Solution to PP-1

Q1 $P(A\cap B) = P(A)+P(B)-P(A\cup B) \geq P(A)+P(B)-1$ (As $P(A\cup B)\leq 1$ Kolmogorov Axiom i))

$\therefore P(A\cup B) \geq \frac{3}{4}+\frac{1}{3}-1 = \frac{1}{12}$

Now $A\cap B \subseteq A$, $A\cap B \subseteq B$

$\Rightarrow P(A\cap B)\leq P(A)$ & $P(A\cap B)\leq P(B)$

$\Rightarrow$ ~~am~~ $P(A\cap B)\leq \min\{P(A), P(B)\} = \frac{1}{3}$

$\Rightarrow P(A\cap B) \leq \frac{1}{3}$

Q2 Let A be an event that no head ever. Let A_n be the ~~vent~~ event that no head in n tosses

$A_n \subseteq A_{n+1}$ here $\bigcup_{n=1}^{\infty} A_n = A$. Then from Theorem 2.1 in Lecture 2 we have

$$\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right), \text{ ie}$$

$$P(A) = \lim_{n\to\infty} P(A_n)$$

$$\therefore P(A) = \lim_{n\to\infty}\frac{1}{2^n} = 0$$

Thus A^c is the event that at least one head occurs. i.e.

$P(A^c) = 1-P(A)$, i.e $P(A^c)=1$ &

hence head eventually appears atleast once.

Q3. The event $A\cup B$ means either A occurs or B occurs or both occurs. So if exactly one of the events occur, then such an event

$A\cap B$

$$A\Delta B = (A\cup B)\setminus(A\cap B)$$

$$P(A\Delta B) = P[(A\cup B)\setminus \cancel{B}\ \cancel{A\cup B}\ A\cap B] = P(A\cup B) - P(A\cap B)$$

~~$= P[(A\setminus B)\cup(B\setminus A)]$~~ using this

Let A_1 and A_2 are two events $A_1\cap A_2=\phi$, then

$P(A_1\cup A_2) = P(A_1)+P(A_2)$.

Hint. $P(A_2\setminus A_1) = P(A_2)-P(A_1)$ if $A_1 \subset A_2$.

$A_2 = A_1 + (A_2\setminus A_1)$

$P(A_2) = P(A_1) + P(A_2\setminus A_1)$

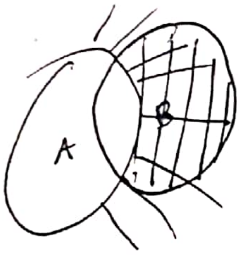
$\Rightarrow P(A_2\setminus A_1) = P(A_2) - P(A_1)$

Q6

$$\therefore \quad P(A \triangle B) = P(A \cup B) - P(A \cap B)$$
$$= P(A) + P(B) - P(A \cap B) - P(A \cap B)$$
$$= P(A) + P(B) - 2P(A \cap B).$$

Q 4: Hint: Just use the definition of conditional probability.

Q 5: Note that $A^c \cap B = B \setminus (A \cap B)$



$$P(A^c \cap B) = P(B \setminus (A \cap B))$$

Since $A \cap B \subseteq B \Rightarrow P(B \setminus (A \cap B)) = P(B) - P(A \cap B)$

$$\therefore \quad P(A^c \cap B) = P(B) - P(A \cap B)$$
$$= P(B) - P(A)P(B) \quad \text{[As A and B are independent]}$$

$$\therefore \quad P(A^c \cap B) = P(B)(1 - P(A))$$
$$= P(B)P(A^c)$$

Hence A^c & B are independent.

$A^c \cap B^c = (A \cup B)^c$ by De Morgan's Law

$$\therefore \quad P(A^c \cap B^c) = P((A \cup B)^c)$$
$$= \cancel{P(} 1 - P(A \cup B)$$
$$= 1 - (P(A) + P(B) - P(A \cap B))$$
$$\cancel{= (1 - P(A)) - P(B)(1 - P(A))}$$
$$= 1 - P(A) - P(B) + P(A)P(B) \quad \text{(As A and B are independent)}$$
$$= (1 - P(A)) - P(B)(1 - P(A))$$
$$= (1 - P(A))(1 - P(B))$$
$$= P(A^c)P(B^c).$$

$\therefore$ A^c & B^c are independent events.

