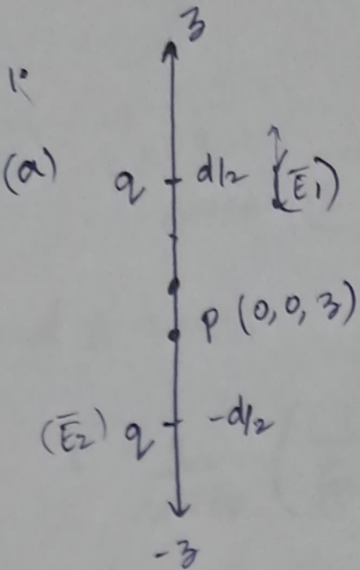
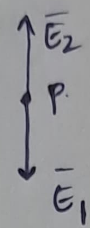
Assignment-1

1.

(a)



For $z \in [-d/2, d/2]$,



For $z \in (0, d/2)$,

$|\bar{E}_1| > |\bar{E}_2|$

$\therefore$ Net electric field: $(\bar{E}_1 - \bar{E}_2)(-\hat{k})$

$$= \frac{kq}{(\frac{d}{2}-z)^2} - \frac{kq}{(\frac{d}{2}+z)^2} (-\hat{k})$$

$$= \frac{-q}{4\pi\epsilon_0}\left(\frac{1}{(\frac{d}{2}-z)^2} - \frac{1}{(\frac{d}{2}+z)^2}\right)\hat{k}$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(\frac{d}{2}+z)^2} - \frac{1}{(\frac{d}{2}-z)^2}\right)\hat{k}$$

For $z \in (0, -\frac{d}{2})$:

$|\bar{E}_2| > |\bar{E}_1|$

$\therefore$ Net electric field: $(\bar{E}_2 - \bar{E}_1)\hat{k}$

$$= \left(\frac{kq}{(\frac{d}{2}+z)^2} - \frac{kq}{(\frac{d}{2}-z)^2}\right)\hat{k} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(\frac{d}{2}+z)^2} - \frac{1}{(\frac{d}{2}-z)^2}\right)\hat{k}$$

For $p = (0,0,0)$ $\bar{E}_1 = -\bar{E}_2 \Rightarrow \vec{E}_{net} = 0$

~~For $z > \frac{d}{2}$ and $z < -\frac{d}{2}$~~

~~$\bar{E}_1 + \bar{E}_2 = \frac{kq}{4(d+z}$~~

For $z > d/2$, $\bar{E}_1 \approx 0$,

$E_{net} = \bar{E}_2 = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{d^2}\hat{k}$

$z > -d/2$, $\bar{E}_2 \approx 0$,

$\bar{E}_{net} = \bar{E}_2 = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{d^2}(-\hat{k})$

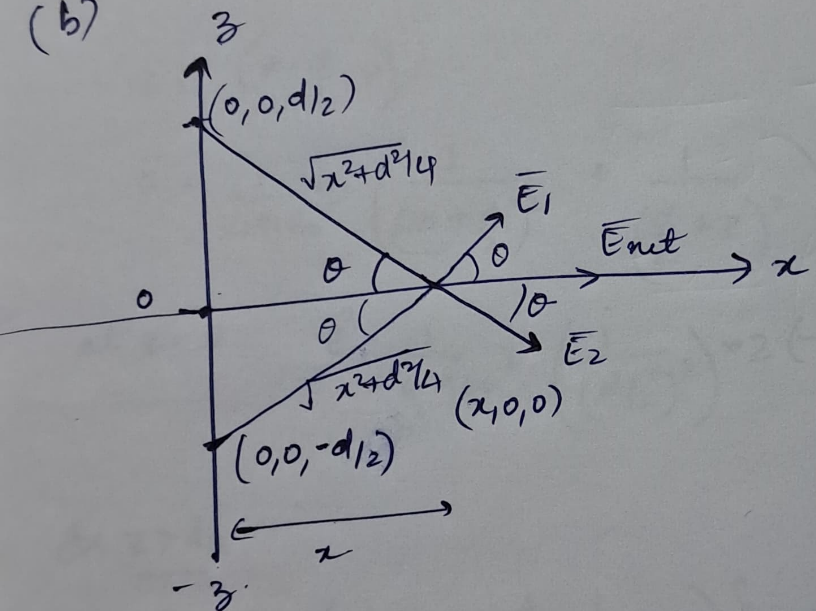
For $z > \frac{d}{2}$ $\cancel{\text{and } z < -d/2,}$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z-d/2)^2} + \frac{1}{(z+d/2)^2}\right)\hat{k}$$

