GRE Math: Revision Notes

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1 Calculus

$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))|\frac{dr}{dt}|dt$

Flux across a smooth closed plane curve:

Flux of $F = M\hat{\imath} + N\hat{j}$ across $C = \oint_C Mdy - Ndx$ (hint to remember: $\begin{vmatrix} M & N \\ dx & dy \end{vmatrix}$

calculation: reparametrize in t and use above relation **OR use Green's**

Flux over any surface:

$\iint_S F\cdot n d\sigma = \iint_R (F \cdot \frac{\nabla(f)}{|\nabla(f)|})\frac{\nabla(f)}{|\nabla(f)\cdot\rho|}dxdy$ (ρ is normal of shadow of surface on X-Y.)

Green's Theorem: Flux (single integral) to double integral

$\oint_C Mdy - Ndx = \iint_R (\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})dxdy$

(for this to work, first partials should be continous, C is simple closed.)

Surface area:

$f(x,y,z) = c$

Area of this level surface over a closed and bounded plane R is given by

$\iint_R \frac{|\nabla(f)|}{|\nabla(f)\cdot\rho|}dA$

if some value needs to be evaluated over the surface when g is the density function over the surface then it is given by:

$\iint_R g(x,y)\frac{|\nabla(f)|}{|\nabla(f)\cdot\rho|}dA$

Curl:

$$\nabla\times F = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Stoke's Theorem:

$\int_C f \cdot dr = \iint_S \nabla \times F \cdot nd\sigma$ $(d\sigma = \frac{|\nabla(f)|}{|\nabla(f)\cdot\rho|}dA)$

Divergence Theorem:

$\iint_S \nabla \times F \cdot nd\sigma = \iiint_D \nabla \cdot FdV$

Coordinate Systems:

Cartesian (x,y,z)

Cylindrical $(r, \theta, z) - (rcos\theta\ , rsin\theta, z)$

Spherical (ρ,ϕ,θ) $-(\rho sin\phi cos\theta, \rho sin\phi sin\theta, \rho cos\phi)$

Jacobian

J(u,v,w)= $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

something to remember:

$\int_{-\infty}^{\infty} e^{x^2} dx = \sqrt{\pi}$

$x^x = e^{xlnx}$ (helps in differentiating)

Max-Min

Critical point if: $f_x = f_y = 0$ or if one of the partials does not exist.

Hessian, $H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

1. local maxima at (a,b) if $f_{xx}(a,b) < 0$ and $H > 0$
2. local minima at (a,b) if $f_{xx}(a,b) > 0$ and $H > 0$
3. Saddle point if $H < 0$
4. Inconclusive if $H = 0$

For absolute extrema, check boundary vaues as well.

Lagrange Multiplier Method:

maximize **f** subject to constraints $g_i = 0$; $i = 1, 2, 3, ..., n$

concept: at maximum rate of growth of $f = c_i$,

$\nabla f \cdot \mathrm{v}_c = 0$ and since $\nabla g \cdot v_c = 0$ (because gradient is perpendicular to tangent(velocity))

gradient f and gradient g must be parallel:

$\nabla f = \lambda \nabla g$ and for more than one constraint, we can write:

$\nabla f = \sum_{i=1}^{n} \lambda_i \nabla g_i \ \ldots (1)$

$g_i = 0 \ldots (2)$

solve (1) and (2) to maximize f subject to g.

Tangent Plane at P_o on a level curve $f(x,y,z)=c$:

$f_x|_{P_o}(x-x_o)+f_x|_{P_o}(y-y_o)+f_z|_{P_o}(z-z_o)=0$

Linearization: $L(x,y)=f(x_o,y_o)+f_x(x_o,y_o)(x-x_o)+f_y(x_o,y_o)(y-y_o)$

Error in Linearization: $E(x,y)\leq\frac{1}{2}M(|x-x_o|+|y-y_o|)^2$ where M is, $\max(f_{xx},f_{yy},f_{xy})$

Mixed Partial Theorem: If first partials exist and continuous, then $f_{xy}=f_{yx}$.

