Exercise sheet 4

1. Let X be a uniform $\{1, 2, \cdots, N\}$ random variable; that is X takes any of the N values with equal probability $\frac{1}{N}$. Find $E[X]$ and $\mathrm{Var}(X)$.

2. Suppose r balls are randomly distributed in n urns. Let X be the number of empty urns. With n fixed, write $p_r(m) = P(X = m)$ for the distribution of X parametrised by r. Show that the recurrence
$$p_{r+1}(m) = p_r(m)\frac{n-m}{n} + p_r(m+1)\frac{m+1}{n}$$
holds for all m and r and solve it.
Show that the expected number of empty urns is $n\left(1 - \frac{1}{n}\right)^r$.

3. An urn contains r red balls and b blue balls. Balls are removed sequentially from the urn (without replacement). What is the expected number of balls left in the urn at the *first* instant at which all the remaining balls are of the same colour?

4. An urn contains balls numbered 1 to N. Let X be the largest number drawn in n drawings when random sampling with replacement is used. Find $E[X]$.

5. Two dice are thrown. Let X be the score on the first dies and Y be the larger of the two scores. Write down the joint distribution of X and Y. Find the means, the variances and the covariance.

6. Let X, Y, Z be independent random variables with the same geometric distribution $\{q^k p\}$. Find $P(X = Y)$, $P(X \geq 2Y)$ and $P(X + Y \leq Z)$.
Let U be the smaller of X and Y, and put $V = X - Y$. Show that U and V are independent.

7. Let X_1 and X_2 be independent Poisson random variables with parameters λ_1 and λ_2 respectively. What is the conditional distribution of X_1 given $X_1 + X_2$?

8. $X_1, X_2, \cdots X_n$ be mutually independent random variables, each having the uniform distribution on $\{1, 2, \cdots, N\}$. Let U_n be the smallest of $X_1, \cdots X_n$ and V_n be the largest. Find the distributions of U_n and V_n.

9. Given a biased coin such that the probability of heads is α, we simulate a perfect coin as follows. Throw the biased coin twice. Interpret HT as success and TH as failure; if neither event occurs repeat the throws until a decision is reached.
   - Show that this model leads to Bernoulli trials with $p = \frac{1}{2}$.
   - Find the distribution and expectation of the number of throws required to reach a decision.

10. [HW 4, due October 4] In a sequence of Bernoulli trials let X be the length of run (of either successes or failures) started by the first trial.
    - Find the distribution of X, $E(X)$, $\mathrm{Var}(X)$.
    - Let Y be the length of the second run. Find the distribution of Y, $E(Y)$, $\mathrm{Var}(Y)$, and the joint distribution of X and Y

11. The cumulative distribution function (CDF) of a random variable X is the function $F_X : \mathbf{R} \to [0, 1]$ given by $F_X(x) = P(X \leq x)$. Show the following:
    - F_X is a nondecreasing function: that is, if $a < b$, then $F_X(a) \leq F_X(b)$.
    - $\lim_{b\to\infty} F_X(b) = 1$.
    - $\lim_{b\to-\infty} F_X(b) = 0$.
    - F_X is right continuous. That is, for any b and any decreasing sequence $b_n, n \geq 1$ that converges to b, $\lim_{n\to\infty} F_X(b_n) = F_X(b)$.