Microeconomic Theory: A Notebook

A book of my own

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Check the Github Page of this project, or email me!

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HERE WE GO!

This is my learning notebook of microeconomic theories. As two of the core courses in an economic Ph.D. curriculum, Microeconomic Theory I and II are beyond important to my research. Therefore, I would love to use this notebook as a commitment mechanism, to document lecture notes, discuss session and office hour intuitions, reading summaries, my personal questions regarding the topics and more. By building a file from scratch, hopefull I could have a more systematic and sophisticated understanding on the content of this course.

I thank Prof. Jonathan Libgober and Prof. Michael Magill at USC Economics for leading the discussion of the courses and providing intuitive ways to understand microeconomic theories. Please check their webpages, they are such fun.

Following the structure of the course, this notebook will cover three aspects of microeconomic theories: (a) individual decision making, (b) game theory, (c) mechanism design and contract theory. Apart from Jonathan's lecture notes, I will also summarize the reading materials, including: Mas-Colell et al. (1995)'s *Microeconomic Theory*, Mailath (2018)'s *Modelling Strategic Behavior*[1], Fudenberg and Tirole (1991)'s *Game Theory*, Myerson (1991)'s *Game Theory: Analysis of Conflicts*, Bolton and Dewatripont (2005)'s *Contract Theory*, Mailath and Samuelson (2006)'s *Repeated Games and Reputation* and Osborne and Rubinstein (1994)'s *A Course in Game Theory*. Other materials will also be referred to along the way.

Building this notebook is truly a memorable journey for me. I would love to share this review and all the related materials to anyone that finds them useful. And unavoidably, I would make some typos and other minor mistakes (hopefully not big ones). So I'd really appreciate any correction. If you find any mistakes, please send the mistakes to this email address saizhang.econ@gmail.com or start a branch on Github. BIG thanks in advance!

[1]Latest version (May 2021) available here.

Contents

Part I

Individual Decision Making

CHAPTER 1

PREFERENCES AND CHOICES, UTILITIES

Contents

The first chapter summarizes the basic setting of individual decision making: preferences, choices and utilities. The main reference is Chapter 1 of Mas-Colell et al. (1995).

In this chapter, we will focus on 3 domains:

choice	given a set A, what choice from A is made
preference	given alternatives x, y, which does the decision maker prefers
utility	given an object X, how much does the DM likes X (as a number)

The starting point of individual decision problem is a *set of possible (mutually exclusive) alternatives* from which the individual must choose. To model decision making process on this set of alternatives, one can:

- either start from the tastes, i.e., *preference relations* of individuals, and set up the patterns of decision making with preferences
- or, start from the actual actions of individuals, i.e. *choices*, to deduct a pattern of decision making

With this two major approaches in mind, we know what's coming: the ***rationality*** of preferences and the central assumption of choices, the ***Weak Axiom of Revealed Preference (WARP)***. And of course, the two approaches and two basic assumptions are parallel, so we need to figure out how link the (underlying) preferences and (observed) choices.

1.1 Preference Relations

We start from the basic: *weak preference relation*, $\succsim$.

Definition 1.1.1: weak preference $\succsim$

A weak preference relation $\succsim$ on a set X is a subset of $X \times X$. If $(x, y) \in \succsim \Rightarrow x$ is at least as good as y, written as $x \succsim y$

A weak preference relation will induce two other types of relations on X:

Definition 1.1.2: strict preference $\succ$ and indifference $\sim$

With $\succsim$ defined by Def. 1.1.1, we have

- the *strict preference relation*, $\succ$ can be induced from $\succsim$ as: $x \succ y \Leftrightarrow x \succsim y \wedge y \not\succsim x$, or in words, x if preferred to y.
- the *indifference relation*, $\sim$ can be induced from $\succsim$ as: $x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x$, or in words, x is indifferent to y.

With the definition of these relations, we now define the central assumption of relations: ***rationality***.

Definition 1.1.3: Rationality of $\succsim$

A weak preference relation $\succsim$ is *rational* if it is:

- Complete: $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$ or both
- Transitive: $\forall x, y, z \in X$, $x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$

How to understand them? They are both strong assumptions:

- Completeness of $\succsim$ means it is well-defined between any two possible alternatives. From the perspective of an individual, completeness means that she will make choices, and only meditated choices.
- Transitivity of $\succsim$ implies that the decision maker will not have a preference cycle, since whoever has a preference cycle would suffer economically for it[1].

With the definition of rational $\succsim$ in Def. 1.1.3, we can prove the following properites of $\succ$ and $\sim$ *induced* by $\succsim$:

[1]There are 2 types of violations of transitivity: irrational and mechanical. Irrational violations are easy to understand: decision makers simply do not follow transivity assumption, many reasons have been raised, including mental account, framing, menu effect, attraction effect, etc. Mechanical violations means that decision makers are "forced" to violate transitivity. One example of this type of violation is aggregation of considerations: decision makers may aggregate several sub-preferences as together to make the choice, leading to violation of transitivity. Another example is when the preference is only defined for differences above a certain level (problem of perceptible differences). See Mas-Colell et al. (1995, Page 7-8), Rubinstein (2012, Page 4-5) for details

Theorem 1.1.4: Properties of $\succ$ and $\sim$

If $\succsim$ is rational, then:

i. $\succ$ is irreflexive ($x \succ x$ never holds) and transitive ($x \succ y \wedge y \succ z \Rightarrow x \succ z$)
   Proof:
   - irreflexive: by Def. 1.1.2, $x \succ x \Rightarrow x \succsim x \wedge x \not\succsim x$, self contracdiction.
   - transitive: $x \succ y \Rightarrow x \succsim y \wedge y \not\succsim x$, $y \succ z \Rightarrow y \succsim z \wedge z \not\succsim y$. By transitivity of $\succsim$, $x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$. If $z \succsim x$, by transitivity of $\succsim$ and $x \succsim y$, we would have $z \succsim y$, contradicting $y \succ z$. Therefore $x \succsim z \wedge z \not\succsim x \Rightarrow x \succ z$.

ii. $\sim$ is reflexive ($x \sim x, \forall x$), transitive ($x \sim y \wedge y \sim z \Rightarrow x \sim z$) and symmetric ($x \sim y \Rightarrow y \sim x$)
   Proof:
   - reflexive: by completeness of $\succsim$, $\forall x, x \succsim x \Rightarrow x \sim x$
   - transitive: $x \sim y \Rightarrow x \succsim y \wedge y \succsim x$, $y \sim z \Rightarrow y \succsim z, z \succsim y$, by the transitivity of $\succsim$, we have $x \succsim z \wedge z \succsim x$, hen $x \sim z$
   - symmetric: $x \sim y \Rightarrow x \succsim y \wedge y \succsim x \Leftrightarrow y \succsim x \wedge x \succsim y \Rightarrow y \sim x$

iii. $x \succ y \succsim z \Rightarrow x \succ z$
   Proof: $x \succ y \Rightarrow x \succsim y \wedge y \not\succsim x$, hence $x \succ y \succsim z \Rightarrow x \succsim z$. If $z \succsim x$, by transitivity of $\succsim$, $y \succsim x$, contradicting $x \succ y$. Therefore, $z \not\succsim x$

We can also directly define a *rational* $\succ$ (see Kreps (1990, Page 19-21)):

Definition 1.1.5

A strict preference ralation $\succ$ is rational if it is:
- asymmetric: $\nexists x, y \in X$ s.t. $x \succ y \wedge y \succ x$
- negatively transitive: $x \succ y \Rightarrow \forall z \in X \setminus \{x, y\}, x \succ z \vee z \succ y \vee$ both.

With Def. 1.1.5 and Def. 1.1.3, we can prove that $\succsim$ is rational iff $\succ$ is rational:

Theorem 1.1.6

$\succsim$ is rational $\Leftrightarrow$ $\succ$ is rational, specifically:
- $\succsim$ is complete $\Leftrightarrow$ $\succ$ is asymmetric
- $\succsim$ is transitive $\Leftrightarrow$ $\succ$ is negatively transitive

Now we prove this theorem:

Step 1 proof $\succsim$ is rational $\Rightarrow$ $\succ$ is rational

- **asymmetric**
  if $\exists x, y$ s.t. $x \succ y$ and $y \succ x$, then by the definition of induced strict preference, the pair x, y must satisfy

$$\begin{cases} x \succsim y \text{ and } y \not\succsim x & (x \succ y) \\ y \succsim x \text{ and } x \not\succsim y & (y \succ x) \end{cases}$$

  which is, by completeness of rational $\succsim$, impossible. Therefore, such pair x, y don't exist. $\succ$ is proved to be asymetric.

- **negatively transitive**
  First, $\forall z \notin \{x, y\}$, by completeness of rational $\succsim$, the relation between x and z is either $x \succsim z$ or $z \succsim x$. Similarly, the relation between y and z is either $y \succsim z$ or $z \succsim y$.
  Second, given $x \succ y$, x, y satisfies $x \succsim y$ and $y \not\succsim x$.
  Also, it is easy to prove that: $x \succ y \wedge y \succsim z \Rightarrow x \succ z$, $x \succ y \wedge z \succsim x \Rightarrow z \succ y$; and $x \succ y \wedge z \sim x \Rightarrow z \succ y$, $x \succ y \wedge y \sim z \Rightarrow x \succ z$
  Now we have the following scenarios:
  1. if $z \succsim x$ and $y \succsim z$, by transitivity of rational $\succsim$, $y \succsim x$, contradicting the definition of $x \succ y$. This scenario doesn't exist.
  2. if $x \succsim z$ and $y \succsim z$, since $x \succ y$, with the auxilary result proved above, we have $x \succ z$
  3. if $z \succsim x$ and $z \succsim y$, since $x \succ y$, with the auxilary result proved above, we have $z \succ y$
  4. if $x \succsim z$ and $z \succsim y$, since $x \succ y$, suppose:
     (a) $z \succsim x$ as well, then $x \sim z$, in this case $z \succ y$;
     (b) $z \not\succsim x$, then $x \succ z$
     (c) $y \succsim z$ as well, then $y \sim z$, in this case $x \succ z$
     (d) $y \not\succsim z$, then $z \succ y$
     therefore, a complete summary of (a) to (d) would give:

	$z \succsim x$	$z \not\succsim x$
$y \succsim z$	$z \succ y$ & $x \succ z$	$x \succ z$
$y \not\succsim z$	$z \succ y$	$x \succ z$ & $z \succ y$

  Combining all above, we have proved negative transitivity of $\succ$.

With asymmetry and negative transitivity proved, we've proved that $\succsim$ is rational $\Rightarrow \succ$ is rational

Step 2 proof $\succ$ is rational $\Rightarrow \succsim$ is rational.

- Complete: with a rational $x \succ y$, we know $\nexists x, y$ s.t. $x \succ y$ and $y \succ x$ by asymmetry. Therefore, $\forall x, y$, we have two possibilities.
  – $x \succ y$ and $y \not\succ x$, which would naturally induce a weak preference $x \succsim y$
  – $y \succ x$ and $x \not\succ y$, which would naturally induce a weak preference $y \succsim x$
  therefore, $\forall x, y$, either $x \succsim y$ or $y \succsim x$ completeness of $\succsim$ is proven.
- Transitive: with a rational $x \succ y$, negative transivity gives $\forall z \notin \{x, y\}$, either $x \succ z$, $z \succ y$, or both. By negative transitivity, we have:
  – $x \succ z$: following same procedure, we know $x \succsim z$. If:
    * $y \succsim z$, since $x \succ z \Rightarrow z \not\succsim x$, by completeness we have $x \succsim z$, thus $x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$
    * $z \succsim y$, since $x \succ y \Rightarrow x \not\succsim y$, by completeness we have $x \succsim y$, thus $x \succsim z \wedge z \succsim y \Rightarrow x \succsim y$
  – $z \succ y$: again, we know $z \succsim y$. If:
    * $x \succsim z$, since $x \succ y \Rightarrow y \not\succsim x$, by completeness we have $x \succsim y$, thus $z \succsim y \wedge x \succsim z \Rightarrow x \succsim y$

- $z \succsim x$, with $x \succsim y$, suppose $y \succsim z$, this contradicts $z \succ y$, thus $z \succsim x \wedge x \succsim y \Rightarrow z \succsim y$

– $x \succ z$ and $z \succ y$: again we know $x \succsim z$ and $z \succsim y$. Suppose $y \succsim x$, this contradicts $x \succ y$, therefore $x \succsim z \wedge z \succsim y \Rightarrow x \succsim y$

In all three scenarios, transitivity is proved.

With completeness and transitivity proved, we've proved that $\succ$ is rational $\Rightarrow \succsim$ is rational.

Notice that negative positivity in Def. 1.1.5, is logically equivalent to its *contrapositive*: $\exists z \in X \setminus \{x, y\}$ s.t. $x \not\succ z \wedge z \not\succ y \Rightarrow x \not\succ y$. This is percisely why the definition is called negative transitivity.

1.2 Choice Rules

Next, we approach the theory of decision making from choice behavior itself. Formally, choice behavior is represented by means of a *choice structure* $(\mathcal{B}, C(\cdot))$. Now, we define choice structure $(\mathcal{B}, C(\cdot))$:

Definition 1.2.1: choice structure $(\mathcal{B}, C(\cdot))$

A choice structure $(\mathcal{B}, C(\cdot))$ has two ingredients:

- $\mathcal{B} \subset \mathcal{P}(X) \setminus \varnothing$, where $\mathcal{P}(X)$ is the power set of X. This means, every element $B \in \mathcal{B}$ is a subset of X[a].
- $C(\cdot)$ is a *choice rule correspondence* that assigns a nonempty set of chosen elements $C(B) \subset B, \forall B \in \mathcal{B}$[b].

[a]The elements $B \in \mathcal{B}$ are so-called *budget sets*. The budget sets in $\mathcal{B}$ should be thought of as an exhaustive listing of all the choice experiments that can be achieved, but it is possible that some subsets of X are not achievable.

[b]The choice set $C(B)$ can contain a single element, which is the choice among the alternatives in B. BUT, $C(B)$ can contain multiple elements, then elements of $C(B)$ are the *acceptable alternatives* in B.

Now we discuss the CORE assumption in this section: the Weak Axiom of Revealed Preference (WARP):

Definition 1.2.2: WARP

A choice set $(\mathcal{B}, C(\cdot))$ satisfies WARP if:

- $\forall B, B'$ and $x, y \in B \cap B', x \in C(B), y \in C(B') \Rightarrow x \in C(B')$

Or in words, WARP requires that if x is chosen from some alternatives where y is also available, then there can be NO budget set containing both x and y but only y is chosen.

Following WARP, define the *reveal preference relation* $\succsim^*$ as:

Definition 1.2.3: revealed preference $\succsim^*$

Given a choice structure $(\mathcal{B}, C(\cdot))$, $x \succsim^* y \Leftrightarrow \exists B \in \mathcal{B}$ s.t. $x, y \in B \wedge x \in C(B)$

In words, x is revealed at least as good as y.

With revealed preference defiend, we can rephrase WARP as: *If x is revealed at least as good as y, then y **cannot** be revealed preferred to x.* Hence, $\succsim^*$ is not symmetric.

One thing to remember is that $\succsim^*$ need not be either complete or transitive. For $\succsim^*$ to be comparable, for a $B \in \mathcal{B}$ and $x, y \in B$, we must have either $x \in C(B)$, $y \in C(B)$ or both. An example is:

Example: revealed preference $\succsim^*$

Consider $(\mathcal{B}, C(\cdot))$ from $X = \{x, y, z\}$, where $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$. Under WARP, $C\{x, y\} = \{x\} \Rightarrow y \notin C\{x, y, z\}$. BUT, we can have $z \in C(\{x, y, z\})$.

This is why the induced preference is called *revealed*: you don't know what else is going on.

There are 3 interesting properties of choice rules to keep in mind:

- Property α: $x \in A \subseteq B, x \in C(B) \Rightarrow x \in C(A)$.
- Property β: $x, y \in C(A), A \subseteq B, y \in C(B) \Rightarrow x \in C(B)$
- Property γ: $x \in C(B_1), x \in C(B_2) \cdots x \in C(B_n) \Rightarrow x \in C(B_1 \cup B_2 \cup \cdots \cup B_n)$

For a rational preference relation $\succsim$, its induced choice function $C(\cdot, \succsim)$ satisfies the three properties:

α: For $x \in A \subseteq B, x \in C(B, \succsim), x \succsim y, \forall y \in B$, and $A \subseteq B \Leftrightarrow \forall z \in A, z \in B$, hence $\forall z \in A, x \succsim z \Rightarrow x \in C(A, \succsim)$

β: For $x, y \in C(A, \succsim), A \subseteq B, y \in C(B, \succsim), x \succsim y$ and $\forall z \in B, y \succsim z$, by transitivity, $\forall z \in B, x \succsim z \Rightarrow x \in C(B, \succsim)$

γ: $x \in C(B_i, \succsim), i = 1, \cdots, n \Rightarrow \forall b_k^i \in B_i, x \succsim b_i \Rightarrow \forall b \in \{b_k^i\}, i = 1, \cdots, n, x_i \succsim b_i \Rightarrow x \in C(B_1 \cup B_2 \cup \cdots \cup B_n, \succsim)$

With Sen's α and β defined, we have:

Theorem 1.2.4: Sen's α and β

For a non-empty C, C satisfies α and $\beta \Leftrightarrow C$ satisfies WARP

Now, let's prove this.

- **Sen's α and $\beta \Rightarrow$ WARP**: Suppose $x, y \in A \cap B, x \in C(A), y \in C(B)$. Let $\tilde{A} = A \cap B$, since $\tilde{A} \subset A$ and $\tilde{A} \subset B$, Sen's α gives $x, y \in C(\tilde{A})$. Now we have $\tilde{A} \subset B, y \in C(B), x, y \in C(\tilde{A})$, by Sen's β, $x \in C(B)$. Thus, WARP holds.
- **WARP $\Rightarrow$ Sen's α and β**: WARP gives $x, y \in A \cap B, x \in C(B), y \in C(A) \Rightarrow x \in C(A)$, in the special case of $A \subset B$, WARP will be reduced to Sen's α and β.[2]

[2]Another way of proving this is to introduce rationalizing preference relation (Def.1.3.5). In fact, for $\mathcal{B}$ includes all subsets of X of up to 3 elements, WARP $\Leftrightarrow$ C is rationalizable $\Leftrightarrow$ Sen's α and β are satisfied.

1.3 Linking Preferences with Choices

Now we have two major approaches of decision making process: preference relations in Section 1.1 and choice rules in Section 1.2, what we need to do is to link them. This linkage will emerge when we examine two central assumptions: **rationality** and **WARP**. So the major question here is:

$$\textbf{rational} \succsim \overset{???}{\Longleftrightarrow} (\mathcal{B}, C(\cdot)) \textbf{ satisfies WARP}$$

And the answer is: *YES!* but not exactly. Now let's dig in.

Rational $\succsim \Rightarrow (\mathcal{B}, C(\cdot))$ satisfies WARP

First, **rational** $\succsim \Rightarrow (\mathcal{B}, C(\cdot))$ **satisfies WARP** is a big YES. To prove this, we need to define *induced choice correspondence*:

Definition 1.3.1: induced choice correspondence $C^*(B, \succsim)$

Given a **rational** $\succsim$ on X, if the decision maker faces a nonempty subset of alternatives $B \subset X$, by maximizing her preference, she would choose any one of the elements in the *induced choice correspondence*: $C^*(B, \succsim) = \{x \in B : x \succsim y, \forall y \in B\}$

The induced choice correspondence $C^*(B, \succsim)$ has an important property:

Theorem 1.3.2: nonemptiness of $C^*(B, \succsim)$

if X is finite, $C^*(B, \succsim)$ will be **nonempty**.

A brief proof of this proposition is: If X is finite, B is finite as well. We will prove by induction. Starting from $|B| = 1$, the only element of B is in $C^*(B, \succsim)$. Now suppose $C^*(B, \succsim)$ is nonempty when $|B_n| = n$, let $x^* \in C^*(B_n, \succsim)$; when $|B_{n+1}| = n + 1$, let the $n + 1$th element y ($\{y\} = B_{n+1} \setminus B_n$). By the completeness of a rational $\succsim$, either $y \succsim x^*$ or $x^* \succsim y$:

i. $y \succsim x^*$: since $x^* \in C^*(B_n, \succsim) \Rightarrow x^* \succsim x, \forall x \in B_n$. By transitivity of $\succsim$, $y \succsim x, \forall \in B_n$. By completeness, $y \succsim y$ as well. Hence, $y \in C^*(B_{n+1}, \succsim)$.

ii. $x^* \succsim y$: since $x^* \in C^*(B_n, \succsim) \Rightarrow x^* \succsim x, \forall x \in B_n$, hence $x^* \succsim x, \forall x \in B_n \cup y \Rightarrow x^* \in C^*(B_{n+1}, \succsim)$

Notice that when B is finite, a stronger condition of $\succsim$ being acyclic and complete is equilavent to an induced choice rule $C^*(B, \succsim) \neq \varnothing$:

Theorem 1.3.3: acyclic $\succsim$ and nonemptiness of $C^*(B, \succsim)$

For a finite B, $\succsim$ is complete and **acyclic** $\Leftrightarrow C^*(B, \succsim) \neq \varnothing$

$\succsim$ is acyclic mean that: $b_1 \succsim b_2, b_2 \succsim b_3, \cdots, b_{n-1} \succsim b_n \Rightarrow b_n \not\succsim b_1$. An example of transitive but not *acyclic* relations is indifference $\sim$: $a_1 \sim a_2 \sim \cdots \sim a_n \Rightarrow a_n \sim a_1$. A brief proof of Theorem 1.3.3 is:

i. acyclic $\succsim \Rightarrow C^*(B, \succsim) \neq \varnothing$: Suppose if $C^*(B, \succsim) = \varnothing$, for $b_1 \in B, b_1 \notin C^*(B, \succsim) \Rightarrow \exists b_2$ s.t. $b_2 \succsim b_1$. Continue this process, we can generate a sequence of $\cdots \succsim b_2 \succsim b_1$, since B is finite, this sequence must end at b_n. If $\succsim$ is acyclic, $b_1 \not\succsim b_n$, this gives $b_n \succ b_1$, which would mean b_n must be in $C^*(B, \succsim)$, contradicting.

ii. $C^*(B, \succsim) \neq \varnothing \Rightarrow$ acyclic $\succsim$: Suppose $\succsim$ is not acyclic, then there exists $b_1 \succsim b_2 \succsim \cdots \succsim b_n \succsim b_1$, then for set $B = \{b_1, b_2, \cdots, b_n\}$, $\nexists b^*$ s.t. $b^* \succsim b_i \forall b_i \in B$, i.e., $C^*(B, \succsim) = \varnothing$.

Of course, we want to extend this to the situation where B is infinite. However, in general, it is possible that $C^*(B, \succsim)$ is empty (if you set the most preferred side of B open, it would be impossible to choose based on the preferences). However, we can add some assumptions to finite non-emptiness and choice coherence, to ensure that choice function $C(\cdot)$ is well-behaved on infinite sets as well.[3]

With induced choice correspondence $C^*(B, \succsim)$ defined and non-emptyness proved, we can then say:

Theorem 1.3.4: rational $\succsim \Rightarrow$ WARP

If $\succsim$ is a rational preference relation, then the choice structure generated by $\succsim$, $(\mathcal{B}, C^*(\cdot, \succsim))$, satisfies WARP

We can prove this theorem quite easily: $\forall B, B'$ suppose we have $x, y \in B \cap B'$ and $x \in C^*(B, \succsim), y \in C^*(B', \succsim)$, then $x \succsim a, \forall a \in B$ and $y \succsim b, \forall b \in B'$. Naturally, we have $x \succsim y$ since $y \in B$. By rationality (transitivity) of $\succsim$, we have $x \succsim y \succsim b, \forall b \in B'$, which means $x \in C^*(B', \succsim)$. This is precisely the definition of WARP

$(\mathcal{B}, C(\cdot))$ satisfies WARP $\Rightarrow$ Rational $\succsim$

The proof of this direction is more subtle, and is NOT necessarily a yes. Again, we start from a auxilary definition:

Definition 1.3.5: $\succsim$ rationalizing $C(B)$

For a choice structure $(\mathcal{B}, C(\cdot))$, a rational preference relation $\succsim$ **rationalizes** $C(\cdot)$ relative to $\mathcal{B}$ if $C(B) = C^*(B, \succsim), \forall B \in \mathcal{B}$.

In words, if for all budget sets $B \in \mathcal{B}$, the choices generated by a rational $\succsim$, is

[3] The simplest assumption is: For a choice function $C(\cdot)$ that satisfies finite non-emptiness and choice coherence, let the corresponding preference relation be $\succsim_C$, this preference will generate a no-better-than subset for x: $NBT_{\succsim_C}(x) = \{x' \in X : x \succsim_C x'\}$. With this setting, we can have:

$$x \in A \subseteq NBT_{\succsim_C}(x) \Rightarrow C(A) \neq \varnothing$$

This means that if everything in A is revealed to be no better than x and x is also in A, *some* choice will be made out of A. This assumption is equivalment to the conclusion that for the infinite set A, we also have $C(\cdot) \equiv C(\cdot)_{\succsim_C}$. And of course, if we reverse the assumption above, we would have:

$$\forall x \in A, A \nsubseteq NBT_{\succsim_C}(x) \Rightarrow C(A) = \varnothing$$

. This is easy to understand: in this case, x is the least preferred one in A, since A is infinite, you can never find an alternative that is preferred over all others.

just the choice rule $C(\cdot)$, $C(\cdot)$ is rationalized by $\succsim$. This is, in a sense, constructing an explanation of decision making behavior with preferences.

We already proved that $C^*(B, \succsim)$ satisfies WARP, which means that if a rationalizing preference relation to exist, WARP must be satisfied. However, if WARP is satisfied, a rationalizing preference relation does **NOT** necessarily exist.[4] Intuitiviely, more budget sets $B \in \mathcal{B}$ would mean that, to satisfy WARP, choice behavior would be restricted more, and it is easier to be self-contradicting. Therefore, to pin down a rational preference relation to rationalize $C(\cdot)$ relative to $\mathcal{B}$, we need to put some **restrictions on** $\mathcal{B}$.

Theorem 1.3.6: Existence of rationalizing $\succsim$

If $(\mathcal{B}, C(\cdot))$ is a choice structure that:

i. WARP is satisfied

ii. $\mathcal{B}$ includes **all** subsets of X of **up to 3** elements

then there exists a rational preference relation $\succsim$ s.t. $C(B) = C^*(B, \succsim), \forall B \in \mathcal{B}$. And this rational $\succsim$ is the **only** preference relation that can rationalize $(\mathcal{B}, C(\cdot))$[a].

