Portfolio optimization

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Abstract

This project investigates systematic portfolio construction in the cryptocurrency market using constrained mean–variance Markowitz optimization. We consider a universe of 20 liquid crypto-assets augmented with a risk-free allocation and a portfolio initially valued at 1,000,000 \$. The portfolio is rebalanced under realistic risk controls with long-only positions, a volatility cap, a minimum diversification requirement, and a limit on cumulative fee costs. Model inputs are estimated from historical price data. We benchmark a baseline model considering only one asset, then a naive Markowitz implementation against an enhanced variant aimed at improving robustness and risk-adjusted performance in the presence of noisy estimates and regime changes typical of crypto markets. Performance is assessed using standard portfolio metrics such as total return, volatility, maximum drawdown, diversification, Sharpe ratio, and realized fees over the evaluation window (Jan 2022–Jan 2024), providing a practical comparison between classical optimization and more robust portfolio design under constraints.

Keywords: cryptocurrency; Markowitz optimization; constraints; rebalancing; transaction costs; backtesting, diversification, volatility, Ledoit-Wolf covariance shrinkage.

Contents

1 Introduction

Cryptocurrency markets have become a major asset class for both retail and institutional investors, offering high growth potential but also exhibiting extreme volatility, frequent regime shifts, and strong co-movements across tokens. From a portfolio-construction perspective, these characteristics make the classical risk–return trade-off particularly challenging: expected returns are difficult to estimate reliably, covariance structures change over time, and naive diversification across many coins may fail to reduce risk when correlations are high. As a result, systematic allocation methods must be evaluated not only on raw performance, but also on robustness and the ability to respect realistic risk controls.

In this project, we study systematic portfolio construction in a universe of $n = 20$ liquid crypto-assets, augmented with a risk-free cash account. Our goal is to design a rebalanced allocation strategy over the period January 2022–January 2024, starting from an initial wealth of Harry Markowitz \$1,000,000. We adopt the mean–variance framework of [2] as a baseline, where at each rebalancing date we estimate expected returns and a covariance matrix from a rolling historical window, then compute optimal portfolio weights by solving a convex optimisation problem. However, applying classical Markowitz directly to crypto data is known to be fragile: small estimation errors in expected returns or covariances can lead to unstable, overly concentrated portfolios and large turnover.

To address these challenges, we incorporate practical risk-management and implement a sequence of enhancements that improve stability while keeping the optimisation problem convex and tractable in `CVXPY`. We model proportional transaction costs, enforce explicit risk budgets (volatility, diversification, fee budget), and introduce concentration controls per-asset cap. We also improve the robustness of the risk model using covariance shrinkage of Ledoit–Wolf [1], and finally integrate a convex diversification constraint.

The remainder of the report is organised as follows. Section 2 introduces the market setting and notation, and Section 3 describes the data range and backtest parameters. Section 4 defines the performance metrics, while Section 5 formalises the risk constraints used throughout. Section 6 analyses the correlation structure of the asset universe. Section 7 and 8 present the Markowitz theoretical model and its implementation in CVXPY, followed by robustness improvements (concentration caps, covariance shrinkage) and transaction costs. We conclude by showing how a second order cone (SOC) diversification constraint can be integrated directly into the optimisation to better control concentration in the risky sleeve, and we discuss the resulting trade-offs between diversification, turnover, and net performance.

2 Framework

2.1 Market setting and notation

We consider a market observed on a discrete time grid $t \in \{0, 1, \ldots, T\}$ (typically trading days). The investment universe is composed of:

(i) n risky assets, indexed by $i \in \{1, \ldots, n\}$, with price process $\{p_t^i\}_{t=0}^T$;

(ii) one risk-free asset (cash account), indexed by $i = 0$, yielding a constant annual interest rate r.

The investor starts with an initial wealth $V_0 > 0$ and dynamically allocates wealth across the $n+1$ assets over time.

To work at daily frequency, we introduce a time step Δt. In a standard convention, $\Delta t = 1/365$ and the risk-free growth factor is:

$$B_{t+1} = B_t(1 + r\Delta t), \qquad B_0 = 1. \tag{1}$$

For each risky asset, the return is defined by:

$$R_{t+1}^i = \frac{p_{t+1}^i - p_t^i}{p_t^i}. \tag{2}$$

2.2 Portfolio representation

At any time t, the portfolio is described by the holdings (quantities):

$$\delta_t^i \text{ for } i \in \{1, \ldots, n\}, \qquad \delta_t^0 \text{ for the risk-free asset.} \tag{3}$$

The dollar amount invested in asset i is:

$$s_t^i = \delta_t^i p_t^i \quad (i = 1, \dots, n), \qquad s_t^0 = \delta_t^0 B_t. \tag{4}$$

The total portfolio value is therefore:

$$V_t = \sum_{i=0}^{n} s_t^i. \tag{5}$$

We represent the allocation through portfolio weights exposures :

$$x_t^i = \frac{s_t^i}{V_t}, \qquad i \in \{0, 1, \dots, n\}. \tag{6}$$

By construction, the portfolio is fully invested:

$$\sum_{i=0}^{n} x_t^i = 1. \tag{7}$$

In the long-only setting (no short positions), we impose:

$$x_t^i \geq 0, \qquad i \in \{0, 1, \dots, n\}. \tag{8}$$

In practice, weights are the key state variables: prices move continuously, while the investor only changes $(x_t^i)_i$ at rebalancing dates.

