Infinitely lived agent

AD equilibrium

AD equilibrium

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- regulating mechanism: $\{\hat{p}_t\}_{t=0}^{\infty}$, with numeraire $\hat{p}_0 = 1$

such that:
- given $\{\hat{p}_t\}_{t=0}^{\infty}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i, c_t^i \ge 0$$

- market clearing (disposal of unused goods is costly):

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, \forall t$$

AD equilibrium: social planner solves PO

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- utility weight: $\left\{\hat{\alpha}^1, \hat{\alpha}^2\right\}$

such that:
- given $\left\{\hat{\alpha}^1, \hat{\alpha}^2\right\}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

$$\max \alpha^1 \sum_{t=0}^{\infty} \beta^t u(c_t^1) + \alpha^2 \sum_{t=0}^{\infty} \beta^t u(c_t^2)$$

s.t. $c_t^1 + c_t^2 \le w_t^1 + w_t^2, \alpha^1 + \alpha^2 = 1, \alpha^i, c_t^i \ge 0$
- market clearing is the budget constraint.

AD equilibrium with transfer

- allocation: $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$
- lifetime transfer: $\left\{\hat{t}^1, \hat{t}^2\right\}$
- regulating mechanism: $\{\hat{p}_t\}_{t=0}^{\infty}$, with numeraire $\hat{p}_0 = 1$

such that:
- given $\{\hat{p}_t\}_{t=0}^{\infty}$, $\left\{\hat{c}_t^1, \hat{c}_t^2\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i) \;\text{ s.t. }\; \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}_t^i, c_t^i \ge 0$$

- market clearing: $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2, \forall t$

How to solve: just solve the zero lifetime transfer condition:

$$t^i(\alpha) \equiv \sum_{t=0}^{\infty} \mu_t \left(c_t^i(\alpha) - w_t^i\right) = \sum_{t=0}^{\infty} \alpha^i \beta^t u'(c_t^i)\left(c_t^i(\alpha) - w_t^i\right) = 0$$

where μ_t is the Lagrange multiplier of social planner's problem.

SM equilibrium

SM equilibrium

- allocation: $\left\{\left\{\tilde{c}_t^1, \tilde{c}_t^2\right\}, \left\{\tilde{a}_{t+1}^1, \tilde{a}_{t+1}^2\right\}\right\}_{t=0}^{\infty}$
- regulating mechanism: $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$

such that:
- given $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$, $\left\{\left\{\tilde{c}_t^1, \tilde{c}_t^2\right\}, \left\{\tilde{a}_{t+1}^1, \tilde{a}_{t+1}^2\right\}\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

s.t. $c_t^i + \frac{a_{t+1}^i}{1+\tilde{r}_{t+1}} \le w_t^i + a_t^i,\ c_t^i \ge 0,\ a_{t+1}^i \ge -\overline{A}^i > -\infty$
- market clearing (disposal of unused goods is costly):

$$\tilde{c}_t^1 + \tilde{c}_t^2 = w_t^1 + w_t^2,\ \tilde{a}_{t+1}^1 + \tilde{a}_{t+1}^2 = 0,\ \forall t$$

- asset holding at t $\left\{\tilde{a}_t^1, \tilde{a}_t^2\right\}_{t=0}^{\infty}$ with:

$$\tilde{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - w_{t+\tau}^i\right)}{\hat{p}_{t+1}}$$

AD ⇔ SM

Proof of Position 1: AD⇒SM:

- **Step 1: Define the SM interest rate** as

$$\frac{1}{1+\tilde{r}_{t+1}} = \frac{\hat{p}_{t+1}}{\hat{p}_t}$$

- **Step 2: Derive** $\tilde{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - w_{t+\tau}^i\right)}{\hat{p}_{t+1}}$
- **Step 3: Find debt limit** $\overline{A}^i$ since $\tilde{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - w_{t+\tau}^i\right)}{\hat{p}_{t+1}} \ge -\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} w_{t+\tau}^i}{\hat{p}_{t+1}} \ge -\sum_{\tau=1}^{\infty} \xi^{\tau-1} \hat{p}_{t+1} > -\infty$, let $-\overline{A}^i = -\sup \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}}{\hat{p}_{t+1}} w_{t+\tau}^i$
- **Step 4: Check utility maximization** FOCs align

Proof of Position 2: AD⇐SM:
- **Step 1: Construct price series**

$$\hat{p}_{t+1} = \frac{\hat{p}_t}{1+\tilde{r}_{t+1}}$$

as
- **Step 2: SM equilibrium allocation is utility maximizing within AD budget** since SM has one more constraint.

Overlapping generation

AD equilibrium

AD equilibrium

- allocation: $\left\{\hat{c}_1^0, \left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^{\infty}\right\}$
- regulating mechanism: $\{\hat{p}_t\}_{t=1}^{\infty}$, with m or p_1 (when $m = 0$) as the numeraire.

such that:
- given $\{\hat{p}_t\}_{t=1}^{\infty}$, $\left\{\hat{c}_1^0, \left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^{\infty}\right\}$ solves:

$$\max_{c^t \ge 0} u(c_t^t) + \beta u(c_{t+1}^t)$$

s.t.

$$\hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \le \hat{p}_t w_t^t + \hat{p}_{t+1} w_{t+1}^t$$

$$\max_{c_1^0 \ge 0} u(c_1^0) \;\text{ s.t. }\; \hat{p}_1 c_1^0 \le \hat{p}_1 w_1^0 + m$$

- market clearing (disposal of unused goods is costly):

$$\hat{c}_t^{t-1} + \hat{c}_t^t = w_t^{t-1} + w_t^t, \forall t \ge 1$$

SM equilibrium

SM equilibrium

- allocation: $\left\{\tilde{c}_1^0, \left\{\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{s}_t^t\right\}_{t=1}^{\infty}\right\}$
- regulating mechanism: interest rates $\{\tilde{r}_t\}_{t=1}^{\infty}$

such that:
- given $\{\tilde{r}_{t+1}\}_{t=1}^{\infty}, \forall t > 1$, $\left\{\tilde{c}_t^t, \tilde{c}_{t+1}^t\right\}_{t=1}^{\infty}$ solves:

$$\max_{c^t \ge 0} u(c_t^t) + \beta u(c_{t+1}^t) \;\text{ s.t. }\; c_t^t + s_t^t \le w_t^t,$$

$$c_{t+1}^t \le w_{t+1}^t + (1+\tilde{r}_{t+1}) s_t^t$$

$$\max_{c_1^0 \ge 0} u(c_1^0) \;\text{ s.t. }\; c_1^0 \le w_1^0 + m(1+\tilde{r}_1)$$

- good market clearing (disposal of unused goods is costly):

$$\tilde{c}_t^{t-1} + \tilde{c}_t^t = w_t^{t-1} + w_t^t, \forall t \ge 1$$

- asset market clearing: the budget constraint gives

$$\tilde{c}_{t+1}^{t+1} + \tilde{c}_{t+1}^t + s_{t+1}^{t+1} = w_{t+1}^{t+1} + w_{t+1}^t + (1+\tilde{r}_{t+1}) s_t^t$$

plug the good market clearing condition, get

$$s_{t+1}^{t+1} = (1+\tilde{r}_{t+1}) s_t^t$$

iterate this backwards to $s_0^0 = m$, get

$$s_t^t = \prod_{\tau=1}^{t} (1+\tilde{r}_\tau) m$$

AD ⇔ SM

The interest rate and price stream are still inter-determined:

$$\frac{1}{1+\tilde{r}_{t+1}} = \frac{\hat{p}_{t+1}}{\hat{p}_t}, \quad \frac{1}{1+\tilde{r}_1} = \hat{p}_1$$