[a]The existence of a rationalizing preference relation $\succsim$ brings many interesting properties, one of them is *path-invariance*: $\forall B_1, B_2 \in \mathcal{B}, B_1 \cup B_2 \in \mathcal{B} \wedge C(B_1) \cup C(B_2) \in \mathcal{B} \Rightarrow C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$, meaning that the decision problem can safely be subdivided. A proof is: for $x \in C(B_1 \cup B_2)$ and $y \in C(B_1) \cup C(B_2)$, since $C(B_1) \cup C(B_2) \subset B_1 \cup B_2$, thus $x \in C(B_1 \cup B_2) \Rightarrow x \in C(C(B_1) \cup C(B_2))$; for $x \in C(C(B_1) \cup C(B_2))$ and $y \in B_1 \cup B_2$, we have $x \succsim z, \forall z \in C(B_1), C(B_2) \Rightarrow x \succsim w, \forall w \in B_1 \cup B_2 \Rightarrow x \in C(B_1 \cup B_2)$

Now let's prove it, by examing the natural candidate for a rationalizing preference relation: the **revealed preference relation** $\succsim^*$:

Step 1 Prove that $\succsim^*$ is rational

- Completeness: By (ii) of Def.1.3.5, all binary subsets of X are in $\mathcal{B}$. Hence, $\{x, y\} \in \mathcal{B}$. For this binary menu, $C(\{x, y\})$ must contain either x or y, therefore, $x \succsim^* y$ or $y \succsim^* x$ or both. Completeness proved.
- Transitivity: $\forall \{x, y, z\} \in \mathcal{B}$, $C(\{x, y, z\}) \neq \varnothing$. Suppose $x \succsim^* y, y \succsim^* z$, which implies that $x \in C(\{x, y\}), y \in C(\{y, z\})$, we then have three cases for $C(\{x, y, z\})$:
  a. $x \in C(\{x, y, z\})$, WARP gives that $x \in C(\{x, z\}) \Rightarrow x \succsim^* z$
  b. $y \in C(\{x, y, z\})$, we have $x \in C(\{x, y\})$. WARP gives $x \in C(\{x, y, z\}) \Rightarrow x \succsim^* z$
  c. $z \in C(\{x, y, z\})$, we have $y \in C(\{y, z\})$. WARP gives $y \in C(\{x, y, z\})$, and $x \in C(\{x, y\})$, WARP gives $x \in C(\{x, y, z\}) \Rightarrow x \succsim^* z$

  Hence, $x \succsim^* y, y \succsim^* z \Rightarrow x \succsim^* z$

Step 2 Prove that $\succsim^*$ rationalizes $C(\cdot)$ on $\mathcal{B}$

Now, we need to show $\forall B \in \mathcal{B}, C(B) = C^*(B, \succsim^*)$. Logically, this means the revealed preference $\succsim^*$ inferred from $C(\cdot)$ actually generates $C(\cdot)$. Formally, we prove it in 2 steps:

a. Suppose $x \in C(B)$, which means that $\forall y \in B, x \succsim^* y$ (by Def.1.2.3), hence

[4]A simple example is: $X = \{x, y, z\}, \mathcal{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$. Since $\mathcal{B}$ contains 3 binary menus, the choice structure $C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x, z\}) = \{z\}$ vacuously satisfy WARP. But, this choice structure cannot be rationalized since it contradicts transitivity.

$x \in C^*(B, \succsim^*)$ (by Def.1.3.1). This proves $C(B) \subseteq C^*(B, \succsim^*)$

b. Suppose $x \in C^*(B, \succsim^*)$, which means that $\forall y \in B, x \succsim^* y$ (by Def.1.3.1). Therefore, $\forall y \in B$, there must exist a set $B_y \in \mathcal{B}$ s.t. $x, y \in B_y \Rightarrow x \in C(B_y)$. Since $C(B) \neq \emptyset$, suppose $z \in C(B)$, since $x \in C(B_z)$, WARP implies that $x \in C(B)$. This proves $C^*(B, \succsim^*) \subseteq C(B)$

Together, we have $C(B) = C^*(B, \succsim^*)$.

Step 3 Prove $\succsim^*$ is the unique choice

Since $\mathcal{B}$ includes all two-element subsets of X, the choice behavior in $C(\cdot)$ completely determines the pairwise preference relations over X of any rationalizing preference.

Now, it is **proved**! Notice that the main assumption(restriction) here is $\mathcal{B}$ **includes all subsets of** X **of up to 3 elements**, this gives completeness, which is fundamental.

Things to keep in mind

We have proved the twoway links of preferences and choices:

- Rational $\succsim \Rightarrow (\mathcal{B}, C^*(\cdot, \succsim))$ satisfies WARP (see Thm.1.3.4)
- A WARP-satisfying, up-to-3-element $(\mathcal{B}, C(\cdot))$ can be uniquely rationalized by a rational $\succsim$ (see Thm.1.3.6)

However, there are still something to keep in mind.

First, for a given choice structure $(\mathcal{B}, C(\cdot))$, there may be **more than one** rationalizing preference relation $\succsim$ in general. Here is the simplest example: For $X = \{x, y\}, \mathcal{B}\{\{x\}, \{y\}\}$ and the choice structure $C(\{x\}) = \{x\}, C(\{y\} = \{y\})$. In this case, **ANY** relation preference relation of X can rationalize $C(\cdot)$ This is related to both Def.1.3.5 and (ii) of Thm.1.3.6. Thm.1.3.6 gives that if $\mathcal{B}$ contains **ALL binary** menus of X, then there could be at most one rationalizing preference relation.

Second, the restriction for WARP$\Rightarrow$ rational $\succsim$, namely $\mathcal{B}$ containing all subsets of up to 3 elements, is too strong. For many economic problems, we will not consider all possible subsets, or limit ourselves to up-to-3-element ones. A strengthened version of WARP will be introduced later for that purpose.

Finally, up till now, we define a rationalizing preference as one: $C(B) = C^*(B, \succsim)$ (Def.1.3.5). A common alternative would be to require only $C(B) \subset C^*(B, \succsim)$: if $C(B)$ is a **subset** of the most preferred choices generated by $\succsim$, i.e., $C^*(B, \succsim)$. This will allow indifferences to be more than the situation of anything might be picked. And it is empirically intuitive in a sense that observed choices will never fully reveal decision makers' entire preferencing maximizing choice set. Naturally, $C(B) \subset C^*(B, \succsim)$ is weaker than $C(B) = C^*(B, \succsim)$. But $C(B) \subset C^*(B, \succsim)$ has an interesting property: the all-indifferent preference will be able to rationalize *any* choice behavior. Therefore, when $C(B) \subset C^*(B, \succsim)$ is used, you would always need to put some additional restrictions on the rationalizing preference relation for the specific economic context.

1.4 Introducing Utility

Now, with preferences and choices defined, and the linkage between the two established, we need to transfer these concepts into math for analytic studies. This is exactly why utility functions are introduced: to assign a number and rank the elements in X according to preferences.

Definition 1.4.1: utility function $u(\cdot)$

A function $u : X \to \mathbb{R}$ is a *utility function representing relation* $\succsim$ if $\forall x, y \in X, x \succsim y \Leftrightarrow u(x) \geq u(y)$

Notice that a utility function representing a preference relation $\succsim$ is NOT unique. **Rank-preserving** is the only requirement, hence, any strictly increasing function $f : \mathbb{R} \to \mathbb{R}, v(x) = f(u(x))$ will also represent $\succsim$ as $u(\cdot)$. The logic is quite straight forward: for $x, y \in X$ and $u(\cdot)$ represents $\succsim$, then $x \succsim y \Leftrightarrow u(x) \geq u(y)$, for a strictly increasing $f(\cdot)$, $u(x) \geq u(y) \Leftrightarrow f(u(x)) \geq f(u(y)) \Leftrightarrow v(x) \geq v(y)$, hence $v(\cdot)$ represents $\succsim$ as well. The major requirement here is **strictly increasing** $f(\cdot)$.

Two concepts to keep in mind:

1. **Ordinal** properties of utility functions: the **invariant** properties of $u(\cdot)$ across all of its strictly increasing transformations $f(u(\cdot))$. Ranking (i.e. the preference represented by utility functions) is ordinal.
2. **Cardinal** properties of utility functions: the **variant** properties of $u(\cdot)$ across all of its strictly increasing transformation $f(u(\cdot))$. Numerical values associated with the alternatives in X (i.e. the magnitude of the differences between alternatives) is cardinal.

The numerical value, or even the size of relative differences have no particular meaning. Only ranking is "real", since the "level of utility" is **unobservable** and anything required to know the "level of utility" is **untestable**.

The central theorem of utility functions is closely linked to rationality:

Theorem 1.4.2: representable by $u(\cdot) \Rightarrow$ rational $\succsim$

A preference relation $\succsim$ can be represented by a utility function $\Rightarrow \succsim$ is rational

The proof is

- *Completeness*. Since $u(\cdot)$ represents preference relations between alternatives, and $u : X \to \mathbb{R}$, thus $\forall x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. By Def.1.4.1, we have either $x \succsim y$ or $y \succsim x$, hence $\succsim$ is complete.
- *Transitivity*. For $x \succsim y, y \succsim z$. By Def.1.4.1, $u(x) \geq u(y), u(y) \geq u(z)$, hence $u(x) \geq u(z) \Rightarrow x \succsim z$.

What about the other way? It is true, subject to some prerequisites:

Theorem 1.4.3: rational $\succsim$ $\Rightarrow$ representable by $u(\cdot)$ (finite)

$\succsim$ is rational and X is **finite** $\Rightarrow$ there is a utility function representing $\succsim$.

The major prerequisite here is X being **finite**. The proof is done by induction: Suppose there are N elements in X:

- When $N = 1$, any number could be assigned to that element as its utility.
- Suppose a rational $\succsim$ on $X = x_1, x_2, \cdots, x_{N-1}$ could be represented by a utility function $u(\cdot)$. Without losing generality, we can assume $u(x_1) \leq u(x_2) \leq \cdots \leq u(x_{N-1})$. For the Nth element x_N, by the rationality of $\succsim$, we have three scenarios:
  i $\forall i \in 1, \cdots, N-1, x_N \succsim x_i$: by Def.1.4.1, $u(x_N) \geq u(x_i)$.
  ii $\forall i \in 1, \cdots, N-1, x_i \succsim x_N$:, $u(x_N) \leq u(x_i)$.
  iii $\exists i, j \in 1, \cdots, N-1, i \neq j, x_j \succsim x_N \succsim x_i$: $u(x_j) \geq u(x_N) \geq u(x_i)$. By completeness and transitivity, $x_1, x_2, \cdots, x_{N-1}$ can be "divided" by x_N, meaning that for $I = \{i : x_N \succsim x_i\}$ and $J = \{j : s_j \succsim x_N\}$, $I \cup J = \{1, \cdots, N-1\}$. Note that we have assumed the index as the ranking, hence let $i^* = \max I$, $j^* = \min J$, $i^*+1 = j^*$, hence we must have $u(x_i) \leq u(x_{i^*}) \leq u(x_N) \leq u(x_{j^*}) \leq u(x_j)$

  In all 3 scenarios, $u(\cdot)$ represents $\succsim$ on $X = \{x_1, \cdots, x_{N-1}, x_N\}$ as well.

With this induction, we prove Thm.1.4.3[5].

Now extend Thm.1.4.3 from finite X to countable infinite X:

Theorem 1.4.4: rational $\succsim$ $\Rightarrow$ representable by $u(\cdot)$ (countably infinite)

$\succsim$ is rational and X is **countable** $\Rightarrow$ there is a utility function representing $\succsim$.

To prove Thm.1.4.4, we can construct a utility function: for any set S, its emuneration $\{s_1, s_2, \cdots\}$ (which exists if S is countable), define an auxilary function $d : S \longrightarrow \mathbb{R}$ as $d(s_i) = \left(\frac{1}{2}\right)^n$, then for a countable set $X = \{x_1, x_2, \cdots\}$, the utility of any element $\tilde{x} \in X$ can be defined as

$$u(x^*) = \sum_{\tilde{x}_i \in NBT(x^*)} d(\tilde{x}_i)$$

[6] where $NBT(x^*)$ is the set of all elements that are **n**ot **b**etter **t**han x^*, i.e. $NBT(x^*) = \{\tilde{x}_i : \tilde{x}_i \in X \wedge x^* \succsim \tilde{x}_i\}$. It is easy to see that $NBT(x^*)$ is a countable subset of X. Suppose $NBT(x^*)$ has k elements ($k < n$), we can calculate the utility $u(x^*) = \sum_{i=1}^{k} \left(\frac{1}{2}\right)^i$. After this construction, rest of the proof is trivial: $\forall x, y \in X$, $x \succsim y \Rightarrow NBT(x) \supseteq NBT(y)$, which means that $NBT(x)$ has at least as many elements as $NBT(y)$, by the constructed utility function, it is easy to see $u(x) \geq u(y)$; Conversely, $u(x) \geq u(y)$ simply means that $NBT(x)$ contains at least as many elements as $NBT(y)$ does, which directly leads to $x \succsim y$.

Now, let's figure out the difficult question: what about uncountable sets? Here is a very general proposition:

[5] Another way of proof is: Start with $x^{start} \in X$, define $W_x = \{y : y \prec x^{start}\}$ then W_x is either empty or not: If not empty, pick $\tilde{x} \in W_x$, shrink W_x to $\{y : y \prec \tilde{x}\}$ and repeat this procedure till $a \sim x^{stop}$ where $u(a) = 0$, then x^{stop} is the "lower bound" of the set. With this process, we can generate a utility function for any finite set X that is rational.

[6] Notice that $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = 1$, utility is bounded to $[0, 1)$.

Theorem 1.4.5: rational $\succsim$ $\Rightarrow$ representable by $u(\cdot)$ (infinite)

For a rational preference $\succsim$ on a set X, $\succsim$ can be represented by $u(\cdot)$ if and only if some countable set X^* of X has the property that $\forall x, y \in X, x \succ y \Rightarrow \exists x^* \in X^*$ s.t. $x \succsim x^* \succ y$.

The proof of Thm.1.4.5 is not that difficult:

Step 1: such X^* exists $\Rightarrow$ rational $\succsim$ can be represented by $u(\cdot)$.

- Given such X^*, $x \succsim y \Rightarrow u(x) \geq u(y)$
  Suppose X^* exists, let it be $X^* = \{x_1^*, x_2^*, \cdots\}$. Again, define $d(x_n^*) = \frac{1}{2^n}$, we can then construct a utility function as
  $$\forall x \in X, u(x) = \sum_{\tilde{x}_i^* \in X^* \cap NBT(x)} d(\tilde{x}_i^*)$$
  Since $x \succsim y \Leftrightarrow NBT(x) \supseteq NBT(y)$, hence $NBT(x) \cap X^* \supseteq NBT(y) \cap X^*$, which, by the construction of $\sum \frac{1}{2^n}$, leading to $u(x) \geq u(y)$.
- Given such X^*, $u(x) \geq u(y) \Rightarrow x \succsim y$
  We can prove the contrapositive: $y \not\succsim x \Rightarrow u(y) \not\geq u(x)$. Given the rationality of $\succsim$, $y \not\succsim x \Rightarrow x \succ y$, then $\exists x^* \in X^*$ s.t. $x \succsim x^* \succ y$, hence we know $NBT(x)$ is strictly larger then $NBT(y)$ ($NBT(x)$ includes x^*), therefore, by definition, $u(x) > u(y) \Rightarrow u(y) \not\geq u(x)$.

Step 2: rational $\succsim$ can be represented by $u(\cdot) \Rightarrow$ such X^* exists.

We want to prove this, but it is very difficult to prove in general, so we construct a special case: Let $\{I_n\}$ be a set of all closed intervals with rational endpoints, that is, each I_n is an interval of $[\underline{q}_n, \bar{q}_n]$ where $\bar{q}_n > \underline{q}_n$ are rational numbers. The set of rational numbers is countable, the cross product of two countable sets is also countable, hence $\{I_n\}$ is countable as well. Let $u(X)$ denote the set of real numbers $\{r \in \mathbb{R} : \exists x \in X, r = u(x)\}$, there will be 3 possible scenarios:

i. $u(X) \cap I_n \neq \emptyset$: for each I_n, pick one $x \in X$ s.t. $u(x) \in I_n$ and name it x_n
ii. $u(X) \cap I_n = \emptyset$: let $\bar{r}_n = \inf\{r \in u(X) : r > \bar{q}_n\}$. If $\exists x \in X$ s.t. $u(x) = \bar{r}_n$, choose one such x and name it x_n
iii. $u(X) \cap I_n = \emptyset \wedge \forall x \in X, \bar{r}_n \neq u(x)$, no x will be defined as x_n

If we define X^* as the collection of all x_n in case i) and ii). Notice there is less than one x_n for each I_n and I_n is countable, hence X^* is countable as well.
Suppose $x \succ y$ for $x, y \in X$, we have $u(\cdot)$ representing $\succsim$, hence $u(x) > u(y)$. Choose some rational number q in the open interval $(u(y), u(x))$ and let $\bar{r} = \inf\{r \in u(X) : r > q\}$. Given this setup, we have $u(x) \geq \bar{r}$ since $u(x) > q$. If:

- $u(x) > \bar{r}$: we can always find a rational number q' s.t. $u(x) > q' > \bar{r}$. Let n be the index of the interval $[q, q']$, since $q < \bar{r} < q'$, $\bar{r} \in u(X) \cap [q, q'] \Rightarrow u(X) \cap [q, q'] \neq \emptyset$. Therefore, $\exists x^* \in X^*$, namely x_n, s.t. $u(x^*) \in [q, q']$, leading to $u(x) > u(x^*) > u(y)$.
- $u(x) = \bar{r}$: we can always find a rational number q' s.t. $q > q' > u(y)$. Let n be the index of the interval $[q, q']$. If $u(X) \cap [q', q] \neq \emptyset$, then $\exists x^* \in X^*$ s.t. $u(x) \geq q \geq u(x^)) \geq q' \geq u(y)$, then $x \succsim x^* \succ y$. If $u(X) \cap [q, q'] = \emptyset$, then $[q, q']$ fits into category (ii) above, and $\exists x^* \in X^*$, namely x_n, such that $u(x^*) = \bar{r} = u(x)$. With this x^*, we have $u(x^*) = u(x) > u(y) \Rightarrow x \succsim x^* \succ y$.

This is a very smart proof, and it is very general as well. However, you would have to make sure that the countable seubset X^* exists, which is not very practical. This problem leads to topological $\gtrsim$, which will be covered later.

It is natural that if we start from $\succ$, we would have the same logic (See Kreps, 1990, Page 30):

- similar definition: $x \succ y \Leftrightarrow u(x) > u(y)$
- similar theorems:
  i $\exists u(\cdot)$ representing $\succ \Rightarrow$ rational $\succ$ (asymmetric and negatively transitive)
  ii If X is finite or at least countably infinite, $\exists u(\cdot)$ representing $\succ \Leftrightarrow$ rational $\succ$

1.5 Commentary

In this section, I discuss some of common commentaries on the standard preference model presented above.

Preference model as a descriptive model

A common complaint about the standard utility maximization/preference ranking model is that no one in reality actually calculates a number as utility before making choices. This comment has a lot of sense to it since we rarely care about utility, let alone doing some math, before grocery shopping. But this observation does NOT invalidate the usefulness of preference/utility model.

The standard model does NOT regulate agents to consciously maximize utility, instead, it assumes individuals act *as if* they maximize utility. Mathematically, we have already proven that if choice behavior satisfies finite nonemptiness and WARP, then something will be chosen, and agents' choice behavior is just *as if* it were preference driven, or the choice behavior can be linked to a preference. If the set of choices is countable, then the preference-driven choice can be indexed by numbers, hence, becomes a mathematical question.

Utility/preference/choice system is considered as a description of choice behavior. Long as people do make a choice, and that choice satisfies WARP, we can always find a numerical way to *describe* the behavioral pattern.

Empirical limits

To verify utility maximization as a model of choices over the choice space X, we need to check every subset A of it. And we also need to know all of $C(A)$. Of course we have already managed to verify the two-way link between preference, utility and choice for all choice menus with no more than 3 elements, but above that, it would be extremely difficult. Empirically, we will observe (at best) $C(A)$ for finitely many subsets of X, we would most likely observe only the *one* element that is selected out of $C(A)$ while failing to identify equally-preferred alternatives simply because they are not observed to be chosen. In these scenarios, how can we tell whether our observations are aligned with utility maximization? In later chapters, we will come back to this problem.

Framing

The way bundles are framed/presented can affect how they are perceived, hence influce individuals' decision making process. One of the most cited economic research by Kahneman and Tversky (1979) discussed this problem in a very clear and innovative fashion. Framing will be a problem if it induces violence of WARP: a is picked when comparing to b, but when c is available as well, b will be chosen instead. This may look silly and will never happen in real life but numerous examples of violation of WARP can be raised due to the framing problem. Designers/publicists are actually trained to take advantage of this "irrationality" to influence consumers' decision making process. A hugh strand of literature in behavioral economics discuss and explore the framing problem, limited attention, heuristics, impatience are introduced to explain such phenomena.

Indecision

Another big problem is that agents may just NOT be able to make a decision. Sometimes the differences between alternatives are trivial or too complicated to measure, the problem of indecision could rise. Rational preferences gives that for each pair of objects x and y, an agent can choose between: x is better than y, y is better than x, x and y are equal. However, if we add another option *I can't tell which is better*, the transitivity would be violated quite easily. Consider it this way, the choice of "I can't decide" allows $C(A) = \varnothing$ even for a finite set of alternatives, this simply goes against the model we have built up.

Inconsistency and probabilistic choice

It is widely documented that an agent could be inconsistent about her choices. This could be an issue of framing, anchoring, indecision, or just unjustifiable inconsistency. This brings the stochastic side of choices: agents' choices may be subject to many random factors hence not deterministic. This will be discussed more thoroughly in later chapters.

Determinants of preference

Since the model is a description of choice behavior, it does not provide any intuition on how a decision is made and what are the determinants of preferences. Later, we will talk about dynamic choice, where an agent's experiences affect her subsequent choices. The standard model needs to be adjusted to incorporating the evolving decision making process through time. Another situation is welfare analysis. Institutional factors would need to be included in the models: preferences could be partially determined socially, different social classes, countries, religions, cultures will likely lead agents to have different preferences. This has been examined more and more by institiutional economists, I will include some inspiring works later.

Range of choices as a value

Nobel Laureate Amartya Sen is very vocal on standard theories being too ends-oriented and not attentive enough to process. In standard theory, suppose $x \in C(A)$, then the individual is equally well off between choosing x from A and being given a x directly. The opportunity to choose (or the ordeal of choosing in the opposite sense) has been ignored. A sufficient amount of psychological evidence has suggested that the right to choose is indeed valuable. The benefit/cost of the choosing process certainly deserves to be considered. Later, in the discussion of random choice, we will consider an example of what would happen with the choosing process taken into consideration.

For the content of this chapter, my main reference is Chapter 1 of Mas-Colell et al. (1995). Section 1, Chapter 2 of Kreps (1990) covers similar content but starts from strict preference $\succ$, it is a very good complement to Mas-Colell et al. (1995). Chapter 1 of Kreps (2013) explores choice and preferences on infinite sets. Lecture 1-3 of Rubinstein (2012) give a well organized, lecture-structured summary of these contents, it is a very good read.

CHAPTER 2

FUNDAMENTALS OF CONSUMER THEORY

Contents

The second chapter focuses on the most fundamental decision unit of microeconomic theory: *consumer*. The main reference is Chapter 2 and 3 of Mas-Colell et al. (1995).

The basic setting of consumer demand study is *market economy*, where the goods and services that the consumer may acquire and consume are available for purchase at known prices (or trade for other goods at know exchange rates).

In this chapter, we will focus on 2 major aspects of the consumer theory: choice and demand.

choice	individual decision making analysis based on choice
demand	individual decision making analysis based on preference

The starting point of individual decision problem is a *set of possible (mutually exclusive) alternatives* from which the individual must choose. To model decision making process on this set of alternatives, one can:

- either start from the tastes, i.e., *preference relations* of individuals, and set up the patterns of decision making with preferences
- or, start from the actual actions of individuals, i.e. *choices*, to deduct a pattern of decision making

The two aspects of consumer theory are actually closely related to each other. Just like choices and preferences in Chapter 1, they are two sides of the same coin. However, they are NOT equivalent. The major conclusion of choice-based consumer theory is that WARP is essentially equivalent to the *compensated law of demand*, but WARP imposes fewer restrictions on demand than preference-based theory, hence, does NOT necessarily guarantee the existence of a rationalizing preference relation for consumer demand, therefore, *strong axiom of revealed preference* is introduced.

2.1 Basic Setting

First, we introduce the basic settings of a consumer's problem in a market economy. These concepts will keep reoccuring in the following sections.

2.1.1 Commodities

First, we need to define the goods and services the consumers consume. We do not actually care about what they specifically are, instead, we use a very abstract concept *commodities* to summarize and analyze them.

Definition 2.1.1: commodity bundle

Assume there are $L < \infty$ different commodities, a *commodity vector* or *commodity bundle* is a list of amounts of the different commodities:

$$x = [x_1, \cdots, x_L]^T$$

x can be view as a point in an $\mathbb{R}^L$ space, i.e., the commodity space. Each entry x_l of x ($l = 1, \cdots, L$) represents the amount of commodity l consumed, hence, the vector is referred to as *consumption vector* or *consumption bundle*.

Three things to keep in mind:

- Time can be incorporated into this setting, namely, today's commodity is distinct from tomorrow's commodity, even if they are otherwise the same. The value of time will come back in later chapters and is crucial in a large strand of behavioral economic literature. Same logic applies to other limitations that are easily neglected, like geographic ones.
- Negative entries can exists in a commodity vector, indicating debits or net outflows of goods. In a producing problem or exchange problem, negative entries of commodity vectors are not rare.
- Consumption is quite flexible and comes in many format empirically, for the sake of data collection conveniency, consumption data are often aggregated monthly, quarterly for even annually. Meanwhle, some consumptions in the commodity vectors may not actually occur in the market.

2.1.2 The Consumption Set

Consumptions are limited by a number of constraints, which will form a subset of commodity space $X \subset \mathcal{R}^L$. With in this subset, all possible commodity bundles can be consumed, this is exactly the definition of consumption sets (see Mas-Colell et al. (1995, Page 19-20) for some simple examples of consumption sets).

For now, we will focus on the simplest consumption set: all possible non-negative commodity bundles:

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_l \geq 0, \forall l = 1, \cdots, L\}$$

It is easy to show that

$\mathbb{R}^L_+$ is a **convex** set

A brief proof: $\forall \vec{x}, \vec{y} \in \mathbb{R}^L_+$ and $\forall \alpha \in [0,1]$, $\alpha\vec{x} + (1-\alpha)\vec{y} = [\alpha x_1 + (1-\alpha)y_1, \cdots, \alpha x_L + (1-\alpha)y_L]^T$. Since $x_i \geq 0, y_i \geq 0, \alpha x_i + (1-\alpha)y_i \geq 0 \Longrightarrow \alpha\vec{x} + (1-\alpha)\vec{y} \in \mathbb{R}^L_+$.

Convexity of consumption sets is an essential assumption here, but some of the results do survive without the assumption fo convexity.

Although consuption sets are formed due to some constraints, but these constraints have nothing to do with consumers' budget (exogenous constraints). It is intuitive that with a large enough budget (infinitely large if you may), you can always afford any consumption bundle in a give consumption set. But what if consumers, as in reality, do have a budget constraint and cannot afford every bundle in the consumption set?

