# Analysis of Several Variables

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# Chapter 1

# Topology of Euclidean Space

## 1.1 Introduction

**Definition 1.**

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \cdots, x_n) | x_1, \cdots, x_n \in \mathbb{R}\}$$

## 1.2 $\mathbb{R}^n$ as a Vector Space

- $\langle x, y \rangle = \sum_{i=1}^{n} \ \forall \ x, y = \mathbb{R}^n$
- $\{e_i\}_{i=1}^{n}$ is an orthonormal basis for $\mathbb{R}^n$
- Simplest maps $\mathbb{R}^n \to \mathbb{R}^m$? **Linear maps: It sends lines to lines**

**Example 1.** *Linear map* $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = xf(1) \ \forall \ x \in \mathbb{R}$$

**Corollary 1.** *if* $c \in \mathbb{R}$, *then* $x \mapsto cx$ *is a liner map*
$therfore \{f : \mathbb{R} roundback Linear\} \iff \mathbb{R}$

**Remark 1.** *Let* $l : \mathbb{R}^n - > \mathbb{R}^m$ *be a linear map.*

$$le_j = \sum_{i=1}^{m} a_{ij} e_i \ \forall \ j = 1, \cdots, n$$

*we write* $[L]_{\{e_i\}_{i=1}^{n}}^{\{e_j\}^{m}{j-1} = (a_{ij})_{m \times n}}$

**Definition 2.** *Define distance function* $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \geq 0$ *by* $d(x, y) = ||x - y||$

**Definition 3** (Scalar Product)**.**

**Remark 2.**

- $|x| = \langle x, y \rangle^{1/2}$
- $<, >$ *is liner w.r.t 1st and second slot*

**Theorem 1** (Cauchy Schwarz inequality )**.**

$$\forall \ x, y \in \mathbb{R}^n, < x, y > \leq |x| \, |y|$$

*Proof.*

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j - x_j y_i)^2 = 2 \left[ \sum \sum x_i^2 y_i^2 - \sum_{i,j} x_i x_j y_i y_j \right] \implies < x, y > \leq |x| \, |y|$$
$$= \left[ |x|^2 \, |y|^2 - < x, y >^2 \right]$$

□

**Remark 3.** *equality occurs if and only if* $x_i y_j = x_j y_i \ \forall\ i, j$

**Theorem 2.** *Let* $L : \mathbb{R}^n \to \mathbb{R}^m$ *a linear map. Then* $\exists M > 0$ *such that* $|Lx| \leq M\,|x| \ \ \forall\ x \in \mathbb{R}^n$

*Proof.* As $x = \sum_{i=1}^{n} x_i e_i$,
so $Lx = \sum_{i=1}^{n} x_i L e_i$

$$\implies |Lx| = \left| \sum_{i=1}^{n} x_i L e_i \right| \leq \sum_{i=1}^{n} |x_i L e_i| = \sum |x_i|\,|Le_i| \leq |x|\,(|Le_i|)$$

$$\left\langle x, y^{22} \right\rangle$$

$\square$

## 1.3 Connectedness

$$\circlearrowleft$$

$$\left\| e^{e^{e^{e^{e^{e^{e}}}}}} \right\| \left\| x^2 \frac{x}{e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{e}}}}}}}}}}}} \right\|$$

# Chapter 2

# Differentiation

## 2.1 The Derivative as a Linear Map

## 2.2 The Chain Rule

## 2.3 Inverse and Implicit Function Theorems

# Chapter 3

# Integration

## 3.1 Multiple Integrals

## 3.2 Change of Variables

## 3.3 Fubini's Theorem

# Chapter 4

# Manifolds

## 4.1 Submanifolds of Euclidean Space

## 4.2 Tangent Spaces

## 4.3 Differential Forms