# Microeconomic Theory I: A Notebook

*With Jonathan Libgober*

**Sai Zhang**

Check the Github Page of this project, or email me!

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# HERE WE GO!

This is my learning notebook of Microeconomic Theory I (Course number: ECON601 at USC Economics). As one of the core courses in an economic Ph.D. curriculum, Microeconomic Theory I is beyond important to my research. Therefore, I would love to use this notebook as a commitment mechanism, to document lecture notes, discuss session and office hour intuitions, reading summaries, my personal questions regarding the topics and more. By building a file from scratch, hopefull I could have a more systematic and sophisticated understanding on the content of this course.

I thank Prof. Jonathan Libgober at USC Economics for leading the discussion of the course and providing intuitive ways to understand microeconomic theory. Please check his webpage here, he is such fun.

I also appreciate the time and effort my TA Qitong Wang put into this course, guiding me through discussing sessions and problem sets. When I have questions, he is always there to help.

Following the structure of the course, this notebook will cover three aspects of microeconomic theories: (a) individual decision making, (b) game theory, (c) mechanism design and contract theory. Apart from Jonathan's lecture notes, I will also summarize the reading materials, including: Mas-Colell et al. (1995)'s *Microeconomic Theory*, Mailath (2018)'s *Modelling Strategic Behavior*[1], Fudenberg and Tirole (1991)'s *Game Theory*, Myerson (1991)'s *Game Theory: Analysis of Conflicts*, Bolton and Dewatripont (2005)'s *Contract Theory*, Mailath and Samuelson (2006)'s *Repeated Games and Reputation* and Osborne and Rubinstein (1994)'s *A Course in Game Theory*. Other materials will also be referred to along the way.

Building this notebook is truly a memorable journey for me. I would love to share this review and all the related materials to anyone that finds them useful. And unavoidably, I would make some typos and other minor mistakes (hopefully not big ones). So I'd really appreciate any correction. If you find any mistakes, please send the mistakes to this email address saizhang.econ@gmail.com or start a branch on Github. BIG thanks in advance!

[1]Latest version (May 2021) available here.

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# Part I

# Individual Decision Making

# CHAPTER 1

## PREFERENCES AND CHOICES, UTILITIES

### Contents



The first chapter summarizes the basic setting of individual decision making: preferences, choices and utilities. The main reference is Chapter 1 of Mas-Colell et al. (1995).

In this chapter, we will focus on 3 domains:

| | |
|---|---|
| **choice** | given a set $A$, what choice from $A$ is made |
| **preference** | given alternatives $x, y$, which does the decision maker prefers |
| **utility** | given an object $X$, how much does the DM likes $X$ (as a number) |

The starting point of individual decision problem is a *set of possible (mutually exclusive) alternatives* from which the individual must choose. To model decision making process on this set of alternatives, one can:

- either start from the tastes, i.e., *preference relations* of individuals, and set up the patterns of decision making with preferences
- or, start from the actual actions of individuals, i.e. *choices*, to deduct a pattern of decision making

With this two major approaches in mind, we know what's coming: the ***rationality*** of preferences and the central assumption of choices, the ***Weak Axiom of Revealed Preference (WARP)***. And of course, the two approaches and two basic assumptions are parallel, so we need to figure out how link the (underlying) preferences and (observed) choices.

## 1.1 Preference Relations

We start from the basic: *weak preference relation*, $\succsim$.

**Definition 1.1.1.** A weak preference relation $\succsim$ on a set $X$ is a subset of $X \times X$. If $(x, y) \in \succsim \Rightarrow x$ is at least as good as $y$, written as $x \succsim y$

A weak preference relation will induce two other types of relations on $X$:

**Definition 1.1.2.** With $\succsim$ defined by Def. 1.1.1, we have

- the *strict preference relation*, $\succ$ can be induced from $\succsim$ as: $x \succ y \Leftrightarrow x \succsim y \wedge y \not\succsim x$, or in words, $x$ if preferred to $y$.
- the *indifference relation*, $\sim$ can be induced from $\succsim$ as: $x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x$, or in words, $x$ is indifferent to $y$.

With the definition of these relations, we now define the central assumption of relations: ***rationality***.

**Definition 1.1.3.** A weak preference relation $\succsim$ is *rational* if it is:

- Complete: $\forall x, y \in X, x \succsim y$ or $y \succsim x$ or both
- Transitive: $\forall x, y, z \in X, x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$

How to understand them? They are both strong assumptions:

- Completeness of $\succsim$ means it is well-defined between any two possible alternatives. From the perspective of an individual, completeness means that she will make choices, and only meditated choices.
- Transitivity of $\succsim$ implies that the decision maker will not have a preference cycle, since whoever has a preference cycle would suffer economically for it[1].

With the definition of rational $\succsim$ in Def. 1.1.3, we can prove the following properites of $\succ$ and $\sim$ *induced* by $\succsim$:

**Theorem 1.1.1.** If $\succsim$ is rational, then:

i. $\succ$ is irreflexive ($x \succ x$ never holds) and transitive ($x \succ y \wedge y \succ z \Rightarrow x \succ z$)
   Proof:
   - irreflexive: by Def. 1.1.2, $x \succ x \Rightarrow x \succsim x \wedge x \not\succsim x$, self contracdiction.
   - transitive: $x \succ y \Rightarrow x \succsim y \wedge y \not\succsim x$, $y \succ z \Rightarrow y \succsim z \wedge z \not\succsim y$. By transitivity of $\succsim$, $x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$. If $z \succsim x$, by transitivity of $\succsim$ and $x \succsim y$, we would have $z \succsim y$, contradicting $y \succ z$. Therefore $x \succsim z \wedge z \not\succsim x \Rightarrow x \succ z$.

ii. $\sim$ is reflexive ($x \sim x, \forall x$), transitive ($x \sim y \wedge y \sim z \Rightarrow x \sim z$) and symmetric ($x \sim y \Rightarrow y \sim x$)