2.1.3 Prices and Consumption Cost

Budget constraints are an important economic constraint faced by consumers: one can only consume the commodity bundles that she can afford.

To formalize this constraint, we need to introduce the *price vector*:

$$\vec{p} = \left[p_1, \cdots, p_L\right] \in \mathbb{R}^L$$

This price vector contains unit price information for each of the L commodities. They are all traded in the market and the price information of them is publicly quoted (the *principle of completeness of markets*). For simplicity, we assume $\vec{p} \gg 0$ i.e. $\forall l, p_l > 0$[1].

Another important assumption is the *price-taking assumption*: consumers do NOT have the power to influence the prices. Here, each consumer only buys a small (neglectable) fraction of the total demand for commodities.

With prices defined, we can finally define the **economic-affordability constraint** of consumers: For a consumer with wealth w, a consumption bundle $\vec{x} \in \mathbb{R}_+^L$ is affordable if its total cost does NOT exceed the consumer's wealth level w, formally,

$$\vec{p} \cdot \vec{x} = p_1x_1 + p_2x_2 + \cdots + p_Lx_L \leq w$$

With the two core assumptions stated above, consumers face a *linear price schedule*.

2.2 Choice Based Consumer Theory

2.2.1 Walrasian Budgets

We have already defined the economic-affordability constraint of consumers, if we also limit consumption bundle x to be non-negative, we would have the Walrasian, or competitive budget:

Definition 2.2.1: Walrasian budget

The Walrasian, or competitive budget set

$$B_{\mathbf{p},w} = \{x \in \mathbb{R}_+^L : \mathbf{p} \cdot \mathbf{x} \leq w\}$$

is the set of all feasible consumption bundles give market prices $\mathbf{p}$ and wealth w.

From a Walrasian budget's point of view, a consumer can only choose a consumption bundle x from $B_{\mathbf{p},w}$. An underlining assumption here is $w > 0$, otherwise consumers cannot afford anything. We can also separately define the "edge" of a Walrasian budget set as:

Definition 2.2.2: budget hyperplane

The *budget hyperplane* is the set $\{x \in \mathbb{R}_+^L : \mathbf{p} \cdot \mathbf{x} = w\}$

It determines the upper bound of the budget set: with prices of other commodities ($\mathbf{p}_{-i}$) and wealth level w fixed, the change of commodity i's price p_i will enlarge/shrink the budget set by moving the budget hyperplane. Geometrically, the price vector $\mathbf{p}$ must be orthogonal to the budget hyperplane, we can think it this way: for any two bundles $\mathbf{x}$ and $\mathbf{x}'$ one the budget hyperplane, we must have $\mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}' = w$, hence $\mathbf{p} \cdot \Delta\mathbf{x} = \mathbf{p} \cdot (\mathbf{x}' - \mathbf{x}) = 0$ is always true.

[1]Of course, price can be negative, meaning that consumers are actually paid to consume the "bad "commodity, such as polution.

A core feature of the Walrasian budget set is that it is **convex**: $\forall \mathbf{x}, \mathbf{y} \in B_{\mathbf{p},w}, \alpha \in [0,1], \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in B_{\mathbf{p},w}$. This is very easy to prove: $\mathbf{x} \in \mathbb{R}^L_+ \wedge \mathbf{y} \in \mathbb{R}^L_+ \Rightarrow \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \mathbb{R}^L_+$, $\mathbf{p}\cdot\mathbf{x} \leq w \wedge \mathbf{p}\cdot\mathbf{y} \leq w \Rightarrow \alpha(\mathbf{p}\cdot\mathbf{x}) + (1-\alpha)(\mathbf{p}\cdot\mathbf{y}) \leq w$. Notice that the Walrasian budget set is not automatically convex. Its convexity is induced from the convexity of its superset (the consumption set), in this case $\mathbb{R}^L_+$. In general, it is easy to show that the Walrasian budget set will convex as long as its corresponding consumption set is convex.

Of course, it is perfectly possible that a consumer's budget is NOT convex (and not Walrasian, in that sense), the brilliant work of Deaton and Muellbauer (1980) has documented and discussed many complicated consumption sets that are not convex.

2.2.2 Walrasian Demand Function

With Walrasian budgets defined, we can define *Walrasian demand correspondence* and *Walrasian demand function* as:

Definition 2.2.3: Walrasian demand correspondence/function

For each *price-wealth* pair $(\mathbf{p}, w)$, Walrasian demand correspondence is the set of chosen consumption bundles, written as $x(\mathbf{p}, w)$. When $x(\mathbf{p}, w)$ is single-valued, it will be referred to as a **Walrasian demand function**.

The two main assumptions of $x(\mathbf{p}, w)$ are:

Definition 2.2.4: homogeneity of degree zero and Walras' law

For a Walrasian demand correspondence $x(\mathbf{p}, w)$, we assume it is:

1. **Homogeneous of degree zero**: $\forall \mathbf{p}, w$ and $\alpha > 0$, $x(\alpha\mathbf{p}, \alpha w) = x(\mathbf{p}, w)$. This means that if wealth and prices change in the same proportion at the same time, consumer would not change her choice.
   There are two implications of homogeneous of degree zero assumption:
   - **Dimension reduction**: With this assumption, we can reduce $x(\mathbf{p}, w)$ to $x(\tilde{\mathbf{p}}, 1)$ where $\tilde{\mathbf{p}} = \frac{1}{w}\mathbf{p}$, hence to a L-argument problem.
   - **Choice structure**: By homogeneity of degree zero, $x(\mathbf{p}, w)$ depends only on $\mathbf{p}$ and w, i.e. the budget set, then for the family of Walrasian budget sets $\mathcal{B}^* = \{B_{\mathbf{p},w} : \mathbf{p} \gg 0, w > 0\}$, $(\mathcal{B}^*, x(\cdot))$ is a choice structure. This choice structure does NOT include all possible subsets of X, in particular, not all two- and three-element subsets of X[a], therefore, requires more assumptions to have a rationalizing preference.

   ← X is the set of all possible bundles
2. **Walras' law**: $\forall \gg 0, w > 0$, $\mathbf{p}\cdot\mathbf{x} = w$ for every $\mathbf{x} \in x(\mathbf{p}, w)$. Walras' law means that a consumer want to spend all her wealth for consumption, every consumption bundle hence will exhaust w. Walras' law implies that goods are continuous.
   Notice that this is implicitly intertemporal, meaning that Walras' law consider lifetime resource allocation.

[a] This relates to the argument that when one consumption bundle $\mathbf{x}^*$ is infeasible, it not being chosen does NOT mean it is less preferred. This will be dealt with more carefully with more

assumptions of preference-based demand.

Next, we disucess several basic definitions induced from the Walrasian demand function. For a demand function

$$x(\mathbf{p}, w) = [x_1(\mathbf{p}, w), x_2(\mathbf{p}, w), \cdots, x_L(\mathbf{p}, w)]^T$$

we have the following two effects:

Wealth effects Fix $\mathbf{p}$, get a function of $x(w)$.

Take the partial derivative of demand function on wealth:

$$D_w x(\mathbf{x}, w) = \left[\frac{\partial x_1(\mathbf{p}, w)}{\partial w}, \frac{\partial x_2(\mathbf{p}, w)}{\partial w}, \cdots, \frac{\partial x_L(\mathbf{p}, w)}{\partial w}\right]^T$$

, then $\frac{\partial x_l(\mathbf{p},w)}{\partial w}$ is the *wealth effect* for the lth commodity. Wealth effects can classify commodities into two types:

- **normal**: $\partial x_l(\mathbf{p}, w)/\partial w \geq 0$
- **inferior**: $\partial x_l(\mathbf{p}, w)/\partial w < 0$

Price effects For commodity l, fix $\mathbf{p}_{-l}$ and w, get a function of $x(p_l)$.

Take the partial derivative of demand function on price vector:

$$D_{\mathbf{p}} x(\mathbf{p}, w) = \begin{bmatrix} \nabla_{\mathbf{p}} x_1(\mathbf{p}, w) \\ \vdots \\ \nabla_{\mathbf{p}} x_L(\mathbf{p}, w) \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_1(\mathbf{p},w)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_L(\mathbf{p},w)}{\partial p_L} \end{bmatrix}$$

For most goods, the price effects would be negative: you would buy more if the price is lower. However, there *Giffen* goods (often low quality) and *Veblen* goods (often luxurious) that have positive price effects.

Regarding wealth effects $\partial x_l(\mathbf{p}, w)/\partial w$ and price effects $\partial x_l(\mathbf{p}, w)/\partial p_k$, we have the following two theorems:

Theorem 2.2.5: price effect and wealth effect cancel out

If the Walrasian demand function $x(\mathbf{p}, w)$ is **homogeneous of degree zero**, the $\forall \mathbf{p}, w$:

$$\sum_{k=1}^{L} \frac{\partial x_l(\mathbf{p}, w)}{\partial p_k} p_k + \frac{\partial x_l(\mathbf{p}, w)}{\partial w} w = 0, \forall l = 1, \cdots, L$$

This is easy to proof: homogeneity of degree zero gives $x(\alpha\mathbf{p}, \alpha w) - x(\mathbf{p}, w) = 0$, differentiating with respect to α, get:

$$\mathrm{D}_{\alpha\mathbf{p}} x(\alpha\mathbf{p}, \alpha w) \cdot \mathbf{p} + \mathrm{D}_{\alpha w} x(\alpha\mathbf{p}, \alpha w) w = 0$$

this is true for any α, if we take $\alpha = 1$, we get $\mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w) \cdot \mathbf{p} + \mathrm{D}_w x(\mathbf{p}, w) w = \mathbf{0}$, which is just the matrix notation of Thm.2.2.5.

Intuitively, this implies the price (substitution) effects and wealth (income) effects induced by the price change of one commodity, when weighted by the prices of other commodities and wealth, will cancel out.

Another theorem is induced from Walras' law:

Theorem 2.2.6: change of total expenditure

If the Walrasian demand function $x(\mathbf{p}, w)$ satisfies **Walras' law**, then for all $\mathbf{p}, w$:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(\mathbf{p}, w)}{\partial p_k} + x_k(\mathbf{p}, w) = 0, \forall k = 1, \cdots, L$$

and

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(\mathbf{p}, w)}{\partial w} = 1$$

The proof is also easy: Walras' law gives $\mathbf{p} \cdot x(\mathbf{p}, w) = w$, take derivatives with respect to $\mathbf{p}$, get $\mathbf{p} \cdot \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w) + x(\mathbf{p}, w)^T = \mathbf{0}^T$[2]; take derivatives with repsect to w, get $\mathbf{p} \cdot \mathrm{D}_w(\mathbf{p}, w) = 1$. The intuition is: a change in prices do NOT change the total expenditure, and the total expenditure will change by the same amount with the change in wealth.

If we define elasticities as:

$$\epsilon_{lk} = \frac{\partial x_l(\mathbf{p}, w)/x_l(\mathbf{p}, w)}{\partial p_k/p_k} \quad \text{\% change in demand for } l \text{ per \% change in the price of } k$$
$$\epsilon_{lw} = \frac{\partial x_l(\mathbf{p}, w)/x_l(\mathbf{p}, w)}{\partial w/w} \quad \text{\% change in demand for } l \text{ per \% change in wealth } w$$

We can rewrite Thm.2.2.5 as $\sum_{k=1}^{L} \epsilon_{lk}(\mathbf{p}, w) + \epsilon_{lw}(\mathbf{p}, w) = 0$ for $l = 1, \cdots, L$: this directly expresses that an equal % change in all prices and wealth leads to no change in demand, i.e., the homogeneity of degree zero. And we can rewrite the two equations in Thm.2.2.6 as $\sum_{l=1}^{L} \frac{p_l x_l(\mathbf{p},w)}{w} \epsilon_{lk}(\mathbf{p}, w) + \frac{p_k x_k(\mathbf{p},w)}{w} = 0$ for $k = 1, \cdots, L$ and $\sum_{l=1}^{L} \frac{p_l x_l(\mathbf{p},w)}{w} \epsilon_{lw}(\mathbf{p}, w) = 1$.

2.2.3 WARP and Law of Demand

Since for the family of Walrasian budget sets $\mathcal{B}^* = \{B_{\mathbf{p},w} : \mathbf{p} \gg 0, w > 0\}$, $(\mathcal{B}^*, \mathbf{p}(\cdot))$ is a choice structure, naturally, we would like to check when WARP holds for this choice structure.

[2] This will give: $\mathbf{p} \cdot \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w)\mathbf{p} + x(\mathbf{p}, w)^T \mathbf{p} = \mathbf{0}^T \Rightarrow \mathbf{p} \cdot \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w)\mathbf{p} = -w$

Definition 2.2.7: WARP of $x(\mathbf{p}, w)$

A Walrasian demand function $x(\mathbf{p}, w)$ satisfies WARP if any two price-wealth conditions $(\mathbf{p}, w)$ and $(\mathbf{p}', w')$ satisfies:

$$\mathbf{p} \cdot x(\mathbf{p}', w') \leq w \text{ and } x(\mathbf{p}', w') \neq x(\mathbf{p}, w) \Rightarrow \mathbf{p}' \cdot x(\mathbf{p}, w) > w'$$

The intuition is quite straightforward: If $\mathbf{p} \cdot x(\mathbf{p}', w') \leq w$ and $x(\mathbf{p}', w') \neq x(\mathbf{p}, w)$, the consumer chooses $x(\mathbf{p}, w)$ even when $x(\mathbf{p}', w')$ is affordable. Hence, $x(\mathbf{p}, w)$ is preferred over $x(\mathbf{p}', w')$, which means that the only reason why she chooses $x(\mathbf{p}, w)$ instead of $x(\mathbf{p}', w')$ is that she can not afford $x(\mathbf{p}, w)$ at $(\mathbf{p}', w')$, i.e. $\mathbf{p}'x(\mathbf{p}, w) > w'$. An easier way to understand WARP is that we **CANNOT** have both $\mathbf{p}' \cdot x(\mathbf{p}, w) \leq w'$ and $\mathbf{p} \cdot x(\mathbf{p}', w') \leq w$, unless $x(\mathbf{p}', w') = x(\mathbf{p}, w)$.

It is easy to show that this definition is a special case of Def.1.2.2: here we consider single-valued $C(\cdot)$ (function), then Def.1.2.2 gives that $\forall B, B'$ and $x, y \in B \cap B'$, $x = C(B)$, $y = C(B') \Rightarrow x \in C(B') \Rightarrow x = y$. Rewrite this in the context of Walrasian demand functions, we have: for any $(\mathbf{p}, w), (\mathbf{p}', w')$, if $\mathbf{p} \cdot x(\mathbf{p}', w') \leq w$ and $\mathbf{p}' \cdot x(\mathbf{p}, w) \leq w'$, then $x(\mathbf{p}, w) = x(\mathbf{p}', w')$. This is the contrapositive statement of Def.2.2.7. Hence the two definitions are equivalent.

WARP and compensated price changes

WARP can also be stated in terms of compensated price changes. At $(\mathbf{p}, w)$, the consumer chooses $x(\mathbf{p}, w)$, if she still want to afford $x(\mathbf{p}, w)$ at a new price $\mathbf{p}'$, she would need to adjust her wealth to $w' = \mathbf{p}' \cdot x(\mathbf{p}, w)$, this gives *Slutsky wealth compensation* $\Delta w = w' - w = \Delta \mathbf{p} \cdot x(\mathbf{p}, w) = (\mathbf{p}' - \mathbf{p})x(\mathbf{p}, w)$, $\Delta \mathbf{p}$ is referred to as *compensated price changes*. With these concepts defined, WARP implies:

Theorem 2.2.8: WARP $\Rightarrow$ law of demand

For any $\Delta \mathbf{p}$ from initial situation $(\mathbf{p}, w)$ to $(\mathbf{p}', w') = (\mathbf{p}', \mathbf{p}' \cdot (\mathbf{x}, w))$, if WARP holds, we have

$$(\mathbf{p}' - \mathbf{p}) \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)] \leq 0$$

with strict inequality unless $x(\mathbf{p}, w) = x(\mathbf{p}', w')$.

Here is a proof: rewrite the left-side, get

$$(\mathbf{p}' - \mathbf{p}) \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)] = \mathbf{p}' \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)] - \mathbf{p} \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)]$$

By Walras' law, $\mathbf{p}' \cdot x(\mathbf{p}', w') = w'$, $\mathbf{p} \cdot x(\mathbf{p}, w) = w$, also by assumption of compensated price changes, $\mathbf{p}' \cdot x(\mathbf{p}, w) = w'$, if if WARP holds, since $x(\mathbf{p}, w)$ is affordable at $(\mathbf{p}', w')$, $x(\mathbf{p}', w')$ must NOT be affordable at $(\mathbf{p}, w)$, i.e., $\mathbf{p} \cdot x(\mathbf{p}', w') > w$, therefore, we have:

$$(\mathbf{p}' - \mathbf{p}) \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)] = \underbrace{\mathbf{p}' \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)]}_{=0} - \underbrace{\mathbf{p} \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)]}_{>0} < 0$$

This Theorem goes both way, that is

Theorem 2.2.9: law of demand ⇒ WARP

If for any compensated price changes, $(\mathbf{p}'-\mathbf{p})\cdot[x(\mathbf{p}',w')-x(\mathbf{p},w)]\leq 0$ holds, the WARP is satisfied.

We proof the contrapositive: if WARP is violated, there exists a compensated price change such that $(\mathbf{p}'-\mathbf{p})\cdot[x(\mathbf{p}',w')-x(\mathbf{p},w)]>0$. A violation of WARP gives that for $(\mathbf{p},w)$ and $(\mathbf{p}',\mathbf{w}')$ such that $x(\mathbf{p},w)\neq x(\mathbf{p}',w')$, $\mathbf{p}'\cdot x(\mathbf{p},w)\leq w'$ and $\mathbf{p}\cdot x(\mathbf{p}',w')\leq w$ can both be satisfied. The proof is done in 2 steps:

Step 1 Prove the fact that:

If for any two price-wealth pairs $(\mathbf{p},w),(\mathbf{p}',w')$, $\mathbf{p}\cdot x(\mathbf{p}',w')=w, x(\mathbf{p}',w')\neq x(\mathbf{p},w)\Rightarrow \mathbf{p}'\cdot x(\mathbf{p},w)>w'$, then WARP holds.

Again, we prove the contrapositive of this proposition: If WARP is violated, there exists a compensated price change violating WARP. Let $(\mathbf{p}',w')$ and $(\mathbf{p}'',w'')$ violates WARP such that $x(\mathbf{p}',w')\neq x(\mathbf{p}'',w'')$, $\mathbf{p}'\cdot x(\mathbf{p}'',w'')\leq w'$ and $\mathbf{p}''\cdot x(\mathbf{p}',w')\leq w''$. We have two scenarios:

- $\mathbf{p}'\cdot x(\mathbf{p}'',w'')=w'$ or $\mathbf{p}''\cdot x(\mathbf{p}',w')=w''$ or both: It is easy to show that the condition of the price-wealth pairs are violated.
- $\mathbf{p}'\cdot x(\mathbf{p}'',w'')<w'$ and $\mathbf{p}''\cdot x(\mathbf{p}',w')<w''$: we can construct a price-wealth pair $(\mathbf{p},w)$ such that both $x(\mathbf{p}',x')$ and $x(\mathbf{p}'',x'')$ are affordable. By picking an $\alpha\in(0,1)$ to linearly combine p' and p'', we can have:

$$(\alpha\mathbf{p}'+(1-\alpha)\mathbf{p}'')\cdot x(\mathbf{p}',w')=(\alpha\mathbf{p}'+(1-\alpha)\mathbf{p}'')\cdot x(\mathbf{p}'',w'')$$

let $\mathbf{p}=\alpha\mathbf{p}'+(1-\alpha)\mathbf{p}''$, $w=(\alpha\mathbf{p}'+(1-\alpha)\mathbf{p}'')\cdot x(\mathbf{p}',w')$[3]. Now, both $x(\mathbf{p},w)$ and $\mathbf{x}'(\mathbf{p}',w')$ are on the constructed budget line $(\mathbf{p},w)$. It is easy to show that

$$\begin{aligned}\alpha w'+(1-\alpha)w'' &> \alpha\mathbf{p}'\cdot x(\mathbf{p}',w')+(1-\alpha)\mathbf{p}''\cdot x(\mathbf{p}',w')\\ &= w=\mathbf{p}\cdot x(\mathbf{p},w)\\ &= [\alpha\mathbf{p}+(1-\alpha)\mathbf{p}'']\cdot x(\mathbf{p},w)\end{aligned}$$

hence, either $\mathbf{p}'\cdot x(\mathbf{p},w)<w'$ or $\mathbf{p}''\cdot x(\mathbf{p},w)<w''$. If $\mathbf{p}'\cdot x(\mathbf{p},w)<w'$, we know $x(\mathbf{p}',w')$ is on the constructed line $(\mathbf{p},w)$, therefore, $\mathbf{p}\cdot x(\mathbf{p}',w')=w$. However, we already have $\mathbf{p}'\cdot x(\mathbf{p},w)<w'$, the two conditions constitute a violation of WARP.

Now we know, we can only consider compensated price changes to test WARP, we can move to Step 2.

Step 2 With Step 1 done, we know that if WARP does NOT hold, there exists a compensated price change from $(\mathbf{p}',w')$ to $(\mathbf{p},w)$ such that $x(\mathbf{p},w)\neq x(\mathbf{p}',w')$, $\mathbf{p}\cdot x(\mathbf{p}',w')=w$ and $\mathbf{p}'\cdot x(\mathbf{p},w)\leq w'$. Meanwhile, Walras' law gives $w=\mathbf{p}\cdot x(\mathbf{p},w), w'=\mathbf{p}'\cdot x(\mathbf{p}',w')$, leading to

$$\mathbf{p}\cdot[x(\mathbf{p}',w')-x(\mathbf{p},w)]=0,\ \mathbf{p}'\cdot[x(\mathbf{p}',w')-x(\mathbf{p},w)]\geq 0$$

Hence, combined with the condition $x(\mathbf{p},w)\neq x(\mathbf{p}',w')$, we have

$$(\mathbf{p}'-\mathbf{p})\cdot[x(\mathbf{p}',w')-x(\mathbf{p},w)]>0$$

[3]Why this construction works? Since $A=\mathbf{p}'\cdot x(\mathbf{p}',w')=w'>\mathbf{p}'\cdot x(\mathbf{p}'',w'')=A'$, $B=\mathbf{p}''\cdot x(\mathbf{p}',w')<w''=\mathbf{p}''\cdot x(\mathbf{p}'',w'')=B'$, therefore if α is properly chosen, we can achieve $\alpha A+(1-\alpha)B=\alpha A'+(1-\alpha)B'$ with $A>A', B<B'$.

Now we have completed the prove of the contrapositive of Thm.2.2.9.

Notice that so far, we have only linked WARP to **compensated** price changes, which means that we are focusing on the case that the budget line rotates through one point (not the end point) $x(\mathbf{p}, w)$ from $(\mathbf{p}, w)$ to $(\mathbf{p}', w')$. A brilliant graphic presentation can be found in Figure 2.F.4 on Mas-Colell et al. (1995, Page 33): after a compensated price change, WARP requires that the demand for the commodity that experiences a price drop must increase, which is exactly the law of demand:

Theorem 2.2.10: compensated law of demand

For a compensated price change from $\mathbf{p}$ to $\mathbf{p}'$ (wealth changing accordingly from w to w' s.t. $\mathbf{p}'x(\mathbf{p}, w) = w'$), the law of demand requires demand and price move in opposite directions, i.e.

$$\Delta \mathbf{x} \cdot \Delta \mathbf{p} = (\mathbf{p}' - \mathbf{p}) \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)] \leq 0$$

And of course, if the price changes are NOT compensated, meaning that wealth does NOT change accordingly, in this case (Mas-Colell et al., 1995, Figure 2.F.5, Page 33), the new budget line would be either completely above the original one (more generous) or completetly below (less generous), and WARP would impose NO restrictions on the new consumption bundle.[4]

WARP and multivalued $x(\mathrm{p}, w)$

Here, we (briefly) extended WARP into multivalued $x(\mathbf{p}, w)$, Walrasian demand **correspondences**, which is a generalization of single-valued demand functions.

First, we have the generalized version of WARP:

Definition 2.2.11: generalized WARP of $x(\mathbf{p}, w)$

(Generalizing 2.2.7) For any $(\mathbf{p}, w)$ and $(\mathbf{p}', w')$, if $\mathbf{x} \in x(\mathbf{p}, w)$, $\mathbf{x}' \in x(\mathbf{p}', w')$, $\mathbf{p}' \cdot \mathbf{x} \leq w'$ and $\mathbf{x}' \notin x(\mathbf{p}, w)$ lead to $\mathbf{p} \cdot \mathbf{x}' > w$, then the Walrasian demand correspondence satisfies WARP.

and we have a generalized version of the compensated law of demand:

Theorem 2.2.12: generalized compensated law of demand

(Generalizing 2.2.10) From any initial position $(\mathbf{p}, w)$ with demand $\mathbf{x} \in x(\mathbf{p}, w)$ for compensated price change to new prices $\mathbf{p}'$ and wealth level $w' = \mathbf{p}' \cdots \mathbf{x}$, we have

$$(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) \begin{cases} < 0, & \mathbf{x}' \notin x(\mathbf{p}, w) \\ = 0, & \mathbf{x}' \in x(\mathbf{p}, w) \end{cases}$$

[4] In In fact, if $x(\cdot, \cdot)$ is homogeneous of degree one w.r.t. w, then the law of demand actually still holds for *uncompensated* price changes: $\mathrm{d}p \cdot \mathrm{D}_{\mathbf{p}} x(p, w) \mathrm{d}\mathbf{p} \leq 0, \forall \mathrm{d}\mathbf{p}$. A brief proof is: by homogeneity of degree one w.r.t. w, $x(\mathbf{p}, \alpha w) = \alpha x(\mathbf{p}, w)$, differentiate w.r.t. α and evaluate at $\alpha = 1$, get $\mathrm{D}_w x(\mathbf{p}, w) w = x(\mathbf{p}, w)$, then the Slutsky matrix is $S(\mathbf{p}, w) = \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w) + \mathrm{D}_w x(\mathbf{p}, w) \cdot x(\mathbf{p}, w)^T = \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w) + \frac{1}{w} x(\mathbf{p}, w) x(\mathbf{p}, w)^T$. Hence $\mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w) = S(\mathbf{p}, w) - \frac{1}{w} x(\mathbf{p}, w) x(\mathbf{p}, w)^T$ is a negative semidefinite matrix, the proof is complete.

