frac{1}{3}\left(\frac{e^{3i\pi x} - e^{-3i\pi x}}{2i}\right) + \ldots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \ldots\right]$$