Name: Sahil T Chaudhary
Andrew ID: stchaudh

MCT HW-5

Exercise 1.

(a) $$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

Eigen values of A are,

$\lambda_1 = 1$ & $\lambda_2 = 0.5$

$\lambda_1 = 1$, & is not defective

$\Rightarrow$ Stable i.s.L.

$\lambda_2 < 1 \Rightarrow$ Asymptotic stable

Hence, the system is Stable i.s.L.

(b) $$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$(-7-\lambda)\left[-7+2\lambda+\lambda^2\right]+2\left[-2-2\lambda\right]$$
$$+6\left[-10-2\lambda\right] = 0$$

$$49-14\lambda-7\lambda^2+7\lambda-2\lambda^2-\lambda^3-4-4\lambda$$
$$-60-12\lambda = 0$$
$$\lambda^3+9\lambda^2+23\lambda+15=0$$

$$\lambda_1 = -1, \quad \lambda_2 = -5, \quad \lambda_3 = -3$$

$\because$ All eigen values < 0,

the system is asymptotic stable.

Excercise 2:

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$\text{rank}(P) = 2 \rightarrow$ not full rank

$$\therefore M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \hat{A} = M^{-1} A M = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\therefore A_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\hat{B} = M^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{C} = CM = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$$

$$\therefore C_c = \begin{bmatrix} 3 & 3 \end{bmatrix}$$

$\therefore$ controllable form is,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 3 \end{bmatrix} x$$

Now,

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

Hence, it is unobservable

Since all modes in the reduced form are controllable, the reduced form is stabilizable.

Now, to check detectability we need to find the unobservable modes.

Doing observable decomposition on the reduced form.

$$\therefore M^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix}$$

$\because \hat{A} = M^{-1} A M = \begin{bmatrix} 0 & 3 \\ 0 & \boxed{-1} \end{bmatrix}$

This is the unobservable mode.

$\because \lambda < 0$, it is asymptotic stable, & hence stable i.s.L.

Since the unobservable mode is stable i.s.L., the system is <u>detectable</u>.

Exercise 3.

$$\ddot{x} = \frac{u_1}{m}\sin\theta + \frac{\varepsilon u_2}{m}\cos\theta$$

$$\ddot{y} = \frac{u_1}{m}\cos\theta + \frac{\varepsilon u_2}{m}\sin\theta - g$$

$$\ddot{\theta} = u_2/J$$

At equilibrium pts.,

$\tilde{x}(t) = 0$, $\tilde{y}(t) = 0$, $\tilde{\theta}(t) = 0$,

$\tilde{u}_1(t) = mg$, $\tilde{u}_2(t) = 0$

$$z = \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\therefore \dot{z} = \begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix}$$

$\therefore$ Jacobian,

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -\frac{u_1}{m}\cos\theta - \frac{\varepsilon u_2}{m}\sin\theta & 0 & 0 & 0 \\ -\frac{u_1}{m}\sin\theta + \frac{\varepsilon u_2}{m}\cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\& \frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ -\frac{\sin\theta}{m} & \frac{\varepsilon}{m}\cos\theta \\ \frac{\cos\theta}{m} & \frac{\varepsilon}{m}\sin\theta \\ 0 & 1/J \end{bmatrix}$$

At equilibrium pts.,

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon/m \\ 1/m & 0 \\ 0 & 1/J \end{bmatrix} = B$$

$\therefore$ The linearized system is

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon/m \\ 1/m & 0 \\ 0 & 1/J \end{bmatrix} u$$

The eigen values of A

are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

& 2 linearly independent eigen vectors exist,

$\therefore m = 4$ & $q = 2$

$$\therefore J = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & 0 \\ 0 & e^{J_2 t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{J_2 t} \end{bmatrix}$$

Now,
$$J_2 = D + N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore e^{tN} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

As $t \to \infty$, the system $\to \infty$.
Hence, the system is <u>not stable</u>.

Exercise 4.

