Topic 3: *Moving the Goalposts* Approach

by Sai Zhang

Key points:

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3.1 Finite Sample Bias-Variance Tradeoffs

3.1.1 Setup

Consider the fixed design regression model

$$y_i = w_i \beta(z_i) + h(z_i) + \epsilon_i \tag{3.1}$$

where

- w_i, z_i are treated as **fixed**
- ϵ_i is **independent**, with $\mathbb{E}[\epsilon_i] = 0, \mathbb{E}[\epsilon_i^2] = \sigma_i^2$
- observation: $\left\{ \left(y_i, w_i, z_i' \right)' \right\}_{i=1}^n$

one example is the case where w_i is **binary** , then

$$\beta(z) = f(1, z) - f(0, z)$$

which is just the ATE conditional on z under the unconfoundedness assumption. This includes the RD design, where z_i is the running variable and w_i is the treatment assignment.

Now, consider for the weighted average treatment effect

$$L_\mu[\beta(\cdot)] = \int \beta(z) \mathrm{d}\mu(z)$$

where $\int \mu(z) = 1$ is a **signed** measure (weight, allowing **negative** weights), construct a linear estimator

$$\hat{L}_a = \sum_{i=1}^n a_i y_i$$

where the estimation weights a_i can depend on $\left\{ z_i, w_i, \sigma_i^2 \right\}_{i=1}^n$, but **not** on y_i. Together, the bias of $\hat{L}_a$ for $L_\mu[\beta(\cdot)]$, given the regression function $\beta(\cdot), h(\cdot)$, is

$$\mathbb{E}_{\beta(\cdot),h(\cdot)}\left[\hat{L}_a\right] - L_\mu[\beta(\cdot)] = \sum_{i=1}^n a_i \left[w_i\beta(z_i) + h(z_i)\right] - \int \beta(z)\mathrm{d}\mu(z)$$

and its variance, given the regression function $\beta(\cdot), h(\cdot)$, is just

$$\text{Var}_{\beta(\cdot),h(\cdot)}\left[\hat{L}_a\right] = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

To bound the bias, assume $h(\cdot)$ is known to belong in a class of functions $\mathcal{H}$, then two approaches can be adopted, for the regularity of $\beta(\cdot)$ and the choice of $\mu(\cdot)$:

1 arbitrary $\beta(\cdot)$, optimizing weights μ by *moving the goalposts*, s.t. $L_\mu\left[\beta(\cdot)\right]$ is easy to estimate (Crump et al., 2006; Imbens and Wager, 2019) which gives the worst-case bias

$$\inf_{\mu} \sup_{\beta(\cdot),h(\cdot)} \left| \sum_{i=1}^{n} a_i \left[w_i \beta(z_i) + h(z_i) \right] - \int \beta(z)\mathrm{d}\mu(z) \right| \qquad \text{s.t. } h(\cdot) \in \mathcal{H}, \int \mathrm{d}\mu(z) = 1 \qquad (3.2)$$

2 assume constant treatment effects, i.e., $\beta(z) = \beta, \forall z$, which means that $L_\mu\left[\beta(\cdot)\right] = \beta$ regardless of μ (Armstrong et al., 2020), and the worst-case bias is

$$\sup_{\beta,h(\cdot)} \left| \sum_{i=1}^{n} a_i \left[w_i \beta + h(z_i) \right] - \beta \right| \qquad \text{s.t. } h(\cdot) \in \mathcal{H} \qquad (3.3)$$

And, the two approaches can be linked as such:

• If $\sum_{i=1}^{n} a_i w_i = 1$, 3.2 and 3.3 are both equal to

$$\sup_{h(\cdot)} \left| \sum_{i=1}^{n} a_i h(z_i) \right| \text{ s.t. } h(\cdot) \in \mathcal{H} \qquad (3.4)$$