The two Euler equations are:

$$u_t'(c_t^t) = \beta u_{t+1}'(c_{t+1}^t)(\hat{p}_t/\hat{p}_{t+1}) \quad \cdots \text{ AD}$$

$$u_t'(c_t^t) = \beta u_{t+1}'(c_{t+1}^t)(1+\tilde{r}_{t+1}) \quad \cdots \text{ SM}$$

An easy proof:

- ***AD ⇒ SM***: AD equilibrium allocation satisfies SM FOC and the SM budget constraints:

$$\tilde{\mu}_t^t = \hat{\lambda} \hat{p}_t, \tilde{\mu}_{t+1}^t = \hat{\lambda} \hat{p}_{t+1}, \forall t \ge 0;\ \tilde{s}_t^t = w_t^t - \hat{c}_t^t, \forall t \ge 1$$

- ***SM ⇒ AD***: SM equilibrium allocation satisfies AD FOC and the AD budget constraints:

$$\hat{\lambda}^t = \frac{\tilde{\mu}_t^t}{\prod_{\tau=0}^{t-1}(1+\tilde{r}_{t-\tau})} = \frac{\tilde{\mu}_{t+1}^t}{\prod_{\tau=0}^{t-1}(1+\tilde{r}_{t+1-\tau})}, \forall t \ge 1$$

$$\hat{\lambda}^0 = \tilde{\mu}_1^0 (1+\tilde{r}_1)$$

Offer curve

Use

$$u'(c_t^t) = \lambda^t p_t$$

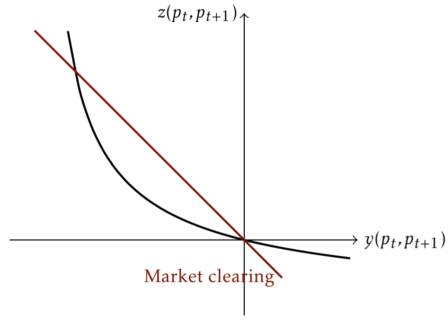
$$\beta u'(c_{t+1}^t) = \lambda^t p_{t+1}$$

and

$$y(p_t, p_{t+1}) = c_t^t(p_t, p_{t+1}) - w_1$$

$$z(p_t, p_{t+1}) = c_{t+1}^t(p_t, p_{t+1}) - w_2$$

get $z = f(y)$



- **Autarky equilibrium**
  - In *knife-edge economy*, **Autarky is PE**
  - In *classical economy*, **Autarky is PE**
  - In *Samuelson economy*, **Autarky is NOT PE**
- **Monetary stationary equilibrium**
  - In *Samuelson economy*, **MSE is PE** and **Pareto dominating Autarky**
  - In *classical economy*, **MSE is PE** but **NOT Pareto dominating Autarky**
- **Monetary non-stationary equilibria**
  - In *Samuelson economy*, **MNSE is not PE** (converge to Autarky)
  - In *classical economy*, **every MNSE is PE**
  - In *kife-edge economy*, **every MNSE is PE**

Balasko-Shell condition

Define

$$\frac{1}{1+r_{t+1}} = \frac{\beta u'\left(\hat{c}_{t+1}^t\right)}{u'\left(\hat{c}_t^t\right)} = \frac{\hat{p}_{t+1}}{\hat{p}_t}$$

Then the allocation is Pareto efficient if and only if

$$\sum_{t=1}^{\infty} \prod_{\tau}^{t} (1+r_{\tau+1}) = +\infty$$

This **includes** two scenarios (PE):
- **AD prices falling**, i.e., **positive interest rate** r_{t+1}
  - classic Autarky
- **AD prices constant**, i.e., **zero interest rate** r_{t+1}
  - knife-edge Autarky, all knife-edge MNSE (converge to Autarky)
  - classic MSE, all classic MNSE (converge to classic MSE)
  - Samuelson MSE

This **excludes** one scenario (not PE):
- **AD prices increasing**, i.e., **negative interest rate** r_{t+1}
  - Samuelson Autarky, all Samuelson MNSE (converge to Samuelson Autarky).

Neoclassical growth model
social planner's problem

$$\max_{\{c_t, i_t, k_{t+1}, l_t, n_t, y_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (u(c_t) + \psi(l_t))$$

s.t.

$$y_t = c_t + i_t \qquad \text{market clearing}$$
$$k_{t+1} = (1-\delta)k_t + i_t \qquad \text{capital accumulation}$$
$$1 = n_t + l_t \qquad \text{time endowment}$$
$$y_t = A\left(\alpha k_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right)^{\frac{1}{1-\frac{1}{\gamma}}} \qquad \text{production function}$$
$$0 \le c_t, i_t, k_t, n_t, l_t, y_t, \forall t \qquad \text{non-negativity}$$

solve it, get

Social planner's problem: equilibrium

The equilibrium in this economy is characterized by the following equations:

$$u'(c_t) = (F_k(k_{t+1}, n_{t+1}) + 1 - \delta)\beta u'(c_{t+1}) \qquad \text{Euler}$$
$$u'(c_t) = \frac{\psi'(1-n_t)}{F_n(k_t, n_t)} \qquad \text{labor}$$
$$k_{t+1} = F(k_t, n_t) + (1-\delta)k_t - c_t \qquad \text{budget}$$

and steady state

$$\frac{1}{\beta} - (1-\delta) = F_k(k^*, n^*) \qquad \text{Euler}$$
$$u'(c^*) = \frac{\psi'(1-n^*)}{F_n(k^*, n^*)} \qquad \text{labor}$$
$$c^* = F(k^*, n^*) - \delta k^* \qquad \text{budget}$$

The Pareto efficient condition for an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ is (by Stokey and Lucas):
- satisfying the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})[F_k(k_t, n_t) + (1-\delta)]$$

- : Transversality condition:

$$\lim_{t\to\infty} \lambda_t F_k(k_t, n_t) k_t = 0$$

where λ_t, the social planner's Lagrange multiplier, is just $\beta^t u'(c_t)$.

competitive equilibrium
AD equilibrium

numaraire: period 1 good price $p_0 = 1$
- HH's problem

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$\sum_{t=0}^{\infty} p_t (c_t + (k_{t+1} - (1-\delta)k_t)) = \sum_{t=0}^{\infty} p_t \left(w_t n_t^s + r_t k_t^s\right) + \pi$$

- firm's problem

$$\max p_t (F(k_t^s, n_t^s) - r_t k_t^s - w_t n_t^s)$$

- market clearing:

$$F(k_t^s, n_t^s) = c_i + k_{t+1} - (1-\delta)k_t$$
$$k_t^s = k_t^d = k_t$$
$$n_t^s = n_t^d = n_t$$

Euler equation

$$\frac{\beta u'(c_{t+1})}{\beta u'(c_t)} = \frac{1}{F_k(k_{t+1}, 1) + (1-\delta)}$$

TVC condition for Pareto efficiency

$$\lim_{t\to\infty} p_t k_{t+1} = 0$$

SM equilibrium
- HH's problem

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t + \frac{a_{t+1}}{1 + r_{t+1}} = r_t k_t + w_t + a_t + \pi$$

