Microeconomic Model Cheat Sheet

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Modelling essential: what we need to decide

- **Primitive assumptions**:
  - Who the **agents** are, what are their **preferences** and objective functions
  - What **technology** agents can access
  - What **endowment** agents have
- **Decision problems**: resource allocation problem (among agents, over time, etc.).
- **Information sets**: what do agents know, how will their knowledge change, what is their **expectation**.
- **Allocation mechanism**: how agents interact and achieve equilibrium. 2 main mechanisms are:
  - **price system** in competitive equilibrium
  - benevolent **central planner** maximizes a social welfare function.

Infinitely Lived Agent Model

Features

- **discrete** time, indexed by t
- economy lives **infinitely**, $t = 0, 1, 2, \cdots$
- single commodity exogenously produced, indexed by t, pure **exchange/endowment** economy.
- no firms/government, only **two types of households**.
- each type of households is continuum of **identical** households of that type, they are **price takers**, can be represented by a **representative** household

Agents' preferences

Utility of type i household is

$$U(c^i) = \sum_{t=0}^{\infty} \beta_i^t u(c_t^i)$$

where $\left(c^i\right) = \left\{c_t^i\right\}_{t=0}^{\infty}$, $\beta_i \in (0, 1)$,

The utility function $u(c_t^i)$ is assumed to be:

- **continuously differentiable** of the second order
- **monotonically increasing, strictly concave**: $u'(c_t^i) > 0, u''(c_t^i) < 0$
- **satisfies Inada conditions** (never 0 or infinity consumption): $\lim_{c_t^i \to \infty} u'(c_t^i) = 0, \lim_{c_t^i \to 0} u'(c_t^i) = \infty$
- **time additivity**: $u(c_t^i)$ is independent of c_{t+j}^i, c_{t-j}^i.
- **impatient discounting** $\beta_i < 1$: households value today's consumption more than future's.
- Constant relative risk aversion (**CRRA**): $u(c_t^i) = \frac{c^{1-\sigma}-1}{1-\sigma}$ $\left(\lim_{\sigma\to 1} \frac{c^{1-\sigma}-1}{1-\sigma} = \lim_{\sigma \to 1} \frac{e^{(1-\sigma)\ln(c)}-1}{1-\sigma} = \ln(c)\right)$, **The RRA coefficient** $\sigma(c) = \frac{-u''(c_t^i)c}{u'(c_t^i)} = \frac{-\left(-\sigma c^{-(1+\sigma)}c\right)}{c^{-\sigma}} = \sigma$. Higher RRA means higher risk aversion.
- Constant intertemporal elasticity of substitution (**IES**):

  $$IES = -\frac{\mathrm{d}\ln(c_{t+1}/c_t)}{\mathrm{d}\ln(u'(c_{t+1}/u'(c_t))} = \frac{1}{\sigma}$$

  hence higher RRA (more risk-averse), lower IES (consumption variation over time).

Agents' endowment

A deterministic endowment stream of the consumption good for type i household is

$$w^i = \left(w_0^i, w_1^i, \cdots\right) = \left\{w_t^i\right\}_{t=0}^{\infty}$$

Arrow-Debreu Market (AD) approach

Market structure: Basic Case

Households trade just **once** in $t = 0$ market, they trade all future consumption and deliver the promised amount in $t = 1, 2, \cdots$ market.
Households have perfect information of the entire endowment sequence, all information is public.

Equilibrium: Basic Case

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- regulating mechanism: $\{\hat{p}_t\}_{t=0}^{\infty}$, with numeraire $\hat{p}_0 = 1$

such that:

- given $\{\hat{p}_t\}_{t=0}^{\infty}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

  $$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i) \text{ s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i, c_t^i \ge 0$$

- market clearing (disposal of unused goods is costly):

  $$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, \forall t$$

How to solve:

- ***Step 1***: solve the Lagrangian

  $$\max \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) + \lambda^i \left(\sum_{t=0}^{\infty} p_t w_t^i - \sum_{t=0}^{\infty} p_t c_t^i\right)$$