and $z < -d/2$,

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z-d/2)^2} + \frac{1}{(z+d/2)^2}\right)-\hat{k}$$

(b)



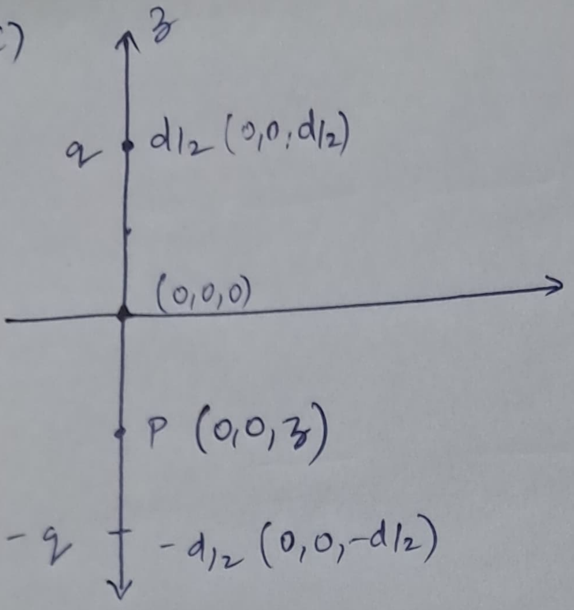
$$|\bar{E}_{net}| = (|\bar{E}_1| + |\bar{E}_2|)\cos\theta$$

$$= \frac{q}{4\pi\epsilon_0} \times \frac{1}{(x^2+d^2/4)} \cdot \frac{x}{\sqrt{x^2+d^2/4}}$$

(∵ When x is +ve, $\bar{E}_{net}$ is along $+\hat{i}$ and when it's −ve, $\bar{E}_{net}$ is along $-\hat{i}$)

$$\vec{E}_{net} = \frac{q(x)}{4\pi\epsilon_0 (x^2+d^2/4)^{3/2}}\hat{i}$$

(1) c)



~~For $z \in (0, d/2)$:~~

For $z \in (0, d/2)$:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{\left(\frac{d}{2} - z\right)^2} + \frac{1}{\left(\frac{d}{2} + z\right)^2} \right) (-\hat{k})$$

$z \in \left(-\frac{d}{2}, 0\right)$:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{\left(z + \frac{d}{2}\right)^2} + \frac{1}{\left(\frac{d}{2} + z\right)^2} \right) (-\hat{k})$$

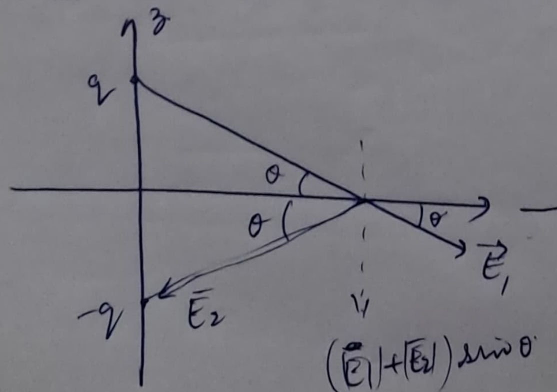
at $z = 0$: $\vec{E} = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{(d/2)^2} \right) \times 2 (-\hat{k})$

For $z > d/2$:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(z - d/2)^2} + \frac{1}{(z + d/2)^2} \right) \hat{k}$$

