Problems on Scalar potential

EX 1: A vector field is given by $\overline{F} = (x^2 + xy^2)i + (y^2 + x^2y)j$
Show that $\overline{F}$ is irrotational and find its Scalar potential.

Solution: : We have

$$\text{curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2 + x^2y)\right] i - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 + xy^2)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(y^2 + x^2y) - \frac{\partial}{\partial y}(x^2 + xy^2)\right] k$$

$$= 0\,i + 0\,j + (2xy - 2xy)\,k = 0\,i + 0\,j + 0\,k$$

Hence, $\overline{F}$ is irrotational.

If Φ is the scalar potential then $\overline{F} = \nabla\Phi$.

$$\therefore \quad (x^2 + xy^2)\,i + (y^2 + x^2y)\,j + 0\,k = \frac{\partial\Phi}{\partial x} i + \frac{\partial\Phi}{\partial y} j + \frac{\partial\Phi}{\partial y} k$$

$$\therefore \quad \frac{\partial\Phi}{\partial x} = x^2 + xy^2$$

$$\frac{\partial\Phi}{\partial y} = y^2 + x^2y$$

$$\frac{\partial\Phi}{\partial z} = 0$$

$$\text{But } d\Phi = \frac{\partial\Phi}{\partial x} dx + \frac{\partial\Phi}{\partial y} dy + \frac{\partial\Phi}{\partial z} dz$$

$$= \left[x^2 + xy^2\right] dx + \left[y^2 + x^2y\right] dy + 0\,dz$$

$$= x^2\,dx + y^2\,dy + (xy^2\,dx + x^2y\,dy)$$

$$\text{By integration } \Phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{1}{2}x^2y^2$$

By integration $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} + \frac{x^2y^2}{2} + c$

EX2: A vector field $\overline{F}$= $(y\sin z-\sin x)i+(x\sin z+2yz)j+(xy\cos z+y^2)k$ prove that it is irrotational and hence find its scalar potential.

Solution: We have

$$\text{curl } \overline{F}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\sin z-\sin x & x\sin z+2yz & xy\cos z+y^2 \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(xy\cos z+y^2)-\frac{\partial}{\partial z}(x\sin z+2yz)\right]i$$

$$+\left[\frac{\partial}{\partial z}(y\sin z-\sin x)-\frac{\partial}{\partial x}(xy\cos z+y^2)\right]j$$

$$+\left[\frac{\partial}{\partial x}(x\sin z+2yz)-\frac{\partial}{\partial y}(y\sin z-\sin x)\right]k$$

$$=[x\cos z+2y-x\cos z-2y]\,i$$

$$+[y\cos z-y\cos z]\,j+[\sin z-\sin z]\,k$$

$$=0i+0j+0k=\overline{0}$$

Hence, $\overline{F}$ is irrotational.

If Φ is the scalar potential then $\overline{F}=\nabla\Phi$.

$$\therefore\quad (y\sin z-\sin x)\,i+(x\sin z+2yz)\,j+(xy\cos z+y^2)\,k$$

$$=\frac{\partial\Phi}{\partial x}i+\frac{\partial\Phi}{\partial y}j+\frac{\partial\Phi}{\partial z}k$$

$$\therefore\quad \frac{\partial\Phi}{\partial x}=y\sin z-\sin x$$

$$\frac{\partial\Phi}{\partial y}=x\sin z+2yz$$

$$\frac{\partial\Phi}{\partial z}=xy\cos z+y^2$$

But $d\Phi=\dfrac{\partial\Phi}{\partial x}dx+\dfrac{\partial\Phi}{\partial y}dy+\dfrac{\partial\Phi}{\partial z}dz$

$$=(y\sin z-\sin x)\,dx+(x\sin z+2yz)\,dy+(xy\cos z+y^2)\,dz$$

$$=[y\sin z\,dx+x\sin z\,dy+xy\cos z\,dz]+(-\sin x)\,dx+(2yz\,dy+y^2dz)$$

By integration $\phi=xy\sin z+\cos x+y^2z+\mathrm{c}$