5) Goal is to maximize $u^t C u$ (unit vector u) & $\perp$ to e

$\because$ C is a square symmetric positive definite matrix

$C = Q D Q^T$ (eigen value decomposition)

$u^t C u = u^t Q D Q^T u$ $\quad D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$

$= (Q^T u)^T D (Q^T u)$ $\quad \lambda_1 \geq \lambda_2 \geq \dots \lambda_d$

Let $v = Q^T u$

$v = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_d]^T$

where $\alpha_i = u \cdot \hat{e_i}$ $\quad$ ($\hat{e_i}$ - i^{th} eigen vector)

clearly $\alpha_1 = 0$ ($\because u \perp e_1$)

$\Rightarrow v = [0 \ \alpha_2 \ \dots \ \alpha_d]^T$

$\therefore u^t C u = [0 \ \alpha_2 \ \dots \ \alpha_d] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_d \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix}$

$= [0 \ \lambda_2\alpha_2 \ \dots \ \lambda_d\alpha_d] \begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix}$

$= \lambda_2 \alpha_2^2 + \lambda_3 \alpha_3^2 + \dots \lambda_d \alpha_d^2$

$\because v = Q^T u$

$\Rightarrow v^T v = u^T Q Q^T u$

$= u^T u$ $\Rightarrow \sum_{i=1}^{d} \alpha_i^2 = 1$

$= 1$

$\because \lambda_2 > \lambda_3 \cdots > \lambda_n$

if $\alpha_2 = 1-h \quad 1 \geq |h| \geq 0$

$\Rightarrow u^t C u = \lambda_2 (1-h)^2 + \lambda_3 \alpha_3^2 + \cdots$

$= \lambda_2 (1+h^2) - 2h\lambda_2 + \lambda_3 \alpha_3^2 + \cdots$

$= \lambda_2 + \lambda_2 h^2 - 2h\lambda_2 - h\lambda_2 + \lambda_3 \alpha_3^2 + \cdots$

$= \lambda_2 + \lambda_2 h(h-1) - (\alpha_3 + \alpha_4 \cdots \alpha_d)\lambda_2 + \lambda_3 \alpha_3^2 + \cdots$

$= \lambda_2 + \underbrace{\lambda_2 h(h-1)}_{\leq 0} + \alpha_3 \underbrace{(\lambda_3 \alpha_3 - \lambda_2)}_{\leq 0} + \alpha_4 (\lambda_4 \alpha_4 - \lambda_2) \cdots$

$\because \lambda_3 \leq \lambda_2$

$\alpha_3 \leq 1$

$\leq \lambda_2$

with equality occuring only at $h=0$

$\Rightarrow \alpha_2 = 1$ to maximize $u^t C u$.

$\Rightarrow \quad v = [0 \; 1 \; 0 \; 0 \cdots 0] \quad (\because \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \cdots \alpha_d^2 = 1)$

$\Rightarrow \quad u = Qv$

$\Rightarrow \quad u = e_2$. (corresponding to λ_2).

Hence the direction f perpendicular to e for which $f^t C f$ is maximized is eigenvector C with second highest eigen value