2.3 Wealth dynamics

Between two rebalancing dates, the portfolio is held constant in quantities $(\delta_t^i)_i$ (self-financing, ignoring transaction costs in this section). The wealth at time $t + 1$ satisfies:

$$V_{t+1} = \delta_t^0 B_{t+1} + \sum_{i=1}^{n} \delta_t^i p_{t+1}^i. \tag{9}$$

Expressed in terms of weights, this becomes:

$$V_{t+1} = V_t \left[x_t^0 (1 + r\Delta t) + \sum_{i=1}^{n} x_t^i (1 + R_{t+1}^i) \right]. \tag{10}$$

Equivalently, the portfolio one-period return is:

$$R_{t+1}^p = \frac{V_{t+1} - V_t}{V_t} = x_t^0 \, r\Delta t + \sum_{i=1}^{n} x_t^i R_{t+1}^i. \tag{11}$$

2.4 Optimization objective

The goal of the project is to design a rebalanced allocation rule $t \mapsto x_t$ that maximizes portfolio performance over the backtest horizon, while enforcing risk-management constraints. The next sections specify the optimization model used to compute target weights and the performance/risk metrics used to evaluate the strategy.

3 Backtest setting and parameters

3.1 Asset universe

We consider a universe of $n = 20$ crypto-assets, along with a risk-free asset (cash). In the following, risky assets are indexed by $i \in \{1, \dots, n\}$ and cash by $i = 0$.

3.2 Time horizon and data range

Historical price data spans from March 28, 2021 to March 26, 2024. The backtest itself is performed over January 1, 2022 to January 1, 2024. This interval is designed to include a regime of high instability followed by the start of an upward trend.

3.3 Risk-free rate

The risk-free asset is modeled with a constant annual yield $r = 5\%$. At daily frequency, we use the corresponding daily rate $r/365$ when computing excess returns.

4 Performance metrics

Let $\{V_t\}_{t=0}^{T}$ denote the portfolio value over the backtest period, where T is the number of trading days in the backtest. We define the daily portfolio return:

$$r_t = \frac{V_t - V_{t-1}}{V_{t-1}}, \qquad t = 1, \ldots, T, \tag{12}$$

and its empirical mean:

$$\bar{r} = \frac{1}{T} \sum_{t=1}^{T} r_t. \tag{13}$$

4.1 Annualized return

The annualized return is defined as the average total return per year over the backtest:

$$R = \frac{365}{T} \frac{V_T - V_0}{V_0}. \tag{14}$$

4.2 Annualized volatility

The annualized volatility is the standard deviation of daily returns scaled to a yearly basis:

$$\sigma = \sqrt{365} \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} (r_t - \bar{r})^2}. \tag{15}$$

4.3 Sharpe ratio

The Sharpe ratio [3] measures risk-adjusted performance using the average daily excess return, where the daily risk-free rate is $r/365$:

$$\mathrm{SR} = \frac{\sqrt{365}\left(\bar{r} - \frac{r}{365}\right)}{\sqrt{\frac{1}{T-1} \sum_{t=1}^{T} (r_t - \bar{r})^2}} = \frac{365\left(\bar{r} - \frac{r}{365}\right)}{\sigma} = \frac{365\bar{r} - r}{\sigma}. \tag{16}$$

4.4 Diversification

Diversification measures concentration through the dispersion of risky-asset weights (the risk-free weight is excluded). For a given day t, let x_t^i be the portfolio weight of risky asset $i \in \{1, \ldots, n\}$. Define:

$$H(x_t) = \frac{\sum_{i=1}^{n} \left(x_t^i\right)^2}{\left(\sum_{i=1}^{n} x_t^i\right)^2}, \qquad D(x_t) = \frac{1}{n} \frac{1}{H(x_t)}. \tag{17}$$

Intuitively, $H(x_t)$ increases when weights are concentrated, while $D(x_t)$ increases when weights are more evenly spread across assets.

4.5 Maximum drawdown

Maximum drawdown quantifies the largest peak-to-trough loss in portfolio value:

$$\mathrm{MDD} = \max_{t \in \{0, \ldots, T\}} \left(1 - \frac{V_t}{\max_{0 \le u \le t} V_u}\right). \tag{18}$$

5 Risk constraints

The objective of this project is to maximize strategy performance while ensuring that the resulting allocation remains within a predefined risk budget over the whole backtest period. The constraints are:

(i) Long-only: no negative positions.

(ii) Volatility cap: maximum 25%.

(iii) Diversification floor: minimum 70%.

(iv) Fee-cost budget : maximum 0.5% of initial capital.

5.1 Long-only feasibility

Let $x_t = (x_t^0, x_t^1, \dots, x_t^n)$ be the portfolio weight vector at date t. We enforce:

$$x_t^i \geq 0 \quad \forall i \in \{0, 1, \dots, n\}, \qquad \sum_{i=0}^{n} x_t^i = 1, \tag{19}$$

which prevents leverage and short-selling and ensures a fully invested portfolio.

