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# Topic 18: Eigenvalue and Spike Models

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**Key points**: .

**Disclaimer**: *The note is built on Prof. Jinchi Lv's lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.*

## 18.1 Motivation

Consider $n$ independent observations $\mathbf{X}_i \in \mathbb{R}^p$ drawn from a $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, then the covariance can be decomposed into 2 parts, white noise and low rank

$$\mathbf{\Sigma} = \mathrm{Cov}(\mathbf{X}_i) = \mathbf{I} + \sum_{k=1}^{M} \theta_k \nu_k \nu_k' = \mathbf{\Sigma}_0 + \mathbf{\Phi}$$

where $M$ denotes the **number of spikes** in the distribution of eigenvalues. The idea is: spikes deviate from a reference model along a **small fixed number** of unknown directions. If $\mathbf{\Phi} = \mathbf{0}$, then none of the sample eigenvalues is separated from the bulk.

**Why a spike model is interesting?** A spike model can help determine the latent dimension of the data, some examples being

- Principal component analysis (PCA): spikes are related to the directions of the most variations of the data, i.e., the principal components
- Clustering model: $M$ spikes is equivalent to $M + 1$ clusters
- Economic significance: $M$ is related to the number of factor loadings

Then the question is threefold:

- How to determine $M$
- How to estimate $\nu_k$
- How to test $\theta_k$

Under rank one alternative, we would like to test the hypothesis

$$H_1 : \mathbf{\Sigma} = \mathbf{I}_p + \theta \boldsymbol{\nu}\boldsymbol{\nu}', \theta > 0$$

against the null

$$H_0 : \mathbf{\Sigma} = \mathbf{I}_p$$

with the key assumptions:

A1 Gaussian error
A2 large $p$: $p \leq n$ but allows $p/n \to \gamma \in (0, 1)$

Under these assumptions, for the $n \times p$ data matrix $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1' & \cdots & \mathbf{X}_n' \end{pmatrix}'$, $\mathbf{X}'\mathbf{X}$ has a $p-$dimensional **Wishart** distribution $W_p(n, \boldsymbol{\Sigma})$ with the degree of freedom $n$ and covariance matrix $\boldsymbol{\Sigma}$, which is a *random matrix*.

If $\mathbf{Y} = \mathbf{M} + \mathbf{X}$, that is, the sum of the *random matrix* $\mathbf{X}$ and a *deterministic matrix* $\mathbf{M}$ (also $n \times p$), then $\mathbf{Y}'\mathbf{Y}$ has a $p-$dimensional Wishart distribution $W_p(n, \boldsymbol{\Sigma}, \boldsymbol{\Psi})$ with $n$ degrees of freedom, covariance matrix $\boldsymbol{\Sigma}$ and non-centrality matrix $\boldsymbol{\Psi} = \boldsymbol{\Sigma}^{-1}\mathbf{M}'\mathbf{M}$.

**Definition 18.1.1: Density of Wishart Distribution**

The PDF of Wishart distribution is defined as

$$f(\mathbf{X}) = \frac{1}{2^{np/2}\Gamma_p\left(\frac{n}{2}\right)|\boldsymbol{\Sigma}|^{n/2}}|\mathbf{X}|^{(n-p-1)/2}\exp\left(-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)\right)$$

where $\mathbf{X}$ is a symmetric positive semidefinite and $\Gamma_p\left(\frac{n}{2}\right)$ is a multivariate gamma function such that

$$\Gamma_p\left(\frac{n}{2}\right) = \pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma\left(\frac{n}{2} - \frac{j-1}{2}\right)$$

Notice that the sample covariance matrix $\mathbf{S} = \frac{1}{n}\mathbf{X}'\mathbf{X}$ is just a scaled version of Wishart distribution

$$n\mathbf{S} = \mathbf{X}'\mathbf{X} \sim W_p(n, \boldsymbol{\Sigma})$$

For $\boldsymbol{\Sigma} = \mathbf{I}_p$, the empirical distribution fo eigenvalues converges to Marcenko-Pastur distribution

$$f^{\mathrm{MP}}(x) = \frac{1}{2\pi\gamma x}\sqrt{(b_+ - x)(x - b_-)}$$

where $b_\pm = (1 \pm \sqrt{\gamma})^2$. Then:

