RESEARCH STATEMENT

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1. Introduction

My research interests lie in the intersection of homotopy theory and higher category theory. Specifically, I am interested in chromatic homotopy theory, computations of Picard spectra and their connections to number theory and in particular to the theory of modular forms. I use algebraic and homotopy theoretic techniques combined with computational tools like spectral sequences to compute Picard groups of E_∞-ring spectra.

Picard groups were first studied in number theory as class groups of number fields. Later this notion was generalised to define the Picard group for a scheme X. Abstractly, one can define the Picard groups for any symmetric monoidal category by taking the group of isomorphism classes of objects invertible under the tensor product. Symmetric monoidal categories abound in homotopy theory, and some natural examples are the category of modules over an $\mathbb{E}_\infty$-ring R and the $K(n)$-local category of spectra, where $K(n)$ is the Morava K-theory spectrum at height n. In homotopy theory, interest in Picard groups arose once Hopkins, Mahowald and Sadofsky[HMS94] noted that the Picard group of the $K(n)$-local category of spectra is non-trivial and may contain "exotic elements". Categorifying the Picard group to a Picard spectrum, $\mathcal{P}ic$ as in [MS16] allows us to reinterpret the computation of the $K(1)$-local Picard group in [HMS94] as a descent result for the Picard spectrum of $K(1)$-local spectra. In [GHMR15], Goerss, Henn, Mahowald and Rezk computed the Picard group of the $K(2)$-local category at $p = 3$ using the resolution of the $K(2)$-local sphere developed in [GHMR05]. However, contrary to the height 1 case, this resolution is not a resolution in the category of E_∞-rings. This presents a significant obstacle to the use of descent techniques to compute the Picard group in the height 2 case.

For l a topological generator of $\mathbb{Z}_p^\times$, where p is any prime, Behrens [Beh06] introduced a semi cosimplicial spectrum $Q(l)$ which has a resolution constructed using topological modular forms, tmf and related spectra. In the case $p = 3$ and $l = 2$, $Q(2)$ is closely related to the $K(2)$-local sphere, $L_{K(2)}S^0$ and in fact satisfies a cofiber sequence [Beh06]

$$DQ(2) \to L_{K(2)}S^0 \to Q(2), \tag{1}$$

where $DQ(2)$ is the $K(2)$-local Spanier-Whitehead dual of $Q(2)$.

In my thesis, I investigate the Picard group of $Q(2)$. This group is more amenable to computation via descent techniques. Using descent, I calculate some elements in the Picard group of $Q(2)$. I also prove detection results for the elements of Picard group of the $K(2)$-local category of spectra.

2. Background

In the following, we work at the prime 3 and assume that all spectra are $K(2)$-localised. In [Beh06], Behrens defined $Q(2)$ as the global sections of a semi simplicial stack $\mathcal{M}_\bullet$, defined as

$$\mathcal{M}_\bullet := \lim\Big(\mathcal{M}_{ell} \leftleftarrows \mathcal{M}_0(2) \amalg \mathcal{M}_{ell} \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \mathcal{M}_0(2) \Big), \tag{2}$$

where $\mathcal{M}_{ell}$ is the moduli stack of elliptic curves and $\mathcal{M}_0(2)$ is the moduli stack of elliptic curves with $\Gamma_0(2)$ level structures.

The Goerss-Hopkins-Miller theorem produces an etale sheaf of E_∞-ring spectra on $\mathcal{M}_{ell}$ and $\mathcal{M}_0(2)$. The global sections of $\mathcal{M}_{ell}$ and $\mathcal{M}_0(2)$ are called TMF and $TMF_0(2)$ respectively.

Taking the global sections defines $Q(2)$ as the homotopy limit of a semi cosimplicial diagram

$$Q(2) := \lim\Big(TMF \rightrightarrows TMF_0(2) \times TMF \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} TMF_0(2) \Big). \tag{3}$$

Working with $Q(2)$ instead of $L_{K(2)}S^0$ has many advantages. Firstly, the maps in the resolution of $Q(2)$ arise from maps of elliptic curves, which allows the use of number theoretic techniques. For example, one of the maps in the resolution of $Q(2)$ is given by

$$\begin{aligned} \psi_d :&\mathcal{M}_0(2) \to \mathcal{M}_0(2) \\ &(C,H) \to (C/H, Ker(\hat{\phi}_H)), \end{aligned} \tag{4}$$