Again, if the multivalued demand correspondence $x(\mathbf{p}, w)$ satisfies Walras' law and compensated law of demand (Def.2.2.12), this $x(\mathbf{p}, w)$ satisfies the generalized WARP (Def.2.2.11), the proof follows the same logic of the single-valued version: We only need to show for all compensated price changes, generalized WARP holds. We can prove its contrapositive, if WARP does not hold, there exists a compensated price violates the generalized compensated law of demand. Suppose $\mathbf{x} \in x(\mathbf{p}, w), \mathbf{x}' \in x(\mathbf{p}', w')$, $\mathbf{p}' \cdot \mathbf{x} = w'$ and $\mathbf{p} \cdot \mathbf{x}' \leq w$, then it is easy to show that $(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) = w - \mathbf{p} \cdot \mathbf{x}' \geq 0$, it violates the compensated law of demand, except for the case that $\mathbf{x}' \in x(\mathbf{p}, w)$, where $(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) = 0$.

2.2.4 Slutsky Matrix

When consumer demand $x(\mathbf{p}, w)$ is differentiable with respect to both prices and wealth, we can rewrite the inequality in Thm.2.2.10: $\mathrm{d}\mathbf{p} \cdot \mathrm{d}\mathbf{x} \leq 0$. In the spirit of compensated price changes, we know the change in wealth related to $\mathrm{d}\mathbf{p}$ is $\mathrm{d}w = x(\mathbf{p}, w) \cdot \mathrm{d}\mathbf{p}$. Use the chain rule, we can get

$$\mathrm{d}\mathbf{x} = \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w)\mathrm{d}\mathbf{p} + \mathrm{D}_w x(\mathbf{p}, w)\mathrm{d}w$$

and we can plug in the induced wealth difference $\mathrm{d}w = x(\mathbf{p}, w)\mathrm{d}\mathbf{p}$, get

$$\begin{aligned}\mathrm{d}\mathbf{x} &= \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w)\mathrm{d}\mathbf{p} + \mathrm{D}_w x(\mathbf{p}, w) \cdot [x(\mathbf{p}, w)\mathrm{d}\mathbf{p}] \\ &= \left[\mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w) + \mathrm{D}_w x(\mathbf{p}, w) x(\mathbf{p}, w)^T\right] \mathrm{d}\mathbf{p}\end{aligned}$$

and, plug this back into the law of demand $\mathrm{d}\mathbf{p} \cdot \mathrm{d}\mathbf{x} \leq 0$, we will have a very important expression:

$$\mathrm{d}\mathbf{p} \underbrace{\left[\mathrm{D}_{\mathbf{p}} x(\mathbf{p}, w) + \mathrm{D}_w x(\mathbf{p}, w) x(\mathbf{p}, w)^T\right]}_{\equiv S(\mathbf{p}, w)} \mathrm{d}\mathbf{p} \leq 0$$

where $S(\mathbf{p}, w)$ is the **Slutsky** matrix, or *substitution* matrix.

To write it explicitly:

Definition 2.2.13: Slutsky matrix

The Slutsky matrix $S(\mathbf{p}, w)$ is defined as:

$$S(\mathbf{p}, w) = \begin{bmatrix} s_{11}(\mathbf{p}, w) & \cdots & s_{1L}(\mathbf{p}, w) \\ \vdots & \ddots & \vdots \\ s_{L1}(\mathbf{p}, w) & \cdots & s_{LL}(\mathbf{p}, w) \end{bmatrix}$$

where

$$s_{lk}(\mathbf{p}, w) = \underbrace{\frac{\partial x_l(\mathbf{p}, w)}{\partial \mathbf{p}_k}}_{\text{direct effect of } \mathrm{d}p_k} + \underbrace{\frac{\partial x_l(p, w)}{\partial w} x_k(\mathbf{p}, w)}_{\text{effect through wealth compensation due to } \mathrm{d}p_k}$$

This term measures the differential change in x_l (consumption of l) due to a differential change in p_k (the price of k), when wealth is adjusted so that the consumer can still afford the pre-change consumption bundle. We can interpret the two sub-terms as:

- $\frac{\partial x_l(\mathbf{p},w)}{\partial p_k}$: the change in demand for l due to the change of k's price is $\frac{\partial x_l(\mathbf{p},w)}{\partial p_k}\mathrm{d}p_k$
- $\frac{\partial x_l(p,w)}{\partial w}x_k(\mathbf{p},w)$: the wealth change that makes the consumer able to just afford the original consumption bundle is $x_k(\mathbf{p},w)\mathrm{d}p_k$, the effect of this wealth change on the demand for l is $\frac{\partial x_l(p,w)}{\partial w}x_k(\mathbf{p},w)\mathrm{d}p_k$

In short, the price change of k influences the demand of l in 2 ways: directly (through partial derivatives) and indirectly (through wealth compensation). Slutsky matrix documents this influence, i.e., **substitution effect**, for each possible pair of goods.

Here, we have the core feature of Slutsky matrix:

Theorem 2.2.14: Slutsky matrix is negative semidefinite

If a differentiable Walrasian demand function $x(\mathbf{p},w)$ satisfies Walras' law, homogeneity of degree 0, and the weak axiom, then at any $(\mathbf{p},w)$, $S(\mathbf{p},w)$ satisfies

$$\mathbf{v}\cdot S(\mathbf{p},w)\mathbf{v}\leq 0,\forall\mathbf{v}\in\mathbb{R}^L$$

i.e., Slutsky matrix is negative semidefinite[a].

[a] Notice that "semidefnite" is an abbreviation, in fact, Slutsky is not required to be symmetric here. Symmetry of $S(\mathbf{p},w)$ has more meanings later. See

There are several implications worth noticing:

Theorem 2.2.15: features induced by negative semidefinite Slutksy matrix

If $S(\mathbf{p},w)$ is negative semidefinite, then

(a) $s_{ll}(\mathbf{p},w)\leq 0$: the *substitution effect* of good l with respect to its own price is always **non-positive**.

(b) Giffen good must be inferior: $s_{ll}(\mathbf{p},w)=\frac{\partial x_l(\mathbf{p},w)}{\partial p_l}+\frac{\partial x_l(\mathbf{p},w)}{\partial w}x_l(\mathbf{p},w)\leq 0$, then it is easy to see that at $(\mathbf{p},w)$, $\frac{\partial x_l(\mathbf{p},w)}{\partial p_l}>0\Rightarrow\frac{\partial x_l(\mathbf{p},w)}{\partial w}<0$.

(c) If $x(\mathbf{p},w)$ is differentiable, homogeneous of degree 0 and satisfies Walras' law, then $\forall(\mathbf{p},w),\mathbf{p}\cdot S(\mathbf{p},w)=0,S(\mathbf{p},w)\mathbf{p}=0$. This means that $S(\mathbf{p},w)$ is always singular (its rank is less than L).
It is easy to prove:
Thm.2.2.6 gives $\mathbf{p}\cdot S(\mathbf{p},w)=\mathbf{p}\cdot\mathrm{D}_{\mathbf{p}}x(\mathbf{p},w)+\mathbf{p}\cdot\mathrm{D}_w x(\mathbf{p},w)x(\mathbf{p},w)^T=\mathbf{p}\mathrm{D}_{\mathbf{p}}x(\mathbf{p},w)+x(\mathbf{p},w)^T=0$; Thm.2.2.5 gives $S(\mathbf{p},w)\mathbf{p}=\mathrm{D}_{\mathbf{p}}x(\mathbf{p},w)\mathbf{p}+\mathrm{D}_w x(\mathbf{p},w)x(\mathbf{p},w)^T\mathbf{p}=\mathrm{D}_{\mathbf{p}}x(\mathbf{p},w)\mathbf{p}+\mathrm{D}_w x(\mathbf{p},w)w=0$.

Finally, we can have a little discussion on WARP and the negative semidefiniteness of the Slutsky matrix.

- WARP $\Rightarrow$ the Slutsky matrix being negative semidefinite.
  The Slutsky matrix is constructed from WARP and compensated law of demand $\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{p}\leq 0$, hence WARP implying semi-definite Slutsky matrices is vacuous.
- the Slutsky matrix being negative semidefinite $\nRightarrow$ WARP

Negative semidefiniteness of the Slutsky matrix itself is NOT sufficient for WARP, the Slutsky matrix must be negative definite for all vectors other than those that are proportional to $\mathbf{p}$, that is $\forall \mathbf{v} \neq \alpha \mathbf{p}, \mathbf{v} \cdot S(\mathbf{p}, w)\mathbf{v} < 0$.

For the 5 conditions:

(1) **WARP**: $\mathbf{p} \cdot x(\mathbf{p}', w') \leq w$ and $x(\mathbf{p}', w') \neq x(\mathbf{p}, w) \Rightarrow \mathbf{p}' \cdot x(\mathbf{p}, w) > w'$
(2) **WWARP** (weaker WARP): $\mathbf{p} \cdot x(\mathbf{p}', w') < w$ and $x(\mathbf{p}', w') \neq x(\mathbf{p}, w) \Rightarrow \mathbf{p}' \cdot x(\mathbf{p}, w) > w'$
(3) Slutsky matrix being **negative semidefinite** (NSD): $\mathbf{v} \cdot S(\mathbf{p}, w)\mathbf{v} \leq 0, \forall \mathbf{v} \in \mathbb{R}^L$
(4) Slutsky matrix being **negative definite** (ND): $\mathbf{v} \cdot S(\mathbf{p}, w)\mathbf{v} < 0, \forall \mathbf{v} \in \mathbb{R}^L$
(5) Slutksy matrix being **symmetric**

they have the following relations:

- **WWARP** $\Leftrightarrow$ Slutsky matrix is **NSD**
- Slutsky matrix is ND $\Rightarrow$ **WARP**
- Slutsky matrix is **symmetric** and **NSD** $\Leftrightarrow$ **WWARP** $\Leftrightarrow$ **WARP**

This is discussed rather thoroughly by Kihlstrom et al. (1976). Symmetry of Slutsky matrix will play an important role in the preference based consumer theory.

2.3 Preference Based Consumer Theory

2.3.1 Properties of Preferences Required

To analyze demand with preferences (and later, utility), we first need some assumptions on preferences $\succsim$:

$\succsim$ is rational

- **completeness**: $\forall \mathbf{x}, \mathbf{y} \in X$, it must be either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$ or both
- **transitivity**: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ $\mathbf{x} \succsim \mathbf{y}, \mathbf{y} \succsim \mathbf{z} \Rightarrow \mathbf{x} \succsim \mathbf{z}$

$\succsim$ is desirable It is reasonable to assume that larger amounts of commodities are preferred. So basically, the more the better.

Here, we have two common assumption, a stronger one, and a weaker one:

- **monotone** (*stronger*): if $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{y} \gg \mathbf{x} \Rightarrow \mathbf{y} \succ \mathbf{x}$
  strongly monotone (*even stronger*): if $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x} \Rightarrow \mathbf{y} \succ \mathbf{x}$
- **locally nonsatiated** (*weaker*): $\forall \mathbf{x} \in X, \forall \varepsilon > 0, \exists \mathbf{y} \in X$ s.t. $\|\mathbf{y} - \mathbf{x}\| \leq \varepsilon$ and $\mathbf{y} \succ \mathbf{x}$

Local nonsatiation is the common used one, since it poses minimal constraints. It is easy to see that **strongly monotone** $\Rightarrow$ **monotone** $\Rightarrow$ **locally nonsatiated**, here is a proof: If $\succsim$ is strongly monotone and $\mathbf{x} \gg \mathbf{y}$, then $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ and $x \succ y$, thus $\succsim$ is monotone; if $\succsim$ is monotone, $\mathbf{x} \in X$ and $\varepsilon > 0$, let $\mathbf{y} = \mathbf{x} + \frac{\varepsilon}{\sqrt{L}}\mathbf{e}$ where $\mathbf{e} = (1, \cdots, 1) \in \mathbb{R}^L$, then $\|\mathbf{y} - \mathbf{x}\| \leq \varepsilon$ and $y \succ x$, hence $\succsim$ is locally nonsatiated.

$\succsim$ will divide X into 3 sets relative to $\mathbf{x}$:

- **upper contour set**: $\{\mathbf{y} \in X \mid \mathbf{y} \succsim \mathbf{x}\}$
- **lower contour set**: $\{\mathbf{y} \in X \mid \mathbf{x} \succsim \mathbf{y}\}$
- **indifferent set**: $\{\mathbf{y} \in X \mid \mathbf{x} \sim \mathbf{y}\}$

and **local nonsatiation** guarantees that the indifference set is a line.

$\succsim$ is convex $\succsim$ is **convex** if $\forall \mathbf{x} \in X$, $\mathbf{y} \succsim \mathbf{x}$ and $\mathbf{z} \succsim \mathbf{x} \Rightarrow \alpha\mathbf{y} + (1-\alpha)\mathbf{z} \succsim \mathbf{x}, \forall \alpha \in [0,1]$; that is, the upper contour set of $\mathbf{x}$, $\{\mathbf{y} \in X \mid \mathbf{y} \succsim \mathbf{x}\}$ is convex.

Convexity assumption is one of the central assumption, it expresses two intuitive observations of economic agents:

(i) diminishing marginal rates of substitution
(ii) inclination for diversification

$\succsim$ is **strictly convex** if $\forall \mathbf{x} \in X$, $\mathbf{y} \succsim \mathbf{x}$ and $\mathbf{z} \succsim \mathbf{x} \Rightarrow \alpha\mathbf{y} + (1-\alpha)\mathbf{z} \succ \mathbf{x}, \forall \alpha \in [0,1]$.

$\succsim$ is continuous So far, we have rationality, local nonsatiation and convexity, but these are NOT enough to guarantee a preference $\succsim$ to be representable by a utility funciton. An example is **lexicographic preference relation**.

Example: lexicographic preferences

The lexicographic preference relation is $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow (x_1 > y_1) \vee (x_1 = y_1 \wedge x_2 > y_2) \vee \cdots \vee (x_1 = y_1 \wedge \cdots x_{L-1} = y_{L-1} \wedge x_L \geq y_L)$. It is complete, transitive, strongly monotone and strictly convex; BUT, there is **NO** utility function representing it[a].

[a] Here is a brief proof: Suppose there is a utility function $u(\cdot)$, then for every x_1, we can pick a rational number $r(x_1)$ s.t. $u(x_1,2) > r(x_1) > u(x_1,1)$, if $x_1 > x_1'$, then $r(x_1) > u(x_1,1) > u(x_1',2) > r(x_1')$. Hence $r(\cdot)$ is a one-to-one function from the set of real numbers (uncountably infinite) to the set of rational numbers (countably infinite), which is mathematically impossible. Another way of proving it is: suppose there is a utility function $u(\cdot)$, $[\inf u(x,\cdot), \sup u(x,\cdot)]$ is an interval, hence if $x < y$, $\sup u(x,\cdot) < \inf u(y,\cdot)$, which is impossible

To guarantee utility representability, $\succsim$ needs to be continuous: $\succsim$ is continuous if it is preserved under limits, that is, $\forall \left\{\{x^n, y^n\}\right\}_{n=1}^{\infty}$ with $x^n \succsim y^n, \forall n, x = \lim_{n\to\infty} x^n, y = \lim_{n\to\infty} y^n \Rightarrow x \succsim y$.

Continuity can also be expressed as: $\forall \mathbf{x}$, the upper contour set $\{\mathbf{y} \in X \mid \mathbf{y} \succsim \mathbf{x}\}$ and the lower contour set $\{\mathbf{y} \in X \mid \mathbf{x} \succsim \mathbf{y}\}$ are both **closed**. Here is the proof of this equivalence:

- continuity $\Rightarrow$ contour set closedness: $\forall \left\{y^n\right\}_{n=1}^{\infty}$, let $x^n = x, \forall n$, then if $x^n = x \succsim y^n$, by continuity, $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$, hence the lower contour set is closed; the upper contour set can be proved the same way.
- contour set closedness $\Rightarrow$ continuity: Suppose $\exists \{x^n\}, \left\{y^n\right\} \in X$ s.t. $\forall n, x^n \succsim y^n$, $x^n \to x \in X, y^n \to y \in X$ and $y \succ x$. We know both $\left\{z \in X \mid y \succ z\right\}$ and $\{z \in X \mid z \succ x\}$ are open, hence $\exists N_1, N_2 \in \mathbb{Z}^+$ s.t. $y \succ x^n, \forall n > N_1; y^n \succ x, \forall n > N_2$, conceivably, there are two cases on $\left\{y^n\right\}$:

- $\exists N_3 \in \mathbb{Z}^+$ s.t. $y^n \succsim y, \forall n > N_3$: then we have $y^n \succ x^n, \forall n > \max\{N_1, N_3\}$, contradiction.
- $\exists \{y^{k(n)}\}$ s.t. $y \succ y^{k(n)}, \forall n$: then $\exists m \in \mathbb{Z}^+$ s.t. $k(m) > N_2$, we know $\{z \in X \mid z \succ y^{k(m)}\}$ is open, $\exists N_4 \in \mathbb{Z}^+$ s.t. $y^n \succ y^{k(m)}, \forall n > N_4$. Since $\{z \mid z \succsim y^{k(m)}\}$ is closed, $x \succsim y^{k(m)}$, however, since $k(m) > N_2$, $y^{k(m)} \succ x$ is assumed, hence contradiction.

Another way of proving this is to assume $\succsim$ to be monotone.

2.3.2 Utility Representing Preference

Given the assumption of continuity, we can finally have the following thoerem (Debreu representation theorem):

Theorem 2.3.1: rational and continuous $\succsim \Leftrightarrow$ continuous $u(\cdot)$

If a preference relation $\succsim$ is rational and **continuous**, then there is a continuous utility function $u(x)$ that represents $\succsim$.

Here is the proof:

- **continuous $u(\cdot) \Rightarrow$ continuous preference**:
  suppose $\{x_n\} \to x^*, \{y_n\} \to y^*$, $u(\cdot)$ represents $\succsim$, hence $u(x_n) \geq u(y_n)$. If $u(\cdot)$ is continuous, $u(x_n) \to u(x^*), u(y_n) \to u(y^*)$, hence $u(x^*) \geq u(y^*)$, leading to $x^* \succsim y^*$.
- **continuous preference $\Rightarrow$ continuous $u(\cdot)$**:
  The proof is done in 3 steps:
  Step 1 First, we construct the utility function.
  Since preferences are continuous, and if monotone is assumed, for each $\mathbf{x} \in \mathbb{R}^L$, define a function $\alpha(\mathbf{x}) : \mathbb{R}^L \to \mathbb{R}$ such that $\alpha(\mathbf{x}) = \inf\{a \mid a \cdot \mathbf{e} \succsim \mathbf{x}\}$ where $\mathbf{e} = (1, \cdots, 1)^T$. By the continuity of $\succsim$, $\{a \mid a \cdot \mathbf{e} \succsim \mathbf{x}\}$ is non-empty and bounded below, hence inf exists. Also, continuity of $\succsim$ implies the upper contour set of $\mathbf{x}$ is closed, hence $\exists \underline{a} \in \mathbb{R}$ s.t. $\underline{a} = \inf\{a \mid a \cdot \mathbf{e} \succsim \mathbf{x}\}$. Let $\alpha(\mathbf{x}) = \underline{a}$, $\alpha(\mathbf{x})\mathbf{e} \succsim \mathbf{x}$, also $\mathbf{x} \succ (\alpha(\mathbf{x}) - \frac{1}{n})\mathbf{e}$, by continuity, $\mathbf{x} \succsim \alpha(\mathbf{x})$. Hence, $\mathbf{x} \sim \alpha(\mathbf{x})\mathbf{e}$[5], then, we can take $\alpha(\mathbf{x})$ as the utility function $u(\mathbf{x}) = \alpha(\mathbf{x})$.
  Step 2 Next, we prove $\alpha(\mathbf{x}) \geq \alpha(\mathbf{y}) \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$
  - $\alpha(\mathbf{x}) \geq \alpha(\mathbf{y}) \Rightarrow \mathbf{x} \succsim \mathbf{y}$: Suppose $\alpha(\mathbf{x}) \geq \alpha(\mathbf{y})$, by monotonicity, $\alpha(\mathbf{x})\mathbf{e} \succsim \alpha(\mathbf{y})\mathbf{e}$. Since $\mathbf{x} \sim \alpha(\mathbf{x})\mathbf{e}, \mathbf{y} \sim \alpha(\mathbf{y})\mathbf{e}$, we have $\mathbf{x} \succsim \mathbf{y}$
  - $\alpha(\mathbf{x}) \geq \alpha(\mathbf{y}) \Leftarrow \mathbf{x} \succsim \mathbf{y}$: Suppose $\mathbf{x} \succsim \mathbf{y}$, then $\alpha(\mathbf{x})\mathbf{e} \sim \mathbf{x} \succsim \mathbf{y} \sim \alpha(\mathbf{y})\mathbf{e}$, by monotonicity, $\alpha(\mathbf{x}) \geq \alpha(\mathbf{y})$

  Step 3 Finally, we prove $\alpha(\mathbf{x})$ is continuous: $\lim_{n\to\infty} \mathbf{x}^n = \mathbf{x} \Rightarrow \lim_{n\to\infty} \alpha(\mathbf{x}^n) = \alpha(\mathbf{x}), \forall \{\mathbf{x}^n\}_{n=1}^{\infty}$
  - $\{\alpha(\mathbf{x}^n)\}_{n=1}^{\infty}$ must have a convergent subsequence: by monotonicity, $\forall \epsilon >$

[5] $\mathbf{x} \sim \alpha(\mathbf{x})\mathbf{e}$ can also be proven as: by continuity, $A^{up} = \{\alpha \in \mathbf{R}_+ \mid \alpha\mathbf{e} \succsim \mathbf{x}\}$ and $A^{low}\{\alpha \in \mathbf{R}_+ \mid \mathbf{x} \succsim \alpha\mathbf{e}\}$ are both non-empty and closed. By completeness of $\succsim$, $\mathbb{R}_+ \subset A^{up} \cup A^{low}$; $\mathbb{R}_+$ is connected, hence $A^{up} \cap A^{low} \neq \emptyset$, thus there exists α s.t. $\alpha\mathbf{e} \sim \mathbf{x}$. And by monotonicity, $\alpha_1 > \alpha_2 \Rightarrow \alpha_1\mathbf{e} \succ \alpha_2\mathbf{e}$, hence the scalar $\alpha\mathbf{e} \sim \mathbf{x}$ is unique.

$0, \forall \mathbf{x}'$ s.t. $\|\mathbf{x}' - \mathbf{x}\| \leq \epsilon$, $\alpha(\mathbf{x}')$ lies in a compact subset of $\mathbb{R}_+$, $[\alpha_0, \alpha_1]$. Since $\{\mathbf{x}^n\}_{n=1}^{\infty}$ converges to $\mathbf{x}$, then $\exists N$ s.t. $\forall n > N$, $\alpha(\mathbf{x}^n)$ lies in this compact set, hence, this infinite sequence must have a convergent subsequence.
- all convergent subsequences of $\{\alpha(\mathbf{x}^n)\}_{n=1}^{\infty}$ converge to $\alpha(\mathbf{x})$: suppose otherwise, then there is a strictly increasing function $m(\cdot)$ that assigns to each n a positive integer $m(n)$, and the subsequence $\left\{\alpha(\mathbf{x}^{m(n)})\right\}_{n=1}^{\infty}$ converges to $\alpha' \neq \alpha(\mathbf{x})$. Without losing generality, let $\alpha' > \alpha(\mathbf{x})$, then by monotonicity, $\alpha'\mathbf{e} \succ \alpha(\mathbf{x})\mathbf{e}$. Let $\hat{\alpha} = \frac{1}{2}[\alpha' + \alpha(\mathbf{x})]$, then $\hat{\alpha}(\mathbf{e}) \succ \alpha(\mathbf{x})\mathbf{e}$. Since $\alpha(\mathbf{x}^{m(n)}) \to \alpha' > \hat{\alpha}$, then $\exists \bar{N}$ s.t. $\forall n > \bar{N}$, $\alpha(\mathbf{x}^{m(n)}) > \hat{\alpha}$, thus $\mathbf{x}^{m(n)} \sim \alpha(\mathbf{x}^{m(n)})\mathbf{e} \succ \hat{\alpha}\mathbf{e}$. By continuity of $\succsim$, we get $\mathbf{x} \sim \alpha(\mathbf{x})\mathbf{e} \succsim \hat{\alpha}\mathbf{e}$, leading to a contradiction. With the same logic, $\alpha' < \alpha(\mathbf{x})$ is ruled out as well. Hence, all convergent subsequences $\{\alpha(\mathbf{x}^n)\}_{n=1}^{\infty}$ converge to $\alpha(\mathbf{x})$. Hence, $\lim_{n\to\infty} \alpha(\mathbf{x}^n) = \alpha(\mathbf{x})$.

Now a continuous utility function can represent a continuous preference relation, there are several results on how they relate to each other:

- continuous $u(\cdot)$ can represent $\succsim$, but any strictly increasing yet discontinuous transformation of $u(\cdot)$ may also represents $\succsim$
- preference $\succsim$ is monotone $\Rightarrow$ utiity function $u(\cdot)$ is increasing: $\mathbf{x} \gg \mathbf{y} \Rightarrow u(\mathbf{x}) > u(\mathbf{y})$
- preference $\succsim$ is convex $\Rightarrow$ utility function $u(\cdot)$ is quasiconcave: the upper contour set of $\mathbf{x}$, $\left\{\mathbf{y} \in \mathbb{R}_+^L \mid \mathbf{y} \succsim \mathbf{x}\right\}$ is convex $\Rightarrow u(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \min\{u(\mathbf{x}), u(\mathbf{y})\}$. It is **quasiconcave**, NOT concave.

Increasingness and quasiconcavity are both ordinal properties of $u(\cdot)$, preserved for any increasing transformation.

2.3.3 UMP (Utility Maximizing Problem)

A consumer's problem is to choose a feasible consumption bundle given a positive price-wealth combination $(\mathbf{p}, w)$ to maximize her utility:

$$\max_{\mathbf{x} \geq 0} u(\mathbf{x}), \text{ s.t. } \mathbf{p} \cdot \mathbf{x} \leq w$$

if $\mathbf{p} \gg 0$, $u(\cdot)$ is continuous, then this problem has a solution: the Walrasian budget set $B_{\mathbf{p},w} = \left\{\mathbf{x} \in \mathbb{R}_+^L \mid \mathbf{p} \cdot \mathbf{x} \leq w\right\}$ is a compact set, hence a continuous function on it always has a maximum value.

This problem will induce two objects: Walrasian demand correspondence/function $x(\mathbf{p}, w) \in \mathbb{R}_+^L$ and the value function $V(\mathbf{p}, w)$.

Walrasian demand $x(\mathbf{p}, w)$

Suppose continuous $u(\cdot)$, representing locally nonsatiated $\succsim$, $x(\mathbf{p}, w)$ has the following properties:

Theorem 2.3.2: properties of $x(\mathbf{p}, w)$

- ***homogeneity of degree zero*** in $(\mathbf{p}, w)$: $x(\alpha\mathbf{p}, \alpha w) = x(\mathbf{p}, w), \forall \mathbf{p} \gg 0, w > 0, \alpha > 0$
- ***Walras' law***: $\mathbf{p} \cdot x(\mathbf{p}, w) = w$
- ***convexity/uniqueness***:
  (a) if $\succsim$ is convex, then $u(\cdot)$ is quasiconcave, hence $x(\mathbf{p}, w)$ is a convex set
  (b) if $\succsim$ is strictly convex, then $u(\cdot)$ is strictly quasiconcave, hence $x(\mathbf{p}, w)$ is single-valued
- ***continuity***: Walrasian demand $x(\mathbf{p}, w)$ is continuous (preference is continuous).