For $z < -d/2$:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(z - d/2)^2} + \frac{1}{(z + d/2)^2} \right) (-\hat{k})$$



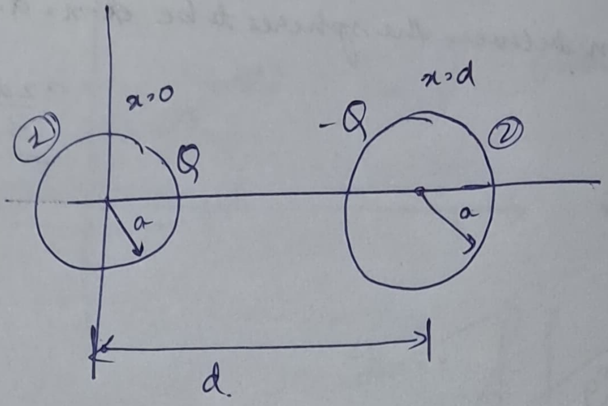
$(|E_1| - |E_2|)\cos\theta = 0$

$(|E_1| + |E_2|)\sin\theta$

$$\therefore \vec{E} = \frac{q}{4\pi\epsilon_0} \times \frac{1}{\left(x^2 + \frac{d^2}{4}\right)} \times 2 \times \frac{d}{2\left(x^2 + \frac{d^2}{4}\right)^{1/2}}$$

$$\boxed{\vec{E} = \frac{qd}{4\pi\epsilon_0 \left(x^2 + \frac{d^2}{4}\right)^{3/2}} (-\hat{j})}$$

② A crude device: measures charge



$F_{res} = \underline{kx}$. Let x be the displacement of sphere ① from $x=0$.

Force on sphere ① due to charge on sphere ②:

$$\vec{F} = \frac{Q^2}{4\pi\epsilon_0} \times \frac{1}{(d-x)^2}$$

~~for equilibrium~~

$$\cancel{\frac{Q^2}{}}$$

$$\frac{Q^2}{4\pi\epsilon_0} \times \frac{1}{(d-x)^2} = kx$$

$$Q^2 = 4\pi\epsilon_0 kx \cdot (d-x)^2$$

$$Q = \sqrt{4\pi\epsilon_0 kx (d-x)^2}$$

For maximum charge, $\frac{dQ}{dx} = 0$

$$\frac{d}{dx}\left(\sqrt{x}\,(d-x)\right) = \frac{1}{2\sqrt{x}}(d-x) + \sqrt{x}(-1) = 0$$

$$d-x+2x(-1) = 0$$

$$\boxed{x = \frac{d}{3}}$$

$$q_{max} = \sqrt{4\pi k \epsilon_0 \frac{d}{3} \cdot \left(\frac{4d^2}{9}\right)}$$

$$q_{max} = \sqrt{\frac{16\pi}{27} k\epsilon_0 d^3}$$

$=$

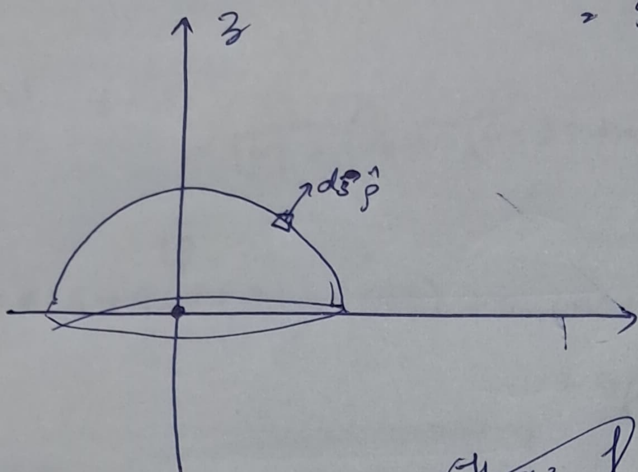
wo the separation between the spheres to be $d - x = d - \frac{d}{3}$