where (C,H) is an elliptic curve with level structure and $\hat{\phi}_H$ is the dual isogeny corresponding to the map $\phi_H : C \to C/H$. The map induced by ψ_d on modular forms is classically known as the Atkin-Lehner involution and is amenable to study via the theory of modular forms. Secondly, $Q(2)$ is the limit of a diagram in the category of E_∞-rings. This allows the use of descent theoretic techniques to understand the category of modules over $Q(2)$, $Mod_{Q(2)}$ in terms of the category of modules over TMF and $TMF_0(2)$ denoted as Mod_{TMF} and $Mod_{TMF_0(2)}$ respectively. In particular, we have a functor

$$F : Mod_{Q(2)} \to \lim\Big(Mod_{TMF} \rightrightarrows Mod_{TMF_0(2)} \times Mod_{TMF} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} Mod_{TMF_0(2)} \Big), \tag{5}$$

where we denote by L the limit on the RHS of equation (5). Using [Lur, Theorem 7.2], we get that F is a fully faithful functor. Applying $\mathcal{P}ic$ and using the fact that $\mathcal{P}ic$ commutes with limits, we get a functor

$$F|_{\mathcal{P}ic} : \mathcal{P}ic(Q(2)) \hookrightarrow \lim\Big(\mathcal{P}ic(TMF) \rightrightarrows \mathcal{P}ic(TMF_0(2)) \times \mathcal{P}ic(TMF) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{P}ic(TMF_0(2)) \Big). \tag{6}$$

3. Results

The goal of my thesis is to construct objects in $Pic(Q(2))$ using elements of $Pic(L)$. The strategy I employ can be summarized as follows. Firstly, I compute $Pic(L)$ in Theorem 3.1. Secondly, I calculate an explicit formula for the right adjoint G of F in Theorem 3.2. G however does not necessarily map $Pic(L)$ into $Pic(Q(2))$. In Theorem 3.3 we use invertible elements in the $K(2)$-local category to construct some non-trivial elements in $Pic(Q(2))$.

The objects in $Pic(L)$ up to an equivalence are given by the 0-th homotopy group of $\mathcal{P}ic(L)$, $\pi_0(\mathcal{P}ic(L))$. The group $\pi_0(\mathcal{P}ic(L))$ is computable via a Bousfield-Kan Spectral Sequence with E_1-page given by

$$E_1^{s,t} = \begin{cases} \pi_t(\mathcal{P}ic(TMF)), & \text{for } s = 0 \\ \pi_t(\mathcal{P}ic(TMF_0(2))) \times \pi_t(\mathcal{P}ic(TMF)), & \text{for } s = 1 \\ \pi_t(\mathcal{P}ic(TMF_0(2))), & \text{for } s = 2 \\ 0 & \text{for } s > 2. \end{cases} \tag{7}$$

Due to the vanishing line above $s = 2$, the spectral sequence degenerates at page 3 and $E_3^{s,t} = E_\infty^{s,t}$. Since, we are only interested in computing $\pi_0(\mathcal{P}ic(L))$, we can restrict to computations around $t - s = 0$ in the E_1-page. Therefore, the main relevant terms are $\pi_1(\mathcal{P}ic(TMF)) \simeq \pi_0(TMF)^\times$ and $\pi_1(\mathcal{P}ic(TMF_0(2))) \simeq \pi_0(TMF_0(2))^\times$. We now describe the $Pic(L)$.

Theorem 3.1. *There is a short exact sequence*

$$0 \to E_2^{1,1} \to Pic(L) \to Pic(TMF) \to 0, \tag{8}$$

where $Pic(TMF) \simeq \mathbb{Z}/72$ is generated by ΣTMF.

$E_2^{1,1}$ has representatives of the form $\{(a,b) : \psi_d(a) \cdot a = b\}$, where ψ_d is the Atkin-Lehner involution in equation (4), $a \in \pi_0(TMF_0(2))^\times$ and $b \in \pi_0(TMF)^\times$. As a group, $E_2^{1,1} \simeq \mathbb{Z}_3 \oplus G_1$, where G_1 is a part of the short exact sequence

$$0 \to \frac{\pi_0(TMF_0(2))^\times}{\pi_0(TMF)^\times Ker(\psi_d - 1)} \to G_1 \to \mathbb{Z}/2 \to 0. \tag{9}$$

The proof of this theorem relies on the explicit description of the action of the Atkin-Lehner involution ψ_d on $q-$expansions of modular forms. In particular , if f is a modular form with a q-expansion $f(q) = \sum a_n q^n$, then $\psi_d(f)(q) = 2^k f(q^2)$, where k is the weight of the modular form. The main complication which arises in this computation is the fact that the rings, $\pi_0(TMF_0(2))$ and $\pi_0(TMF)$ are complete local rings.

Given an object in $Pic(L)$ we would like to construct an object in $Pic(Q(2))$. Since F preserves small colimits, the adjoint functor theorem gives a right adjoint G, which allows us to produce potential elements in $Pic(Q(2))$. In what follows we describe an explicit construction of the right adjoint G.