We can prove these properties:

- ***homogeneity of degree zero*** in $(\mathbf{p}, w)$: $\forall \alpha > 0$, we always have $\{\mathbf{x} \mid \alpha\mathbf{p} \cdot \mathbf{x} \leq \alpha w\} = \{\mathbf{x} \mid \mathbf{p} \cdot \mathbf{x} \leq w\}$, this is, the set of feasible $\mathbf{x}$ does NOT change when both $\mathbf{p}$ and w are multiplied by α. The utility-maximizing $x(\mathbf{p}, w)$ is in these sets, hence $x(\mathbf{p}, w) = x(\alpha\mathbf{p}, \alpha w)$.
- ***Walras' law***: This is guaranteed by the local nonsatiation assumption. If $\exists\mathbf{x}$ s.t $\mathbf{p} \cdot \mathbf{x} < w$, then in the sufficiently small neighborhood of $\mathbf{x}$, $\exists\mathbf{y}$ s.t. $\mathbf{py} < w$ and $\mathbf{y} \succ \mathbf{x}$, contradicting $\mathbf{x}$ being optimal.
- ***convexity/uniqueness***:
  (a) convex $\succsim \Rightarrow$ convex $x(\mathbf{p}, w)$: suppose two maximizers $\mathbf{x}_1, \mathbf{x}_2 \in x(\mathbf{p}, w)$, since Walrasian budget set is convex, hence $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ is feasible. $u(\mathbf{x}_1) = u(\mathbf{x}_2) = u_{\max}$, hence by quasiconcavity of $u(\cdot)$, $u(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \geq u_{\max}$, hence $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in x(\mathbf{p}, w)$.
  (b) strictly convex $\succsim \Rightarrow$ unique $x(\mathbf{p}, w)$: suppose two maximizers $\mathbf{x}_1, \mathbf{x}_2 \in x(\mathbf{p}, w)$, again $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ is feasible. $u(\mathbf{x}_1) = u(\mathbf{x}_2) = u_{\max}$, hence by stirct quasiconcavity of $u(\cdot)$, $u(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) > u_{\max}$, hence $\mathbf{x}_1, \mathbf{x}_2$ are not maximizers, i.e., there are at most one maximizer (and there must be a maximizer in the compact Walrasian set).
- ***continuity***: Walrasian demand $x(\mathbf{p}, w)$ is the solution of the utility maximizing problem, then by maximum theorem, if the utility function is continuous (preference is continuous), its maximizer $x(\mathbf{p}, w)$ is also continuous.
  This property can also be proved in a more "economic" way: If $x(\mathbf{p}, w)$ is not continuous, then $\exists \left\{p^n\right\} \to p^*$ such that $x(\mathbf{p}^*, w) = \mathbf{x}^*$, but $x(p^n, w) \nrightarrow \mathbf{x}^*$, or in an ϵ representation: $\exists\epsilon > 0$ s.t. $\|x(\mathbf{p}^n, w), \mathbf{x}^*\| > \epsilon$. We know $x(\mathbf{p}^n, w)$ is in a compact set (p^n is bounded away from 0 and by wealth w), hence we can assume, without loss of generality, $\exists\mathbf{y}^* \neq \mathbf{x}^*, x(\mathbf{p}^n, w) \to \mathbf{y}^*$. Since $\forall n, \mathbf{p}^n x(\mathbf{p}^n, w) \leq w$, $\mathbf{p}^*\mathbf{y}^* \leq w$, at the same time, $\mathbf{x}^*$ is the utility maximizer given the Walrasian budget, hence $\mathbf{x}^* \succ \mathbf{y}^*$. By the continuity of preferences, this $\succ$ is preserved in the small neighborhoods of $\mathbf{x}^*$ and $\mathbf{y}^*$: for sufficiently large n, $x(\mathbf{p}^n, w)$ is in the neighborhodd of y^*; we can choose a bundle $\mathbf{z}^*$ in the neighborhood of $\mathbf{x}^*$ s.t. $\mathbf{p}^*\mathbf{z}^* < w$. For sufficiently large n, we have $\mathbf{p}^n\mathbf{z}^* < w$; and at the same time $\mathbf{z}^* \succ x(\mathbf{p}^n, w)$. These two: $\mathbf{p}^n\mathbf{z}^* < w, \mathbf{z}^* \succ x(\mathbf{p}^n, w)$ are directly contradicting each other (monotonicity).

In general, with the Lagrangean $\mathcal{L} = u(\mathbf{x}) + \lambda(w - \mathbf{p} \cdot \mathbf{x})$, there are two cases of optimum $x(\mathbf{p}, w)$:

- interior optimum ($\forall x_l > 0$): by solving the Lagrangean, get FOC: $\partial u(\mathbf{x}^*)/\partial x_l = \lambda p_l, \forall l \in \{1, \cdots, L\}$. Or, write in matrix notation: $\nabla u(\mathbf{x}^*) = \lambda \mathbf{p}$ where the gradient vector of $u(\cdot)$ at $\mathbf{x}$ is $\nabla u(\mathbf{x}) = [\partial u(\mathbf{x})/\partial x_1, \cdots, \partial u(\mathbf{x})/\partial x_L]$.
  At the interior optimum, we have the marginal rate of substitution of l for k at $\mathbf{x}^*$ as $MRS_{lk}(\mathbf{x}^*) = \frac{\partial u(\mathbf{x}^*)/\partial x_l}{\partial u(\mathbf{x}^*)/\partial x_k} = \frac{p_l}{p_k}$, it is equal to the price ratio. Graphically, $\mathbf{x}^*$ is the tangent point of the utility curve to the budget line.
- border optimum ($\exists x_l = 0$): when there is no interior optimum, $\nabla u(\mathbf{x}^*)$ is not proportional to prices, particularly, $\partial u(\mathbf{x}^*)/\partial x_l \begin{cases} \leq 0, & x_l^* = 0 \\ = 0, & x_l^* > 0 \end{cases}$. At the border, due to nonnegativity of consumption, even when the agent wants to decrease the consumption of x_l, she won't be able to.

The Lagrange multiplier λ itself is very interesting as well. λ gives the marginal value of relaxing the constraint in the UMP, hence it is the consumer's **marginal utility value of wealth** at the optimum:

$$\nabla u(\mathbf{x}(\mathbf{p}, w)) \cdot \mathrm{D}_w x(\mathbf{p}, w) = \underbrace{\lambda \mathbf{p} \cdot \mathrm{D}_w x(\mathbf{p}, w) = \lambda}_{\text{By Walras' law: } \mathbf{p} \cdot x(\mathbf{p}, w) = w}$$

where $\mathrm{D}_w x(\mathbf{p}, w) = [\partial x_1(\mathbf{p}, w)/\partial w, \cdots, \partial x_L(\mathbf{p}, w)/\partial w]$

But here we assume the Lagrangean multiplier is fixed across all commodities, which is a very strong assumption. Hence, we could perhaps think of another way to select λ: Think Lagrangean multiplier as a penalty mechanism such that any deviation from the constraint will decrease utility, hence, λ can only be chosen from $\left\{\frac{\partial u(\mathbf{x})}{\partial x_l} \cdot \frac{1}{p_l}\right\}_{l=1}^{L}$; at the same time, λ reflects the marginal utility of wealth, so it is reasonable to choose a value that is as large as possible. Therefore, we can choose $\lambda = \max_l \frac{\partial u(\mathbf{x})}{\partial x_l} \cdot \frac{1}{p_l}$. Choosing such λ, we have the Lagrangean as:

$$\mathcal{L} = u(\mathbf{x}) + \frac{\partial u(\mathbf{x})}{\partial x_{l^*}} \cdot \frac{1}{p_{l^*}}(w - \mathbf{p} \cdot \mathbf{x}) = \underbrace{\frac{\partial u(\mathbf{x})}{\partial x_{l^*}} \cdot \frac{1}{p_{l^*}} w}_{\equiv U^*} + u(\mathbf{x}) - \frac{\partial u(\mathbf{x})}{\partial x_{l^*}} \cdot \frac{1}{p_{l^*}} \cdot \mathbf{p} \cdot \mathbf{x}$$

If $u(\mathbf{x})$ is homogeneous of degree m, then by Euler's theorem[6], we have

$$\mathcal{L} = U^* + u(\mathbf{x}) - \frac{\partial u(\mathbf{x})}{\partial x_{l^*}} \cdot \frac{1}{p_{l^*}} \cdot \left(\sum_{l=1}^{L} p_l x_l\right) \leq U^* + u(\mathbf{x}) - \left(\sum_{l=1}^{L} \frac{\partial u(\mathbf{x})}{\partial x_l} x_l\right) = U^* + (1-m)u(\mathbf{x})$$

this will be an upper bound for utility, it is achieved when the agent spends everything on commodity l^*, i.e., purchase $\frac{w}{p_{l^*}}$. This is feasible, and achieves the upper bound.

[6]Euler's theorem: For a function $f(\mathbf{x})$ homogeneous of degree m (i.e. $f(t\mathbf{x}) = t^m f(\mathbf{x}), \forall t \neq 0$), if f has all partial derivatives of first order, then $\sum_{l=1}^{L} x_l \frac{\partial f}{\partial x_l} = m f(\mathbf{x})$. A brief proof is: define $x_l' = x_l t, \forall l$, then $t^m f(\mathbf{x}) = f(t\mathbf{x}) \overset{\partial t}{\Rightarrow} m t^{m-1} f(\mathbf{x}) = \sum_{l=1}^{L} \frac{\partial f}{\partial x_l'} \frac{\partial x_l'}{\partial t} = \sum_{l=1}^{L} \frac{\partial f}{\partial x_l'} x_l$. If we choose $t = 1$, thus $x_l' = x_l$, we would have Euler's theorem.

In general, solving for $x(\mathbf{p}, w)$ is a constrained optimization problem, I will discuss it more thoroughly in the later chapter.

Indirect utility function $v(\mathbf{p}, w)$

The indirect utility function $v(\mathbf{p}, w) \in \mathbb{R}$ is the maximized utility, i.e.,

$$v(\mathbf{p}, w) = u(\mathbf{x}^*), \forall \mathbf{x}^* \in x(\mathbf{p}, w)$$

it has the following properties, closely related to those of Walrasian demand $x(\mathbf{p}, w)$ in Thm.2.3.2:

Theorem 2.3.3: properties of $v(\mathbf{p}, w)$

1. ***homogeneity of degree zero*** in $(\mathbf{p}, w)$: $v(\alpha\mathbf{p}, \alpha w) = v(\mathbf{p}, w), \forall \mathbf{p} \gg 0, w > 0, \alpha > 0$
2. ***strictly increasing in w, nonincreasing in p_l***, $\forall l$
3. ***quasiconvexity***: $\forall \bar{v}$, the set $\{(\mathbf{p}, w) : v(\mathbf{p}, w) \leq \bar{v}\}$ is convex
4. ***continuity***: $v(\mathbf{p}, w)$ is continuous in $\mathbf{p}$ and w.

The proof of these properties relies on the the properties of $x(\mathbf{p}, w)$:

1. ***homogeneity of degree zero*** in $(\mathbf{p}, w)$: it follows the homogeneity of Walrasian demand $x(\mathbf{p}, w)$.
2. ***strictly increasing in w, nonincreasing in p_l***, $\forall l$: this follows Walras' law $x(\mathbf{p}, w) \cdot \mathbf{p} = w$ and $v(\mathbf{p}, w) = u(x(\mathbf{p}, w))$ where $u(\cdot)$ is increasing.
3. ***quasiconvexity***: suppose $v(\mathbf{p}, w), v(\mathbf{p}', w') \leq \bar{v}$. Then $\forall \alpha \in [0, 1]$, consider $(\mathbf{p}'', w'') = (\alpha\mathbf{p} + (1 - \alpha)\mathbf{p}', \alpha w + (1 - \alpha)w')$, first, we have
$$\mathbf{p}'' \cdot \mathbf{x} \leq w'' \Rightarrow \alpha\mathbf{p} \cdot \mathbf{x} + (1 - \alpha)\mathbf{p}' \cdot \mathbf{x} \leq \alpha w + (1 - \alpha)w'$$
hence, either $\mathbf{p} \cdot \mathbf{x} \leq w$ or $\mathbf{p}' \cdot \mathbf{x} \leq w'$ or both. Intuitively, if $\mathbf{x}$ can be afforded at $(\mathbf{p}'', w'')$, it must be affordable at either $(\mathbf{p}, w)$ or $(\mathbf{p}', w')$ or both. And $v(\mathbf{p}, w)$ is the maximum utility under $(\mathbf{p}, w)$ (same for $v(\mathbf{p}', w')$ and $(\mathbf{p}', w')$), hence $u(\mathbf{x}) \leq v(\mathbf{p}, w) \leq \bar{v}$ or $u(\mathbf{x}) \leq v(\mathbf{p}', w') \leq \bar{v}$ must hold. Therefore $v(\mathbf{p}'', w'') \leq \bar{v}$.
4. ***continuity***: since $\succsim$ is strictly convex, we have a unique $x(\mathbf{p}, w)$, and $v(\mathbf{p}, w) = u(x(\mathbf{p}, w))$. We have proved $x(\mathbf{p}, w)$ is continuous, we know $u(\cdot)$ is continuous. Hence, $v(\mathbf{p}, w)$ is continuous.

2.3.4 EMP (Expenditure Minimizing Problem)

A consumer's problem can also be to choose the cheapest consumption bundle that achieves a certain level of utility, given a positive price wealth combination $(\mathbf{p}, w)$:

$$\min_{\mathbf{x} \geq 0} \mathbf{p} \cdot \mathbf{x} \text{ s.t. } u(\mathbf{x}) \geq u$$

As UMP, when $\mathbf{p} \gg 0$, only very general conditions are needed for EMP to have a solution, that is, **nonemptiness**: $\exists \mathbf{x}$ s.t. $u(\mathbf{x}) = u$.

And as UMP, EMP will also induce two objects: Hicksian demand $h(\mathbf{p}, u)$ and expenditure function $e(\mathbf{p}, u)$.

Hicksian demand $h(\mathbf{p}, u)$

The optimal commodity vector in EMP is $h(\mathbf{p}, u) \subset \mathbb{R}^L_+$, a.k.a., the Hicksian demand correspondence (function, or single-valued ones). Hicksian demand $h(\mathbf{p}, u)$ parallels Walrasian demand $x(\mathbf{p}, w)$, hence it has parallel properties as those of $x(\mathbf{p}, w)$ in Thm.2.3.2:

Theorem 2.3.4: properties of $h(\mathbf{p}, u)$

1. ***homogeneity of degree zero*** in $\mathbf{p}$: $h(\alpha\mathbf{p}, u) = x(\mathbf{p}, w), \forall \mathbf{p} \gg 0, w > 0, \alpha > 0$
2. ***No excess utility***: $\forall \mathbf{x} \in h(\mathbf{p}, u), u(\mathbf{x}) = u$
3. ***convexity/uniqueness***:
   (a) if $\succsim$ is convex, $h(\mathbf{p}, u)$ is a convex set
   (b) if $\succsim$ is strictly convex, $h(\mathbf{p}, u)$ is single-valued
4. ***continuity***: $h(\mathbf{p}, u)$ is continuous in $\mathbf{p}$ and u

Here is the proof:

- ***homogeneity of degree zero*** in $\mathbf{p}$: minimizing $\mathbf{p} \cdot \mathbf{x}$ subject to $u(\mathbf{x}) \geq u$ is the same as minimizing $\alpha\mathbf{p} \cdot \mathbf{x}$ since $\alpha > 0$
- ***No excess utility***: suppose $\exists \mathbf{x} \in h(\mathbf{p}, u)$ s.t. $u(\mathbf{x}) > u$.
- ***convexity/uniqueness***:
  (a) convex $\succsim \Rightarrow$ convex $h(\mathbf{p}, u)$: for $\mathbf{x}, \mathbf{x}' \in h(\mathbf{p}, u)$, then $u(\mathbf{x}), u(\mathbf{x}') \geq u$ and $\mathbf{p}\cdot\mathbf{x} = \mathbf{p}\cdot\mathbf{x}'$. Then $\forall \alpha \in [0, 1]$, let $\alpha\mathbf{x}+(1-\alpha)\mathbf{x}'$, $\mathbf{p}\cdot\mathbf{x}'' = \mathbf{p}\cdot\mathbf{x} = \mathbf{p}\cdot\mathbf{x}'$; meanwhile, $\succsim$ is convex, $u(\cdot)$ is quasiconcave, hence $u(\mathbf{x}'') \geq u$, hence $\mathbf{x}'' \in h(\mathbf{p}, u)$
  (b) strictly convex $\succsim \Rightarrow$ single-valued $h(\mathbf{p}, u)$: for $\mathbf{x}, \mathbf{x}' \in h(\mathbf{p}, u)$, then again $u(\mathbf{x}), u(\mathbf{x}') \geq u$ and $\mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}'$. Then $\forall \alpha \in [0, 1]$, let $\alpha\mathbf{x} + (1 - \alpha)\mathbf{x}'$, $\mathbf{p} \cdot \mathbf{x}'' = \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}'$; meanwhile, $\succsim$ is strictly convex, $u(\cdot)$ is strictly quasiconcave, hence $u(\mathbf{x}'') > u$, then we could find a β close enough to 1 s.t. $u(\beta\mathbf{x}'') > u$ and $\mathbf{p} \cdot (\beta\mathbf{x}'') < \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}'$, hence neither $\mathbf{x}$ nor $\mathbf{x}'$ is EMP optimizer, therefore, $h(\mathbf{p}, u)$ must be a singleton.
- ***continuity***: continuity of $h(\mathbf{p}, u)$ follows the continuity of $u(\cdot)$.

And again, just like UMP, we use the $\mathcal{L} = \mathbf{p} \cdot \mathbf{x}$ Lagrangean to solve EMP. The FOC is: $\exists \lambda \geq 0$ s.t.

$$\mathbf{p} \geq \lambda \nabla u(\mathbf{x}^*) \text{ and } \mathbf{x}^* \cdot [\mathbf{p} - \lambda \nabla u(\mathbf{x}^*)] = 0$$

this result comes from the Kuhn-Tucker condition, which will be discussed later.

A very important property of Hicksian demand is that it satisfies **compensated law of demand**:

Theorem 2.3.5: $h(\mathbf{p}, u)$ satisfies compensated law of demand

For a continuous $u(\cdot)$ representing a locally nonsatiated $\succsim$, and a single-valued Hicksian demand $h(\mathbf{p}, u), \forall \mathbf{p} \gg 0$, then $\forall \mathbf{p}', \mathbf{p}''$:

$$(\mathbf{p}'' - \mathbf{p}') \cdot [h(\mathbf{p}'', u) - h(\mathbf{p}', u)] \leq 0$$

The proof is extremely easy: since $h(\mathbf{p}, u)$ is the expenditure minimizer, hence, we have

$$\begin{cases} \mathbf{p}'' \cdot h(\mathbf{p}'', u) \leq \mathbf{p}'' \cdot h(\mathbf{p}', u) & (1) \\ \mathbf{p}' \cdot h(\mathbf{p}'', u) \geq \mathbf{p}' \cdot h(\mathbf{p}', u) & (2) \end{cases} \xRightarrow{(1)-(2)} (\mathbf{p}'' - \mathbf{p}') \cdot [h(\mathbf{p}'', u) - h(\mathbf{p}', u)] \leq 0$$

An immediate implication of Thm.2.3.5 is that own-price effects are always non-positive: $(\mathbf{p}''_l - \mathbf{p}'_l)[h_l(\mathbf{p}'', u) - h_l(\mathbf{p}', u)] \leq 0$. This looks familiar, as in Thm.2.2.10 and Thm.2.2.12 for Walrasian demand, however, notice that Walrasian demand does **NOT** necessarily satisfy the law of demand: for Walrasian demand to do so, it must satisfy WARP as well.

← See the discussion of Thm.2.2.8 and Thm.2.2.9.

Expenditure function $e(\mathbf{p}, u)$

Given $\mathbf{p} \gg 0$ and target utility level $u > u(0)$, we can write the value of EMP as $e(\mathbf{p}, u)$, which is the expenditure of the EMP solving bundle $\mathbf{x}^*$, i.e., the Hicksian demand: $e(\mathbf{p}, u) = \mathbf{p} \cdot h(\mathbf{p}, u)$. Parelleling Thm.2.3.3, $e(\mathbf{p}, u)$ has the following properties (still, continuous $u(\cdot)$ representing locally nonsatiated $\succsim$):

Theorem 2.3.6: properties of $e(\mathbf{p}, u)$

1. ***homogeneity of degree one*** in $\mathbf{p}$
2. ***strictly increasing*** in u, ***non-decreasing in*** in $p_l, \forall l$
3. ***concave*** in $\mathbf{p}$
4. ***continuous*** in $\mathbf{p}$ and u

Here is the proof:

- ***homogeneity of degree one*** in $\mathbf{p}$: The constraint set of EMP, $u(\mathbf{x}) \geq u$ has **nothing** to do with $price changes$, therefore, $\alpha\mathbf{p} \cdot \mathbf{x}$ and $\mathbf{p} \cdot \mathbf{x}$ have the same optimizer $\mathbf{x}^*$, hence the expenditures in $\alpha\mathbf{p}$ and $\mathbf{p}$ at $\mathbf{x}^*$ are: $e(\alpha\mathbf{p}, u) = \alpha\mathbf{p} \cdot \mathbf{x} = \alpha e(\mathbf{p}, u)$.
- first-order properties:
  - ***strictly increasing*** in u: suppose $e(\mathbf{p}, u)$ were not strictly increasing in u, then let $\mathbf{x}^1$ be the optimizer for u^1 and $\mathbf{x}^2$ for u^2, where $u^2 > u^1$ and $\mathbf{p}\cdot\mathbf{x}^1 \geq \mathbf{p}\cdot\mathbf{x}^2 > 0$. Then we can find a bundle $\alpha\mathbf{x}^2$ ($\alpha \in (0, 1)$) that is close enough to $\mathbf{x}^2$ s.t. $\mathbf{p} \cdot \alpha\mathbf{x}^2 < \mathbf{p} \cdot \mathbf{x}^1$ and $u(\alpha\mathbf{x}^2) > u(\mathbf{x}^1) = u^1$, this contradicts the premise of $\mathbf{x}^1$ being the EMP optimizer.
  - ***non-decreasing in*** in $p_l, \forall l$: suppose two price vectors $\mathbf{p}^1$ and $\mathbf{p}^2$ where $\exists l, p^1_l \geq p^2_l$ and $\forall k, p^1_k = p^2_k$. Let $\mathbf{x}^2$ be the EMP optimizer for $(\mathbf{p}^2, u)$, then, $e(\mathbf{p}^2, u) = \mathbf{p}^2 \cdot \mathbf{x}^2 \geq \underbrace{\mathbf{p}^1 \cdot \mathbf{x}^2 \geq e(\mathbf{p}^1, u)}_{\text{definition of } e(\mathbf{p},u)}$.
- ***concave*** in $\mathbf{p}$: for a fixed utility level $\bar{u}$, then for a price vector $\mathbf{p} = \alpha\mathbf{p}^1 + (1 - \alpha)\mathbf{p}^2$,

suppose that $\mathbf{x}$ is the EMP optimizer with $(\mathbf{p}, \bar{u})$, hence

$$\begin{aligned} e(\mathbf{p}, \bar{u}) = \mathbf{p} \cdot \mathbf{x} &= \left(\alpha \mathbf{p}^1 + (1-\alpha)\mathbf{p}^2\right) \cdot \mathbf{x} \\ &= \alpha \mathbf{p}^1 \cdot \mathbf{x} + (1-\alpha)\mathbf{p}^2 \cdot \mathbf{x} \\ &\geq \alpha e(\mathbf{p}^1, u) + (1-\alpha) e(\mathbf{p}^2, u) \end{aligned}$$

A graphic way of understanding the concavity of $e(\mathbf{p}, u)$ w.r.t. $\mathbf{p}$ is: for any given $\bar{\mathbf{p}}$ and its EMP optimizer $\bar{\mathbf{x}}$, if prices changes while holding consumption fixed at $\bar{\mathbf{x}}$, the expenditure would be linear in $\mathbf{p}$: $\mathbf{p} \cdot \bar{\mathbf{x}}$. But by the nature of EMP, we know the expenditure function must be the minimum expenditure, i.e. $e(\mathbf{p}, u) \leq p \cdot \bar{\mathbf{x}}$, hence the expenditure function lies under the straight line $\mathbf{p} \cdot \bar{\mathbf{x}}$ and touching it only at the EMP optimizers. And this has to be true for any $\mathbf{x}$, therefore, $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$ (Mas-Colell et al., 1995, Figure 3.E.2, Page 60).

- ***continuous***: again, continuity of $e(\mathbf{p}, u)$ follows the continuity of $u(\cdot)$.

