Assignment-2

S. Sai Ashish,
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(1) By efron stein inequality:

$$var(Z) \leq \sum_{i=1}^{n} E((Z - E^i(Z))^2]$$

the RHS, can be manipulated as

$$E((Z - E^i(Z))^2] = E(E^i(Z - E^i(Z))^2]$$

Let,

$$Z_{i=1} = f(x_1, \dots x_i = 1, \dots x_n)$$

$$Z_{i=-1} = f(x_1, \dots x_i = -1, \dots x_n)$$

$$\boxed{E^i(Z) = P Z_{i=1} + (1-P) Z_{i=-1}}$$

$$E((Z - E^i(Z))^2) = E(E^i[(Z - P_i Z_{i=1} - (1-P_i) Z_{i=-1})^2]]$$

$$= E(E^i[(Z - Z_{i=-1}) + P_i(Z_{i=-1} - Z_{i=1})]^2]$$

$$= E[E^i[(Z - Z_{i=-1})^2 + P_i^2(Z_{i=-1} - Z_{i=1})^2 + 2P_i(Z_{i=-1} - Z_{i=1})(Z - Z_{i=-1})]$$

on expanding expectation $E^i[f(z)]$ [redacted])

$$= E(P_i((Z_1 - Z_1 P_i - (1-P_i) Z_{-1}))^2 + (1-P_i)(Z_{-1} - Z_1 P_i - (1-P_i) Z_{-1})^2]$$

$$= E(P_i(1-P_i)^2(Z_1 - Z_{-1})^2 + (1-P_i)P_i^2(Z_1 - Z_{-1})^2)$$

Now

$$|Z_1 - Z_7| = |f(x_1, \ldots, x_i = 1, \ldots x_n) - f(x_1, \ldots x_{i=1}, \ldots x_n)|$$
$$\leq c_i$$

by using this

$$E(Z - E^i(Z))^2 \leq P_i(1-P_i)^2 c_i^2 + (1-P_i)P_i^2 c_i^2$$
$$\leq P_i(1-P_i) c_i^2$$

Substituting back.

$$\boxed{Var(Z) \leq \sum_{i=1}^{n} P_i(1-P_i) c_i^2}$$

Efron-Stein inequality

Let $X_1, \cdots X_n$ be independent RV's. Let $f: X^n \to \mathbb{R}$ be a square integrable function

Let $Z = f(X_1, \cdots X_n)$,

(•) $E_i(Z) = E(Z \mid X_1, X_2, \cdots X_n)$

(•) $E^i(Z) = \int_{x_i \in X} f(x_1, \cdots x_i, \cdots x_n)\, dP(x_i)$

(•) $\Delta_i = E_i(Z) - E_{i-1}(Z)$

(•) $\sum_{i=1}^{n} \Delta_i = E_n(Z) - \cancel{E_{n-1}(Z)} + \cancel{E_{n-1}(Z)} - \cancel{E_{n-2}(Z)} + \cdots + \cancel{E_1(Z)} - E(Z)$

$\sum_{i=1}^{n} \Delta_i = E_n(Z) - E(Z)$

$\boxed{E_n(Z) = E(Z \mid X_1, \cdots X_n) = Z}$

$\boxed{\sum_{i=1}^{n} \Delta_i = Z - E(Z)}$ — ①

(a) Show that

$\mathrm{var}(Z) = E\left(\sum_{i=1}^{n} \Delta_i^2\right)$

$$\mathrm{var}(Z) = E\left((Z - E(Z))^2\right)$$

$$= E\left(\left(\sum_{i=1}^{n} \Delta_i\right)^2\right)$$

$$= E\left(\sum_{i=1}^{n} \Delta_i^2 + 2\sum_{j>i} \Delta_i \Delta_j\right)$$

$$= E\left(\sum_{i=1}^{n} \Delta_i^2\right) + 2\sum_{j>i} E(\Delta_i \Delta_j)$$

Now

$$E(\Delta_i \Delta_j) = E\left[E_i(\Delta_i \Delta_j \mid x_i)\right] \quad \because j > i$$

$$= E\left[\Delta_i \, E_i(\Delta_j)\right] \qquad \text{since } E_i(\Delta_i) = \Delta_i$$

