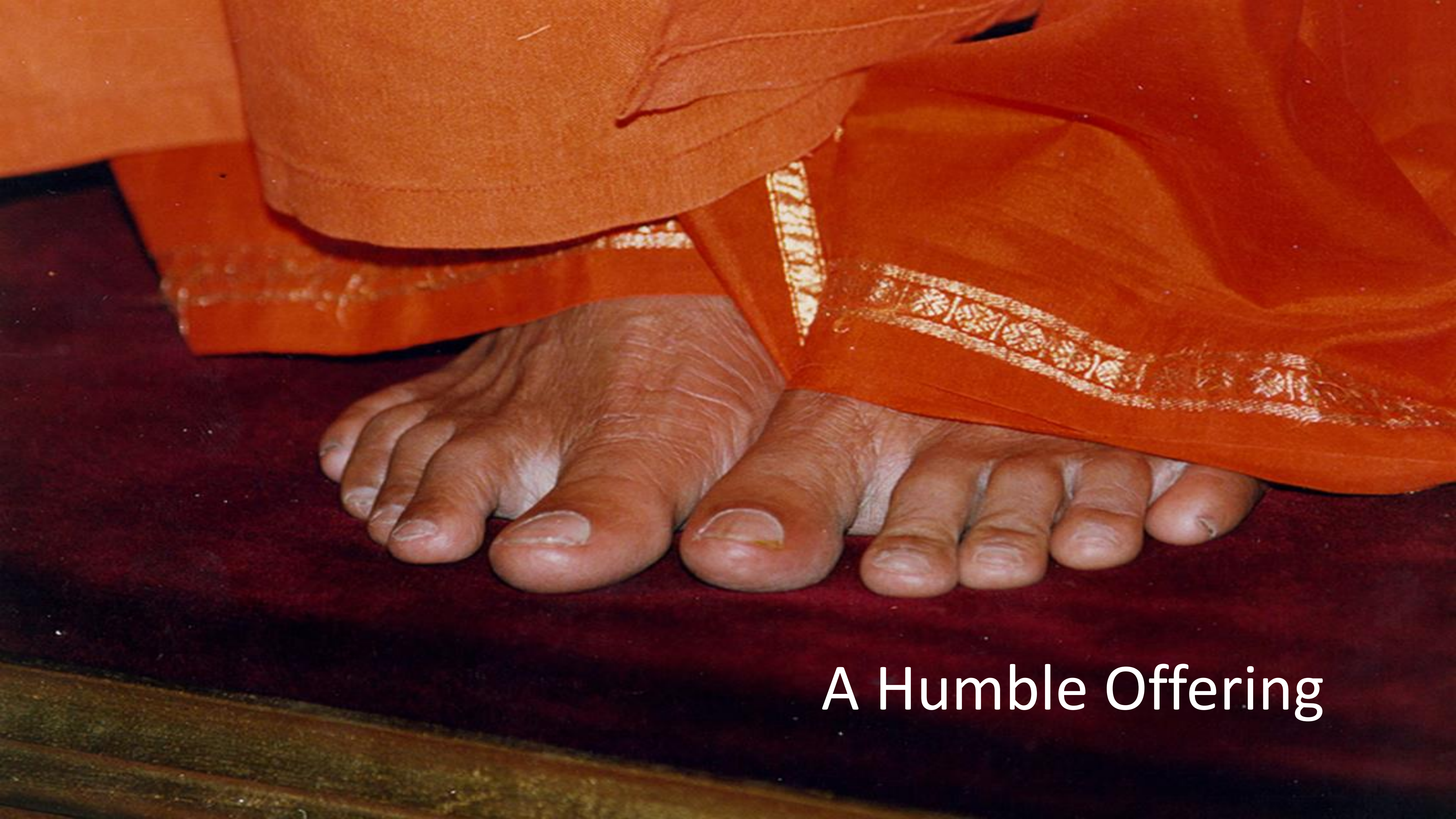

A Humble Offering

DIGITAL ROOTS

By Ilhan M. Izmirli
Dept of statistics,George Mason University, USA

CONTENTS

Introduction

- The digital root concept is the idea that any number greater than 9 can be reduced to a single digit.
- The digital root of a natural number is obtained by computing the sum of the digits, then computing the sum of the digits of the resulting number, and so on,

  till a single digit number is obtained.
- In Vedic Mathematics the digital root is known as Beejank.
- It is denoted by B(n).
- The concept of digital root has been known for sometime before the development of computer devices.
- This idea was mostly used by accountants to check their results.
- It is a well established and useful part of recreational mathematics which materializes in as diverse applications as computer programming.

