Math: From An Economist's Perspective

For Personal Reference, happy to circulate

Sai Zhang

Check the Github Page for this project, or email me!

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HERE WE GO!

Math is fascinating, certainly. It is clean, organized, beautiful, philosophical, but it is also hard to grasp. I started this project for one simple purpose: As an Economic Ph.D. student, math was not my strongest suit and I NEED to change that. Hence, this math-learning notebook will be tailored according to the need of an economist, instead of being, you know, math math.

Here, I cover the math knowledge ranging from basic concepts, to fundamental theories including linear algebra and real analysis, and more integrated topics including optimization, dynamic methods stochastic control, etc. There are several valuable sources I referred to in the process of making this notebook. Two general aspects are reviewed here: math theories and their application in economic research. I organize the theoretical contents based on Hoy et al. (2011)'s *Mathematics of Economics*, Carter (2001)'s *Fundations of Mathematical Economics* and Eichhorn and Gleißner (2016)'s *Mathematics and Methodology for Economics*; Intriligator (2002)'s *Mathematical Optimization and Economic Theory*, Vali (2014)'s *Principles of Mathematical Economics* and De la Fuente (2000)'s *Mathematical Methods and Models for Economists* are my main references for the application of math thoeries in specific economic questions. Although the above listed books are rather thorough and well-organized, I referred to other great ones, when in need, for specific topics.

I thank Prof. Brijesh Pinto at USC Economics for reminding me the importance of math and the pleasure of playing with it in the math camp prior to my Ph.D. study. Though brief and abstract, the math camp had actually inspired me to go back to the beginning, really dive in and put together this personal learning notes.

Since this notebook approaches math in an application perspective, I will not only review the theoretical aspects of each topic, but include some necessary modelling simulation techniques and codes as well. All the codes, including the LaTeX file of this notebook can be found on my Github page. Building this review is truly a memorable journey for me. I would love to share this review and all the related materials to anyone that finds them useful. And unavoidably, I would make some typos and other minor mistakes (hopefully not big ones). So I'd really appreciate any correction. If you find any mistakes, please either set up a branch on Github or send the mistakes to this email address saizhang.econ@gmail.com, BIG thanks in advance!

Contents

1 Basic Concepts **3**

1.1 Set Theory and Space Theory 3

1.1.1 Subsets and Set Operations 3

1.1.2 Relations and Order 5

1.2 Chap1Sec2 6

1.3 Chap1Sec3 6

1.4 Chap1Sec4 6

2 Linear Algebra **7**

3 Real Analysis **8**

4 Optimization **9**

5 Dynamic Method **10**

Bibliography **11**

CHAPTER 1

BASIC CONCEPTS

Contents

1.1 Set Theory and Space Theory 3
1.1.1 Subsets and Set Operations 3
1.1.2 Relations and Order 5
1.2 Chap1Sec2 6
1.3 Chap1Sec3 6
1.4 Chap1Sec4 6

In this chapter, I review the fundamental concepts that will reoccur constantly in this notebook.

In Section 1.1, I summarize the basic elements of set theory,

1.1 Set Theory and Space Theory

A **set** is a collection of elements, normally denoted as

$$S = \{x(\in A) : P(x)\}$$

where S is the set, x represents elements (add $x \in A$ if x must also belong to set A), $P(x)$ represents the property of x.

1.1.1 Subsets and Set Operations

Definition 1.1. If all elements of a set X are also of another set Y, then X is a **subset** of Y, which can be written as $X \subseteq Y$.

After defining subsets, we have two extended definitions: **proper subset** and **set equality**.

Definition 1.2. If all elements of a set X are also of another set Y, but not all elements of Y are in X, then X is a **proper subset** of Y, written as $X \subset Y$

Definition 1.3. If two sets X and Y contain exactly the same elements, X and Y are **equal**, written as $X = Y$.