- firm's problem

$$\max F(k_t^d, n_t^d) - r_t k_t^d - w_t n_t^d$$

-
- market clearing

$$F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t$$
$$k_t = k_t^d \qquad 1 = n_t^d$$
$$a_t = 0$$

recursive form
social planner

$$v(k) = \max_{c, k'} u(c) + \beta v(k')$$
$$= \max_{k' \le f(k) + (1-\delta)k} \left\{u(F(k) - k' + (1-\delta)k) + \beta v(k')\right\}$$

competitive equilibrium

Use **SM equilibrium**. The HH's Bellman equation is

$$v(k, K) = \max_{c, k'} \left\{u(c) + \beta v(k', K')\right\}$$

s.t.

$$c + k' - (1-\delta)k = w(K) + kr(K), \; K' = H(K)$$

and the recursive competitive equilibrium is
- value function: $v(k, K)$
- policy function: $c = C(k, K)$, $k' = G(k, K)$
- pricing function: $w(K), r(K)$
- aggregate law of motion: $K' = H(K)$

s.t.
- given pricing functions, **value function solves the Bellman equation**, with C, G the associated policy functions
- **pricing function maximize firm's profit**
- **consistency**: $H(K) = K' = G(K, K)$
- **market clearing**: $C(K, K) + G(K, K) - (1-\delta)K = F(K, 1)$

Balanced growth
3 conditions to ensure balanced growth:
- **strictly positive growth**: $\gamma_c > 0$
- **life-time utility bounded**: $\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t u(\gamma^t c_0) > \infty$
- **transversality**: $\lim_{t\to\infty} \beta^t u'(c_t) k_t = 0$

Romer externality
- CE

-firms Firms solve

$$\max F(L_{i,t}, K_{i,t}, \overline{K}_t) - w_t L_{i,t} - r_t K_{i,t}$$

s.t. $\sum_{i=1}^{N} K_{i,t} = \overline{K}_t$, $\sum_{i=1}^{N} L_{i,t} = 1$. solve it, get $\frac{K_{i,t}}{L_{i,t}} = \frac{K_{j,t}}{L_{j,t}} = \overline{K}_t, \forall i, j$, and $w_t = (1-\alpha)A\overline{K}_t$, $r_t = \alpha A$, $Y_{i,t} = A\overline{K}_t L_{i,t}$, aggregate output $\overline{Y}_t = A\overline{K}_t$.
- consumers:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t = r_t k_t + w_t$$

- SP

$$\max \beta^t \frac{c^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + \overline{K}_{t+1} - (1-\delta)\overline{K}_t = Y_t = A\overline{K}_t$$

the inefficiency of externality (lower growth rate):

$$g_c^{CE} = [\beta(\alpha A + 1 - \delta)]^{\frac{1}{\sigma}} < [\beta(A + 1 - \delta)]^{\frac{1}{\sigma}} = g_c^{SP}$$

Two capital model

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + i_{K,t} + i_{H,t} = Y_t$$

where
- $Y_t = K_t^{\alpha} H_t^{1-\alpha}$
- $i_{K,t} = K_{t+1} - (1-\delta_K)K_t$
- $i_{H,t} = H_{t+1} - (1-\delta_H)H_t$

solve it get

$$\alpha\left(\frac{H_{t+1}}{K_{t+1}}\right)^{1-\alpha} + (1-\delta_K) = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = (1-\alpha)\left(\frac{H_{t+1}}{K_{t+1}}\right)^{-\alpha} + (1-\delta_H)$$

hence H_t/K_t is constant.

Two sector model

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + K_{t+1} - (1-\delta_K)K_t = K_t^{\alpha}(\phi_t H_t)^{1-\alpha} \qquad \phi_t H_t \text{ for output}$$
$$H_{t+1} - H_t = A(1-\phi_t)H_t \qquad (1-\phi_t)H_t \text{ for accumulation}$$
$$c_t, \phi_t, H_{t+1}, K_{t+1} \ge 0$$

Euler equation

$$\left(\frac{\phi_t H_t}{K_t}\right)^{-\alpha} c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left(\frac{\phi_{t+1} H_{t+1}}{K_{t+1}}\right)^{-\alpha} (1 + A)$$

gives

$$g_c = \left\{\beta \cdot \left[\alpha\left(\frac{\phi_{t+1} H_{t+1}}{K_{t+1}}\right)^{1-\alpha} + (1-\delta_K)\right]\right\}^{1/\sigma}$$

again, H_t/K_t constant.

3 sector endogenous growth
social planner

Step 1: solve the **static choice of production inputs**

$$\max L_{1,t}^{1-\alpha} \int_0^{A_t} x_t^{\alpha}(i) \mathrm{d}i$$

s.t.

$$\int_0^{A_t} x_t(i) \mathrm{d}i = K_t, \; L_{1,t} = \phi_t L$$

get

$$Y_t = L_{1,t}^{1-\alpha} \int_0^{A_t} x_t^{\alpha}(i) \mathrm{d}i = (\phi_t L)^{1-\alpha} A_t \left(\frac{K_t}{A_t}\right)^{\alpha} = K_t^{\alpha} (\phi_t L A_t)^{1-\alpha}$$

Step 2, solve **intertemporal planning problem**.

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + K_{t+1} - (1-\delta)K_t = K_t^{\alpha} (\phi_t L A_t)^{1-\alpha}$$
$$A_{t+1} - A_t = \kappa(1-\phi_t) L A_t$$

get Euler equation:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[\alpha\left(\frac{\phi_{t+1} L A_{t+1}}{K_{t+1}}\right)^{1-\alpha} + (1-\delta)\right]$$

decentralize
- Firm's problem:
  - **final good firm**:

$$\max_{x_t(i), L_{1,t}} \left\{L_{1,t}^{1-\alpha} \int_0^{A_t} x_t^{\alpha}(i) \mathrm{d}i - w_t L_{1,t} - \int_0^{A_t} p_t(i) x_t(i) \mathrm{d}i\right\}$$

  - **intermediate good producer**:

$$\max_{x_t(i)} \{p_t(i) x_t(i) - r_t K_t(i)\} \text{ s.t. } K_t(i) = x_t(i)$$

  - **R&D sector producer**:

$$\max_{A_{t+1}, L_{2,t}} \left\{p_t^P (A_{t+1} - A_t) - w_t L_{2,t}\right\}$$

s.t.

$$A_{t+1} - A_t = \kappa A_t L_{2,t}$$

  - closing the model

$$p_t^P \cdot p_t = \sum_{\tau = t+1}^{\infty} \pi_{\tau}(i) p_{\tau}$$

where
  - price of patent p_t^P $w_t = \kappa A_t L_{2,t} p_t^P$
  - positive profit of intermediate good producer $\pi_t(i) = \alpha(1-\alpha) L_{1,t} (\alpha^2 / r_t)^{\frac{\alpha}{1-\alpha}}$
  - p_t is the consumption price (time t price p_0 as numeraire), faced by **households**
- HH's problem

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$\sum_{t=0}^{\infty} p_t (c_t + i_t) = \sum_{t=0}^{\infty} p_t \left(w_t L_t + r_t K_t + \int_0^{A_t} \pi_t(i) \mathrm{d}i\right)$$
$$i_t = K_{t+1} - (1-\delta)K_t$$

get Euler $\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta[r_{t+1} + (1-\delta)]$ where $r_t = \alpha^2 L_{1,t}^{1-\alpha} (x_t)^{\alpha-1} \xLongrightarrow[x_t(i) = K_t(i)]{\int_0^{A_t} K_t(i) = K_t} \alpha^2 (\phi_t L)^{1-\alpha} \left(\frac{A_t}{K_t}\right)^{1-\alpha}$, together

