The Mathematics behind Dynamic Collateralization

Mosaic protocol

Preamble

The Bernoulli Distribution: A Rigorous Framework for Binary Experiments

The Bernoulli distribution, a cornerstone of discrete probability theory, elegantly captures the essence of a single binary experiment. Named after the illustrious Jakob Bernoulli, this distribution provides a powerful framework for analyzing scenarios with precisely two possible outcomes.

Formally, we define a Bernoulli trial as an experiment yielding exactly two outcomes: success, denoted by 1, and failure, denoted by 0. The critical aspect lies in the associated probabilities. Let p represent the probability of success, where $0 \leq p \leq 1$. Consequently, $q = 1 - p$ signifies the probability of failure, ensuring a complete probability space.

A random variable X is said to follow a Bernoulli distribution with parameter p if its probability mass function (PMF) is given by:

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ q & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

This equation embodies the core principle of the Bernoulli distribution. It asserts that the probability of success ($x = 1$) is governed by p, while the probability of failure ($x = 0$) is determined by q. Any other outcome for X has a probability of zero.

The intuitive appeal of the Bernoulli distribution lies in its wide range of applications. Consider the following classic examples:

* **Coin Flipping:** Imagine a fair coin toss. Heads represents success ($x = 1$) with probability $p = 0.5$, and tails represents failure ($x = 0$) with probability $q = 1 - p = 0.5$. * **Die Rolling:** Rolling a single die, where success ($x = 1$) denotes rolling a specific number (e.g., 6) with probability p, and failure ($x = 0$) encompasses all other possibilities with probability $q = 1 - p$.

However, the power of the Bernoulli distribution extends beyond these elementary examples. It serves as a fundamental building block for more complex probability distributions, particularly the Binomial distribution. The Binomial distribution models a sequence of independent Bernoulli trials, allowing us to analyze the probability of obtaining a specific number of successes within this series.

Properties and Characterizations

The Bernoulli distribution possesses several noteworthy properties that solidify its theoretical foundation.

Theorem 1. *The expected value, also known as the mean, of a random variable X following a Bernoulli distribution with parameter p is:*

$$E[X] = p \cdot 1 + q \cdot 0 = p \tag{2}$$

This signifies that the expected value reflects the average outcome of a large number of Bernoulli trials, aligning with the intuitive notion that the average success rate over numerous trials converges towards the probability of success in a single trial (p).

Theorem 2. *The variance of X is given by:*

$$Var[X] = p(1 - p) = pq \tag{3}$$

This showcases the inherent variability associated with the Bernoulli distribution. A higher variance implies a greater spread of potential outcomes around the expected value.

Theorem 3. *The Moment Generating Function (MGF) of X is:*

$$M_X(t) = pe^t + qe^0 = p(e^t) + q \tag{4}$$

The MGF offers a valuable tool for generating moment-based characterizations of the distribution, enabling the derivation of higher-order moments.

Theorem 4. *The Characteristic Function (CF) of X is:*

$$\phi_X(t) = pe^{it} + qe^0 = p\cos(t) + q + ip\sin(t) \tag{5}$$

The CF provides a unique way to identify the distribution based on its Fourier transform properties.

Connection to Indicator Functions

The Bernoulli distribution enjoys a strong connection with indicator functions. An indicator function, denoted by $\mathbb{I}(A)$, takes a value of 1 if the event A occurs and 0 otherwise. In the context

The Poisson Distribution: A Rigorous Framework for Count-Based Events

The Poisson distribution is a fundamental discrete probability distribution that describes the likelihood of a given number of events occurring within a fixed interval of time or space. Named after the French mathematician Siméon Denis Poisson, this distribution is particularly useful for modeling events that occur independently and with a known constant mean rate.

Formally, we define a random variable X to follow a Poisson distribution with parameter $\lambda > 0$, where λ represents the average number of occurrences in the given interval. The probability mass function (PMF) of a Poisson-distributed random variable X is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots \tag{6}$$

This equation captures the essence of the Poisson distribution, asserting that the probability of observing k events in the interval is determined by λ.