The equality of two sets can also be expressed in another way:

$$X \subseteq Y \text{and} Y \subseteq X \Leftrightarrow X = Y$$

. Now with the basic definitions, we can move to the set operations. Here, all the set operations are based subsets of a **universal set** U.

Definition 1.4. For two subsets of U, X and Y:

- Intersection: $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$, if X and Y don't share any common elements, the intersection would be an **empty/null set** $\varnothing$, X and Y are said to be **disjoint**
- Union: $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$. The union of two sets strictly contain their intersection, or $X \cap Y \subset X \cup Y$
- Complement: $X^C(\bar{X}) = \{x \in U : x \notin X\}$. The complement of the universal set $U^C = \varnothing$
- Relative difference: $X - Y = x \in U : x \in X \text{ and } x \notin Y$.

With Definition 1.4 in mind, we have:

- $(X \cap Y)^C = X^C \cup Y^C, (X \cup Y)^C = X^C \cap Y^C, X - Y = X \cap Y^C, X \subseteq Y \to Y^C \subseteq X^C$
- $X \subseteq Y \to X \cup (Y - X) = Y, X - Y \subseteq X \cup Y, X \cap Y = \varnothing \to Y \cap X^C = Y$
- $X - Y = X - (X \cap Y) = (X \cup Y) - Y, (X - Y) - Z = X - (Y \cup Z), X - (Y - Z) = (X - Y) \cup (X \cap Z), (X \cup Y) - Z = (X - Z) \cup (Y - Z), X - (Y \cup Z) = (X - Y) \cap (X - Z)$

Definition 1.4 can be extended to collections of sets.

Definition 1.5. For a collection of sets $\mathcal{S}$:

- Union: $\bigcup_{S \in \mathcal{S}} S = \{x : x \in S \text{ for some } S \in \mathcal{S}\}$
- Intersection: $\bigcup_{S \in \mathcal{S}} S = \{x : x \in S \text{ for every } S \in \mathcal{S}\}$

Apart from union, intersection, complement and relative difference (subtract), we can also take the **product** of two sets:

Definition 1.6. The **Cartesian product** of two sets X and Y is the set of **ordered** pairs $X \times Y = \{(x, y) : x \in X, y \in Y\}$

Notice that **order** is important, meaning that (x, y) is different from (y, x). Equally, we can generalize this definition as $\prod_{k=1}^{n} S_k = \{(s_1, \cdots, s_n) | s_1 \in S_1, \cdots, s_n \in S_n\}$. If we want to denote the product of all BUT the ith set, we can use: $X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$.

Another two definitions focus on two important collections of subsets:

Definition 1.7.
- Partition: a collection of **disjoint** subsets of U, of which the union is U, i.e., for U, if the collection of its n subsets $\{X_i\}, i = 1, \cdots, n$ is a partition of U, $\{X_i\}$ must satisfy: $X_i \cap X_j = \varnothing, \forall i, j = 1, \cdots, n, i \neq j$

and $\bigcup_{i=1}^{n} X_i = U$. On element level, each element of U lies in one and ONLY one subset of the partition.

- Power set: the power set $\mathcal{P}(X)$ of a set X is the set of ALL its subsets, i.e., $\mathcal{P}(X)\{A : A \subseteq X\}$[1]

1.1.2 Relations and Order

For two sets X and Y, any subset R of their product $X \times Y$ is called a *binary relation*, any pair of elements $(x, y) \in R \subseteq X \times Y$ can be described as x is **related** to y, denoted as $x\mathcal{R}y$. A relation is a subset of the product, expressing the relationship between elements.