[1] There are 2 types of violations of transitivity: irrational and mechanical. Irrational violations are easy to understand: decision makers simply do not follow transivity assumption, many reasons have been raised, including mental account, framing, menu effect, attraction effect, etc. Mechanical violations means that decision makers are "forced" to violate transitivity. One example of this type of violation is aggregation of considerations: decision makers may aggregate several sub-preferences as together to make the choice, leading to violation of transitivity. Another example is when the preference is only defined for differences above a certain level (problem of perceptible differences). See Mas-Colell et al. (1995, Page 7-8), Rubinstein (2012, Page 4-5) for details

Proof:
- reflexive: by completeness of $\succsim$, $\forall x, x \succsim x \Rightarrow x \sim x$
- transitive: $x \sim y \Rightarrow x \succsim y \wedge y \succsim x$, $y \sim z \Rightarrow y \succsim z, z \succsim y$, by the transitivity of $\succsim$, we have $x \succsim z \wedge z \succsim x$, hen $x \sim z$
- symmetric: $x \sim y \Rightarrow x \succsim y \wedge y \succsim x \Leftrightarrow y \succsim x \wedge x \succsim y \Rightarrow y \sim x$

iii. $x \succ y \succsim z \Rightarrow x \succ z$
Proof: $x \succ y \Rightarrow x \succsim y \wedge y \not\succsim x$, hence $x \succ y \succsim z \Rightarrow x \succsim z$. If $z \succsim x$, by transitivity of $\succsim$, $y \succsim x$, contradicting $x \succ y$. Therefore, $z \not\succsim x$

We can also directly define a *rational* $\succ$ (see Kreps (1990, Page 19-21)):

**Definition 1.1.4.** A strict preference ralation $\succ$ is rational if it is:

- asymmetric: $\nexists x, y \in X$ s.t. $x \succ y \wedge y \succ x$
- negatively transitive: $x \succ y \Rightarrow \forall z \in X \setminus \{x, y\}, x \succ z \vee z \succ y \vee$ both.

With Def. 1.1.4 and Def. 1.1.3, we can prove that $\succsim$ is rational iff $\succ$ is rational:

**Theorem 1.1.2.** $\succsim$ is rational $\Leftrightarrow$ $\succ$ is rational, specifically:

- $\succsim$ is complete $\Leftrightarrow$ $\succ$ is asymmetric
- $\succsim$ is transitive $\Leftrightarrow$ $\succ$ is negatively transitive

Now we prove this theorem:

**Step 1** proof $\succsim$ is rational $\Rightarrow$ $\succ$ is rational

- **asymmetric**
if $\exists x, y$ s.t. $x \succ y$ and $y \succ x$, then by the definition of induced strict preference, the pair $x, y$ must satisfy

$$\begin{cases} x \succsim y \text{ and } y \not\succsim x & (x \succ y) \\ y \succsim x \text{ and } x \not\succsim y & (y \succ x) \end{cases}$$

which is, by completeness of rational $\succsim$, impossible. Therefore, such pair $x, y$ don't exist. $\succ$ is proved to be asymetric.
- **negatively transitive**
First, $\forall z \notin \{x, y\}$, by completeness of rational $\succsim$, the relation between $x$ and $z$ is either $x \succsim z$ or $z \succsim x$. Similarly, the relation between $y$ and $z$ is either $y \succsim z$ or $z \succsim y$.
Second, given $x \succ y$, $x, y$ satisfies $x \succsim y$ and $y \not\succsim x$.
Also, it is easy to prove that: $x \succ y \wedge y \succsim z \Rightarrow x \succ z$, $x \succ y \wedge z \succsim x \Rightarrow z \succ y$; and $x \succ y \wedge z \sim x \Rightarrow z \succ y$, $x \succ y \wedge y \sim z \Rightarrow x \succ z$
Now we have the following scenarios:
1. if $z \succsim x$ and $y \succsim z$, by transitivity of rational $\succsim$, $y \succsim x$, contradicting the definition of $x \succ y$. This scenario doesn't exist.
2. if $x \succsim z$ and $y \succsim z$, since $x \succ y$, with the auxilary result proved above, we have $x \succ z$
3. if $z \succsim x$ and $z \succsim y$, since $x \succ y$, with the auxilary result proved above, we have $z \succ y$
4. if $x \succsim z$ and $z \succsim y$, since $x \succ y$, suppose:

(a) $z \succsim x$ as well, then $x \sim z$, in this case $z \succ y$;
(b) $z \not\succsim x$, then $x \succ z$
(c) $y \succsim z$ as well, then $y \sim z$, in this case $x \succ z$
(d) $y \not\succsim z$, then $z \succ y$
therefore, a complete summary of (a) to (d) would give:

| | $z \succsim x$ | $z \not\succsim x$ |
|---|---|---|
| $y \succsim z$ | $z \succ y$ & $x \succ z$ | $x \succ z$ |
| $y \not\succsim z$ | $z \succ y$ | $x \succ z$ & $z \succ y$ |

Combining all above, we have proved negative transitivity of $\succ$.

With asymmetry and negative transitivity proved, we've proved that $\succsim$ is rational $\Rightarrow \succ$ is rational