An object in the limit category L is a "compatible system" of objects (M_0, M_1, M_2) where $M_0 \in Mod_{TMF}$, $M_1 \in Mod_{TMF_0(2)} \times Mod_{TMF}$ and $M_2 \in Mod_{TMF_0(2)}$ with various isomorphism data. Following [HY17, Theorem B], in Theorem 3.2 we construct a semi-cosimplicial diagram whose limit is the right adjoint G. Given an isomorphism class of objects in the limit category, we pick a convenient representative of the form (M_0, ϕ), where M_0 is a TMF-module and $\phi : d_{0_*} M_0 \to d_{1_*} M_1$ is an isomorphism satisfying a "cocycle condition".

Theorem 3.2. *With (M_0, ϕ) as above, the right adjoint G is given as the limit*

$$\lim(M_0 \underset{d_1}{\overset{\phi\circ d_0}{\rightrightarrows}} d_{1_*} M_0 \overset{d_{2_*}(\phi)\circ d_0}{\underset{d_2}{\underset{d_1}{\rightarrow}}} d_{2_*} d_{1_*} M_0) \ .$$

The "cocycle condition" on ϕ ensures that this is a semi cosimplicial diagram.

We remark that the formula for a general semi simplicial diagram is more complicated. The case of the $Q(2)$ diagram is less complicated because of the special nature of the face maps in the resolution of $Q(2)$.

Theorem 3.1 allows us to interpret an object of $Pic(L)$ as a tuple (n, a, b), where n is an integer $\pmod{72}$, $a \in \pi_0(TMF_0(2))^\times$ and $b \in \pi_0(TMF)^\times$ satisfying $\psi_d(a) \cdot a = b$. As a corollary of Theorem 3.2, we get the formula for the adjoint G, when restricted to $Pic(L)$.

Corollary 3.1. *The right adjoint G restricted to Pic(L) is given by the formula*

$$G(n, a, b) = \ S^n \wedge \lim(TMF \underset{d_1}{\overset{(a,b)\cdot d_0}{\rightrightarrows}} TMF_0(2) \times TMF \overset{b\cdot d_0}{\underset{d_2}{\underset{d_1}{\rightarrow}}} TMF_0(2))$$

Corollary 3.1 allows us to compute the images of the elements in the Picard group of the $K(2)$-local category of spectra, Pic_2 . We have a map $\eta : Pic_2 \to Pic(Q(2))$ which is induced by the unit map of $Q(2)$. We know that the algebraic part of Pic_2 is generated by $S^0, S^0\langle det\rangle$ and the exotic part is generated by P and Q [GHMR15, Kar10]. Using the construction of the spectra P and $S^0\langle det\rangle$ in [GHMR15] and [BBGS22] respectively, we calculate the images of various elements in Pic_2 in Theorem 3.3.

Theorem 3.3. *The spectra $S^0, S^0\langle det\rangle$ and P are detected in $\mathcal{P}ic(L)$ and therefore are also detected in $\mathcal{P}ic(Q(2))$. The image of $S^0\langle det\rangle$ in $Pic(Q(2))$ is given by $G(0, -4, 16)$. The image of P is given by $G(48, 2^{36}, \frac{\psi_d(\Delta^2)}{\Delta^2})$, where ψ_d is the Atkin-Lehner involution.*

An important consequence of this theorem is that, we have now produced non-trivial elements in $Pic(Q(2))$.

4. Current work

Detection of Picard elements. In general, the right adjoint of a symmetric monoidal functor is only lax monoidal. For an object $X \in Pic(L)$, it is not necessary that $G(X) \in Pic(Q(2))$ and thus we need a detection theorem for when $G(X) \in Pic(Q(2))$. We briefly review a detection result for a height 1 example, where descent does not hold for $\mathcal{P}ic$. At height 1 and $p = 3$, we have a finite resolution for the $K(1)$-local sphere:

$$L_{K(1)}S \longrightarrow L_{K(1)}(BP\langle 1\rangle) \underset{1}{\overset{\Psi^4}{\rightrightarrows}} L_{K(1)}(BP\langle 1\rangle)\ ,$$

where $BP\langle 1\rangle$ is the 1-truncated Brown-Peterson spectrum. Since this is a resolution in the category of E_∞-rings, we get a functor

$$\mathfrak{F} : Mod_{L_{K(1)}S} \to \lim\left(Mod_{L_{K(1)}(BP\langle 1\rangle)} \underset{1}{\overset{\Psi^4}{\rightrightarrows}} Mod_{L_{K(1)}(BP\langle 1\rangle)} \right).$$