   - 3.2 automatically equals 3.4
   - 3.3 is optimized (w.r.t. μ) by setting μ to place weight $a_i w_i$ on observation i, i.e., $\mu(\mathcal{Z}) = \sum_{i:z_i \in \mathcal{Z}} a_i w_i$, which implies $\sum_{i=1}^{n} a_i w_i \beta(z_i) - \int \beta(z)\mathrm{d}\mu(z) = 0$, hence the equality.
• Otherwise, 3.2 and 3.3 are both infinite:

   - 3.3 can be made arbitrarily large by choosing large enough β
   - 3.2 can be made arbitrarily large by making $\beta(\cdot)$ constant (as in 3.3) and large enough

3.1.2 Moving-the-goalpost Approach

3.1.3 Constant-treatment-effect Approach

Armstrong et al. (2020) adopt this approach, focusing on the case where $h(\cdot)$ is a high dimensional linear function, and the penalty function is an l_p norm of the coefficients.

Basic setting: Homoskedastic Gaussian errors

First, consider

$$Y = w\beta + Z\gamma + \epsilon \qquad (3.5)$$

where

- $\beta \in \mathbb{R}$ is the constant treatment effect to be estimated
- $\gamma \in \Gamma$ is the control coefficients, subject to the restriction (i.e., the function class $\mathcal{H}$)

$$\Gamma = \Gamma(C) = \left\{\gamma \in \mathcal{G} : \text{Pen}(\gamma) \le C\right\} \tag{3.6}$$

where $\text{Pen}(\cdot)$ is a seminorm[1] on some linear subspace $\mathcal{G}$ of $\mathbb{R}^k$.

- $w = (w_1, \cdots, w_n)' \in \mathbb{R}^n$ and $Z = (z_1', \cdots, z_n')' \in \mathbb{R}^{n \times k}$ are defined as before
- $\epsilon \sim \mathcal{N}\left(0, \sigma^2 I_n\right)$ is assumed **normal and homoskedastic**, with σ^2 known

For estimation, the goal is to construct estimators and CIs for β:

- estimator $\hat{\beta}$: consider the worst-case performance over the parameter space $\mathbb{R} \times \Gamma$ under the **MSE** criterion

$$R_{MSE}\left(\hat{\beta}; \Gamma\right) = \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} \mathbb{E}_{\beta, \gamma}\left[\left(\hat{\beta} - \beta\right)^2\right]$$

- for CIs, we have 2 requirements:

  A **coverage**: A $100 \cdot (1 - \alpha)\%$ CI with half-length $\hat{\chi} = \hat{\chi}(Y, X)$ is an interval $\left\{\hat{\beta} \pm \hat{\chi}\right\}$ s.t.

$$\inf_{\beta \in \mathbb{R}, \gamma \in \Gamma} P_{\beta, \gamma}\left(\beta \in \left\{\hat{\beta} \pm \hat{\chi}\right\}\right) \ge 1 - \alpha$$

  B **length**: the exepcted length of a CI $\mathbb{E}_{\beta, \gamma}\left[2\hat{\chi}\right]$ should be as short as possible

  notice that length-optimized CIs are **not** necessarily centered at an MSE-centered $\hat{\beta}$.

Linear estimators and CIs

Again, consider estimators that are **linear** in the outcomes Y, $\hat{\beta} = a'Y$, where a is the $n-$vector weights. In the vector form, the worst-case bias (as 3.3) is

$$\overline{\text{bias}}_\Gamma\left(\hat{\beta}\right) = \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} a'\left(w\beta + Z\gamma\right) - \beta \tag{3.7}$$

and the variance, under the assumption of homoskedasticity, is

$$\text{Var}\left(\hat{\beta}\right) = \sigma^2 a'a$$

Then the MSE is

$$R_{MSE}\left(\hat{\beta}; \Gamma\right) = \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} \mathbb{E}_{\beta, \gamma}\left[\left(\hat{\beta} - \beta\right)^2\right] = \overline{\text{bias}}_\Gamma\left(\hat{\beta}\right)^2 + \text{Var}\left(\hat{\beta}\right)$$