  FOCs: $\beta^t / c_t^i = \lambda^i p_t \Rightarrow \frac{\beta^t}{c_t^i p_t} = \frac{\beta^{t+1}}{c_{t+1}^i p_{t+1}} \Rightarrow c_{t+1}^i = \beta \frac{p_t}{p_{t+1}} c_t^i$. FOC gives that price changes p_t/p_{t+1} and subjective discounting β determines consumption smoothing.
- ***Step 2***: Use market clearing condition

  $$c_t^1 + c_t^2 = w_t^1 + w_t^2$$

  and FOC $p_{t+1} c_{t+1}^i = \beta p_t c_t^i$, get $\frac{p_{t+1}}{p_t} = \beta \frac{w_t^1 + w_t^2}{w_{t+1}^1 + w_{t+1}^2}$, combined with the numeraire assumption $p_0 = 1$, solve the price sequence $\{\hat{p}_t\}_{t=0}^{\infty}$.
- ***Step 3***: Plug $\{\hat{p}_t\}_{t=0}^{\infty}$ and $p_{t+1} c_{t+1}^i = \beta p_t c_t^i$ back to budget constraint

  $$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} \beta^t c_0^i = \frac{c_0^i}{1-\beta} = \sum_{t=0}^{\infty} p_t w_t^i$$

  to solve allocation $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$

Pareto efficiency: Basic Case

An allocation $\left(c^1, c^2\right) = \left\{c_t^1, c_t^2\right\}_{t=0}^{\infty}$ is **Pareto efficient** if:

- it is feasible: $\sum_{t=0}^{\infty} p_t c_t^i \le \sum_{t=0}^{\infty} p_t w_t^i$
- no other feasible allocation $(\bar{c}^1, \bar{c}^2)$ such that $\forall i, U(\bar{c}^i) \ge U(c^i)$ and $\exists i, U(\bar{c}^i) \ge U(c^i)$

An AD competitive equilibrium allocation $\left(c^1, c^2\right) = \left\{c_t^1, c_t^2\right\}_{t=0}^{\infty}$ is Pareto efficient.

Proof: Suppose there is an allocation $\left(\bar{c}^1, \bar{c}^2\right) = \left\{\bar{c}_t^1, \bar{c}_t^2\right\}_{t=0}^{\infty}$, Pareto-dominating AD allocation $\left(\hat{c}^1, \hat{c}^2\right) = \left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$.
If the Pareto dominating allocation $\left(\bar{c}^1, \bar{c}^2\right)$ exists, its utility $\overline{U} = U(\bar{c}^1, \bar{c}^2)$ must be bigger than the AD allocation utility $\tilde{U} = U(\tilde{c}^1, \tilde{c}^2)$, therefore, the only reason that it is not chosen as the AD allocation is that it is **infeasible**.
Formally, suppose $\bar{c}^1 > \hat{c}^1$ and $\bar{c}^2 \ge \hat{c}^2$,

- **Step 1**: for household 1 ($\hat{c}^1 < \bar{c}^1$), if $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^1 \le \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 = \sum_{t=0}^{\infty} \hat{p}_t w_t^1$ (the Pareto-dominating allocation is also feasible), the AD equilibrium $(\hat{c}^1, \hat{c}^2)$ will NOT maximize HH1's utility, hence contradiction.
- **Step 2**: for household 2 ($\hat{c}^2 \le \bar{c}^2$), if $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t w_t^2$ (the Pareto-superior allocation cost less for HH2), then $\exists \delta > 0$ s.t. $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^2 + \delta \le \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^c = \sum_{t=0}^{\infty} \hat{p}_t w_t^2$, then there is always an allocation $\left\{\bar{c}_0^2 + \delta, \bar{c}_t^2\right\}$, achieves a strictly higher utility than the AD allocation (which is utility maximizing), hence contradiction.
- **Step 3**: from **Step 1**, $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 = \sum_{t=0}^{\infty} \hat{p}_t w_t^1$; from **Step 2**, $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^2 \ge \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t w_t^2$, then

  $$\sum_{i=1,2} \sum_{t=0}^{\infty} \hat{p}_t w_t^i < \sum_{i=1,2} \sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i$$

Therefore, this Pareto allocation is actually infeasible.
This proof requires **the value of the aggregate endowment is finite**, which is quite intuitive.

PE allocation: Social planner's problem

By **1st welfare theorem**, we can solve the competitive equilibrium allocation by solving Pareto efficient allocation. This is the social planner's problem: a **weighted** utility maximization problem.