$= \frac{2d}{3}$

(3)

Flux density field: $F_1 = 5a_z = 5\hat{k}$

$= 5\cos\phi\,\hat{\rho} - 5\sin\phi\,\hat{\phi}$



~~Flux~~ $\cancel{= \iint \vec{F}\cdot\hat{n}\,ds}$

$\hat{n}\,ds = \rho^2 \sin\phi\, d\phi\, d\theta\, \hat{\rho}$

$$\left[\begin{array}{l}\phi \in [0, \pi/2) \\ \theta \in (0, 2\pi)\end{array}\right]$$

~~Flux~~ $\cancel{= \iint \rho^2 \sin\theta\, d\phi\, d\theta\, (5\cos\theta)}$

~~Flux~~ $\cancel{= \iint}$

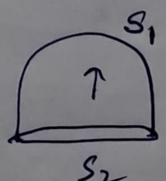
$\hat{n}\,ds = \rho^2 \sin\phi\, d\phi\, d\theta\, \hat{\rho}$

$$\text{Flux} = \iint \vec{F}\cdot\hat{n}\,ds = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} 5\cos\phi\,\rho^2 \sin\phi\, d\phi\, d\theta$$

$$\frac{5}{2}\rho^2 (2\pi) \int_{\phi=0}^{\pi/2} \sin 2\phi\, d\phi$$

$$= \frac{5\rho^2\pi}{\cancel{2}} \left(\cancel{-\frac{\cos 2\phi}{2}}\right) = 5\pi\rho^2 = \underline{5a^2\pi}$$

Simpler way: Since $\vec{E}$ is constant,



Flux leaving from S_1 = flux leaving from S_2:

$5 \times (\pi a^2)$

$= \underline{5\pi a^2}$

③ $F_2 = 5z\hat{k}$

$$5\rho\cos\phi\,(\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi})$$

flux $= \iint \bar{F}_2 \cdot \hat{n}\,ds$

$\hat{n}\,ds = \rho^2 \sin\phi \, d\theta \, d\phi \, \hat{\rho}$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} 5\rho\cos^2\phi \cdot \rho^2 \sin\phi \, d\phi \, d\theta$$

$$= 2\pi(5\rho^3) \cdot \int_{\theta=0}^{\pi/2} \sin\phi\cos^2\phi \, d\phi$$

$\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$t = 1$

$$= 10\pi\rho^3 - \int t^2 \, dt$$

$$\left[-\frac{1}{3}\cos^3(\phi)\right]_0^{\pi/2}$$

$$= \frac{1}{3}$$

$$= \frac{10\pi}{3}\rho^3 = \frac{10\pi a^3}{3}$$

From the bottom surface; flux $= 0$ as $z = 0$

~~div~~

Divergence theorem:

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_D \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial z}(5z) = \underline{\underline{5}}$$

$$\iiint 5\, dV = \iiint_{\rho=0}^{a} 5\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int\int_{\phi=0}^{\pi/2} \frac{5a^3}{3} \sin\phi\, d\phi\, d\theta = \frac{5a^3}{3}\cdot\int_0^{2\pi} d\theta$$

$$= \underline{\underline{\frac{10\pi a^3}{3}}}$$

(Same as calculated above)

5)

$$\vec{F} = \left[\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right]\hat{s} + 3(\cos\phi - \sin\phi)\hat{\phi} - 2\hat{z}$$

(a) $s = 3$:

$$|F|^2 = \left(4 + 3(\cos\phi + \sin\phi)\right)^2 + 4 + \left(3(\cos\phi - \sin\phi)\right)^2$$

$$= 16 + 9\cancel{(1 + 2\sin\phi\cos\phi)} + 24\cos\phi + 24\sin\phi + 4 + 9\cancel{(1 - 2\sin\phi\cos\phi)}$$

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$$\boxed{|\vec{F}| = \sqrt{38 + 24(\sin\phi + \cos\phi)}}$$

