Study Oriented Project

MATH F266

A Project Report on

"An Approach to Prove the Sendov Conjecture"

Submitted to:

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By

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ABSTRACT

Determining the roots of polynomials, or ”solving algebraic equations”, is among the oldest problems in mathematics. Factorizing becomes increasingly difficult as the degree increases. Roots of higher degree polynomials are found through numerical computations and approximation using various results. Location of roots and critical points plays a key role in calculus and hence all its branches. An important example of application is stability criteria for control systems.

The building point of the argument of location of roots and critical points begins from the Gauss Lucas Theorem and has modified and sharpened by various results and has perhaps arrived at the Sendov Conjecture. In this report, we look to explore an approach to prove the conjecture using the Classical Theorem of Grace.

Keywords: Polynomials, Roots, Critical Points, Gauss-Lucas Theorem, Sendov Conjecture, Grace Theorem

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Chapter 1

Introduction

1.1 Theorem Statement and History

"The conjecture of Bl. Sendov, better known as Illief's conjecture, states that if all zeros of a polynomial $z \mapsto P(z)$ lie in the unit disk $|z| \leq 1$ and if z_0 is any one such zero, then the disk $|z - z_0| \leq 1$ contains at least one zero of $z \mapsto P'(z)$".

The conjecture was first brought to attention in 1962 at the International Congress of Mathematicians in Stocholm and published in Hayman's "Research Problems in Function Theory" in 1967 where it was wrongly attributed to Illief and hence came to be popularized by that name itself. As of date, however, it has only been proved for a few special cases for polynomials having degree at most eight. For polynomials of degree three, it was proved by Saff and Twomey, Brannan, Schmeisser. A proof for polynomials with at most four zeroes was given by Cohen and Smith. Brown and Xiang proved the conjecture for degree less than nine in 1999. Dégot has proven the conjecture for large n but this proof requires additional conditions. But the general case still remains open in spite of 80 papers devoted to it.

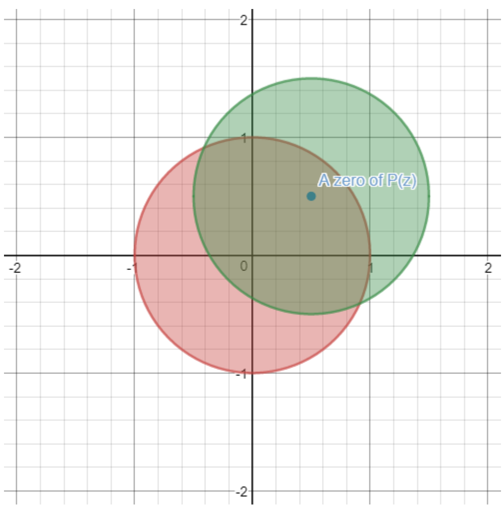

Figure 1.1: Conjecture states that at least one critical point of P(z) lies in the blue region.

1.2 Physical and Geometric Interpretations

1.2.1 Physical Interpretation

Consider a system of point masses under a certain force field where critical points of the polynomial which are not multiple zeros act as equilibrium points of the field. The field is due to the particles placed at zeros of the polynomial, having mass equal to the multiplicity of the zero and attracting with a force inversely proportional to the distance from the particle. The conjecture implies that if all these particles of unit masses and situated on the disk $|z| \leq 1$, then at least one equilibrium point will lie within the unit disk of each particle.

1.2.2 Geometric Interpretation

It is known that the critical points ζ_1 and ζ_2 ($\neq z_1, z_2, z_3$) of the polynomial

$$\text{P(z)} = (z-\zeta_1)^{m_1}(z-\zeta_2)^{m_2}(z-\zeta_3)^{m_3}$$

lie at the foci of the ellipse which touches the line segments (z_1, z_2), (z_2, z_3) and (z_3, z_1) in the points that divide these segments in the ratios m_1/m_2, m_2/m_3, m_3/m_1 respectively. The conjecture implies that if the vertices of the triangle z_1, z_2, z_3 all lie in the unit disk $|z| \leq 1$, each vertex is within unit distance from one of the foci of the inscribed ellipse.