**Step 2** proof $\succ$ is rational $\Rightarrow \succsim$ is rational.

- Complete: with a rational $x \succ y$, we know $\nexists x, y$ s.t. $x \succ y$ and $y \succ x$ by asymmetry. Therefore, $\forall x, y$, we have two possibilities.
  - $x \succ y$ and $y \not\succ x$, which would naturally induce a weak preference $x \succsim y$
  - $y \succ x$ and $x \not\succ y$, which would naturally induce a weak preference $y \succsim x$

  therefore, $\forall x, y$, either $x \succsim y$ or $y \succsim x$ completeness of $\succsim$ is proven.
- Transitive: with a rational $x \succ y$, negative transivity gives $\forall z \notin \{x, y\}$, either $x \succ z$, $z \succ y$, or both. By negative transitivity, we have:
  - $x \succ z$: following same procedure, we know $x \succsim z$. If:
    * $y \succsim z$, since $x \succ z \Rightarrow z \not\succsim x$, by completeness we have $x \succsim z$, thus $x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$
    * $z \succsim y$, since $x \succ y \Rightarrow x \not\succsim y$, by completeness we have $x \succsim y$, thus $x \succsim z \wedge z \succsim y \Rightarrow x \succsim y$
  - $z \succ y$: again, we know $z \succsim y$. If:
    * $x \succsim z$, since $x \succ y \Rightarrow y \not\succsim x$, by completeness we have $x \succsim y$, thus $z \succsim y \wedge x \succsim z \Rightarrow x \succsim y$
    * $z \succsim x$, with $x \succsim y$, suppose $y \succsim z$, this contradicts $z \succ y$, thus $z \succsim x \wedge x \succsim y \Rightarrow z \succsim y$
  - $x \succ z$ and $z \succ y$: again we know $x \succsim z$ and $z \succsim y$. Suppose $y \succsim x$, this contradicts $x \succ y$, therefore $x \succsim z \wedge z \succsim y \Rightarrow x \succsim y$

  In all three scenarios, transitivity is proved.

With completeness and transitivity proved, we've proved that $\succ$ is rational $\Rightarrow \succsim$ is rational.

Notice that negative positivity in Def. 1.1.4, is logically equivalent to its *contrapositive*: $\exists z \in X \setminus \{x, y\}$ s.t. $x \not\succ z \wedge z \not\succ y \Rightarrow x \not\succ y$. This is percisely why the definition is called negative transitivity.

## 1.2 Choice Rules

Next, we approach the theory of decision making from choice behavior itself. Formally, choice behavior is represented by means of a *choice structure* $(\mathcal{B}, C(\cdot))$. Now, we define choice structure $(\mathcal{B}, C(\cdot))$:

**Definition 1.2.1.** A choice structure $(\mathcal{B}, C(\cdot))$ has two ingredients:

- $\mathcal{B} \subset \mathcal{P}(X) \setminus \emptyset$, where $\mathcal{P}(X)$ is the power set of $X$. This means, every element $B \in \mathcal{B}$ is a subset of $X$[2].
- $C(\cdot)$ is a *choice rule correspondence* that assigns a nonempty set of chosen elements $C(B) \subset B, \forall B \in \mathcal{B}$[3].

Now we discuss the CORE assumption in this section: the Weak Axiom of Revealed Preference (WARP):

**Definition 1.2.2.** A choice set $(\mathcal{B}, C(\cdot))$ satisfies WARP if:

- $\forall B, B'$ and $x, y \in B \cap B', x \in C(B), y \in C(B') \Rightarrow x \in C(B')$

Or in words, WARP requires that if $x$ is chosen from some alternatives where $y$ is also available, then there can be NO budget set containing both $x$ and $y$ but only $y$ is chosen.

Following WARP, define the *reveal preference relation* $\succsim^*$ as:

**Definition 1.2.3.** Given a choice structure $(\mathcal{B}, C(\cdot))$, $x \succsim^* y \Leftrightarrow \exists B \in \mathcal{B}$ s.t. $x, y \in B \wedge x \in C(B)$

In words, $x$ is revealed at least as good as $y$.

With revealed preference defiend, we can rephrase WARP as: *If $x$ is revealed at least as good as $y$, then $y$ **cannot** be revealed preferred to $x$.* Hence, $\succsim^*$ is not symmetric.

One thing to remember is that $\succsim^*$ need not be either complete or transitive. For $\succsim^*$ to be comparable, for a $B \in \mathcal{B}$ and $x, y \in B$, we must have either $x \in C(B)$, $y \in C(B)$ or both.

An example is:

**Example 1.2.1.** Consider a choice structure $(\mathcal{B}, C(\cdot))$ from $X = \{x, y, z\}$, where $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$. Under WARP, $C\{x, y\} = \{x\} \Rightarrow y \notin C\{x, y, z\}$. BUT, we can have $z \in C(\{x, y, z\})$.

This is why the induced preference is called *revealed*: you don't know what else is going on.

There are 3 interesting properties of choice rules to keep in mind:

- Property $\alpha$: $x \in A \subseteq B, x \in C(B) \Rightarrow x \in C(A)$.
- Property $\beta$: $x, y \in C(A), A \subseteq B, y \in C(B) \Rightarrow x \in C(B)$

[2]The elements $B \in \mathcal{B}$ are so-called *budget sets*. The budget sets in $\mathcal{B}$ should be thought of as an exhaustive listing of all the choice experiments that can be achieved, but it is possible that some subsets of $X$ are not achievable.

[3]The choice set $C(B)$ can contain a single element, which is the choice among the alternatives in $B$. BUT, $C(B)$ can contain multiple elements, then elements of $C(B)$ are the *acceptable alternatives* in $B$.

- Property $\gamma$: $x \in C(B_1), x \in C(B_2) \cdots x \in C(B_n) \Rightarrow x \in C(B_1 \cup B_2 \cup \cdots \cup B_n)$

For a rational preference relation $\succsim$, its induced choice function $C(\cdot, \succsim)$ satisfies the three properties:

$\alpha$: For $x \in A \subseteq B, x \in C(B, \succsim)$, $x \succsim y, \forall y \in B$, and $A \subseteq B \Leftrightarrow \forall z \in A, z \in B$, hence $\forall z \in A, x \succsim z \Rightarrow x \in C(A, \succsim)$

$\beta$: For $x, y \in C(A, \succsim), A \subseteq B, y \in C(B, \succsim)$, $x \succsim y$ and $\forall z \in B, y \succsim z$, by transitivity, $\forall z \in B, x \succsim z \Rightarrow x \in C(B, \succsim)$

$\gamma$: $x \in C(B_i, \succsim), i = 1, \cdots, n \Rightarrow \forall b_k^i \in B_i, x \succsim b_i \Rightarrow \forall b \in \{b_k^i\}, i = 1, \cdots, n, x_i \succsim b_i \Rightarrow x \in C(B_1 \cup B_2 \cup \cdots \cup B_n, \succsim)$

With Sen's $\alpha$ and $\beta$ defined, we have:

**Theorem 1.2.1.** For a non-empty $C$, $C$ satisfies $\alpha$ and $\beta \Leftrightarrow C$ satisfies WARP

Now, let's prove this.