(b) $\phi = 45^\circ$

$$|\vec{F}| = \sqrt{\left(\frac{40}{s^2+1} + 3\sqrt{2}\right)^2 + 4}$$

(c) $$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{z}\right) \cdot (\vec{F})$$

$$\frac{-40(2s)}{(s^2+1)^2} + \frac{1}{s} \cdot (3(-\sin\phi - \cos\phi))$$

$$= \frac{-80s}{(s^2+1)^2} - \frac{3}{s}(\sin\phi + \cos\phi)$$

(d) $\nabla \times \vec{F}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial s} & \frac{1}{s}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_s & F_\theta & F_z \end{vmatrix}$$

=

(d) $\nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial s} & \frac{1}{s}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_s & F_\theta & F_z \end{vmatrix} = \hat{i}\left(\frac{1}{s}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right) - \hat{j}\left(\frac{\partial}{\partial s}F_z - \frac{\partial}{\partial z}F_s\right) + \hat{k}\left(\frac{\partial F_\theta}{\partial s} - \frac{1}{s}\frac{\partial F_s}{\partial \theta}\right)$$

$$= \hat{i}\left(\overset{0}{\cancel{\frac{1}{s}3(-\sin\phi + \cos\phi)}}\right) - \hat{j}(0) + \hat{k}\left(0 - \frac{3}{s}(-\sin\phi + \cos\phi)\right)$$

$$= \hat{k}\left(-\frac{3}{s}(-\sin\phi + \cos\phi)\right)$$

Non-conservative field.

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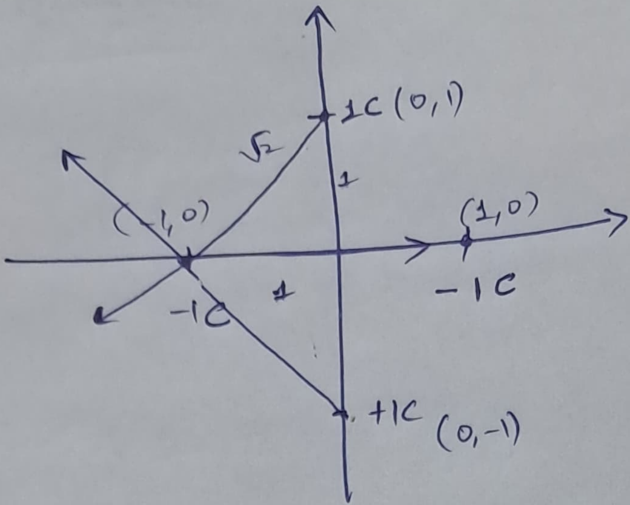

(c) Using the summation method,

$$W = 0 + 1\cdot\left(\frac{-1}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{2}}\right) + -1\left(\frac{1}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{2}} + \frac{-1}{4\pi\epsilon_0}\cdot\frac{1}{2}\right)$$

$$+1\cdot\left(\frac{-1}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{2}}\times 2 + \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{2}\right)$$

$$= \frac{1}{4\pi\epsilon_0(\sqrt{2})}\left(-1 - 1 + \frac{1}{\sqrt{2}} - 2 + \frac{1}{\sqrt{2}}\right)$$

$$\frac{(-4+\sqrt{2})}{\sqrt{2}\,(4\pi\epsilon_0)}$$

$$\therefore \cancel{P}\, E = \frac{(-4+\sqrt{2})}{\sqrt{2}(4\pi\epsilon_0)}$$

(d) Guass law: $\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$

$\nabla\cdot E = \frac{\rho}{\epsilon_0}$

(d) Guass law:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$\boxed{\rho(r) = q\,\delta(r - r_0)}$ for a point charge at r_0.