$f'(x,y)=\lim_{h\to0,k\to0}(f(x+h,y+k)-f(x,y)-hf_x-kf_y)/(\sqrt{h^2+k^2})=0$

Continuity of partials $\implies$ Differenitiability

Differentiability $\implies$ Continuity.

Chain Rule for 3 independent variables

$\frac{df}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}+f_z\frac{dz}{dt}$

Directional Derivative: $\nabla f\cdot\mathbf{u}$; $\mathbf{u}$ is unit vector.

Normal Line of Level curve at P_o:

$x=x_o+f_x|_{P_o}t$

$y=y_o+f_y|_{P_o}t$

$z=z_o+f_z|_{P_o}t$

Linear Differential Equation:

$y'+P(x)y=Q(x)$

solution: $y\cdot e^{\int Pdx}=\int(Q\cdot e^{\int Pdx})dx+C$

Leibniz Rule of differentiation over an integral:

$g(x)=\int_{a(x)}^{b(x)}f(x,t)dt$

*****important** $g'(x)=f(x,b(x))b'(x)-f(x,a(x))a'(x)+\int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)dt$

2 Complex

Necessary and Sufficient Condition for differentiability of $f(z)=u(x,y)+iv(x,y)$

1. CRE:

$u_x=v_y$ and $u_y=-v_x$

2. u_x, u_y, v_x, v_y are all continuous.

CRE in polar form:

$ru_r=v_\theta$ and $u_\theta=-rv_r$

A typical question on this is when f is given to be analytic and u is given and v is unknown v(a,b)=c and we are asked to find v(p,q).
use CRE to find v_x and v_y and then integrate v_y wrt to y to find v in the form:
v=$\int udy$+$\phi(x)$ and then use v_x to find $\phi(x) + C$ and evaluate C using v(a,b)=c
Harmonic:
H(x,y) is harmonic if throughout the domain if first partials exist and are continuous and satisfy the Laplace equation:
$H_{xx} + H_{yy} = 0$
If f is analytic, u and v are harmonic.
If u is harmonic, then u is the real part of an analytic function.
Line Integral:
$\int_C \ f(z)dz = \int_a^b f(z(t))z'(t)dt$; C is anticlockwise oriented
simple curve is also called a Jordan Curve
ML Inequality:
$|\int_C \ f(z)dz| \leq M \int_a^b |z'(t)|dt$; M= max$|f(z)|$ over the domain of integration
Cauchy Goursat Theorem:
$\oint_C f(z)dz = 0$ when f is analytic inside C.

there are some results related to this when a curve is oriented clockwise inside an anti clockwise curve.

Liouville's Theorem:
I don't think the theorem is required for this test but a result which follows from the theorem is strong:
If f is entire and bounded in the complex plane, then f(z) is constant throughout the domain.
Cauchy Residue Theorem:
$\int_C \ f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z)$
Types of Singular Points

1. Non-Isolated
2. Isolated
   a. Removable- If in the Laurent series expansion of the function there are no negative powers.
   b. Pole- If in the Laurent series expansion of the function there are finite negative powers.
   c. Essential- If in the Laurent series expansion of the function there are infinite negative powers.

Order of pole is the number of negative terms in the Laurent series expansion. Pole of order 1 is called a simple pole.