The $t-$statistic is

$$\frac{\hat{\beta} - \beta}{\sqrt{\text{Var}\left(\hat{\beta}\right)}} \sim \mathcal{N}(b, 1), \ |b| \le \frac{\overline{\text{bias}}_\Gamma\left(\hat{\beta}\right)}{\sqrt{\text{Var}\left(\hat{\beta}\right)}}$$

and a two-sided CI can then be formed as

$$\hat{\beta} \pm \chi, \qquad \text{where } \chi = \sqrt{\text{Var}\left(\hat{\beta}\right)} \cdot \text{cv}_\alpha\left(\frac{\overline{\text{bias}}_\Gamma\left(\hat{\beta}\right)}{\sqrt{\text{Var}\left(\hat{\beta}\right)}}\right) \tag{3.8}$$

and the $\text{cv}_\alpha(B)$ denotes the $1 - \alpha$ quantile of a $|\mathcal{N}(B, 1)|$. This is a **fixed-length confidence interval** (FLCI), with a fixed length of 2χ. It depends on X and σ^2, but not on Y or $(\beta, \gamma)'$.

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[1]Seminorm satisfies **triangle inequality** $\text{Pen}\left(\gamma + \tilde{\gamma}\right) \le \text{Pen}(\gamma)$ and **homogeneity** $\text{Pen}\left(c\gamma\right) = |c|\text{Pen}(\gamma), \forall c$, but **NOT** necessarily positive definite ($\text{Pen}(\gamma) = 0$ does not imply $\gamma = 0$). Essentially, any convex set Γ that is symmetric satisfies this definition.

Optimal weights

We have two optimization goals

- minimizing MSE: $R_{MSE}\left(\hat{\beta};\Gamma\right) = \overline{\text{bias}}_\Gamma\left(\hat{\beta}\right)^2 + \text{Var}\left(\hat{\beta}\right)$
- minimizing CI length: $\chi = \sqrt{\text{Var}\left(\hat{\beta}\right)} \cdot \text{cv}_\alpha\left(\overline{\text{bias}}_\Gamma\left(\hat{\beta}\right) / \sqrt{\text{Var}\left(\hat{\beta}\right)}\right)$

They both increasing in $\text{Var}\left(\hat{\beta}\right)$ and $\overline{\text{bias}}_\Gamma\left(\hat{\beta}\right)$, hence to find the optimal weights, it suffices to minimize variance subject to a bound B on worst-case bias, which can be written as:

$$\min_{a \in \mathbb{R}} a'a \text{ s.t. } \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} a'\left(w\beta + Z\gamma\right) - \beta \leq B \tag{3.9}$$

The optimal weight is then given by:

Theorem 3.1.1: Optimal Weight

Let π_λ^* be a solution to[a]

$$\min_\pi \|w - Z\pi\|_2^2 \text{ s.t. } \text{Pen}(\pi) \leq t_\lambda \tag{3.10}$$

and suppose that $\|w - Z\pi\|_2 > 0$, $\text{Pen}(\cdot)$ is continuous, then the optimal weight solving 3.9 is

$$a_\lambda^* = \frac{w - Z\pi_\lambda^*}{\left(w - Z\pi_\lambda^*\right)' w}$$

with the bound

$$B = \frac{C}{t_\lambda} \cdot \frac{\left(w - Z\pi_\lambda^*\right)' Z\pi_\lambda^*}{\left(w - Z\pi_\lambda^*\right)' w}$$

Consequently, we have
- estimator

$$\hat{\beta}_\lambda = a_\lambda^{*\prime} Y = \frac{\left(w - Z\pi_\lambda^*\right)' Y}{\left(w - Z\pi_\lambda^*\right)' w}$$

