HW - 1

EE16BTECH11038

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(1)

From plots, it can be observed that, Chi-square is not sub gaussian and from previous assignment we found out that it is sub exponential.

But as n increases and tends to infinity from central limit theorem the distribution goes to a gaussian thus the tail probability as $N \to$ infinity tends to that of a gaussian.

(3) $f(t) = \sup(\lambda t - \log[E(e^{\lambda x})])$

Let X be a Bernoulli random variable such that

$$X = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

$\boxed{E(X) = p}$

Now Let

$$Y = X - p = \begin{cases} 1-p, & p \\ -p, & 1-p \end{cases}$$

$E(Y_i) = 0$

Thus Y is the centered Bernoulli RV with parameter p.

$$E(e^{\lambda y}) = e^{-\lambda p}(1-p+pe^{\lambda})$$

Now plugging it back to Cramer's expression:

$$I(t) = \sup_{\lambda} \{ \lambda t + \lambda p - \log(1-p+pe^{\lambda}) \}$$

Now Maximum $g(\lambda) = \lambda t + \lambda p - \log(1-p+pe^{\lambda})$ wrt λ.

$$g'(\lambda) = (t+p) - \frac{pe^{\lambda}}{(1-p+pe^{\lambda})} = 0$$

$$(t+p) = \frac{pe^{\lambda}}{(1-p+pe^{\lambda})}$$

$$(pe^{\lambda}) = (p+t)(1-p) + (t+p)\, pe^{\lambda}$$

$$pe^{\lambda} = \frac{(p+t)(1-p)}{(1-t-p)}$$

$$\lambda = \log\left[\frac{(p+t)(1-p)}{p(1-p-t)}\right]$$

Plugging λ back

we get

$$f(t) = (p+t)\log(L) - \log\left(1-p+p(L)\right)$$

where $\boxed{L = \frac{(p+t)(1-p)}{p(1-p-t)}}$

$$f(t) = (p+t)\log\left(\frac{(p+t)(1-p)}{p(1-p-t)}\right) - \log\left(1-p+p\left(\frac{(p+t)(1-p)}{p(1-p-t)}\right)\right)$$

$$f(t) = \log\left(\left(\frac{(p+t)(1-p)}{p(1-p-t)}\right)^{p+t}\right) + \log\left[\frac{(p)(1-p-t)}{p-p^2}\right]$$

$$f(t) = \log\left[\left(\frac{(p+t)(1-p)}{p(1-p-t)}\right)^{p+t}\left[\frac{(p)(1-p-t)}{p(1-p)}\right]\right]$$

$$f(t) = \log\left[\frac{(p+t)^{p+t}(1-p)^{p+t-1}}{p^{p+t}(1-p-t)^{p+t-1}}\right]$$

$$f(t) = \log\left[\left(1+\frac{t}{p}\right)^{p+t}\left(\frac{1-p}{1-p-t}\right)^{p+t-1}\right]$$

$$f(t) = (p+t)\log\left(1+\frac{t}{p}\right) - (p+t-1)\log\left(1-\left(\frac{t}{1-p}\right)\right)$$

(4) Bennett's inequality

$x_1, x_2 \dots x_n$ are centered

$x_i \le b$; $b > 0$

$v = \sum_{i=1}^{n} \mathbb{E}(x_i^2)$; $S = \sum_{i=1}^{n} (x_i - \mathbb{E}(x_i))$

$$\log(\mathbb{E}[e^{\lambda S}]) \le n \log\left(1 + \frac{v}{nb^2}\phi(\lambda b)\right) \le \frac{v}{b^2}\phi(\lambda b)$$

$$\phi(u) = e^u - u - 1$$

Now e^x is an monotonically increasing function

$$\log[\mathbb{E}(e^{\lambda S})] \le \frac{v}{b^2}\phi(\lambda b)$$

$$\mathbb{E}(e^{\lambda S}) \le e^{\frac{v}{b^2}\phi(\lambda b)} \quad \text{— (a)}$$

Now,

$$P(S \ge t) = P(e^{\lambda S} \ge e^{\lambda t}) \le \frac{\mathbb{E}(e^{\lambda S})}{e^{\lambda t}}$$

From chernoff's bound.

$$P(S \ge t) \le e^{-\lambda t}\, \mathbb{E}(e^{\lambda S})$$

Now we know from (a)

$$P(S \ge t) \le e^{-\lambda t} e^{\frac{v}{b^2}\phi(\lambda b)}$$

$$\le e^{-\lambda t + \frac{v}{b^2}\phi(\lambda b)}$$

Now we find minima of $f(\lambda) = -\lambda t + \frac{v}{b^2}\phi(\lambda b)$ wrt λ.

$$f'(\lambda) = -t + \frac{d}{d\lambda}\left((e^{\lambda b} - \lambda b - 1)\frac{v}{b^2}\right)$$

$$= -t + \frac{\cancel{b}ve^{\lambda b}}{b} - \frac{\cancel{b}v}{b} = 0$$

$$\Rightarrow \lambda = \frac{1}{b}\log\left(1 + \frac{tb}{v}\right)$$

substituting it back,

$$P(S \geq t) \leq \exp\left\{\frac{-t}{b}\log\left(1+\frac{tb}{v}\right) + \frac{v}{b^2}\left(1+\frac{tb}{v} - \log\left(1+\frac{tb}{v}\right) - 1\right)\right.$$

$$P(S \geq t) \leq \exp\left\{\frac{-t}{b}\log\left(1+\frac{tb}{v}\right) + \frac{v}{b^2}\left[1+\frac{tb}{v} - 1 - \log\left(1+\frac{tb}{v}\right)\right]\right\}$$

$$P(S \geq t) \leq \exp\left\{-\frac{t}{b}\log\left(1+\frac{tb}{v}\right) + \frac{t}{b} - \frac{v}{b^2}\log\left(1+\frac{tb}{v}\right)\right\}$$

$$P(S \geq t) \leq \exp\left\{\frac{-v}{b^2}\left[\frac{tb}{v}\log\left(1+\frac{tb}{v}\right) - \frac{tb}{v} + \log\left(1+\frac{tb}{v}\right)\right]\right\}$$

$$P(S \geq t) \leq \exp\left\{\frac{-v}{b^2}\left[u\log(1+u) - u + \log(1+u)\right]\right\}$$

where $\boxed{u = tb/v}$

$$P(S \geq t) \leq \exp\left\{-\frac{v}{b}\left[(1+u)\log(1+u) - u\right]\right\}$$

where $u = \frac{tb}{v}$

Now Let

$$h(u) = \left[\log(1+u)\right](1+u) - u$$

Thus

$$P(S \geq t) \leq \exp\left[-\frac{v}{b^2} h\left(\frac{tb}{v}\right)\right]$$

Hence proved