Macroeconomic Model Cheat Sheet

Author: Sai Zhang (email me or check my Github page)

Modelling essential: what we need to decide

- **Primitive assumptions**:
  - Who the **agents** are, what are their **preferences** and objective functions
  - What **technology** agents can access
  - What **endowment** agents have
- **Decision problems**: resource allocation problem (among agents, over time, etc.).
- **Information sets**: what do agents know, how will their knowledge change, what is their **expectation**.
- **Allocation mechanism**: how agents interact and achieve equilibrium. 2 main mechanisms are:
  - **price system** in competitive equilibrium
  - benevolent **central planner** maximizes a social welfare function.

Infinitely Lived Agent Model

Features

- **discrete** time, indexed by t
- economy lives **infinitely**, $t = 0, 1, 2, \cdots$
- single commodity exogenously produced, indexed by t, pure **exchange/endowment** economy.
- no firms/government, only **two types of households**.
- each type of households is continuum of **identical** households of that type, they are **price takers**, can be represented by a **representative** household

Agents' preferences

Utility of type i household is

$$U(c^i) = \sum_{t=0}^{\infty} \beta_i^t u(c_t^i)$$

where $\left(c^i\right) = \left\{c_t^i\right\}_{t=0}^{\infty}$, $\beta_i \in (0,1)$,

The utility function $u(c_t^i)$ is assumed to be:

- **continuously differentiable** of the second order
- **monotonically increasing, strictly concave**: $u'(c_t^i) > 0, u''(c_t^i) < 0$
- **satisfies Inada conditions** (never 0 or infinity consumption): $\lim_{c_t^i \to \infty} u'(c_t^i) = 0, \lim_{c_t^i \to 0} u'(c_t^i) = \infty$
- **time additivity**: $u(c_t^i)$ is independent of c_{t+j}^i, c_{t-j}^i.
- **impatient discounting** $\beta_i < 1$: households value today's consumption more than future's.
- Constant relative risk aversion (**CRRA**): $u(c_t^i) = \frac{c^{1-\sigma}-1}{1-\sigma}$ $\left(\lim_{\sigma\to 1} \frac{c^{1-\sigma}-1}{1-\sigma} = \lim_{\sigma\to 1} \frac{e^{(1-\sigma)\ln(c)}-1}{1-\sigma} = \ln(c)\right)$. **The RRA coefficient** $\sigma(c) = \frac{-u''(c_t^i)c}{u'(c_t^i)} = \frac{-\left(-\sigma c^{-(1+\sigma)}c\right)}{c^{-\sigma}} = \sigma$. Higher RRA means higher risk aversion.
- Constant intertemporal elasticity of substitution (**IES**):

$$IES = -\frac{\mathrm{d}\ln(c_{t+1}/c_t)}{\mathrm{d}\ln(u'(c_{t+1}/u'(c_t))} = \frac{1}{\sigma}$$

  hence higher RRA (more risk-averse), lower IES (consumption variation over time).

Agents' endowment

A deterministic endowment stream of the consumption good for type i household is

$$w^i = \left(w_0^i, w_1^i, \cdots\right) = \left\{w_t^i\right\}_{t=0}^{\infty}$$

Arrow-Debreu Market (AD) approach

Market structure: Basic Case

Households trade just **once** in $t = 0$ market, they trade all future consumption and deliver the promised amount in $t = 1, 2, \cdots$ market.

Households have perfect information of the entire endowment sequence, all information is public.

Equilibrium: Basic Case

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- regulating mechanism: $\{\hat{p}_t\}_{t=0}^{\infty}$, with numeraire $\hat{p}_0 = 1$

such that:

- given $\{\hat{p}_t\}_{t=0}^{\infty}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i) \text{ s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i, c_t^i \ge 0$$

- market clearing (disposal of unused goods is costly):

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, \forall t$$

How to solve:

- ***Step 1***: solve the Lagrangian

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t^i) + \lambda^i \left(\sum_{t=0}^{\infty} p_t w_t^i - \sum_{t=0}^{\infty} p_t c_t^i\right)$$

  FOCs: $\beta^t / c_t^i = \lambda^i p_t \Rightarrow \frac{\beta^t}{c_t^i p_t} = \frac{\beta^{t+1}}{c_{t+1}^i p_{t+1}} \Rightarrow c_{t+1}^i = \beta \frac{p_t}{p_{t+1}} c_t^i$. FOC gives that price changes p_t/p_{t+1} and subjective discounting β determines consumption smoothing.
- ***Step 2***: Use market clearing condition

$$c_t^1 + c_t^2 = w_t^1 + w_t^2$$

  and FOC $p_{t+1} c_{t+1}^i = \beta p_t c_t^i$, get $\frac{p_{t+1}}{p_t} = \beta \frac{w_t^1 + w_t^2}{w_{t+1}^1 + w_{t+1}^2}$, combined with the numeraire assumption $p_0 = 1$, solve the price sequence $\{\hat{p}_t\}_{t=0}^{\infty}$.
- ***Step 3***: Plug $\{\hat{p}_t\}_{t=0}^{\infty}$ and $p_{t+1} c_{t+1}^i = \beta p_t c_t^i$ back to budget constraint

$$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} \beta^t c_0^i = \frac{c_0^i}{1-\beta} = \sum_{t=0}^{\infty} p_t w_t^i$$

  to solve allocation $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$

Pareto efficiency: Basic Case

An allocation $\left(c^1, c^2\right) = \left\{c_t^1, c_t^2\right\}_{t=0}^{\infty}$ is **Pareto efficient** if:

- it is feasible: $\sum_{t=0}^{\infty} p_t c_t^i \le \sum_{t=0}^{\infty} p_t w_t^i$
- no other feasible allocation $(\tilde{c}^1, \tilde{c}^2)$ such that $\forall i, U(\tilde{c}^i) \ge U(c^i)$ and $\exists i, U(\tilde{c}^i) \ge U(c^i)$

An AD competitive equilibrium allocation $\left(c^1, c^2\right) = \left\{c_t^1, c_t^2\right\}_{t=0}^{\infty}$ is Pareto efficient.

Proof: Suppose there is an allocation $\left(\bar{c}^1, \bar{c}^2\right) = \left\{\bar{c}_t^1, \bar{c}_t^2\right\}_{t=0}^{\infty}$, Pareto-dominating AD allocation $\left(\hat{c}^1, \hat{c}^2\right) = \left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$.

If the Pareto dominating allocation $\left(\bar{c}^1, \bar{c}^2\right)$ exists, its utility $\overline{U} = U(\bar{c}^1, \bar{c}^2)$ must be bigger than the AD allocation utility $\tilde{U} = U(\tilde{c}^1, \tilde{c}^2)$, therefore, the only reason that it is not chosen as the AD allocation is that it is **infeasible**.

Formally, suppose $\bar{c}^1 > \hat{c}^1$ and $\bar{c}^2 \ge \hat{c}^2$,

- **Step 1**: for household 1 ($\hat{c}^1 < \bar{c}^1$), if $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^1 \le \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 = \sum_{t=0}^{\infty} \hat{p}_t w_t^1$ (the Pareto-dominating allocation is also feasible), the AD equilibrium $(\hat{c}^1, \hat{c}^2)$ will NOT maximize HH1's utility, hence contradiction.
- **Step 2**: for household 2 ($\hat{c}^2 \le \bar{c}^2$), if $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t w_t^2$ (the Pareto-superior allocation cost less for HH2), then $\exists \delta > 0$ s.t. $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^2 + \delta \le \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^c = \sum_{t=0}^{\infty} \hat{p}_t w_t^2$, then there is always an allocation $\left\{\bar{c}_0^2 + \delta, \bar{c}_t^2\right\}$, achieves a strictly higher utility than the AD allocation (which is utility maximizing), hence contradiction.
- **Step 3**: from **Step 1**, $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 = \sum_{t=0}^{\infty} \hat{p}_t w_t^1$; from **Step 2**, $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^2 \ge \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t w_t^2$, then

$$\sum_{i=1,2} \sum_{t=0}^{\infty} \hat{p}_t w_t^i < \sum_{i=1,2} \sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i$$

Therefore, this Pareto allocation is actually infeasible.