2.3.5 Duality: EMP and UMP

Here, we try to establish the link between UMP and EMP:

Theorem 2.3.7: Relationship between UMP and EMP

For a continuous $u(\cdot)$ representing a locally nonsatiated $\succsim$ on $X = \mathbb{R}^L_+$ and a price vector $\mathbf{p} \gg 0$, we have:

- $\mathbf{x}^*$ is optimal in UMP with wealth $w > 0 \Rightarrow \mathbf{x}^*$ is optimal in EMP with required utility $u(\mathbf{x}^*)$, and $e(\mathbf{p}, u(\mathbf{x}^*)) = w$
- $\mathbf{x}^*$ is optimal in EMP with utility $u > u(0) \Rightarrow \mathbf{x}^*$ is optimal in UMP with wealth $\mathbf{p} \cdot \mathbf{x}^*$, and $v(\mathbf{p}, \mathbf{p} \cdot \mathbf{x}^*) = u$

Here is the proof:

- UMP$\Rightarrow$EMP: Suppose $\mathbf{x}^*$ is not optimal in EMP with required utility level u($\mathbf{x}^*$), then $\exists \mathbf{x}'$ s.t. $u(\mathbf{x}') \geq u(\mathbf{x}^*)$ and $\mathbf{p} \cdot \mathbf{x}' < \mathbf{p} \cdot \mathbf{x}^* \leq w$, by local nonsatiation, we can find an affordable $\mathbf{x}''$ that is close to $\mathbf{x}'$ but has strictly higher utility, that is $u(\mathbf{x}'') > u(\mathbf{x}')$ and $\mathbf{p}\mathbf{x}'' < w$, this leads to $u(\mathbf{x}'') > u(\mathbf{x}^*)$, contradicting to $\mathbf{x}^*$ maximizing utility. Hence $\mathbf{x}$ is optimal in EMP with $u = u(\mathbf{x}^*)$, and $e(\mathbf{p}, u(\mathbf{x}^*)) = \mathbf{p}\mathbf{x}^* = w$
- EMP$\Rightarrow$UMP: With $u > u(0) \Rightarrow \mathbf{x}^* \neq 0 \Rightarrow \mathbf{p}\mathbf{x}^*$, suppose $\mathbf{x}^*$ is not optimal in UMP with wealth $\mathbf{p}\mathbf{x}^*$, then $\exists \mathbf{x}'$ s.t. $\mathbf{p}\mathbf{x}' \leq \mathbf{p}\mathbf{x}^*$, then consider $\mathbf{x}'$'s scaled-down bundle $\mathbf{x}'' = \alpha \mathbf{x}', \alpha \in (0,1)$, by continuity of $u(\cdot)$, if α is close enough to 1, we have $\mathbf{p}\mathbf{x}'' < \mathbf{p}\mathbf{x}^*$ and $u(\mathbf{x}'') > u(\mathbf{x}^*)$, contradicting to $\mathbf{p}^*$ minimizing expenditure. Hence $\mathbf{p}^*$ is optimal in UMP with $w = \mathbf{p}\mathbf{x}^*$, and $v(\mathbf{p}, \mathbf{p}\mathbf{x}^*) = u(\mathbf{x}^*) = w$

A very important and direct connection between the two problem is

Proposition 2.3.8: link between $e(\mathbf{p}, u)$ and $v(\mathbf{p}, w)$

$\forall \mathbf{p} \gg 0, w > 0, u > u(0)$, we have

$$e(\mathbf{p}, v(\mathbf{p}, w)) = w, \;\; v(\mathbf{p}, e(\mathbf{p}, u)) = w$$

With this link, we can establish a direct correspondence between the properties of the indirect utility function $v(\mathbf{p}, w)$ (Thm.2.3.3) and the expenditure function $e(\mathbf{p}, u)$ (Thm.2.3.6). Here, assume $\mathbf{p}, \mathbf{p}' \gg 0, u, u' \in \mathbb{R}$:

- Properties of $v(\mathbf{p}, w) \Rightarrow$ Properties of $e(\mathbf{p}, u)$
  i **Homogeneity**. $e(\mathbf{p}, u)$ is ***homogeneous of degree one*** in $\mathbf{p}$: define $w = e(\mathbf{p}, u)$, then $u = v(\mathbf{p}, w)$. Let $\alpha > 0$, then $e(\alpha\mathbf{p}, u) = e(\alpha\mathbf{p}, v(\mathbf{p}, w))$, since $v(\cdot, \cdot)$ is ***homogeneous of degree zero*** in $(\mathbf{p}, w)$, hence $e(\alpha\mathbf{p}, v(\mathbf{p}, w)) = e(\alpha\mathbf{p}, v(\alpha\mathbf{p}, \alpha w)) = \alpha w = \alpha e(\mathbf{p}, u)$.
  ii **Monotonicity**. $e(\mathbf{p}, u)$ is
    - ***strictly increasing*** in u: let $u' > u$, define $w = e(\mathbf{p}, u), w' = e(\mathbf{p}, u')$, then $u = v(\mathbf{p}, w), u' = v(\mathbf{p}, w')$. Since $v(\cdot, \cdot)$ is strictly increasing in w, $u' > u \Rightarrow w' > w \Leftrightarrow e(\mathbf{p}', u) > e(\mathbf{p}, u)$.
    - ***non-decreasing in*** in $\mathbf{p}$: let $\mathbf{p}' \geq \mathbf{p}$, define $w = e(\mathbf{p}, u), w' = e(\mathbf{p}', u)$, then $u = v(\mathbf{p}, w) = v(\mathbf{p}', w')$. Again, since $v(\cdot, \cdot)$ is non-increasing in p, that is, $\mathbf{p}' \geq \mathbf{p} \Rightarrow v(\mathbf{p}', w) \leq v(\mathbf{p}, w)$, for $v(\mathbf{p}, w) = v(\mathbf{p}', w')$ to hold, $w' \geq w$, that is $e(\mathbf{p}', u) \geq e(\mathbf{p}, u)$.
  iii **Concavity**. $e(\mathbf{p}, u)$ is ***concave*** in $\mathbf{p}$: let $\alpha \in [0, 1]$, define $w = e(\mathbf{p}, u)$ and $w' = e(\mathbf{p}', u)$, then $u = v(\mathbf{p}, w) = v(\mathbf{p}', w')$. Define $\mathbf{p}'' = \alpha\mathbf{p} + (1 - \alpha)\mathbf{p}'$ and $w'' = \alpha w + (1 - \alpha)w'$, since $v(\cdot, \cdot)$ is ***quasiconvex***, $v(\mathbf{p}'', w'') \leq u$. By the ***monotonicity*** of $v(\cdot, \cdot)$ in w, we get $w'' \leq e(\mathbf{p}'', u)$, that is $e(\alpha\mathbf{p}+(1-\alpha)\mathbf{p}', u) \geq \alpha w + (1 - \alpha)w' = \alpha e(\mathbf{p}, u) + (1 - \alpha)e(\mathbf{p}', u)$.
  iv **Continuity**. $e(\mathbf{p}, u)$ is ***continuous*** in $\mathbf{p}$ and u: to prove continuity, we need to prove that $\forall \{(\mathbf{p}^n, u^n)\}_{n=1}^{\infty} \to (\mathbf{p}, u)$, $\forall w$, we can show that if $e(\mathbf{p}^n, u^n) \leq w$, by the ***monotonicity*** of $v(\cdot, \cdot)$, we know $e^n \leq v(\mathbf{p}^n, w)$, then by the ***continuity*** of $v(\cdot, \cdot)$, $e \leq v(\mathbf{p}, w)$; by the same logic, we can show $e(\mathbf{p}^n, u^n) \geq w \Rightarrow e(\mathbf{p}, u) \geq w$. Hence, $e(\mathbf{p}, u)$ is continuous.
- Properties $e(\mathbf{p}, u)$ of $\Rightarrow$ Properties of $v(\mathbf{p}, w)$
  i **Homogeneity**. $v(\mathbf{p}, w)$ is ***homogeneous of degree zero*** in $(\mathbf{p}, w)$: define $u = v(\mathbf{p}, w)$ then $e(\mathbf{p}, u) = w$, hence $v(\alpha\mathbf{p}, \alpha w) = v(\alpha\mathbf{p}, \alpha e(\mathbf{p}, u))$, since $e(\mathbf{p}, u)$ is ***homogeneous of degree one*** in $\mathbf{p}$, we have $v(\alpha\mathbf{p}, \alpha e(\mathbf{p}, u)) = v(\alpha\mathbf{p}, e(\alpha\mathbf{p}, u)) = u = v(\mathbf{p}, w)$.
  ii **Monotonicity**. $v(\mathbf{p}, w)$ is
    - ***strictly increasing*** in w: let $w' > w$, and $u = v(\mathbf{p}, w), u' = v(\mathbf{p}, w')$, then $e(\mathbf{p}, u) = w, e(\mathbf{p}, u') = w'$, since $e(\cdot, \cdot)$ is strictly increasing in u, $w' > w \Rightarrow u > u' \Rightarrow v(\mathbf{p}, w') > v(\mathbf{p}, w)$.
    - ***non-increasing*** in $\mathbf{p}$: let $\mathbf{p}' \geq \mathbf{p}$, and again $u = v(\mathbf{p}', w), u' = v(\mathbf{p}', w)$, then $e(\mathbf{p}, u) = e(\mathbf{p}', u') = w$, and again, since $e(\cdot, \cdot)$ is non-decreasing in $\mathbf{p}$, $\mathbf{p}' \geq \mathbf{p} \Rightarrow u' \leq u \Rightarrow v(\mathbf{p}, w) \geq v(\mathbf{p}', w)$
  iii **Quasiconvexity**. $\forall \bar{v}$, the set $\{(\mathbf{p}, w) : v(\mathbf{p}, w) \leq \bar{v}\}$ is ***convex***. Let $\alpha \in [0, 1]$, let $u = v(\mathbf{p}, w), u' = v(\mathbf{p}', w')$, then $v(\mathbf{p}, u) = w, v(\mathbf{p}, u') = w'$. Without lossing generality, assume $u' \geq u$, then since $e(\cdot, u)$ is concave and strictly increasing in u, we have $e(\alpha\mathbf{p} + (1 - \alpha)\mathbf{p}', u') \geq \alpha e(\mathbf{p}, u') + (1 - \alpha)e(\mathbf{p}', u') \geq \alpha e(\mathbf{p}, u) + (1 - \alpha)e(\mathbf{p}', u') = \alpha w + (1 - \alpha)w'$, let $\mathbf{p}'' = \alpha\mathbf{p} + (1 - \alpha)\mathbf{p}'$ and $w'' = \alpha w + (1 - \alpha)w'$, then $e(\mathbf{p}'', u') \geq w''$. Again by the monotonicity of

$e(\cdot, u)$, we have $v(\mathbf{p}'', w'') \leq u'$.

iv **Continuity**. $v(\mathbf{p}, w)$ is ***continuous*** in $\mathbf{p}$ and w. Again, we need to prove that $\forall \{(\mathbf{p}^n, w^n)\}_{n=1}^{\infty}$ with $(\mathbf{p}^n, w^n) \to (\mathbf{p}, w)$, $v(\mathbf{p}^n, w^n) \leq u, \forall n \Rightarrow v(\mathbf{p}, w) \leq u$ and $v(\mathbf{p}^n, w^n) \geq u, \forall n \Rightarrow v(\mathbf{p}, w) \geq u$. If $v(\mathbf{p}^n, w^n) \leq u, \forall n$, then by the monotonicity of $e(\cdot, \cdot)$, $w^n \leq e(\mathbf{p}^n, u), \forall n$. Then by the continuity of $e(\cdot, \cdot)$, $w \leq e(\mathbf{p}, u)$. The same logic applies to the proof of the other direction.

For continuity and concavity/quasi-convexity, an simpler alternative is to use the idea of **epigraph**:

- Concavity-quasiconvexity: For any u
  - concavity of $e(\cdot, u)$ is equivalent to convexity of $\{(\mathbf{p}, w) \mid e(\mathbf{p}, u) \geq w\}$
  - quasi-convexity of $v(\cdot)$ is equivalent to convexity of $\{(\mathbf{p}, w) \mid v(\mathbf{p}, w) \leq u\}$

  By Prop.2.3.8 and monotonicity, $v(\mathbf{p}, w) \leq u \Leftrightarrow e(\mathbf{p}, u) \geq w$, hence, the two sets are equivalent for any u.
- Continuity: For any u
  - the continuity of $e(\cdot)$ is equivalent to both $\{(\mathbf{p}, w, u) \mid e(\mathbf{p}, u) \leq w\}$ and $\{(\mathbf{p}, w, u) \mid e(\mathbf{p}, u) \geq w\}$ being closed sets
  - the continuity of $v(\cdot)$ is equivalent to both $\{(\mathbf{p}, w, u) \mid v(\mathbf{p}, u) \geq u\}$ and $\{(\mathbf{p}, w, u) \mid v(\mathbf{p}, u) \leq u\}$ being closed sets

  again, by Prop.2.3.8 and monotonicity, $v(\mathbf{p}, w) \leq u \Leftrightarrow e(\mathbf{p}, u) \geq w$ and $v(\mathbf{p}, w) \geq u \Leftrightarrow e(\mathbf{p}, u) \leq w$, hence the two continuity conditions are equivalent.

EMP is the dual problem of **UMP**. For the discussion of their relationships, we need the Duality Theorem:

Theorem: Duality Theorem

For a nonempty closed set $K \subset \mathbb{R}^L$ and its support function $\mu_K(\cdot)$[a], there exists a unique $\bar{\mathbf{x}} \in K$ s.t. $\bar{\mathbf{p}} \cdot \bar{\mathbf{x}} = \mu_K(\bar{\mathbf{p}})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{\mathbf{p}}$, and

$$\nabla \mu_K(\bar{\mathbf{p}}) = \bar{\mathbf{x}}$$

[a] The support function is defined as: $\mu_K(\mathbf{p}) = \inf \{\mathbf{p} \cdot \mathbf{x} : \mathbf{x} \in K\}$

The proof of this theorem, and more about duality, will be discussed more elaborately in Chap.3.

With this duality between UMP and EMP established, and the basic assumption of continuous $u(\cdot)$ representing locally nonsatiated, strictly convex $\succsim$, and $\mathbf{p} \gg 0$, we can further establish the relationships between the four induced objects: $x(\mathbf{p}, w)$, $v(\mathbf{p}, w)$ from UMP; $h(\mathbf{p}, u)$, $e(\mathbf{p}, u)$ from EMP.

$h(\mathbf{p}, u)$ and $e(\mathbf{p}, u)$

By definition, the expenditure function can be naturally written as $e(\mathbf{p}, u) = \mathbf{p} \cdot h(\mathbf{p}, u)$, hence, in the opposite direction, we have:

Theorem 2.3.9: $h(\mathbf{p}, u) = \nabla e(\mathbf{p}, u)$

With a continuous $u(\cdot)$ representing a locally nonsatiated and strictly convex $\succsim$, the Hicksian demand $h(\mathbf{p}, u)$ is the derivative vector of $e(\mathbf{p}, u)$ w.r.t. prices $\mathbf{p}$:

$$h(\mathbf{p}, u) = \nabla_{\mathbf{p}} e(\mathbf{p}, u)$$

that is, $h_l(\mathbf{p}, u) = \partial e(\mathbf{p}, u)/\partial p_l, \forall l = 1, \cdots, L$

This theorem (and others below) will be proved with 3 different approaches:

Proof 1 duality argument: by the Duality Theorem, for the set $K = \left\{\mathbf{x} \in \mathbb{R}_+^L : u(\mathbf{x}) \geq u\right\}$, $e(\mathbf{p}, u) = \min \mathbf{p} \cdot \mathbf{x}$ is exactly its support function and $h(\mathbf{p}, u)$ is the optimizer of $e(\mathbf{p}, u)$, by the Duality Theorem, $h(\mathbf{p}, u) = \nabla_{\mathbf{p}} e(\mathbf{p}, u)$.

Proof 2 FOC argument: assume $h(\mathbf{p}, u) \gg 0$ and differentiable at $(\mathbf{p}, u)$, then by the chain rule, we have

$$\begin{aligned}
\nabla_{\mathbf{p}} e(\mathbf{p}, u) &= \nabla_{\mathbf{p}} \left[\mathbf{p} \cdot h(\mathbf{p}, u)\right] = h(\mathbf{p}, u) + \left[\mathbf{p} \cdot \mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)\right]^T \\
&= h(\mathbf{p}, u) + \underbrace{\left[\underbrace{\lambda \nabla u(h(\mathbf{p}, u))}_{\text{by FOC of EMP: } \mathbf{p} = \lambda \nabla u(h(\mathbf{p}, u))} \cdot \mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)\right]^T}_{\xRightarrow{\forall \mathbf{p}, u(h(\mathbf{p}, u)) = u} = 0} \\
&= h(\mathbf{p}, u)
\end{aligned}$$

Proof 3 Envelope Theorem argument: assume $h(\mathbf{p}, u) \gg 0$ and differentiable at $(\mathbf{p}, u)$, then by the Envelope Theorem, we have

$$\nabla_{\mathbf{p}} e(\mathbf{p}, u) = \nabla_{\mathbf{p}} (\mathbf{p} \cdot \mathbf{x}^{\text{optimal}}) = \mathbf{x}^{\text{optimal}} = h(\mathbf{p}, u)$$

The same idea is behind all three proofs: **the changes in demand caused by price changes has no first order effect on expenditure**. This is because no matter how the demand changes with price changes, it will always be expenditure minimizing subject to the same target utility, leading to the same expenditure. If see breakdown the FOC proof:

$$\nabla_{\mathbf{p}} e(\mathbf{p}, u) = \underbrace{h(\mathbf{p}, u)}_{\text{direct effect}} + \underbrace{\left[\lambda \nabla u(h(\mathbf{p}, u)) \cdot \mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)\right]^T}_{\text{indirect effect induced by change in demand}}$$

the indirect effect is just the marginal utility w.r.t. price changes, which should be 0 since each optimal prices in EMP should achieve the same utility u.

From the FOC proof, we have $\nabla_{\mathbf{p}} e(\mathbf{p}, u) = h(\mathbf{p}, u) + \left[\mathbf{p} \cdot \mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)\right]^T$, this leads to some interesting properties of the price derivatives of the Hicksian demand $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$:

Theorem 2.3.10: properties of $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$

For a continuous $u(\cdot)$ representing a locally nonsatiated and strictly convex $\succsim$ defined on $\mathbb{R}_+^L$, suppose $h(\cdot, u)$ is continuously differentiable at $\mathbf{p}$, then its $L \times L$ derivative matrix $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$ has the following properties:

(i) $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u) = \mathrm{D}_{\mathbf{p}}^2 e(\mathbf{p}, u)$

(ii) $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$ is a **negative semidefinite** matrix

(iii) $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$ is a **symmetric** matrix

(iv) $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u) \cdot \mathbf{p} = 0$

These properties are quite straight forward, since we have $h(\mathbf{p}, u) = \nabla_{\mathbf{p}} e(\mathbf{p}, u)$, and $e(\mathbf{p}, u)$ is **twice continuously differentiable**, and **concave**, its second derivative matrix, i.e. Hessian matrix, is negative semidefinite[7] and *symmetric*[8].

$h(\mathbf{p}, u)$ and $x(\mathbf{p}, w)$

Mirroring Prop.2.3.8, Hicksian demand $h(\mathbf{p}, u)$ and Walrasian demand $x(\mathbf{p}, w)$ can be linked directly as

Proposition 2.3.11: link between $h(\mathbf{p}, u)$ and $x(\mathbf{p}, w)$

$\forall \mathbf{p} \gg 0, w > 0, u > u(0)$, we have

$$h(\mathbf{p}, u) = x(\mathbf{p}, e(\mathbf{p}, u)),\ x(\mathbf{p}, w) = h(\mathbf{p}, v(\mathbf{p}, w))$$

The interpretations of Prop.2.3.11 are:

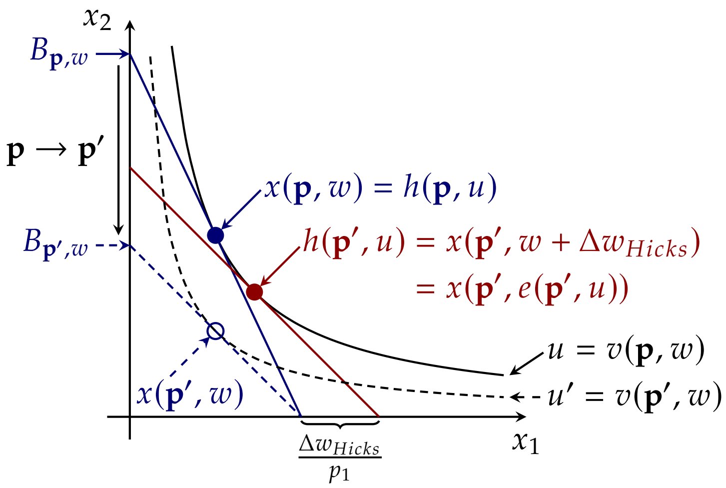
- $h(\mathbf{p}, u) = x(\mathbf{p}, e(\mathbf{p}, u))$
  $h(\mathbf{p}, u)$, **compensated demand correspondence**, gives how much the demand would change if the consumer's wealth were adjusted to keep the utility level at u (i.e., $h(\mathbf{p}', u) - h(\mathbf{p}, u) = x(\mathbf{p}', e(\mathbf{p}', u)) - x(\mathbf{p}, e(\mathbf{p}, u))$). The wealth change required to keep utility at u, **Hicksian wealth compensation**, is then $\Delta w_{\text{Hicksian}} = e(\mathbf{p}', u) - e(\mathbf{p}, u) = e(\mathbf{p}', u) - w$. This is reflected in Fig.2.3.1
- $x(\mathbf{p}, w) = h(\mathbf{p}, v(\mathbf{p}, w))$
  $x(\mathbf{p}, w)$, in contrast, gives how much the utility would change if the consumer's wealth level w stays the same while the price changes from $\mathbf{p}$ to $\mathbf{p}'$.

Hicksian demand $h(\mathbf{p}, u)$ is **not directly observable** (utility is one of its arguments), but it can be computed from the observable Walrasian demand $x(\mathbf{p}, w)$, with the **Slutsky Equation**:

[7] This is the differential analog of Thm.2.3.5: $\mathrm{d}\mathbf{p} \cdot \mathrm{d}h(\mathbf{p}, u) \leq 0 \xRightarrow{\mathrm{d}h(\mathbf{p},u) = \mathrm{D}_{\mathbf{p}} h(\mathbf{p},u)\mathrm{d}\mathbf{p}} \mathrm{d}\mathbf{p} \cdot \mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u) \mathrm{d}\mathbf{p} \leq 0$, which means that $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$ is negative semidefinite.

[8] The symmetry of $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$ is less intuitive economically, it is closely linked to non-cycling of rational preferences.

Figure 2.3.1: Hicksian wealth compensation



Theorem 2.3.12: the Slutsky equation

For a continuous $u(\cdot)$ representing a locally nonsatiated and strictly convex $\succsim$ defined on $X = \mathbb{R}^L_+$, then $\forall(\mathbf{p}, w)$ and $u = v(\mathbf{p}, w)$, we have:

$$\frac{\partial h_l(\mathbf{p}, u)}{\partial p_k} = \frac{\partial x_l(\mathbf{p}, w)}{\partial p_k} + \frac{\partial x_l(\mathbf{p}, w)}{\partial w} x_k(\mathbf{p}, w), \forall l, k$$

or in matrix notation:

$$\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u) = \underbrace{\mathrm{D}_{\mathbf{p}} w(\mathbf{p}, w) + \mathrm{D}_w x(\mathbf{p}, w) \cdot x(\mathbf{p}, w)^T}_{\equiv S(\mathbf{p}, w) \text{ in Def.2.2.13}}$$

Here is the proof: Consider the price-wealth pair $(\bar{\mathbf{p}}, \bar{w})$ and the target utility level $\bar{u}$, naturally, $\bar{w} = e(\bar{\mathbf{p}}, \bar{u})$. By Prop.2.3.11, we know $h_l(\mathbf{p}, u) = x_l(\mathbf{p}, e(\mathbf{p}, u)), \forall(\mathbf{p}, u)$. Differentiate this w.r.t. p_k at $(\bar{\mathbf{p}}, \bar{u})$, get

$$\left.\frac{\partial h_l(\mathbf{p}, u)}{\partial p_k}\right|_{(\bar{\mathbf{p}}, \bar{u})} = \frac{\partial x_l(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u}))}{\partial w} \cdot \frac{\partial e(\bar{\mathbf{p}}, \bar{u})}{\partial p_k}$$

$$\xRightarrow{\text{Thm.2.3.9}} \frac{\partial h_l(\bar{\mathbf{p}}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u}))}{\partial w} \cdot h_k(\bar{\mathbf{p}}, \bar{u})$$

$$\xRightarrow{\bar{w} = (\bar{\mathbf{p}}, \bar{u})} \frac{\partial h_l(\bar{\mathbf{p}}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{\mathbf{p}}, \bar{w})}{\partial p_k} + \frac{\partial x_l(\bar{\mathbf{p}}, \bar{w})}{\partial w} \cdot w_k(\bar{\mathbf{p}}, \bar{w})$$

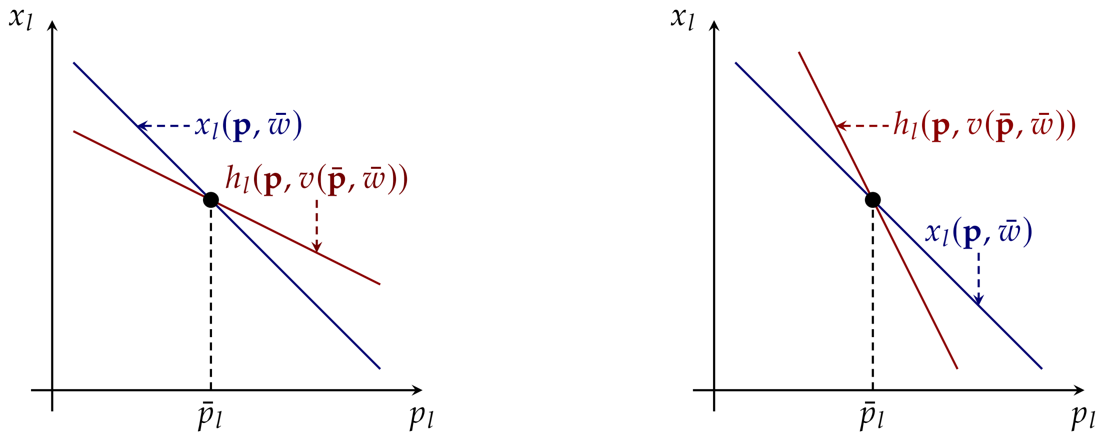
Here, Walrasian demand function is observable, and the Slutsky matrix is calculable, we can therefore use this equation to calculate $\mathrm{D}_{\mathbf{p}} h(\mathbf{p}, u)$. And for an arbitrary good l, its demand curve w.r.t. p_l given prices of other goods p_{-l} fixed is depicted in Fig.2.3.2[9].

← observable in principle, since $\mathbf{p}$ and w are both observable

When we focus on one good l, we have $\frac{\partial h_l(\bar{\mathbf{p}}, \bar{u})}{\partial p_l} = \frac{\partial x_l(\bar{\mathbf{p}}, \bar{w})}{\partial p_l} + \frac{\partial x_l(\bar{\mathbf{p}}, \bar{w})}{\partial w} \cdot w_l(\bar{\mathbf{p}}, \bar{w})$, the

[9] Here, I plot prices on the x-axis as it makes more sense to me in this context :)

Figure 2.3.2: Hicksian and Walrasian demand curve



Normal good   Inferior good

relationship between the Hicksian demand and the Walrasian demand becomes much more clear:

- If good l is normal good, then $\partial x_l(\bar{\mathbf{p}}, \bar{w})/\partial w > 0$, this means that if its price p_l increases, Hicksian demand (with wealth increased accordingly to match the initial utility level $\bar{u} = v(\bar{\mathbf{p}}, \bar{w})$) for good l will decrease, but by a lower amount than the decrease of Walrasian demand (without wealth adjustment).
- If good l is inferior good, then $\partial x_l(\bar{\mathbf{p}}, \bar{w})/\partial w < 0$, the opposite logic follows.