(zero everywhere except at r_0)

∞ at r_0

$$\nabla \cdot E = \frac{q}{\epsilon_0}\,\delta(r - r_0)$$ $\Rightarrow$ divergence of electric field is zero everywhere except at the location of the charge

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(e) In electrostatics, the electric field is irrotational. (conservative)

$\Rightarrow$ curl of E is zero everywhere.

$$\nabla \times E = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial E_y}{\partial z}\hat{i} + \frac{\partial E_x}{\partial z}\hat{j} + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$E_z = 0$

$$\frac{\partial E_y}{\partial z} = \frac{\partial E_x}{\partial z} = 0$$

$\frac{\partial E_y}{\partial x}$ and $\frac{\partial E_x}{\partial y}$ cancel each other.

Here ~~when~~ we get zero when we subtract.

$\therefore\ \nabla \times E = 0$

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$V = V_0 e^{-r/a}$

(a) charge density ρ_v at $r = a$.

$$E = -\nabla V = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}\right)$$

$$\boxed{\vec{E} = \frac{V_0 a}{a} e^{-r/a}\hat{r}}$$

$$\nabla \cdot \bar{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(E_\phi \sin\phi) + \frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta}$$

$$\bar{\nabla}\cdot\bar{E} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{V_0 a e^{-r/a}}{a}\right)$$

$$= \frac{V_0 a}{a r^{\cancel{2}}}\left(2\cancel{r} e^{-r/a} + r^{\cancel{2}}\left(-\frac{1}{a}\right)e^{-r/a}\right)$$

$$= \frac{V_0 a}{r a}\left(2 - \frac{r}{a}\right) e^{-r/a} = \frac{\rho}{\epsilon_0}$$

at $r = a$, $\rho = \epsilon_0\left(\frac{V_0}{a^2}(2-1)\right)e^{-1}$

$$\boxed{\rho = \frac{V_0 \epsilon_0}{a^2 e}}$$

(b) $\vec{E}$ at $r = a = \frac{V_0}{a} e^{-a/a}\hat{r} = \frac{V_0}{ae}\hat{r}$

(c)

$$\rho = \frac{V_0 \epsilon_0}{a}\left(\frac{2}{r} - \frac{1}{a}\right) e^{-r/a}$$

Total charge = $\iiint \rho \, dv$

$dv = r^2 \sin\phi \, dr \, d\phi \, d\theta$

~~$dv = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$~~

$\therefore$ Total charge = $\iiint r^2 \sin\phi \, dr \, d\phi \, d\theta \cdot \left(\frac{V_0 \epsilon_0}{a}\right)\left(\frac{2}{r} - \frac{1}{a}\right) e^{-r/a}$

$$= \frac{V_0 \epsilon_0}{a} \iint \left(\sin\phi \int_{r=0}^{a} \left(2r - \frac{r^2}{a}\right) e^{-r/a} \, dr \right) d\phi \, d\theta$$

$$= \cancel{\frac{V_0 \epsilon_0}{a} \iint \sin\phi} \; \frac{V_0 \epsilon_0}{a} \int \int_{\phi=0}^{\pi} \sin\phi \left(a^2 e^{-a/a}\right) d\phi \, d\theta$$

$$= \frac{V_0 \epsilon_0 a^2}{ae} \int \left[-\cos\phi\right]_{\phi=0}^{\pi} d\theta = \frac{2 V_0 \epsilon_0 a}{e} \theta \Big]_{\theta=0}^{2\pi}$$

$$= \frac{2 V_0 \epsilon_0 a (2\pi)}{e}$$

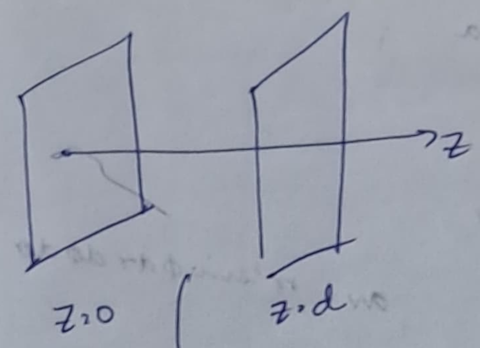
$$= \frac{4\pi V_0 a \epsilon_0}{e},$$

$\therefore$ Charge enclosed with sphere of radius $a = \frac{4\pi V_0 a \epsilon_0}{e}$

