0.1 Asymptotic Notations

0.1.1 Big-Oh $O(n)$

$T(n) \in O(n)$, there exists $c > 0$ and the value $n_0 \geq 0$ such that $0 \leq T(n) \leq c \cdot f(n)$ for all $n \geq n_0$. Note that $O(f(n))$ is set of functions.

0.1.2 Omega $\Omega(n)$

$T(n) \in \Omega(n)$, there exists $c > 0$ and the value $n_0 \geq 0$ such that $T(n) \geq c \cdot f(n)$ for all $n \geq n_0$. Note that $\Omega(f(n))$ is set of functions.

0.1.3 Theta $\Theta(n)$

$T(n) \in \theta(n)$, there exists $c_1 > 0$, $c_2 > 0$ and the value $n_0 \geq 0$ such that $c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)$ for all $n \geq n_0$. Note that $\Omega(f(n))$ is set of functions.

Example to prove $\in$

$$\textbf{let} \quad f(n) = \frac{(65n^4 + 2n + 3)}{(n+1)}$$

$$\textbf{prove} \quad \frac{(65n^4 + 2n + 3)}{(n+1)} = \Theta(n^3)$$

According to the definition of $\Theta(f(n))$, if there exists constants $c_1 \geq 0$, $c_2 \geq 0$, $n_0 \geq 0$ such that $c_1 \times f(n) \leq T(n) \leq c_2 \times g(n)$ for all $n \geq n_0$.

$$\frac{(65n^4)}{(n+4n)} \leq \frac{(65n^4 + 2n + 3)}{(n+1)} \leq \frac{(65n^4 + 2n^4 + 3n^4)}{n}$$

$$\frac{(65n^4)}{(5n)} \leq \frac{(65n^4 + 2n + 3)}{(n+1)} \leq \frac{(70n^4)}{n}$$

$$13n^3 \leq \frac{(65n^4 + 2n + 3)}{(n+1)} \leq 70n^3$$

$$c_1 = 13, c_2 = 70, n_0 = 1$$

Therefore, $f(n) = \theta(n^3)$

$$\textbf{let} \quad g(n) = 21 \times n\log(n) + 2n + 1$$

$$\textbf{prove} \quad 21 \times n\log(n) + 2n + 1 = O(n\log(n))$$

According to the definition of $\Theta(f(n))$, if there exists constants $c_1 \geq 0$, $c_2 \geq 0$, $n_0 \geq 0$ such that $c_1 \times f(n) \leq T(n) \leq c_2 \times g(n)$ for all $n \geq n_0$.

$$n\log(n) \leq 21 \times n\log(n) + 2n + 1 \leq 21 \times n\log(n) + 2 \times n\log(n) + n\log(n)$$

$$n\log(n) \leq 21 \times n\log(n) + 2n + 1 \leq 24 \times n\log(n)$$

$$c_1 = 1, c_2 = 24, n_0 = 2$$

Therefore, $g(n) = \Theta(n\log(n))$

Example to prove $\notin$

Using ONLY the definition of O(f(n))/(f(n)) prove that the following statements are FALSE. Your proofs using Limits will not get a mark:

$$\textbf{Prove} \quad n^2 \notin \Theta(\log(n))$$

In order to prove this statement, we will use proof by contradiction:

$$\textbf{Assume} \quad n^2 \in O(\log(n))$$

This implies that there are constants c, n_0, such that the below inequality holds for all $n \geq n_0$.

$$n^2 \leq c \times \log(n)$$

let us consider $n = \lceil c \times n_0 \rceil$. Substituting $n = \lceil c \times n_0 \rceil$ in the above equation, we get:

$$(\lceil c \times n_0 \rceil)^2 \leq c \times (\log(\lceil c \times n_0 \rceil))$$

This is a Contradiction, since we know via algebraic manipulation, the L.H.S is in fact greater than the R.H.S for all $n \geq n_0$

By disproving $n^2 \notin O(\log(n))$, therefore, $n^2 \notin \Theta(\log(n))$

Useful Facts

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq \infty$, then $f(n) \in O(g(n))$
- $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$, then $f(n) \in \Omega(g(n))$
- $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$ and $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq \infty$, then $f(n) \in \Theta(g(n))$
- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ then $f(n) \sim (g(n))$

Helpful Results

- $O(f(n)) + O(g(n)) = O(f(n) + g(n)) = O(\mathbf{max}(f(n), g(n))$
- $O(f(n)) * O(g(n)) = O(f(n) * g(n)) = O(f(n)g(n))$

Example:

$$O(n^{10000}) = n^{10000+1} = n^{10001}$$

$$O(2^n + n^{10000}) = 2^n$$