5.2 Volatility limit

Volatility is computed from daily portfolio returns over the backtest window using (15). We impose:

$$\sigma_{\text{ann}} \leq 0.25. \tag{20}$$

5.3 Diversification limit

Using (17), diversification is evaluated as the daily average:

$$\overline{D} = \frac{1}{T} \sum_{t=1}^{T} D(_t), \qquad \overline{D} \geq 0.70. \tag{21}$$

5.4 Transaction fee-cost budget

Let Fee_t be the transaction fees paid at rebalancing date t (paid in cash), and let V_0 be the initial wealth. Over the whole backtest, total fee cost is defined as:

$$\text{Fees} \; = \; \frac{\sum_{t=1}^{T} \text{Fee}_t}{V_0}. \tag{22}$$

The constraint is:

$$\text{Fees} \leq 0.005. \tag{23}$$

6 Covariance analysis

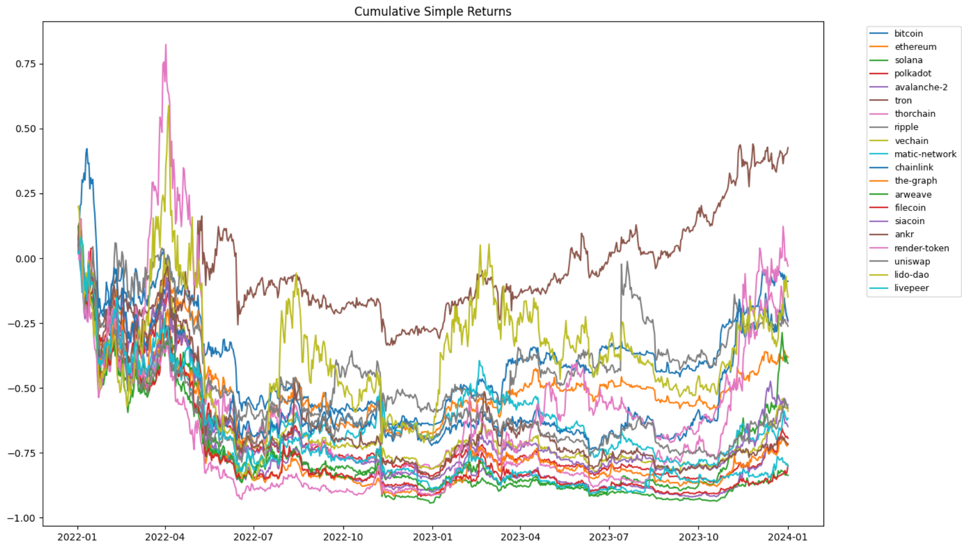

Figure 1: Cumulative Simple Returns.

As we can observe, many of the curves seem to follow the same pattern of variation over time. Since several assets move in a similar way, it is natural to study their correlation structure in order to quantify these co-movements and assess the true level of diversification.

Correlation matrix

We compute the Pearson correlation matrix of daily returns and represent it as a heatmap.

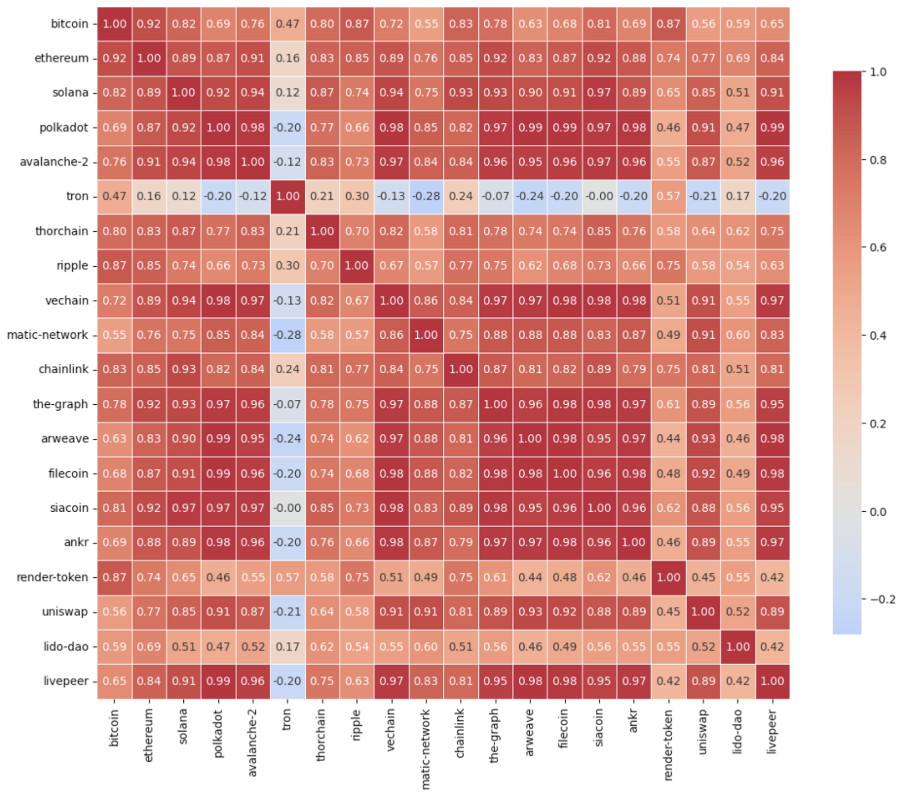

Figure 2: Correlation matrix of daily returns for the 20 crypto-assets.