Q6 : No. $P(\text{all alike}) = \frac{2}{8} = \frac{1}{4}$. When we toss a coin three times there are eight possible outcomes. HHH & TTT are two of them. Galton thought if the first two ~~are~~ positions are fixed i.e. they are either HH. or TT, then the next position will be a head for HH or TT with ~~prop~~ probability ½.

Q7. Here $A = \bigcup_{j=1}^{k} \{\omega_j\}$, $k \le n$.

Define $P(A) = \sum_{j=1}^{k} P(\{\omega_j\}) = \sum_{j=1}^{k} p_j$. We shall show that P is a probability measure

It is clear from the problem that $0 \le P(A) \le 1$. So the first axiom holds.

$$P(\Omega) = \sum_{j=1}^{n} p_j = 1. \quad \text{(2nd axiom holds)}$$

~~Let $A_1 \cup A_2 \cup \ldots \cup A_m$ be an event where $\{A_j\}_{j=1}^{m}$ is a mutually exclusive set of events, then~~

Let $\{A_n\}$ be a sequence of mutually disjoint events. Since Ω is a finite set, there can be only finite number of mutually ~~disjoint~~ exclusive events. Thus there exists n_0, s.t. $\forall n > n_0$ $A_{n_0} = \phi$. Thus

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P(A_1 \cup A_2 \cup \ldots \cup A_{n_0})$$

$$= \sum_{\{j : j \in A_1 \cup A_2 \cup \ldots \cup A_{n_0}\}} p_j$$

Since ~~$A_1 \cup A_2 \cup \ldots \cup$~~ $A_1 \ldots A_{n_0}$ are mutually disjoint

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{\{j : \omega_j \in A_1\}} p_j + \ldots + \sum_{\{j : \omega_j \in A_{n_0}\}} p_j$$

$$= P(A_1) + P(A_2) + \ldots + P(A_{n_0}) + P(\phi) + \ldots + P(\phi) + \ldots$$

Thus the third axiom holds. If $p_j = \frac{1}{n}$, then, we are reduced to the classical case.

(Hint, : If Ω is countably infinite, then $A = \bigcup_{j \geq k}^{m} \{\omega_j\}$ for some $k \in \mathbb{N}$ where $m \in \mathbb{N}$ or ~~m ∈~~ $m = \infty$. The same argument holds)

Home work

Q.8: If either $P(A) = 0$ or $P(B) = 0$, then $P(A \cap B) = P(\phi) = 0$
$= P(A).P(B)$

Thus ~~P(A)~~ A and B are independent. If neither $P(A)$ or $P(B)$ is 0, then $P(A \cap B) = 0 \neq P(A)P(B)$. Thus A and B are not independent. This a key idea.

Q9: <u>Polya Urn Model:</u> Let us look at the problem at the j-th stage. We say that R_j is the event that j-th ball drawn is red and B_j is the event that the j-th ball drawn is blue

$$P(B_j) = P(B_{j-1} \cap B_j) + P(R_{j-1} \cap B_j)$$

Note that

$$P(R_1) = \frac{r}{r+b} \qquad P(B_1) = \frac{b}{r+b}$$

$$P(B_1 \cap B_2) = P(B_1) P(B_2 | B_1) = \frac{b}{r+b} \cdot \frac{b+d}{r+b+d}$$

$$P(R_1 \cap B_2) = P(R_1) P(B_2 | R_1) = \frac{r}{r+b} \cdot \frac{b}{r+b+d}$$

$$\therefore \quad P(B_2) = P(B_1 \cap B_2) + P(R_1 \cap B_2)$$
$$= \frac{b(b+d)}{(r+b)(r+b+d)} + \frac{rb}{(r+b)(r+b+d)}$$
$$= \frac{b(b+d)+rb}{(r+b)(b+r+d)} = \frac{b(b+d+r)}{(r+b)(b+d+r)}$$

$$\therefore \quad P(B_2) = \frac{b}{r+b}$$

In fact it can be shown that using mathematical induction.