A similar interpretation can be given for the critical points of the polynomial

$$\text{P(z)} = (z-z_1)(z-z_2)\dots(z-z_n)$$

as the foci of a curve Γ of class n, which is tangent to the sides of the polygon with vertices at the points z_j; j = 1, 2, …, n. The conjecture implies that within unit distance of each vertex lies at least one focus of Γ ".

Chapter 2

Building Blocks

2.1 Theorems and Results Used

2.1.1 Gauss-Lucas Theorem

Theorem 1: "All the critical points of a non-constant polynomial P lie in the convex hull H of the set of zeros of P. If the zeros of P are not collinear, no critical point of P lies on the boundary ∂H of H unless it is a multiple zero of P."
Since the class of polynomials under consideration for the conjecture are only those whose roots lie in the disk $|z| \leq 1$, all critical points must also lie in the same disk.

2.1.2 Apolar Polynomials

Consider two polynomials

$$\text{p(z)=} \sum_{k=0}^{n} \binom{n}{k} A_k\, z^k, \quad \text{q(z)=} \sum_{k=0}^{n} \binom{n}{k} B_k\, z^k, \quad A_n B_n \neq 0$$

p(z) and q(z) are said to be apolar if their coefficients satisfy the following condition:

$$\text{A(p,q):} \ \sum_{k=0}^{n} (-1)^k \binom{n}{k} A_k\, B_{n-k} = 0$$

Theorem 2: **(Classical Theorem of Grace)** If p(z) and q(z) are apolar polynomials and if one of them has all its zeros in a circular region C, then the other will have at least one zero in C.

Chapter 3

Approach for the Proof

3.1 Inspiration

After reading the Grace's Theorem, it becomes clear as how to proceed further. The objective now becomes to construct a polynomial q(z) of degree n-1, which is apolar to P'(z) such that all zeros of q(z) lie inside the unit disk $|z - z_0| \leq 1$, where z_0 is any zero of P(z). Note that P'(z) is the first derivative of P(z) and all zeros of P(z) lie inside the disk $|z| \leq 1$. The Grace Theorem implies that at least one zero of P'(z) (critical point of P(z)) lies in the disk $|z - z_0| \leq 1$. Hence, Sendov's conjecture is complete.

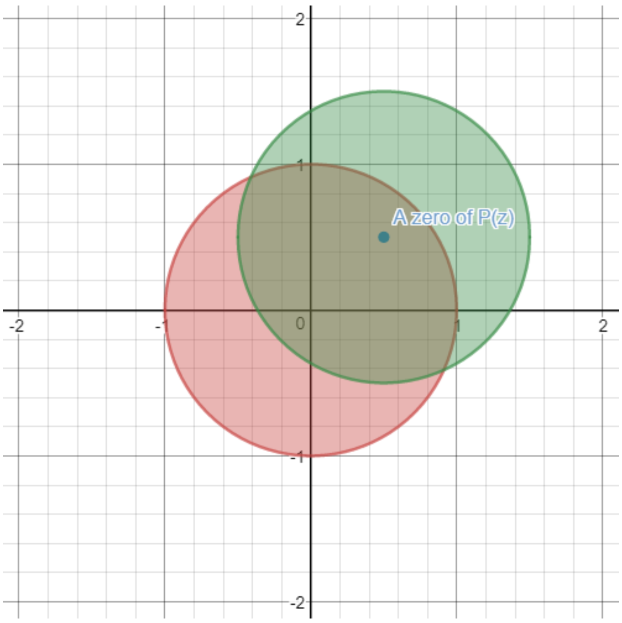

Figure 3.1: Theorem 1 states that all critical points of P(z) lie in the pink region. Combined with the conjecture, it means that at least one critical point lies in the overlapping region.