- worst-case bias

$$\overline{\text{bias}}_\Gamma\left(\hat{\beta}_\lambda\right) = C\overline{B}_\lambda = \frac{C}{\text{Pen}\left(\pi_\lambda^*\right)} \frac{\left(w - Z\pi_\lambda^*\right)' Z\pi_\lambda^*}{\left(w - Z\pi_\lambda^*\right)' w}$$

- variance of estimator

$$V_\lambda = \frac{\sigma^2 \|w - Z\pi_\lambda^*\|_2^2}{\left[\left(w - Z\pi_\lambda^*\right)' w\right]^2}$$

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[a]This regression can be refered to as a regularized propensity score regression (but w_i need not be binary) with penalty $\text{Pen}(\pi)$

This result follows by applying Donoho (1994), Low (1995) and Armstrong and Kolesár (2018), rewriting 3.9

as a convex optimization problem.

A Proofs

A.1 Proof of Theorem 3.1.1

Following Armstrong and Kolesár (2018, Equation (25)), the modulus of continuity is given by

$$\omega(\delta) = \sup_{\beta,\gamma} 2\beta \qquad \text{s.t. } \|w\beta + Z\gamma\|_2 \leq \frac{\delta}{2}, \qquad \text{Pen}(\gamma) \leq C$$

Introducing a substitution (rescaling γ by β) $\pi = -\frac{\gamma}{\beta}$, get

$$\omega(\delta) = \sup_{\beta,\pi} 2\beta \qquad \text{s.t. } \beta\|w - Z\pi\|_2 \leq \frac{\delta}{2}, \qquad \beta\text{Pen}(\pi) \leq C \qquad (3.11)$$

recall the optimization problem in Theorem 3.1.1:

$$\min_{\pi}\|w - Z\pi\|_2^2 \quad \text{s.t. } \text{Pen}(\pi) \leq t_\lambda$$

We can relate the two problems via the following logic: we want to make $\|w - Z\pi\|_2$ and $\text{Pen}(\pi)$ small so that large values of β satisfy the constraint of 3.11. Formally:

Lemma A.1

- If $\exists \pi \in \mathcal{G}$ s.t. $w = Z\pi$ and $\text{Pen}(\pi) = 0$, then $w(\delta) = \infty, \forall \delta \geq 0$ (automatic)
- **Otherwise**:

  (i) $\forall \delta > 0$, the problem 3.11 has a solution $\beta_\delta^{mod}, \pi_\delta^{mod}$ with $\beta_\delta^{mod} > 0$. For $t_\lambda = \frac{C}{\beta_\delta^{mod}} = \frac{2C}{w(\delta)}$, π_δ^{mod} is also a solution to the penalized regression (3.10)

  $$\min_{\pi}\|w - Z\pi\|_2^2 \quad \text{s.t. } \text{Pen}(\pi) \leq t_\lambda$$

  with optimized objective

  $$\|w - Z\pi\|_2 = \frac{\delta}{2\beta_\delta^{mod}} = \frac{\delta}{w(\delta)} > 0$$

  (ii) $\forall t_\lambda > 0$, the penalized regression above has a solution $\pi_\lambda^\star$. Setting

  $$\beta_\lambda^* = \frac{C}{t_\lambda}$$

  $$\delta_\lambda = 2\beta_\lambda^*\|w - Z\pi\|_2 = \frac{2C}{t_\lambda}\|w - Z\pi\|_2$$

  the pair $\left(\beta_\lambda^*, \pi_\lambda^*\right)$ solves the modulus problem 3.11 at $\delta = \delta_\lambda$, with optimized objective $w(\delta_\lambda) = \frac{2C}{t_\lambda}$, as long as $\|w - Z\pi\|_2 > 0$

Proof of Lemma A.1: we prove the lemma with the following steps:

A <u>the penalized problem 3.10 has a solution</u>:

Let $\mathcal{G}^{(0)}$ denote the linear subspace of vectors $\pi \in \mathcal{G}$ s.t. $Z\pi = 0$, $\text{Pen}(\pi) = 0$; let $\mathcal{G}^{(1)}$ be a subspace s.t. $\mathcal{G} = \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)}$. Then, we can write $\pi \in \mathcal{G}$ uniquely as $\pi = \pi^{(0)} + \pi^{(1)}$ where $\pi^{(0)} \in \mathcal{G}^{(0)}, \pi^{(1)} \in \mathcal{G}^{(1)}$.