$$P(B_j) = \frac{b}{r+b} \quad \text{for all } j \in \mathbb{N}$$

In the second part, we have to find. So here we use Bayes Thm

$$\cancel{P(B_2 \mid B_1)}$$

$$P(B_1 \mid B_2) = \frac{P(B_1 \cap B_2)}{P(B_2)}$$

$$= \frac{P(B_1)\,P(B_2 \mid B_1)}{P(B_2)}$$

$$= \frac{\frac{b}{r+b} \cdot \frac{b+d}{r+b+d}}{\frac{b}{r+b}}$$

$$= \frac{b+d}{r+b+d}$$

Q10. Do not solve in the class, ~~but~~ kindly

Hint: Use Theorem of Total Probability & idea of independence

a) Ans: $\frac{\alpha+\beta}{2}$

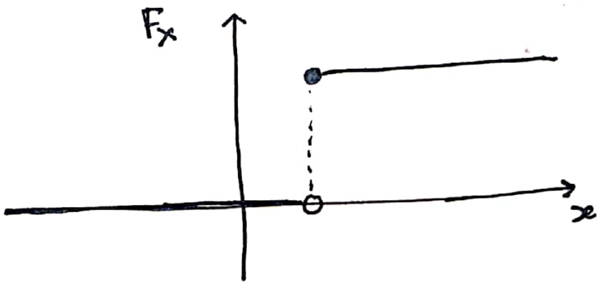
b) Ans $\frac{\alpha^2+\beta^2}{2}$.

Solutions and Hints to Practice Problem (Set-2)

1) If $x < c$, then $P(X \le x) = 0$, Since $P(X = c) = 1$.

If $x \ge c$, then $P(X \le c) = 1$.

$$\therefore \quad F_X(x) = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x \ge c \end{cases}$$



2) Let us consider X to be continuous random variable.

$$P\left(\left[y, y+\tfrac{1}{n}\right]\right) = F_X\left(y+\tfrac{1}{n}\right) - F_X(y)$$

Now $\left[y, y+\frac{1}{n}\right]$, is a decreasing sequence of intervals.

Hence

$$\lim_{n\to\infty} P\left(\left[y, y+\tfrac{1}{n}\right]\right) = \lim_{n\to\infty} F_X\left(y+\tfrac{1}{n}\right) - F_X(y)$$

↓ Using the fact that F_X is cont. at y.

$$\Rightarrow \quad P\left(\lim_{n\to\infty}\left[y, y+\tfrac{1}{n}\right]\right) = F_X(y) - F_X(y)$$

↓

(Last result in Lecture 2) also holds for monotonically decreasing events

$$\Rightarrow \quad P(\{y\}) = 0$$

(The student should do the converse. Kindly do not discuss in class) The reverse is ~~given over leaf~~ obvious since X is continuous. It needs some working if nothing is mentioned about X.

3. $Var(aX) = E[aX - E(aX)]^2$

$$= E\left[a^2(X - E(X))^2\right]$$ using the fact that $E(aX) = aE(X)$

$$= a^2 E(X - E(X))^2$$

$$= a^2 Var(X).$$

Kindly do not solve in class.

4. We shall give only solutions of 4 i) & 4 ii)
4 iii) & 4 iv) can be solved ~~accord~~ in a similar fashion.

i) $(a, b] = (-\infty, b] \setminus (-\infty, a]$

Further $(-\infty, a] \subset (-\infty, b]$, thus $\{\omega : X(\omega) \le a\} \subseteq \{\omega : X(\omega) \le b\}$

$$\therefore P((a,b]) = P((-\infty, b] \setminus (-\infty, a])$$

$$= P(\{\omega: X(\omega) \le b\} \setminus \{\omega: X(\omega) \le a\})$$

$$= P(\{\omega: X(\omega) \le b\}) - P(\{\omega: X(\omega) \le b\})$$

$$= P(\{\omega: X(\omega) \le b\}$$

(If $A \subset B$, $P(A \setminus B) = P(A) - P(B)$)

Actually we should write $P((a,b])$ as $P_X((a,b])$. But we will write as we are writing

$$= P(X \le b) - P(X \le a)$$

$$= F_X(b) - F_X(a)$$

ii) $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b]$

$$\therefore P([a,b]) = P\left(\bigcap_{n=1}^{\infty} (a - \tfrac{1}{n}, b]\right)$$

$$= \lim_{n\to\infty} P((a - \tfrac{1}{n}, b])$$

$$= \lim_{n\to\infty} \left(F_X(b) - F_X(a - \tfrac{1}{n})\right)$$

$$= F_X(b) - \lim_{y \uparrow a} F_X(y)$$ [$y \uparrow a, \Rightarrow y < a$ & $y \to a$]

[F_X is right cont. not left cont.].