- **Sen's $\alpha$ and $\beta$ $\Rightarrow$ WARP**: Suppose $x, y \in A \cap B, x \in C(A), y \in C(B)$. Let $\tilde{A} = A \cap B$, since $\tilde{A} \subset A$ and $\tilde{A} \subset B$, Sen's $\alpha$ gives $x, y \in C(\tilde{A})$. Now we have $\tilde{A} \subset B, y \in C(B), x, y \in C(\tilde{A})$, by Sen's $\beta$, $x \in C(B)$. Thus, WARP holds.
- **WARP $\Rightarrow$ Sen's $\alpha$ and $\beta$**: WARP gives $x, y \in A \cap B, x \in C(B), y \in C(A) \Rightarrow x \in C(A)$, in the special case of $A \subset B$, WARP will be reduced to Sen's $\alpha$ and $\beta$.[4]

## 1.3 Linking Preferences with Choices

Now we have two major approaches of decision making process: preference relations in Section 1.1 and choice rules in Section 1.2, what we need to do is to link them. This linkage will emerge when we examine two central assumptions: **rationality** and **WARP**. So the major question here is:

$$\textbf{rational } \succsim \overset{???}{\Longleftrightarrow} (\mathcal{B}, C(\cdot)) \textbf{ satisfies WARP}$$

And the answer is: *YES!* but not exactly. Now let's dig in.

### Rational $\succsim \Rightarrow (\mathcal{B}, C(\cdot))$ satisfies WARP

First, **rational $\succsim \Rightarrow (\mathcal{B}, C(\cdot))$ satisfies WARP** is a big YES. To prove this, we need to define *induced choice correspondence*:

**Definition 1.3.1.** Given a **rational** $\succsim$ on $X$, if the decision maker faces a nonempty subset of alternatives $B \subset X$, by maximizing her preference, she would choose any one of the elements in the *induced choice correspondence*: $C^*(B, \succsim) = \{x \in B : x \succsim y, \forall y \in B\}$

The induced choice correspondence $C^*(B, \succsim)$ has an important property:

[4] Another way of proving this is to introduce rationalizing preference relation (Def.1.3.2). In fact, for $\mathcal{B}$ includes all subsets of $X$ of up to 3 elements, WARP $\Leftrightarrow$ $C$ is rationalizable $\Leftrightarrow$ Sen's $\alpha$ and $\beta$ are satisfied.

**Theorem 1.3.1.** if $X$ is finite, $C^*(B,\succsim)$ will be **nonempty**.

A brief proof of this proposition is: If $X$ is finite, $B$ is finite as well. We will prove by induction. Starting from $|B|=1$, the only element of $B$ is in $C^*(B,\succsim)$. Now suppose $C^*(B,\succsim)$ is nonempty when $|B_n|=n$, let $x^*\in C^*(B_n,\succsim)$; when $|B_{n+1}|=n+1$, let the $n+1$th element $y$ ($\{y\}=B_{n+1}\setminus B_n$). By the completeness of a rational $\succsim$, either $y\succsim x^*$ or $x^*\succsim y$:

i. $y\succsim x^*$: since $x^*\in C^*(B_n,\succsim)\Rightarrow x^*\succsim x,\forall x\in B_n$. By transitivity of $\succsim$, $y\succsim x,\forall\in B_n$. By completeness, $y\succsim y$ as well. Hence, $y\in C^*(B_{n+1},\succsim)$.
ii. $x^*\succsim y$: since $x^*\in C^*(B_n,\succsim)\Rightarrow x^*\succsim x,\forall x\in B_n$, hence $x^*\succsim x,\forall x\in B_n\cup y\Rightarrow x^*\in C^*(B_{n+1},\succsim)$

Notice that when $B$ is finite, a stronger condition of $\succsim$ being acyclic and complete is equilavent to an induced choice rule $C^*(B,\succsim)\neq\varnothing$:

**Theorem 1.3.2.** For a finite $B$, $\succsim$ is complete and **acyclic** $\Leftrightarrow C^*(B,\succsim)\neq\varnothing$

$\succsim$ is acyclic mean that: $b_1\succsim b_2,b_2\succsim b_3,\cdots,b_{n-1}\succsim b_n\Rightarrow b_n\not\succsim b_1$. An example of transitive but not *acyclic* relations is indifference $\sim$: $a_1\sim a_2\sim\cdots\sim a_n\Rightarrow a_n\sim a_1$. A brief proof of Theorem 1.3.2 is:

i. acyclic $\succsim\Rightarrow C^*(B,\succsim)\neq\varnothing$: Suppose if $C^*(B,\succsim)=\varnothing$, for $b_1\in B,b_1\notin C^*(B,\succsim)\Rightarrow\exists b_2$ s.t. $b_2\succsim b_1$. Continue this process, we can generate a sequence of $\cdots\succsim b_2\succsim b_1$, since $B$ is finite, this sequence must end at $b_n$. If $\succsim$ is acyclic, $b_1\not\succsim b_n$, this gives $b_n\succ b_1$, which would mean $b_n$ must be in $C^*(B,\succsim)$, contradicting.
ii. $C^*(B,\succsim)\neq\varnothing\Rightarrow$ acyclic $\succsim$: Suppose $\succsim$ is not acyclic, then there exists $b_1\succsim b_2\succsim\cdots\succsim b_n\succsim b_1$, then for set $B=\{b_1,b_2,\cdots,b_n\}$, $\nexists b^*$ s.t. $b^*\succsim b_i\forall b_i\in B$, i.e., $C^*(B,\succsim)=\varnothing$.