3.1.1 Construction of Apolar Polynomial

In this report, we will try to closely observe the construction for polynomial of degree four and will look to extend and generalize the procedure.

$$\text{P(z)} = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$
$$\Rightarrow P'(z) = 4\text{a}_4 z^3 + 3a_3 z^2 + 2a_2 z + a_1$$

Let z_0 be a zero of P(z), $|z_0| \leq 1$.
Converting P'(z) to Apolar form by multiplying and dividing the k^{th} term by $\binom{3}{k}$ which gets us,

$$\text{P'(z)} = \binom{3}{3} A_3 z^3 + \binom{3}{2} A_2 z^2 + \binom{3}{1} A_1 z + \binom{3}{0} A_0$$

where $A_3 = 4a_4$, $A_2 = a_3$, $A_1 = 2a_2/3$, $A_0 = a_1$

Construct q(z) as follows

$$\text{q(z)} = (z - z_0/\alpha)(z - z_0/\beta)(z - z_0/\gamma)$$

where α, β, γ are our control parameters with which we can set zeros of q(z) relative to z_0.
Expanding the terms and multiplying and dividing k^{th} term of q(z) with $\binom{3}{k}$ to get Apolar form,

$$\text{q(z)} = \binom{3}{3} B_3 z^3 + \binom{3}{2} B_2 z^2 + \binom{3}{1} B_1 z + \binom{3}{0} B_0 \qquad \text{where,}$$

$$B_3 = 1 \qquad B_2 = z_0(1/\alpha + 1/\beta + 1/\gamma)/3$$
$$B_1 = z_0^2(1/\alpha\beta + 1/\beta\gamma + 1/\gamma\alpha)/3 \qquad B_0 = z_0^3/\alpha\beta\gamma$$

Applying the apolar condition to P'(z) and q(z),

$$\text{A(P',q)}: \sum_{k=0}^{3} (-1)^k \binom{3}{k} A_k \, B_{n-k} = 0$$

On simplifying further, we get

$$4a_4 z_0^3 + 2a_2 z_0(\alpha\beta + \beta\gamma + \gamma\alpha)/3 = a_1 \alpha\beta\gamma + a_3 z_0^2(\alpha + \beta + \gamma)$$

Using Mathematica to solve the equation for α, β and γ

In[1]:= `Solve[4 a4 z0^3 + 2 a2 z0 (α β + β γ + α γ) / 3 == a1 α β γ + a3 z0^2 (α + β + γ), {α, β, γ}]`

Solve: Equations may not give solutions for all "solve" variables.

Out[1]=

$$\left\{\left\{\gamma \to \frac{z0\,(12\,a4\,z0^2 - 3\,a3\,z0\,\alpha - 3\,a3\,z0\,\beta + 2\,a2\,\alpha\,\beta)}{3\,a3\,z0^2 - 2\,a2\,z0\,\alpha - 2\,a2\,z0\,\beta + 3\,a1\,\alpha\,\beta}\right\},\right.$$

$$\left\{\alpha \to \frac{1}{2\,(4\,a2^2 - 9\,a1\,a3)}\left(6\,a2\,a3\,z0 - 36\,a1\,a4\,z0 + \sqrt{(-6\,a2\,a3\,z0 + 36\,a1\,a4\,z0)^2 - 4\,(4\,a2^2 - 9\,a1\,a3)\,(9\,a3^2\,z0^2 - 24\,a2\,a4\,z0^2)}\right),\right.$$

$$\beta \to \frac{1}{4\,a2^2 - 9\,a1\,a3}\left(6\,a2\,a3\,z0 - \frac{12\,a2^3\,a3\,z0}{4\,a2^2 - 9\,a1\,a3} + \frac{27\,a1\,a2\,a3^2\,z0}{4\,a2^2 - 9\,a1\,a3} - 36\,a1\,a4\,z0 + \frac{72\,a1\,a2^2\,a4\,z0}{4\,a2^2 - 9\,a1\,a3} - \frac{162\,a1^2\,a3\,a4\,z0}{4\,a2^2 - 9\,a1\,a3} - \frac{2\,a2^2\sqrt{(-6\,a2\,a3\,z0 + 36\,a1\,a4\,z0)^2 - 4\,(4\,a2^2 - 9\,a1\,a3)\,(9\,a3^2\,z0^2 - 24\,a2\,a4\,z0^2)}}{4\,a2^2 - 9\,a1\,a3} + \frac{9\,a1\,a3\sqrt{(-6\,a2\,a3\,z0 + 36\,a1\,a4\,z0)^2 - 4\,(4\,a2^2 - 9\,a1\,a3)\,(9\,a3^2\,z0^2 - 24\,a2\,a4\,z0^2)}}{2\,(4\,a2^2 - 9\,a1\,a3)}\right)\Bigg\},$$