Residue Calculation for simple pole:

If f(z) is of the form f(z)=p(z)/q(z)

Let z_o be a zero of q(z)

$\implies q(z_0) = 0$

and if $q'(z_o) \neq 0$

$\implies f(z_{zo})$ has a pole of order 1 at z_o

$\underset{z=z_0}{Res} f(z) = p(z_o)/q'(z_o)$

3 Linear Algebra

System has unique solution if

$Rank(A) = R(A|b) = n$=number of variables ($A|b$ is the augmented matrix)

No solution if

R(A)$< R(A|b)$

Infinite Solutions if

$R(A) = R(A|b) < n$ (number of unknowns)

If A is invertible then:

$\exists$ B such that $AB = BA = I$

$|A| \neq 0$

$(AB)^{-1} = B^{-1}A^{-1}$

$|P^{-1}| = 1/|P|$

$(A^{-1})^T = (A^T)^{-1}$

A quick note on calculation of Adjoint of a 2×2 matrix:

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ then } Adj(A) = \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

To prove that a non-empty subset is a vector subspace, then show that:

It is closed under vector operation and it is closed under scalar multiplication.

Let S_1 and S_2 be two finite subsets of a vector space V and let $S_1 \subset S_2$ then:

If S_1 is linearly dependent $\implies S_2$ is also.

If S_2 is linearly independent $\implies S_1$ is also.

A linearly independent subset of V us called the Basis, β of V.

The number of elements in β is the dimension of V.
Dimension of $\{0\}$ is 0.

Eigenvalues:
$|A - \lambda I| = 0$
A and A^T have the same eigenvalues.
If λ is the eigenvalue of A, then the eigenvalue of A^2 is λ^2 and so on with exponents.
$1/\lambda$ is the eigenvalue of A^{-1}.
$det(A) = \prod_{i=1}^{n} \lambda_i$
$Trace(A) = \sum_{i=1}^{n} \lambda_i$

A and B are similar if $\exists$ a non-singular matrix P such that $A = P^{-1}BP$.
Similar matrices have same eigenvalues and characteristic polynomials.

It may happen that repeated eigenvalues result in linearly independent eigenvectors. The following example illustrates how to find them:

$$A = \begin{vmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 24 \end{vmatrix} ; \lambda = 1, 2, 2$$

For $\lambda = 2$

$$[A - \lambda I|0]$$

$$= \begin{vmatrix} 1 & -4/3 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{vmatrix}$$

$\implies x - 4y/3 + 2z = 0$

E_2 = {(x,y,z) | above equation holds}
= {(4y/3-2z, y, z) | y,z $\in \mathbb{R}$}
= {y(4/3, 1, 0), z(-2, 0, 1) | y,z $\in \mathbb{R}$}
the two vectors are independent and span the solution space and another vector can be obtained from $\lambda = 0$. So we have three linearly independent eigenvectors.

Diagonalization:

P is a matrix formed by the columns of eigenvectors and if it is non-singular (eigenvectors are linearly independent) then the diagonalized matrix of A is given by:

$$D = P^{-1}AP = \begin{vmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & \lambda_n \end{vmatrix}$$

$A^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}D^2P$

$\vdots$

$$A^n = (P^{-1}AP)\dots(P^{-1}AP) = P^{-1}D^nP \text{ and } D^n = \begin{vmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & \lambda_n^n \end{vmatrix}$$

Diagonalizable matrix may not be invertible, for example if some eigenvalue is zero then det(A)=$\prod_{i=1}^{n} \lambda_i = 0 \implies$ singular.

Invertible matrix may not be diagonalizable. example (1 1; 0 1)

Linear Tranformation, LT:

If an LT, L : $V \to W$, then zero of V, 0_v must map to zero of W, 0_w.

L is a one-one LT $\iff Ker(L) = \{0_v\}$ Ker(L) is subspace of V

Rank Nullity Theorem

For a finite dimensional vector space V,

L : $V \to$W then,

Dim(Range(L))+Dim(Ker(L))=Dim(V)

It follows from this that **when L is one-one**, Ker(L)={0} and Dim(Ker(L))=0

$\implies$ Dim(Range(L))=Dim(V)

$\implies$ **L is onto**.

A glance at any example on coordinatization should be sufficient at this stage.

Symmertric Matrix:

$A^T = A$

all real eigenvalues.

always diagonalizable if non-singular.