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

$$V = x_1^2 + x_2^2$$

$$\therefore \dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \quad \text{—} \ (1)$$

Now,

$$\dot{x}_1 = a x_1$$

$$\dot{x}_2 = x_1 - x_2$$

Plugging in eqn (1)

$$\dot{V} = 2x_1(a x_1) + 2x_2(x_1 - x_2) \leq 0$$

$$\Rightarrow 2a x_1^2 + 2x_1x_2 - 2x_2^2 \leq 0$$

Dividing by x_2^2

$$\Rightarrow \quad a\left(\frac{x_1}{x_2}\right)^2 + \frac{x_1}{x_2} - 1 \leq 0$$

Let $\frac{x_1}{x_2} = \mu$

$\Rightarrow ax^2 + x - 1 \leq 0$

Taking the discriminant of the eq^n,

$$b^2 - 4ac < 0$$

$$1 + 4a < 0$$

$$\boxed{\therefore a < -0.25}$$

Excercise 5:

$$\dot{x}_1 = x_2 - x_1 x_2^2$$
$$\dot{x}_2 = -x_1^3$$

(a) Linearizing the system,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -x_2^2 & 1-2x_1x_2 \\ -3x_1^2 & 0 \end{bmatrix}$$

Now, equilibrium pts. are

$$\dot{x}_1 = 0 \Rightarrow x_2(1-x_1x_2) = 0$$

& $$\dot{x}_2 = 0 \Rightarrow x_1^3 = 0$$

$$\therefore x_1 = 0$$
$$x_2 = 0$$

$\therefore$ at equilibrium pts.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

Eigen values of A are $\lambda_1 = 0$ & $\lambda_2 = 0$

Since the eigen values are zero for an approximated system, it is risky.

(b) $V(x_1, x_2) = x_1^4 + 2x_2^2$

$$\dot{V} = 4x_1^3 \dot{x}_1 + 4x_2 \dot{x}_2$$

$$= 4x_1^3 (x_2 - x_1 x_2^2) + 4x_2(-x_1^3)$$

$$= \cancel{4x_1^3 x_2} - 4x_1^4 x_2^2 - \cancel{4x_1^3 x_2}$$

$$= -4x_1^4 x_2^2 < 0 \quad \forall x \neq 0$$

Hence, the system is stable

Q5.py

Q5.py > ...

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Exercise 5 (c)
def originalSys(X):
    x1,x2 = X
    return [x2 - x1*(x2**2), -x1**3]

def linearizedSys(X):
    x1,x2 = X
    return [x2,0]

x1 = np.linspace(-4,4,20)
x2 = np.linspace(-4,4,20)
X1,X2 = np.meshgrid(x1,x2)
u1,v1 = np.zeros(X1.shape),np.zeros(X2.shape)
u2,v2 = np.zeros(X1.shape),np.zeros(X2.shape)
A,B = X1.shape

for i in range(A):
    for j in range(B):
        x = X1[i,j]
        y = X2[i,j]
        x_dot = originalSys([x,y])
        xlin_dot = linearizedSys([x,y])
        u1[i,j] = x_dot[0]
        v1[i,j] = x_dot[1]
        u2[i,j] = xlin_dot[0]
        v2[i,j] = xlin_dot[1]

plot1 = plt.quiver(X1, X2, u1, v1, color='r')
plt.title("Original Non-Linear System")
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.xlim([-4, 4])
plt.ylim([-4, 4])
plt.show()

