Propositional and First-Order Logic

Chapter 7.4─7.8, 8.1─8.3, 8.5

Knowledge Representation and Propositional Logic

Knowledge Representation

- Logic
- Rules like if then
- Semantic Net like Google graph
- Frames e.g. state and fillers
- Scripts

Big Ideas

- Logic is a great knowledge representation language for many AI problems
- **Propositional logic** is the simple foundation and fine for some AI problems
- **First order logic** (FOL) is much more expressive as a KR language and more commonly used in AI
- There are many variations: horn logic, higher order logic, three-valued logic, probabilistic logics, etc.

Logic roadmap overview

- Propositional logic (review)
- Problems with propositional logic
- First-order logic (review)
  - Properties, relations, functions, quantifiers, …
  - Terms, sentences, wffs, axioms, theories, proofs, …
- Extensions to first-order logic
- Logical agents
  - Reflex agents
  - Representing change: situation calculus, frame problem
  - Preferences on actions
  - Goal-based agents

Propositional logic

- **Logical constants**: true, false
- **Propositional symbols**: P, Q,... (**atomic sentences**)
- Wrapping **parentheses**: (…)
- Sentences are combined by **connectives**:

  $\wedge$ and [conjunction]
  $\vee$ or [disjunction]
  $\Rightarrow$ implies [implication / conditional]
  $\Leftrightarrow$ is equivalent[biconditional]
  $\neg$ not [negation]
- **Literal**: atomic sentence or negated atomic sentence

  P, $\neg$ P

Examples of PL sentences

- $(P \wedge Q) \rightarrow R$

  "If it is hot and humid, then it is raining"
- $Q \rightarrow P$

  "If it is humid, then it is hot"
- Q

  "It is humid."
- We're free to choose better symbols, btw:

  Ho = "It is hot"

  Hu = "It is humid"

  R = "It is raining"

Propositional logic (PL)

- Simple language for showing key ideas and definitions
- User defines set of propositional symbols, like P and Q
- User defines **semantics** of each propositional symbol:
  – P means "It is hot", Q means "It is humid", etc.
- A sentence (well formed formula) is defined as follows:
  – A symbol is a sentence
  – If S is a sentence, then $\neg$S is a sentence
  – If S is a sentence, then (S) is a sentence
  – If S and T are sentences, then (S $\vee$ T), (S $\wedge$ T), (S $\rightarrow$ T), and (S $\leftrightarrow$ T) are sentences
  – A sentence results from a finite number of applications of the rules

Some terms

- The meaning or **semantics** of a sentence determines its **interpretation**
- Given the truth values of all symbols in a sentence, it can be “evaluated” to determine its **truth value** (True or False)
- A **model** for a KB is a *possible world* – an assignment of truth values to propositional symbols that makes each sentence in the KB True

More terms

- A **valid sentence** or **tautology** is a sentence that is True under all interpretations, no matter what the world is actually like or what the semantics is. Example: "It's raining or it's not raining"
- An **inconsistent sentence** or **contradiction** is a sentence that is False under all interpretations. The world is never like what it describes, as in "It's raining and it's not raining."
- **P entails Q**, written P |= Q, means that whenever P is True, so is Q. In other words, all models of P are also models of Q.

Truth tables

- Truth tables are used to define logical connectives
- and to determine when a complex sentence is true given the values of the symbols in it

Truth tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	*False*	*True*	*False*	*False*	*True*	*True*
False	*True*	*True*	*False*	*True*	*True*	*False*
True	*False*	*False*	*False*	*True*	*False*	*False*
True	*True*	*False*	*True*	*True*	*True*	*True*

Example of a truth table used for a complex sentence

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	*False*	*False*	*False*	*True*
False	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
True	*True*	*True*	*False*	*True*

On the implies connective: P → Q

- Note that → is a logical connective
- So $P \rightarrow Q$ is a logical sentence and has a truth value, i.e., is either true or false
- If we add this sentence to the KB, it can be used by an inference rule, *Modes Ponens*, to derive/infer/prove Q if P is also in the KB
- Given a KB where P=True and Q=True, we can also derive/infer/prove that P→Q is True