This proof requires **the value of the aggregate endowment is finite**, which is quite intuitive.

PE allocation: Social planner's problem

By **1st welfare theorem**, we can solve the competitive equilibrium allocation by solving Pareto efficient allocation. This is the social planner's problem: a **weighted** utility maximization problem.

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- utility weight: $\left\{\hat{\alpha}^1, \hat{\alpha}^2\right\}$

such that:

- given $\left\{\hat{\alpha}^1, \hat{\alpha}^2\right\}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

$$\max \alpha^1 \sum_{t=0}^{\infty} \beta^t u(c_t^1) + \alpha^2 \sum_{t=0}^{\infty} \beta^t u(c_t^2)$$

  s.t. $c_t^1 + c_t^2 \le w_t^1 + w_t^2, \alpha^1 + \alpha^2 = 1, \alpha^i, c_t^i \ge 0$
- market clearing is the budget constraint.

How to solve:

solve the Lagrangian

$$\alpha^1 \sum_{t=0}^{\infty} \beta^t u(c_t^1) + \alpha^2 \sum_{t=0}^{\infty} \beta^t u(c_t^2) + \sum_{t=0}^{\infty} \mu_t \left(w_t^1 + w_t^2 - c_t^1 - c_t^2\right)$$

FOC gives $\alpha^1 \beta^t u'(c_t^1) = \mu_t = \alpha^2 \beta^t u'(c_t^2) \Rightarrow \alpha^1 u'(c_t^1) = \alpha^2 u'(c_t^2)$, plug them back to $c_t^1 + c_t^2 = w_t^1 + w_t^2$, solve $(c_t^1, c_t^2) = \left(c_t^1(\alpha_1, \alpha_2), c_t^2(\alpha_1, \alpha_2)\right)$.

The Lagrangian multiplier μ_t is the AD equilibrium prices, normalized by the total endowment each period: $\hat{\alpha}^i \beta^t u'(\hat{c}_t^i) = \hat{\mu}_t, \beta^t u'(\hat{c}_t^i) = \hat{\lambda}^i \hat{p}_t \Rightarrow \frac{\hat{\mu}_t}{\hat{\alpha}^i} = \hat{\lambda}^i \hat{p}_t$

All the PE allocations are Pareto efficient, but only one is AD equilibrium allocation, that allocation needs to satisfy the AD budget constraint, achieved by ***transfer***. This procedure, Negishi "trick", follows the second welfare theorem: every Pareto-efficient allocation can be decentralized as an equilibrium with transfers.

The AD equilibrium with transfer is:

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- lifetime transfer: $\left\{\hat{t}^1, \hat{t}^2\right\}$
- regulating mechanism: $\{\hat{p}_t\}_{t=0}^{\infty}$, with numeraire $\hat{p}_0 = 1$

such that:

- given $\{\hat{p}_t\}_{t=0}^{\infty}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i) \text{ s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}^i, c_t^i \ge 0$$

- market clearing: $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, \forall t$

But the solving is actually much easier, we just solve the zero lifetime transfer condition:

$$t^i(\alpha) \equiv \sum_{t=0}^{\infty} \mu_t \left(c_t^i(\alpha) - w_t^i\right) = \sum_{t=0}^{\infty} \alpha^i \beta^t u'(c_t^i) \left(c_t^i(\alpha) - w_t^i\right) = 0$$

In general, transfer function $t^i(\alpha)$ satisfies **zero sum**: $t^1(\alpha) + t^2(\alpha) = 0$; and **homogeneous of degree 1** $t^i(k\alpha) = kt^i(\alpha)$.

Sequential Market (SM) approach

Market structure

Households trade in spot markets for immediate delivery of consumption goods at every t, **bond** is traded, (purchasing at t denoted by a^i_{t+1}), they are traded at t, representing one unit of consumption at $t+1$. The interest rate of bonds r_{t+1} regulates the market: bond of 1 unit of consumption at t will be compensated by $(1+r_{t+1})$ units of consumption at $t+1$.
Households have perfect information of the entire endowment sequence, all information is public.

Equilibrium

- allocation: $\left\{\left\{\tilde{c}^1_t,\tilde{c}^2_t\right\},\left\{\tilde{a}^1_{t+1},\tilde{a}^2_{t+1}\right\}\right\}_{t=0}^{\infty}$
- regulating mechanism: $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$

such that:

- given $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$, $\left\{\left\{\tilde{c}^1_t,\tilde{c}^2_t\right\},\left\{\tilde{a}^1_{t+1},\tilde{a}^2_{t+1}\right\}\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t u(c^i_t)$$

s.t. $c^i_t + \frac{a^i_{t+1}}{1+\tilde{r}_{t+1}} \le w^i_t + a^i_t,\ c^i_t \ge 0,\ a^i_{t+1} \ge -\overline{A}^i > -\infty$

- market clearing (disposal of unused goods is costly):

$$\tilde{c}^1_t + \tilde{c}^2_t = w^1_t + w^2_t,\ \tilde{a}^1_{t+1} + \tilde{a}^2_{t+1} = 0,\ \forall t$$

How to solve:

- *Step 1*: Take advantage the fact that the SM equilibrium allocation $\left\{\tilde{c}^1_t,\tilde{c}^2_t\right\}_{t=0}^{\infty}$ and the AD equilibrium $\left\{\hat{c}^1_t,\hat{c}^2_t\right\}_{t=0}^{\infty}$ are equivalent.
- *Step 2*: Solve the asset holdings $\left\{\tilde{a}^1_t,\tilde{a}^2_t\right\}_{t=0}^{\infty}$ with:

$$\tilde{a}^i_{t+1} = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}^i_{t+\tau} - w^i_{t+\tau}\right)}{\hat{p}_{t+1}}$$

i.e., the asset holding at t is the sum of all future excess demands, discounted by the $t+1$ price.

The existence of a SM equilibrium requires No-Ponzi scheme: $\overline{A}^i < \infty$.

Proof by contradiction: Suppose there is no debt limit.
Without debt limit, agent i can consume more at $t=0$ and keep borrowing to keep the consumption level in the future, formally:

$$c^i_0 = \tilde{c}^i_0 + \frac{\epsilon}{1+\tilde{r}_1},\ c^i_t = \tilde{c}^i_t$$

$$a^i_1 = \tilde{a}^i_1 + \epsilon,\ a^i_{t+1} = \tilde{a}^i_{t+1} - \prod_{\tau=1}^{t}(1+\tilde{r}_{\tau+1})\epsilon$$

This allocation satisfies the budget constraint, and can achieve a strictly higher utility, hence contradicting utility maximization. At the same time, $\prod_{\tau=1}^{t}(1+\tilde{r}_{\tau+1}\epsilon) \xrightarrow{t\to\infty} \infty$, contradicting to the limited resource nature of the economy.