5) $\lambda f_X(x) + (1-\lambda) h_X(x) \geq 0, \quad \forall x$ Since $\lambda \geq 0$ & f_X and h_X are densities

$$\int_{-\infty}^{\infty} [\lambda f_X(x) + (1-\lambda) h_X(x)] \, dx$$

$$= \lambda \int_{-\infty}^{\infty} f_X(x) \, dx + (1-\lambda) \int_{-\infty}^{+\infty} h_X(x) \, dx$$

$$= \lambda + (1-\lambda) = 1.$$ Since $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{\infty} h_X(x)\,dx.$

6) $$f_X(x) = \begin{cases} c\, x^{-d}, & x > 1 \\ 0 & \text{otherwise} \end{cases}$$

$f(x) \geq 0$ is obvious, if $c \geq 0$.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{1}^{\infty} \frac{c}{x^d}\,dx = \lim_{M\to\infty} c \left[\frac{x^{1-d}}{1-d}\right]_1^M$$

$$= c \lim_{M\to\infty} \left[\frac{M^{1-d}}{1-d} - \frac{1}{1-d}\right]$$

If $d > 1 \Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx = \frac{c}{d-1} \left(\because M^{1-d} \to 0 \text{ as } M \to \infty\right)$

If $d < 1$ then $\int_{-\infty}^{\infty} f_X(x)\,dx$ then it is divergent

So if $d > 1$, we have f_X to be a density function

we must have $d - 1 = c$ or $c + 1 = d$.

$F_X(x) = 1 - x^{-(d-1)}, \quad d > 1.$ (Obtained directly from definition)

7) We will provide only solution of 7(c) in the class

$$m_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}t}\, m_X\left(\frac{t}{b}\right)$$

$$m_{\frac{X+a}{b}}(t) = E\left[e^{t\left(\frac{X+a}{b}\right)}\right]$$

$$= E\left[e^{t\frac{X}{b}} \cdot e^{\frac{a}{b}t}\right]$$

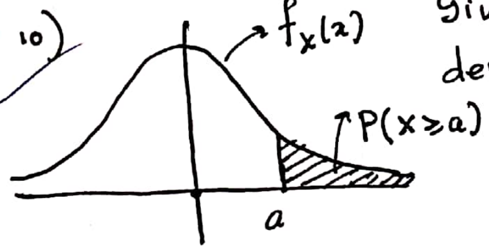
$$\cancel{= e^{a/b\, t}\left[E\left[e^{\frac{t}{b}X}\right]\right]}$$

$$= e^{\frac{a}{b}t}\, E\left[e^{t/b\, X}\right] \quad \left(\because e^{\frac{a}{b}t} \text{ is free of } X\right)$$

$$\therefore \quad m_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}t}\, m_X\left(\frac{t}{b}\right).$$

Q8) and Q9) should be tried by the students. Kindly do not solve in class.

Q. 10)



Give $F_X(0) = \frac{1}{2} = P(X \leq 0)$. So the density function is symmetric around $x = 0$.

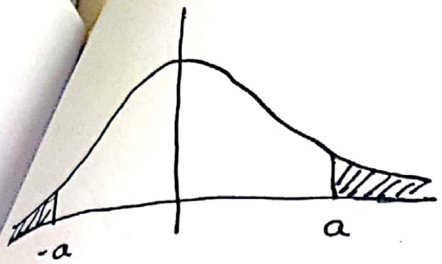
$$P(X \geq a) = \int_a^{\infty} f_X(x)\,dx$$

$$\therefore \quad P(X \geq a) = \int_0^{\infty} f_X(x)\,dx - \int_0^{a} f_X(x)\,dx$$

Since the density function is symmetric around $x = 0$, we have

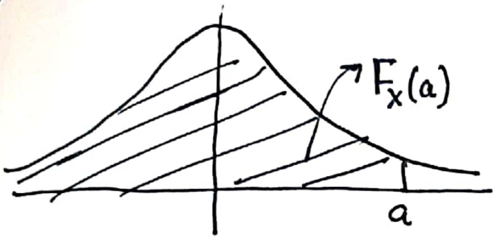
$$\int_0^{\infty} f_X(x)\,dx = \int_{-\infty}^{0} f_X(x)\,dx = F_X(0) = \frac{1}{2}$$

$$\therefore \quad P(X \geq a) = \frac{1}{2} - \int_0^{a} f_X(x)\,dx$$



Again as the curve is symmetric about $x=0$, we have

$$\int_a^{\infty} f_X(x)\,dx = \int_{-\infty}^{-a} f_X(x)\,dx$$



$$\Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx = F_X(-a)$$

$$\Rightarrow 1 - F_X(a) = F_X(-a)$$

$$\Rightarrow F_X(a) + F_X(-a) = 1.$$

(The standard normal distribution has these features as we will learn soon) → Kindly mention this in the class.