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[\alpha^2\left(\frac{\phi_{t+1} L A_{t+1}}{K_{t+1}}\right)^{1-\alpha} + (1-\delta)\right]$$

Real business cycle

2-period

- CE
  - HH:

$$\max_{c_1,c_2,l,a} u(c_1)+v(l)+\beta u(c_2)$$

  s.t.

$$c_1+a=w(1-l)+\pi_1+\pi^I$$
$$c_2=(1+r)a+\pi_2$$

  solve it, get

$$u'(c_1)=\beta(1+r)u'(c_2) \quad \text{intertemporal Euler equation}$$
$$v'(l)=u'(c_1)w \quad \text{intratemporal labor supply}$$

  - Firm:

$$\pi_1=\max_L F_1(L)-wL \quad \text{period 1}$$
$$\pi_2=\max_K F_2(K)-r^K K \quad \text{period 2}$$

  - Investment firm solve

$$\max_I \pi^I=\frac{r^K}{1+r}I-I$$

- planner:

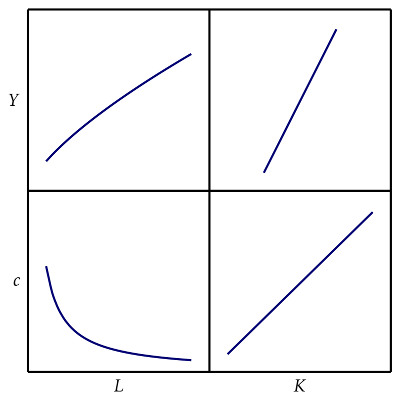
$$\max_{c_1,c_2,l,Y_1,Y_2,L,K,I} u(c_1)+v(l)+\beta u(c_2)$$

s.t.

$$Y_1=F_1(L)$$
$$l=1-L$$
$$Y_2+F_2(K)$$
$$KI$$
$$Y_1=c_1+I$$
$$Y_2=c_2$$

RBC: characteristic equations

$$Y_1 F_1(L) \quad \Rightarrow \frac{\mathrm{d}Y}{\mathrm{d}L}>0$$
$$v'(1-L)=F_1'(L)u'(c_1) \quad \Rightarrow \frac{\mathrm{d}c_1}{\mathrm{d}L}<0$$
$$u'(c_1)=\beta F_2'(K)u'(F_2(K)) \quad \Rightarrow \frac{\mathrm{d}c_1}{\mathrm{d}K}>0$$
$$Y_1=u'^{-1}\left[\beta F_2'(K)u'(F_2(K))\right]+K \quad \Rightarrow \frac{\mathrm{d}Y_1}{\mathrm{d}K}>0$$



The model can incorporate several shocks:

- **short-lived productivity shock**: $Y_1=AF_1(L)$

$$Y_1=AF_1(L) \quad \Rightarrow Y-L \text{ curve} \uparrow$$
$$v'(1-L)=AF_1'(L)u'(c_1) \quad \Rightarrow c-L \text{ curve} \uparrow$$

  there are income and substitution effects on labor supply, if **substitution effect** dominates, it looks like a business cycle.

- **impatience**: β increases

$$u'(c_1)=\beta F_2'(K)u'(F_2(K)) \quad \Rightarrow c-K \text{ curve} \leftarrow$$
$$Y_1=u'^{-1}\left[\beta F_2'(K)u'(F_2(K))\right]+K \quad \Rightarrow Y-K \text{ curve} \leftarrow$$

- **laziness or taxes**:

$$v'(1-L)=(1-\tau)F_1'(L)u'(c_1) \quad \Rightarrow c-L \text{ curve} \downarrow$$
$$\theta v'(1-L)=F_1'(L)u'(c_1) \quad \Rightarrow c-L \text{ curve} \downarrow$$

- **optimism about the future**: one example is positive productivity shock of $F_2(K)$

$$u'(c_1)=\beta AF_2'(K)u'(AF_2(K)) \quad \Rightarrow c-K \text{ curve} \downarrow$$
$$Y_1=u'^{-1}\left[\beta AF_2'(K)u'(AF_2(K))\right]+K \quad \Rightarrow Y-K \text{ curve} \downarrow$$

Infinite, stochastic

$$\max_{c(s^t),L(s^t),K(s^t)} \sum_{t=0}^{\infty}\beta^t\sum_{s^t}\Pr(s^t)u\left(c(s^t),L(s^t)\right)$$

s.t.

$$c(s^t)+K(s^{t+1})\le A(s_t)F(K(s^t),L(s^t))+(1-\delta)K(s^t),\ K_0 \text{ given}$$

Character equations

the intra-temporal consumption-labor condition:

$$-u_L(c(s^t),L(s^t))=u_c(c(s^t),L(s^t))A(s^t)F_L(K(s^t),L(s^t))$$

and the intertemporal Euler equation:

$$u_c(c(s^t),L(s^t))=\beta\mathbb{E}_t\left\{u_c(c(s^{t+1}),L(s^{t+1}))\cdot[A(s_{t+1})F_K(K(s^{t+1}),L(s^{t+1}))+(1-\delta)]\right\}$$

and the transversality condition

$$\lim_{t\to\infty}\beta^t\mathbb{E}_t\left[u_c\left(c(s^t),L(s^t)\right)\cdot A(s_t)\cdot F_K\left(K(s^t),L(s^t)\right)K(s^t)\right]=0$$

Recursive form

- state variables: s,K
- Bellman equation:

$$V(K,s)=\max_{c,L,K'}u(c,L)+\beta\sum\Pr(s'\mid s)V(K',s')$$

  s.t.

$$c+K'\le A(s)F(K,L)+(1-\delta)K$$

- FOC w.r.t. K' is

$$\beta\sum\Pr(s'\mid s)V_K(K'(K,s),s')-\lambda(K,s)=0$$

  with envelope theorem

$$V_K(K,s)=\lambda(K,s)[A(s)F_K(K,L(K,s))+(1-\delta)]$$

  get Euler equation:

$$\begin{aligned}&u_c(c(K,s),L(K,s))\\=&\beta\sum\Pr(s'\mid s)u_c(c(K'(K,s),s'),L(K'(K,s),s'))\\&\cdot\left[A(s')F_K\left(K'(K,s),L(K'(K,s),s')\right)+(1-\delta)\right]\end{aligned}$$

In summary, RBC model has 3 fundamental equations:

RBC equations

- Intertemporal Euler equation:

$$u_c(c_t,L_t)=\beta\mathbb{E}_t\left(u_c(c_{t+1},L_{t+1})\cdot[A_{t+1}F_K(K_{t+1},L_{t+1})+(1-\delta)]\right)$$

- Intratemporal consumption-labor equation:

$$-u_L(c_t,L_t)=u_c(c_t,L_t)A_tF_L(K_t,L_t)$$

- Resource constraint:

$$c_t+K_{t+1}-(1-\delta)K_t\le A_tF(K_t,L_t)$$

Q-theory: with condition on I_t

If $K_{t+1}=g(K_t,I_t)$, then social planner solves

$$\max_{c_t,K_t,I_t,L_t}\mathbb{E}\sum\beta^t u(c_t,L_t)$$

s.t.