Applying $\mathcal{P}ic$, we get the functor

$$\mathfrak{F}|_{\mathcal{P}ic} : Pic(L_{K(1)}Sp) \to \lim\left(Pic_{K(1)}(L_{K(1)}(BP\langle 1\rangle)) \underset{1}{\overset{\Psi^4}{\rightrightarrows}} Pic_{K(1)}(L_{K(1)}(BP\langle 1\rangle)) \right).$$

However, $\mathfrak{F}|_{\mathcal{P}ic}$ is not an equivalence. Since $\mathfrak{F}$ preserves small colimits, we get a right adjoint $\mathfrak{G}$. For X in the limit category, it is not necessary that $\mathfrak{G}(X) \in Pic(L_{K(1)}Sp)$. In fact, we can show that, for X in the limit category, $\mathfrak{G}(X) \in Pic(L_{K(1)}Sp)$ if and only if $L_{K(1)}(\mathfrak{G}(X) \wedge BP\langle 1\rangle) \simeq L_{K(1)}(BP\langle 1\rangle)$ as $L_{K(1)}(BP\langle 1\rangle)$-modules. Inspired by this, we conjecture

Conjecture 4.1. *If $X \in Pic(L)$ then $G(X) \in Pic(Q(2))$ if and only if $G(X) \wedge_{Q(2)} TMF \simeq TMF$ as TMF-modules.*

The condition $G(X) \wedge_{Q(2)} TMF \simeq TMF$ as TMF modules is clearly necessary . To prove the converse, I plan to use techniques from [Beh06, Lemma 2.10.5] to compute $\pi_*(G(X))$ and $K(2)_*(G(X))$ and to compare them to $\pi_*(Q(2))$ and $K(2)_*(Q(2))$ respectively.

Injectivity of images of invertible $K(2)$-local spectra. We have a map from $\eta : Pic_2 \to Pic(L)$. One can ask if this injective. Theorem 3.3 shows that the group generated by spectra $S^0, S^0\langle det\rangle$ and P maps injectively into $Pic(L)$. To determine whether it is injective on Pic_2 it suffices to calculate the image of the "exotic" invertible spectrum Q defined in [GHMR15]. The image of Q should follow from computing the effect of smashing with Q on the Atkin-Lehner involution ψ_d in (4).

Hidden extensions. Preliminary work shows that the short exact sequence (8) is not split. Recall that $E_2^{1,1}$ consists of two parts $\mathbb{Z}_3$ and G_1, and therefore there are two possible ways to get an extension. First using $\mathbb{Z}_3$ and second using G_1. The extension with respect to $\mathbb{Z}_3$ should be computable using techniques from [HMS94]. However, to compute the extension with respect to G_1, we first need to compute the 3-torsion in the group G_1, which is part of my future plans.

5. Future work

Computation of 3-torsion in $Pic(L)$. 3-torsion in $Pic(L)$ is primarily concentrated in the group G_1 which satisfies a short exact sequence (9). The term $\frac{\pi_0(TMF_0(2))^\times}{\pi_0(TMF)^\times Ker(\psi_d - 1)}$ in equation (9) is hard to compute by hand. However, the term $Ker(\psi_d - 1)$ can be replaced by the closely related term $Im(\psi_d + 1)$, which would allow us to use a multiplicative basis of $\pi_0(TMF_0(2))^\times$. Using this I plan to write a computer program to calculate the group.

Image of $Mod_{Q(2)}$ and $\mathcal{S}p_{K(2)}$. The spectrum $Q(2)$ has a representation as $E_2^{h\Gamma}$, where E_2 is a Lubin Tate theory at height 2 and the group Γ is an infinite discrete group [Beh07]. Γ is abstractly isomorphic to the free product of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ and is dense in the Morava stabilizer group, $\mathbb{S}_2$. Let $\mathcal{S}p_{K(2)}$ be the category of $K(2)$-local spectra. We have maps

$$\mathcal{S}p_{K(2)} \to Mod_{Q(2)} \hookrightarrow L \tag{10}$$

The maps in Equation (10) are not equivalences. However, we could gain insights into the category of $K(2)$-local spectra by calculating the images of these functors following the case of height 1 in [BCM20, Proposition 3.10]. Roughly, they identify the $K(1)$-local spectra with category of K-modules with a "continuous" $\mathbb{Z}$-action. I would like to understand if similar descriptions hold at height 2 as well.

Analogues for prime 2. In [BO16], Behrens and Ormsby consider spectra $Q(3)$ and $Q(5)$ at the prime $p = 2$. They show that $Q(3)$ detects the divided β-family [MRW77], an important part of the $L_{K(2)}S^0$. I would like to investigate if analogues of Theorem 3.1 and Theorem 3.3 hold in the case of $Q(3)$.

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