11) $P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| > \varepsilon\right)$

[This problem can be discussed in the starting of Tutorial on 27th or 9 will discuss in the class after Binomial distribution done.

$$= P\left(|S_n - n\tfrac{1}{2}| > n\varepsilon\right)$$

$$= P\left(|S_n - E(S_n)| > n\varepsilon\right)$$

$$= P\left((S_n - E(S_n))^2 > n^2\varepsilon^2\right)$$

$$\leq \frac{Var(S_n)}{n^2\varepsilon^2} \quad \text{(By Chebyshev's inequality)}$$

$$\therefore\ \cancel{\lim}\ P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| > \varepsilon\right) \leq \frac{n\frac{1}{2}\cdot\frac{1}{2}}{n^2\varepsilon^2} \quad \left(Var(S_n) = npq = n\tfrac{1}{2}\cdot\tfrac{1}{2}\right)$$

for Binomial dist

$$\therefore\ P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| > \varepsilon\right) \leq \frac{1}{4n\varepsilon^2}$$

$$\therefore\ \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| > \varepsilon\right) \leq 0 \longrightarrow (A)$$

But $P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| > \varepsilon\right) \geq 0\ \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| > \varepsilon\right) \geq 0 \rightarrow (B)$

Combine (A) & (B) to ~~get~~ reach the conclusion

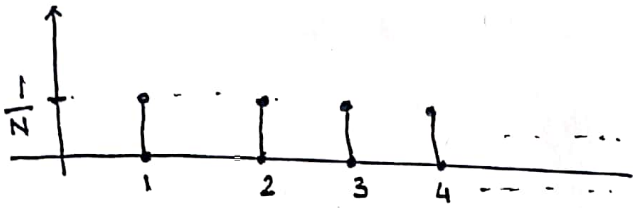
Solution and Hints (Problem set-3)

1. $f_X(x) \geq 0$ is obvious

$$\sum_{x=1}^{N} f_X(x) = \sum_{x=1}^{N} \frac{1}{N} = \frac{N}{N} = 1$$

Hence $f_X(x)$ is a pmf.

Graph of f_X :



Statisticians call it ~~to be~~ discrete uniform distribution.

$$E[X] = \sum_{x=1}^{N} x \cdot \frac{1}{N} = \frac{N+1}{2}$$

$$\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= \sum_{x=1}^{N} x^2 \frac{1}{N} - \left(\frac{N+1}{2}\right)^2 \\
&= \frac{N(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2 \\
&= \frac{(N+1)(N-1)}{12}.
\end{aligned}$$

2.

$$f_X(0) = 1-p$$

$$f_X(1) = p$$

$$\therefore E(X) = 0.(1-p) + 1.p = p$$

$$E(X^2) = 0^2(1-p) + 1^2.p$$

$$\therefore Var(X) = p - p^2 = p(1-p)$$

Since each trial having ~~Sucess~~ success and failure as the outcome, and each success has value 1 and failure zero. Thus it is obvious that

$$X_n = X_1 + \cdots + X_n$$

3.

$$G_X(s) = E[s^X], \quad s \in [-1, +1]$$

$$\therefore \; G_X(s) = \sum_{n=0}^{\infty} s^n P(X=n) \qquad \Big[\; s \in [-1,+1], \text{ is the range where } G_X(s) \text{ is finite, ie the series converges absolutely} \Big]$$