- In recreational number theory A.Grey was the first to apply digital root.
- Let n be a natural number and let S(n) denote the sum of the digits of n . In a finite number of steps , the sequence S(n),S(S(n)),S(S(S(n))),------becomes a constant which is called digital root.
- We define digital root of a number as

$B : N \to D$ where N is the set of natural numbers and D={1,2,3,4,5,6,7,8,9} and B(a)=sum of the digits of the number until single digit is obtained.

e.g. B(123)=1+2+3=6

- To show that the function is well defined

Let a , b ϵ N and a=b

therefore the sum of the digits of a = the sum of the digits of b.

- Two numbers a and b are equivalent if and only if both have same digital root.

$$a \sim b \Leftrightarrow B(a)=B(b).$$

Digital root partition the set of non negative integers

- Let the relation be defined as aRb$\Leftrightarrow$a $\sim$ b

R is reflexive

$B(a)=B(a) \Leftrightarrow a \sim a \Leftrightarrow aRa$

R is symmetric

$aRb \Rightarrow B(a)=B(b) \Rightarrow B(b)=B(a) \Rightarrow b \sim a \Rightarrow bRa$

R is transitive

aRb and $bRc \Rightarrow B(a)=B(b)$ and $B(b)=B(c) \Rightarrow B(a)=B(b)=B(c) \Rightarrow B(a)=B(c) \Rightarrow a \sim c \Rightarrow aRc$

So it is an equivalence relation and it partitions the set of non negative integers.

Properties of Digital roots

1. If 1 ≤ n ≤ 9 then B(n) = n.

2. B(n) = n - 9$\left\lfloor \frac{n}{9} \right\rfloor$ where $\lfloor x \rfloor$ stands for the greatest integer less than or equal to x.

3. B(n)=$\begin{cases} 9 & if\ n\ mod 9 = 0 \\ n\ mod9 & otherwise \end{cases}$

e.g. 998 →8, 3198 →3, 6566 →5

4.The difference between n and B(n) is a multiple of 9; i.e., n-B(n) = 9k for some non-negative integer k.

It suffices to show that the difference between n and S(n) is a multiple of 9.where

S(n) is a sum of digits of n.

n=$a_0 + 10a_1 + 10^2a_2 + 10^3a_3 +$--------------,

S(n)=$a_0 + a_1 + a_2 +$---------,

n – S(n) =(10-1)$a_1 + (10^2 - 1)a_2 + (10^3 - 1)a_3$+----------

therefore n-S(n) is a multiple of 9.

- The difference between n and B(n) is a multiple of 3.
- A number having digital root ‘a’ can be represented as a+(9*j) : j=0,1,2,3,------

5. For all pairs m , n we have

(i) $B(m+n) = B(B(m)+B(n))$

(ii) $B(mn) = B(B(m)B(n))$

$$
\begin{aligned}
B(B(m)+B(n)) &= B(m)+B(n)-(9*s) \\
&= m-(9*k)+n-(9*l)-(9*s) \\
&= m+n-9*(k+l+s) \\
&= B(m+n)
\end{aligned}
$$

$$
\begin{aligned}
B(B(m).B(n)) &= B(m).B(n)-(9*s) \\
&= (m-(9*k)).(n-(9*l))-(9*s) \\
&= mn-9ml-9kn+81kl-9s \\
&= mn-9*(ml+kn-9kl+s) \\
&= B(mn)
\end{aligned}
$$