$$c_t+I_t\le A_tF(K_t,L_t),\ K_{t+1}\le g(K_t,I_t)$$

Lagrange:

$$\begin{aligned}\mathcal{L}=&\mathbb{E}\sum\beta^t u(c_t,L_t)+\lambda_t\left(A_tF(K_t,L_t)-c_t-I_t\right)\\&+\mu_t\left(g(K_t,I_t)-K_{t+1}\right)\end{aligned}$$

Log-linearization

Log linearization formula

- $X_t=X_{ss}(1+x_t)$
- $X_tY_t=X_{ss}Y_{ss}(1+x_t+y_t)$
- $f(X_t)=f(X_{ss})\left(1+\frac{f'(X_{ss})}{f(X_{ss})}X_{ss}x_t\right)$
- $X_t^{\theta}Y_t=\alpha Z_t\Rightarrow\theta x_ty_t=z_t$

Log linearization of RBC model with

$$u(c,L)=\frac{c^{1-\gamma}}{1-\gamma}-\psi\frac{L^{1+\phi}}{1+\phi} \qquad F(K,L)=AK^{\alpha}L^{1-\alpha}$$

get

- Euler equation $C_t^{-\gamma}=\beta\mathbb{E}_t\left(C_{t+1}^{\gamma}\left[A_{t+1}K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha}+(1-\delta)\right]\right)$:

$$\begin{aligned}-\gamma c_t=\mathbb{E}_t[&-\gamma c_{t+1}\\&+(1-\beta(1-\delta))(a_{t+1}+(\alpha-1)k_{t+1}+(1-\alpha)l_{t+1})]\end{aligned}$$

- Intratemporal consumption-labor equation:

$$\phi l_t=-\gamma c_t+a_t+\alpha k_t-\alpha l_t\Rightarrow l_t=-\frac{1}{\phi+\alpha}(-\gamma c_t+a_t+\alpha k_t)$$

- Resource constraint $C_t+K_{t+1}-(1-\delta)K_t=Y_t$

$$\frac{C_{ss}}{Y_{ss}}c_t+\frac{K_{ss}}{Y_{ss}}k_{t+1}-(1-\delta)\frac{K_{ss}}{Y_{ss}}k_t=y_t$$

- Production $Y_t=A_tK_t^{\alpha}L_t^{1-\alpha}$:

$$y_t=a_t+\alpha k_t+(1-\alpha)l_t$$

Asset pricing

Household's problem is

$$\max_{c(s^t),a(s^{t+1})}\sum_{t,s^t}\beta^t\Pr(s^t)u(c(s^t))$$

s.t.

$$\begin{aligned}&c(s^t)+q(s^t)\cdot a_{t+1}(s^t)\le W(s^t)\\&W(s^{t+1})=y(s^{t+1})+\left(q(s^{s+1})+d(s^{t+1})\right)\cdot a_{t+1}(s^t)\end{aligned}$$

complete market and AD security ($d=1,q=0$):

$$c(s^t)+q(s^t)\cdot a_{t+1}(s^t)\le y(s^t)+a_t(s^{t-1})$$

get Euler

$$u'(c(s^t))=\beta\sum_{s_{t+1}}\Pr(s_{t+1}\mid s^t)u'(c(s^t,s_{t+1}))\frac{q(s^{t+1})+d(s^{t+1})}{q(s^t,s_{t+1})}$$

Pricing kernel

let $R(s^t,s_{t+1})\equiv\frac{q(s^{t+1})+d(s^{t+1})}{q(s^t,s_{t+1})}$, then get the Euler equation

$$1=\beta\mathbb{E}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}R_{t+1}\right]$$

This is true for all asset, hence:

- Pricing kernel: $m_{t+1}=\beta\frac{u'(c(s^t,s_{t+1}))}{u'(c(s^t))}$
- risk-free rate: $R_{t+1}^f=\frac{1}{\mathbb{E}_t[m_{t+1}]}$
- pricing formula: $\mathbb{E}_t(R_{t+1})=R_{t+1}^f-R_{t+1}^f\mathrm{Cov}(m_{t+1},R_{t+1})$
- risk-neutral probabilities:

$$Q(s_{t+1}\mid s^t)=\frac{\Pr(s_{t+1}\mid s^t)u'\left(c'(s_{t+1},s^t)\right)}{\sum_{s_{t+1}}\Pr(s_{t+1}\mid s^t)u'\left(c'(s_{t+1},s^t)\right)}$$

- Sharpe ratio: $s^j=\frac{\mathbb{E}(R_{t+1}^j)-R_{t+1}^f}{\sigma(R_{t+1}^j)}$ and $|s_j|\le\frac{\sigma(m_{t+1})}{\mathbb{E}(m_{t+1})}$

Welfare cost

Lucas calculation

If $c_t=c_0e^{gt}\exp(\epsilon_t)$ where $\mathbb{E}(\exp(\epsilon_t))=1$, then the cost of business cycle is the number λ that solves:

$$\mathbb{E}\left(\sum_{t=0}^{\infty}\beta^t u\left(c_0(1+\lambda)e^{gt}\exp(\epsilon_t)\right)\right)=\mathbb{E}\left(\sum_{t=0}^{\infty}\beta^t u\left(c_0e^{gt}\right)\right)$$

Alvarez calculation

- $\{c_t\}$: business-cycle consumption process
- $\{C_t\}$: an alternative consumption process

consider the utility of the consumption process $\{c_t\}$: $U(\{c_t\})$ (standard utility is $U(\{c_t\})=\sum_{s_t}\beta^t\Pr(s^t)u(c(s^t))$). Welfare cost $\lambda(\alpha)$ solves

$$U[(1+\lambda(\alpha))\{c_t\}]=U[(1-\alpha)\{c_t\}+\alpha\{C_t\}]$$

Money

LM equation

LM equation:

$$\frac{M^S}{p}=m^D(Y,i)$$

that is, money demand equals money supply

HH's problem is

$$\max\mathbb{E}\sum_{t=0}^{\infty}\beta^t u\left(c_t,L_t,\frac{M_t}{P_t}\right)$$

s.t.

$$P_tc_t+P_tk_{t+1}+B_{t+1}+M_{t+1}+T_t\le W_tL_t+(P_t(1-\delta)+R_t^K)k_t+(1+i_t)B_t+M_t$$

$$P_tc_t\le M_t$$

the characteristic equations of this economy:

$$u_{L,t}=-\frac{W_t}{P_t}u_{c,t}\equiv-w_tu_{c,t}$$

$$u_{c,t}=\beta u_{c,t+1}\left(1-\delta+\frac{R_{t+1}^K}{P_{t+1}}\right)\equiv\beta u_{c,t+1}(1-\delta+r_{t+1}^K)$$

$$u_{c,t}=\beta u_{c,t+1}\frac{1+i_{t+1}}{P_{t+1}/P_t}\equiv\beta u_{c,t+1}(1+r_{t+1})\qquad\text{bond}$$

$$i_{t+1}=\frac{u_{M,t+1}}{u_{c,t+1}}\qquad\text{money}$$

A nice form of money supply

$$\frac{M_t}{P_t}=\left(\frac{P_{t+1}}{P_t}\right)^{-\theta}\Rightarrow\log P_t=\frac{1}{1+\theta}[\log M_t+\theta\log P_{t+1}]$$

iterate forward, get

$$\log P_t=\frac{1}{1+\theta}\sum_{s=0}^{T-1}\left(\frac{\theta}{1+\theta}\right)^s\log M_{t+s}+\left(\frac{\theta}{1+\theta}\right)^T p_{t+T}$$

$$\xrightarrow{T\to\infty}\log P_t=\frac{1}{1+\theta}\sum_{s=0}^{\infty}\left(\frac{\theta}{1+\theta}\right)^s\log M_{t+s}$$