Hence the coefficient of s^n is $P(X=n)$

a)

$$G_X(s) = e^{\lambda(s-1)}$$

$$= e^{\lambda s} e^{-\lambda}$$

$$= e^{-\lambda} e^{\lambda s}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda s}{1!} + \frac{\lambda^2 s^2}{2!} + \cdots + \frac{\lambda^n s^n}{n!} + \cdots\right]$$

$$= e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} s + \frac{\lambda^2 e^{-\lambda}}{2!} s^2 + \cdots + \frac{\lambda^n e^{-\lambda}}{n!} s^n + \cdots$$

$$\therefore \quad \boxed{P(X=n) = \frac{e^{-\lambda}\lambda^n}{n!}}$$

Hence $X \sim \text{Poisson}(\lambda)$.

b)

$$G_X(1) = \sum_{n=0}^{\infty} P(X=n) = 1.$$

If we consider $\boxed{0^0 = 1}$ → a convention, then

$$G_X(0) = P(X=0).$$

c)

$$\cancel{G'_X(s) = \sum_{n=0}^{\infty} s^{n-1} P(X-}$$

$$\cancel{G'_X(s) = \sum_{n=0}^{\infty} (n-1) s^{n-1} P(X=n)}$$

$$G'_X(s) = \sum_{n=0}^{\infty} n s^{n-1} P(X=n)$$

$$\therefore \; G'_X(1) = \sum_{n=0}^{\infty} n P(X=n) = E[X].$$

3) Will be solved in the class when independent random variables are discussed.

4) Here the sampling is with ~~replace me~~ replacement. At each draw we observe the item drawn and replace it back. Thus at every draw, the probability of getting a defective item is $\frac{K}{M}$ which is fixed for each draw. Thus X can be modelled as a Binomial random variable. Thus

$$f_X(x) = \binom{n}{x}\left(\frac{K}{M}\right)^x\left(1-\frac{K}{M}\right)^{n-x}.$$

5) Here the sampling is with replacement. So each step we draw a light bulb, see whether it is defective or not. So we have to choose the bulbs in a seperate manner. The proper bulbs has to be chosen from N bulbs while the defective is chosen from M − N bulbs. Let us draw a sample of size n
Let X be the random variable denoting the number of non-defective bulbs drawn when we draw a sample of size n.

$$\therefore\ f_X(x) = \frac{\binom{N}{x}\binom{M-N}{n-x}}{\binom{M}{n}} \qquad x = 0, 1, 2 \ldots n.$$

In the literature this is called the hypergeometric distribution.
(The students should carefully think as to why one gets such an expression for f_X: Explain them if they are stuck).

$$E[X] = \sum_{x=0}^{n} x\,\frac{\binom{N}{x}\binom{M-N}{n-x}}{\binom{M}{n}} = n\cdot\frac{N}{M}\sum_{x=1}^{n}\frac{\binom{N-1}{x-1}\binom{M-N}{n-x}}{\binom{M-1}{n-1}}$$

Set $x-1=y$

$$\therefore\ E[X] = n\cdot\frac{N}{M}\sum_{y=0}^{n-1}\frac{\binom{K-1}{y}\binom{M-1-N+1}{n-1-y}}{\binom{M-1}{n-1}} \qquad n\binom{N}{n} = N\binom{N-1}{n-1}$$

$$\therefore \quad E[X] = n\frac{N}{M} \cdot \frac{1}{\binom{M-1}{n-1}} \sum_{x=1}^{n} \binom{N-1}{y}\binom{M-1-k+1}{n-1-y}$$

- We will now use the following fact

$$\sum_{j=0}^{n} \binom{a}{j}\binom{b}{n-j} = \binom{a+b}{n}$$

This follows from the fact

$$(1+x)^a (1+x)^b = (1+x)^{a+b}$$

Now expanding both sides and equating the coefficients of x^n.

$$\therefore \quad E[X] = n\frac{N}{M} \cdot \frac{1}{\binom{M-1}{n-1}} \cdot \binom{M-1}{n-1}$$

$$\therefore \quad \boxed{E[X] = n \cdot \frac{N}{M}}$$

(This is what we will get if we sample without replacement)

Variance is a bit tricky.

- $$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$
$$= E[X^2 - X + X] - (E[X])^2$$
$$= E[X(X-1)] + E[X] - (E[X])^2$$

So we have to compute.

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1) \frac{\binom{N}{x}\binom{M-N}{n-x}}{\binom{M}{n}}$$

$$= n(n-1)\frac{N(N-1)}{M(M-1)} \sum_{x=2}^{n} \frac{\binom{N-2}{x-2}\binom{M-N}{n-x}}{\binom{M-2}{n-2}}$$

set $x-2=y$.