7.B(m+n)=B($m' + n'$)
B(mn)=B($m'n'$)

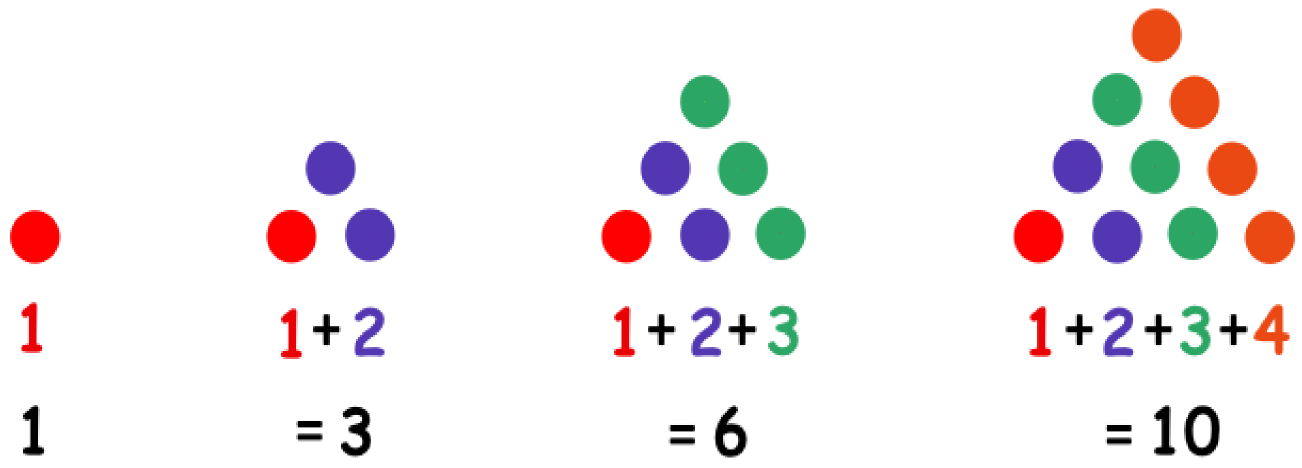
Where m, m' leave the same remainder under division by 9,and n, n' leave the same remainder under division by 9.

8.A prime number exceeding 3 cannot have a digital root equal to 3,6 or 9.

9.The digital root of a triangular number is one of the following numbers: 1,3,6,9

<u>Triangular number</u> : It counts the objects arranged in a equilateral triangle.

The n^{th} triangular number is the number of dots in the triangular arrangement with n dots on each side , and is equal to the sum of the n natural numbers from 1 to n.



The triangular numbers are 0,1,3,6,10,15,21,-----------

$$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

We consider different forms that a triangular number n= $\frac{m(m+1)}{2}$ can have , depending upon the remainder that m leaves under division by 6.

The different forms of m are 6k,6k+1,6K+2,6k+3,6k+4,6k+5

If m=6k or 6k+2 or 6k+3 or 6k+5 the product $\frac{m(m+1)}{2}$ is a multiple of 3. hence the digital root of it will be one of the numbers 3,6,9. If m=6k+1 or 6k+4 then the product $\frac{m(m+1)}{2}$ is of the form 9m+1 .hence the digital root is 1.

10.If n is a perfect square then B(n) ϵ {1,4,7,9}

We consider different forms of a perfect square n=m^2 can have depending on the remainder that m leaves under division by 9. The different forms for m are : 9k,9k$\pm$1,9k $\pm$2,9k $\pm$3,9k $\pm$4.

we find that B(n) ϵ {1,4,7,9}.

11. If n is a perfect cube , then B(n) ϵ {1,8,9}.

12. If n is a perfect sixth power , then B(n) ϵ{1,9}.

13.Digital root of an even perfect number(except 6) is 1

Every even perfect number m is of the form m=$2^{p-1}(2^p - 1)$

by putting x = 2^{p-1},we have B(n)=B($2x^2 - x$)=2B(x)B(x)-B(x)

B(x)=4,7,1,7,1,------

2B(x)B(x)=5,8,2,8,2,------

therefore the digital root is 1.

