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MCT HW-3

Exercise 1.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

For controllability,

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$|P| = -1 \Rightarrow \text{full rank}$$

$\therefore$ The system is controllable

For observability,

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$|Q| = 0 \Rightarrow \text{not full rank}$$

$\therefore$ The system is not observable

Exercise 2.

$$\hat{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\hat{B}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \Rightarrow r(\hat{B}^2) = 3$$

$$\hat{B}^1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow r(\hat{B}^1) = 2$$

$$\hat{C}^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow r(\hat{C}^2) = 2$$

$\Rightarrow$ not observable

$$\hat{C}^1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow r(\hat{C}^1) = 2$$

$\therefore$ The system is controllable, but not observable.

Exercise 3.

CASE-I:

$$\therefore A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P = [B \quad AB]$$

$$= \begin{bmatrix} 1 & -0.1 \\ 0 & 0.1 \end{bmatrix}$$

$$r(P) = 2$$

$\therefore$ The system is controllable because the input u could affect both state variables x_1 & x_2 & hence the system could reach any desired state.

CASE – II :–

$$A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & -0.2 \end{bmatrix}$$

$$r(P) = 1$$

∴ The system is not controllable because the input u has no effect on x_1 & hence the system cannot reach any desired state.

Exercise 4.

1. a)

$$x + y + z = 3$$
$$x + 2y + 3z = 0$$
$$x + 3y + 2z = 3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right]$$

$$R_2 \to R_2 - R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -3 \\ 1 & 3 & 2 & 3 \end{array}\right]$$

$$R_3 \to R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -3 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$R_3 \to R_3 + 2R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$$\therefore\ x + y + z = 3$$

$$-y + z = -3$$

$$3z = -6$$

$$\boxed{\begin{array}{l} \therefore\ z = -2 \\ y = 1 \\ x = 4 \end{array}}$$

b)

$$x + 2y - z = 1$$
$$2x + 5y - z = 3$$
$$x + 3y + 2z = 6$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$\therefore$
$$x + 2y - z = 1$$
$$y - z = 1$$
$$4z = z$$

$$\therefore \boxed{\begin{aligned} z &= 1 \\ y &= 0 \\ x &= 2 \end{aligned}}$$

c)

$$x_1 + x_2 - x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + x_3 + 0x_4 = 4$$
$$3x_1 + 5x_2 + 3x_3 - x_4 = 5$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right]$$

$$R_3 \rightarrow R_3 - (R_1 + R_2)$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 0 & 1 & 3 & 0 & 0 \end{array}\right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & 3 & 0 & 0 \end{array}\right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 2 & -2 \end{array}\right]$$

$$x_1 + x_2 - x_3 + x_4 = 1$$
$$x_2 + x_3 - 2x_4 = 2$$
$$2x_4 = -2$$

$\therefore$ This system has <u>no solution</u>.

2.
$$x_1 + 2x_2 + 4x_3 = 3$$
$$3x_1 + 8x_2 + 14x_3 = 13$$
$$2x_1 + 6x_2 + 13x_3 = 4$$

Using LU Decomposition

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$u_{11} = 1, \quad u_{12} = 2, \quad u_{13} = 4$$
$$l_{21} = 3, \quad u_{22} = 2, \quad u_{23} = 2$$
$$l_{31} = 2, \quad l_{32} = 1, \quad u_{33} = 3$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Now,

$$\because Lz = y$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$z_1 = 3$$

$$3z_1 + z_2 = 13 \Rightarrow z_2 = 4$$

$$2z_1 + z_2 + z_3 = 4$$

$$\Rightarrow z_3 = -6$$

Now,

$$\because Ux = z$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$x_1 + 2x_2 + 4x_3 = 3$$
$$2x_2 + 2x_3 = 4$$
$$3x_3 = -6$$

$$\boxed{\begin{aligned} \therefore x_3 &= -2 \\ x_2 &= 4 \\ x_1 &= 3 \end{aligned}}$$

Exercise 6.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix}$$

For the system to be controllable,

$$\det(P) \neq 0$$

$$\gamma \neq 0$$

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3+\gamma & -1 \end{bmatrix}$$

For the system to be observable,

$$\det(Q) \neq 0$$

$$-1 + 3 - \gamma \neq 0$$

$$\therefore \gamma \neq 2$$

1. For the system to be controllable but not observable,

$$\gamma = 2$$

2. For the system to be observable but not controllable,

$$\gamma = 0$$

Exercise 7.

1.

$$x_{(k+1)} = A x(k) + B u(k)$$

$$y(k) = x(k+1)$$

$$x(k+1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

2.

$$P = \begin{bmatrix} B & AB \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{Full rank}$$

$\therefore$ the system is controllable

Intuitively too the system is controllable, as the input can change & control all state variables.

For example, you can achieve any desired combinations of LED brightness by giving the input in the appropriate order.
Ex: The 5 the LED will reach the desired brightness after 5 timesteps. So if we plan 5 timesteps in advance we can make the system reach any desired state.

Exercise 5.