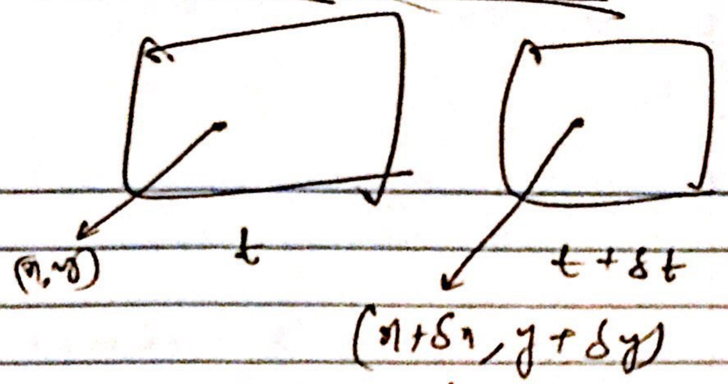
Optical flow Constraint function



disp = $(\delta x, \delta y)$ Optical flow $(u,v) = \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}\right)$

umption 1:
Brightness

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t)$$

umption 2:
displacement $(\delta x, \delta y)$ and time step δt are small

$I(x,y,t)$ · $I(x+\delta x, y+\delta y, t+\delta t)$

Taylor series Expansion
Expand a function as an infinite sum of its derivatives.

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + O(\delta x^2) \rightarrow \text{almost zero}$$

For a function of 3 variables with small $\delta x, \delta y, \delta t$:

$$f(x+\delta x, y+\delta y, t+\delta t) \approx f(x,y,t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial t}\delta t$$

from
Assumption #2:

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t$$

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) + I_x \delta x + I_y \delta y + I_t \delta t$$

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) \quad \text{--- } (1)$$

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) + I_x \delta x + I_y \delta y + I_t \delta t \quad \text{--- } (2)$$

Sub (1) from (2) : $I_x \delta x + I_y \delta y + I_t \delta t = 0$

divide by δt and take limit as $\delta t \to 0$: $I_x \frac{\partial x}{\partial t} + I_y \frac{\partial y}{\partial t} + I_t = 0$

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$

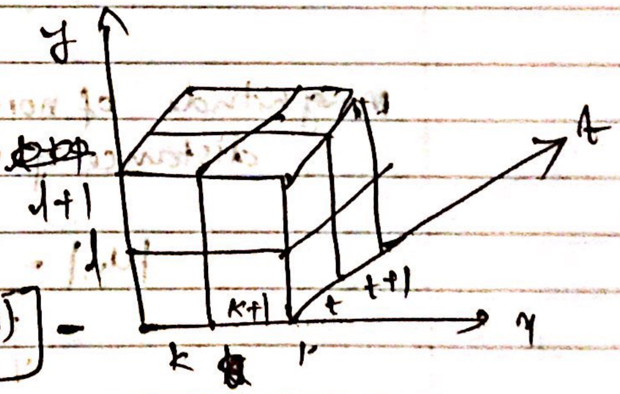
(u,v) : optical flow

Computation partial derivatives (I_x, I_y, I_t) can be easily computed from 2 frames

$I_x(k,l,t) =$

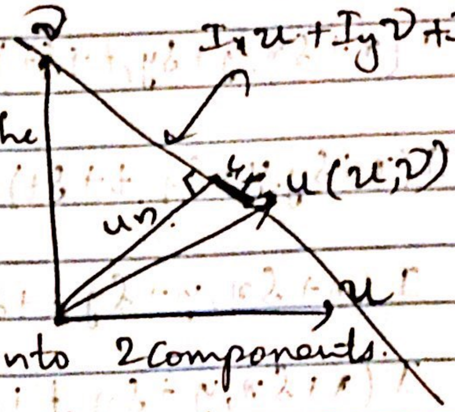
$$\frac{1}{4}\left[I(k+1,l,t) + I(k+1,l,t+1) + I(k+1,l+1,t) + I(k+1,l+1,t+1)\right] -$$

$$\frac{1}{4}\left[I(k,l,t) + I(k,l,t+1) + I(k,l+1,t) + I(k,l+1,t+1)\right]$$



Similarly for $I_y(k,l,t)$ and $I_t(k,l,t)$

Geometric Interpretation



for any point (x,y) in the image
its optical flow (u,v) lies on the line:

$$I_x u + I_y v + I_t = 0$$

optical flow can be split into 2 components.

$$u = u_n + u_p$$

u_n = Normal flow
u_p = Parallel flow

Normal flow:

direction of normal flow
unit vector $\perp^{ar}$ to the constraint line.

$$\hat{u}_n = \frac{(I_x, I_y)}{\sqrt{I_x^2 + I_y^2}}$$

Magnitude of normal flow:
distance of origin from the constraint line:

$$|u_n| = \frac{|I_t|}{\sqrt{I_x^2 + I_y^2}}$$

$$\boxed{u_n = \frac{|I_t|}{(I_x^2 + I_y^2)} (I_x, I_y)}$$

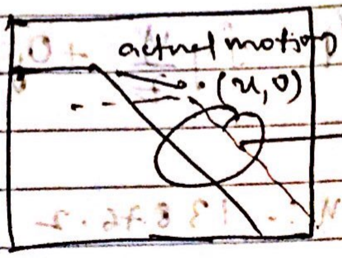
Parallel flow

we cannot determine u_p the optical flow component parallel to the constraint line.

Aperture problem

Constraint Equation:

$$\boxed{I_x u + I_y v + I_t = 0}$$



perpendicular normal flow.

2 unknowns, 1 Equation
we need additional Constraint