Therefore, we have $Z\pi = Z\pi^{(1)}$, and

$$\begin{aligned}
\text{Pen}(\pi^{(1)}) &= \text{Pen}(\pi^{(1)}) - \text{Pen}(-\pi^{(0)}) \le \text{Pen}(\pi) \\
\text{Pen}(\pi^{(1)}) &= \text{Pen}(\pi^{(1)}) + \text{Pen}(\pi^{(0)}) \ge \text{Pen}(\pi)
\end{aligned} \quad \Rightarrow \text{Pen}(\pi^{(1)}) = \text{Pen}(\pi)$$

THen, 3.10 can be wriiten in terms of $\pi^{(1)} \in \mathcal{G}^{(1)}$ only. The level sets of this optimization problem are bounded and closed (by continuity of the seminorm $\text{Pen}(\cdot)$), so it has a solution, which is also the solution to the original problem.

B <u>the modulus problem 3.11 has a solution:</u> for the problem 3.11, feasible values of β are bounded as:

$$\beta \le \frac{\delta}{2} \cdot \frac{1}{\|w - Z\pi\|_2} \qquad\qquad \beta \le C \cdot \frac{1}{\text{Pen}(\pi)}$$

i.e., β is bounded by the inverse of the minimum of $\max\{\|w - Z\pi\|_2, \text{Pen}(\pi)\}$ over π, and it is strictly positive. Hence, $\beta, \tilde{\pi}^{(1)}$ can be restricted to a compact set without changing the optimization problem.

C <u>proof of statement (i):</u> Proof by contradiction, if it's not true, then $\exists \tilde{\pi}$ s.t.

$$\text{Pen}(\tilde{\pi}) \le \frac{C}{\beta_\delta^{mod}} \equiv t_\lambda, \qquad\qquad \|w - Z\tilde{\pi}\|_2 \le \|w - Z\pi_\delta^{mod}\|_2 - \nu$$

then for some η, let $\tilde{\pi}_\eta = (1-\eta)\tilde{\pi}$, we have

$$\begin{aligned}
\|w - Z\tilde{\pi}_\eta\|_2 &= \|w - Z(1-\eta)\tilde{\pi}\|_2 \\
&\le \|w - X\tilde{\pi}\|_2 + \eta\|Z\tilde{\pi}\|_2 \\
&\le \|w - X\pi_\delta^{mod}\|_2 - v + \eta\|Z\tilde{\pi}\|_2 \\
&\le \frac{\delta}{2\beta_\delta^{mod}} - v + \eta\|Z\tilde{\pi}\|_2
\end{aligned}$$

Hence, $\exists \eta$ small enough, s.t.

$$\|w - Z\tilde{\pi}_\eta\|_2 < \frac{\delta}{2\beta_\delta^{mod}}, \qquad\qquad \text{Pen}(\tilde{\pi}_\eta) \le (1-\eta)\frac{C}{\beta_\delta^{mod}} < \frac{C}{\beta_\delta^{mod}}$$

therefore, by setting $\pi = \tilde{\pi}_\eta$, we can allow a strictly bigger β, which is a contradiction.

D <u>proof of statement (ii):</u> This result follows immediately.

Next, use Lemma A.1 to prove Theorem 3.1.1. Following Armstrong and Kolesár (2018), the class of bias-variance optimizing estimators is

$$\frac{\left(w\beta_\delta^{mod} + Z\gamma_\delta^{mod}\right)' Y}{\left(w\beta_\delta^{mod} + Z\gamma_\delta^{mod}\right)' w}$$

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