$\mathbf{P \rightarrow Q}$

- When is $P \rightarrow Q$ true? Check all that apply
  - ❑ P=Q=true
  - ❑ P=Q=false
  - ❑ P=true, Q=false
  - ❑ P=false, Q=true

$P \rightarrow Q$

- When is $P \rightarrow Q$ true? Check all that apply
  - ☑ P=Q=true
  - ☑ P=Q=false
  - ☐ P=true, Q=false
  - ☑ P=false, Q=true
- We can get this from the truth table for $\rightarrow$
- Note: in FOL it's much harder to prove that a conditional true.
  - Consider proving prime(x) $\rightarrow$ odd(x)

Inference rules

- **Logical inference** creates new sentences that logically follow from a set of sentences (KB)
- An inference rule is **sound** if every sentence X it produces when operating on a KB logically follows from the KB
  - i.e., inference rule creates no contradictions
- An inference rule is **complete** if it can produce every expression that logically follows from (is entailed by) the KB.
  - Note analogy to complete search algorithms

Sound rules of inference

- Here are some examples of sound rules of inference
- Each can be shown to be sound using a truth table

RULE	PREMISE	CONCLUSION
Modus Ponens	$A, A \rightarrow B$	B
And Introduction	A, B	$A \wedge B$
And Elimination	$A \wedge B$	A
Double Negation	$\neg\neg A$	A
Unit Resolution	$A \vee B, \neg B$	A
Resolution	$\mathbf{A \vee B, \neg B \vee C}$	$\mathbf{A \vee C}$

Soundness of modus ponens

A	B	A → B	OK?
True	True	True	√
True	False	False	√
False	True	True	√
False	False	True	√

Resolution

- **Resolution** is a valid inference rule producing a new clause implied by two clauses containing *complementary literals*
  - A literal is an atomic symbol or its negation, i.e., P, ~P
- Amazingly, this is the only interference rule you need to build a sound and complete theorem prover
  - Based on proof by contradiction and usually called resolution refutation
- The resolution rule was discovered by Alan Robinson (CS, U. of Syracuse) in the mid 60s

Resolution

- A KB is actually a set of sentences all of which are true, i.e., a conjunction of sentences.
- To use resolution, put KB into *conjunctive normal form* (CNF), where each sentence written as a disjunction of (one or more) literals

Tautologies

$(A \rightarrow B) \leftrightarrow (\sim A \vee B)$

$(A \vee (B \wedge C)) \leftrightarrow (A \vee B) \wedge (A \vee C)$

Example

- KB: $[P \rightarrow Q \text{ , } Q \rightarrow R \wedge S]$
- KB in CNF: $[\sim P \vee Q \text{ , } \sim Q \vee R \text{ , } \sim Q \vee S]$
- Resolve KB(1) and KB(2) producing: $\sim P \vee R$ *(i.e., $P \rightarrow R$)*
- Resolve KB(1) and KB(3) producing: $\sim P \vee S$ *(i.e., $P \rightarrow S$)*
- New KB: $[\sim P \vee Q \text{ , } \sim Q \vee \sim R \vee \sim S \text{ , } \sim P \vee R \text{ , } \sim P \vee S]$

Soundness of the resolution inference rule

α	β	γ	$\alpha \vee \beta$	$\neg\beta \vee \gamma$	$\alpha \vee \gamma$
False	*False*	*False*	*False*	*True*	*False*
False	*False*	*True*	*False*	*True*	*True*
False	*True*	*False*	*True*	*False*	*False*
False	*True*	*True*	*True*	*True*	*True*
True	*False*	*False*	*True*	*True*	*True*
True	*False*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*False*	*True*
True	*True*	*True*	*True*	*True*	*True*

From the rightmost three columns of this truth table, we can see that

$$(\alpha \vee \beta) \wedge (\beta \vee \gamma) \leftrightarrow (\alpha \vee \gamma)$$

is valid (i.e., always true regardless of the truth values assigned to α, β and γ