plot2 = plt.quiver(X1, X2, u2, v2, color='g')
plt.title("Linearized System")
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.xlim([-4, 4])
plt.ylim([-4, 4])
plt.show()
```

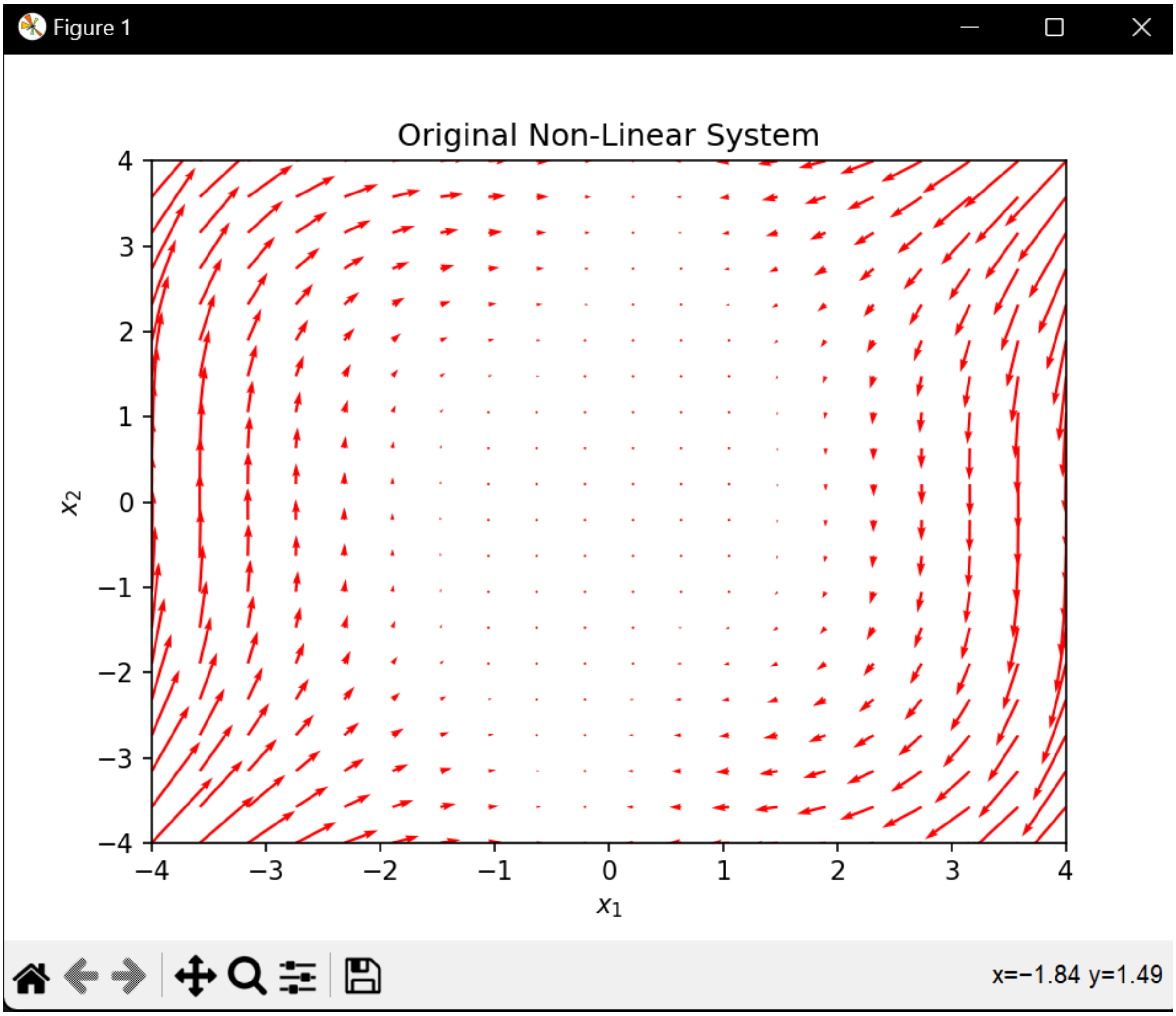
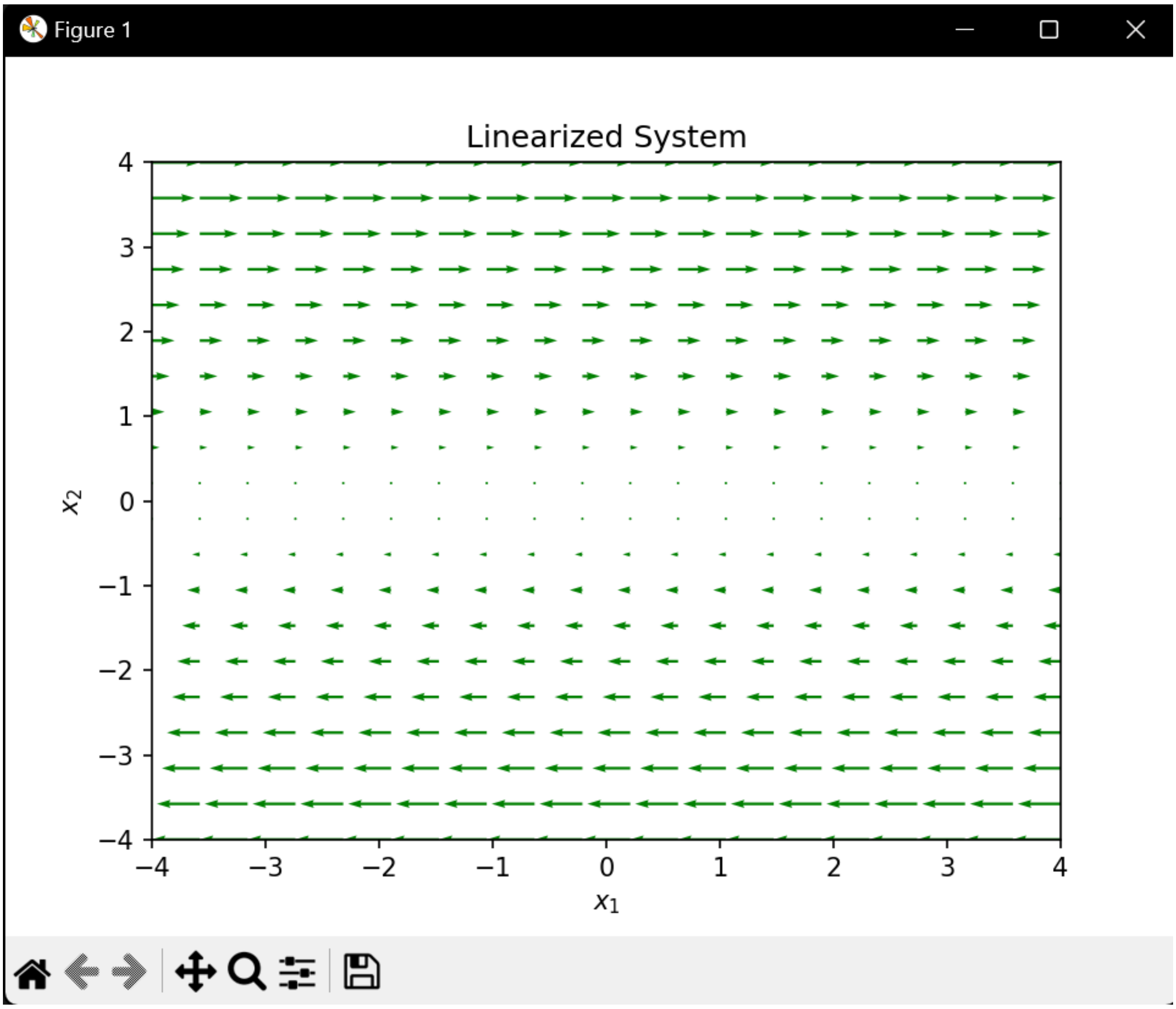


Figure 1