$$\left\{\alpha \to \frac{1}{2\,(-4\,a2^2 + 9\,a1\,a3)}\left(-6\,a2\,a3\,z0 + 36\,a1\,a4\,z0 + \sqrt{(-6\,a2\,a3\,z0 + 36\,a1\,a4\,z0)^2 - 4\,(4\,a2^2 - 9\,a1\,a3)\,(9\,a3^2\,z0^2 - 24\,a2\,a4\,z0^2)}\right),\right.$$

$$\beta \to \frac{1}{4\,a2^2 - 9\,a1\,a3}\left(6\,a2\,a3\,z0 + \frac{12\,a2^3\,a3\,z0}{-4\,a2^2 + 9\,a1\,a3} - \frac{27\,a1\,a2\,a3^2\,z0}{-4\,a2^2 + 9\,a1\,a3} - 36\,a1\,a4\,z0 - \frac{72\,a1\,a2^2\,a4\,z0}{-4\,a2^2 + 9\,a1\,a3} + \frac{162\,a1^2\,a3\,a4\,z0}{-4\,a2^2 + 9\,a1\,a3} - \frac{2\,a2^2\sqrt{(-6\,a2\,a3\,z0 + 36\,a1\,a4\,z0)^2 - 4\,(4\,a2^2 - 9\,a1\,a3)\,(9\,a3^2\,z0^2 - 24\,a2\,a4\,z0^2)}}{-4\,a2^2 + 9\,a1\,a3} + \frac{9\,a1\,a3\sqrt{(-6\,a2\,a3\,z0 + 36\,a1\,a4\,z0)^2 - 4\,(4\,a2^2 - 9\,a1\,a3)\,(9\,a3^2\,z0^2 - 24\,a2\,a4\,z0^2)}}{2\,(-4\,a2^2 + 9\,a1\,a3)}\right)\Bigg\}\Bigg\}$$

Figure 3.2: Solution of the Apolar Condition

Only the highlighted solution is relevant, note that it is an equation in three variables, and only this solution is dependent on other two variables. This means that we can choose α and β arbitrarily such that α, β and γ satisfy the constraint of q(z) having all its zeros inside the disk $|z - z_0| \leq 1$, i.e,

$$|z_0 - z_0/\alpha| \leq 1,\ |z_0 - z_0/\beta| \leq 1 \text{ and } |z_0 - z_0/\gamma| \leq 1$$

Numerically approach values of α β and γ over all sets of coefficients of P(z) such that zeros lie inside the unit disk centered at origin.

Chapter 4

Conclusion and Notes

Firstly, this procedure is not limited to polynomials of fourth degree. On observing the equation which sets the condition of apolarity of q(z), we can see that it will be symmetric in higher degree polynomials. Note how the second term is the cyclic sum of product of every pair of parameters, first term on the R.H.S. is product of parameters and the next term is sum of parameters. There will be similar terms and more combination of sum of products for higher degree polynomials. In each case, there will be n-1 parameters and n-2 will be under our control.

Without loss of generality, a_n can be taken to be 1, this will make the analysis simpler to compute.

This proof methodology opens the question for the possibility of existence of contrary Sendov's conjecture, i.e existence of at least one critical point outside the disk $|z-z_0| \leq 1$. This question is worth pursuing as hinted by Dégot in his paper on "Sendov Conjecture for High Degree Polynomials" which concludes that the conjecture is true for a degree greater than N, which depends on the choice of zero about which we construct the disk. Choose α and β such that q(z) is apolar to P'(z) but all zeros of q(z) lie outside the disk $|z-z_0| \leq 1$.

Before pursuing the path of apolar polynomials to prove the conjecture, there were other similar ideas through which a constructional proof seemed possible.
1. Using Rouche's Theorem, construct q(z) having at least one zero inside C,
C: $|z-z_0| \leq 1$, such that $|P'(z)-q(z)| \leq |P'(z)|$ on C, then it follows that q(z) and P'(z) have same number of zeros inside C.
2. Exploring Jensen's Theorem for real polynomials can also lead to sharpening the disk of $|z| \leq 1$ for location of critical points.

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