Project Report

Design Project

MATH F377

A Refinement of Cauchy's Bound for Polynomials of Degree 3n

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Abstract

Determining the roots of polynomials, or ”solving algebraic equations”, is among the oldest problems in mathematics. Factorizing becomes increasingly difficult as the degree increases. Roots of higher degree polynomials are found through numerical computations and approximation using various results. Location of roots and critical points plays a key role in calculus and hence all its branches.

Cauchy’s bound for region containing all the zeros of complex polynomials is a classic result and stands as a milestone marking the beginning point of development in the theory of the field. In this report, we develop a refinement of the Cauchy’s bound by grouping the terms differently within the same proof structure and offer a result for polynomials whose degree is a multiple of three.

Key Words: refinement, zeros, polynomial, bound, Cauchy.

Introduction

For a given complex polynomial, $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, Cauchy's bound gives the upper bound for magnitude of all zeros of $p(z)$ as follows:

$$R = 1 + \max_{0 \le i \le n-1} |a_i|$$

This bound has been refined by Jain and Tewari[1] and given as:

$$R = \max_{1 \le j \le \lfloor \frac{n+1}{2} \rfloor} \left\{ \frac{|a_{n-2j+1} + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}}{2} \right\}.$$

A study of the proof given to establish this refinement gives motivation to develop more such variants that can give better bounds. The general idea being that higher degree polynomial equations don't have formulae for finding zeros so we group the terms of these polynomials into smaller degree polynomials which have exact formulae for zeros and use these to find the overall bound for zero of the given polynomial.

To arrive at our refinement, we group the terms in polynomials of degree 3 and use the formula first given by Cardano[2].

Proof

Let $p(z)$ be a complex polynomial given by, $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and n is a multiple of 3. Then,

$$|p(z)| = |z^n + a_{n-1}z^{n-1} + \cdots + a_0|$$

$$\geq |z^n| - |a_{n-1}z^{n-1}| - |a_{n-2}z^{n-2}| - \cdots - |a_1 z| - |a_0|$$

$$\geq |z^n| - |a_{n-1}z^{n-1}| - |a_{n-2}z^{n-2}| - |a_{n-3}z^{n-3}| - |z^{n-3}| + |z^{n-3}| - \cdots$$

$$\geq |z^{n-3}|\left(|z^3| - |a_{n-1}||z^2| - |a_{n-2}||z| - (|a_{n-3}| + 1)\right) + |z^{n-6}|\left(|z^3| - |a_{n-4}||z^2| - |a_{n-5}||z| - (|a_{n-6}| + 1)\right) + \cdots + |z^{n-3j}|\left(|z^3| - |a_{n-3j+2}||z^2| - |a_{n-3j+1}||z| - (|a_{n-3j}| + 1)\right) + \cdots + \left(|z^3| - |a_2 z^2| - |a_1 z| - (|a_0| + 1)\right) + 1.$$

So,

$|p(z)| > 0$ if $\left(|z^3| - |a_{n-3j+2}||z^2| - |a_{n-3j+1}||z| - (|a_{n-3j}| + 1)\right) > 0$ for $j = 1 \ldots n/3$

$$\Rightarrow |z| \geq \max_{1\leq j\leq n/3} \zeta\left(|z^3| - |a_{n-3j+2}||z^2| - |a_{n-3j+1}||z| - (|a_{n-3j}| + 1)\right)$$

where ζ function finds the zeros of the polynomial.

Hence, we arrive at the refinement that is given by:

$$R = \max_{1\leq j\leq n/3} \zeta\left(|z^3| - |a_{n-3j+2}||z^2| - |a_{n-3j+1}||z| - (|a_{n-3j}| + 1)\right)$$

Comparative Analysis of Bounds

Consider the polynomial, $p(z) = z^6 + 2z^5 + 3z^4 + 4z^3 + 5z^2 + 6z + 7$.

Actual Bound	Cauchy's Bound	Jain&Tewari's	New Bound
1.43615	**8**	**7**	**6.18028**

The calculations for completing this table are presented in the appendix.

We find that the new bound improves on Jain and Tewari's, as predicted because we have grouped a higher degree polynomial in our approach, which converges the value of the bound towards the actual bound. In this case, the bound offers an improvement of 11.71% from Jain & Tewari's bound and 22.746% improvement from Cauchy's bound.