Here, Slutsky (substitution) matrix appears once again. However, different from the choice-based, WARP-regulated demand, which requires the Slutsky matrix to be **negative semidefinite** and to satisfy $S(\mathbf{p}, w)\mathbf{p} = 0$ (see Thm.2.2.15), one extra condition is required for preference maximization: S being **symmetric**. The more essential distinction between these two approaches is that they adopt two different wealth compensation approaches: For an initial price-wealth pair $(\bar{\mathbf{p}}, \bar{w})$ and its corresponding utility level $\bar{u} = u(\bar{x})$, and the price after change $\mathbf{p}'$:

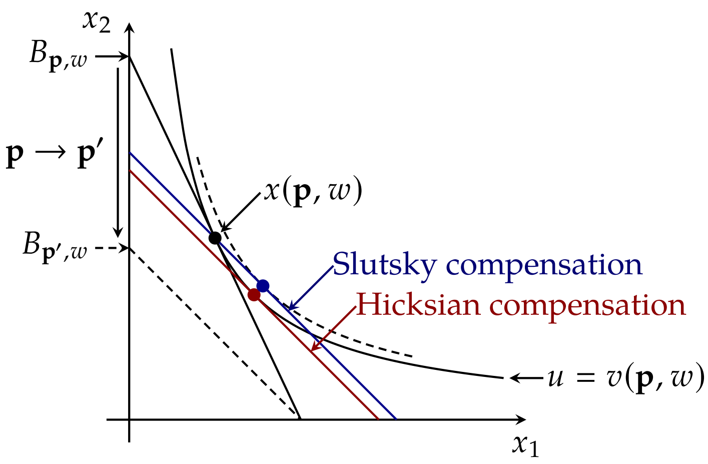
- Slutsky wealth compensation (Section2.2.4): wealth is adjusted from $\bar{w}$ to w^S s.t. $\bar{x}$ could still be afforded, $w^S = \bar{x} \cdot \mathbf{p}'$, giving: $\Delta w_{\text{Slutsky}} = \mathbf{p}' \cdot x(\bar{\mathbf{p}}, \bar{w}) - \bar{w}$
- Hicksian wealth compensation (this section): wealth is adjusted from from $\bar{w}$ to w^H s.t. $\bar{u}$ could still be achieved, $v(\mathbf{p}', w^H) = \bar{u}$, giving: $\Delta w_{\text{Hicks}} = e(\mathbf{p}', \bar{u}) - \bar{w}$

The two compensation approaches are closely linked: $\Delta w_{\text{Hicks}} \leq \Delta w_{\text{Slutsky}}$, and since $\nabla_{\mathbf{p}} e(\bar{\mathbf{p}}, \bar{u}) = h(\bar{\mathbf{p}}, \bar{u}) = x(\bar{\mathbf{p}}, \bar{w})$, the equality holds **only** when the price change is a differential change ($\mathrm{d}p$):

$$
\begin{aligned}
\Delta w_{\text{Slutsky}} &= \mathbf{p}' \cdot x(\bar{\mathbf{p}}, \bar{w}) - \bar{w} \\
&\qquad \mathbf{p}' \cdot x(\bar{\mathbf{p}}, \bar{w}) - \bar{\mathbf{p}} \cdot x(\bar{\mathbf{p}}, \bar{w}) = x(\bar{\mathbf{p}}, \bar{w}) \cdot (\mathbf{p}' - \bar{\mathbf{p}}) \\
&\qquad = \nabla_{\mathbf{p}} e(\bar{\mathbf{p}}, \bar{u}) \cdot (\mathbf{p}' - \bar{\mathbf{p}}) = e(\mathbf{p}', \bar{u}) - e(\bar{\mathbf{p}}, \bar{u}) \\
&\qquad = e(\mathbf{p}', \bar{u}) - \bar{w} = \Delta w_{\text{Hicks}}
\end{aligned}
$$

the intuition is, for a very small change of price for good l, the total effect on the expenditure required to still achieve $\bar{u}$ is just the direct effect of the price change. But if the price change is not sufficiently small (discrete price changes), we would always have $\Delta w_{\text{Hicks}} < \Delta w_{\text{Slutsky}}$ (see Fig.2.3.3).

Figure 2.3.3: Hicksian and Slutsky wealth compensation



$x(\mathbf{p}, w)$ **and** $v(\mathbf{p}, w)$

Mirroring Thm.2.3.9, we can link the Walrasian demand $x(\mathbf{p}, w)$ and indirect utility function $v(\mathbf{p}, w)$ through **Roy's identity**:

Theorem 2.3.13: Roy's identity

With a continuous $u(\cdot)$ representing a locally nonsatiated and strictly convex $\succsim$, and an indirect utility function differentiable at $(\bar{\mathbf{p}}, \bar{w}) \gg 0$, the Walrasian demand $x(\mathbf{p}, w)$ is the (negative) derivative vector of $v(\mathbf{p}, w)$ w.r.t. prices $\mathbf{p}$, **normalized** by marginal indirect utility of wealth:

$$x(\bar{\mathbf{p}}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{\mathbf{p}}, \bar{w})} \nabla_{\mathbf{p}} v(\bar{\mathbf{p}}, \bar{w})$$

that is, $x_l(\bar{\mathbf{p}}, \bar{w}) = -\frac{\partial v(\bar{\mathbf{p}}, \bar{w})/\partial p_l}{\partial v(\bar{\mathbf{p}}, \bar{w})/\partial w}, \forall l = 1, \cdots, L$

Again, Roy's identity will be proved with 3 different approaches:

Proof 1 : Let $\bar{u} = v(\bar{\mathbf{p}}, \bar{w})$. Since $v(\mathbf{p}, e(\mathbf{p}, \bar{u})) = \bar{u}, \forall \mathbf{p}$, differentiate the equation w.r.t. $\mathbf{p}$ and evaluating at $\mathbf{p} = \bar{\mathbf{p}}$, get

$$\begin{aligned}
&\nabla_{\mathbf{p}} v(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u})) + \frac{\partial v(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u}))}{\partial w} \nabla_{\mathbf{p}} e(\bar{\mathbf{p}}, \bar{u}) = 0 \\
\xRightarrow{\text{Thm.2.3.9}} &\nabla_{\mathbf{p}} v(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u})) + \frac{\partial v(\bar{\mathbf{p}}, e(\bar{\mathbf{p}}, \bar{u}))}{\partial w} h(\bar{\mathbf{p}}, \bar{u}) = 0 \\
\xRightarrow[\bar{w}=e(\bar{\mathbf{p}},\bar{u})]{h(\bar{\mathbf{p}},\bar{u})=x(\bar{\mathbf{p}},\bar{w})} &\nabla_{\mathbf{p}} v(\bar{\mathbf{p}}, \bar{w}) + \frac{\partial v(\bar{\mathbf{p}}, \bar{w})}{\partial w} x(\bar{\mathbf{p}}, \bar{w}) = 0 \\
\Rightarrow &x(\bar{\mathbf{p}}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{\mathbf{p}}, \bar{w})} \nabla_{\mathbf{p}} v(\bar{\mathbf{p}}, \bar{w})
\end{aligned}$$

Proof 2 FOC argument: assume $x(\mathbf{p}, w) \gg 0$ and differentiable at $(\bar{(\mathbf{p})}, \bar{u})$, then by the chain rule, we have

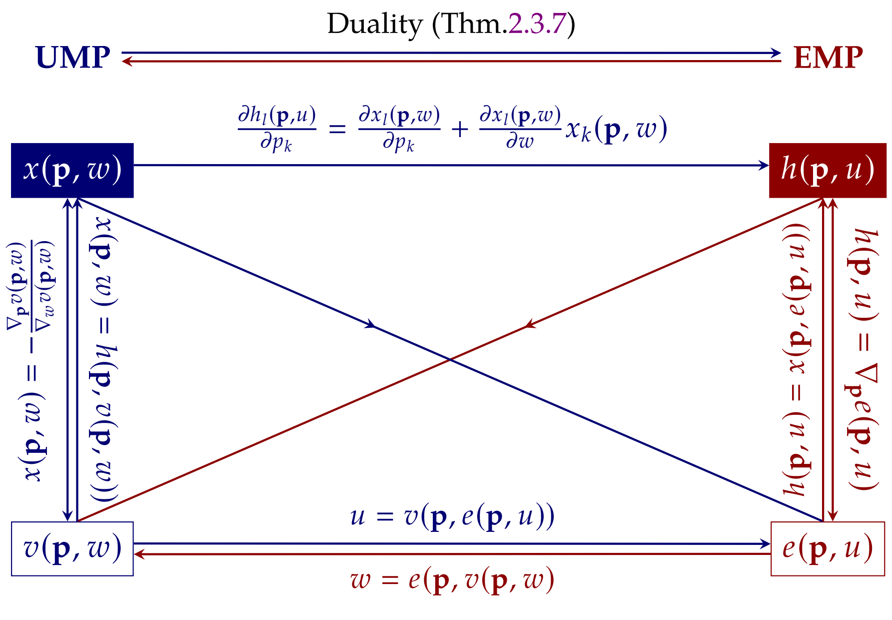
$$\begin{aligned}
\left.\frac{\partial v(\mathbf{p}, w)}{\partial p_l}\right|_{(\bar{\mathbf{p}}, \bar{w})} &= \sum_{k=1}^{L} \underbrace{\frac{\partial u\left(x(\bar{\mathbf{p}}, \bar{w})\right)}{\partial x_k}}_{\equiv \lambda p_k \text{ (Lagrange)}} \frac{\partial x_k(\bar{\mathbf{p}}, \bar{w})}{\partial p_l} \\
&= \underbrace{\lambda}_{=\frac{\partial v(\bar{\mathbf{p}}, \bar{w})}{\partial w}} \underbrace{\sum_{k=1}^{L} p_k \frac{\partial x_k(\bar{\mathbf{p}}, \bar{w})}{\partial p_l}}_{\text{by Thm.2.2.6 } = -x_l(\bar{\mathbf{p}}, \bar{w})} \\
&= -\frac{\partial v(\bar{\mathbf{p}}, \bar{w})}{\partial w} \cdot x_l(\bar{\mathbf{p}}, \bar{w}) \\
\Rightarrow x_l(\bar{\mathbf{p}}, \bar{w}) &= -\frac{\partial v(\bar{\mathbf{p}}, \bar{w})/\partial p_l}{\partial v(\bar{\mathbf{p}}, \bar{w})/\partial w}
\end{aligned}$$

Proof 3 Envelope Theorem argument: by the Envelope Theorem, we have $\partial v(\bar{\mathbf{p}}, \bar{w})/\partial p_l = -\lambda x_l(\bar{\mathbf{p}}, \bar{w})$, and $\partial v(\bar{\mathbf{p}}, \bar{w})/\partial w = \lambda$, combining the two equations yields the proposition.

With Roy's identity, it will be much easier to compute Walrasian demand from indirect utility, one only need to take derivatives instead of solving the system of FOC equations.

In summary, the duality between UMP and EMP have been established, and the four objects are interlinked with each other through the aforementioned properties (see Fig.2.3.4).

Figure 2.3.4: Duality of UMP and EMP



Empirically, Walrasian demand $x(\mathbf{p}, w)$ can be observed (in principle), or relatively easily computed from the indirect utility $v(\mathbf{p}, w)$ via Roy's Identity; Hicksian demand

$h(\mathbf{p}, u)$ can be derivated from the observable $x(\mathbf{p}, w)$ through the Slutsky equation, and the expenditure function $e(\mathbf{p}, u)$ can be derived from $h(\mathbf{p}, u)$ by integration w.r.t. prices. Therefore, all objects of interest in the preference-based consumer theory can indeede be inferred (or justified) by a rational preference.

2.4 Problems of Revealed Preferences

After examining the preference-based consumer theory, we have concluded that if a continuously differentiable demand function $x(\mathbf{p}, w)$ is generated by rational preferences, this demand function must have certain properties (see Thm.2.3.2) and have a symmetric and negative semidefinite substitution matrix $S(\mathbf{p}, w)$. In this section, we would examine the reverse: if a demand function $x(\mathbf{p}, w)$ has these properties, can it be rationalized by some preferences?

2.4.1 Recover Preferences from Demand Functions

The recovering $\gtrsim$ from $x(\mathbf{p}, w)$ will be done in 2 steps:

- **Step 1**: recover $e(\mathbf{p}, u)$ from $x(\mathbf{p}, w)$
- **Step 2**: recover $\gtrsim$ from $e(\mathbf{p}, u)$

Recover $e(\mathbf{p}, u)$ from $x(\mathbf{p}, w)$

The first step is to recover $e(\mathbf{p}, u)$ given a Walrasian demand function $x(\mathbf{p}, w)$ that has the assumed properties: satisfies Walras' law, homogeneous of degree 0(see Thm.2.3.2). And, demand is assumed to be single-valued.

The recovering of $e(\mathbf{p}, u)$ relies on the duality property $h(\mathbf{p}, u) = x(\mathbf{p}, e(\mathbf{p}, u))$ and the proposition $h(\mathbf{p}, u) = \nabla_{\mathbf{p}} e(\mathbf{p}, u)$, combine these two, get:

$$\frac{\partial e(\mathbf{p})}{\partial p_i} = x_i(\mathbf{p}, e(\mathbf{p})), \forall i = 1, \cdots, L$$

or, in a familiar form of expression

$$\nabla_{\mathbf{p}} e(\mathbf{p}) = x(\mathbf{p}, e(\mathbf{p}))$$

For this system of differential equations to have a solution, we must have $e(\mathbf{p})$ to be **twice continuously differentiable**, i.e., $e(\mathbf{p})$'s Hessian matrix is **symmetric**. Do the twice differentiation of $e(\mathbf{p})$:

$$\begin{aligned} \mathrm{D}^2_{\mathbf{p}} e(\mathbf{p}) &= \mathrm{D}_{\mathbf{p}} x(\mathbf{p}, e(\mathbf{p})) + \mathrm{D}_w x(\mathbf{p}, e(\mathbf{p})) \cdot x(\mathbf{p}, e(\mathbf{p}))^T \\ &= S(\mathbf{p}, e(\mathbf{p})) \end{aligned}$$

we actually get the Slutsky matrix again. Now we can conclude

- $\nabla_{\mathbf{p}} e(\mathbf{p}) = x(\mathbf{p}, e(\mathbf{p}))$ has a solution $\Leftrightarrow$ the Slutsky matrix $S(\mathbf{p}, w)$ being **symmetric**

  - $\Rightarrow$: proved by the above equation, which requires the Slutsky matrix be the Hessian matrix of $e(\mathbf{p})$, which aligns with Thm.2.3.10
  - $\Leftarrow$: by Frobenius theorem (see below for details), the symmetry of $\nabla_{\mathbf{p}}$ at all points of its domain is equivalent to the existence of a solution
- the solution $e(\mathbf{p}, u)$ has the properties of an expenditure function (Thm.2.3.6) $\Leftrightarrow$ the Slutsky matrix $S(\mathbf{p}, w)$ being **negative semidefinite**

Together, we have

Proposition 2.4.1: conditions of $S(\mathbf{p}, w)$ to recover $e(\mathbf{p}, u)$

$S(\mathbf{p}, w)$ being **symmetric** and **negative semidefinite** is the necessary and sufficient condition to recover $e(\mathbf{p}, u)$

Prop.2.4.1 is always true, and it can be linked to the discussion of WARP in the following way:

$L = 2$ when there are only 2 goods, the Slutsky matrix is automatically symmetric, hence as long as the demand function $x(p_1, p_2, w)$ satisfies WARP (negative semidefiniteness guaranteed), $e(\mathbf{p}, u)$ will be recovered:
  - step 1: normalize $p_2 = 1$, pick an arbitrary price-wealth point $(p_1^0, 1, w^0)$ and assign utility value u^0 to
  - step 2: solve the differential equation

$$\frac{\mathrm{d}e(p_1)}{\mathrm{p}_1} = x_1(p_1, e(p_1))$$

  where $e(p_1) = e(p_1, 1, u^0)$, $x_1(p_1, w) = x_1(p_1, 1, w)$, with the initial condition $e(p_1^0) = w^0$

  Fig.2.4.5 illustrates the essence of this procedure intuitively: for any (p_1, w), the demand function at the point $x_1(p_1, w)$ is the slope of some expenditure function, and for any given initial condition (p_1^0, w^0), there will be an expenditure curve starts at it.

$L > 2$ when there are more than 2 goods, WARP does not guarantee the symmetry of the Slutsky matrix, hence, symmetry condition must also be satisfied for the recovery of $e(\mathbf{p}, u)$

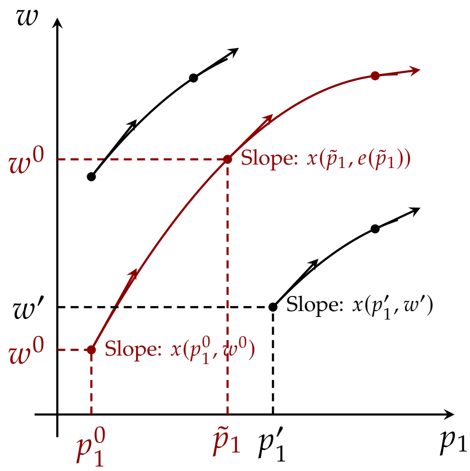
Recover $\succsim$ from $e(\mathbf{p}, u)$

The second step is to recover $\succsim$ given an expenditure function $e(\mathbf{p}, u)$ that has the assumed properties: continuous, strictly increasing in u, non-decreasing, homogeneous of degree 1, and concave in $\mathbf{p}$ (see Thm.2.3.6). Since demand is single-valued, $e(\mathbf{p}, u)$ is also differentiable.

Here, let $V_u \subset \mathbb{R}^L$ be an at-least-as-good-as set for each utility level u s.t. $e(\mathbf{p}, u)$ is the minimal expenditure required for a bundle in V_u at price $\mathbf{p} \gg 0$, i.e.

$$e(\mathbf{p}, u) = \min_{\mathbf{x} \geq 0} \mathbf{p} \cdot \mathbf{x} \ \text{ s.t. } \mathbf{x} \in V_u$$

Figure 2.4.5: Recover $e(\mathbf{p}, u)$ from $x(\mathbf{p}, w)$



Intuitively, the set $V_u = \left\{\mathbf{x} \in \mathbb{R}_+^L \mid \mathbf{p} \cdot \mathbf{x} \geq e(\mathbf{p}, u), \forall \mathbf{p} \gg 0\right\}$ satisfies the requirement, that is

Proposition 2.4.2: At-least-as-good-as set V_u

For $e(\mathbf{p}, u)$ that is strictly increasing in u, continuous, non-decreasing, homogeneous of degree 1, concave and differentiable in $\mathbf{p}$, then $\forall u$, $e(\mathbf{p}, u)$ is the expenditure function associated with the at-least-as-good-as set

$$V_u = \left\{\mathbf{x} \in \mathbb{R}_+^L \mid \mathbf{p} \cdot \mathbf{x} \geq e(\mathbf{p}, u), \forall \mathbf{p} \gg 0\right\}$$

i.e. $e(\mathbf{p}, u) = \min\left\{\mathbf{p} \cdot \mathbf{x} \mid \mathbf{x} \in V_u\right\}, \forall \mathbf{p} \gg 0$.

Proof: Immediately by definitions, $e(\mathbf{p}, u) \leq \min\left\{\mathbf{p} \cdot \mathbf{x} \mid \mathbf{x} \in V_n\right\}$. Next, prove $e(\mathbf{p}, u) \geq \min\left\{\mathbf{p} \cdot \mathbf{x} \mid \mathbf{x} \in V_n\right\}$: since $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$, we have

$$e(\mathbf{p}', u) \leq e(\mathbf{p}, u) + \nabla_{\mathbf{p}} e(\mathbf{p}, u) \cdot (\mathbf{p}' - \mathbf{p}), \forall \mathbf{p}, \mathbf{p}'$$

since $e(\mathbf{p}, u)$ is homogeneous of degree 1, by Euler's formula, $e(\mathbf{p}, u) = \mathbf{p} \cdot \nabla_{\mathbf{p}} e(\mathbf{p}, u)$, hence $e(\mathbf{p}', u) \leq \mathbf{p}' \cdot \nabla_{\mathbf{p}} e(\mathbf{p}, u), \forall \mathbf{p}'$, plus the fact that $\nabla_{\mathbf{p}(\mathbf{p},u)} \geq 0$, gives that $\nabla_{\mathbf{p}(\mathbf{p},u)} \in V_u$. By the definition of the at-least-as-good-as set V_u, we have $\min\left\{\mathbf{p} \cdot \mathbf{x} \mid \mathbf{x} \in V_u\right\} \leq \mathbf{p} \cdot \nabla_{\mathbf{p}(\mathbf{p},u)} = e(\mathbf{p}, u)$. The proof is finished.

With Prop.2.4.2, we can find a set of V_u for each level of u, and since $\partial e(\mathbf{p}, u)/\partial u > 0$, we have $u' > u \Rightarrow V_{u'} \subset V_u$. Each V_u is closed, convex, and bounded below, hence these at-least-as-good-as sets can define a preference $\succsim$ (represented by utility levels u) that has $e(\mathbf{p}, u)$ as its expenditure function (see Fig.2.4.6).

This conclusion can be extended to the case that the underlying preferences of $e(\mathbf{p}, u)$ is **non-convex**. A non-convex $\succsim$ will generate a non-convex at-least-as-good-as set, as shown in Fig.2.4.7.

For this non-convex at-least-as-good-as set, we can always find its convex hull (the solid line) that also generates the expenditure function $e(\mathbf{p}), u)$. Here we have the correspondence between the differentiability of $e(\mathbf{p}, u)$ and the , for a specific price-utility pair $(\mathbf{p}^*, u^*)$, there would be more than one expenditure minimizers, and at this

Figure 2.4.6: Recover $\succsim$ from $e(\mathbf{p}, u)$

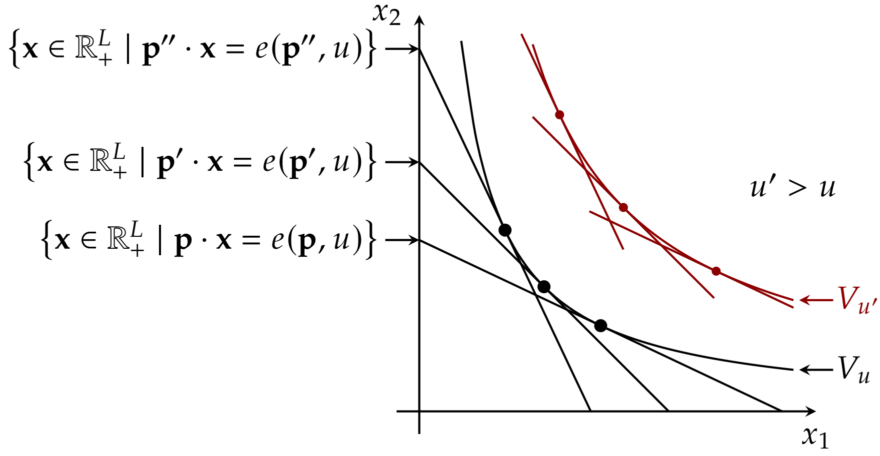
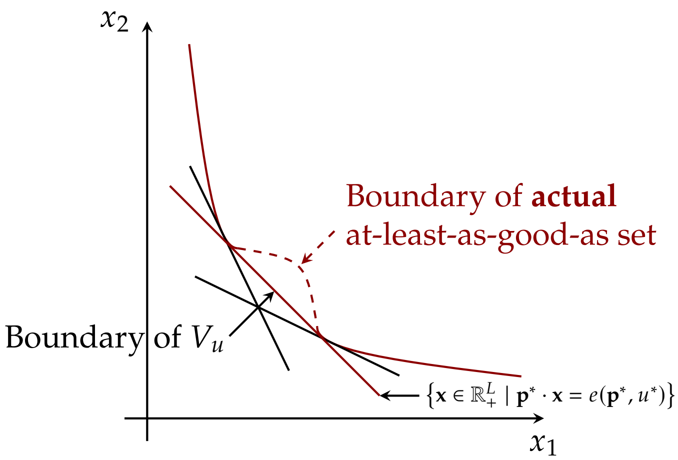


Figure 2.4.7: Recover non-convex $\succsim$ from $e(\mathbf{p}, u)$



price-utility pair $(\mathbf{p}^*, u^*)$, the generated $e(\mathbf{p}, u)$ would **NOT** be differentiable. This can be summarized as:

$$e(\mathbf{p}, u) \text{ is differentiable} \Rightarrow \succsim \text{ is convex}$$

With the two steps, $\succsim$ can be recovered from demand function $x(\mathbf{p}, w)$ through expenditure function $e(\mathbf{p}, u)$. This problem, especially the problem of recovering $e(\mathbf{p}, u)$ from $x(\mathbf{p}, u)$, which involves the differential equation $h(\mathbf{p}, u) = \nabla_{\mathbf{p}} e(\mathbf{p}, u)$, is the **integrability** problem.

2.4.2 Discussion on integrability

The discussion on recovering preferences from demand functions is based on Hurwicz and Uzawa's work, to summarize, we have

- **(B)**: **budget** exhaustion: $\mathbf{p} \cdot x(\mathbf{p}, w) = w$
- **(D)**: the demand function $x(\mathbf{p}, w)$ is **differentiable** everywhere
- **(S)**: the Slutsky matrix $S(\mathbf{p}, w)$ is **symmetric**

- **(NSD)**: the Slutsky matrix $S(\mathbf{p}, w)$ is **negative semidefinite**
- **(IB)**: the demand function $x(\mathbf{p}, w)$ satisfies the **boundedness** condition:

$$\left|\frac{\partial x_i(\mathbf{p}, w)}{\partial w}\right| \leq M < \infty, \forall i = 1, \cdots, n, \forall m \geq 0$$

These assumptions are required for the process of recovering preferences from demand functions discussed above: If all 5 of Hurwicz-Uzawa assumptions are satisfied, then $\exists u$ s.t. $\forall(\mathbf{p}, w)$, $x(\mathbf{p}, w)$ is the **unique** maximizer of u over the budget set $\{\mathbf{x} : \mathbf{p} \cdot \mathbf{x} \leq w\}$.

To integrate from demand $x(\mathbf{p}, w)$ to a utility function, it is sufficient to integrate from $x(\mathbf{p}, w)$ to the indirect utility function $v(\mathbf{p}, w)$, then $x(\cdot)$ and $v(\cdot)$ must satisfy the Hotelling's lemma:

$$-\frac{v_{\mathbf{p}}(\mathbf{p}, w)}{v_w(\mathbf{p}, w)} = \mathbf{x}(\mathbf{p}, w)$$

From this, we have two separate aspects to understand integrability:

- *mathematical* integrability: the existence of $v(\cdot)$ to satisfy the differential equation above.
- *economic* integrability: $v(\cdot)$ should be a well-behaved indirect utility function as well, corresponding to a direct utility function.

This result is quite general and is one of the fundamental results, some recent work seeks to extend it to reflect the nuance of empirical observations. I have searched some papers on this topic and will briefly discussed the results here. For a rather thorough discussion on the development of this topic, see Hands (2006).

Incomplete systems of demand functions

Hurwicz and Uzawa's result is theoretically exhaustive, in the sense that as long as the 5 integrability conditions are satisfied, with suitable regularity conditions assumed, **any** set of demand functions can be retionalized by some utility function. Epstein (1982) discussed the problem of incomplete sets of demand functions.