Original Non-Linear System
4
3
2
1
0
−1
−2
−3
−4
x_2
−4
−3
−2
−1
0
1
2
3
4
x_1
x=−1.84 y=1.49

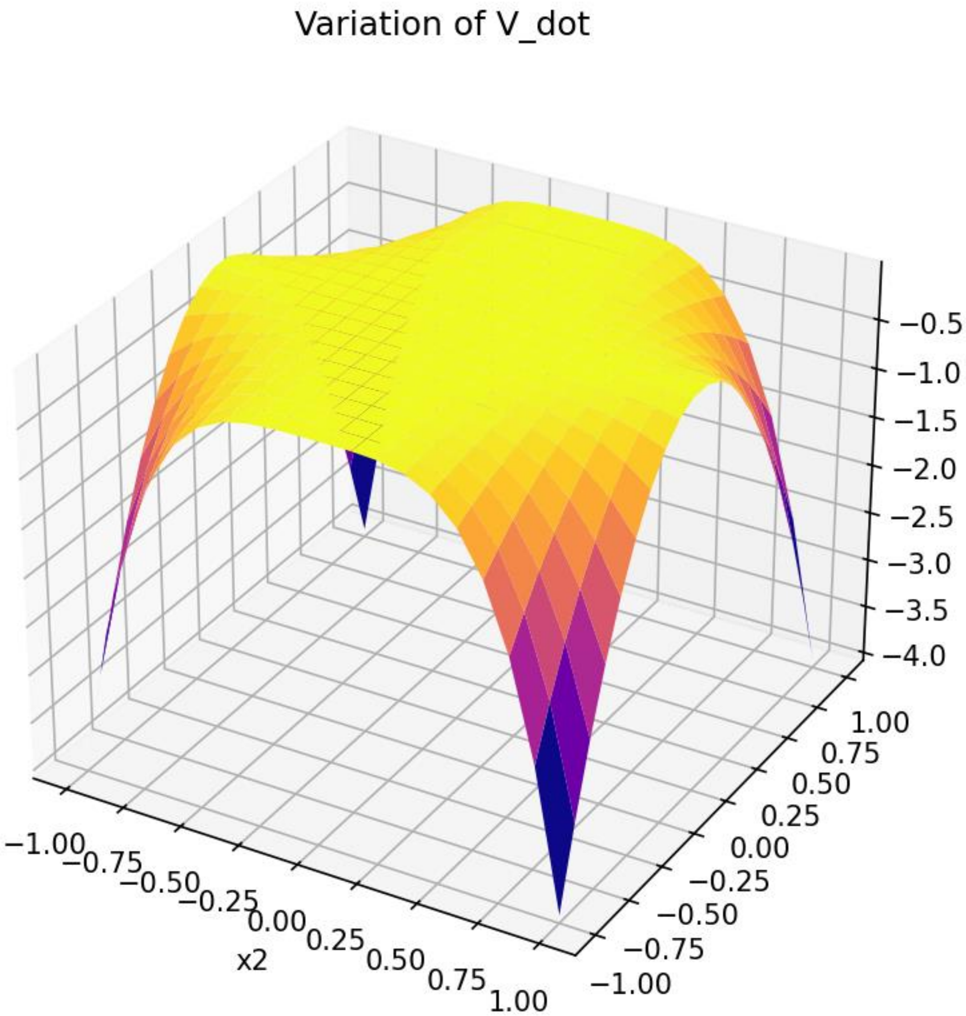



```python
# Exercise 5 (d)
xd1 = np.linspace(-1,1,20)
xd2 = np.linspace(-1,1,20)

XD1,XD2 = np.meshgrid(xd1,xd2)

V_dot = -4*(XD1**4) * (XD2**2)

fig = plt.figure(figsize=(10,10))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(XD1,XD2,V_dot,cmap = 'plasma')
ax.set_xlabel('x1')
ax.set_xlabel('x2')
plt.title("Variation of V_dot")
plt.show()
```


Variation of V_dot
x2
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0

Exercise 6.

(a) $$x(k+1) = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

$$y(k) = [5 \quad 5]\, x(k)$$

$$SS \rightarrow TF$$

$$G_D(z) = C(zI - A)^{-1} B + D$$

Using MATLAB to solve the above eq^n.

$$G_D = \frac{5}{z-1} - \frac{5}{(2z-1)(z-1)} - \frac{10}{2z-1}$$

$$= \frac{10z - 5 - 5 - 10z + 10}{(2z-1)(z-1)} = 0$$

The poles are $z = \frac{1}{2}$ & $z = 1$

Since the first pole lies inside the unit circle, & the second pole lies on the unit circle (marginally stable), the system is BIBO stable.

(b) $$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

SS $\rightarrow$ TF

$$G_c(s) = C(sI - A)^{-1} B + D$$

Using MATLAB to solve the above

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s^2+2s-7}{s^3+9s^2+23s+15} & \dfrac{-2}{s^2+4s+3} & \dfrac{2(3s+11)}{s^3+9s^2+23s+15} \\[2ex] \dfrac{2}{s^2+8s+15} & \dfrac{1}{s+3} & \dfrac{-2}{s^2+8s+15} \\[2ex] \dfrac{-2}{s^2+4s+3} & \dfrac{-2}{s^2+4s+3} & \dfrac{s+5}{s^2+4s+3} \end{bmatrix}$$

$$= \frac{1}{(s+5)(s+3)(s+1)} \begin{bmatrix} s^2+2s-7 & -2(s+5) & 2(3s+11) \\ 2(s+1) & (s+5)(s+1) & -2(s+1) \\ -2(s+5) & -2(s+5) & (s+5)^2 \end{bmatrix}$$

$\therefore$ the denominator of the transfer fn is

$$\frac{1}{(s+5)(s+3)(s+1)}$$

Hence, the poles are $-1, -3 \& -5$

Hence, ~~the~~ system is BIBO stable.

Exercise 7.

1. $V_C = V_H = 1$, $\beta = 0.2$

$f_C = f_H = 0.1$

$$\therefore \quad \dot{T}_C = 0.1(T_{Ci} - T_C) + 0.2(T_H - T_C)$$

$$\dot{T}_H = 0.1(T_{Hi} - T_H) + 0.2(T_C - T_H)$$

$$\Rightarrow \dot{T}_C = 0.1\,T_{Ci} - 0.3T_C + 0.2T_H$$

$$\& \quad \dot{T}_H = 0.1\,T_{Hi} + 0.2T_C - 0.3T_H$$

$$\boxed{\begin{bmatrix} \dot{T}_C \\ \dot{T}_H \end{bmatrix} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} T_C \\ T_H \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} T_{Ci} \\ T_{Hi} \end{bmatrix}}$$

$\Rightarrow \dot{x} = A x + B u$

$$\boxed{y = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} T_c \\ T_H \end{bmatrix}}$$

2. When $u = 0$

$\therefore y(t) = C e^{At} x(0)$

Now, for calculating e^{At},

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -0.3-\lambda & 0.2 \\ 0.2 & -0.3-\lambda \end{vmatrix} = 0$$

$\therefore \lambda_1 = -0.5 \quad , \lambda_2 = -0.1$

Using C-H theorem,

$e^{\lambda_1 t} = \beta_1 \lambda_1 + \beta_0$

$\Rightarrow e^{-0.5t} = -0.5\beta_1 + \beta_0$

& $e^{\lambda_2 t} = \beta_1 \lambda_2 + \beta_0$

$\Rightarrow e^{-0.1t} = -0.1\beta_1 + \beta_0$

$$\therefore \beta_0 = 1.25e^{-0.1t} - 0.25e^{-0.5t}$$

$$\beta_1 = 2.5e^{-0.1t} - 2.5e^{-0.5t}$$

$$\therefore e^{At} = \beta_1 A + \beta_0 I$$

$$= 0.5\begin{bmatrix} e^{-0.1t} + e^{-0.5t} & e^{-0.1t} - e^{-0.5t} \\ e^{-0.1t} - e^{-0.5t} & e^{-0.1t} + e^{-0.5t} \end{bmatrix}$$

$$\boxed{\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.5e^{-0.1t} + 0.5e^{-0.5t} & 0.5e^{-0.1t} - 0.5e^{-0.5t} \\ 0.5e^{-0.1t} - 0.5e^{-0.5t} & 0.5e^{-0.1t} + 0.5e^{-0.5t} \end{bmatrix} \begin{bmatrix} T_c(0) \\ T_H(0) \end{bmatrix}}$$

3. SS $\rightarrow$ TF

$$G(s) = C(sI - A)^{-1}B + D$$

where $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}$

$$B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

Using MATLAB to solve the above eqn:

$$G_c(s) = \begin{bmatrix} \dfrac{10s+3}{100s^2+60s+5} & \dfrac{2}{100s^2+60s+5} \\ \dfrac{2}{100s^2+60s+5} & \dfrac{10s+3}{100s^2+60s+5} \end{bmatrix}$$

$$= \frac{1}{100s^2+60s+5} \begin{bmatrix} 10s+3 & 2 \\ 2 & 10s+3 \end{bmatrix}$$

$$\Delta s = 100s^2 + 60s + 5$$

$\therefore$ the poles are

$$s = -0.1 \ \& \ -0.5$$

Hence the system is BIBO stable!