Link AD and SM equilibrium

The link between AD equilibrium and SM equilibrium is built on 2 propositions:

Propositions of AD≡SM

1 For an AD equilibrium allocation $\left\{\hat{c}^1_t,\hat{c}^2_t\right\}_{t=0}^{\infty}$ and prices $\{\hat{p}_t\}_{t=0}^{\infty}$ with $\frac{\hat{p}_{t+1}}{\hat{p}_t} = \xi < 1, \forall t$, then there exists debt limits $\left(\overline{A}^1,\overline{A}^2\right)$ and a corresponding SM equilibrium, with allocation $\left\{\tilde{c}^1_t,\tilde{c}^2_t\right\}_{t=0}^{\infty}$ and interest rates $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$, such that

$$\hat{c}^i_t = \tilde{c}^i_t, \forall i,t$$

2 For a SM equilibrium allocation $\left\{\left\{\tilde{c}^1_t,\tilde{c}^2_t\right\},\left\{\tilde{a}^1_{t+1},\tilde{a}^2_{t+1}\right\}\right\}_{t=0}^{\infty}$ and interest rates $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ where $\tilde{a}^i_t \ge -\overline{A}^i$, $\tilde{r}_{t+1} > 0$, there exists a corresponding AD equilibrium with allocation $\left\{\hat{c}^1_t,\hat{c}^2_t\right\}_{t=0}^{\infty}$ and prices $\{\hat{p}_t\}_{t=0}^{\infty}$ such that

$$\hat{c}^i_t = \tilde{c}^i_t, \forall i,t$$

Proof of Position 2: AM⇐SM

- **Step 1: Construct interest rate**

  Define the SM interest rate as $\frac{1}{1+\tilde{r}_{t+1}} = \frac{\hat{p}_{t+1}}{\hat{p}_t}$, then the SM budget will be $\hat{p}_t\tilde{c}^i_t + \hat{p}_{t+1}\tilde{a}^i_{t+1} = \hat{p}_t w^i_t + \hat{p}_t\tilde{a}^i_t$, iterate this, get $\sum_{t=0}^{\infty}\hat{p}_t\tilde{c}^i_t + \lim_{T\to\infty}\hat{p}_{T+1}\tilde{a}^i_{T+1} = \sum_{t=0}^{\infty}\hat{p}_t w^i_t$, since $\lim_{T\to\infty}\hat{p}_{T+1}\tilde{a}^i_{T+1} = \lim_{T\to\infty}\prod_{\tau=1}^{T+1}\frac{\tilde{a}^i_{T+1}}{1+\tilde{r}^i_\tau} \ge 0$, **SM budget satisfaction leads to AD budget satisfaction**.
- **Step 2: Derive $\tilde{a}^i_{t+1}$**

  By plug in $\frac{1}{1+\tilde{r}_{t+1}} = \frac{\hat{p}_{t+1}}{\hat{p}_t}$ to SM budget constraint, get $\tilde{a}^i_t = c^i_t - w^i_t + \tilde{a}^i_{t+1}\frac{\hat{p}_{t+1}}{\hat{p}_t}$, do an iteration of this equation, derive asset holding as

$$\tilde{a}^i_{t+1} = \sum_{\tau=1}^{\infty}\frac{\hat{p}_{t+\tau}\left(\hat{c}^i_{t+\tau} - w^i_{t+\tau}\right)}{\hat{p}_{t+1}}$$

  plug it in the SM equilibrium budget constraint $\hat{c}^i_t + \frac{\tilde{a}^i_{t+1}}{1+\tilde{r}_{t+1}} = w_t + \tilde{a}^i_t$, the constraint is satisfied.
- **Step 3: Find debt limit $\overline{A}^i$**

  It is easy to show that the asset holding $\tilde{a}^i_{t+1} = \sum_{\tau=1}^{\infty}\frac{\hat{p}_{t+\tau}\left(\hat{c}^i_{t+\tau}-w^i_{t+\tau}\right)}{\hat{p}_{t+1}} \ge -\sum_{\tau=1}^{\infty}\frac{\hat{p}_{t+\tau}w^i_{t+\tau}}{\hat{p}_{t+1}} \ge -\sum_{\tau=1}^{\infty}\xi^{\tau-1}\hat{p}_{t+1} > -\infty$, therefore, households will **never** choose $\tilde{a}^i_{t+1}$ to exceed the debt limit $-\overline{A}^i = -\sup\sum_{\tau=1}^{\infty}\frac{\hat{p}_{t+\tau}}{\hat{p}_{t+1}}w^i_{t+\tau}$. This debt limit can also be derived by setting consumption c^i_t as 0 in SM budget constraint and a backward iteration.
- **Step 4: Check utility maximization**

  This constructed SM allocation must satisfy AD budget as well, and it is an AD equilibrium allocation, hence it is lifetime utility maximizing allocation.

Proof of Position 2: AM⇐SM

- **Step 1: Construct price series**

  From the construction before,

$$\hat{p}_{t+1} = \frac{\hat{p}_t}{1+\tilde{r}_{t+1}}$$

  SM equilibrium consumption satisfies AD budget constraint, market clearing and non-negative consumption condition.
- **Step 2: SM equilibrium allocation is utility maximizing within AD budget**

  Prove by contradiction: suppose there exist an alternative satisfying AD budget and yields a higher utility. Optimization within SM budget has one more constraint than optimization within AD budget: **No Ponzi scheme**.

  Since no Ponzi scheme constraints ever bind in the SM equilibrium, SM equilibrium allocation will be optimizer within AD budget set as well.

Overlapping Generation Model

Features

- **discrete** time, indexed by t
- For each t, a new generation t of identical individuals is born and live for two periods $t, t+1$.
- There is an initial old generation (generation 0), they are born before $t=1$, they can be endowed with m units of **fiat maoney**.
- single commodity exogenously produced, indexed by t, pure **exchange/endowment** economy.
- There is exogenously given **net** asset/liability of the entire private sector

Agents' preferences

utility of a generation t agent is

$$U_t(c^t) = u(c^t_t) + \beta u(c^t_{t+1})$$

utility of the initial old is

$$U_0(c^0) = u(c^0_1)$$

Again, $u(\cdot)$ is assumed to be strictly increasing, strictly concave, twice continuously differentiable, typically satisfying Inada condition and CRRA as well.

AD Equilibrium

- allocation: $\left\{\hat{c}^0_1,\left\{\hat{c}^t_t,\hat{c}^t_{t+1}\right\}_{t=1}^{\infty}\right\}$
- regulating mechanism: $\{\hat{p}_t\}_{t=1}^{\infty}$, with m or p_1 (when $m=0$) as the numeraire.

such that:

- given $\{\hat{p}_t\}_{t=1}^{\infty}$, $\left\{\hat{c}^0_1,\left\{\hat{c}^t_t,\hat{c}^t_{t+1}\right\}_{t=1}^{\infty}\right\}$ solves:

$$\max_{c^t\ge 0} u(c^t_t) + \beta u(c^t_{t+1}) \text{ s.t. } \hat{p}_t c^t_t + \hat{p}_{t+1}c^t_{t+1} \le \hat{p}_t w^t_t + \hat{p}_{t+1}w^t_{t+1}$$

$$\max_{c^0_1\ge 0} u(c^0_1) \text{ s.t. } \hat{p}_1 c^0_1 \le \hat{p}_1 w^0_1 + m$$

- market clearing (disposal of unused goods is costly):

$$\hat{c}^{t-1}_t + \hat{c}^t_t = w^{t-1}_t + w^t_t, \forall t \ge 1$$

How to solve:

- *Step 1*: solve the Lagrangian

$$u_t(c^t_t) + \beta u_{t+1}(c^t_{t+1}) + \lambda^t\left(p_t w^t_t + p_{t+1}w^t_{t+1} - p_t c^t_t - p_{t+1}c^t_{t+1}\right)$$

FOCs: $\frac{u'_t(c^t_t)}{p_t} = \lambda^t = \frac{\beta u'_{t+1}(c^t_{t+1})}{p_{t+1}} \Rightarrow \frac{p_{t+1}}{p_t} = \frac{\beta u'_{t+1}(c^t_{t+1})}{u'_t(c^t_t)} \Rightarrow c^t_t = f(c^t_{t+1}, w^t_t, w^t_{t+1})$. FOC gives that price changes p_t/p_{t+1} and subjective discounting β determines consumption smoothing.
- *Step 2*: Solve the initial old's Lagrangian:

$$u(c^0_1) + \lambda^0\left(w^0_1 + m - c^0_1\right)$$

  get the initial old's consumption $\hat{c}^0_1 = w^0_1 + m$.
- *Step 3*: Plug $\hat{c}^0_1$ into the market clearing condition $c^{t-1}_t + c^t_t = w^{t-1}_t + w^t_t$, get

$$\hat{c}^1_1 = w^0_1 + w^1_1 - \hat{c}^0_1 = w^1_1 - m$$

- *Step 4*: Plug $\hat{c}^1_1 = w^1_1 - m$ and the FOC back to the budget constraint

$$\hat{p}_1 c^1_1 + \hat{p}_2 c^1_2 = \hat{p}_1 w^1_1 + \hat{p}_2 w^1_2$$

  get $\hat{c}^1_2$. Iterate this process forward, solve the allocation $\left\{\hat{c}^0_1,\left\{\hat{c}^t_t,\hat{c}^t_{t+1}\right\}_{t=1}^{\infty}\right\}$ and price stream $\{\hat{p}_t\}_{t=1}^{\infty}$.