Multiplication table

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Vedic square

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	1	3	5	7	9
3	3	6	9	3	6	9	3	6	9
4	4	8	3	7	2	6	1	5	9
5	5	1	6	2	7	3	8	4	9
6	6	3	9	6	3	9	6	3	9
7	7	5	3	1	8	6	4	2	9
8	8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9	9

Theorem :Let p(x) be any n degree polynomial whose roots are only integers , If a be its root then B(p(B(a)))=9.

Proof: Let p(x) is n degree polynomial and has only integral root.

Then let $\alpha_1, \alpha_2, \alpha_3$,------, α_n be n roots then p(x)=(x- α_1)(x- α_2)(x- α_3)--(x- α_n).

now B(α_1)- α_1 must be a multiple of 9

Therefore B(p(B(α_1)))=B[(B(α_1)- α_1)(B(α_1)- α_2)(B(α_1)- α_3)----(B(α_1)- α_n)]

=9

Inverse of the theorem is not true.

Significance : Let us suppose a polynomial p(x) of n degree is given whose roots are integers.Let us check the digital roots of p(1),p(2),p(3),--------p(9).If B(p(d)) $\neq 9$ for some d ϵ D then any number lying in the equivalence class of d , can never satisfy p(x) and if B(p(d))=9 for some d ϵ D then numbers equivalent to d can be its root.

Digital Roots of powers of numbers in an Arithmetic Progression

Let k,m and n be three consecutive terms in an arithmetic progression with common difference d. Let x=$k^3+m^3+n^3$. If d is not a multiple of 3,then B(x)=9

Proof: Let k=m-d and n=m+d. Then x= $k^3+m^3+n^3$=3 m^3+6md^2 =3m(m^2 +2d^2)

We must prove m(m^2 +2 d^2) is divisible by 3 for any natural number m and for any natural number d that is not a multiple of 3.

if m is divisible by 3 then result follows

Case 1: m=3r+1

m(m^2 +2 d^2)=(3r+1)($9r^2+6r+1+2d^2$)

d=3s $\pm$1

2 d^2=18s^2 $\pm$12s+2

Thus m(m^2 +2 d^2)=3(3r+1)(3 r^2+2r+6 s^2 $\pm$4s+1)

Case 2: m=3r+2

m(m^2 +2d^2) =(3r+2)(9r^2+12r+4+2d^2)

By substituting $2d^2 = 18s^2 \pm$ 12s+2

m(m^2 +2d^2) =3(3r+2)($3r^2 + 4r + 6s^2$ $\pm$4s+2)

therefore B(x)=9

Application: Prove that $1^3 + 2^3 + 3^3$ +-----------+ 50^3 is divisible by 9?

(50^3+ 49^3 + 48^3)+(47^3+ 46^3+ 45^3)+--------+(5^3+ 4^3 + 3^3)+ 2^3 +1 .

<u>Theorem</u>

Let q be a multiple of three .Let n_1, n_2, n_3,----, n_q be any q consecutive terms of an arithmetic progression whose common difference d is not multiple of three . Let

x= ${n_1}^3 + {n_2}^3$ +------+${n_q}^3$ Then , B(x)=9.

Theorem : Let n_1, n_2, n_3,----, n_9 be nine consecutive terms in an arithmetic progression with common difference d . Let X = $n_1{}^9 + n_2{}^9$ +------+$n_9{}^9$ then B(x)=9

Proof :

$$n_1 = n_5 - 4d$$
$$n_2 = n_5 - 3d$$
$$n_3 = n_5 - 2d$$
$$n_4 = n_5 - d$$
$$n_6 = n_5 + d$$
$$n_7 = n_5 + 2d$$
$$n_8 = n_5 + 3d$$
$$n_9 = n_5 + 4d$$

X= $n_1{}^9 + n_2{}^9$ +------+$n_9{}^9$=9($n_5{}^9 + 240n_5{}^7d^2$ +--------+144,708 n_5d^8)

Therefore B(n)=9

Let q be a multiple of 9 .Let n_1, n_2, n_3,----, n_q be any q consecutive terms of an arithmetic progression whose common difference d . Let

x = $n_1{}^9 + n_2{}^9$ +------+$n_q{}^9$ Then B(x)=9.

