

Topic 14: Regularization Methods in Thresholded Parameter Space

by Sai Zhang

Key points: The connections and differences of all regularization methods and some interesting phase transition phenomena.

Disclaimer: The note is built on Prof. [Jinchi Lv](#)'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

14.1 Model Setup

Now, consider a generalized linear model (GLM) linking a p -dimensional predictor \mathbf{x} to a scalar response Y . With canonical link, the conditional distribution of Y given \mathbf{x} has density

$$f(y; \theta, \phi) = \exp [y\theta - b(\theta) + c(y, \phi)]$$

where $\theta = \mathbf{x}'\boldsymbol{\beta}$ with $\boldsymbol{\beta}$ a p -dimensional regression coefficient vector, $b(\cdot)$ and $c(\cdot, \cdot)$ are known functions and ϕ is dispersion parameter. Again, $\boldsymbol{\beta} = (\beta_{0,1}, \dots, \beta_{0,p})'$ is sparse with many zero components, and $\log p = O(n^a)$ for some $0 < a < 1$.

The penalized negative log-likelihood is

$$Q_n(\boldsymbol{\beta}) = -n^{-1} [\mathbf{y}'\mathbf{X}\boldsymbol{\beta} - \mathbf{1}'\mathbf{b}(\mathbf{X}\boldsymbol{\beta})] + \|p_\lambda(\boldsymbol{\beta})\|_1$$

where

- $\mathbf{y} = (y_1, \dots, y_n)'$, $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)'$, each column of \mathbf{X} is rescaled to have L_2 -norm \sqrt{n}
- $\mathbf{b}(\boldsymbol{\theta}) = (b(\theta_1), \dots, b(\theta_n))'$ with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)'$
- $\|p_\lambda(\boldsymbol{\beta})\|_1 = \sum_{j=1}^p p_\lambda(|\beta_j|)$

Next, define **robust spark** κ_c

Definition 14.1.1: Robust spark κ_c

The robust spark κ_c of the $n \times p$ design matrix \mathbf{X} is defined as the smallest possible positive integer s.t. there exists an $n \times \kappa_c$ submatrix of $\frac{1}{\sqrt{n}}\mathbf{X}$ having a singular value less than a given positive constant c ([Zheng et al., 2014](#)), and

$$\kappa_c \leq n + 1$$

Bounding sparse model size can control collinearity and ensure model identifiability and stability, and as $c \rightarrow 0+$, κ_c approaches the spark. Robust spark can be some large number diverging with n :

Proposition 14.1.2: Order of κ_c

Assume $\log p = o(n)$ and that the rows of the $n \times p$ random design matrix \mathbf{X} are i.i.d. as $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ has smallest eigenvalue bounded from below by some positive constant. Then there exist

positive constants c and \tilde{c} s.t. with asymptotic probability one, $\kappa_c \geq \frac{\tilde{c}n}{\log p}$

Next, we define a thresholded parameter space

Definition 14.1.3: Thresholded parameter space

$$\mathcal{B}_{\tau,c} = \left\{ \beta \in \mathbb{R}^p : \|\beta\|_0 < \frac{\kappa_c}{2}, \text{ and for each } j, \beta_j = 0 \text{ or } |\beta_j| \geq \tau \right\}$$

where $\beta = (\beta_1, \dots, \beta_p)'$. τ is some positive threshold on parameter magnitude:

Here, τ is very important:

- τ is key to distinguishing between important covariates and noise covariates for the purpose of variable selection
- τ typically needs to satisfy $\tau \sqrt{n/\log p} \xrightarrow{n \rightarrow \infty} \infty$

It turns out that the solution to the regularization problem has the (very natural) hard-thresholding property:

Proposition 14.1.4: Hard-thresholding property

or the L_0 -penalty $p_\lambda(t) = \lambda \mathbf{1}_{t \neq 0}$, the global minimizer $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_p)'$ of the regularization problem over \mathbb{R}^p satisfies that each component $\hat{\beta}_j$ is either 0 or has magnitude larger than some positive threshold

This hard-thresholding property is shared by many other penalties such as SICA penalties. This property guarantees sparsity of the model: weak signals are generally difficult to stand out comparing to noise variables due to impact of high dimensionality

14.2 Asymptotic Equivalence of Regularization Methods

For a universal $\lambda = c_0 \sqrt{\log p/n}$ with $c_0 > 0$ and p implicitly as $n \vee p$, consider 2 key events:

$$\mathcal{E} = \left\{ \|n^{-1} \mathbf{X}' \epsilon\|_\infty \leq \lambda/2 \right\} \quad \mathcal{E}_0 = \left\{ \|n^{-1} \mathbf{X}'_{\alpha_0} \epsilon\|_\infty \leq c_0 \sqrt{\log n/n} \right\}$$

where $\epsilon = \mathbf{y} - \mathbb{E}\mathbf{y}$, \mathbf{X}_α is a submatrix of \mathbf{X} consisting of columns in α . Here, let $\alpha_0 = \text{supp}(\beta_0)$ (non-zero variables in the true model).

For this setting, consider the following technical conditions:

- **Error tail distribution:** $\Pr(\mathcal{E}^c) = O(p^{-c_1})$ and $\Pr(\mathcal{E}_0^c) = O(n^{-c_1})$ for some positive constant c_1 that can be sufficiently large for large enough c_0
- **Bounded variance:** $b(\theta)$ satisfies that $c_2 \leq b''(\theta) \leq c_2^{-1}$ in its domain, where c_2 is some positive constant
- **Concave penalty function:** $p_\lambda(t)$ is increasing and concave in $t \in [0, \infty)$ with $p_\lambda(0) = 0$, and is differentiable with $p'_\lambda(0+) = c_3 \lambda$ for some positive constant c_3

we have that the global minimizer exists and satisfies oracle inequalities:

Theorem 14.2.1: Oracle Inequalities

Assume

References

Zemin Zheng, Yingying Fan, and Jinchi Lv. High dimensional thresholded regression and shrinkage effect. *Journal of the Royal Statistical Society: Series B: Statistical Methodology*, pages 627–649, 2014.