SM Equilibrium

- allocation: $\left\{\tilde{c}^0_1,\left\{\tilde{c}^t_t,\tilde{c}^t_{t+1},\tilde{s}^t_t\right\}_{t=1}^{\infty}\right\}$
- regulating mechanism: interest rates $\{\tilde{r}_t\}_{t=1}^{\infty}$

such that:

- given $\{\tilde{r}_{t+1}\}_{t=1}^{\infty}, \forall t > 1$, $\left\{\tilde{c}^t_t,\tilde{c}^t_{t+1}\right\}_{t=1}^{\infty}$ solves:

$$\max_{c^t\ge 0} u(c^t_t) + \beta u(c^t_{t+1}) \text{ s.t. } c^t_t + s^t_t \le w^t_t,$$
$$c^t_{t+1} \le w^t_{t+1} + (1+\tilde{r}_{t+1})s^t_t$$
$$\max_{c^0_1\ge 0} u(c^0_1) \text{ s.t. } c^0_1 \le w^0_1 + m(1+\tilde{r}_1)$$

- good market clearing (disposal of unused goods is costly):

$$\tilde{c}^{t-1}_t + \tilde{c}^t_t = w^{t-1}_t + w^t_t, \forall t \ge 1$$

- asset market clearing: the budget constraint gives

$$\tilde{c}^{t+1}_{t+1} + \tilde{c}^t_{t+1} + s^{t+1}_{t+1} = w^{t+1}_{t+1} + w^t_{t+1} + (1+\tilde{r}_{t+1})s^t_t$$

  plug the good market clearing condition, get

$$s^{t+1}_{t+1} = (1+\tilde{r}_{t+1})s^t_t$$

  iterate this backwards to $s^0_0 = m$, get

$$s^t_t = \prod_{\tau=1}^{t}(1+\tilde{r}_\tau)m$$

How to solve:

- *Step 1*: solve the Lagrangian

$$u_t(c^t_t) + \beta u_{t+1}(c^t_{t+1}) + \mu^t_t\left(w^t_t - c^t_t - s^t_t\right)$$
$$+ \mu^t_{t+1}\left(w^t_t + (1+\tilde{r}_{t+1})s^t_t - c^t_{t+1}\right)$$

FOCs: $u'_t(c^t_t) = \mu^t = \beta u'_{t+1}(c^t_{t+1})(1+\tilde{r}_{t+1}) \Rightarrow \tilde{r}_{t+1} = \frac{u'_t(c^t_t)}{\beta u'_{t+1}(c^t_{t+1})} - 1 \Rightarrow c^t_t = f(c^t_{t+1}, w^t_t, w^t_{t+1})$.

- *Step 2*: Solve the initial old's Lagrangian:

$$u(c_1^0)+\lambda^0\left(w_1^0+m(1+r_1)-c_1^0\right)$$

get the initial old's consumption $\hat{c}_1^0 = w_1^0 + m(1+\tilde{r}_{t+1})$.
- *Step 3*: Plug $\hat{c}_1^0$ into the market clearing condition $c_t^{t-1}+c_t^t = w_t^{t-1}+w_t^t$, get

$$\hat{c}_1^1 = w_1^0+w_1^1-\hat{c}_1^0 = w_1^1 - m(1+\tilde{r}_{t+1})$$

- *Step 4*: Plug $\hat{c}_1^1 = w_1^1 - m(1+\tilde{r}_{t+1})$ and the FOC back to the budget constraint

$$w_t^t + \frac{1}{1+\tilde{r}_{t+1}} w_{t+1}^t = c_t^t + \frac{1}{1+\tilde{r}_{t+1}} c_{t+1}^t$$

get $\hat{c}_2^1$. Iterate this process forward, solve the allocation $\left\{\hat{c}_1^0, \left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^\infty\right\}$ and price stream $\{\tilde{r}_{t+1}\}_{t=1}^\infty$.

Propositions of AD≡SM

1 For an AD equilibrium allocation $\left\{\hat{c}_1^0, \left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^\infty\right\}$ and prices $\{\hat{p}_t\}_{t=1}^\infty$ with $\hat{p}_t > 0$, then there exists a corresponding SM equilibrium, with allocation $\left\{\tilde{c}_1^0, \left\{\tilde{c}_t^t, \tilde{c}_{t+1}^t\right\}_{t=1}^\infty\right\}$ and interest rates $\{\tilde{r}_{t+1}\}_{t=0}^\infty$, such that

$$\tilde{c}_t^{t-1} = \hat{c}_t^{t-1}, \tilde{c}_t^t = \hat{c}_t^t \forall t \geq 1$$

2 For a SM equilibrium allocation $\left\{\tilde{c}_1^0, \left\{\tilde{c}_t^t, \tilde{c}_{t+1}^t\right\}_{t=1}^\infty\right\}$ and interest rates $\{\tilde{r}_{t+1}\}_{t=0}^\infty$ where $\tilde{r}_{t+1} > -1$, there exists a corresponding AD equilibrium with allocation $\left\{\hat{c}_1^0, \left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^\infty\right\}$ and prices $\{\hat{p}_t\}_{t=1}^\infty$ such that

$$\hat{c}_t^{t-1} = \tilde{c}_t^{t-1}, \hat{c}_t^t = \tilde{c}_t^t \forall t \geq 1$$

The interest rate and price stream are still inter-determined:

$$\frac{1}{1+\tilde{r}_{t+1}} = \frac{\hat{p}_{t+1}}{\hat{p}_t}, \quad \frac{1}{1+\tilde{r}_1} = \hat{p}_1$$

The two Euler equations are:

$$u_t'(c_t^t) = \beta u_{t+1}'(c_{t+1}^t)(\hat{p}_t/\hat{p}_{t+1}) \cdots \text{AD}$$
$$u_t'(c_t^t) = \beta u_{t+1}'(c_{t+1}^t)(1+\tilde{r}_{t+1}) \cdots \text{SM}$$

An easy proof:
- ***Prop. 1***: AD equilibrium allocation satisfies SM FOC and the SM budget constraints:

$$\tilde{\mu}_t^t = \hat{\lambda}\hat{p}_t, \tilde{\mu}_{t+1}^t = \hat{\lambda}\hat{p}_{t+1}, \forall t \geq 0; \ \tilde{s}_t^t = w_t^t - \hat{c}_t^t, \forall t \geq 1$$

- ***Prop. 2***: SM equilibrium allocation satisfies AD FOC and the AD budget constraints:

$$\hat{\lambda}^t = \frac{\tilde{\mu}_t^t}{\prod_{\tau=0}^{t-1}(1+\tilde{r}_{t-\tau})} = \frac{\tilde{\mu}_{t+1}^t}{\prod_{\tau=0}^{t-1}(1+\tilde{r}_{t+1-\tau})}, \forall t \geq 1$$

$$\hat{\lambda}^0 = \tilde{\mu}_1^0(1+\tilde{r}_1)$$

Offer curve
Assume the economy is stationary, the endowments at each period of life by successive generations are constant:

$$w_t^t = w_1, w_{t+1}^t = w_2, \forall t \geq 1; \ w_1^0 = w_2$$

By the Lagrangian of AD equilibrium:

$$u'(c_t^t) = \lambda^t p_t$$
$$\beta u'(c_{t+1}^t) = \lambda^t p_{t+1}$$

solve c_t^t, c_{t+1}^t as functions of p_t, p_{t+1}, get $c_t^t(p_t, p_{t+1}), c_{t+1}^t(p_t, p_{t+1})$, then define y as the excess demand at t, define z as the excess demand at $t+1$:

$$y(p_t, p_{t+1}) = c_t^t(p_t, p_{t+1}) - w_1$$
$$z(p_t, p_{t+1}) = c_{t+1}^t(p_t, p_{t+1}) - w_2$$