References:

[1] Jain, Tewari “A refinement of Cauchy’s bound for the moduli of zeros of a polynomial”. Bull. Math. Soc. Sci. Math. Roumanie Tome 61 (109) No. 2, 2018, 173–185

[2] Weisstein, Eric W. "Cubic formula." *omega* 86 (2002): 87

Appendix

Calculations have been performed using Mathematica as follows:

In[2]:= Solve[z^6 + 2 z^5 + 3 z^4 + 4 z^3 + 5 z^2 + 6 z + 7 == 0, z]

Out[2]= $\{\{z \to \text{Root}[7 + 6 \#1 + 5 \#1^2 + 4 \#1^3 + 3 \#1^4 + 2 \#1^5 + \#1^6 \&, 1]\},$
$\{z \to \text{Root}[7 + 6 \#1 + 5 \#1^2 + 4 \#1^3 + 3 \#1^4 + 2 \#1^5 + \#1^6 \&, 2]\},$
$\{z \to \text{Root}[7 + 6 \#1 + 5 \#1^2 + 4 \#1^3 + 3 \#1^4 + 2 \#1^5 + \#1^6 \&, 3]\},$
$\{z \to \text{Root}[7 + 6 \#1 + 5 \#1^2 + 4 \#1^3 + 3 \#1^4 + 2 \#1^5 + \#1^6 \&, 4]\},$
$\{z \to \text{Root}[7 + 6 \#1 + 5 \#1^2 + 4 \#1^3 + 3 \#1^4 + 2 \#1^5 + \#1^6 \&, 5]\},$
$\{z \to \text{Root}[7 + 6 \#1 + 5 \#1^2 + 4 \#1^3 + 3 \#1^4 + 2 \#1^5 + \#1^6 \&, 6]\}\}$

In[3]:= N[%2]

Out[3]= {{z → -1.30787 - 0.593295 i}, {z → -1.30787 + 0.593295 i}, {z → -0.402509 - 1.34167 i}, {z → -0.402509 + 1.34167 i}, {z → 0.710379 - 1.10685 i}, {z → 0.710379 + 1.10685 i}}

In[4]:= Abs[-1.3078697439524358` - 0.5932947074145877` i]

Out[4]= 1.43615

In[5]:= Abs[-0.4025091253602798` - 1.3416668277593835` i]

Out[5]= 1.40074

In[6]:= Abs[0.7103788693127157` - 1.106845298383849` i]

Out[6]= 1.3152

In[16]:= Solve[Abs[z]^3 - 2 Abs[z]^2 - 3 Abs[z] - 5 == 0, z]

Out[16]= $\left\{\left\{z \to \frac{1}{3}\left(-2 - 13\left(\frac{2}{205 - 3\sqrt{3693}}\right)^{1/3} - \left(\frac{1}{2}\left(205 - 3\sqrt{3693}\right)\right)^{1/3}\right)\right\},\right.$
$\left.\left\{z \to \frac{1}{3}\left(2 + \left(\frac{205}{2} - \frac{3\sqrt{3693}}{2}\right)^{1/3} + \left(\frac{1}{2}\left(205 + 3\sqrt{3693}\right)\right)^{1/3}\right)\right\}\right\}$

In[17]:= $N\left[\left\{\left\{z \to \frac{1}{3}\left(-2 - 13\left(\frac{2}{205 - 3\sqrt{3693}}\right)^{1/3} - \left(\frac{1}{2}\left(205 - 3\sqrt{3693}\right)\right)^{1/3}\right)\right\},\right.\right.$
$\left.\left.\left\{z \to \frac{1}{3}\left(2 + \left(\frac{205}{2} - \frac{3\sqrt{3693}}{2}\right)^{1/3} + \left(\frac{1}{2}\left(205 + 3\sqrt{3693}\right)\right)^{1/3}\right)\right\}\right\}\right]$

Out[17]= {{z → -3.34417}, {z → 3.34417}}

In[18]:= Solve[Abs[z]^3 - 5 Abs[z]^2 - 6 Abs[z] - 8 == 0, z]

In[19]:= $N\left[\left\{\left\{z \to \frac{1}{3}\left(-5 - \frac{43}{\left(368 - 3\sqrt{6213}\right)^{1/3}} - \left(368 - 3\sqrt{6213}\right)^{1/3}\right)\right\},\right.\right.$

$\left.\left.\left\{z \to \frac{1}{3}\left(5 + \left(368 - 3\sqrt{6213}\right)^{1/3} + \left(368 + 3\sqrt{6213}\right)^{1/3}\right)\right\}\right\}\right]$

Out[19]= $\{\{z \to -6.18028\}, \{z \to 6.18028\}\}$

In[21]:= Solve[Abs[z]^2 - 2 Abs[z] - 4 == 0, z]

Out[21]= $\{\{z \to -1 - \sqrt{5}\}, \{z \to 1 + \sqrt{5}\}\}$

In[22]:= $N[\{\{z \to -1 - \sqrt{5}\}, \{z \to 1 + \sqrt{5}\}\}]$

Out[22]= $\{\{z \to -3.23607\}, \{z \to 3.23607\}\}$

In[23]:= Solve[Abs[z]^2 - 4 Abs[z] - 6 == 0, z]

Out[23]= $\{\{z \to -2 - \sqrt{10}\}, \{z \to 2 + \sqrt{10}\}\}$

In[24]:= $N[\{\{z \to -2 - \sqrt{10}\}, \{z \to 2 + \sqrt{10}\}\}]$

Out[24]= $\{\{z \to -5.16228\}, \{z \to 5.16228\}\}$

In[25]:= Solve[Abs[z]^2 - 6 Abs[z] - 7 == 0, z]

Out[25]= $\{\{z \to -7\}, \{z \to 7\}\}$