Of course, we want to extend this to the situation where $B$ is infinite. However, in general, it is possible that $C^*(B,\succsim)$ is empty (if you set the most preferred side of $B$ open, it would be impossible to choose based on the preferences). However, we can add some assumptions to finite non-emptiness and choice coherence, to ensure that choice function $C(\cdot)$ is well-behaved on infinite sets as well. The simplest assumption is: For a choice function $C(\cdot)$ that satisfies finite non-emptiness and choice coherence, let the corresponding preference relation be $\succsim_C$, this preference will generate a no-better-than subset for $x$: $NBT_{\succsim_C}(x)=\{x'\in X:x\succsim_C x'\}$. With this setting, we can have:

$$x\in A\subseteq NBT_{\succsim_C}(x)\Rightarrow C(A)\neq\varnothing$$

This means that if everything in $A$ is revealed to be no better than $x$ and $x$ is also in $A$, *some* choice will be made out of $A$. This assumption is equivalment to the conclusion that for the infinite set $A$, we also have $C(\cdot)\equiv C(\cdot)_{\succsim_C}$. And of course, if we reverse the assumption above, we would have:

$$\forall x\in A,A\not\subseteq NBT_{\succsim_C}(x)\Rightarrow C(A)=\varnothing$$

. This is easy to understand: in this case, $x$ is the least preferred one in $A$, since $A$ is infinite, you can never find an alternative that is preferred over all others.

With induced choice correspondence $C^*(B,\succsim)$ defined and non-emptyness proved, we can then say:

**Theorem 1.3.3.** If $\succsim$ is a rational preference relation, then the choice structure generated by $\succsim$, $(\mathcal{B}, C^*(\cdot, \succsim))$, satisfies WARP

We can prove this theorem quite easily: $\forall B, B'$ suppose we have $x, y \in B \cap B'$ and $x \in C^*(B, \succsim), y \in C^*(B', \succsim)$, then $x \succsim a, \forall a \in B$ and $y \succsim b, \forall b \in B'$. Naturally, we have $x \succsim y$ since $y \in B$. By rationality (transitivity) of $\succsim$, we have $x \succsim y \succsim b, \forall b \in B'$, which means $x \in C^*(B', \succsim)$. This is precisely the definition of WARP

### $(\mathcal{B}, C(\cdot))$ satisfies WARP $\Rightarrow$ Rational $\succsim$

The proof of this direction is more subtle, and is NOT necessarily a yes. Again, we start from a auxilary definition:

**Definition 1.3.2.** For a choice structure $(\mathcal{B}, C(\cdot))$, a rational preference relation $\succsim$ **rationalizes** $C(\cdot)$ relative to $\mathcal{B}$ if $C(B) = C^*(B, \succsim), \forall B \in \mathcal{B}$.

In words, if for all budget sets $B \in \mathcal{B}$, the choices generated by a rational $\succsim$, is just the choice rule $C(\cdot)$, $C(\cdot)$ is rationalized by $\succsim$. This is, in a sense, constructing an explanation of decision making behavior with preferences.

We already proved that $C^*(B, \succsim)$ satisfies WARP, which means that if a rationalizing preference relation to exist, WARP must be satisfied. However, if WARP is satisfied, a rationalizing preference relation does **NOT** necessarily exist.[5] Intuitiviely, more budget sets $B \in \mathcal{B}$ would mean that, to satisfy WARP, choice behavior would be restricted more, and it is easier to be self-contradicting. Therefore, to pin down a rational preference relation to rationalize $C(\cdot)$ relative to $\mathcal{B}$, we need to put some **restrictions on** $\mathcal{B}$.

**Theorem 1.3.4.** If $(\mathcal{B}, C(\cdot))$ is a choice structure that:

i. WARP is satisfied
ii. $\mathcal{B}$ includes **all** subsets of $X$ of **up to 3** elements

then there exists a rational preference relation $\succsim$ s.t. $C(B) = C^*(B, \succsim), \forall B \in \mathcal{B}$. And this rational $\succsim$ is the **only** preference relation that can rationalize $(\mathcal{B}, C(\cdot))$[6].

Now let's prove it, by examing the natural candidate for a rationalizing preference relation: the **revealed preference relation** $\succsim^*$:

**Step 1** Prove that $\succsim^*$ is rational

- Completeness: By (ii) of Def.1.3.2, all binary subsets of $X$ are in $\mathcal{B}$. Hence, $\{x, y\} \in \mathcal{B}$. For this binary menu, $C(\{x, y\})$ must contain either $x$ or $y$, therefore, $x \succsim^* y$ or $y \succsim^* x$ or both. Completeness proved.
- Transitivity: $\forall \{x, y, z\} \in \mathcal{B}$, $C(\{x, y, z\}) \neq \varnothing$. Suppose $x \succsim^* y, y \succsim^* z$, which implies that $x \in C(\{x, y\}), y \in C(\{y, z\})$, we then have three cases for

[5] A simple example is: $X = \{x, y, z\}, \mathcal{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$. Since $\mathcal{B}$ contains 3 binary menus, the choice structure $C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x, z\}) = \{z\}$ vacuously satisfy WARP. But, this choice structure cannot be rationalized since it contradicts transitivity.

