Probability and Statistics for Economics Cheatsheet

Random experiments

The outcome in a random experiment is **unpredictable**:

- outcome is too complicates or poorly understood
- outcome is designed to be unpredictable
- coincidences, or independent chains of events

Random card shuffle experiment: take top card from a deck and insert randomly, to complete the shuffle of n cards, we need $T = n + \frac{n}{2} + \cdots + \frac{n}{n-1} + 1 = n \log n$ shuffles.

Random number generator: $x_{n+1} = \frac{ax_n+b}{c} - \left[\frac{ax_n+b}{c}\right]$, the remainder after dividing by c, hence $x_{n+1} \in [0, c-1]$, let $u_{n+1} = \frac{x_{n+1}}{c}$, x_0, a, b, c all be integers. For very large a and good choice of b, c, the sequence $u_1, u_2, \cdots$ is like a sequence of numbers randomly picked from $[0, 1]$

Probabilities

Ways of assigning probabilities:

- symmetry: assume all outcomes are equally likely
- experimental method: relative frequency in repeated random experiment
- subjective method: assign probabilities using knowledge of random experiment
- market method

Elements of $\boxed{\textbf{probability space}}$:

- outcome space Ω and outcomes $\omega \in \Omega$
- event E, $E \subset \Omega$
- probability function/measure $P : \mathcal{A} \to [0, 1]$: a function from **a collection** $\mathcal{A}$ of subsets of Ω to the interval $[0, 1]$.

Classes of events

Events $E_1, E_2, \cdots$ are just sets. They also follow the algebras of sets, such as:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup A^C = U$	$A \setminus B = A \cap B^C$
$(\bigcup_{i=1}^{\infty} A_i)^C = \bigcap_{i=1}^{\infty} A_i^C$	$(\bigcap_{i=1}^{\infty} A_i)^C = \bigcup_{i=1}^{\infty} A_i^C$
$(A^C)^C = A$	$A \cup B = U, A \cap B = \varnothing \Leftrightarrow B = A^C$

$A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B \Leftrightarrow A \setminus B = \varnothing \Leftrightarrow B^C \subseteq A^C$

Two special relations:

- **disjoint**: $E_1 \cap E_2 = \varnothing$
- **partition**: $\bigcup_{i=1}^{\infty} E_i = \Omega$

$\sigma-$field and Borel $\sigma-$field

Definition of $\sigma-$field: $\mathcal{A}$ (a collection of subsets of Ω) is a $\sigma-$field if

(i) $\varnothing \in \mathcal{A}$

(ii) $E \in \mathcal{A} \Rightarrow E^C \in \mathcal{A}$

(iii) $E_1, E_2, \cdots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$

smallest $\sigma-$field:$\mathcal{A} = \{\varnothing, \Omega\}$

Variation of Parameters

$$F(x) = y'' + y'$$

$$y_h = b_1 y_1(x) + b_2 y_2(x), y_1 y_2 \text{ are L.I.}$$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$u_1 = \int^t -\frac{y_2 F(t) dt}{w[y_1, y_2](t)}$$

$$u_2 = \int^t \frac{y_1 F(t) dt}{w[y_1, y_2](t)}$$

$$y = y_h + y_p$$

ODEs

1st Order Linear	Use integrating factor, $I = e^{\int P(x) dx}$
Separable:	$\int P(y) dy/dx = \int Q(x)$
HomogEnEous:	$dy/dx = f(x, y) = f(xt, yt)$ sub $y = xV$ solve, then sub $V = y/x$
Exact:	If $M(x, y) + N(x, y) dy/dx = 0$ and $M_y = N_x$ i.e. $\langle M, N \rangle = \nabla F$ then $\int_x M + \int_y N = F$
3) Distinct complex roots	$y = B_1 x^a \cos(b \ln x) + B_2 x^a \sin(b \ln x)$

Series Solution

$y'' + p(x) y' + q(x) y = 0$

Useful when $p(x), q(x)$ not constant

Guess $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

e^x	$\sum_{n=0}^{\infty} x^n / n!$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Systems

$\vec{x}' = A\vec{x}$

A is diagonalizable	$\vec{x}(t) = a_1 e^{\lambda_1 t} \vec{v_1} + \cdots + a_n e^{\lambda_n t} \vec{v_n}$
A is not diagonalizable	$\vec{x}(t) = a_1 e^{\lambda_1 t} \vec{v_1} + a_2 e^{\lambda t} (\vec{w} + t\vec{v})$ where $(A - \lambda I)\vec{w} = \vec{v}$; $\vec{v}$ is an Eigenvector w/ value λ; i.e. $\vec{w}$ is a generalized Eigenvector
$\vec{x}' = A\vec{x} + \vec{B}$	Solve y_h; $\vec{x_1} = e^{\lambda_1 t} \vec{v_1}, \vec{x_2} = e^{\lambda_2 t} \vec{v_2}$; $\vec{X} = [\vec{x_1}, \vec{x_2}]$; $\vec{X} \vec{u}' = \vec{B}$; $y_p = \vec{X} \vec{u}$; $y = y_h + y_p$

Matrix Exponentiation

$A^n = S D^n S^{-1}$

D is the diagonalization of A

Laplace Transforms

$L[f](s) = \int_0^{\infty} e^{-sx} f(x) dx$

$f(t) = t^n, n \geq 0$	$F(s) = \frac{n!}{s^{n+1}}, s > 0$
$f(t) = e^{at}, a$ *constant*	$F(s) = \frac{1}{s-a}, s > a$
$f(t) = \sin bt, b$ *constant*	$F(s) = \frac{b}{s^2+b^2}, s > 0$
$f(t) = \cos bt, b$ *constant*	$F(s) = \frac{s}{s^2+b^2}, s > 0$
$f(t) = t^{-1/2}$	$F(s) = \frac{\pi}{s^{1/2}}, s > 0$
$f(t) = \delta(t - a)$	$F(s) = e^{-as}$
f'	$L[f'] = sL[f] - f(0)$
f''	$L[f''] = s^2 L[f] - sf(0) - f'(0)$
$L[e^{at} f(t)]$	$L[f](s - a)$
$L[u_a(t) f(t - a)]$	$L[f] e^{-as}$

Gaussian Integral

$\int_{-\infty}^{+\infty} e^{-1/2(\vec{x}^T A \vec{x})} = \frac{\sqrt{2\pi}^n}{\sqrt{\det A}}$

Complex Numbers

Systems of equations	If $\vec{w_1} = u(\vec{t}) + iv(\vec{t})$ is a solution, $\vec{x_1} = u(\vec{t}), \vec{x_2} = v(\vec{t})$ are solutions i.e. $\vec{x_h} = c_1\vec{x_1} + c_2\vec{x_2}$
Euler's Identity	$e^{ix} = \cos x + i \sin x$

Vector Spaces

$v_1, v_2 \in V$

1. $v_1 + v_2 \in V$
2. $k \in \mathbb{F}, kv_1 \in V$
3. $v_1 + v_2 = v_2 + v_1$
4. $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
5. $\forall v \in V, 0 \in V \mid 0 + v_1 = v_1 + 0 = v_1$
6. $\forall v \in V, \exists - v \in V \mid v + (-v) = (-v) + v = 0$
7. $\forall v \in V, 1 \in \mathbb{F} \mid 1 * v = v$
8. $\forall v \in V, k, l \in \mathbb{F}, (kl)v = k(lv)$
9. $\forall k \in \mathbb{F}, k(v_1 + v_2) = kv_1 + kv_2$
10. $\forall v \in V, k, l \in \mathbb{F}, (k + l)v = kv + lv$