Thus

$$E[X] = n(n-1)\,\frac{N(N-1)}{M(M-1)} \sum_{y=0}^{n-2} \frac{\binom{N-2}{y}\binom{M-2-(N-2)}{n-2-y}}{\binom{M-2}{n-2}}$$

$$= n(n-1)\,\frac{N(N-1)}{M(M-1)} \quad \left[\text{Arguing as in the case of } E[X]\right]$$

$$\therefore \quad Var(X) = \frac{nN}{M}\left[\frac{(M-N)(M-n)}{M(M-1)}\right]$$

$$\boxed{Var(X) = \frac{nN(M-N)(M-n)}{M^2(M-1)}}$$

6) This problem will be discussed in the class.

—x—

Solution Set - (Problem Set-4)

1. We will first show that it is a pdf. $f_X(x) \geq 0$, is obvious.

Now $\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{1}{\pi(x^2+1)} dx$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx$$

$$= \frac{1}{\pi} \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b \frac{1}{(x^2+1)} dx$$

$$= \frac{1}{\pi} \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left[\tan^{-1}b - \tan^{-1}a\right]$$

$$\cancel{= \frac{1}{\pi} \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left[\quad\right]}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\cdot\pi = 1.$$

For finding the mean or expectation let us first compute the following

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{\pi(x^2+1)} dx$$

$$= \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left[\int_a^b \frac{x}{\pi(x^2+1)} dx\right]$$

Now consider: $\int_a^b \frac{x}{(x^2+1)} dx = \frac{1}{2}\int_{a^2+1}^{b^2+1} \frac{dz}{z} = \frac{1}{2}\left[\ln z\right]_{a^2+1}^{b^2+1}$

Put $z = x^2+1$

$$= \cancel{\frac{1}{2}\left[\ln(b^2+1) - \right]}$$

$$= \frac{1}{2}\left[\ln(b^2+1) - \ln(a^2+1)\right]$$

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b \frac{x}{(x^2+1)}\,dx = \infty - \infty, \text{ which is an undefined quantity.}$$

So $E[X]$ is not well-defined or does not exist for this distribution.

It is often called the Cauchy distribution.

Alternative approach

A more convincing approach is the following. Let us try to find the moment generating function

$$mgf = E\left[e^{tX}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\pi(1+x^2)}\,dx \geq \int_0^{\infty} \frac{\cancel{e^{tx}}\, e^{tx}}{\pi(1+x^2)}\,dx \quad \left(\because \frac{e^{tx}}{\pi(1+x^2)} \geq 0, \forall x \in \mathbb{R}\right)$$

$$\geq \int_0^{\infty} \frac{tx}{\pi(1+x^2)}\,dx, \quad \cancel{(\because e^{tx} \geq tx \ \forall x \geq 0)}$$

$$[\because e^{tx} \geq tx, \forall x \geq 0]$$

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to\infty} \int_0^a \frac{tx}{x^2+1}\,dx$$

$$\cancel{= \lim_{a\to\infty} [\tan}$$

Now
$$\int_0^a \frac{tx}{x^2+1}\,dx = t \int_0^a \frac{x\,dx}{x^2+1}, \quad \text{Set } z = x^2+1$$

$$= \frac{t}{2} \int_1^{a^2+1} \frac{\cancel{z}\,dz}{z}$$

$$= \frac{t}{2}\left[\ln(a^2+1) - \ln 1\right]$$

$$= \frac{t}{2} \ln(a^2+1)$$

$$\therefore \int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to\infty} \frac{t}{2} \ln(a^2+1) = \infty. \quad \square$$

2. $m_X(t) = e^{t^2/2}$

$\Rightarrow m_X(t) = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \cdots\cdots$

Now $E(X)$ = Coefficient of t in the series expression of $m_X(t)$

$\therefore E(X) = 0$

while $E(X^2) = 1$ (i.e. cofficient of $\frac{t^2}{2!}$)

$\therefore Var(X) = 1.$ Here $X \sim N(0,1)$.

3. We have to find ~~P(X >~~ $P(X \leq \gamma)$. [Note ~~$\gamma < 1$~~ $\gamma < 1$]

$$P(X \leq \gamma) = \frac{1}{B(a,b)} \int_0^{\gamma} x^{a-1}(1-x)^{b-1}\,dx .$$

For various values of a & b, various values of the probability emerges. (Can the students find this for say $a = 2$, $b = 12$.)