[6] The existence of a rationalizing preference relation $\succsim$ brings many interesting properties, one of them is *path-invariance*: $\forall B_1, B_2 \in \mathcal{B}, B_1 \cup B_2 \in \mathcal{B} \wedge C(B_1) \cup C(B_2) \in \mathcal{B} \Rightarrow C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$, meaning that the decision problem can safely be subdivided. A proof is: for $x \in C(B_1 \cup B_2)$ and $y \in C(B_1) \cup C(B_2)$, since $C(B_1) \cup C(B_2) \subset B_1 \cup B_2$, thus $x \in C(B_1 \cup B_2) \Rightarrow x \in C(C(B_1) \cup C(B_2))$; for $x \in C(C(B_1) \cup C(B_2))$ and $y \in B_1 \cup B_2$, we have $x \succsim z, \forall z \in C(B_1), C(B_2) \Rightarrow x \succsim w, \forall w \in B_1 \cup B_2 \Rightarrow x \in C(B_1 \cup B_2)$

$C(\{x,y,z\})$:

a. $x \in C(\{x,y,z\})$, WARP gives that $x \in C(\{x,z\}) \Rightarrow x \succsim^* z$

b. $y \in C(\{x,y,z\})$, we have $x \in C(\{x,y\})$. WARP gives $x \in C(\{x,y,z\}) \Rightarrow x \succsim^* z$

c. $z \in C(\{x,y,z\})$, we have $y \in C(\{y,z\})$. WARP gives $y \in C(\{x,y,z\})$, and $x \in C(\{x,y\})$, WARP gives $x \in C(\{x,y,z\}) \Rightarrow x \succsim^* z$

Hence, $x \succsim^* y, y \succsim^* z \Rightarrow x \succsim^* z$

**Step 2** Prove that $\succsim^*$ rationalizes $C(\cdot)$ on $\mathcal{B}$

Now, we need to show $\forall B \in \mathcal{B}, C(B) = C^*(B, \succsim^*)$. Logically, this means the revealed preference $\succsim^*$ inferred from $C(\cdot)$ actually generates $C(\cdot)$. Formally, we prove it in 2 steps:

a. Suppose $x \in C(B)$, which means that $\forall y \in B, x \succsim^* y$ (by Def.1.2.3), hence $x \in C^*(B, \succsim^*)$ (by Def.1.3.1). This proves $C(B) \subseteq C^*(B, \succsim^*)$

b. Suppose $x \in C^*(B, \succsim^*)$, which means that $\forall y \in B, x \succsim^* y$ (by Def.1.3.1). Therefore, $\forall y \in B$, there must exist a set $B_y \in \mathcal{B}$ s.t. $x, y \in B_y \Rightarrow x \in C(B_y)$. Since $C(B) \neq \varnothing$, suppose $z \in C(B)$, since $x \in C(B_z)$, WARP implies that $x \in C(B)$. This proves $C^*(B, \succsim^*) \subseteq C(B)$

Together, we have $C(B) = C^*(B, \succsim^*)$.

**Step 3** Prove $\succsim^*$ is the unique choice

Since $\mathcal{B}$ includes all two-element subsets of $X$, the choice behavior in $C(\cdot)$ completely determines the pairwise preference relations over $X$ of any rationalizing preference.

Now, it is **proved**! Notice that the main assumption(restriction) here is $\mathcal{B}$ **includes all subsets of** $X$ **of up to 3 elements**, this gives completeness, which is fundamental.

## Things to keep in mind

We have proved the twoway links of preferences and choices:

- Rational $\succsim \Rightarrow (\mathcal{B}, C^*(\cdot, \succsim))$ satisfies WARP (see Thm.1.3.3)
- A WARP-satisfying, up-to-3-element $(\mathcal{B}, C(\cdot))$ can be uniquely rationalized by a rational $\succsim$ (see Thm.1.3.4)

However, there are still something to keep in mind.

First, for a given choice structure $(\mathcal{B}, C(\cdot))$, there may be **more than one** rationalizing preference relation $\succsim$ in general. Here is the simplest example: For $X = \{x,y\}, \mathcal{B}\{\{x\},\{y\}\}$ and the choice structure $C(\{x\}) = \{x\}, C(\{y\} = \{y\})$. In this case, **ANY** relation preference relation of $X$ can rationalize $C(\cdot)$ This is related to both Def.1.3.2 and (ii) of Thm.1.3.4. Thm.1.3.4 gives that if $\mathcal{B}$ contains **ALL binary** menus of $X$, then there could be at most one rationalizing preference relation.

Second, the restriction for WARP$\Rightarrow$ rational $\succsim$, namely $\mathcal{B}$ containing all subsets of up to 3 elements, is too strong. For many economic problems, we will not consider all possible subsets, or limit ourselves to up-to-3-element ones. A strengthened version of WARP will be introduced later for that purpose.

Finally, up till now, we define a rationalizing preference as one: $C(B) = C^*(B, \succsim)$

(Def.1.3.2). A common alternative would be to require only $C(B) \subset C^*(B, \succsim)$: if $C(B)$ is a **subset** of the most preferred choices generated by $\succsim$, i.e., $C^*(B, \succsim)$. This will allow indifferences to be more than the situation of anything might be picked. And it is empirically intuitive in a sense that observed choices will never fully reveal decision makers' entire preferencing maximizing choice set. Naturally, $C(B) \subset C^*(B, \succsim)$ is weaker than $C(B) = C^*(B, \succsim)$. But $C(B) \subset C^*(B, \succsim)$ has an interesting property: the all-indifferent preference will be able to rationalize *any* choice behavior. Therefore, when $C(B) \subset C^*(B, \succsim)$ is used, you would always need to put some additional restrictions on the rationalizing preference relation for the specific economic context.

## 1.4 Introducing Utility

Now, with preferences and choices defined, and the linkage between the two established, we need to transfer these concepts into math for analytic studies. This is exactly why utility functions are introduced: to assign a number and rank the elements in $X$ according to preferences.

**Definition 1.4.1.** A function $u : X \to \mathbb{R}$ is a *utility function representing relation* $\succsim$ if $\forall x, y \in X, x \succsim y \Leftrightarrow u(x) \geq u(y)$

Notice that a utility function representing a preference relation $\succsim$ is NOT unique. **Rank-preserving** is the only requirement, hence, any strictly increasing function $f : \mathbb{R} \to \mathbb{R}, v(x) = f(u(x))$ will also represent $\succsim$ as $u(\cdot)$. The logic is quite straight forward: for $x, y \in X$ and $u(\cdot)$ represents $\succsim$, then $x \succsim y \Leftrightarrow u(x) \geq u(y)$, for a strictly increasing $f(\cdot)$, $u(x) \geq u(y) \Leftrightarrow f(u(x)) \geq f(u(y)) \Leftrightarrow v(x) \geq v(y)$, hence $v(\cdot)$ represents $\succsim$ as well. The major requirement here is **strictly increasing** $f(\cdot)$.