New Keynesian model

2-period for IS-LM

From

$$u'(c_1)=\beta(1+r)u'(c_2)\qquad\text{Euler equation}$$
$$F_2'(K)=1+r\qquad\text{investment FOC}$$
$$Y_1=c_1+K\qquad t=1\text{ market clearing}$$
$$Y_2=c_2=F_2(K)\qquad t=2\text{ market clearing}$$

get **IS** equation (**I**nvestment = **S**avings)

$$u'(Y_1-K(r))=\beta(1+r)u'(F(K(r)))$$

where $K(r)=(F_2')^{-1}(1+r)$. In nominal term

$$\Delta=u'(Y_1-K(i-\pi))-\beta(1+i-\pi)u'(F(K(i-\pi)))$$

hence

2-period LM-IS

$$u'(Y_1-K(i-\pi))=\beta(1+i-\pi)u'(F(K(i-\pi)))\qquad\text{IS}$$
$$\frac{M^S}{p}=m^D(i,Y)\qquad\text{LM}$$

where

- IS: $\frac{di}{dY}=-\frac{\partial\Delta/\partial Y_1}{\partial\Delta/\partial i}<0$, **downward**
- LM: $-\frac{\partial m^D/\partial Y_1>0}{\partial m^D/\partial i<0}>0$ with M^S fixed and p sticky

Adding

- production shock of $t=2$, $AF_2(K)\Rightarrow K(r,A)=(F_2')^{-1}\left(\frac{1+r}{A}\right)$
- government spending: $G\Rightarrow c_1=Y_1-K(r)-G$

get

$$u'(Y_1-G-K(i-\pi,A))=\beta(1+i-\pi)u'(AF(K(i-\pi,A)))\qquad\text{IS}$$
$$\frac{M^S}{p}=m^D(i,Y)\qquad\text{LM}$$

we can analyze the following shocks:

- productivity shock in $t=1$, laziness, tax: **NO** effect
- productivity shock in $t=2$, $A\uparrow$: IS $\rightarrow$
- impatience $\beta\downarrow$: IS curve $\rightarrow$
- higher expected inflation $\pi\uparrow$: IS curve $\uparrow$
- higher government spending $G\uparrow$: IS curve $\rightarrow$
- money supply increase $M^S\uparrow$: LM curve $\downarrow$

NK model

assume $U(C_t,L_t)=u(C_t)-v(L_t)$ where $u(C_t)=\frac{C_t^{1-\gamma}-1}{1-\gamma}$, and $v(L)=\psi\frac{L_t^{1+\phi}}{1+\phi}$

- Euler equation: $C_t^{-\gamma}=\beta\mathbb{E}\left[C_{t+1}^{-\gamma}\frac{1+i_{t+1}}{1+\pi_{t+1}}\right]$
- labor condition: $\psi L_t^{\phi}=\frac{W_t}{P_t}C_t^{-\gamma}$
- market clearing: $Y_t=C_t$
- GDP with misallocation: $Y_t=\frac{A_tL_t}{\Delta_t}$ where the productivity/misallocation $\Delta_t=(1-\mu)\left(\frac{1-\mu(1+\pi_t)^{\epsilon-1}}{1-\mu}\right)^{\frac{\epsilon}{\epsilon-1}}+(1+\pi_t)^{\epsilon}\mu\Delta_{t-1}$
- firm pricing: $\frac{P_{j,t}}{P_t}=\frac{\epsilon}{\epsilon-1}\frac{X_t}{Z_t}$ where $X_t\equiv\chi_tY_t^{1-\gamma}+\beta\mu\mathbb{E}_t\left[X_{t+1}\left(\frac{P_{t+1}}{P_t}\right)^{\epsilon}\right]$, $Z_t\equiv Y_t^{1-\gamma}+\beta\mu\mathbb{E}_t\left[Z_{t+1}\left(\frac{P_{t+1}}{P_t}\right)^{\epsilon-1}\right]$
- real marginal cost: $\chi_t=\frac{W_t}{P_tA_t}$
- inflation: $\frac{1-\mu(1+\pi_t)^{\epsilon-1}}{1-\mu}=\left(\frac{\epsilon}{\epsilon-1}\frac{X_t}{Z_t}\right)^{1-\epsilon}$

NK model summary

Hence, in NK model, we have:

$$y_t=\mathbb{E}\left[y_{t+1}-\frac{1}{\gamma}(i_{t+1}-\pi_{t+1}-\rho)\right]\qquad\text{IS equation}$$
$$\pi_t=\frac{(1-\mu)(1-\beta\mu)}{\mu}\hat{\chi}_t+\beta\mathbb{E}_t(\pi_{t+1})\qquad\text{Phillips curve}$$
$$w_t-p_t=(\gamma+\phi)y_t-\phi a_t\qquad\text{real wage}$$
$$\hat{\chi}_t=w_t-p_t-a_t=(\gamma+\phi)y_t-(1+\phi)a_t\qquad\text{marginal cost}$$

If prices are perfectly flexible ($w_t-p_t=a_t$), have

$$y_t^n=\frac{1+\phi}{\gamma+\phi}a_t\qquad\text{nature level of output}$$
$$l_t^n=\frac{1-\gamma}{\gamma+\phi}a_t\qquad\text{nature level of employment}$$
$$r_{t+1}^n=\rho+\gamma\frac{1+\phi}{\gamma+\phi}(\mathbb{E}[a_{t+1}]-a_t)\qquad\text{natural real interest rate}$$

NK IS-Phillips

define $x_t=y_t-y_t^n$ as the **output gap**, we have

$$x_t=\mathbb{E}\left[x_{t+1}-\frac{1}{\gamma}(i_{t+1}-\pi_{t+1}-r_{t+1}^n)\right]\qquad\text{IS equation}$$
$$\pi_t=\kappa x_t+\beta\mathbb{E}_t(\pi_{t+1})\qquad\text{Phillips curve}$$

where

$$\kappa(\mu)=\frac{(1-\mu)(1-\beta\mu)}{\mu}(\gamma+\phi)$$

- $\kappa'(\mu)<0$: less flexible (higher μ), lower κ
- $\kappa'(\gamma),\kappa'(\phi)>0$: $\hat{\chi}_t=(\gamma+\phi)x_t$, higher γ,ϕ means that marginal cost is sensitive to output gap

With Taylor rule (monetary policy):

$$i_{t+1}=\rho_{t+1}+\lambda_\pi\pi_t+\lambda_xx_t$$

where ρ_{t+1} is the target real rate, get a 2-unknown 2-equation system:

$$\pi_t=\kappa x_t+\mathbb{E}(\pi_{t+1})$$
$$(\gamma+\lambda_x)x_t+\lambda_\pi\pi_t=\mathbb{E}[\gamma x_{t+1}+\pi_{t+1}+r_{t+1}^n]-\rho_{t+1}$$

Optimal monetary policy

$$\max_{y_t,\pi_t}-\frac{1}{2}\mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left[x_t^2+\frac{\epsilon}{\kappa}\pi_t^2\right]$$

s.t. Phillips curve. Two way to frame:

- **discretion**: no commitment, take $\mathbb{E}_t[x_{t+\tau}]$ as given, hence no intertemporal decision:

$$\max_{x_t,\pi_t}x_t^2+\frac{\epsilon}{\kappa}\pi_t^2+\mathbb{E}_0\sum_{\tau=1}^{\infty}\beta^\tau\left[x_\tau^2+\frac{\epsilon}{\kappa}\pi_\tau^2\right]$$

s.t.

$$\pi_t=\kappa x_t+\beta\mathbb{E}_t[\pi_{t+1}]$$

FOCs give

$$x_t=\epsilon\pi_t\qquad\pi_t=\frac{1}{1+\kappa\epsilon}[\beta\mathbb{E}_t(\pi_{t+1})]$$

- **commitment** This is an intertemporal question:

$$\max_{x_t,\pi_t}-\frac{1}{2}\mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left[x_t^2+\frac{\epsilon}{\kappa}\pi_t^2\right]$$

s.t.

$$\pi_t=\kappa x_t+\beta\mathbb{E}[\pi_{t+1}]+u_t$$

FOCs are

$$\beta^tx_t=\lambda_t\kappa$$
$$\beta^t\frac{\epsilon}{\kappa}\pi_t+\lambda_t-\beta\lambda_{t-1}=0$$

Optimal monetary policy: continuous time

In continuous time:

$$x_t=\mathbb{E}\left[x_{t+1}-\frac{1}{\gamma}(i_{t+1}-\pi_{t+1}-r_{t+1}^n)\right]\Rightarrow\dot{x}_t=\frac{1}{\gamma}(i_t-\pi_t-r_t^n)$$
$$\pi_t=\kappa x_t+\beta\mathbb{E}_t(\pi_{t+1})\Rightarrow\dot{\pi}=\rho\pi_t-\kappa x_t$$

the welfare problem is

$$\max-\int_0^\infty e^{-\rho t}(x_t^2+\chi\pi_t^2)$$

s.t.

$$\dot{x}_t=\frac{1}{\gamma}(i_t-\pi_t-r_t^n)\qquad\text{IS equation}$$
$$\dot{\pi}_t=\rho\pi_t-\kappa x_t\qquad\text{Phillips curve}$$
$$i_t\ge0\qquad\text{ZLB}$$

Incomplete market

Aiyagari

Household solves

$$\max_{c(s^t),A(s^{t+1})}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\Pr(s^t)u(c(s^t))$$

s.t.

$$A(s^{t+1})=y(s_t)-c(s^t)+RA(s^t)$$
$$A(s^t)\ge-b\qquad\text{borrowing bound}$$

a natural borrowing constraint is: the lowest-income state $\underline{s}$ forever $b=-\frac{1}{1-R}y(\underline{s})$.

Recursively, state variables are A,s, then

$$V(A,s)=\max_{c,A'}u(c)+\beta\sum_{s'}\Pr(s'\mid s)V(A',s')$$

s.t.

$$A'\le y(s)-c+RA,\ A\ge-b$$

define **cash on hand** $x\equiv RA+y$, rewrite the problem with x,s as state variables:

$$V(x,s)=\max_{c,x'(s')}u(c)+\beta\sum_{s'}\Pr(s'\mid s)V(x'(s'),s')$$

s.t.

$$x'(s')\le R(x-c)+y(s'),\ c\le x+b$$

FOCs are

$$u'(c)-\sum_s\lambda(s)R-\mu=0$$
$$\beta\Pr(s'\mid s)\frac{\partial V(x'(s'),s')}{\partial x}-\lambda(s)=0$$

gives

$$u'(c)=\beta R\sum_{s'}\Pr(s'\mid s)\frac{\partial V(x'(s'),s')}{\partial x}+\mu$$

and envelope theorem

$$\frac{\partial V(x,s)}{\partial x}=u'(c(x,s))$$

gives Euler equation

$$u'(c)=\beta R\sum_{s'}\Pr(s'\mid s)u'(c(x'(s'),s'))+\mu$$
$$\ge\beta R\sum_{s'}\Pr(s'\mid s)u'(c(x'(s'),s'))$$

Insurance contract

Planner chooses effort $\{e_t\}$ and consumption $\{c_t\}$ to **maximize profits s.t. to household's utility**

Frictionless benchmark

Planner's problem

$$V(w_0) = \max_{c,e} \sum_{s^t} \Pr(s^t \mid e)\beta^t (y(s_t) - c(s^t))$$

s.t.

$$\sum_{s^t} \Pr(s^t \mid e)\beta^t \left[u(c(s^t)) - e(s^{t-1})\right] = w_0$$

The recursive form is

$$V(w) = \max_{e,c(s),w'(s)} \sum_s \Pr(s \mid e)\left[y(s) - c(s) + \beta V(w'(s))\right]$$

s.t.

$$\sum_s \Pr(s \mid e)\left[u(c(s)) - e + \beta w'(s)\right] = w$$

FOCs are

$$-\Pr(s \mid e) + \mu \Pr(s \mid e) u'(c(s)) = 0 \Rightarrow u'(c(s)) = \frac{1}{\mu}$$

$$\Pr(s \mid e)\beta V'(w'(s)) + \mu \Pr(s \mid e)\beta = 0 \Rightarrow V'(w'(s)) = -\mu$$

plus envelope condition $V'(w(s)) = -\mu$ then **continuation utility is the same as current utility**.

Limited condition

Better than autarky $v_{aut} = \sum_{t=0}^{\infty} \beta^t (\sum_s \Pr(s) u(c(s))) = \frac{\sum_s \Pr(s)u(c(s))}{1-\beta}$. The recursive problem is

$$V(w) = \max_{c(s),w'(s)} \sum_s \Pr(s)\left[y(s) - c(s) + \beta V(w'(s))\right]$$

s.t.

$$\sum_s \Pr(s)\left[u(c(s)) + \beta w'(s)\right] = w$$

$$u(c(s)) + \beta w'(s) \geq u(y(s)) + \beta v_{aut}, \forall s$$

FOCs give

$$u'(c(s)) = -\frac{1}{V'(w'(s))} \qquad V'(w'(s)) = V'(w) - \frac{\lambda(s)}{\Pr(s)}$$

Two-sided limited commitment

Planner needs a positive profit, the recursive problem is

$$V(w) = \max_{c(s),w'(s)} \sum_s \Pr(s)\left[y(s) - c(s) + \beta V(w'(s))\right]$$

s.t.

$$\sum_s \Pr(s)\left[u(c(s)) + \beta w'(s)\right] = w$$

$$u(c(s)) + \beta w'(s) \geq u(y(s)) + \beta v_{aut}, \forall s$$

$$y(s) - c(s) + \beta V(w'(s)) \geq 0, \forall s$$

FOCs give

$$u'(c(s)) = -\frac{1}{V'(w'(s))} \qquad V'(w'(s)) = \frac{V'(w)\Pr(s) - \lambda(s)}{\Pr(s) + \eta(s)}$$

moral hazard

Both parties commit, but household exerts unobserved effort, the recursive problem is

$$V(w) = \max_{e,c(s),w'(s)} \sum_s \Pr(s \mid e)\left[y(s) - c(s) + \beta V(w'(s))\right]$$

s.t.