Taking

$$E_i \Delta_j = E_i\left[E_j(Z) - E_{j-1}(Z)\right]$$

$$= E_i\left[E\left[f(x_1 \cdots x_n) \mid x_1, x_2 \cdots x_i, (x_{i+1}, x_{i+2} \cdots x_j)\right]\right] - E_i\left(E\left(f(x_1, \cdots x_n) \mid x_1, \cdots x_i, (x_{i+1}, \cdots, x_{j-1})\right)\right)$$

$$= \int\int \cdots \int f(x^n) \, \mathrm{pdf}(x_{i+1}) \, \mathrm{pdf}(x_{i+2}) \cdots \mathrm{pdf}(x_n) \, dx_{i+1} \, dx_{i+2} \cdots dx_n$$

$$- \int \cdots \int f(x^n) \, \mathrm{pdf}(x_{i+1}) \, \mathrm{pdf}(x_{i+2}) \cdots \mathrm{pdf}(x_n) \, dx_{i+1} \, dx_{i+2} \cdots dx_n$$

$$= 0$$

$$E(\Delta_i \Delta_j) = E(\Delta_i \, E_i(\Delta_j)) = \underline{\underline{0}}$$

Thus

$$\operatorname{var}(z) = E\left(\sum_{i=1}^{n} \Delta_i^2\right)$$

(b) $$E_i\left[E^i(z)\right] = E_i\left(f(x^n) \mid X_i^n\right)$$

$$\int_{x_i \in \mathcal{X}} \int_{x_{i+1}^n \in \mathcal{X}^{n-i}} f(x_1, \dots x_{i-1}, x_i, x_{i+1}, \dots x_n)\, dP(x_{i+1}^n)\, dP(x_i)$$

$$= \int_{x_i^n \in \mathcal{X}^{n-i+1}} f(x_1, \dots x_{i-1}, x_i, x_{i+1}, \dots x_n)\, dP(x_i^n)$$

$$= E_{i-1}(z)$$

Thus

$$E_i\left[E^i(z)\right] = E_{i-1}(z)$$

(c)

$$\Delta_i = E_i(z) - E_{i-1}(z)$$

$$= E_i(z) - E_i\left(E^i(z)\right)$$

$$= E_i\left(z - E^i(z)\right)$$

Thus

$$\Delta_i^2 = \left[E_i\left[Z - E^i(Z)\right]\right]^2 \quad \text{— (a)}$$

From Jensen's inequality,

If $g(x)$ is convex, let T be random variable

Then $\boxed{g(E(T)) \le E(g(T))}$

Here

$g(x) = x^2$ and $T = (Z - E^i(Z))$

Thus

$$E_i\left((Z - E^i(Z))^2\right) \ge \left[E_i(Z - E^i(Z))\right]^2 \quad \text{— (b)}$$

By this property. use the (b) in (a)

$$\Delta_i^2 = \left[E_i(Z - E^i(Z))\right]^2 \le E_i\left[(Z - E^i(Z))^2\right]$$

$$\boxed{\Delta_i^2 \le E_i\left((Z - E^i(Z))^2\right)}$$

(c) From 2(a) soln

we got

$$Var(Z) = \sum_{i=1}^{n} E(\Delta_i^2)$$

we know $\Delta_i^2 \le E_i(Z - E^i(Z))^2$

$$\mathrm{Var}(Z) \leq \sum_{i=1}^{n} \underline{\underline{E\left(E_i\left[(Z - E^i(Z))^2\right]\right)}}$$

$$\to$$

$$E\left(E_i\left[(Z - E^i(Z))^2\right]\right)$$

$$= \left[E\left(E\left[(Z - E^i(Z))^2\right] \middle| X_i \right)\right]$$

$$= E\left(\int_{x_{i+1}^n \in \mathcal{X}^{n-i-1}} \left(Z - E^i(Z)\right)^2 dP_{X_{i+1}^n} \right)$$

$$= \int_{[x_1, \dots, x_i] \in \mathcal{X}^i} \left[\int_{x_{i+1}^n \in \mathcal{X}^{n-i}} \left(Z - E^i(Z)\right)^2 dP(X_{i+1}^n) \right] dP(x_1) \cdots dP(x_i)$$

$$= \int_{x_1^n \in \mathcal{X}^n} \left(Z - E^i(Z)\right)^2 dP(X_1^n) = E\left[(Z - E^i(Z))^2\right]$$

Thus,

$$\boxed{\mathrm{Var} \leq \sum_{i=1}^{n} E\left[\left(Z - E^i(Z)\right)^2\right]}$$