- under $H_0 : \boldsymbol{\Sigma} = \mathbf{I}_p$, we have

$$n^{2/3}\left(\frac{\lambda_1 - \mu(\gamma)}{\sigma(\gamma)}\right) \xrightarrow{d} \mathrm{TW}_1$$

  where $\mathrm{TW}_1$ is the Tracy-Widom distribution
- under $H_1 : \boldsymbol{\Sigma} = \mathbf{I}_p + \theta\boldsymbol{\nu}\boldsymbol{\nu}', \theta > 0$, if $\theta$ is strong ($\theta \gg \sqrt{\gamma}$), then

$$n^{1/2}\left(\frac{\lambda_1 - \rho(\theta, \gamma)}{\tau(\theta, \gamma)}\right) \xrightarrow{d} \mathcal{N}(0, 1)$$

Here, the largest eigenvalue test is the best test. **But** when the signal is weak ($0 \le \theta < \sqrt{\gamma}$), the largest eigenvalue under the alternative converges to the same distribution as null:

$$n^{2/3}\left(\frac{\lambda_1 - \rho(\theta, \gamma)}{\tau(\theta, \gamma)}\right) \xrightarrow{d} \mathrm{TW}_1$$

which means that the largest eigenvalue test *fails*. On top of this, **resampling** also fails when $p$ is large.

Next, we develop another test to cope with these problems.

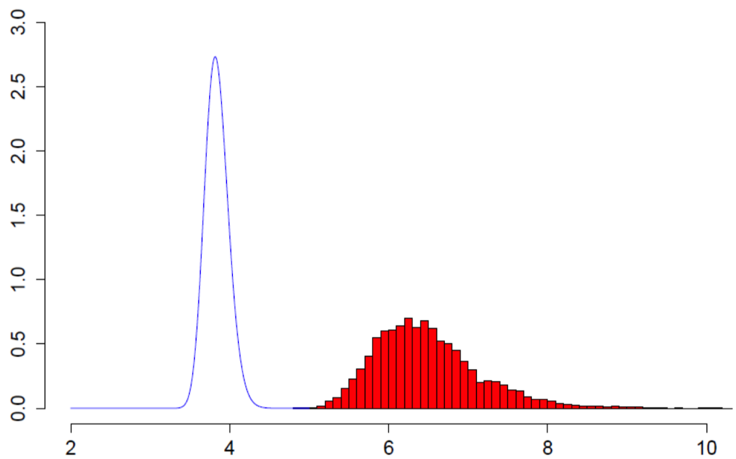



Figure 18.1: Failure of Resampling Test ($n = p = 100$)

## 18.2 Johnstone and Onatski (2020)

Consider the basic equation of classical multivariate statistics:

$$\det(\mathbf{H} - \mathbf{x}\mathbf{E}) = 0 \tag{18.1}$$

with $p \times p$ matrices

$$n_1 \mathbf{H} = \sum_{k=1}^{n_1} \mathbf{x}_k \mathbf{x}_k' \qquad \textit{hypothesis SS}$$

$$n_1 \mathbf{E} = \sum_{k=1}^{n_1} \mathbf{z}_k \mathbf{z}_k' \qquad \textit{error SS}$$

The solution $\mathbf{x}$ is generalized eigenvalues $\{\lambda_i\}_{i=1}^p$, which are the eigenvalue of F-ratio $\mathbf{E}^{-1}\mathbf{H}$. Johnstone and Onatski (2020) summarized 5 topics using $\mathbf{E}^{-1}\mathbf{H}$ relying on the five most common hypergeometric functions[1] ${}_p\mathcal{F}_q$

[1]Hypergeometric functions are:

- scalar inputs

$${}_p\mathcal{F}_q(a, b; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k} \frac{x^k}{k!}$$

  where $(a_j)_k$ are generalized Pochhammer symbols
- single matrix inputs, where $\mathbf{S}$ is symmetric and usually diagonal

$${}_p\mathcal{F}_q(a, b; \mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_p)_\kappa} \frac{C_\kappa(\mathbf{S})}{k!}$$

  where $C_k$ are the zonal polynomials. Easily, ${}_0\mathcal{F}_0(\mathbf{S}) = e^{\mathrm{tr}(\mathbf{S})}$, ${}_1\mathcal{F}_0(a, \mathbf{S}) = |\mathbf{I} - \mathbf{S}|^{-a}$
- two matrix inputs, where $\mathbf{S}, \mathbf{T}$ are both symmetric

$${}_p\mathcal{F}_q(a, b; \mathbf{S}, \mathbf{T}) = \int_{O(p)} {}_p\mathcal{F}_q(a, b; \mathbf{SUTU'})(d)\mathbf{U}$$

