Analysis of Several Variables

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Chapter 1

Topology of Euclidean Space

1.1 Introduction

Definition 1.

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \cdots, x_n) | x_1, \cdots, x_n \in \mathbb{R}\}$$

1.2 $\mathbb{R}^n$ as a Vector Space

- $\langle x, y \rangle = \sum_{i=1}^{n} \ \forall \ x, y = \mathbb{R}^n$
- $\{e_i\}_{i=1}^{n}$ is an orthonormal basis for $\mathbb{R}^n$
- Simplest maps $\mathbb{R}^n \to \mathbb{R}^m$? **Linear maps: It sends lines to lines**

Example 1. *Linear map $f : \mathbb{R} \to \mathbb{R}$*

$$f(x) = xf(1) \ \forall \ x \in \mathbb{R}$$

Corollary 1. *if $c \in \mathbb{R}$, then $x \mapsto cx$ is a liner map*
$therfore \{f : \mathbb{R} roundback Linear\} \iff \mathbb{R}$

Remark 1. *Let $l : \mathbb{R}^n - > \mathbb{R}^m$ be a linear map.*

$$le_j = \sum_{i=1}^{m} a_{ij} e_i \ \forall \ j = 1, \cdots, n$$

we write $[L]_{\{e_i\}_{i=1}^{n}}^{\{e_j\}^{m} j-1 = (a_{ij})_{m \times n}}$

Definition 2. *Define distance function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \geq 0$ by $d(x, y) = ||x - y||$*

Definition 3 (Scalar Product)**.**

Remark 2.
- $|x| = \langle x, y \rangle^{1/2}$
- *$<, >$ is liner w.r.t 1st and second slot*

Theorem 1 (Cauchy Schwarz inequality)**.**

$$\forall \ x, y \in \mathbb{R}^n, < x, y > \leq |x| \, |y|$$

Proof.

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j - x_j y_i)^2 = 2 \left[\sum \sum x_i^2 y_i^2 - \sum_{i,j} x_i x_j y_i y_j \right] \implies < x, y > \leq |x| \, |y|$$
$$= \left[|x|^2 \, |y|^2 - < x, y >^2 \right]$$

$\square$

Remark 3. *equality occurs if and only if* $x_i y_j = x_j y_i \ \forall \ i, j$

Theorem 2. *Let* $L : \mathbb{R}^n \to \mathbb{R}^m$ *a linear map. Then* $\exists M > 0$ *such that* $|Lx| \leq M\,|x| \ \ \forall \ x \in \mathbb{R}^n$

Proof. As $x = \sum_{i=1}^{n} x_i e_i$,
so $Lx = \sum_{i=1}^{n} x_i L e_i$

$$\implies |Lx| = \left| \sum_{i=1}^{n} x_i L e_i \right| \leq \sum_{i=1}^{n} |x_i L e_i| = \sum |x_i|\,|Le_i| \leq |x|\,(|Le_i|)$$

$$\left\langle x, y^{22} \right\rangle$$

□

1.3 Connectedness

$$\circlearrowleft$$

$$\left\| e^{e^{e^{e^{e^{e^{e}}}}}} \right\| \left\| x^2 \frac{x}{e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{e}}}}}}}}}}}} \right\|$$

Chapter 2

Differentiation

2.1 The Derivative as a Linear Map

2.2 The Chain Rule

2.3 Inverse and Implicit Function Theorems

Chapter 3

Integration

3.1 Multiple Integrals

3.2 Change of Variables

3.3 Fubini's Theorem

Chapter 4

Manifolds

4.1 Submanifolds of Euclidean Space

4.2 Tangent Spaces

4.3 Differential Forms