Overall, the correlation matrix reveals a strong common market component: a large fraction of pair-wise correlations are high (often above 0.6 and frequently close to 0.9). This indicates that many assets tend to move together, especially during large market swings, which is typical in the crypto market. Several practical implications follow: Even if portfolio weights are spread across many coins, high correlations imply that portfolio volatility may not decrease as much as in a weakly-correlated universe. However, there is evidence of potential diversifiers such as Tron, Lido DAO, Render, and Ripple, which appear to be less correlated with the rest of the crypto universe. In other words, a few assets exhibit noticeably lower correlations with most others. But, correlation regimes may change. Correlations tend to increase during stress and decrease in calmer periods; the matrix is a descriptive snapshot rather than a structural constant.

7 Markowitz

7.1 Theoretical framework

Markowitz mean–variance theory provides a principled way to trade off expected return and risk [2]. At each rebalancing date t, we form an estimate of (i) the expected returns of the n risky assets and (ii) their joint risk through a covariance matrix, using only past observations from a rolling window. Let $r_t \in \mathbb{R}^n$ denote the vector of daily simple returns of the risky assets at day t. Given a lookback window of length T days, we estimate

$$\hat{\mu}_t = \frac{1}{T}\sum_{k=1}^{T} r_{t-k}, \tag{24}$$

$$\hat{\Sigma}_t = \frac{1}{T-1}\sum_{k=1}^{T}\big(r_{t-k}-\hat{\mu}_t\big)\big(r_{t-k}-\hat{\mu}_t\big)^\top. \tag{25}$$

The portfolio decision variable is the weight vector $x_t = (x_t^0, x_t^1, \ldots, x_t^n)$ where x_t^0 is the cash (risk-free) weight and x_t^i is the weight in risky asset i, $i > 0$. The portfolio is fully invested and long-only:

$$\sum_{i=0}^{n} x_t^i = 1, \qquad x_t^i \geq 0 \;\; \forall i. \tag{26}$$

Cash is modeled as a risk-free asset growing deterministically at rate r (daily factor $1 + r\Delta t$), while risky assets are random through their return vector r_{t+1}.

Mean–variance objective. Under the mean–variance paradigm, the (one-period) conditional expected return and variance of the portfolio at date t are approximated by

$$\widehat{\mathbb{E}}[R_{p,t+1} \mid \mathcal{F}_t] \approx x_t^0\, r\Delta t + \sum_{i=1}^{n} x_t^i\, \hat{\mu}_t^i, \qquad \widehat{\mathrm{Var}}(R_{p,t+1} \mid \mathcal{F}_t) \approx x_t^\top \hat{\Sigma}_t\, x_t, \tag{27}$$

where $x_t = (x_t^1, \ldots, x_t^n)$ denotes the risky-asset subvector and cash has zero variance/covariance. A standard Markowitz allocation is obtained by maximizing a quadratic utility (expected return penalized by risk):

$$\max_{x_t} \quad x_t^0\, r\Delta t + x_t^\top \hat{\mu}_t \; - \; \frac{\gamma}{2}\, x_t^\top \hat{\Sigma}_t\, x_t, \tag{28}$$

where $\gamma > 0$ is a risk-aversion parameter: larger γ yields a more conservative (lower-risk) allocation, while smaller γ allows more risk to chase return.

Equivalent constrained formulations. Problem (28) is equivalent to the classical efficient-frontier problems, such as:

$$\min_{x_t} x_t^\top \hat{\Sigma}_t\, x_t \quad \text{s.t.} \quad x_t^0\, r\Delta t + x_t^\top \hat{\mu}_t \geq m, \;\; \sum_{i=0}^{n} x_t^i = 1, \;\; x_t^i \geq 0, \tag{29}$$

for a target return level m, or alternatively maximizing expected return under a risk (volatility) budget:

$$\max_{x_t} x_t^0 \, r\Delta t + x_t^\top \hat{\mu}_t \quad \text{s.t.} \quad \sqrt{x_t^\top \hat{\Sigma}_t \, x_t} \leq \sigma_{\max}, \quad \sum_{i=0}^{n} x_t^i = 1, \quad x_t^i \geq 0. \tag{30}$$

These formulations make explicit that Markowitz does not aim to "maximize returns" alone, but rather to produce the best expected return given a controlled level of risk.

8 Implementation

8.1 Benchmark

We begin with two intuitive benchmarks: a single-asset portfolio and an equally weighted portfolio.

8.1.1 Single-asset portfolio

In this part, we'll consider the bitcoin only.

