Group Theory

V Sai Prabhav

ISI Year II, Semester I

Chapter 1

Introduction to Sets

Set Theory, there is a symbol $'\epsilon'$ called belongs to. A statement in set theory can look like this $x\epsilon y$.

Axiom 1. *There exist a set with no elements.*

Axiom 2. *Two sets with same elements are equal*

Definition 1. $A \subseteq B$ *if all elements of* A *are in* B

Axiom 3. *if* X, Y *are sets then* $\{x, y\}$

Axiom 4. A *is a set then* $unionA = \{x \in y | y \in A\}$

Axiom 5 (Power set). *Given a set* A *there exist* $P(A) = \{B \mid \ \forall \ B \subseteq A\}$

Axiom 6 (Infinite axiom). *There exist* $I \ \phi \in I \ \forall \ y \in I(P(y) \in I)$

Axiom 7. *A, B are sets then* $A \times B = \{(x, y) \mid x \in A, y \in B\}$ *is a set.*

Definition 2 (Relation). *Relation* R *on a set ea is a subset* $R \subseteq A \times A$

$$(x, y) \in R \text{ then } xRy$$

Axiom 8 (Axiom of choice). *Let* A *be a collection of non empty disjoint sets then there exist a set* C *consisting of exactly one element from each set in* A.

Definition 3. *A relation* R *is called reflexive if* $xRx \ \forall \ x \in A$.
A relation R *is called symmetric if* xRy *then* yRx.
A relation R *is called anti-symmetric if* xRy, yRy *then* $y = x$.
A relation R *is called transitive if* xRy, yRz *then* xRz.

Definition 4 (Partial order). *A partial order on set* A *is a reflexive transitive anti-symmetric relation.*

Definition 5 (Total order). *Total order* R *on a set* A *is a partial order such that* $\forall \ x, y \in A$ *either* xRy *or* yRx

Definition 6 (Minimal element). *A total order on a set* A *is called a well ordered if given any non empty subset* B *of* A.

Theorem 1 (Well-ordering Principle). *Every set can be well ordered. (It is equvivalent to Axiom 8)*

Lemma 1 (Zorn's Lemma). *Let* A *be a set and* $\leq$ *be a partial order on* A *such that every chain in* A *has an upper bound in* A *then* A *has a maximal element.*

Definition 7 (Chain). *A chain in* A *is a subset which is a totally ordered with respect to* $<$

Definition 8 (Upper Bound). *Let* $C \subseteq A$. $x \in A$ *is an upper bound of* C *if* $\forall \ y \in Cy < x$.

Definition 9 (Maximal element). $x \in A$ *is called a maximal element if* $\forall \ y \in Ax < y \implies x = y$

Theorem 2. *TFAE*

- *Axiom of choice*
- *Well- ordering principle*
- *Zorin's lemma*

Definition 10. *A relation R is called an equivalence relation if it is reflexive ,symmetric and transitive.*

Definition 11 (Equivalence class)**.** *Let $x \in A$ then $[x] = \{yRx \mid y \in A\} \subseteq$ called the equivalence class of x.*

Definition 12. $\bigcup_{x\in A}[x] = A$ *and for* $x, y \in A$ $[x] \cap [y] = \phi$ *or* $[x] = [y]$. *That is we get a partition of A.*

Axiom 9. *I be an indexing set. Then* $\times_{i\in I} A_i = \{f : I \to UA_i \mid$ *such that* $f(i) \in A_i\}$

Definition 13 (Principle of Induction)**.** *let $S(n), n \in \mathbb{N}$ be statements. Suppose $S(1)$ is true and $S(n) \implies S(n+1)$ then $S(n)$ is true for all n.*

Definition 14 (Transfinite Induction)**.** *Let I be a well ordered set and $S(i)$ be a statement for $\forall\ i \in I$. Suppose $s(j)\ \forall\ j < i \implies s(i)$ then $S(i)$ is true $\forall\ i \in I$*

Definition 15 (Group)**.** *A group is a triple $(G, ., e)$ where G is a set, . is a binary operator on G and $e \in G$ satisfying the following axioms.*

- *for* $a, b, c \in G(a \cdot b).c = a \cdot (b \cdot c)$.
- $a \cdot e = e \cdot a = a\ \forall\ a \in G$.
- $\forall\ a \in G\ \exists\ b \in G$ *such that* $a \cdot b = b \cdot a = e$.

Definition 16. $G, \cdot, e$ *is called abelian if* $\forall\ a, b \in G a \cdot b = b \cdot a$

1.1 Binary Operations

1.2 Groups and Examples

1.3 Elementary Properties of Groups

Chapter 2

Subgroups and Cosets

2.1 Subgroups

2.2 Cyclic Groups

2.3 Cosets and Lagrange's Theorem

Chapter 3

Homomorphisms and Isomorphisms

3.1 Homomorphisms

3.2 Kernel and Image

3.3 Isomorphism Theorems

Chapter 4

Group Actions

4.1 Group Actions and Permutation Representations

4.2 Orbits and Stabilizers

4.3 The Class Equation