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MCT HW-2

Exercise 1.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, Eigen values are 1, 0, 1
(since it is a triangular matrix)

$\therefore$ characteristic eqn is

$$\lambda(\lambda-1)(\lambda-1) = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

Now, from C-H

$$f(\lambda) = e^{\lambda t} = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0$$

$$\Rightarrow e^t = \beta_2 + \beta_1 + \beta_0 \quad \text{--- (1)}$$

& $\beta_0 = 1$ --- (2)

& $\frac{d f(\lambda)}{dx} = te^{\lambda t} = 2\beta_2 \lambda + \beta_1$

$\Rightarrow te^{t} = 2\beta_2 + \beta_1$ —③

Solving eq^n s ①, ② & ③

$$
\begin{aligned}
\cancel{\beta_1} + \beta_2 &= e^{t} - 1 \\
\cancel{\beta_1} + 2\beta_2 &= te^{t} \\
- \quad - \quad & \quad - \\
\hline
+\beta_2 &= -e^{t} + 1 + te^{t}
\end{aligned}
$$

$\beta_2 = e^{t}(t-1) + 1$

$\therefore \beta_1 = e^{t} - 1 + e^{t} - 1 - te^{t}$

$= e^{t}(2-t) - 2$

$\therefore \beta_0 = 1$

$\beta_1 = 2e^{t} - te^{t} - 2$

$\beta_2 = te^{t} - e^{t} + 1$

Now,

$\therefore f(A) = e^{At} = \beta_2 A^2 + \beta_1 A + \beta_0 I$

$$= \begin{bmatrix} te^t - e^t + 1 & te^t - e^t + 1 & te^t - e^t + 1 \\ 0 & 0 & te^t - e^t + 1 \\ 0 & 0 & te^t - e^t + 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 2e^t - te^t - 2 & 2e^t - te^t - 2 & 0 \\ 0 & 0 & 2e^t - te^t - 2 \\ 0 & 0 & 2e^t - te^t - 2 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\therefore e^{At} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}}$$

Now, from characteristic eq^n & C-H

$$A^3 - 2A^2 + A = 0$$

$$\therefore A^3 = 2A^2 - A$$

$$\Rightarrow A^4 = 2A^3 - A^2$$

$$= 3A^2 - 2A$$

$$\Rightarrow A^5 = 3A^3 - 2A^2$$
$$= 4A^2 - 3A$$

$$\Rightarrow A^6 = 4A^3 - 3A^2$$
$$= 5A^2 - 4A$$

Continuing the pattern,

$$A^{10} = 9A^2 - 8A$$

$$\boxed{A^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Exercise 2:

$$\dot{x}_1 = -\alpha x_1 + u$$
$$\dot{x}_2 = \alpha x_1 - \beta x_2$$

The system is linear

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\therefore A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}$$

Eigen values are -0.1 & -0.2

$$\therefore f(\lambda) = e^{\lambda t} = \beta_1 \lambda + \beta_0$$

$$\Rightarrow e^{-0.1t} = -0.1 \beta_1 + \beta_0$$

$$\& \quad e^{-0.2t} = -0.2 \beta_1 + \beta_0$$

$$\therefore \beta_0 = 2e^{-0.1t} - e^{-0.2t}$$

$$\beta_1 = 10e^{-0.1t} - 10e^{-0.2t}$$

Now, using C-H

$$e^{At} = \beta_1 A + \beta_0 I$$

$$= \begin{bmatrix} -e^{-0.1t} + e^{-0.2t} & 0 \\ e^{-0.1t} - e^{-0.2t} & -2e^{-0.1t} + 2e^{-0.2t} \end{bmatrix}$$

$$+ \begin{bmatrix} 2e^{-0.1t} - e^{-0.2t} & 0 \\ 0 & 2e^{-0.1t} - e^{-0.2t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}$$

Now,

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

$u = 1, \ t_0 = 0, \ x_1(0) = 2, \ x_2(0) = 1$

$$\therefore x(t) = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$+\int_0^t \begin{bmatrix} e^{-0.1(t-\tau)} & 0 \\ e^{-0.1(t-\tau)}-e^{-0.2(t-\tau)} & e^{-0.2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t}-e^{-0.2t} \end{bmatrix}$$

$$+\int_0^t \begin{bmatrix} e^{-0.1(t-\tau)} \\ e^{-0.1(t-\tau)}-e^{-0.2(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t}-e^{-0.2t} \end{bmatrix} + \begin{bmatrix} 10e^{-0.1(t-\tau)} \\ 10e^{-0.1(t-\tau)}-5e^{-0.2(t-\tau)} \end{bmatrix} \Bigg|_0^t$$

$$= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t}-e^{-0.2t} \end{bmatrix} + \begin{bmatrix} 10-10e^{-0.1t} \\ 5-10e^{-0.1t}+5e^{-0.2t} \end{bmatrix}$$

$$\therefore x(t) = \begin{bmatrix} 10-8e^{-0.1t} \\ 5-8e^{-0.1t}+4e^{-0.2t} \end{bmatrix}$$

at $t=5$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5.14 \\ 1.62 \end{bmatrix}$$

Exercise 3.