Table 18.1: 5 Statistical Methods

| | | Statistical method | $n_1\mathbf{H}$ | $n_2\mathbf{E}$ | Univariate Analog |
|---|---|---|---|---|---|
| ${}_0\mathcal{F}_0$ | PCA | Principal components analysis | $W_p(n_1, \boldsymbol{\Sigma}+\boldsymbol{\Phi})$ | $n_2\boldsymbol{\Sigma}$ | $\chi^2$ |
| ${}_1\mathcal{F}_0$ | SigD | Signal detection | $W_p(n_1, \boldsymbol{\Sigma}+\boldsymbol{\Phi})$ | $W_p(n_2, \boldsymbol{\Sigma})$ | non-central $\chi^2$ |
| ${}_0\mathcal{F}_1$ | $\text{REG}_0$ | Multivariate regression, with known error | $W_p(n_1, \boldsymbol{\Sigma}, n_1\boldsymbol{\Phi})$ | $n_2\boldsymbol{\Sigma}$ | $F$ |
| ${}_1\mathcal{F}_1$ | REG | Multivariate regression, with unknown error | $W_p(n_1, \boldsymbol{\Sigma}, n_1\boldsymbol{\Phi})$ | $W_p(n_2, \boldsymbol{\Sigma})$ | non-central $F$ |
| ${}_2\mathcal{F}_1$ | CCA | Canonical correlation analysis | $W_p(n_1, \boldsymbol{\Sigma}, \boldsymbol{\Phi}(\mathbf{Y}))$ | $W_p(n_2, \boldsymbol{\Sigma})$ | $\frac{r^2}{1-r^2}$ |

For ${}_0\mathcal{F}_0$ and ${}_0\mathcal{F}_1$, $\mathbf{E}$ is deterministic, $\boldsymbol{\Sigma}$ is known, $n_2$ disppears, otherwise $\mathbf{E}$ is independent of $\mathbf{H}$.

### 18.2.1 Definitions and global assumptions

Let $\mathbf{Z}$ be an $n \times p$ data matrix with rows (observations) drawn **i.i.d.** from $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$, and a deterministic matrix $\mathbf{M}$ of $n \times p$, then for $\mathbf{Y} = \mathbf{M} + \mathbf{Z}$,

- $\mathbf{H} = \mathbf{Y}'\mathbf{Y}$ has a $p$ dimensional Wishart distribution $W_p(n, \boldsymbol{\Sigma}, \boldsymbol{\Psi})$ with $n$ degrees of freedom, covariance matrix $\boldsymbol{\Sigma}$ and non-centrality matrix $\boldsymbol{\Psi} = \boldsymbol{\Sigma}^{-1}\mathbf{M}'\mathbf{M}$
- the corresponding central Wishart distribution with $\mathbf{M} = \mathbf{0}$ is $W_p(n, \boldsymbol{\Sigma})$

Johnstone and Onatski (2020) assume a relative low dimensionality $p \le \min\{n_1, n_2\}$ where $n_1, n_2$ are the degrees of freedom as in Table 18.1, where

- $p \le n_2$ ensures almost sure invertibility of matrix $\mathbf{E}$ in Equation 18.1
- $p \le n_1$ is not essential, but reduces the number of various situations of consideration.

### 18.2.2 5 classes of problems

With these assumptions, they established a unified statistical problem **symmetric matrix denoising (SMD)** that can be linked to the 5 classes of problems:

**PCA** $n_1$ i.i.d. observations drawn from $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Omega})$ to test the hull hypothesis that the population covariance $\boldsymbol{\Omega} = \boldsymbol{\Sigma}$, with the alternative of interest being