The excess demand at $t=0$ is

$$z(m, p_1) = c_1^0 - w_2 = \frac{m}{p_1}$$

Features of offer curve

By the AD budget constraint: $p_t c_t^t + p_{t+1}c_{t+1}^t = p_t w_t + p_{t+1} w_{t+1}^t$, this gives

$$p_t y(p_t, p_{t+1}) + p_{t+1} z(p_t, p_{t+1}) = 0$$

leading to

$$\frac{z(p_t, p_{t+1})}{y(p_t, p_{t+1})} = -\frac{p_t}{p_{t+1}}, \forall t \geq 1$$

Notice that $y(p_t, p_{t+1})$ and $z(p_t, p_{t+1})$ are both functions of $\frac{p_t}{p_{t+1}}$, this way, we can replace $\frac{p_t}{p_{t+1}}$ as a function of y, plug this $\frac{p_t}{p_{t+1}} = g(y)$ into $z(p_t, p_{t+1})$, we will have the offer curve

$$z = f(y)$$

Properties of offer curve:
- The offer curve is bounded by endowment:

$$y(p_t, p_{t+1}) \geq -w_1, z(p_t, p_{t+1}) \geq -w_2$$

- The curve is in the 2^{nd} and 4^{th} Quadrant:

$$y(p_t, p_{t+1}) \cdot z(p_t, p_{t+1}) < 0$$

- The origin $(0,0)$ is on the offer curve: $z^*(p_t, p_{t+1}) = y^*(p_t, p_{t+1}) = 0$.
- for $y(p_t, p_{t+1}) \neq 0$, AD budget constraint is always satisfied:

$$\frac{z(p_t, p_{t+1})}{y(p_t, p_{t+1})} = -\frac{p_t}{p_{t+1}}, \forall t \geq 1$$

hence the slope of the straight line connecting each point on the offer curve to the origin determines the price ratio $\frac{p_t}{p_{t+1}}$

Besides, we also have a market clearing curve:

$$y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0, \ \forall t > 1$$
$$y(p_1, p_2) + z(m, p_1) = 0, \ t = 1$$

this is a 45-degree line through the 2^{nd} and 4^{th} quadrants.

With offer curve and market clearing curve, we can determine the **entire sequence of excess demands** of young and old at every date as following:

- ***Step 1***: for a given m, at $t=1$, the excess demand of the initial old $z^0 = z^0(p_1, m) = m/p_1$
- ***Step 2***: the initial young's excess demand at $t=1$ is determined by the market clearing curve $y^1(p_1, p_2) = -z^0$
- ***Step 3***: y^1 will determine $z^1(p_1, p_2)$ on the offer curve.
- ***Step 4***: repeat this **market clearing curve-offer curve** procedure.

In this procedure, the initial price p_1 must be picked first. If $m \neq 0$, different p_1 can index a continuum of equilibria.

Slope of offer curve

First, we need the slope of the offer curve to be convex:

$$\frac{\partial c_t^t}{\partial (p_t/p_{t+1})} = \frac{\partial c_t^t}{\partial (1+r_{t+1})} < 0$$

- graphically, $-\frac{p_t}{p_{t+1}}$ is the slope of the line connecting a point on the offer curve to the origin, $c_t^t - w_t^t = y(p_t, p_{t+1})$ is the x-axis, hence $\frac{\partial c_t^t}{\partial (p_t/p_{t+1})} < 0$ means that as c_t^t increases, the slope of the connecting line is increasing (less negative, or shallower).
- economically, if interest rates increasing, young period saving will increase, consumption will decrease (substitution effect).

A very important object: <u>Autarkic interest rate $\bar{r}$</u>:
If we assume stationary endowment: $w_t^t = w1, w_{t+1}^t = w_2, \forall t \geq 1; w_1^0 = w_2$, we will the autarkic interest rate as:

$$1+\bar{r} = \frac{\hat{p}_t}{\hat{p}_{t+1}} = \frac{u_t'(w_1)}{\beta u_{t+1}'(w_2)}$$

where
- $\hat{p}_t, \hat{p}_{t+1}$ are the autarky equilibrium price.
- $\bar{r}$ is determined by the relative size of endowment w_1, w_2
- $\bar{r}$ determines the general shape of the offer curve: $1+\bar{r}$ it is the **<u>negative of the slop of the offer curve at the origin</u>**.

Different cases of OLG model
In the overlapping generation economy:
- m and p_1

  m determines the equilibrium sequences, the initial price level p_1 determines the starting point of an equilibrium sequence hence indexes a continuum of equilibria. There is a special initial price p_1^* such that m/p_1^* is exactly **the intersection point** of the offer curve and the market clearing curve. This will be the **stationary monetary equilibrium**.
- $1+\bar{r} = \hat{p}_t/\hat{p}_{t+1} = u_t'(w_1)/\beta u_{t+1}'(w_2)$

  relative size of endowments w_1, w_2 determines autarkic interest rate $1+\bar{r}$, which determines the shape of the offer curve, and the Pareto efficiency and the stationary of the equilibria.

Now we discuss the different cases:

Knife-edge economy $1+\bar{r} = \hat{p}_t/\hat{p}_{t+1} = u_t'(w_1)/\beta u_{t+1}'(w_2) = 1$

- shape of offer curve: the offer curve is tangent to the market clearing line **at the origin**
- equilibrium:
  - the only stationary equilibrium is the origin allocation, autarky. It is Pareto efficiency.
  - there is NO stationary monetary equilibria, i.e., money is not valued; But $\forall m < 0$, there is a continuum of monetary equilibria, converging to autarky.

Classical economy $1+\bar{r} = \hat{p}_t/\hat{p}_{t+1} = u_t'(w_1)/\beta u_{t+1}'(w_2) > 1$

In this economy, the **old** generation is richer: $w_2 > w_1$, since $u' > 0, u'' < 0, u'(w_1) > u'(w_2) > 0$
- shape of offer curve: the slope of the offer curve is **steeper** than the 45-degree market clearing line **at the origin**.
- two intersections: the origin and a point in the **fourth** quadrant $m/p_1^*, m < 0$
- equilibrium:
  - autarky is a stationary equilibrium, and it is **Pareto efficient**.
  - there is a stationary monetary equilibria at the other intersection point m/p_1^*. It is **Pareto efficient**. In this equilibrium, $m < 0$, the initial old has a debt, the economy has a net debt.
  - there is a continuum of dynamic monetary equilibria with $p_1 > p_1^*$. For all of them, $m < 0$. They all converge to the monetary stationary equilibrium.
- AD prices: $\hat{p}_t$ decreases over time.

Samuelson economy $1+\bar{r} = \hat{p}_t/\hat{p}_{t+1} = u_t'(w_1)/\beta u_{t+1}'(w_2) < 1$

In this economy, the **young** generation is richer: $w_1 > w_2$, since $u' > 0, u'' < 0, u'(w_2) > u'(w_1) > 0$
- shape of offer curve: the slope of the offer curve is **shallower** than the 45-degree market clearing line **at the origin**.
- two intersections: the origin and a point in the **second** quadrant $m/p_1^*, m > 0$
- equilibrium:
  - autarky is a stationary equilibrium, and it is **NOT** Pareto efficient.
  - there is a stationary monetary equilibria at the other intersection point m/p_1^*. In this equilibrium, $m > 0$, the initial old has positive fiat money.
  - there are two continuum of dynamic monetary equilibria:
    - $m > 0$: the continuum is between autarky and stationary monetary equilibrium, with $p_1 > p_1^*$, they all converge to autarky.
    - $m < 0$: every possible (m, p_1) is a dynamic monetary equilibrium, they all converge to autarky.
- AD prices: $\hat{p}_t$ increases over time.