1) $A_1 = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

For λ_1, $m_1 = 1$ & $g_1 = 1$

For λ_2, $m_2 = 1$ & $g_2 = 1$

For λ_3, $m_3 = 1$ & $g_3 = 1$

$\therefore$ For v_1,

$$(A_1 - \lambda_1 I) v_1 = 0$$

$$\begin{bmatrix} 0 & 4 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} = 0$$

For Y_2,

$$(A_1 - \lambda_2 I) Y_2 = 0$$

$$\begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}}_{Y_2} = 0$$

For Y_3,

$$(A_1 - \lambda_3 I) Y_3 = 0$$

$$\begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}}_{Y_3} = 0$$

$$\therefore A_1 = MJM^{-1} = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$\det(A-\lambda I)=0$$

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix}\right)=0$$

$$\lambda^3+3\lambda^2+4\lambda+2=0$$

$$\lambda_1=-1,\ \lambda_2=-1+i,\ \lambda_3=-1-i$$

For λ_1, $m_1=1$ & $g_1=1$

For λ_2, $m_2=1$ & $g_2=1$

For λ_3, $m_3=1$ & $g_3=1$

For v_1,

$$(A_2-\lambda_1 I)\, v_1=0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}=0$$

For v_2,

$$(A_2-\lambda_2 I)\, v_2=0$$

$$\begin{bmatrix} 1-i & 1 & 0 \\ 0 & 1-i & 1 \\ -2 & -4 & -2+i \end{bmatrix} \begin{bmatrix} 1 \\ i-1 \\ -2i \end{bmatrix} = 0$$

For Y_3,

$$(A_3 - \lambda_3 I)\, Y_3 = 0$$

$$\begin{bmatrix} 1+i & 1 & 0 \\ 0 & 1+i & 1 \\ -2 & -4 & -2+i \end{bmatrix} \begin{bmatrix} 1 \\ -1-i \\ 2i \end{bmatrix} = 0$$

$$\therefore A_2 = MJM^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & i-1 & -1-i \\ 1 & -2i & 2i \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ -\frac{1}{2}-\frac{1}{2}i & -1-\frac{1}{2}i & -\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2}i & -1+\frac{1}{2}i & -\frac{1}{2} \end{bmatrix}$$

3) $$A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 2$$

For λ_1, $m_1 = 2$ & $q_1 = 2$

For λ_2, $m_2 = 2$ & $q_2 = 2$

For λ_3, $m_3 = 1$ & $q_3 = 1$

For $\lambda_1 = \lambda_2 = 1$,

$$(A_3 - \lambda_1 I)Y_1 = 0$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow Y_1$$

$$\& \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow Y_2$$

For $\lambda_3 = 2$,

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow Y_3$$

$$\therefore A_3 = MJM^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4) $A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \lambda^3 = 0$$

$$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$\therefore m = 3$ & $q = 1$

For v_1,

$$\therefore \underbrace{\begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} = 0$$

For v_2, $(A - \lambda I) v_2 = v_1$

$$\begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_2 + 3x_3 = 1$$
$$20x_2 + 16x_3 = 0$$
$$\therefore V_2 = \begin{bmatrix} 0 \\ 4 \\ -5 \end{bmatrix}$$

For V_3, $(A - \lambda I) V_3 = V_2$

$$\begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$4x_2 + 3x_3 = 0$$
let $x_2 = 3$
$$\therefore x_3 = -4$$
$$\therefore V_3 = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$$

$$\therefore A_4 = M^{-1} J M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & -5 & -4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & -5 & -4 \end{bmatrix}$$

Exercise 4:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t)$$

i)

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\therefore \lambda_1 = -1+i, \quad \lambda_2 = -1-i$$

Now,

$$f(\lambda) = e^{\lambda_1 t} = \beta_1 \lambda_1 + \beta_0$$

$$\Rightarrow e^{(-1+i)t} = \beta_1(-1+i) + \beta_0$$

$$\& \quad e^{\lambda_2 t} = \beta_1 \lambda_2 + \beta_0$$

$$\Rightarrow e^{(-1-i)t} = \beta_1(-1-i) + \beta_0$$

$$\beta_1 = \frac{e^{(-1+i)t} - e^{(-1-i)t}}{2i}$$

Using Euler's formula,

$$e^{ix} = \cos x + i \sin x$$

$$\therefore \beta_1 = e^{-t} \sin t$$

& $$\beta_0 = e^{-t} (\cos t + \sin t)$$

Now,

$$e^{At} = \beta_1 A + \beta_0 I$$

$$= \begin{bmatrix} 0 & e^{-t} \sin t \\ -2e^{-t} \sin t & -2e^{-t} \sin t \end{bmatrix} + \begin{bmatrix} e^{-t}(\cos t + \sin t) & 0 \\ 0 & e^{-t}(\cos t + \sin t) \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-t}(\cos t + \sin t) & e^{-t} \sin t \\ -2e^{-t} \sin t & e^{-t}(\cos t + \sin t) - 2e^{-t} \sin t \end{bmatrix}$$

Now,

$$y(t) = Ce^{A(t-t_0)} x(t_0) + C \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

$$=\begin{bmatrix}2 & 3\end{bmatrix} e^{At}(0) + \begin{bmatrix}2 & 3\end{bmatrix}\int_{t_0}^{t} e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

$$+\,0$$

$$=\begin{bmatrix}2 & 3\end{bmatrix}\int_{t_0}^{t}\begin{bmatrix} e^{-(t-\tau)}[\cos(t-\tau)+\sin(t-\tau)] & e^{-(t-\tau)}\sin(t-\tau) \\ -2e^{-(t-\tau)}\sin(t-\tau) & e^{-(t-\tau)}[\cos(t-\tau)+\sin(t-\tau)] - 2e^{-(t-\tau)}\sin(t-\tau)\end{bmatrix}$$

$$\begin{bmatrix}1\\1\end{bmatrix}u(\tau)\,d\tau$$

$$=\begin{bmatrix}2 & 3\end{bmatrix}\left.\begin{bmatrix} e^{\tau-t}\cos(t-\tau)+\tfrac{1}{2}e^{\tau-t}[\sin(t-\tau)+\cos(t-\tau)] \\ e^{\tau-t}[\sin(t-\tau)+\cos(t-\tau)]+e^{\tau-t}\cos(t-\tau)-e^{\tau-t}[\sin(t-\tau)+\cos(t-\tau)]\end{bmatrix}\right|_{0}^{t}$$

$$=\begin{bmatrix}2 & 3\end{bmatrix}\begin{bmatrix} e^{-t}+\tfrac{1}{2}-\cos t-\tfrac{1}{2}e^{-t}[\sin t+\cos t] \\ 1-e^{-t}[\sin t+\cos t]-e^{-t}\cos t + e^{-t}[\sin t+\cos t]\end{bmatrix}$$

at $t=5$

$$y(5) = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1.5 \\ -1.01 \end{bmatrix}$$

$$= -0.03$$

ii) $T = 1s$

$$A_d = e^{AT} = e^{-T} \begin{bmatrix} \cos T + \sin T & \sin T \\ -2\sin T & \cos T - \sin T \end{bmatrix}$$

at $T = 1$, $A_d = \begin{bmatrix} 0.508 & 0.309 \\ -0.62 & -0.11 \end{bmatrix}$

$$B_d = A^{-1}(e^{AT} - I)\,B$$

$$= \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -0.492 & 0.309 \\ -0.62 & -1.11 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 1.048 \\ -0.183 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \end{bmatrix}, \quad D = 0$$

$$\therefore x(k+1) = \begin{bmatrix} 0.508 & 0.309 \\ -0.62 & -0.11 \end{bmatrix} x(k) + \begin{bmatrix} 1.048 \\ -0.183 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(k)$$

iii) $\because y(k) = CA_d^{k} x(0) + \sum_{m=0}^{k-1} CA_d^{k-m-1} B_d u(m) + Du(k)$

$\because x(0) = 0$ & $D = 0$

$$\therefore y(5) = \begin{bmatrix} 2 & 3 \end{bmatrix} \left(A_d^4 + A_d^3 + A_d^2 + A_d + I \right) \begin{bmatrix} 1.048 \\ -0.183 \end{bmatrix}$$

$$= -0.03$$

Exercise 5:

$$F_{k+2} = F_{k+1} + F_k$$

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

$$\therefore \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find eigen values of A,

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda + x^2 - 2 = 0$$

$$\therefore \lambda_1 = \frac{1-\sqrt{5}}{2}, \lambda_2 = \frac{1+\sqrt{5}}{2}$$

Using C-H,

$$\lambda^{19} = \beta_1 \lambda + \beta_0$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^{19} = \beta_1 \left(\frac{1-\sqrt{5}}{2}\right) + \beta_0$$

$$\& \left(\frac{1+\sqrt{5}}{2}\right)^{19} = \beta_1 \left(\frac{1+\sqrt{5}}{2}\right) + \beta_0$$

$$\therefore \beta_0 = 2584, \beta_1 = 4181$$

Now,

$$A^{19} = \beta_1 A + \beta_0 I$$

$$= 4181 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 2584 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{19} = \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix}$$

$$\therefore \begin{bmatrix} F_{21} \\ F_{20} \end{bmatrix} = \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10946 \\ 6765 \end{bmatrix}$$

$$\boxed{\therefore F_{20} = 6765}$$