Two concepts to keep in mind:

1. **Ordinal** properties of utility functions: the **invariant** properties of $u(\cdot)$ across all of its strictly increasing transformations $f(u(\cdot))$. Ranking (i.e. the preference represented by utility functions) is ordinal.
2. **Cardinal** properties of utility functions: the **variant** properties of $u(\cdot)$ across all of its strictly increasing transformation $f(u(\cdot))$. Numerical values associated with the alternatives in $X$ (i.e. the magnitude of the differences between alternatives) is cardinal.

The numerical value, or even the size of relative differences have no particular meaning. Only ranking is "real", since the "level of utility" is **unobservable** and anything required to know the "level of utility" is **untestable**.

The central theorem of utility functions is closely linked to rationality:

**Theorem 1.4.1.** A preference relation $\succsim$ can be represented by a utility function $\Rightarrow \succsim$ is rational

The proof is

- *Completeness*. Since $u(\cdot)$ represents preference relations between alternatives, and $u : X \to \mathbb{R}$, thus $\forall x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. By Def.1.4.1, we have either $x \succsim y$ or $y \succsim x$, hence $\succsim$ is complete.

- *Transitivity*. For $x \succsim y, y \succsim z$. By Def.1.4.1, $u(x) \geq u(y), u(y) \geq u(z)$, hence $u(x) \geq u(z) \Rightarrow x \succsim z$.

What about the other way? It is true, subject to some prerequisites:

**Theorem 1.4.2.** $\succsim$ is rational and $X$ is **finite** $\Rightarrow$ there is a utility function representing $\succsim$.

The major prerequisite here is $X$ being **finite**. The proof is done by induction: Suppose there are $N$ elements in $X$:

- When $N = 1$, any number could be assigned to that element as its utility.
- Suppose a rational $\succsim$ on $X = x_1, x_2, \cdots, x_{N-1}$ could be represented by a utility function $u(\cdot)$. Without losing generality, we can assume $u(x_1) \leq u(x_2) \leq \cdots \leq u(x_{N-1})$. For the $N$th element $x_N$, by the rationality of $\succsim$, we have three scenarios:
  i $\forall i \in 1, \cdots, N-1, x_N \succsim x_i$: by Def.1.4.1, $u(x_N) \geq u(x_i)$.
  ii $\forall i \in 1, \cdots, N-1, x_i \succsim x_N$:, $u(x_N) \leq u(x_i)$.
  iii $\exists i, j \in 1, \cdots, N-1, i \neq j, x_j \succsim x_N \succsim x_i$: $u(x_j) \geq u(x_N) \geq u(x_i)$. By completeness and transitivity, $x_1, x_2, \cdots, x_{N-1}$ can be "divided" by $x_N$, meaning that for $I = \{i : x_N \succsim x_i\}$ and $J = \{j : s_j \succsim x_N\}$, $I \cup J = \{1, \cdots, N-1\}$. Note that we have assumed the index as the ranking, hence let $i^* = \max I, j^* = \min J, i^*+1 = j^*$, hence we must have $u(x_i) \leq u(x_{i^*}) \leq u(x_N) \leq u(x_{j^*}) \leq u(x_j)$

  In all 3 scenarios, $u(\cdot)$ represents $\succsim$ on $X = \{x_1, \cdots, x_{N-1}, x_N\}$ as well.

With this induction, we prove Thm.1.4.2[7].

Now extend Thm.1.4.2 from finite $X$ to countable infinite $X$:

**Theorem 1.4.3.** $\succsim$ is rational and $X$ is **countable** $\Rightarrow$ there is a utility function representing $\succsim$.

To prove Thm.1.4.3, we can construct a utility function: for any set $S$, its emuneration $\{s_1, s_2, \cdots\}$ (which exists if $S$ is countable), define an auxilary function $d : S \longrightarrow \mathbb{R}$ as $d(s_i) = \left(\frac{1}{2}\right)^n$, then for a countable set $X = \{x_1, x_2, \cdots\}$, the utility of any element $\tilde{x} \in X$ can be defined as

$$u(x^*) = \sum_{\tilde{x}_i \in NBT(x^*)} d(\tilde{x}_i)$$

[8] where $NBT(x^*)$ is the set of all elements that are **n**ot **b**etter **t**han $x^*$, i.e. $NBT(x^*) = \{\tilde{x}_i : \tilde{x}_i \in X \wedge x^* \succsim \tilde{x}_i\}$. It is easy to see that $NBT(x^*)$ is a countable subset of $X$. Suppose $NBT(x^*)$ has $k$ elements ($k < n$), we can calculate the utility $u(x^*) = \sum_{i=1}^{k} \left(\frac{1}{2}\right)^i$. After this construction, rest of the proof is trivial: $\forall x, y \in X$, $x \succsim y \Rightarrow NBT(x) \supseteq NBT(y)$, which means that $NBT(x)$ has at least as many elements as $NBT(y)$, by the constructed utility function, it is easy to see $u(x) \geq u(y)$; Conversely, $u(x) \geq u(y)$ simply means that $NBT(x)$ contains at least as many elements as $NBT(y)$ does, which directly leads to $x \succsim y$.

Now, let's figure out the difficult question: what about uncountable sets? Here is a

[7]Another way of proof is: Start with $x^{start} \in X$, define $W_x = \{y : y \prec x^{start}\}$ then $W_x$ is either empty or not: If not empty, pick $\tilde{x} \in W_x$, shrink $W_x$ to $\{y : y \prec \tilde{x}\}$ and repeat this procedure till $a \sim x^{stop}$ where $u(a) = 0$, then $x^{stop}$ is the "lower bound" of the set. With this process, we can generate a utility function for any finite set $X$ that is rational.