Pareto efficiency of the equilibria
- **<u>Autarky equilibrium</u>**
  - In *knife-edge economy*, **Autarky is PE**: young and old have the same marginal utility. To compensate old by transferring from young, the utility loss of transferring in young cannot be compensated by the compensation in old since utility is concave.
  - In *classical economy*, **Autarky is PE**: young are endowed less, have a higher marginal utility. The utility loss of young from transferring to old can not be compensated by receiving that amount when old, again, it is due to the concavity of utility.
  - In *Samuelson economy*, **Autarky is NOT PE**: young are endowed more, have a lower marginal utility. The utility loss of young from transferring to old actually can be compensated by receiving that amount when old, it is also due to the concavity of utility.
- **<u>Monetary stationary equilibrium</u>**
  - In *Samuelson economy*, **MSE is PE** and **Pareto dominating Autarky**: the initial old, with positive fiat money, are strictly better off, and as well off as they can be in equilibrium for any $m > 0$. For generation $t \geq 1$, MSE allocation is identical to the steady-state utility maximizing allocation, hence every generation is at least as good as in autarky.
  - In *classical economy*, **MSE is PE** but **NOT Pareto dominating Autarky**, MSE allocation is also steady-state utility maximizing, hence, it is Pareto efficient, but Autarky can not be Pareto improved either.
- **<u>Monetary non-stationary equilibria</u>**

- In *Samuelson economy*, **MNSE is not PE**. All the non-stationary equilibria converge to Autarky, and features **rising prices**, hence $\lim_{t\to\infty} p_t = \infty$. All of these non-stationary equilibria are arbitrarily close to each other, they all converge to Autarky, which has a positive, constant price inflation $\hat{p}_t/\hat{p}_{t+1} < 1$, hence a negative autarky interest rate $\overline{r} < 0$, this means NOT PE.
- In *classical economy*, **every MNSE is PE**. All the non-stationary equilibria converge to the monetary stationary equilibrium, which has zero inflation, zero interest rate, hence they are all PE.
- In **knife-edge economy**, the conclusion is the same as the classical economy.

A general theoretical result is given by Balasko and Shell:

Balasko and Shell Pareto-efficiency condition

Assume:
- stationary endowment $w_t^t = w_1 > 0, w_{t+1}^t = w_1^0 = w_2 > 0$
- allocation is bounded away from zero: $\left(\hat{c}_t^{t-1}, \hat{c}_t^t\right) \geq \delta > 0$

Define

$$\frac{1}{1+r_{t+1}} = \frac{\beta U'\left(\hat{c}_{t+1}^t\right)}{U'\left(\hat{c}_t^t\right)} = \frac{\hat{p}_{t+1}}{\hat{p}_t}$$

Then the allocation is Pareto efficient if and only if

$$\sum_{t=1}^{\infty} \prod_{\tau}^{t} (1 + r_{\tau+1}) = +\infty$$

This <u>**includes**</u> two scenarios (PE):
- AD prices falling, i.e., positive interest rate r_{t+1}
  - <u>classic Autarky</u>
- AD prices constant, i.e., zero interest rate r_{t+1}
  - <u>knife-edge Autarky</u>, <u>all knife-edge MNSE</u> (converge to Autarky)
  - <u>classic MSE</u>, <u>all classic MNSE</u> (converge to classic MSE)
  - <u>Samuelson MSE</u>

This <u>**excludes**</u> one scenario (not PE):
- AD prices increasing, i.e., negative interest rate r_{t+1}
  - <u>Samuelson Autarky</u>, <u>all Samuelson MNSE</u> (converge to Samuelson Autarky).

Neoclassical growth model

Features

- **discrete** time, indexed by t
- economy, households, firms live **infinitely**
- single commodity endogenously produced by firms, consumed and invested by households.
- there is a continuum of identical <u>**households**</u>, they are **price takers**, they maximize their lifetime utility, can be represented by a **representative** household.
- there is a continuum of identical <u>**firms**</u>, they are perfectly competitive, maximize their lifetime profits, can be represented by a representative household.

Agents' endowment

households are endowed with
- 1 unit of time, can be used for labor or leisure $n_t + l_t = 1$
- initial capital stock $\overline{k}_0 > 0$
- **NO goods**: all final goods are produced endogenously.

households and firms meet in markets and trade:
- households sell capital and labor, earn rental and wage
- firms sell goods, earn **zero** profit

And, Households are assumed to own firms, hence own profit.

Agents' preferences and equilibrium behavior

household: $\max U(c) = \sum_{t=0}^{\infty} \beta^t [u(c_t) + \psi(l_t)]$
firms: $\max F(k_t, n_t) - r_t k_t - w_t n_t$ where $F(k_t, n_t) = y_t = c_t + i_t$

assumptions of $u(\cdot)$ and $F(\cdot)$	
$u(c_t)$:	continuously differentiable strictly increasing and concave satisfying Inada condition time separable, $\beta \in (0,1)$
$\psi(l_t)$:	value of leisure
$F(k_t, n_t)$:	continuously differentiable strictly increasing and concave both F_k and F_n satisfying Inada condition homogeneous of degree 1, constant return to scale

Households and firms are perfectly competitive: they are price takers, earn zero economic profit, and achieve a perfectly **competitive equilibrium environment**.

Agents' technology

households: households are endowed with initial capital k_0 and produce new capital via:

$$k_{t+1} = (1-\delta)k_t + i_t$$

where δ is the depreciation ratio $\delta \in [0,1]$

firms: the most common production functions is the family of the constant elasticity of substitution (CES) function:

$$y_t = F(k_t, n_t) = A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}}$$

it has the following properties:

Properties of CES function:
- Convergence:
  - $\gamma \to 1$: converges to Cobb-Douglas function
  - $\gamma \to 0$: converges to Leontief function
- constant return to scale
- positive and diminishing marginal productivity
- inputs are complements (cross partial derivatives > 0)

Here is the proof of the properties of CES functions:
- $\gamma \to 1$ leading to Cobb-Douglas:
  - **L'Hopital rule:** rewrite the function as

$$\ln y_t = \ln A + \frac{\ln\left[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right] \xrightarrow{\gamma\to 1} 0}{1 - \frac{1}{\gamma} \xrightarrow{\gamma\to 1} 0}$$

$$\text{(L'Hopital)} \to \ln A + \frac{\alpha \ln k_t \cdot k_t^{1-\frac{1}{\gamma}} + (1-\alpha)\ln n_t \cdot n_t^{1-\frac{1}{\gamma}}}{\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right)}$$

$$\Rightarrow \lim_{\gamma\to 1} \ln y_t = \ln A + \alpha \ln k_t + (1-\alpha)\ln n_t$$

$$\Rightarrow \lim_{\gamma\to 1} y_t = e^{\ln A + \alpha \ln k_t + (1-\alpha)\ln n_t} = A k_t^{\alpha} n_t^{1-\alpha}$$