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma} + \boldsymbol{\Phi}, \quad \text{with } \boldsymbol{\Phi} = \theta\boldsymbol{\phi}\boldsymbol{\phi}'$$

where $\theta > 0$, $\boldsymbol{\phi}$ are unknown, and $\boldsymbol{\phi}$ is normalized s.t. $\left\|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\phi}\right\| = 1$. W.L.O.G., assume $\boldsymbol{\Sigma} = \mathbf{I}_p$, then under the alternative, the first principal component explains a larger portion of the variation than the other principal components. Re-formulate the hypotheses in terms of the spectral *spike* parameter $\theta$, we have

$$H_0 : \theta_0 = 0 \qquad\qquad H_1 : \theta_0 = \theta > 0 \tag{18.2}$$

where $\theta_0$ is the true value of the *spike*. A **maximal invariant statistic** consists of the solutions $\lambda_1 \ge \cdots \ge \lambda_p$ of Equation 18.1 with

- $n_1\mathbf{H}$ equal to the sample covariance matrix
- $\mathbf{E} = \boldsymbol{\Sigma}$

**SigD** Now consider testing the **equality** of covariance matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Sigma}$, corresponding to 2 independent $p-$dimensional mean-zero Gaussian samples of size $n_1$ and $n_2$, with the alternative still

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma} + \boldsymbol{\Phi}, \quad \text{with } \boldsymbol{\Phi} = \theta\boldsymbol{\phi}\boldsymbol{\phi}'$$

and again, assume $\mathbf{\Sigma} = \mathbf{I}_p$ (but NOT necessarily known), here, instead of Equation 18.1, consider

$$\det\left(\mathbf{H} - \lambda\left(\mathbf{E} + \frac{n_1}{n_2}\mathbf{H}\right)\right) = 0 \tag{18.3}$$

naturally, SigD reduces to PCA as $n_2 \to \infty$ while $n_1$ and $p$ held constant.

**REG$_0$** Next, consider a linear regression with multivariate response

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

with known covariance matrix $\mathbf{\Sigma}$ of the i.i.d. Gaussian rows of the error matrix $\boldsymbol{\epsilon}$. Here, to test linear restrictions on the matrix of coefficients $\boldsymbol{\beta}$, we can split the matrix of transformed response variables $\mathbf{Y}$ into 3 parts $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$, where

- $\mathbf{Y}_1$ is $n_1 \times p$ where $p$ is the number of response variables, $n_1$ is the number of linear restrictions (per each of the $p$ columns of matrix $\boldsymbol{\beta}$), under the null $H_0 : \mathbb{E}\mathbf{Y}_1 = 0$, versus the alternative

$$\mathbb{E}\mathbf{Y}_1 = \sqrt{n_1\theta}\boldsymbol{\psi}\boldsymbol{\phi}' \tag{18.4}$$

  where $\theta > 0$, $\left\|\mathbf{\Sigma}^{-1/2}\boldsymbol{\phi}\right\| = 1$ and $\|\boldsymbol{\psi}\| = 1$
- $\mathbf{Y}_2$ is $(q - n_1) \times p$, where $q$ is the number of regressors
- $\mathbf{Y}_3$ is $(T - q) \times p$, where $T$ is the number of observations

In this case, tests can be based on the solutions $\lambda_1, \cdots, \lambda_p$ to

$$\det(\mathbf{H} - \lambda\mathbf{E}) = \mathbf{0}$$

where $\mathbf{H} = \mathbf{Y}_1'\mathbf{Y}_1/n_1$ and $\mathbf{E} = \mathbf{\Sigma}$. The solutions represent a multivaraite analog of the difference between the sum of squared residuals in the restircted and unrestricted regressions. Under the null, $n_1\mathbf{H}$ is distributed as $W_p(n_1, \mathbf{\Sigma})$. Here,

$$\begin{aligned} &n_1\mathbf{H} \sim W_p(n_1, \mathbf{\Sigma}) && \text{under } H_0 \\ &n_1\mathbf{H} \sim W_p(n_1, \mathbf{\Sigma}, n_1\mathbf{\Phi}),\ \text{where } \mathbf{\Phi} = \theta\mathbf{\Sigma}^{-1}\boldsymbol{\phi}\boldsymbol{\phi}' && \text{under } H_1 \end{aligned}$$

Again, W.L.O.G, assume $\mathbf{\Sigma} = \mathbf{I}_p$. This **canonical form** of REG$_0$ is essentially equivalent to the setting of **matrix denoising**

$$\mathbf{Y}_1 = \mathbf{M} + \mathbf{Z}$$

**REG** Again, consider the linear regression

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

but **NOT** knowing the covariance matrix $\mathbf{\Sigma}$ of rows of $\boldsymbol{\epsilon}$. Here, the solutions again solve $\det(\mathbf{H} - \lambda\mathbf{E}) = \mathbf{0}$ with

$$\mathbf{H} = \mathbf{Y}_1'\mathbf{Y}_1/n_1,\ \mathbf{E} = \mathbf{Y}_3'\mathbf{Y}_3/n_2$$

this represents a multivariate analog of the $F$ ratio: the difference between the sum of squared residuals in the restricted and unrestricted regressions to the sum of squared residuals in the restricted regression. Again, as $n_2 \to \infty$, REG reduces to REG$_0$.