$$\sum_s \Pr(s \mid e)\left[u(c(s)) - e + \beta w'(s)\right] = w$$

$$\sum_s \Pr(s \mid e)\left[u(c(s)) - e + \beta w'(s)\right] \geq \sum_s \Pr(s \mid \tilde{e})\left[u(c(s)) - \tilde{e} + \beta w'(s)\right], \forall \tilde{e}$$

assume the special case: $e \in \{0,1\}$, FOCs give

$$u'(c(s)) = -\frac{1}{V'(w'(s))} \qquad V'(w'(s)) = V'(w) - \lambda\left[1 - \frac{\Pr(s \mid 0)}{\Pr(s \mid 1)}\right]$$

which gives the inverse Euler equation:

$$\mathbb{E}\left[\frac{1}{u'(c_{t+1})}\right] = \frac{1}{u'(c_t)}$$

$c \to 0$: agents will quit.

hidden income

Define transfer $\tau(s) \equiv c(s) - y(s)$, planner's problem (downward binding)

$$V(w) = \max_{\tau(s),w'(s)} \sum_s \Pr(s)\left[-\tau(s) + \beta V(w'(s))\right]$$

s.t.

$$\sum_s \Pr(s)\left[u(\tau(s) + y(s)) + \beta w'(s)\right] = w$$

$$u(\tau(s) + y(s)) + \beta w'(s) \geq u(\tau(s-1) + y(s)) + \beta w'(s-1), \forall s > 1$$

FOC gives

$$\Pr(s)\left[V'(w'(s)) + \mu\right] = \lambda(s+1) - \lambda(s)$$

sum over all s:

$$\sum_s \Pr(s)\left[V'(w'(s)) + \mu\right] = \sum_s [\lambda(s+1) - \lambda(s)]$$

$$\Rightarrow \sum_s \Pr(s) V'(w'(s)) = -\mu = V'(w(s))$$

$$\Rightarrow \mathbb{E}[V'(w'(s))] = V'(w)$$

Optimal policy

Primal optimal tax approach

- linear tax on labor: τ_t^l
- linear tax on capital: τ_t^k. No arbitrage between bond and capital gives: $R_t = 1 + (1 - \tau_t^k)(r_t^k - \delta)$
- no lump-sum tax
- the price series be $p_t \equiv \begin{cases} \prod_{s=0}^{t-1} \frac{1}{R_{s+1}} & t \geq 1 \\ 1 & t = 0 \end{cases}$

Primal optimal tax policy

Solve $\max \sum_{t=0}^{\infty} \beta^t u(c_t, L_t)$ s.t.

$$\sum_t p_t c_t \leq R_0 A_0 + \sum_t p_t w_t (1 - \tau_t^l) L_t$$

get implementability condition:

$$\sum_t \beta^t [u_c(c_t, L_t)c_t + u_L(c_t, L_t)L_t] \leq R_0(B_0 + K_0)u_c(c_0, L_0)$$

then define $W(c, L) = u(c, L) + \mu[u_c(c, L)c + u_L(c, L)L]$, government solves

$$\max \sum_{t=0}^{\infty} \beta^t W(c_t, L_t) - \mu u_c(c_0, L_0) R_0 (B_0 + K_0)$$

s.t.

$$c_t + K_{t+1} + g_t = F(K_t, L_t) + (1 - \delta)K_t$$

get capital rent

$$R_{t+1} = \frac{W_c(c_{t+1}, L_{t+1})}{u_c(c_{t+1}, L_{t+1})} \frac{u_c(c_t, L_t)}{W_c(c_t, L_t)} R_{t+1}^*$$

Optimal monetary policy

The price series $q_t = \begin{cases} \prod_{s=0}^{t-1} \frac{1}{1+i_{s+1}} & t \geq 1 \\ 1 & t = 0 \end{cases}$

Optimal monetary policy

households solve $\max \sum_t \beta^t u(c_t^1, c_t^2, L_t)$ (c_t^1 cash good, c_t^2 credit good), s.t.

$$p_t^1 c_t^1 + p_t^2 c_t^2 + M_{t+1} + B_{t+1} \leq w_t(1 - \tau_t)L_t + M_t + (1 + i_t)B_t$$

$$p_t^1 c_t^1 \leq M_t$$

get implementability condition:

$$\sum_t \beta^t \left[u_{c^1}(c_t^1, c_t^2, L_t)c_t^1 + u_{c^2}(c_t^1, c_t^2, L_t)c_t^2 + u_L(c_t^1, c_t^2, L_t)L_t\right] = 0$$

then government solves

$$\max \sum_t \beta^W (c_t^1, c_t^2, L_t)$$

s.t.

$$c_t^1 + c_t^2 + g_t = AL_t$$

optimal nominal interest rate:

$$1 + i_t = \frac{u_{c^1}(c_t^1, c_t^2, L_t)}{u_{c^2}(c_t^1, c_t^2, L_t)} = \frac{u_{c^1}(c_t^1, c_t^2, L_t)}{W_{c^1}(c_t^1, c_t^2, L_t)} \frac{W_{c^2}(c_t^1, c_t^2, L_t)}{u_{c^2}(c_t^1, c_t^2, L_t)}$$

Searching

- **number of matches:** $h = m(u, v)$
- **market tightness:** $\theta = \frac{v}{u}$
- **vacancy filling rate:** $\frac{h}{v} = \frac{m(u,v)}{v} = m\left(\frac{1}{\theta}, 1\right) \equiv q(\theta)$
- **job finding rate:** $\frac{h}{u} = \frac{h}{v}\frac{v}{u} = q(\theta)\theta = m(1, \theta)$, $\partial\theta q(\theta)/\partial\theta > 0$

static version

- bargaining: $\max_w (w - b)^{\beta}(y - w)^{1-\beta} \Rightarrow w = \beta y + (1 - \beta)b$
- surplus
  - worker: $w = b + \beta(y - b)$
  - firm: $-c + \frac{h}{v}(1 - \beta)(y - b) = -c + q(\theta)(1 - \beta)(y - b) = 0$
- planner solves $\max_v m(u, v)(y - b) - cv$, FOC gives

$$\frac{\partial m(u,v)}{\partial v} = \frac{c}{y - b} = q(\theta)(1 - \beta) \Rightarrow 1 - \beta = \frac{\partial m(u,v)}{\partial v} \frac{v}{m(u,v)}$$

dynamic version

- **Workers'** value

$$rU = b + \frac{h}{u}(E - U) + \dot{U}$$

$$rE = w + s(U - E) + \dot{E}$$

- **Firms'** value

$$rV = -c + \frac{h}{v}(J - V) + \dot{V}$$

$$rJ = y - w + s(V - J) + \dot{J}$$

- **surplus**

$$S = E - U + J - V$$

- bargaining result:

$$E = U + \beta S \qquad \text{workers}$$

$$J = V + (1 - \beta)S = (1 - \beta)S \qquad \text{firms}$$

Solve these, get

- $q(\theta) = \frac{c}{J}$: **more valuable** employees $J \uparrow$ leads to **tighter** market $\theta \uparrow$
- $S = \frac{c}{(1-\beta)q(\theta)}$
- $w = (1 - \beta)b + \beta y + \beta\theta c$: **tighter** market $\theta \uparrow$ leads to **higher** wage $w \uparrow$
- $\frac{r+s}{q(\theta)} + \beta\theta = \frac{(1-\beta)(y-b)}{c}$: **tightness** θ is **constant** over time.
- the elasticity of θ w.r.t $y - b$, ϵ is smaller than empirical observations