Two basic concepts of a relation $R \subseteq X \times Y$ are **domain** and **range**:

Definition 1.8.
- **domain** of R is set of all $x \in X$ that are related to some $y \in Y$: $\text{domain}_R = \{x \in X : (x, y) \in R\}$
- **range** of R is its corresponding subset of Y: $\text{range}_R = \{y \in Y : (x, y) \in R\}$

Any binary relation $R \subseteq X \times Y$ can be characterized by the following properties:

Definition 1.9. The relation R is:

inversible if $\exists R^{-1} = \{(y, x) \in Y \times X : (x, y) \in R\}$
reflexive if $x\mathcal{R}x$
transitive if $x\mathcal{R}y$ and $y\mathcal{R}z \Rightarrow x\mathcal{R}z$
symmetric if $x\mathcal{R}y \Rightarrow y\mathcal{R}x$
antisymmetric if $x\mathcal{R}y$ and $y\mathcal{R}x \Rightarrow x = y$
asymmetric if $x\mathcal{R}y \nRightarrow y\mathcal{R}x$
complete if x and y always have $x\mathcal{R}y$ or $y\mathcal{R}x$ or both

Inversible is a universal property for all relations. Meanwhile, other properties can be used to categorize relations:

Step1 If a relation R is **reflexive** and **transitive**, it is a ***Preorder*** or ***Quasi-order***
Step2 If a relation R is a ***Preorder*** and:
- **symmetric**, it is an ***Equivalence Relation***
- NOT **symmetric**, it is an ***Order Relation***

Step3 If a relation R is an ***Order Relation*** and:
- **complete**, it is a ***Weak Order***
- **antisymmetric**, it is a ***Partial Order***
- both **complete** and **antisymmetric**, it is a ***Total Order (Chain)***
- both **antisymmetric** and bounded, it is a ***Lattice***

A clearer representation of the relationships between the different relations is shown in Figure .

Next, we study some special properties of the relations listed above.

[1]Fact: For any set X with a finite number n of elements, its power set $\mathcal{P}(X)$ has 2^n elements, including one $\emptyset$. This can be proved as: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (x + y)^n \mid_{x=1,y=1} = 2^n$

Equivalence relation

The most important property of equivalence ralations is related to the partitions of sets (see Def. 1.7):

> **Theorem 1.1.** If R is an equivalence relation on any non-empty set A, then the distinct set of equivalence classes of R forms a partition of A. Conversely, if a relation induced by partition $\mathcal{P} = \{A_i\}$ is defined as:
>
> $$\forall x, y \in A, x\mathcal{R}^{\mathcal{P}} y \Leftrightarrow \exists A_i \in \mathcal{P}(x, y \in A_i)$$
>
> this relation $\mathcal{R}^{\mathcal{P}}$ is an equivalence relation.

What does this mean? Here is an example: for the partition $\{\{a, b\}, \{c\}, \{d\}\}$ of set $S = \{a, b, c, d\}$, the relation (ordered pairs) induced by this partition is: $\mathcal{R} = \{(a, a), (a, b), (b, a), (b, b), (c, c), (d, d)\}$, this relation is reflexive, transitive and symmetric, so it is an equivalence relation.

Now, let's prove Theorem 1.1 formally:

- equivalence relation $\longrightarrow$ partition: Suppose $\mathcal{R}$ is an equivalence relation on any non-empty set A, then equivalence classes are $A_1, A_2, \cdots$, then
- partition $\longrightarrow$ equivalence relation: To prove the R is an equivalence relation, the three properties need to be satisfied.

  Reflexive: Let $x \in A$, since $\bigcup P_i = A$, x must belong to at least one set in $\{P_i\}$. Thus, $\exists i$ s.t. $x \in A_i \wedge x \in A_i$, by the definition of partition induced relations, $x\mathcal{R}x$. Hence, $\mathcal{R}$ is reflexive.

  Symmetric:

  Transitive:

1.2 Chap1Sec2

In this section, I

1.3 Chap1Sec3

1.4 Chap1Sec4

CHAPTER 2

LINEAR ALGEBRA

CHAPTER 3

REAL ANALYSIS

CHAPTER 4

OPTIMIZATION

CHAPTER 5

DYNAMIC METHOD

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