<u>Digital roots of Fermat Numbers</u>

Fermat number : It is defined as F_n = 2^{2^n}+1 , n ≥ 1

F_{n-1}= $2^{2^{n-1}}$+1

F_n= ${F_{n-1}}^2 - 2\ F_{n-1} + 2$

$F_0= 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 655373$,---------

Theorem : Let F_n be the n^{th} Fermat number , Then,

$$B(F_n) = \begin{cases} 5 \ \ if\ n\ is\ odd \\ 8\ if\ n\ is\ even \end{cases}$$

Proof : We will prove by induction

It is true for n$\leq$2, Assume it is true for n-1.Then ,

$$B(F_n)\text{=B}({F_{n-1}}^2 - 2\ F_{n-1} + 2\ )$$

$$\text{=B(B}({F_{n-1}}^2) - \text{B}(2\ F_{n-1}) + \text{B(2))}$$

$$\text{=B(B(B}(F_{n-1})\text{B}(F_{n-1})) - 2\text{B}(F_{n-1}) + 2)$$

Suppose n-1 is odd . Then $B(F_n)$=B$(\mathrm{B}(5^2) - 2 * 5 + 2)$ =8mod9

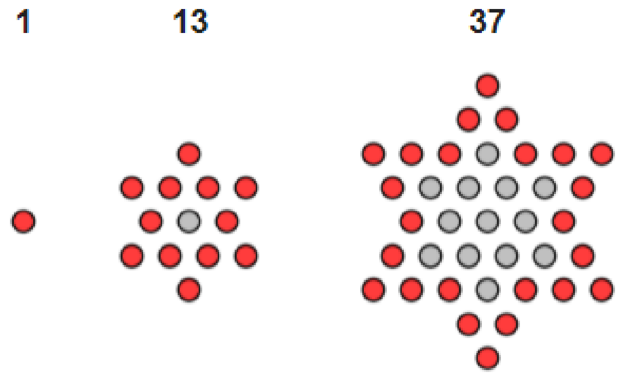
Suppose n-1 is even . Then $B(F_n)$=B(B$(8^2) - 2 * 8 + 2)$=5mod9

Digital roots of star numbers

The j^{th} star number is given by the formula s_j=6j(j-1)+1 for j=1,2,3,-------

So, s_1=1, s_2=13, s_3=37, s_4=73, s_5=121,-----and so on.

s_{j+1}= s_j+12j for j=1,2,3------



B(s_1)=1 , B(s_2)=4 , B(s_3)=1 , B(s_4)=1, B(s_5)=4, B(s_6)=1 ,--------

Therefore the digital root of the star number is always 1 or 4.

Digital roots of a natural number in a given base

Let n be a natural number . For base b>1, we define the digital sum

$$F_b : \mathbb{N} \to \mathbb{N}$$

F_b(n) = $\sum_{i=0}^{k-1} d_i$ where k is the number of digits in the number in base b, and

d_i=$\frac{\mathrm{n \bmod b^{i+1} - n \bmod b^i}}{\mathrm{b^i}}$ is the value of the each digit of the number.

If n$\geq \mathrm{b}$,then n=$\sum_{i=0}^{k-1} d_i b^i$ and therefore F_b(n)= $\sum_{i=0}^{k-1} d_i$ < $\sum_{i=0}^{k-1} d_i b^i$ = n (b>1)

If n < b then F_b(n) = n

therefore the only possible digital roots are the natural numbers 0$< \mathrm{n} < \mathrm{b}$

Example:

In base 12,8 is the digital root of the base 10 number 3110,

$$d_0 = \frac{3110 \bmod 12^{0+1} - 3110 \bmod 12^0}{12^0} = \frac{3110 \bmod 12 - 3110 \bmod 1}{1} = \frac{2-0}{1} = \frac{2}{1} = 2$$

$$d_1 = \frac{3110 \bmod 12^{1+1} - 3110 \bmod 12^1}{12^1} = \frac{3110 \bmod 144 - 3110 \bmod 12}{12} = \frac{86-2}{12} = \frac{84}{12} = 7$$

$$d_2 = \frac{3110 \bmod 12^{2+1} - 3110 \bmod 12^2}{12^2} = \frac{3110 \bmod 1728 - 3110 \bmod 144}{144} = \frac{1382-86}{144} = \frac{1296}{144} = 9$$

$$d_3 = \frac{3110 \bmod 12^{3+1} - 3110 \bmod 12^3}{12^3} = \frac{3110 \bmod 20736 - 3110 \bmod 1728}{1728} = \frac{3110-1382}{1728} = \frac{1728}{1728} = 1$$

$$F_{12}(3110) = \sum_{i=0}^{4-1} d_i = 2+7+9+1 = 19$$

To find digital root of 19

$$d_0 = \frac{19 \bmod 12^{0+1} - 19 \bmod 12^0}{12^0} = \frac{19 \bmod 12 - 19 \bmod 1}{1} = \frac{7-0}{1} = \frac{7}{1} = 7$$

$$d_1 = \frac{19 \bmod 12^{1+1} - 19 \bmod 12^1}{12^1} = \frac{19 \bmod 144 - 19 \bmod 12}{12} = \frac{19-7}{12} = \frac{12}{12} = 1$$

$$F_{12}(19) = \sum_{i=0}^{2-1} d_i = 1 + 7 = 8$$

Direct formula

The formula in base b is :

$$F_b(\text{n}) = \begin{cases} b-1 & if\ n \equiv 0\ mod\ b-1 \\ n\ mod\ (b-1) & if\ n \not\equiv 0\ mod\ b-1 \end{cases}$$

(or)

$$F_b(\text{n})=\text{n-(b-1)} \left\lfloor \frac{\eta-1}{b-1} \right\rfloor$$

$$F_b(a_1 + a_2) = F_b(F_b(a_1) + F_b(a_2))$$

$$F_b(a_1\ a_2) = F_b(F_b(a_1)\ F_b(a_2))$$

$$F_b(\text{-n}) = -\ F_b(\text{n})\text{mod(b-1)}$$

Digital values

- Every number real or imaginary is assigned a digital value.
- The digital values are mostly 1,2,3,4,5,6,7,8,or 9.
- If the number is natural number the digital value is called digital root.
- Digital value is denoted by //x// or by dval(x).
- //0// = 9
  //-1// = 8
  //-2// = 7
  //-3// = 6 and so on.

- A simple way to find out the digital value of a negative integer is to subtract the absolute value of the integer from 9.

Number	Digital value
267	6
266	5
265	4
264	3
263	2
262	1
261	9
260	8
259	7
258	6
257	5
256	4
255	3
254	2
253	1

Some properties of digital values

- //a+b// = // //a// + //b// //
- //a-b// = // //a// - //b// //
- //a * b// = // //a// * //b// //
- // //a+b// +c//= //a+//b+c////
- //9a// = 9
- //8*a// = //-a//
- //9a+b// = //b//
- //a^b//=////a//^b//
- //a/b//= ////a/////b////

So the digital value for any decimal number which is terminating can be found

e.g. //12.321// = //12321/1000//=////12321/////1000////=//9/1//=9

Digital values of irrational number

To find this we will use //a^b//=////a//^b// where a , b are real numbers

e.g. //13^0.5//= //4^0.5/= //2// or //-2// = 2 or 7

Let A be another number such that //a// = //A//

//a^b// = ////a//^b//

and //A^b//= ////A//^b//=////a//^b//

therefore //a^b// = //A^b//

e.g //square root of 7// = //square root of 16//=//4// or //-4// = 4 or 5

Digital values of Imaginary numbers

//a^b//=//A^b// when //a// = //A//

e.g., b=0.5,a= -1, A=8

//i//=//(-1)^0.5//=//8^0.5//

//(-5)^0.5//=//4^0.5//=2 or 7

=//5^0.5*i//

=//5^0.5*8^0.5//

=//(40)^0.5//

=////40//^0.5//

=//4^0.5//

=2 or 7

Digital value in functions

//f(x)// = //f(//x//)//

If two systems of equations have equal digital values of corresponding coefficients of corresponding equations , then the corresponding roots have equal digital values