4. Let X denote the r.v. representing the number of minutes spent in the restaurant.

Here $E(X) = 6 = \frac{1}{\lambda}$ $\therefore \lambda = \frac{1}{6}$

$\therefore f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

$\therefore f_X(x) = \frac{1}{6} e^{-\frac{x}{6}}$

$$P(T > 12) = 1 - P(T \leq 12)$$

$$= 1 - \int_0^{12} \frac{1}{6} e^{-\frac{x}{6}}\,dx$$

$$= 1 - \frac{1}{6}\left[-6 \cdot e^{-x/6}\right]_0^{12}$$

$$= 1 - \frac{1}{6}\left[-6 e^{-\frac{12}{6}} - (-6)\right]$$

$$= 1 + e^{-12/6} - 1 = e^{-2} \approx .1353 .$$

5. Here as before we have $E(x) = 12$ years. $\therefore 12 = \frac{1}{\lambda}$ or $\lambda = \frac{1}{12}$.

Hence $f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$

$$\therefore f_X(x) = \frac{1}{12} e^{-\frac{x}{12}}$$

Suppose Mr J. buys the house when it is already say T years old. So we need to find out

$P(X > T+4 \mid X > T)$, where X is r.v. denoting the working years.

But as exponential distribution is memory-less we have

$$P(X > T+4 \mid X > T) = P(X > 4)$$

$$= 1 - P(X \leq 4)$$

$$= 1 - \int_0^4 \frac{1}{12} e^{-x/12}\, dx$$

$$= e^{-1/3} \simeq .7166\ldots$$

6. The p.d.f. of the Gamma Distribution

$$f_X(x) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x}, \quad \lambda > 0,\ r > 0,\ x \geq 0.$$

For $\lambda = \frac{1}{\beta}$

$$f_X(x) = \frac{1}{\beta \Gamma(r)} \left(\frac{x}{\beta}\right)^{r-1} e^{-x/\beta}$$

$\therefore$ For $r = 2$

$$f_X(x) = \frac{1}{\beta^2} x e^{-x/\beta}$$

So the Weibull distribution has the form

$$f_X(x) = \frac{1}{\beta} \cdot 2x\, e^{-x^2/\beta}$$

which is slightly different from the Gamma distribution.

$\lambda = \frac{1}{\beta}$

$$P(X > t) = 1 - \int_0^t \frac{2x}{\beta} e^{-x^2/\beta}\, dx = e^{-t^2/\beta}$$

Now for given s and t

$$P(X > s+t \mid X > t) = \frac{P(X > s+t)}{P(X > t)}$$

$$= e^{-\frac{s(s+2t)}{\beta}}$$ (The students must compute this)

$$\therefore \quad e^{-\frac{s(s+2t)}{\beta}} = \gamma$$

$$\therefore \quad \frac{s(s+2t)}{\beta} = \ln\left(\frac{1}{\gamma}\right)$$

$$\therefore \quad \beta = \frac{s(s+2t)}{\ln\left(\frac{1}{\gamma}\right)}.$$

7. Here $n = 600$ & $p = \frac{1}{6}$ $\therefore E[X] = np = 100$

$$\mathrm{Var}(X) = npq = 100 \times \frac{5}{6} = \frac{500}{6}.$$

$$P(90 \le X \le 100) = P\left(\frac{90-100}{\sqrt{\frac{500}{6}}} \le \frac{X-np}{\sqrt{npq}} \le \frac{100-100}{\sqrt{\frac{500}{6}}}\right)$$

Since n is large we can approximate

$$\frac{X-np}{\sqrt{npq}}$$

as a standard normal variable.

$$\therefore P(90 \le X \le 100) \approx \Phi(0) - \Phi\left(\frac{-10}{\sqrt{\frac{500}{6}}}\right)$$

$$\approx \frac{1}{2} - \Phi\left(\frac{-10}{\sqrt{\frac{500}{6}}}\right)$$

$$\approx \frac{1}{2} - \left(1 - \Phi\left(\frac{10}{\sqrt{\frac{500}{6}}}\right)\right)$$

$$\approx \Phi\left(\frac{10}{\sqrt{\frac{500}{6}}}\right) - \frac{1}{2}.$$