Proving things

- A **proof** is a sequence of sentences, where each is a premise or is derived from earlier sentences in the proof by an inference rule
- The last sentence is the **theorem** (also called goal or query) that we want to prove
- Example for the "weather problem"

1 Hu	premise	"It's humid"
2 $Hu \rightarrow Ho$	premise	"If it's humid, it's hot"
3 Ho	modus ponens(1,2)	"It's hot"
4 $(Ho \wedge Hu) \rightarrow R$	premise	"If it's hot & humid, it's raining"
5 $Ho \wedge Hu$	and introduction(1,3)	"It's hot and humid"
6 R	modus ponens(4,5)	"It's raining"

Horn sentences

- A **Horn sentence** or **Horn clause** has the form:

  $P1 \wedge P2 \wedge P3 \ldots \wedge Pn \rightarrow Qm$ *where n>=0, m in{0,1}*

- Note: a conjunction of 0 or more symbols to left of $\rightarrow$ and 0-1 symbols to right
- Special cases:
  - n=0, m=1: P (assert P is true)
  - n>0, m=0: $P \wedge Q \rightarrow$ (constraint: both P and Q can't be true)
  - n=0, m=0: (well, there is nothing there!)
- Put in CNF: each sentence is a disjunction of literals with at most one non-negative literal

  $\neg P1 \vee \neg P2 \vee \neg P3 \ldots \vee \neg Pn \vee Q$

$$\boxed{(P \rightarrow Q) = (\neg P \vee Q)}$$

Significance of Horn logic

- We can also have horn sentences in FOL
- Reasoning with horn clauses is much simpler
  - Satisfiability of a propositional KB (i.e., finding values for a symbols that will make it true) is NP complete
  - Restricting KB to horn sentences, satisfiability is in P
- For this reason, FOL Horn sentences are the basis for Prolog and Datalog
- What Horn sentences give up are handling, in a general way, (1) negation and (2) disjunctions

Entailment and derivation

- **Entailment: KB |= Q**
  - Q is entailed by KB (set sentences) iff there is no logically possible world where Q is false while all the sentences in KB are true
  - Or, stated positively, Q is entailed by KB iff the conclusion is true in every logically possible world in which all the premises in KB are true
- **Derivation: KB |- Q**
  - We can derive Q from KB if there's a proof consisting of a sequence of valid inference steps starting from the premises in KB and resulting in Q

Two important properties for inference

Soundness: If KB |- Q then KB |= Q

- If Q is derived from KB using a given set of rules of inference, then Q is entailed by KB
- Hence, inference produces only real entailments, or any sentence that follows deductively from the premises is valid

Completeness: If KB |= Q then KB |- Q

- If Q is entailed by KB, then Q can be derived from KB using the rules of inference
- Hence, inference produces all entailments, or all valid sentences can be proved from the premises

Problems with Propositional Logic

Propositional logic: pro and con

- Advantages
  - Simple KR language sufficient for some problems
  - Lays the foundation for higher logics (e.g., FOL)
  - Reasoning is decidable, though NP complete, and efficient techniques exist for many problems
- Disadvantages
  - Not expressive enough for most problems
  - Even when it is, it can very “un-concise”

PL is a weak KR language

- Hard to identify “individuals” (e.g., Mary, 3)
- Can’t directly talk about properties of individuals or relations between individuals (e.g., “Bill is tall”)
- Generalizations, patterns, regularities can’t easily be represented (e.g., “all triangles have 3 sides”)
- First-Order Logic (FOL) is expressive enough to represent this kind of information using relations, variables and quantifiers, e.g.,
  - *Every elephant is gray:* $\forall$ x (elephant(x) $\rightarrow$ gray(x))
  - *There is a white alligator:* $\exists$ x (alligator(X) ^ white(X))

PL Example

- Consider the problem of representing the following information:
  - Every person is mortal.
  - Confucius is a person.
  - Confucius is mortal.
- How can these sentences be represented so that we can infer the third sentence from the first two?