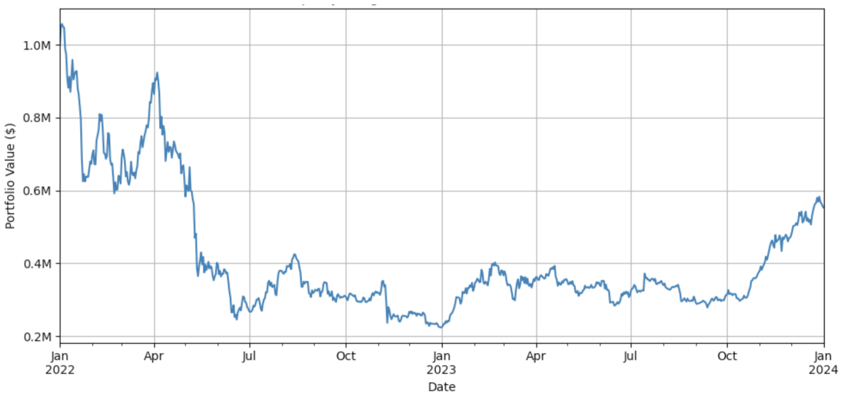

Figure 3: Evolution of the portfolio composed only with bitcoin.

Metric	Value
Final portfolio value	$911,237.48
Total return	-8.88%
Annualized return	-4.44%
Annualized volatility	54.89%
Maximum drawdown	67.08%
Sharpe Ratio	0.10

Table 1: Performance metrics of the asset-only strategy with bitcoin.

Interpretation. The Bitcoin-only benchmark provides a simple reference point: it corresponds to a fully concentrated exposure to the dominant crypto asset, without any diversification or risk control. Over the evaluation period, the strategy ends below its initial value (Table 1), with a negative annualized return and very large risk levels (annualized volatility close to 55% and a maximum drawdown above 67%). The wealth trajectory highlights how strongly performance is driven by the market regime, with deep drawdowns during adverse phases. Overall, this benchmark motivates the need for systematic allocation rules that explicitly control risk, rather than relying on a single-asset directional bet.

8.1.2 Equally weighted portfolio

Now, we want to see what happens when we attribute the same weight to all the assets.

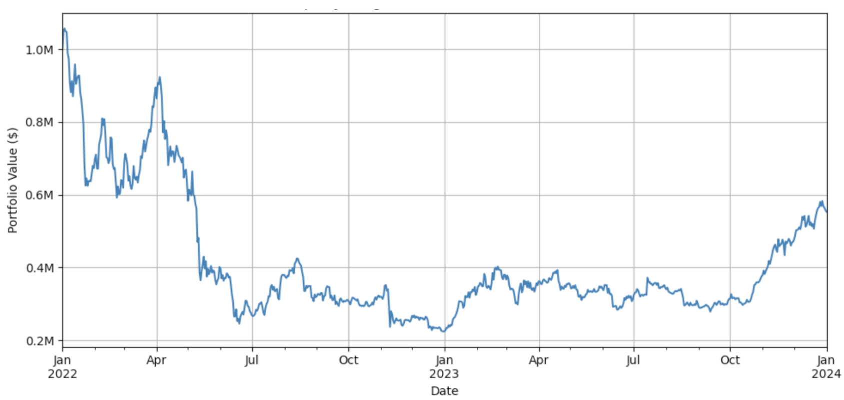

Figure 4: Evolution of the equally weighted portfolio.

Metric	**Value**
Final portfolio value	$553,113.0
Annualized return	-22.3%
Annualized volatility	72.3%
Sharpe Ratio	-0.1
Diversification	100.0%
Maximum drawdown	78.8%

Table 2: Performance metrics of the strategy over the evaluation period.

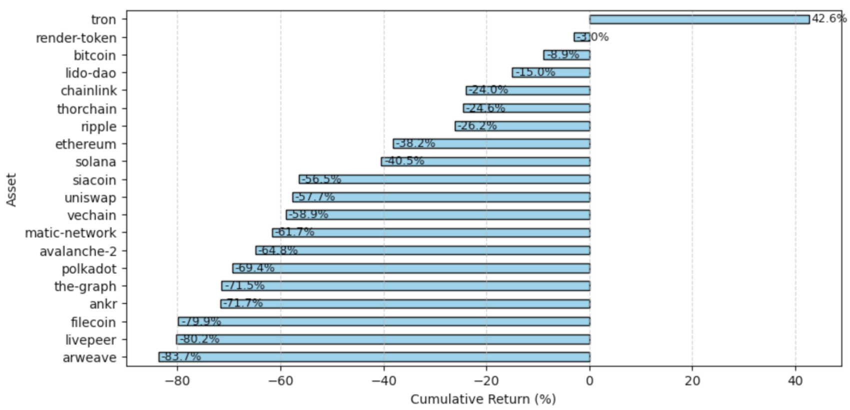

Figure 5: Cumulative returns of the assets over the backtest period

Interpretation. The equally weighted portfolio represents the opposite extreme: it maximizes cross-sectional spreading by assigning the same weight to each asset. Despite this, performance is markedly worse, with a strong negative annualized return, very high volatility (above 70%), and an even larger maximum drawdown. This outcome is consistent with the correlation analysis when most assets share a strong common component, equal-weight diversification does not translate into effective risk reduction. These benchmark results motivate the Markowitz approach used next, where expected returns and the covariance structure are combined with explicit constraints to seek a better risk–return trade-off.

8.2 Practical hyperparameters choices

Before detailing the optimization model itself, the backtest requires a few practical design choices. To avoid overfitting, we restrict ourselves to a very small set of hyperparameters.