The intuitive appeal of the Poisson distribution lies in its wide range of applications. Consider the following classic examples:

Call Centers: The number of phone calls received by a call center per hour can be modeled using a Poisson distribution, where λ is the average call rate. **Traffic Flow:** The number of cars passing through a toll booth in an hour can be described by a Poisson distribution, with λ representing the average traffic rate.

Properties and Characterizations

The Poisson distribution possesses several noteworthy properties that solidify its theoretical foundation.

Theorem 5. *The expected value, or mean, of a random variable X following a Poisson distribution with parameter λ is:*

$$E[X] = \lambda \tag{7}$$

This signifies that the expected number of events in the given interval is λ.

Theorem 6. *The variance of X is given by:*

$$Var[X] = \lambda \tag{8}$$

This indicates that both the mean and the variance of the Poisson distribution are equal, reflecting the inherent property of this distribution.

Theorem 7. *The Moment Generating Function (MGF) of X is:*

$$M_X(t) = \exp(\lambda(e^t - 1)) \tag{9}$$

The MGF provides a powerful tool for deriving the moments of the distribution, facilitating deeper statistical analyses.

Theorem 8. *The Characteristic Function (CF) of X is:*

$$\phi_X(t) = \exp(\lambda(e^{it} - 1)) \tag{10}$$

The CF offers an alternative perspective on the distribution, leveraging the properties of the Fourier transform.

8.1 Connection to the Exponential Distribution

The Poisson distribution is closely related to the Exponential distribution. Specifically, if the number of events in a Poisson process follows a Poisson distribution with rate λ, then the inter-arrival times between consecutive events follow an Exponential distribution with parameter λ. This connection underscores the versatility and utility of the Poisson distribution in modeling real-world phenomena.

Philosophy

In our application, the contribution of a single user towards the collateral follows a Bernoulli distribution. Specifically, if we denote a user's contribution by a random variable X, it takes the value 1 (contributing) with probability p and 0 (not contributing) with probability $1 - p$. As the number of users increases, the cumulative effect of their contributions tends towards a Poisson distribution due to the law of rare events.

Formally, if X_i for $i = 1, 2, \ldots, n$ are independent Bernoulli random variables each with parameter $p = \frac{\lambda}{n}$, then the sum $S_n = \sum_{i=1}^{n} X_i$ approximates a Poisson distribution with parameter λ as

$n \to \infty$. For our purposes, we consider $\lambda = \frac{1}{2}$, thus modeling the aggregate contribution of users as:

$$S_n \sim \text{Poisson}\left(\frac{1}{2}\right) \tag{11}$$

Dynamic Collateralization Curve

To manage collateral dynamically, we utilize a curve that adapts based on the number of contributing users. The dynamic curve is given by:

$$f(x) = 1 + x + \frac{x^2}{2} \tag{12}$$

This curve reflects the accumulation of contributions where x represents a normalized measure of the total contributions. The choice of this specific form ensures that the growth rate of collateral is smooth and manageable.

Our dynamic collateral ratio is then formulated as:

$$R(x) = C \cdot \frac{e^x}{1 + x + \frac{x^2}{2}} \tag{13}$$

where C is a constant and x is expected to remain less than 1. This ensures that the ratio $R(x)$ does not change significantly, providing stability to the system.

Justification and Implications

The transition from a Bernoulli to a Poisson framework allows for the effective aggregation of individual contributions into a coherent probabilistic model. The Poisson distribution, with its parameter $\lambda = \frac{1}{2}$, encapsulates the average contribution rate of users.

The dynamic collateral curve $1 + x + \frac{x^2}{2}$ offers a quadratic adjustment to the exponential growth e^x, balancing rapid initial increases with controlled subsequent growth. This ensures that the collateral ratio $R(x)$ remains stable, even as x increases, providing a robust mechanism for dynamic collateralization in our system.