**CCA** Consider testing for independence between Gaussian vectors $x_t \in \mathbb{R}^p$ and $y_t \in \mathbb{R}^{n_1}$, given zero-mean observations with $t = 1, \cdots, n_1 + n_2$. Partition the population and sample covariance matrices of the observations $(x_t', y_t')'$ into

$$\begin{pmatrix} \mathbf{\Sigma}_{xx} & \mathbf{\Sigma}_{xy} \\ \mathbf{\Sigma}_{yx} & \mathbf{\Sigma}_{yy} \end{pmatrix} \qquad \begin{pmatrix} \mathbf{S}_{xx} & \mathbf{S}_{xy} \\ \mathbf{S}_{yx} & \mathbf{S}_{yy} \end{pmatrix}$$

respsectively. Under $H_0 : \mathbf{\Sigma}_{xy} = \mathbf{0}$, while the alternative is

$$\mathbf{\Sigma}_{xy} = \sqrt{\frac{n_1\theta}{n_1\theta + n_1 + n_2}}\boldsymbol{\phi}\boldsymbol{\psi}' \tag{18.5}$$

where the nuisance parameters $\phi \in \mathbb{R}^p$ and $\psi \in \mathbb{R}^{n_1}$ are normalized s.t.

$$\left\|\mathbf{\Sigma}_{xx}^{-1/2}\boldsymbol{\phi}\right\| = \left\|\mathbf{\Sigma}_{yy}^{-1/2}\boldsymbol{\psi}\right\| = 1$$

And the test can be based on the squared sample canonical correlations $\lambda_1, \cdots, \lambda_p$ that solves

$$\det(\mathbf{H} - \lambda\mathbf{E}) = \mathbf{0}$$

with

$$\mathbf{H} = \mathbf{S}_{xy}\mathbf{S}_{yy}^{-1}\mathbf{S}_{yx} \qquad \mathbf{E}\mathbf{S}_{xx}$$

### 18.2.3 SMD

For a $\mathbf{X} = \mathbf{\Phi} + \mathbf{Z}/\sqrt{p}$ where $\mathbf{Z}$ is a noise matrix from the **Gaussian Orthogonal Ensemble** (GOE)[2] We seek to make inference about a symmetric rank-one *signal* matrix $\mathbf{\Phi} = \theta\boldsymbol{\phi}\boldsymbol{\phi}'$. The null and the alternative is again as in 18.2. The nuisance vector $\boldsymbol{\phi} \in \mathbb{R}^p$ is normalized s.t. $\|\boldsymbol{\phi}\| = 1$.

The problem remains **invariant** under the multiplication of $\mathbf{X}$ from the left by an orthogonal matrix, and from the right by its transpose. A maximal invariant statistic consists of the solutions $\lambda_1, \cdots, \lambda_p$ to $\det(\mathbf{H} - \lambda\mathbf{E}) = \mathbf{0}$ with $\mathbf{H} = \mathbf{X}$ and $\mathbf{E} = \mathbf{I}_p$.

SMD can be viewed as a degenerate version of the 5 classes of problems, as shown in Figure 18.2:

- **SMD**, **PCA**, **REG$_0$**: random $\mathbf{H}$ and deterministic $\mathbf{E}$
- **PCA** and **SigD** are *parallel* to **REG$_0$**
- **CCA** has a different structure of $\mathbf{H}$ and $\mathbf{E}$