[8]Notice that $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = 1$, utility is bounded to $[0, 1)$.

very general proposition:

> **Theorem 1.4.4.** For a rational preference $\succsim$ on a set $X$, $\succsim$ can be represented by $u(\cdot)$ if and only if some countable set $X^*$ of $X$ has the property that $\forall x, y \in X, x \succ y \Rightarrow \exists x^* \in X^*$ s.t. $x \succsim x^* \succ y$.

The proof of Thm.1.4.4 is not that difficult:

Step 1: such $X^*$ exists $\Rightarrow$ rational $\succsim$ can be represented by $u(\cdot)$.

- Given such $X^*$, $x \succsim y \Rightarrow u(x) \geq u(y)$
  Suppose $X^*$ exists, let it be $X^* = \{x_1^*, x_2^*, \cdots\}$. Again, define $d(x_n^*) = \frac{1}{2^n}$, we can then construct a utility function as

$$\forall x \in X, u(x) = \sum_{\tilde{x}_i^* \in X^* \cap NBT(x)} d(\tilde{x}_i^*)$$

  Since $x \succsim y \Leftrightarrow NBT(x) \supseteq NBT(y)$, hence $NBT(x) \cap X^* \supseteq NBT(y) \cap X^*$, which, by the construction of $\sum \frac{1}{2^n}$, leading to $u(x) \geq u(y)$.
- Given such $X^*$, $u(x) \geq u(y) \Rightarrow x \succsim y$
  We can prove the contrapositive: $y \not\succsim x \Rightarrow u(y) \not\geq u(x)$. Given the rationality of $\succsim$, $y \not\succsim x \Rightarrow x \succ y$, then $\exists x^* \in X^*$ s.t. $x \succsim x^* \succ y$, hence we know $NBT(x)$ is strictly larger then $NBT(y)$ ($NBT(x)$ includes $x^*$), therefore, by definition, $u(x) > u(y) \Rightarrow u(y) \not\geq u(x)$.

Step 2: rational $\succsim$ can be represented by $u(\cdot) \Rightarrow$ such $X^*$ exists We want to prove this, but it is very difficult to prove in general, so we construct a special case: Let $\{I_n\}$ be a set of all closed intervals with rantional endpoints, that is, each $I_n$ is an interval of $[\underline{q}_n, \bar{q}_n]$ where $\bar{q}_n > \underline{q}_n$ are rational numbers. The set of rational numbers is countable, the cross product of two countable sets is also countable, hence $\{I_n\}$ is countable as well. Let $u(X)$ denote the set of real numbers $\{r \in \mathbb{R} : \exists x \in X, r = u(x)\}$, there will be 3 possible scenarios:

  i. $u(X) \cap I_n \neq \varnothing$: for each $I_n$, pick one $x \in X$ s.t. $u(x) \in I_n$ and name it $x_n$
  ii. $u(X) \cap I_n = \varnothing$: let $\bar{r}_n = \inf\{r \in u(X) : r > \bar{q}_n\}$. If $\exists x \in X$ s.t. $u(x) = \bar{r}_n$, choose one such $x$ and name it $x_n$
  iii. $u(X) \cap I_n = \varnothing \wedge \forall x \in X, \bar{r}_n \neq u(x)$, no $x$ will be defined as $x_n$

  If we define $X^*$ as the collection of all $x_n$ in case i) and ii). Notice there is less than one $x_n$ for each $I_n$ and $I_n$ is countable, hence $X^*$ is countable as well.
  Suppose $x \succ y$ for $x, y \in X$, we have $u(\cdot)$ representing $\succsim$, hence $u(x) > u(y)$. Choose some rational number $q$ in the open interval $(u(y), u(x))$ and let $\bar{r} = \inf\{r \in u(X) : r > q\}$. Given this setup, we have $u(x) \geq \bar{r}$ since $u(x) > q$. If:

  - $u(x) > \bar{r}$: we can always find a rational number $q'$ s.t. $u(x) > q' > \bar{r}$. Let $n$ be the index of the interval $[q, q']$, since $q < \bar{r} < q'$, $\bar{r} \in u(X) \cap [q, q'] \Rightarrow u(X) \cap [q, q'] \neq \varnothing$. Therefore, $\exists x^* \in X^*$, namely $x_n$, s.t. $u(x^*) \in [q, q']$, leading to $u(x) > u(x^*) > u(y)$.
  - $u(x) = \bar{r}$: we can always find a rational number $q'$ s.t. $q > q' > u(y)$. Let $n$ be the index of the interval $[q, q']$. If $u(X) \cap [q', q] \neq \varnothing$, then $\exists x^* \in X^*$ s.t. $u(x) \geq q \geq u(x^)) \geq q' \geq u(y)$, then $x \succsim x^* \succ y$. If $u(X) \cap [q, q'] = \varnothing$, then $[q, q']$ fits into category (ii) above, and $\exists x^* \in X^*$, namely $x_n$, such that $u(x^*) = \bar{r} = u(x)$. With this $x^*$, we have $u(x^*) = u(x) > u(y) \Rightarrow x \succsim x^* \succ y$.

This is a very smart proof, and it is very general as well. However, you would have to make sure that the countable seubset $X^*$ exists, which is not very practical. This problem leads to topological $\succsim$, which will be covered later.

It is natural that if we start from $\succ$, we would have the same logic (See Kreps, 1990, Page 30):

- similar definition: $x \succ y \Leftrightarrow u(x) > u(y)$
- similar theorems:
  - i $\exists u(\cdot)$ representing $\succ \Rightarrow$ rational $\succ$ (asymmetric and negatively transitive)
  - ii If $X$ is finite or at least countably infinite, $\exists u(\cdot)$ representing $\succ \Leftrightarrow$ rational $\succ$

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