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- utility weight: $\left\{\alpha^1, \alpha^2\right\}$

such that:

- given $\left\{\alpha^1, \alpha^2\right\}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

  $$\max \alpha^1 \sum_{t=0}^{\infty} \beta^t u(c_t^1) + \alpha^2 \sum_{t=0}^{\infty} \beta^t u(c_t^2)$$

  s.t. $c_t^1 + c_t^2 \le w_t^1 + w_t^2, \alpha^1 + \alpha^2 = 1, \alpha^i, c_t^i \ge 0$
- market clearing is the budget constraint.

How to solve:
solve the Lagrangian

$$\alpha^1 \sum_{t=0}^{\infty} \beta^t u(c_t^1) + \alpha^2 \sum_{t=0}^{\infty} \beta^t u(c_t^2) + \sum_{t=0}^{\infty} \mu_t \left(w_t^1 + w_t^2 - c_t^1 - c_t^2\right)$$

FOC gives $\alpha^1 \beta^t u'(c_t^1) = \mu_t = \alpha^2 \beta^t u'(c_t^2) \Rightarrow \alpha^1 u'(c_t^1) = \alpha^2 u'(c_t^2)$, plug them back to $c_t^1 + c_t^2 = w_t^1 + w_t^2$, solve $(c_t^1, c_t^2) = \left(c_t^1(\alpha_1, \alpha_2), c_t^2(\alpha_1, \alpha_2)\right)$.
The Lagrangian multiplier μ_t is the AD equilibrium prices, normalized by the total endowment each period: $\hat{\alpha}^i \beta^t u'(\hat{c}_t^i) = \hat{\mu}_t, \beta^t u'(\hat{c}_t^i) = \hat{\lambda}^i \hat{p}_t \Rightarrow \frac{\hat{\mu}_t}{\hat{\alpha}^i} = \hat{\lambda}^i \hat{p}_t$

All the PE allocations are Pareto efficient, but only one is AD equilibrium allocation, that allocation needs to satisfy the AD budget constraint, achieved by ***transfer***. This procedure, Negishi "trick", follows the second welfare theorem: every Pareto-efficient allocation can be decentralized as an equilibrium with transfers.

The AD equilibrium with transfer is:

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- lifetime transfer: $\left\{\hat{t}^1, \hat{t}^2\right\}$
- regulating mechanism: $\{\hat{p}_t\}_{t=0}^{\infty}$, with numeraire $\hat{p}_0 = 1$

such that:

- given $\{\hat{p}_t\}_{t=0}^{\infty}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

  $$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i) \text{ s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}_t^i, c_t^i \ge 0$$

- market clearing: $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, \forall t$

But the solving is actually much easier, we just solve the zero lifetime transfer condition:

$$t^i(\alpha) \equiv \sum_{t=0}^{\infty} \mu_t \left(c_t^i(\alpha) - w_t^i\right) = \sum_{t=0}^{\infty} \alpha^i \beta^t u'(c_t^i) \left(c_t^i(\alpha) - w_t^i\right) = 0$$

In general, transfer function $t^i(\alpha)$ satisfies **zero sum**: $t^1(\alpha) + t^2(\alpha) = 0$; and **homogeneous of degree 1** $t^i(k\alpha) = kt^i(\alpha)$.

Sequential Market (SM) approach

Market structure

Households trade in spot markets for immediate delivery of consumption goods at every t, **bond** is traded, (purchasing at t denoted by a^i_{t+1}), they are traded at t, representing one unit of consumption at $t+1$. The interest rate of bonds r_{t+1} regulates the market: bond of 1 unit of consumption at t will be compensated by $(1+r_{t+1})$ units of consumption at $t+1$.
Households have perfect information of the entire endowment sequence, all information is public.

Equilibrium

- allocation: $\left\{\hat{c}^1_t, \hat{c}^2_t\right\}_{t=0}^{\infty}$
- regulating mechanism: $\{\hat{p}_t\}_{t=0}^{\infty}$, with numeraire $\hat{p}_0 = 1$

such that:
- given $\{\hat{p}_t\}_{t=0}^{\infty}$, $\left\{\hat{c}^1_t, \hat{c}^2_t\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t u(c^i_t) \ \text{ s.t. } \sum_{t=0}^{\infty} \hat{p}_t c^i_t \leq \sum_{t=0}^{\infty} \hat{p}_t w^i_t, c^i_t \geq 0$$

- market clearing (disposal of unused goods is costly):

$$\hat{c}^1_t + \hat{c}^2_t = w^1_t + w^2_t, \forall t$$

How to solve:
- ***Step 1***:
- ***Step 2***:
- ***Step 3***: