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MCT HW-4

Exercise 1.

$$\frac{Y(s)}{U(s)} = \frac{0s^2+s+3}{s^2+3s+2}$$

$a_0 = 2, a_1 = 3, b_0 = 3, b_1 = 1, b_2 = 0$

The controllable canonical form is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

Exercise 2.

$$\hat{G}_1(s) = \begin{bmatrix} 1/s & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix}$$

$$G(\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore G_{sp} = \hat{G}_1(s) - G(\infty)$$

$$= \begin{bmatrix} 1/s & \frac{2}{s+1} \\ 1/s+3 & -1/s+1 \end{bmatrix}$$

$$d(s) = s(s+1)(s+3) = s^3 + 4s^2 + 3s$$

$$\therefore G_{sp} = \frac{1}{s^3+4s^2+3s} \begin{bmatrix} (s+1)(s+3) & 2s(s+3) \\ s(s+1) & -s(s+3) \end{bmatrix}$$

$$N_1(s) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \quad N_2(s) = \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix}$$

$$N_3(s) = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

Now,

$$A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise 3.

CASE-I:-

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 \end{bmatrix} x$$

$$SS \longrightarrow TF$$

$$G_1(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 2 & 2 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore G_1(s) = \frac{2}{s-2}$$

CASE-II:-

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 0 \end{bmatrix} x$$

SS $\longrightarrow$ TF

$$G_2(s) = \begin{bmatrix} 2 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$G_2(s) = \frac{2}{s-2}$$

$\therefore G_1(s) = G_2(s)$

Hence, the two state eq^{n}s are equivalent

Now,

For system 1,

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$|P| = 0$

$\therefore$ it is uncontrollable.

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

$|O| = 0$

$\therefore$ it is unobservable

Hence, eq^n 1 is not a minimal realization

For system 2,

$$P = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$$

$|P| = -7$

$\therefore$ it is controllable

$$O = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$

$|O| = 0$

$\therefore$ it is unobservable.

Hence, eq^n 2 is also not a minimal realization.

Exercise 4.

$$g(s) = \frac{2s-4}{s^3 - 7s + 6}$$

(a) $a_0 = 6$, $a_1 = -7$, $a_2 = 0$

$b_0 = -4$, $b_1 = 2$, $b_2 = 0$

$\therefore$ controllable canonical form is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -4 & 2 & 0 \end{bmatrix} x + [0] u$$

(b) Canonical observable form is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + [0] u$$

(c) Taking Controllable canonical form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -4 & 2 & 0 \end{bmatrix} x$$

$$\therefore \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ -12 & 14 & -4 \end{bmatrix}$$

$\text{rank}(\mathcal{O}) = 2 \rightarrow$ not full rank

Observable decomposition

$$\therefore M_c^{-1} = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 2 \\ 1 & 1/2 & 4 \end{bmatrix}$$

$$\therefore \hat{A} = M_c^{-1} A M_c = \begin{bmatrix} 0 & 1 & 0 \\ 3 & -2 & 0 \\ 1/2 & 0 & 2 \end{bmatrix}$$

$$\therefore A_{co} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\hat{B} = M^{-1} B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore B_{co} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\therefore \hat{C} = CM = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\therefore C_{co} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$\therefore$ The minimal realization is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Exercise 5.

$$\dot{x} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

$\text{rank}(P) = 1 \rightarrow$ not full rank.

$$\therefore M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \hat{A} = M^{-1} A M = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$$

$$\therefore A_c = \begin{bmatrix} 3 \end{bmatrix}$$

$$\hat{B} = M^{-1} B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore B_c = \begin{bmatrix} 1 \end{bmatrix}$$

$\hat{c} = cM = [2 \quad 1]$

$\therefore c_c = 2$

$\therefore$ controllable form is

$\dot{x} = [3]x + [1]u$
$y = [2]x$

Now,
$\mathcal{O} = [c] = [2]$

rank$(\mathcal{O}) = 1$

$\hookrightarrow$ full rank

$\therefore$ the reduced state eq^n is observable.

Exercise 6.

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1 \quad 1 \quad 0 \quad 1]x$$

For λ_1 in row 2 & λ_2 in row 3, rank(P) & rank$(\mathcal{O}) = 1$, i.e, full rank.

Hence, they are controllable & observable.

$\therefore$ the decomposed eq^{n} is,

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Exercise 7.

$$\frac{\Theta(s)}{V(s)} = \frac{k_T}{(Ls+R)(Js^2+bs)}$$

$$= \frac{k_T}{LJs^3 + (Lb+RJ)s^2 + Rbs}$$

$$= \frac{k_T/LJ}{s^3 + \left(\frac{Lb+RJ}{LJ}\right)s^2 + \frac{Rb}{LJ}s}$$

$$a_0 = 0,\ a_1 = \frac{Rb}{LJ},\ a_2 = \frac{Lb+RJ}{LJ}$$

$$b_0 = \frac{k_T}{LJ},\ b_1 = 0,\ b_2 = 0,\ b_3 = 0$$

$\therefore$ Controllable canonical form is,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{Rb}{LJ} & -\frac{Lb-RJ}{LJ} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{k_I}{LJ} & 0 & 0 \end{bmatrix} x + [0] u$$