(H1) Rolling window of length T. At each rebalancing date, model inputs are estimated using only the most recent T days of past returns. Small T adapts faster but yields noisier estimates; large T is more stable but slower to adapt. We tune T over $T \in \{30, 60, 90\}$.

(H2) Rebalancing frequency. Weights are updated every τ days. More frequent rebalancing reacts faster but increases turnover and fees. We consider $\tau \in \{7, 20, 30\}$ and use τ of the same order as (but smaller than) T as a compromise.

8.3 Optimisation with CVXPY

We implement the Markowitz allocation step using the `CVXPY` library. `CVXPY` lets us specify the portfolio optimization problem by defining the weights as decision variables, an objective, and convex constraints. It then reformulates the problem into a standard conic form and uses a numerical solver to compute the optimal weights.

In our setting, the key constraint is the target volatility requirement, implemented through a quadratic form:

$$w^\top \Sigma_{\text{ann}} w \;\le\; \sigma^2_{\text{target}},$$

where Σ_{ann} is the annualised covariance matrix. This constraint makes the optimisation problem a convex *quadratically-constrained program* (QCQP), which CVXPY typically reduces to a conic form (SOCP). As a consequence, QP-only solvers such as `OSQP` cannot be used directly, because they handle quadratic objectives but require linear constraints.

We therefore rely on conic solvers supported by CVXPY. In practice, both `SCS` and `ECOS` solve the problem successfully and produce very similar allocations and backtest metrics in our experiments. We choose `ECOS` as the default solver for the remainder of the project, as it is well-suited to SOCP problems and was observed to be stable on our instances.

8.4 First Results

We first report baseline Markowitz backtests . The optimiser is re-run on a rolling estimation window of length T days, and the portfolio is updated every τ days. Table 3 summarises three configurations that were tested.

(T, τ)	Ann. Ret.	Ann. Vol.	Sharpe	Max DD	D	Cash avg	Risky avg	Final value
$(30, 7)$	44.97%	26.71%	1.1600	18.34%	0.1270	75.83%	24.17%	1,861,283
$(60, 20)$	11.41%	25.10%	0.3479	30.66%	0.1089	75.70%	24.30%	1,209,188
$(90, 30)$	7.29%	23.80%	0.1450	32.02%	0.1002	76.24%	23.76%	1,127,593

Table 3: Performance sensitivity to the estimation window length T and rebalancing interval τ.

Interpretation. The results are highly sensitive to the choice of (T, τ): annualised returns range from about 7% to 45%, and maximum drawdown varies from 18% to above 30%. This instability is expected in mean–variance optimisation because the inputs $(\hat{\mu}, \hat{\Sigma})$ are estimated from a finite sample and are therefore noisy, especially in a high-volatility asset class such as crypto.

A short window (e.g. $T = 30$) uses fewer observations, which increases estimation error in $\hat{\mu}$ and $\hat{\Sigma}$. Markowitz is particularly sensitive to noise in expected returns: small changes in $\hat{\mu}$ can lead to large changes in the optimal weights. As a result, the optimiser may produce allocations that are effectively "overfit" to the recent sample. This can occasionally generate very strong performance in specific market regimes, but it is typically less robust out-of-sample. Larger windows (e.g. $T = 90$) reduce sampling noise and stabilise the estimates, but they may react more slowly to regime changes, which can reduce realised performance when market conditions shift quickly.

Rebalancing more frequently (smaller τ) makes the strategy more reactive, but it also amplifies the impact of estimation noise because weights are recomputed more often on highly overlapping samples. With larger τ, allocations change less often, which tends to smooth the behaviour but may miss short-term opportunities.

Across all configurations, the risky exposure stays low (about 24%–27%), showing that the volatility constraint is binding and is mostly satisfied by holding a large cash position. As a result, the risky budget is fairly stable; the instability mainly comes from how this risky sleeve is allocated across assets.

To improve stability, we introduce a concentration control such as a per-asset cap ($w_i \le w_{\max}$). In our setting, this cap should not be interpreted as increasing the global diversification metric (which is dominated by the large cash weight), but rather as a regularisation device that prevents extreme bets within the risky sleeve and reduces sensitivity to noisy parameter estimates.

8.5 Empirical impact of a concentration cap $w_{\max}$: robustness across (T, τ)

To assess whether the Markowitz allocation is robust to the choice of the estimation window T and rebalancing interval τ, we compare the baseline formulation (no upper bound on risky weights) with a concentration cap $0 \le w_i \le w_{\max}$, where $w_{\max} = 7\%$.

	No cap (baseline)					**With cap** $w_{\max} = 7\%$				
(T, τ)	Ann. Ret.	Ann. Vol.	Sharpe	Max DD	D	Ann. Ret.	Ann. Vol.	Sharpe	Max DD	D
$(30, 7)$	44.97%	26.71%	1.1600	18.34%	0.1270	36.33%	24.15%	1.0553	17.50%	0.2195
$(60, 20)$	11.41%	25.10%	0.3479	30.66%	0.1089	24.06%	25.17%	0.7217	25.65%	0.2088
$(90, 30)$	7.29%	23.80%	0.1450	32.02%	0.1002	23.02%	22.89%	0.6912	24.57%	0.2120

Table 4: Sensitivity to (T, τ) with and without a per-asset cap.