Total charge enclosed in space $(r \to \infty) = \underline{0}$

(8)

(a)



ρ_0 C/m³ : volume charge density

$$\nabla \cdot E = \frac{\rho_0}{\epsilon_0}$$

$$\bar{E} = \left(\frac{\rho_0}{\epsilon_0} z + C_1\right) \hat{z}$$

$$V = \int \vec{E} \cdot d\ell = \frac{\rho_0 z^2}{2\epsilon} + C_1 z + C_2$$

Since both the plates are grounded,

$V(0) = 0$ and $V(d) = 0$.

$$\frac{\rho_0 (0)^2}{2\epsilon} + C_1(0) + C_2 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$\frac{\rho_0 (d)^2}{2\epsilon} + C_1 d = 0$$

$$C_1 = -\frac{\rho_0 d}{2\epsilon}$$

$$V = \frac{\rho_0 z^2}{2\epsilon} - \frac{\rho_0 d}{2\epsilon} z = \frac{\rho_0 z}{2\epsilon}(z - d)$$

(b) $\vec{E} = \left(\frac{\rho_0 z}{\epsilon} + C_1\right)\hat{z} = \left(\frac{\rho_0 z}{\epsilon} - \frac{\rho_0 d}{2\epsilon}\right)\hat{z}$

(c) Given $V(d) = V_0$

$V(0) = 0 \Rightarrow C_2 = 0$

$V(d) = V_0$

$\frac{\rho_0 z^2}{2\epsilon} + C_1 z = V_0$

$\frac{\rho_0 d^2}{2\epsilon} + C_1 d = V_0$

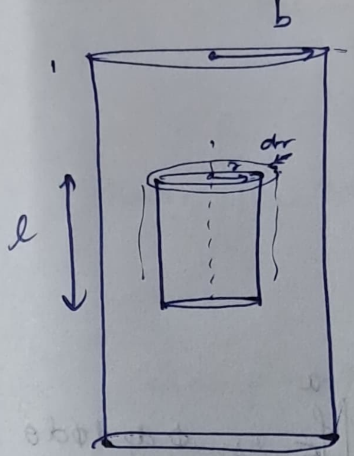
$C_1 d = V_0 - \frac{\rho_0 d^2}{2\epsilon}$

$$\boxed{C_1 = \frac{V_0}{d} - \frac{\rho_0 d}{2\epsilon}}$$

$\therefore V = \frac{\rho_0 z^2}{2\epsilon} + \left(\frac{V_0}{d} - \frac{\rho_0 d}{2\epsilon}\right) z$

$E = \left(\frac{\rho_0 z}{\epsilon} + \left(\frac{V_0}{d} - \frac{\rho_0 d}{2\epsilon}\right)\right)\hat{z}$

4.



$\rho_v = a\rho^2$ (ρ: radius $= r$)

$\rho_v = ar^2$

$E_r \cdot (2\pi r l) = \dfrac{Q_{enc}}{\epsilon_0}$

$Q_{enc} = \iiint \rho_v \, dv = \int_{z=0}^{l}\int_{\theta=0}^{2\pi}\int_{r=0}^{r} ar^2 \, dr (r\,d\theta) dz$

$= a\left(\dfrac{r^4}{4}\right) \cdot (2\pi) \cdot (l)$

$E_r(\cancel{2\pi r l}) \cdot \epsilon_0 = \dfrac{(\cancel{2\pi l})\, a r^{\cancel{4}\,3}}{4}$

$$\boxed{E_r = \frac{ar^3}{4\epsilon_0}}$$

Inside the cylinder

$r < a$

Outside the cylinder:

$E_r(2\pi r l) = \dfrac{a^5}{4}\dfrac{(2\pi)l}{\epsilon_0}$

$$\boxed{E_r = \frac{a^5}{4r\epsilon_0}}$$

outside.

$r > a$