$$a_{11}x + b_{11}y + c_{11} = 0$$

$$a_{12}x + b_{12}y + c_{12} = 0$$

And

$$a_{21}x + b_{21}y + c_{21} = 0$$

$$a_{22}x + b_{22}y + c_{22} = 0$$

Will have same digital values of x as well as y if

// a_{11}// = // a_{21}//

// b_{11}// = // b_{21}//

// c_{11}// = // c_{21}//

Python program to find digital root

Spyder (Python 3.8)

C:\Users\Sahithi Chowdary\digital roots.py

```python
# -*- coding: utf-8 -*-
"""
Created on Sun Dec  5 15:08:08 2021

@author: Sahithi Chowdary
"""
import math
def dr(n):
 if(n=="0"):
     return 0
 else:
     if(n%9==0):
       return 9
     else:
         return (n%9)
n=int(input("Enter the number"))
print(dr(n))
```

Usage

Here you can get help of any object by pressing **Ctrl+I** in front of it, either on the Editor or the Console.

Help can also be shown automatically after writing a left parenthesis next to an object. You can activate this behavior in *Preferences > Help*.

New to Spyder? Read our tutorial

Console 3/A

```
Python 3.8.5 (default, Sep  3 2020, 21:29:08) [MSC v.1916 64 bit (AMD64)]
Type "copyright", "credits" or "license" for more information.

IPython 7.19.0 -- An enhanced Interactive Python.

In [1]: runfile('C:/Users/Sahithi Chowdary/digital roots.py', wdir='C:/Users/Sahithi Chowdary')

Enter the number123
6

In [2]: runfile('C:/Users/Sahithi Chowdary/digital roots.py', wdir='C:/Users/Sahithi Chowdary')

Enter the number65785412
2

In [3]: runfile('C:/Users/Sahithi Chowdary/digital roots.py', wdir='C:/Users/Sahithi Chowdary')

Enter the number1999999999999
1

In [4]:
```

LSP Python: ready | conda: base (Python 3.8.5) | Line 17, Col 18 | UTF-8 | CRLF | RW | Mem 54%

Conclusion

- Digital roots are used to identify whether the number is divisible by 3 or 9 quickly.
- To crosscheck our answers in addition , subtraction, multiplication , and division.

$736 + 245 = 981$ $7 \ + \ 2 \ = \ 9$	$954 - 476 = 478$ $9 - 8 = 1$	$13 \times 14 = 182$ $4 \times 5 = 20 = 2$	$256 \div 9$ $B(Q \times D + R) = B(dvd)$ $B(28 \times 9 + 4) = B(256)$ $1 \times 9 + 4 = 2+5+6$ $13 = 13$ $4 = 4$

- Digital roots are used to verify square roots and cube roots.
- Digital roots are not applicable to approximate values.
- Digital values help us verifying calculations involving not only integers but also complex numbers.

References

- https://people.revoledu.com/kardi/tutorial/DigitSum/significant-digital-root.html
- https://brilliant.org/wiki/digital-root/
- https://www.instructables.com/Mental-Math-Digital-Root-Extraction/
- http://blogannath.blogspot.com/2009/09/vedic-mathematics-lesson-19-digital.html?m=1
- https://en.m.wikipedia.org/wiki/Digital_root#:~:text=The%20digital%20root%20
- https://www.freeonlineresearchpapers.com/digital-values/
- https://www.youtube.com/watch?v=5AyNJK-OPi0
- https://www.youtube.com/watch?v=IhQCknCrAsg&t=1513s
- https://www.hitbullseye.com/Quant/Digital-Root-or-Seed-Number.php

THANK YOU