  - **Total differentials:** rewrite the function as

$$y_t^{1-\frac{1}{\gamma}} = A^{1-\frac{1}{\gamma}}\left[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right]$$

take total differentials, get:

$$\begin{aligned}
\left(1-\frac{1}{\gamma}\right) y_t^{-\frac{1}{\gamma}} \mathrm{d}y_t =& A^{1-\frac{1}{\gamma}}\left(\left(1-\frac{1}{\gamma}\right)\alpha k_t^{-\frac{1}{\gamma}} \mathrm{d}k_t \right.\\
&\left. + \left(1-\frac{1}{\gamma}\right)(1-\alpha) n_t^{-\frac{1}{\gamma}} \mathrm{d}n_t\right)\\
\Rightarrow y_t^{-\frac{1}{\gamma}} \mathrm{d}y_t =& A^{1-\frac{1}{\gamma}}\left(\alpha k_t^{-\frac{1}{\gamma}} \mathrm{d}k_t + (1-\alpha) n_t^{-\frac{1}{\gamma}} \mathrm{d}n_t\right)\\
\xRightarrow{\gamma\to 1} \frac{1}{y_t}\mathrm{d}y_t =& \left(\alpha \frac{1}{k_t}\mathrm{d}k_t + (1-\alpha)\frac{1}{n_t}\mathrm{d}n_t\right)\\
\overset{\int}{\Rightarrow} \ln y_t + c_y =& [\alpha(\ln k_t + c_k) + (1-\alpha)(\ln n_t + c_n)]\\
\Rightarrow y_t =& e^{\alpha c_k + (1-\alpha)c_n - c_y} k_t^{\alpha} n_t^{1-\alpha} = \tilde{A} k_t^{\alpha} n_t^{1-\alpha}
\end{aligned}$$

  - **Taylor expansion:** expand $Q(k_t, n_t) \equiv \alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}$ at $\gamma = 1$, get:

$$\begin{aligned}
Q(k_t, n_t) =& \alpha k_t^{1-\frac{1}{\gamma}}\Big|_{\gamma=1} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\Big|_{\gamma=1}\\
&+ \alpha \ln k_t \cdot k_t^{1-\frac{1}{\gamma}}\Big|_{\gamma=1} \cdot (\gamma - 1)\\
&+ (1-\alpha)\ln n_t \cdot n_t^{1-\frac{1}{\gamma}}\Big|_{\gamma=1} \cdot (\gamma-1) + O\left((\gamma-1)^2\right)\\
=& 1 + (\gamma-1)\ln k_t^{\alpha} n_t^{1-\alpha} + O\left((\gamma-1)^2\right)
\end{aligned}$$

plug $Q(k_t, n_t)$ into y_t, get

$$\begin{aligned}
y_t =& A \cdot Q(k_t, n_t)^{\frac{1}{1-\frac{1}{\gamma}}}\\
=& A\left[1 + (\gamma-1)\ln k_t^{\alpha} n_t^{1-\alpha} + O\left((\gamma-1)^2\right)\right]^{\frac{1}{1-\frac{1}{\gamma}}}\\
\xRightarrow{r=\frac{1}{\gamma-1}} =& A\left[1 + \frac{1}{r}\ln k_t^{\alpha} n_t^{1-\alpha} + O\left(\frac{1}{r^2}\right)\right]^{\gamma r}
\end{aligned}$$

since $\gamma \to 1 \Rightarrow r \to \infty$, we have

$$\begin{aligned}
\lim_{\gamma\to 1} y_t =& \lim_{\gamma\to 1} A\left[1 + \frac{\ln k_t^{\alpha} n_t^{1-\alpha}}{r} + O\left(\frac{1}{r^2}\right)\right]^{\gamma r}\\
=& \lim_{\gamma\to 1} A\left[1 + \frac{\ln k_t^{\alpha} n_t^{1-\alpha}}{r}\right]^{\gamma r}\\
\xRightarrow{(1+\frac{x}{\infty})^{\infty}\to e^x} =& \left(e^{\ln k_t^{\alpha} n_t^{1-\alpha}}\right)^{\gamma} = k_t^{\alpha} n_t^{1-\alpha}
\end{aligned}$$

- $\gamma \to 0$ leading to Leontief:
  - **L'Hopital rule:** rewrite the function as

$$\ln y_t = \ln A + \frac{\ln\left[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right] \xrightarrow{\gamma\to 0} -\infty}{1 - \frac{1}{\gamma} \xrightarrow{\gamma\to 0} -\infty}$$

$$\text{(L'Hopital)} \to \ln A + \frac{\alpha \ln k_t \cdot k_t^{1-\frac{1}{\gamma}} + (1-\alpha)\ln n_t \cdot n_t^{1-\frac{1}{\gamma}} \to 0}{\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right) \to 0}$$

Here, we have a $\frac{0}{0}$ limit. Using L'Hopital rule again yields NOTHING. To proceed, define $x_t = \min\{k_t, n_t\}$, then we have

$$\begin{aligned}
\lim_{\gamma\to 0} \ln y_t =& \ln A + \frac{\alpha \ln k_t \cdot \left(\frac{k_t}{x_t}\right)^{1-\frac{1}{\gamma}} + (1-\alpha)\ln n_t \cdot \left(\frac{n_t}{x_t}\right)^{1-\frac{1}{\gamma}}}{\left(\alpha\left(\frac{k_t}{x_t}\right)^{1-\frac{1}{\gamma}} + (1-\alpha)\left(\frac{n_t}{x_t}\right)^{1-\frac{1}{\gamma}}\right)}\\
=& \ln A + \begin{cases} \ln k_t, & k_t < n_t \\ \ln n_t, & k_t > n_t \\ \ln \alpha \ln k_t + (1-\alpha)\ln n_t, & k_t = n_t \end{cases}\\
=& \ln A + \ln(\min\{n_t, k_t\}) \Rightarrow \lim_{\gamma\to 0} y_t = A\min\{n_t, k_t\}
\end{aligned}$$

  - **Sandwich theorem:** without losing generality, assume $k_t \geq n_t > 0$, consider this inequality:

$$\begin{aligned}
&(1-\alpha) n_t^{1-\frac{1}{\gamma}} \leq \alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}} \leq n_t^{1-\frac{1}{\gamma}}\\
\Rightarrow& (1-\alpha)^{\frac{1}{1-\frac{1}{\gamma}}} n_t \leq \left[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right]^{\frac{1}{1-\frac{1}{\gamma}}} \leq n_t
\end{aligned}$$

notice that

$$\lim_{\gamma\to 0} (1-\alpha)^{\frac{1}{1-\frac{1}{\gamma}}} n_t = n_t$$

then by the Sandwich Theorem, we have

$$\lim_{\gamma\to 0}\left[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right]^{\frac{1}{1-\frac{1}{\gamma}}} = n_t$$

therefore

$$\begin{aligned}
\lim_{\gamma\to 0} y_t =& \lim_{\gamma\to 0} A\left[\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right]^{\frac{1}{1-\frac{1}{\gamma}}}\\
=& A n_t = A\min\{k_t, n_t\}
\end{aligned}$$

- Constant return to scale: $F(\lambda k_t, \lambda n_t) = \lambda F(k_t, n_t)$

  It is very easy to verify directly. Another way to verify this is use Euler's theorem, constant return to scale means homogeneity of degree 1, hence Euler's theorem requires $F(k_t, n_t) = F_k(k_t, n_t)k_t + F_n(k_t, n_t)n_t$, that is:

$$\begin{aligned}
&k_t F_k(k_t, n_t) + n_t F_n(k_t, n_t)\\
=& k_t A \frac{1}{1-\frac{1}{\gamma}} \cdot \left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{\gamma-1}} \alpha\left(1-\frac{1}{\gamma}\right) k_t^{-\frac{1}{\gamma}}\\
&+ n_t A \frac{1}{1-\frac{1}{\gamma}} \cdot \left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{\gamma-1}} (1-\alpha)\left(1-\frac{1}{\gamma}\right) n_t^{-\frac{1}{\gamma}}\\
=& A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right) \cdot \left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{\gamma-1}}\\
=& A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}} = F(k_t, n_t)
\end{aligned}$$

- positive and diminishing marginal productivity.