PL Example

- In PL we have to create propositional symbols to stand for all or part of each sentence, e.g.:

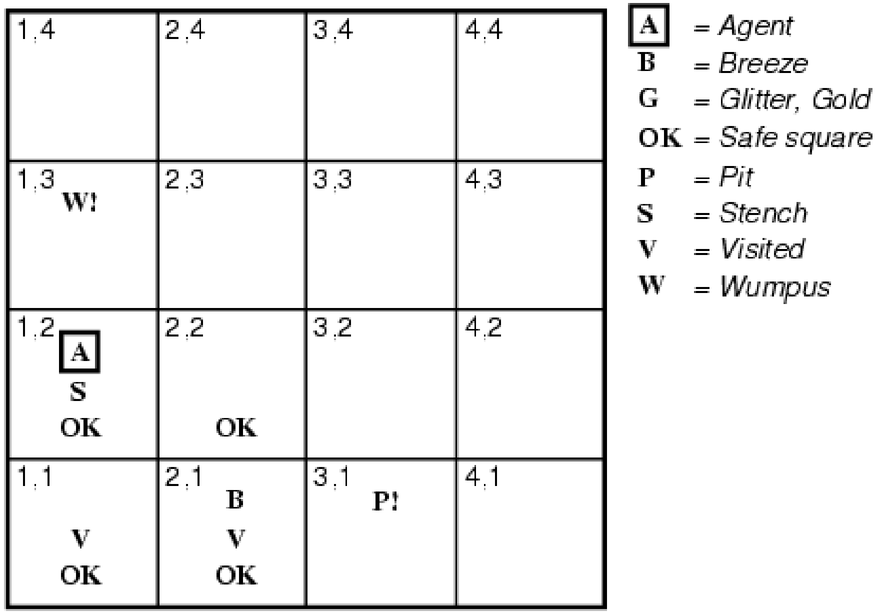
  P = “person”; Q = “mortal”; R = “Confucius”

- The above 3 sentences are represented as:

  $P \rightarrow Q$; $R \rightarrow P$; $R \rightarrow Q$

- The 3rd sentence is entailed by the first two, but we need an explicit symbol, R, to represent an individual, Confucius, who is a member of the classes *person* and *mortal*
- Representing other individuals requires introducing separate symbols for each, with some way to represent the fact that all individuals who are “people” are also “mortal”

Hunt the Wumpus domain



- Some atomic propositions:

  S12 = There is a stench in cell (1,2)

  B34 = There is a breeze in cell (3,4)

  W22 = Wumpus is in cell (2,2)

  V11 = We've visited cell (1,1)

  OK11 = Cell (1,1) is safe.

  …

- Some rules:

  (R1) $\neg S11 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W21$

  (R2) $\neg S21 \rightarrow \neg W11 \wedge \neg W21 \wedge \neg W22 \wedge \neg W31$

  (R3) $\neg S12 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W22 \wedge \neg W13$

