6)

a) $y^t P y = y^t A^t A y = (Ay)^t Ay$

$= \|Ay\|^2 \geq 0$

$\Rightarrow y^t P y \geq 0$

$z^t Q z = z^t A A^t z$

$= (A^t z)^t A^t z$

$= \|A^t z\|^2 \geq 0.$

$\Rightarrow z^t Q z \geq 0.$

b) 'u' is an eigen vector of P with eigen value 'λ'

$\Downarrow$ size of u is $n \times 1$

$\Rightarrow P u = \lambda u$

$\Rightarrow A^T A u = \lambda u$

$\Rightarrow A A^T A u = A(\lambda u)$

$\Rightarrow A A^T (Au) = \lambda (Au)$

$\Rightarrow Q(Au) = \lambda (Au)$

$\Rightarrow$ Au is an eigen vector of Q with eigen value 'λ'.

'v' is an eigen vector of Q with eigen value 'μ'

$\Downarrow$ size of v is $m \times 1$

$\Rightarrow A A^T v = \mu v$

$\Rightarrow A^T A A^T v = \mu (A^T v)$

$\Rightarrow P(A^T v) = \mu (A^T v)$

$\Rightarrow A^T v$ is an eigen vector of P with eigen value 'μ'.

c) $\because$ v_i is an eigen vector of Q

we have $Qv_i = \lambda v_i$ (say $\lambda \neq 0$)

$$v_i^t Q v_i = \lambda v_i^t v_i$$

$$\Rightarrow \beta = \lambda \|v_i\|^2 \qquad \beta \geq 0$$

$$\Rightarrow \lambda = \frac{\beta}{\|v_i\|^2} \geq 0$$

$$\Rightarrow \lambda \geq 0$$

$$\Rightarrow \lambda > 0.$$

$$u_i = \frac{A^T v_i}{\|A^T v_i\|}$$

$$\therefore Au_i = \frac{AA^T v_i}{\|A^T v_i\|}$$

$$= \frac{\lambda v_i}{\|A^T v_i\|} \quad \text{where } \gamma_i = \frac{\lambda}{\|A^T v_i\|^2}$$

$$= \gamma_i v_i \qquad \Rightarrow \gamma_i > 0.$$

$\therefore$ We have a γ_i such that

$$Au_i = \gamma_i v_i.$$

d) Also from above

$$A^T A u_i = \frac{A^T A A^T v_i}{\|A^T v_i\|}$$

$$\Rightarrow Pu_i = \frac{A^T Q v_i}{\|A^T v_i\|} \qquad \Rightarrow Pu_i = \lambda u_i$$

$$= \frac{\lambda A^T v_i}{\|A^T v_i\|} = \lambda u_i$$

If v_i & v_j are not parallel (i.e $v_i \neq \underset{\text{scalar}}{\alpha} v_j$)

and $u_i = \dfrac{A^T v_i}{\|A^T v_i\|}$, $u_j = \dfrac{A^T v_j}{\|A^T v_j\|}$

then $u_i \neq u_j$

if not $u_i = u_j$

$$\Rightarrow A u_i = A u_j$$

$$\Rightarrow \frac{A A^T v_i}{\|A^T v_i\|} = \frac{A A^T v_j}{\|A^T v_j\|}$$

$$\Rightarrow \frac{\lambda_i v_i}{\|A^T v_i\|} = \frac{\lambda_j v_j}{\|A^T v_j\|}$$

$$\Rightarrow v_i = \alpha v_j$$

contradiction.

$\Rightarrow u_i \neq u_j$

$\Rightarrow$ for every v_j we get u_j's such that (unique)

$A u_i = \lambda_i v_i$ $\qquad i \leq \text{Rank}(A^T A) = r$

Let $A^T A = U Q U^T$

where $U = [u_1 | u_2 | \dots u_r | \dots]$

$\underbrace{\mu_1, \mu_2 \dots \mu_r \dots 0}$ → eigen value

Now we have

$A u_1 = \lambda_1 v_1$

$\vdots$

$A u_r = \lambda_r v_r$ $\qquad$ ($\because$ rank of A is $r \leq m \leq n$.

$A u_{r+1} = 0$ $\qquad$ ($u_{r+1}, u_{r+2} \dots u_n$ belong to null space of A.

$\vdots$ $\qquad$ $\Rightarrow A^T A u_{r+i} = 0$

$A u_n = 0$

$$\Rightarrow \begin{bmatrix} Au_1 & Au_2 & \cdots & Au_r & \cdots & Au_n \end{bmatrix}_{m\times n} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_m v_m & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$$

$$\Rightarrow A \underbrace{\begin{bmatrix} u_1 | u_2 \cdots u_r | \cdots u_n \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}}_{V'_{m\times m}} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & \lambda_m \cdots & 0 \\ 0 & & & & \end{bmatrix}_{m\times n}}_{r'}$$

$$\Rightarrow \because U U^T = U^T U = I_{n\times n}$$

$$\Rightarrow AU = Vr'$$

$$\Rightarrow AUU^T = Vr'U^T$$

$$\Rightarrow A_{m\times n} I_{n\times n} = V_{m\times m} r'_{m\times n} U^T$$

$$= V_{m\times m} r'_{m\times n} U^T_{n\times n}$$

$$r'U^T = \begin{bmatrix} \lambda_1 & & \cdots & 0 \\ & \lambda_2 & & \vdots \\ & & \lambda_m \cdots & 0 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_m^T \\ | \\ u_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 u_1^T \\ \lambda_2 u_2^T \\ \vdots \\ \lambda_m u_m^T \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \lambda_m \end{bmatrix}_{m\times m}}_{r} \underbrace{\begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_m^T \end{bmatrix}_{m\times n}}_{W^T}$$

$$\therefore A = V_{m\times m} \cdot r'_{m\times n} U^T$$

$$= \underbrace{V_{m\times m}}_{[v_1 \, v_2 \cdots v_m]} r_{m\times m} \underbrace{W^T_{m\times n}}_{[u_1 \, u_2 \cdots u_m]}.$$