$(AB)^T = B^T A^T$

Caley Hamilton Theorem:

If p(λ) is the characteristic polynomial of A, then A satisfies $p(\lambda) = 0$.

4 Number Theory

$\equiv$ (mod n) is an equivalent relation.

$a \equiv b$ (mod n) $\iff n|a - b$

If $a \equiv b$ (mod n) and $x \equiv y$ (mod n), then:

$a + x \equiv b + y$ (mod n)

$ax \equiv by$ (mod n)

$a^k \equiv b^k$ (mod n)

If $ca \equiv cb$ (mod n), then

$a \equiv b$ (mod n|gcd(n,c))

$ax \equiv b$ (mod n) has a solution $\iff$ gcd(a,n)|b.

It may be helpful to try a problem on Chinese Remainder Theorem.

Fermat's Little Theorem:

If $p \nmid a$, then $a^{p-1} \equiv 1$ (mod p)

Wilson's Theorem:

(p-1)!$\equiv$ -1 (mod p)

Properties of **Euler Totient**:

$\phi(mn) = \phi(m)\phi(n)$

$\phi(p^n) = p^{a-1}(p - 1)$

$\phi(n) = n(1 - 1/p_1)(1 - 1/p_2)\ldots(1 - 1/p_n)$; p_is are the prime factors of n.

Euler Theorem: $e^{\phi(n)} \equiv 1$ (mod n)

5 Topology

Intersection of family of topologies is a topology but union may not be.

$< X, \tau >$: memebers of X in τ are called open.

A subset of X is closed if its compliment is open.

Definition in terms of closed for $< X, \tau >$:

X, ϕ are closed.

finite union of closed is closed.

arbitrary intersection of closed is closed.

Any open set containing x is a nbd of x.

A is open $\iff$ for every x $\in$ A, $\exists$ nbd U of x such that U$\subseteq$A.

Basis: $\mathcal{B}$ a collection of subsets:

1. For every x$\in$X, $\exists$ B $\in \mathcal{B}$ containing x.
2. For any B_1, B_2 containing x, $\exists B_3 \in \mathcal{B}$ such that $B_3 \subseteq (B_1 \cap B_2)$.

U$\subset$X is open in X $\iff$ $\forall$ x $\in$ U $\exists$ B$\in \mathcal{B}$ such that x $\in$ B$\subseteq$U.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the basis for topologies τ_1 and τ_2 on X.

Then $\tau_1 \subseteq \tau_2 \iff$ for every $B_1 \in \mathcal{B}_1$ and every x$\in B_1$, $\exists B_2 \in \mathcal{B}_2$ such that x$\in B_2 \subseteq B_1$.

τ_2 is finer and τ_1 is courser.

Limit points: A$\subseteq$X $< X, \tau >$. A point x $\in$X is called a limit point of A if every nbd of x intersects A at other than x.

closed sets $\iff$ contains all limit points.

$\overline{A \cup B} = \overline{A} \cup \overline{B}$

$\overline{\bigcup_{i=1}^{\infty} A_i} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$ *Similar results do not hold for intersection.

$Bd(A) = \overline{A} \cap (\overline{X - A})$

$\overline{A \times B} = \overline{A} \times \overline{B}$

$\overline{A - B} = \overline{A} - \overline{B}$

Subspace Topology = $\{U \cap Y | U \in \tau\}$

Hausdorff Space (HS): A topological space X is said to be a HS if for every pair of distinct points x,y $\in$ X, $\exists$ disjoint open sets U and V of x and y respectively.

discrete top. is always HS.

$\mathbb{R}$ with finite compliment top. is not HS.

Every singleton set of HS is closed.

Every finite subset of HS is closed.

For a HS: a pt. x $\in$ X is a limit point of A $\iff$ every nbd of x contains all but finitely many points.

HS: limit of convergence of $\{x_n\}$ is unique.

Subspace of HS is also HS.

order $\implies$ HS.