  (R4) $S12 \rightarrow W13 \vee W12 \vee W22 \vee W11$

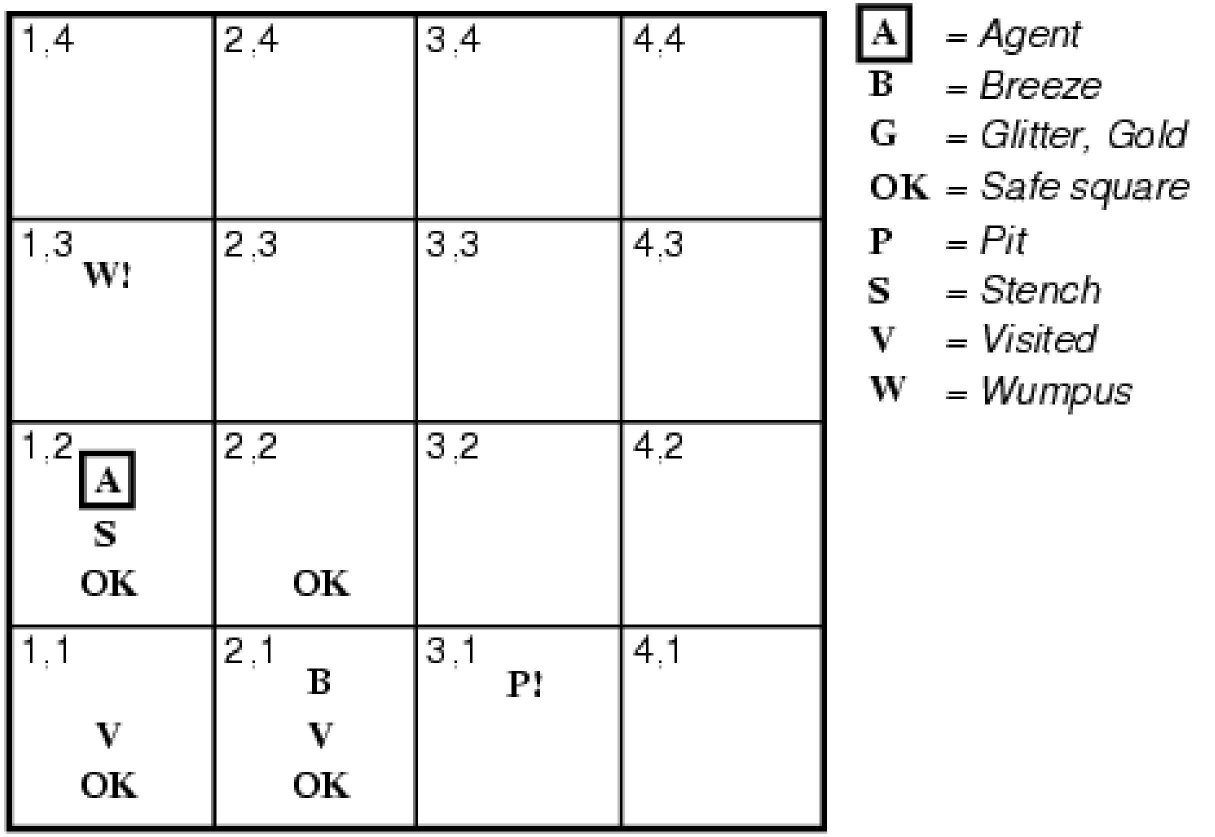
  …

- The lack of variables requires us to give similar rules for each cell!

After the third move

We can prove that the Wumpus is in (1,3) using the four rules given.

See R&N section 7.5



Proving W13

Apply MP with $\neg$S11 and R1:

$\neg W11 \wedge \neg W12 \wedge \neg W21$

Apply And-Elimination to this, yielding 3 sentences:

$\neg W11, \neg W12, \neg W21$

Apply MP to ~S21 and R2, then apply And-elimination:

$\neg W22, \neg W21, \neg W31$

Apply MP to S12 and R4 to obtain:

$W13 \vee W12 \vee W22 \vee W11$

Apply Unit resolution on $(W13 \vee W12 \vee W22 \vee W11)$ and $\neg W11$:

$W13 \vee W12 \vee W22$

Apply Unit Resolution with $(W13 \vee W12 \vee W22)$ and $\neg W22$:

$W13 \vee W12$

Apply UR with $(W13 \vee W12)$ and $\neg W12$:

$W13$

QED

Propositional Wumpus hunter problems

- Lack of variables prevents stating more general rules
  - We need a set of similar rules for each cell
- Change of the KB over time is difficult to represent
  - Standard technique is to index facts with the time when they're true
  - This means we have a separate KB for every time point

Propositional logic summary

- Inference is the process of deriving new sentences from old
  - **Sound** inference derives true conclusions given true premises
  - **Complete** inference derives all true conclusions from a set of premises
- A **valid sentence** is true in all worlds under all interpretations
- If an implication sentence can be shown to be valid, then—given its premise—its consequent can be derived
- Different logics make different **commitments** about what the world is made of and what kind of beliefs we can have
- **Propositional logic** commits only to the existence of facts that may or may not be the case in the world being represented
  - Simple syntax and semantics suffices to illustrate the process of inference
  - Propositional logic can become impractical, even for very small worlds