  Again, fairly straightforward, we verity $F_k(k_t, n_t) > 0, F_n(k_t, n_t) > 0; F_{kk}(k_t, n_t) < 0, F_{nn}(k_t, n_t) < 0$:

- F_k and F_n: take partial derivatives w.r.t. k_t, get

$$F_k = A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}-1} \cdot \alpha k_t^{-\frac{1}{\gamma}} \geq 0$$

for n_t, similarly

$$F_n = A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}-1} \cdot (1-\alpha) n_t^{-\frac{1}{\gamma}} \geq 0$$

- F_{kk} and F_{nn}: take twice partial derivatives w.r.t. k_t, for simplicity, again let $Q_t \equiv \alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}$, get

$$\begin{aligned}
F_{kk} =& AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-1-1} \cdot \left(\frac{1}{1-\frac{1}{\gamma}}-1\right)\cdot\left(1-\frac{1}{\gamma}\right)\alpha k_t^{-\frac{1}{\gamma}} \cdot \alpha k_t^{-\frac{1}{\gamma}} \\
&+ \alpha\left(-\frac{1}{\gamma}\right) k_t^{-\frac{1}{\gamma}-1} \cdot AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-1} \\
=& AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-1-1} \alpha \frac{1}{\gamma} \cdot \alpha k_t^{-\frac{2}{\gamma}} - \alpha\frac{1}{\gamma} k_t^{-\frac{1}{\gamma}-1} AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-1} \\
=& \alpha\frac{1}{\gamma} k_t^{-\frac{1}{\gamma}-1} AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-1} \cdot \left(Q_t^{-1}\alpha k_t^{1-\frac{1}{\gamma}} - 1\right)
\end{aligned}$$

since $\alpha k_t^{1-\frac{1}{\gamma}} Q_t^{-1} = \frac{\alpha k_t^{1-\frac{1}{\gamma}}}{\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}} < 1$, $F_{kk} < 0$. Similarly for F_{nn}:

$$F_{nn} = (1-\alpha)\frac{1}{\gamma} n_t^{-\frac{1}{\gamma}-1} AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-1} \cdot \left(Q_t^{-1}(1-\alpha)n_t^{1-\frac{1}{\gamma}} - 1\right) < 0$$

- $F_{kn} = F_{nk} > 0$: by the symmetry of Hessian matrix, we only need to verifyt F_{kn}, that is

$$\begin{aligned}
F_{kn} =& AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-1-1}\left(\frac{1}{1-\frac{1}{\gamma}}-1\right)\alpha k_t^{-\frac{1}{\gamma}} \cdot (1-\alpha)(1-\frac{1}{\gamma}) n_t^{-\frac{1}{\gamma}} \\
=& AQ_t^{\frac{1}{1-\frac{1}{\gamma}}-2} \frac{1}{\gamma}\alpha(1-\alpha)(k_t n_t)^{-\frac{1}{\gamma}} > 0
\end{aligned}$$

How to solve the model: social planner's problem
This model can be analytically solved only when certain assumptions are imposed on utility functions and production functions. However, there exists a solution: capital must be cleared eventually, that is $k_{T+1} = 0$; and constraint set for k_{t+1} is compact.

Social planner's problem

$$\max_{\{c_t, i_t, k_{t+1}, l_t, n_t, y_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (u(c_t) + \psi(l_t))$$

s.t.

$$\begin{aligned}
y_t &= c_t + i_t && \text{market clearing} \\
k_{t+1} &= (1-\delta)k_t + i_t && \text{capital accumulation} \\
1 &= n_t + l_t && \text{time endowment} \\
y_t &= A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}} && \text{production function} \\
0 &\leq c_t, i_t, k_t, n_t, l_t, y_t, \forall t && \text{non-negativity}
\end{aligned}$$

This problem can be rewritten as:

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (u(c_t) + \psi(1-n_t))$$

s.t.

$$k_{t+1} = (1-\delta)k_t + A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}} - c_t$$

and $c_t, k_t, n_t \geq 0, \forall t$. Then, we can derive the Euler equation with Lagrange method:

FOC of social planner's problem

$$\mathcal{L} = \sum_{t=0}^{\infty}\left[\beta^t (u(c_t) + \psi(1-n_t)) + \lambda_t ((1-\delta)k_t + F(k_t, n_t) - k_{t+1} - c_t)\right]$$

FOC gives:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t u'(c_t) = \lambda_t \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow \beta^t \psi'(1-n_t) = \lambda_t F_n(k_t, n_t) \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Rightarrow \lambda_{t+1}(F_k(k_{t+1}, n_{t+1}) + 1 - \delta) = \lambda_t \qquad (3)$$

(1) and (3) together, gives the Euler equation:

$$u'(c_t) = (F_k(k_{t+1}, n_{t+1}) + 1 - \delta)\beta u'(c_{t+1})$$

Interpretation: today's consumption lost must be compensated by discounted tomorrow's consumption gain.

- $u'(c_t)$: the utility loss of giving up 1 unit of c_t for investment
- $\beta u'(c_{t+1})(F_k(k_{t+1}, n_{t+1}) + 1 - \delta)$:
  - $F_k(k_{t+1}, n_{t+1}) + 1 - \delta$: the capital accumulation $(1-\delta)$ and production $F_k(k_{t+1}, n_{t+1})$ of an extra unit of capital
  - $\beta u'(c_{t+1})$: per-unit discounted utility compensation at $t+1$

(1) and (2) together, gives the consumption-leisure condition:

$$u'(c_t) \cdot F_n(k_t, n_t) = \psi'(1-n_t)$$

Interpretation: the utility loss of giving up 1 unit of l_t ($\psi'(1-n_t)$) must be compensated by the consumption utility gain due to the additional output $F_n(k_t, n_t) \cdot u'(c_t)$

With the FOCS, we can characterize the equilibrium and steady state of this economy:

Social planner's problem: equilibrium

The equilibrium in this economy is characterized by the following equations:

$$\begin{aligned}
u'(c_t) &= (F_k(k_{t+1}, n_{t+1}) + 1 - \delta)\beta u'(c_{t+1}) && \text{Euler equation} \\
u'(c_t) &= \frac{\psi'(1-n_t)}{F_n(k_t, n_t)} && \text{leisure condition} \\
k_{t+1} &= F(k_t, n_t) + (1-\delta)k_t - c_t && \text{budget constraint}
\end{aligned}$$

where $F(k_t, n_t) = A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha)n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}}$

Social planner's problem: steady state

The steady state in this economy is achieved when $c_t \equiv c^*, k_{t+1} \equiv k^*, n_t \equiv n^*$, hence:

$$\begin{aligned}
\frac{1}{\beta} - (1-\delta) &= F_k(k^*, n^*) && \text{Euler equation} \\
u'(c^*) &= \frac{\psi'(1-n^*)}{F_n(k^*, n^*)} && \text{leisure condition} \\
c^* &= F(k^*, n^*) - \delta k^* && \text{budget constraint}
\end{aligned}$$

Plug the output level $y_t = F(k_t, n_t)$ and its partial derivatives back into these three equations, we have the steady state:

$$\begin{aligned}
\frac{1}{\beta} - (1-\delta) &= \alpha A^{1-\frac{1}{\gamma}}\left(\frac{y^*}{k^*}\right)^{\frac{1}{\gamma}} && \text{Euler equation} \\
u'(c^*) &= \frac{\psi'(1-n^*)}{(1-\alpha)A^{1-\frac{1}{\gamma}}\left(\frac{y^*}{n^*}\right)^{\frac{1}{\gamma}}} && \text{leisure condition} \\
c^* &= y^* - \delta k^* && \text{budget constraint}
\end{aligned}$$

And we can have the steady state capital-labor ratio:

$$\frac{k^*}{n^*} = \left(\frac{\alpha}{1-\alpha} \cdot \frac{\psi'(1-n^*)/u'(c^*)}{\frac{1}{\beta} - (1-\delta)}\right)$$

but it is generally impossible to solve analytically.

Social planner's problem: a simple example
In general, social planner's problem of NGM cannot be solved analytically, but for the simplest example, it is possible