As we can see, table 4 highlights that adding $w_{\max} = 7\%$ compresses the dispersion of outcomes and improves robustness. While the cap slightly reduces the upside in the most favourable configuration ($(30, 7)$), it materially improves performance and drawdown in the less favourable ones: for $(60, 20)$ and $(90, 30)$, the annualised return increases from $11.41\% \to 24.06\%$ and from $7.29\% \to 23.02\%$, with drawdowns reduced from $30.66\% \to 25.65\%$ and from $32.02\% \to 24.57\%$, respectively. This pattern is consistent with the expected regularisation effect, the cap reduces sensitivity to estimation noise, produces allocations that are more reliable out-of-sample and it improves the overall stability.

9 Allocation diagnostics

Table 5 summarises realised allocations under the per-asset cap $w_i \le 7\%$. The risky sleeve is structurally small (cash averages $\approx 73\%$), leaving $\approx 27\%$ for risky assets. With a 7% cap, only about $27/7 \approx 4$ assets can be held close to the cap simultaneously, so the risky sleeve is mechanically sparse even before considering correlations.

Beyond this mechanical effect, allocations focus on a subset of assets. This is consistent with Markowitz logic because given a tight risk budget and strong co-movement in crypto, assets are selected based on both estimated returns and marginal diversification benefits. The correlation matrix in Figure 2 helps interpret why some assets are repeatedly used while others are rarely allocated.

Asset	Average (%)	Min (%)	Max (%)	Count > 0
bitcoin	0.75	0.00	7.00	3
ethereum	1.13	0.00	7.00	5
solana	2.24	0.00	7.00	8
polkadot	0.00	0.00	0.00	0
avalanche-2	0.64	0.00	7.00	2
tron	2.59	0.00	7.00	9
thorchain	2.54	0.00	7.00	9
ripple	2.55	0.00	7.00	10
vechain	0.52	0.00	7.00	3
matic-network	1.65	0.00	7.00	6
chain-link	1.57	0.00	7.00	8
the-graph	1.02	0.00	7.00	4
arweave	0.20	0.00	2.37	2
filecoin	0.32	0.00	7.00	1
siacoin	0.46	0.00	3.78	3
ankr	0.92	0.00	7.00	7
render-token	3.50	0.00	7.00	11
uniswap	0.98	0.00	7.00	4
lido-dao	2.47	0.00	7.00	8
livepeer	0.95	0.00	7.00	3

Summary	**Value**
Average allocation in cash (%)	73.00
Average allocation in risky assets (%)	27.00

Table 5: Asset Allocation Analysis (with per-asset cap $w_i \leq 7\%$)

10 Transaction costs: integrating fees in the rebalancing dynamics

So far, the Markowitz step returns target risky weights $w_t^\star \in \mathbb{R}^n$ at each rebalancing date t. In practice, rebalancing generates turnover and must therefore account for proportional transaction costs. As shown in Section 3.3, we assume a proportional fee rate $c = 0.1\%$.

Let V_t denote portfolio value just before rebalancing at date t, and let w_{t^-} be the current risky weights. The risky dollar holdings are $s_{t^-} = V_t^- w_{t^-}$, while target risky dollar holdings are $s_t^\star = V_t^- w_t^\star$. The traded notional is

$$\mathrm{N}_t = \sum_{i=1}^{n} \left| s_t^{i,\star} - s_{t^-}^i \right| = V_t \sum_{i=1}^{n} \left| w_t^{\star\, i} - w_{t^-}^i \right| .$$

Transaction fees paid at t are then

$$\mathrm{Fee}_t = c\,\mathrm{N}_t = c \sum_{i=1}^{n} \left| s_t^{i,\star} - s_{t^-}^i \right| .$$

Fees are paid in cash, so wealth immediately after rebalancing becomes

$$V_t^+ = V_t - \mathrm{Fee}_t,$$

and the effective post-trade weights must be rescaled by V_t^+.

Fee-cost budget. Over the whole backtest, we track cumulative fees and report the fee cost as a fraction of initial wealth V_0:

$$\mathrm{FeeCost} = \frac{\sum_{t \in \mathcal{R}} \mathrm{Fee}_t}{V_0},$$

where $\mathcal{R}$ is the set of rebalancing dates. In addition, we consider a fee budget constraint: $\text{FeeCost} \leq 0.5\%$. This constraint captures the practical idea that overly frequent rebalancing can make an otherwise attractive allocation infeasible once trading costs are included.

Empirical fee impact across (T, τ)**.** Table 6 reports the realised fee cost for the three configurations considered in the sensitivity study. As expected, fee cost increases sharply when the rebalancing interval τ decreases, because turnover grows with more frequent trading.

Configuration (T, τ)	T (days)	τ (days)	Fee cost (% of V_0)
$(90, 30)$	90.00	30.00	0.38
$(60, 20)$	60.00	20.00	0.69
$(30, 7)$	30.00	7.00	1.89

Table 6: Realised fee cost across (T, τ) configurations (fee rate $c = 0.1\%$).