Consider the utility maximization problem of a consumer:

$$\max_{\mathbf{x},\mathbf{y},z} \{u(\mathbf{x}, \mathbf{y}, z) \mid \mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y} + z = w, \mathbf{x}, \mathbf{y}, z \geq 0\}$$

where $\mathbf{x}$ is the set of **observed** demands (priced at $\mathbf{p}$), and $(\mathbf{y}, z)$ is the unobservable set of demands (priced as $(\mathbf{q}, 1)$, z is the numeraire). If the observed $\mathbf{x}$ is part of the solution to the utility maximization problem, it will have some of the properties (notably symmetry and negative semi-definiteness) like that for complete demand systems. However, Hurwicz-Uzawa integrability conditions are **not enough** to guarantee the existence of a utility function for incomplete demand functions. Instead, they proposed that if the observed demand function $\mathbf{x}(w, \mathbf{p}, \mathbf{q}) : \Omega^{1+n+m} \to \Omega^n$ such that $\leftarrow n$ observed, $m+1$ unobserved

1. budget binding: $\forall(w, \mathbf{p}, \mathbf{q})$, $\mathbf{p}x(w, \mathbf{p}, \mathbf{q}) < w, x(w, \mathbf{p}, \mathbf{q}) > 0$
2. $x^i(w, \mathbf{p}, \mathbf{q})$ is **twice continuously differentiable** for all i

3. The $n \times n$ matrix $x_{\mathbf{p}}(w, \mathbf{p}, \mathbf{q}) + x(w, \mathbf{p}, \mathbf{q}) \cdot x_w(w, \mathbf{p}, \mathbf{q})$ (Slutsky matrix for observed demands) is **negative definite**

Then there exists a **continuous, non-decreasing and quasiconcave** utility function u and a **neighborhood** N of $(w^0, \mathbf{p}^0, \mathbf{q}^0)$ s.t. $\forall (w, \mathbf{p}, \mathbf{q}) \in N$, $x(w, \mathbf{p}, \mathbf{q})$ is the unique solver of the aforementioned utility maximizing problem.

The key argument of Epstein (1982) is the **negative definiteness** of the Slutsky matrix of the observed demands, and one thing very important about this result is that it is a **local** result. The proof is skipped here, but duality also plays a crucial role as before.

To extend this local result to possible global results, some extra conditions must be met. Epstein (1982) proposed that if $x_{\mathbf{q}}(w, \mathbf{p}, \mathbf{q}) = 0$, the 3 conditions will guarantee the glocal integrability of $x(\cdot)$, that is, observed demands (the first n goods) are independent of the prices of the remaining $(m+1)$ goods (demands unobserved). This result is quite intuitive: consider the case of $n = 1$, that is, the demand of only one good is observed. If other (normalized) prices $\mathbf{q}$ are constant, then this problem can fit into a two-good framework and the standard integrability results apply. Epstein (1982) also proposed several classes of demand functions $x(\cdot)$ that have global integrability (see Table 1 on Page 419 of the paper for details).

LaFrance and Hanemann (1989) generalized Epstein's result by employing an arbitrary price index for the goods of which the demand is unobserved: $\pi(\mathbf{q})$. This deflator function $\pi(\mathbf{q})$[10] is used to normalize $\mathbf{p}$ and w. In addition to the 3 conditions of Epstein, LaFrance and Hanemann gave a condition on $\pi(\mathbf{q})$ to guarantee the **global** integrability: $\exists \pi : \mathbb{R}^m_{++} \longrightarrow \mathbb{R}_+$ s.t. $\pi \in C^2$ is **increasing**, **homogeneous of degree 1**, **concave in q** and $x(w, \mathbf{p}, \mathbf{q}) = \tilde{x}\left(\frac{1}{\pi(\mathbf{q})} \cdot w, \frac{1}{\pi(\mathbf{q})} \cdot \mathbf{p}\right)$. This condition requires that the observed demands depend upon the prices of other goods **only** through a price index that deflates the prices of the observed goods and total expenditure.

However, this condition depends on the separability of $\mathbf{q}$ from $(\mathbf{p}, w)$ in the indirect utility function: $v(w, \mathbf{p}, \mathbf{q}) \equiv \varphi\left(\frac{w}{\pi(\mathbf{q})}, \frac{w}{\pi(\mathbf{q})}\right)$ and in the expediture function $e(u, \mathbf{p}, \mathbf{q}) \equiv \pi(\mathbf{q})\epsilon\left(\frac{\mathbf{p}}{\pi(\mathbf{q})}, u\right)$. This is equivalent to a homothetically separable utility function: $u(\mathbf{x}, \mathbf{y}) \equiv \bar{u}\left(\mathbf{x}, f(\mathbf{y})\right)$, where $f(\cdot)$ is homogeneous of degree 1. Obviously, this is a very restrictive condition. Therefore, LaFrance and Hanemann introduced the concept of **weak integrability**: If $\forall (w, \mathbf{p}, \mathbf{q})$, there is a function $\nu : \mathbb{R}^{n+m+1}_{++} \longrightarrow \mathbb{R}$ that is **continuous, increasing and quasiconcave** in $(\mathbf{x}, s)$ where $\mathbf{x} = x(w, \mathbf{p}, \mathbf{q})$ and $s = \sigma(w, \mathbf{p}, \mathbf{q}) \equiv y - \mathbf{p} \cdot \mathbf{x}$[11] solves the maximization problem:

← here the notation is modified for convenience, **x** is still the vector of observed demands, **y** is the vector of unobserved demands

$$\max_{\mathbf{x}, s} \{\nu(\mathbf{x}, s, \mathbf{q}) \mid \mathbf{p} \cdot \mathbf{x} + s \leq w, \mathbf{x} \geq 0, s > 0\}$$

and the duality is established in the same way as the complete demand system: **weak** integrability is equivalent to the existence of an expenditure function $e(\mathbf{p}, \mathbf{q}, u)$ that is increasing and concave in $\mathbf{p}$, homogeneous of degree 1 in $(\mathbf{p}, \mathbf{q})$, and satisfies

$$\mathbf{p} \cdot x(e(\mathbf{p}, \mathbf{q}, u), \mathbf{p}, \mathbf{q}) + \sigma(e(\mathbf{p}, \mathbf{q}, w), \mathbf{p}, \mathbf{q}) \equiv e(\mathbf{p}, \mathbf{q}, u)$$

[10] some examples are: $\pi(\mathbf{q}) \equiv q_1$, $\pi(\mathbf{q}) \equiv \sum_{j=1}^m \alpha_j q_j$, $\pi(\mathbf{q}) \equiv \prod_{j=1}^m q_j^{\alpha_j}$, where $\sum_{j=1}^m \alpha_j = 1, \alpha_j \geq 0$.

[11] Here, $\mathbf{x}$ is still the observed demands, and s is the expenditure on all other goods: $s \equiv \mathbf{q} \cdot \mathbf{z}$

and Hotelling's lemma.

LaFrance and Hanemann proved that the following conditions need to be satisfied to achieve (global) weak integrability:

- $x(\cdot) \in C^2$, homogeneous of degree 0 in $(w, \mathbf{p}, \mathbf{q})$
- $x(w, \mathbf{p}, \mathbf{q}) \geq 0$, $\mathbf{p} \cdot x(w, \mathbf{p}, \mathbf{q}) < w$ (positive and not budget-exhausting)
- Slutsky matrix S is **symmetric, negative semidefinite**

These conditions are pretty much the same as the integrability condition for complete demand systems. In addition,

- $e(\cdot) \in C^3$ in $\mathbf{p}$: extra degree of differentiability to guarantee $x(\cdot) \in C^2$
- $\forall(w, \mathbf{p}, \mathbf{q})$, s satisfies $\frac{\partial s_{ij}}{\partial w} = \frac{\partial s_{ji}}{\partial w}$ and $\frac{\partial s_{ij}}{\partial p_k} = \frac{\partial s_{ji}}{\partial p_k}, \forall i, j, k = 1, \cdots, n$: given by the symmetry of Slutsky matrix.

These conditions are much easier to satisfy and verify empircally. LaFrance and Hanemann also discussed welfare implications of this result.

Linear nad quasi-linear integrability

Amir, Erickson, and Jin (2017) surveys the microfoundation of linear demand and discussed the integrability of it. Linear demand is the solution to the quadratic utility maximization problem,

Consider the numeraire as additively separable, the UMP is then:

$$\max U(\mathbf{x}) + y \qquad \text{s.t. } \mathbf{p} \cdot \mathbf{x} + y \leq w$$

with Walrasian demand $(\mathbf{x}^*, y^*)$ as its solution.

Its dual problem (EMP) is

$$\min \mathbf{p} \cdot \mathbf{x} + y \qquad \text{s.t. } U(\mathbf{x}) + y \geq u$$

with Hicksian demand $(\mathbf{x}^h, y^h)$ as its solution. Amir, Erickson, and Jin showed that under quasi-linear utility, both the Walrasian demand and the Hicksian demand satisfy the Law of Demand[12]. Here, the result does NOT reply on the classic assumption that the utility function is quasi-concave.

Now we look at the case of quadratic utility: $U(x) = \mathbf{a}'\mathbf{x} - \frac{1}{2}\mathbf{x}'\mathbf{B}\mathbf{x}$, where $\mathbf{a} \in \mathbb{R}^n_{++}$, $\mathbf{B}$ is an $n \times n$ matrix, normalized to be symmetric and have all its diagonal entries equal to 1. Then the utility maximization problem is

$$\max \left\{ \mathbf{a}'\mathbf{x} - \frac{1}{2}\mathbf{x}'\mathbf{B}\mathbf{x} + y \right\} \text{ s.t. } \mathbf{p}'\mathbf{x} + y = w$$

Here, the demand function is linear , for the consumer's problem to have an **interior** solution and preserve the **linear** nature of the resulting demand function, we need:

The demand function is actually an **affine** function whenever positive and 0 otherwise, to be precise.

$$\mathbf{B}^{-1}(\mathbf{a} - \mathbf{p}) > 0 \qquad \mathbf{p}'\mathbf{B}^{-1}(\mathbf{a} - \mathbf{p}) \leq w$$

[12]For the Hicksian demand: by convex analysis, $\frac{\partial e(\mathbf{p},u)}{\partial p_i} = x_i^h$ (which is just Shepard's Lemma). For Walrasian, since utility is quasi-linear in the numeraire, there are no income effects.

so the essential condition here is **B** is **inversible**, i.e., U is **strictly** concave. This gives a significant result: **B** is positive definite, then the inverse demand the direct demand satisfy the **strict Law of Demand**:

$$[U(\mathbf{x}) - U(\mathbf{x}')] \cdot (\mathbf{x} - \mathbf{x}') < 0, \ \forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_+$$

Nocke and Schutz (2017) extended the integrability problem into the case when observed demand does not vary with income and consumers have quasi-linear preferences. In various applied fields of microeconomics, researchers typically focus on partial equilibrium settings and therefore often make the implicit assumption that preferences are quasi-linear and consumer income is sufficiently high to not be exhausted by the consumption under consideration.

In the quasi-linear case, demand is **not** a function of income,

2.4.3 Frobenius theorem

The Frobenius theorem is one of the fundamental theorems in differential topology. It is the foundation for the disucssion of the integrability problem, which I think it's worth spending some time understanding the basics. The discussion below is by no means thorough or rigorous[13], but my own interpretation of this topic.

Frobenius theorem (vector field version)

Frobenius theorem can be illustrated with the language of vector fields, for which some basic definitions need to be introduced.

Definition 2.4.3: Basic definitions of differntial geometry

1 **smooth differentiable manifold** M is a collection of **open** sets $U_\alpha \subset M$ and a collection of homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^N$ that satisfies
  - $\bigcup_\alpha U_\alpha = M$ (*U_α are the coordinate neighborhoods*)
  - when $U_\alpha \cap U_\beta \neq \varnothing$, the mapping $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is C^∞ (M has a countable base)

2 **coordinates**: define the $k-$th coordinate projection by $r^k\left(x^1, \cdots, x^N\right) = x^k$, then the coordinates on M is

$$x^k = r^k \circ \varphi_\alpha : U_\alpha \to \mathbb{R}$$

if $x \in U_\alpha$, x is uniquely determined by its coordinates $\left(x^1(x), \cdots, x^m(x)\right)$

3 **tangent vectors** for a curve $\gamma : (-a, a) \to M$, its tangent vector X maps a smooth function $f : M \to \mathbb{R}$ to its **directional derivative** $X(f)$ [a]. At a point $x \in M$, the tangent vector is

$$X(f)_x \in \mathbb{R}$$

and it follows the rules of derivations (for smooth f, g)

$$X(fg)_x = X(f)_x g(x) + f(x)X(g)_x \qquad (aX + bY)(f) = aX(f) + bY(f)$$

[13]For more math, check Sternberg (1999), Warner (1983), McCleary (2013)

denote the collection of tangent vectors at x as $\mathrm{T}_x(M)$

4 **vector field** $\frac{\partial}{\partial x^k}$: for every $x \in M$, and the coordinates $x^k = r^k \circ \varphi$, the vector field is defined as

$$\frac{\partial}{\partial x^k}(f)_x = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial r^k}\right|_{\varphi(x)}, \quad k = 1, \cdots, m$$

and these tangent vectors $\left\{\frac{\partial}{\partial x^k}\right\}_{k=1}^{m}$ froms the basis for $\mathrm{T}_x(M)$, that is

$$X = \sum_{k=1}^{m} X^k \frac{\partial}{\partial x^k}, \quad \forall X \in \mathrm{T}_x(M)$$

where $X^k = X(x^k)$[b].

5 **Integral curves**: the solution to the ODE system

$$X(x^k) = X^k = \frac{\mathrm{d}(x^k \circ \gamma)}{\mathrm{d}t}$$

6 **Dual space**: for the vector field $\frac{\partial}{\partial x^k}$, its dual form is $\mathrm{d}x^j$

$$\mathrm{d}x^j\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^k}(x^j) = \left.\frac{\partial(r^j \circ \varphi \circ \varphi^{-1})}{\partial r^k}\right|_{\varphi(x)} = \delta^j_k$$

and $\mathrm{d}x^j$ forms the differential $\mathrm{d}f$ where $f \in C^\infty(M)$ s.t. $\mathrm{d}f(X) = X(f)$:

$$\mathrm{d}f = \sum_{j=1}^{m} \lambda^j \mathrm{d}x^j$$

where $\lambda^j = \frac{\partial f}{\partial x^j}$[c].

[a]derivative of f in the direction X

[b]This is given by:

$$\frac{\partial}{\partial x^k}(x^j) = \left.\frac{\partial(x^j \circ \varphi^{-1})}{\partial r^k}\right|_{\varphi(x)} = \left.\frac{\partial(r^j \circ \varphi \circ \varphi^{-1})}{\partial r^k}\right|_{\varphi(x)} = \left.\frac{\partial r^j}{\partial r^k}\right|_{\varphi(x)} = \delta^j_k \Rightarrow X(x^j) = \sum_{k=1}^{m} X^k \delta^j_k = X^j$$

[c]This is given by:

$$\mathrm{d}f\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^k}(f) = \frac{\partial f}{\partial x^k} = \lambda^k$$

To illustrate the idea of integrability and Frobenius theorem, consider the following example:

Example: Frobenius theorem and integrability

For the manifold $M = \mathbb{R}^3$, and a point $(x,y,z) \in \mathbb{R}^3$. Consider two basis X, Y, they are the tangent vector of curve γ and σ respectively, and they satisfy $\varphi(\gamma(0)) = (x,y,z)$, $\eta(\sigma(0)) = (x,y,z)$. We evaluate the two cases below:

- Case 1:

$$\begin{aligned} X(x,y,z) &= \frac{\partial}{\partial x} + x\frac{\partial}{\partial z} = [1,0,x]' \\ Y(x,y,z) &= \frac{\partial}{\partial y} + y\frac{\partial}{\partial z} = [0,1,y]' \end{aligned}$$

- Case 2:

$$\begin{aligned} X(x,y,z) &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} = [1,0,y]' \\ Y(x,y,z) &= \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} = [0,1,-x]' \end{aligned}$$

The goal here is to find a 2-dimensional surface N that has tangent spaces formed by the basis X, Y. To find N, we must solve $\varphi_t(x,y,z)$ and $\eta_s(x,y,z)$ s.t. the point (x,y,z) will be transformed in the same way through $\varphi_t \to \eta_s \to \varphi_{-t}$ and through η_s[14] should give the same result.

Now, focus on **Case 1**: To solve $\varphi_t(x,y,z)$ (same logic for $\eta_s(x,y,z)$) is to solve the following system of differential equations (for $X = [1,0,x]'$):

$$\frac{\mathrm{d}(x\circ\gamma)}{\mathrm{d}t} = 1 \qquad \frac{\mathrm{d}y\circ\gamma}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}(z\circ\gamma)}{\mathrm{d}t} = x$$

combined with $\gamma(0) = (x,y,z)$, gives

$$\varphi_t(x,y,z) = \begin{bmatrix} x+t & y & z+xt+\frac{1}{2}t^2 \end{bmatrix}'$$

similarly

$$\eta_s(x,y,z) = \begin{bmatrix} x & x+y & z+ys+\frac{1}{2}s^2 \end{bmatrix}'$$

together

$$\Sigma_{t,s}(x,y,z) = \begin{bmatrix} t+x & s+y & z+xt+\frac{1}{2}t^2+ys+\frac{1}{2}s^2 \end{bmatrix}' \equiv \varphi_t \circ \eta_s(x,y,z) \equiv \eta_s \circ \varphi_t(x,y,z)$$

therefore, the two 1-dimensional integral curves for X and Y together form a smooth 2-dimensional manifold.

Meanwhile for **Case 2**:

$$\varphi_t(x,y,z) = \begin{bmatrix} x+t & y & z+yt \end{bmatrix}' \qquad \eta_s(x,y,z) = \begin{bmatrix} x & y+s & z-xs \end{bmatrix}'$$

It's easy to verify that

$$\begin{aligned} \varphi_t \circ \eta_s(x,y,z) &= \begin{bmatrix} x+t & y+s & z+yt-(x+t)s \end{bmatrix}' \\ \eta_s \circ \varphi_t(x,y,z) &= \begin{bmatrix} x+t & y+s & z-xs+(y+s)t \end{bmatrix}' \end{aligned}$$

[14] That is, integrating along X by t then along Y by s and integrating along Y by s then along X by t

they are **different**. Hence for Case 2, the two 1-dimensional solutions can not generate an integral submanifold. This example illustrates the idea of Frobenius theorem. Now we discuss the theorem in more general terms:

Theorem 2.4.4: Frobenius Theorem (vector fields)

Let M be a manifold of dimension m, and $\mathcal{D}$ a $p-$dimensional C^∞ distribution, if $\mathcal{D}$ is **involutive**, then $\forall x \in M$, $\exists$ an **integral** sub-manifold N that contains x. Centered at x, there exists a coordinate neighborhood U s.t. the slices

$$x_i = K_i, \forall i \in \{p+1, \cdots, N\}$$

are integral sub-manifolds of $\mathcal{D}$. If N is a connected integral sub-manifold of U, then $i(N) \cap U$ corresponds to one of these slices.

Several definitions are crucial to the theorem:

- **distribution**: for M ($m-$dimension), a $p-$dimensional distribution $\mathcal{D}$ ($p \leq m$) is a choice of a $p-$dimensional subspace of the tangent space $\mathrm{T}_x(M), \forall x \in M$. In the examples, only in **Case 1** the 2-dimensional distribution is the tangent bundle of a sub-manifold.
  And, for a sub-manifold $N \subset M$ to be an integral manifold, the tangent space of N at x should correspond to the distribution $\mathcal{D}(x)$, i.e. $\mathrm{d}i(T_y(N)) = \mathcal{D}(i(y))$.
- **involutive**: A smooth $\mathcal{D}$ is involutive if $[X, Y] \in \mathcal{D}$ whenever X and Y are smooth vector fields that lie in $\mathcal{D}$, where the Lie Bracket is defined as $[X, Y](f) = X(Y(f)) - Y(X(f))$[15].
  Involution gives that: if $X_1, \cdots, X_p$ are smooth vector fields that span $\mathcal{D}$ in a neighborhood U,
  $$[X_i, X_j] = C_{ij}^k X_k$$

The detailed proof of Frobenius Theorem is not the focus of this notebook, but the intuition is that involution is the key to integrability. Here are two interesting examples illustrating the idea of involution.

Example: Involutive and non-involutive distributions

- **Non-involutive**: The Heisenberg group of matrices (3×3 for simlicity)

$$H = \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix}$$

[15]The Lie Bracket measures the extent to which the derivatives in the directions X and Y do not commute. Let φ_t be a local flow of X around x, then

$$[X, Y]_x = \frac{\mathrm{d}}{\mathrm{d}t} \left(D_x \varphi_t\right)^{-1} Y_{\varphi_t(x)}\Big|_{t=0}$$

the idea is φ_t moves from x in the direction of X, and for Y in this direction, the mapping $D_x\varphi_t$: $\mathrm{T}_x(M) \to \mathrm{T}_{\varphi(x)}$ bring Y back to $T_x(M)$. This means that, $[X, Y]$ will vanish if and only if Y is **invariant** under the flow of X. Therefore, Lie Bracket can be seen as a **derivative**: $\mathcal{L}_X Y$.

and the standard basis at the identity $\mathbf{0} \in \mathbb{R}^3$

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

consider the (left-invariant) vector fields

$$E_1 = He_1 \simeq e_1 \qquad E_2 = He_2 \simeq e_2 + x_1 e_3 \qquad E_3 = He_3 \simeq e_3$$

It is easy to see that for a distribution $\mathcal{D}$ on $\mathbb{R}^3$ spanned by E_1, E_2

$$[E_1, E_2] = E_1E_2 - E_2E_1 = e_3 \simeq E_3 \neq 0$$

hence $\mathcal{D}$ is not involutive.

- **Involutive**: $\forall y \in \mathbb{R}^3$, there exists a **smooth** $\gamma : [0,1] \to \mathbb{R}^3$ with $\gamma'(t) \in \mathcal{D}_{\gamma(t)}, \forall t$ s.t.

$$\gamma(0) = 0, \gamma(1) = y$$

Since $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$, we have

$$\gamma'(t) = \alpha(t)E_1 + \beta(t)E_2$$

which gives the system of equations

$$\begin{aligned} x' &= \alpha \Rightarrow x(t) = \int_0^t \alpha(s)\mathrm{d}s \\ y' &= \beta \Rightarrow y(t) = \int_0^t \beta(s)\mathrm{d}s \qquad \xRightarrow{y=(X,Y,Z)} \begin{aligned} \alpha(t) &= X + af(t) \\ \beta(t) &= Y + bf(t) \end{aligned} \\ z' &= x\beta \Rightarrow \int_0^t \beta(s) \int_0^s \alpha(s')\mathrm{d}s'\mathrm{d}s \end{aligned}$$

and f satisfies $\int_0^1 f(t)\mathrm{d}t = 0$, $\int_0^1 tf(t)\mathrm{d}t = 0$, $\Gamma \equiv \int_0^1 \int_0^t f(t)f(s)\mathrm{d}s\mathrm{d}t \neq 0$, therefore, we have

$$x(1) = X \qquad y(1) = Y \qquad z(1) = \frac{XY}{2} + ab\Gamma = Z$$

which pin down a and b.

The first example is an extreme case: if there were a submanifold tangent to the distribution, then only points **in that submanifold** could be reached along curves **tangent** to the distribution. Meanwhile, the second example is its contrast, by following curves **tangent** to the distribution, **every** point can be reached.

Frobenius theorem (differential forms version)

Frobenius theorem can also be stated in the language of differential forms, that is

Theorem 2.4.5: Frobenius Theorem (differential forms)

Let M be a smooth manifold of dimension m, and $\theta^1, \theta^2, \cdots, \theta^{m-p} \in \Lambda^1(M)$ be **smooth** pointwise **linearly independent** forms. If there exists 1-forms $\alpha^i_j \in \Lambda^1(M)$ s.t.

$$\mathrm{d}\theta^a = \sum_{b=1}^{m-p} \alpha^a_b \wedge \theta^b$$

for all $a = 1, \cdots, m-p$, then $\forall x \in M, \exists$ unique $k-$dimensional manifold $i : N \hookrightarrow M$ s.t. $x \in N$ and $i^*(\theta^a) = 0, \forall a = 1, \cdots, n-k$.
Further, for this $x \in M$, there is a coordinate neighbohood $U \subset M$ with coordinates $(x^1, \cdots, x^m)$ s.t. N has coordinates $(x^1, \cdots, x^p)$ and

$$\theta^a = \sum_{b=p+1}^{n} A^a_b \mathrm{d}x^b$$

so that θ^a are generated by $\mathrm{d}x^a$ for $a = p+1, \cdots, m$ and the **joint kernel** of θ^a corresponds to the joint kernel of $\mathrm{d}x^a$

The proof of Frobenius Theorem in dierrentional forms is quite complicated as well. Here, I present a simple example on $M = \mathbb{R}^3$, which hopefully will facilitate the understanding of the theorem:

Example: Frobenius Theorem on $\mathbb{R}^3$

Let $M = \mathbb{R}^3$ and define a $1-$form by

$$\theta = \theta_1 \mathrm{d}x + \theta_2 \mathrm{d}y + \theta_3 \mathrm{d}z$$

the kernels of this $1-$form are given by vectors

$$X = X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} + X^3 \frac{\partial}{\partial z}$$

that satisfy $\theta_i X^i = 0$. Next, compute $\mathrm{d}\theta$

$$\mathrm{d}\theta = \left(\frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial \theta_1}{\partial z} - \frac{\partial \theta_3}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial \theta_3}{\partial y} - \frac{\partial \theta_2}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z$$

When is $\mathrm{d}\theta = \alpha \wedge \theta$ for some $\alpha \in \Lambda^1(M)$? The condition $\mathrm{d}\theta = \alpha \wedge \theta$ is **equivalent** to the condition $\mathrm{d}\theta \wedge \theta = 0$, then the integrability is given by:

$$\mathrm{d}\theta \wedge \theta = \left(\frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y}\right) \theta_3 + \left(\frac{\partial \theta_1}{\partial z} - \frac{\partial \theta_3}{\partial x}\right) \theta_2 + \left(\frac{\partial \theta_3}{\partial y} - \frac{\partial \theta_2}{\partial z}\right) \theta_1 = 0$$

If the form θ safisfies Frobenius Theorem, we could find coordinates (x^1, x^2, x^3) locally s.t.

$$\theta = A(x) \mathrm{d}x^3$$

which can also be written as $\mathbf{V}_\theta \cdot \mathrm{curl}(\mathbf{V}_\theta) = 0$

An intuitive way of understanding Frobenius Theorem is that integrability is equivalent to a **symmetry** condition. And the integrability guarantees the existence of the solution to a total differential equation (often written as a system of partial differential equations).

One thing to notice is that Frobenius Theorem only guarantees the **local** existence of the solution, which is not enough for the problem of recovering preferences from demand functions. But it does lay the ground for the Hurwicz-Uzawa Theorem discussed previously.

2.5 Welfare Analysis and Money Metrics

For the content of this chapter, my main reference is Chapter 1 of Mas-Colell et al. (1995). Section 1, Chapter 2 of Kreps (1990) covers similar content but starts from strict preference $\succ$, it is a very good complement to Mas-Colell et al. (1995). Chapter 1 of Kreps (2013) explores choice and preferences on infinite sets. Lecture 1 and 2 of Rubinstein (2012) give a well organized, lecture-structured summary of these contents, it is a very good read.

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