Continuity (f:X$\to$Y)

if inverse image of every open set in Y is open in X

$\iff$ inverse image of closed is closed.

$\iff$ for every A$\subset$X, $f(\overline{A}) \subset \overline{f(A)}$.

$\iff$ for every x in X and every nbd U of x, $f(U) \subseteq V$.

I :X'$\to$X is continuous $\iff$ τ' is finer than τ.

$f^{-1}(U \times V) = f^{-1}(U) \cap f^{-1}(V)$.

Homeomorphism:

if f is bijective and f and f^{-1} are continuous.

Homeomorphism is an equivalence relation.

A topological property is one which is preserved under homeomorphism.

Metric Spaces:

1. d(x,y)$\geq$0.
2. d(x,y)=d(y,x)
3. d(x,z)$\leq$d(x,y)+d(y,z).

Metric space is HS.

Co-fintie is not metrizable $\implies$ not HS.

Connected:

if it cannot be expressed as the union of disjoint non-empty open sets.

in metric space: separated if $A \cap \overline{B}$ and $B \cap \overline{A}$ are empty. Connected if it cannot be written as a separation.

X is connected $\implies$ X, ϕ are the only clo-open sets.

Connectedness is a topological property.

Continuous image of connected set is connected.

Union of connected sets with a common point is connected.

Connected $\implies$ interval.

if A is connected, and if $A \subseteq B \subseteq \overline{A} \implies$ B is connected.

product space of X and Y is connected if X, Y are connected.

Topologist's sine curve:A= $\{(x, sin(1/x))|0 < x \leq 1\}$ is path connected but $\overline{A}$ is not.

Compactness is a topological property.

Containment of limit point helps in compactness.

Closed subset of compact is compact.

Compact subspace of HS is closed.

Uniform Continuity: f:X$\rightarrow$Y

defined on set; $d_Y(f(x), f(y)) < \epsilon\forall$ x,y for which $d_X(x, y) < \delta$

A continuous function from a compact metric space to any metric space is uniformly continuous.

6 Group Theory

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A set of n elements has 2^n subsets.

Q is countable.

Transcendental numbers are uncountable as opposed to algebraic numbers.

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Dihedral groups: D_n

o(D_n)=2n.

$D_3 = S_3$

o(H)|o(G)

$\exists$ a subgroup for every factor of G if G is Abelian.

No. of cyclic generators of cyclic group $= \phi(o(G))$

Cauchy's: for finite groups

if p|o(G) then $\exists$ at least one subgroup of order p.

If finite group is of prime order, then it is cyclic.

For Abelian, $(a \cdot b)^n = a^n \cdot b^n$

o(a)|o(G), $a^{o(G)} = e$

HK is a subgroup iff HK=KH

o(G|N)=o(G)|o(N)

Every subgroup of Abelian is normal.

see an example to find number of upto isomorphisms.

Homomorphism: G→G'

e maps to e'

if o(g)=m $\implies$ o(g')=m

If H is subgrp of G $\implies$ ϕ(H) is a subgrp of G'

one-one $\iff$ Kernel has only identity.

7 Real Analysis

p is a limit point of $< X, d >$ for every $\epsilon > 0$ such that d$(p_n, p) < \epsilon, \exists$ n≥N

nbd of p has all but finitely many points.

unique limit.

bounded.

convergence $\implies$ Cauchy sequence

In $\mathbb{R}^k$ every Cauchy sequence converges.

monotonic convergence iff bounded.

Series:

$\sum a_n$ converges $\iff$ $|\sum_{k=n}^{m} a_k| < \epsilon$

Non-negative series converges iff partial sums are bounded.

Comparison Test.

Decreasing non-negative sequence series converges $\iff$ $\sum 2^k a_{2^k}$ converges.

$\sum 1/n^p$ converges if p>1.

Root Test.

Ratio Test:

$\lim_{n\to\infty} |a_{n+1}/a_n|$ converges if this limit is <1, otherwise diverges.