11 Ledoit–Wolf covariance shrinkage

When the estimation window is short relative to the number of assets, the sample covariance

$$S = \frac{1}{T-1} \sum_{t=1}^{T} (r_t - \bar{r})(r_t - \bar{r})^\top$$

is a noisy estimator of the true covariance Σ. This noise is particularly harmful for mean–variance type optimizers because portfolio weights depend (explicitly or implicitly) on Σ^{-1}: small perturbations of S can generate large changes in the inverse, leading to unstable and highly concentrated allocations.

To mitigate this effect, we use a shrinkage estimator of Ledoit–Wolf type [1]:

$$\widehat{\Sigma}_{LW} = (1 - \delta)S + \delta F,$$

where the target is a scaled identity matrix $F = mI$ (equal variances, zero covariances), with $m = \frac{1}{p}\text{tr}(S)$, and the shrinkage intensity δ is estimated to minimize a quadratic (Frobenius-norm) loss :

$$\delta^\star \in \arg\min_{\delta \in [0,1]} \mathbb{E}\left[\|(1-\delta)S + \delta F - \Sigma\|_F^2\right].$$

Intuitively, shrinkage trades a small increase in bias for a substantial reduction in estimation variance, which stabilizes eigenvalues, improves the condition number of the covariance estimate, and yields allocations that are less sensitive to sampling noise.

Empirically, the use of $\widehat{\Sigma}_{LW}$ leads to a more diversified allocation: average diversification increases. As a counterpart, the portion invested in risky assets increases slightly (from 26.49% to 29.21%), which raises the notional reallocated at each rebalancing and therefore transaction costs (from 0.69% to 0.76% of capital). Over the tested period, this turnover effect results in slightly lower net performance (Sharpe 0.706 $\to$ 0.625), while the maximum drawdown remains comparable. Thus, shrinkage primarily constitutes a methodological improvement for the robustness of the risk model, promoting more stable diversification, but its net performance gain depends on the market regime and the explicit consideration of transaction costs.

12 Increasing diversification

12.1 Motivation

In the baseline Markowitz allocation, the risky sleeve can become concentrated in a small subset of assets, which reduces the diversification metric D defined in Section 5.3. In the crypto universe, this effect is amplified by strong correlations and noisy input estimates. To explicitly control concentration, we enforce a lower bound on the diversification proxy $D(\cdot)$ directly at the optimisation step.

12.2 From the diversification metric to a convex ℓ_2 constraint

Diversification is measured from the dispersion of risky-asset weights. For a given day t, letting $w_t = (x_t^1, \dots, x_t^n) \in \mathbb{R}_+^n$ denote the risky-weight subvector, we define

$$H(x_t) = \frac{\sum_{i=1}^n \left(x_t^i\right)^2}{\left(\sum_{i=1}^n x_t^i\right)^2}, \qquad D(x_t) = \frac{1}{n}\frac{1}{H(x_t)} = \frac{1}{n}\frac{\left(\sum_{i=1}^n x_t^i\right)^2}{\sum_{i=1}^n \left(x_t^i\right)^2}.$$

and we impose, at each rebalancing optimisation, a diversification floor

$$D(w_t) \;\geq\; D_{\min}.$$

Using the expression above, this is equivalent to

$$\frac{1}{n}\frac{(\mathbf{1}^\top w_t)^2}{\|w_t\|_2^2} \;\geq\; D_{\min} \quad\Longleftrightarrow\quad \|w_t\|_2^2 \;\leq\; \frac{(\mathbf{1}^\top w_t)^2}{nD_{\min}} \quad\Longleftrightarrow\quad \|w_t\|_2 \;\leq\; \frac{\mathbf{1}^\top w_t}{\sqrt{nD_{\min}}}.$$

This last inequality is a second-order cone constraint (SOC). Therefore, adding it preserves convexity of the Markowitz subproblem, which is already solved in conic form due to the volatility constraint.

Now let's look at empirical results for $(T, \tau) = (90, 30)$ with $D_{\min} = 0.70$. Table 7 summarises the backtest metrics obtained after adding the SOC diversification constraint with $D_{\min} = 0.70$ (fees are included as in Eq. (12) and reported through the fee-cost metric). We observe that the diversification is clearly higher ($D = 0.60$) while risk stays controlled (22.71% vol, 23.53% max drawdown).

Métrique	Valeur
Rendement annualisé (%)	16.93
Volatilité annualisée (%)	22.71
Ratio de Sharpe	0.51
Drawdown maximum (%)	23.53
Diversification moyenne D	0.60
Total fees ($)	3661.22
Fee cost (% de V_0)	0.37
Allocation moyenne en cash (%)	69.32
Allocation moyenne en actifs risqués (%)	30.68
Valeur finale du portefeuille ($)	1 296 470.60

Table 7: Performance with SOC diversification constraint ($D_{\min} = 0.70$), configuration $(T, \tau) = (90, 30)$.

13 Conclusion

This report studied constrained mean–variance portfolio construction in a 20-asset crypto universe with a cash allocation. We showed that classical Markowitz is sensitive to hyperparameters and noisy estimates, and that practical constraints and robustness improvements materially affect both allocations and net performance. In particular, controlling concentration stabilizes outcomes across (T, τ) choices, while fees reveal the cost of turnover. Finally, integrating diversification directly as a convex SOC constraint provides an explicit, tractable way to limit risky-sleeve concentration during optimisation.

References

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