### 18.2.4 The likelihood ratios

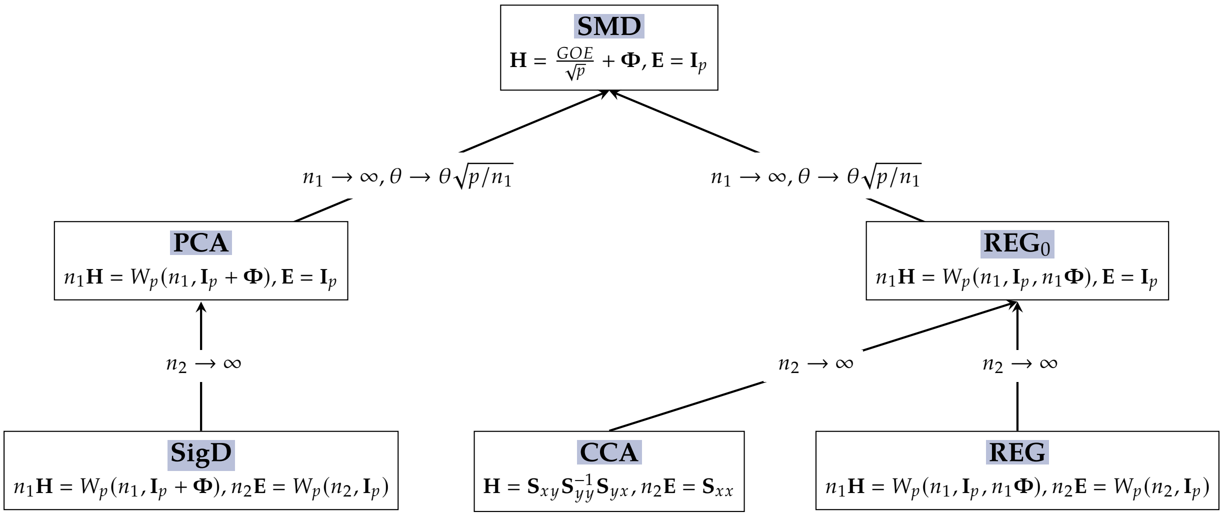
The goal is to study the asymptotic behavior of likelihood ratios based on the observed eigenvalues

$$\Lambda = \operatorname{diag}\left\{\lambda_1, \cdots, \lambda_p\right\}$$

[2] $\mathbf{Z}$ is from the GOE that it is **symmetric** and

$$\mathbf{Z}_{ii} \sim \mathcal{N}(0, 2) \qquad \mathbf{Z}_{ij} \sim \mathcal{N}(0, 1) \text{ if } i > j$$

Figure 18.2: SMD and 5 Classes of Statistical Problems




then the likelihood of the alternative versus the null is given by

$$\mathcal{L}(\theta, \Lambda) = \frac{p(\Lambda; \theta)}{p(\Lambda; 0)} = \alpha(\theta)_{\mathrm{p}}\mathcal{D}_{\mathrm{q}}(\mathbf{a}, \mathbf{b}; \mathbf{\Phi}, \Lambda) \tag{18.6}$$

where $\mathbf{\Phi} = \mathbf{\Phi}(\theta)$ is a $p$-dimensional matrix diag $\{\mathbf{\Phi}_{11}, 0, \cdots, 0\}$. Consider the hypergeometric functions of 2 matrix arguments $\mathbf{\Phi}, \mathbf{\Lambda}$ are defined as

$${}_{\mathrm{p}}\mathcal{F}_{\mathrm{q}}(\mathbf{a}, \mathbf{b}; \mathbf{\Phi}, \mathbf{\Lambda}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa - k} \frac{(a_1)_\kappa \cdots (a_{\mathrm{p}})_\kappa}{(b_1)_\kappa \cdots (b_1)_\kappa} \frac{C_\kappa(\mathbf{\Phi}) C_\kappa(\mathbf{\Lambda})}{C_\kappa(\mathbf{I}_p)}$$

where $\mathbf{a} = (a_1, \cdots, a_{\mathrm{p}})$ and $\mathbf{b} = (b_1, \cdots, b_{\mathrm{q}})$ are parameters, $\kappa$ are partitions of the integer $k$, $(a_j)_\kappa$ and $(b_i)_\kappa$ are the generalized Pochhammer symbols, $C_\kappa$ are the zonal polynomials. For each of the 6 classes of problems, we have the parameters as in Table where $n = n_1 + n_2$.

Table 18.2: Parameters of the Likelihood Ratios in Eq.18.6

| Classes | ${}_{\mathrm{p}}\mathcal{F}_{\mathrm{q}}$ | $\alpha(\theta)$ | $a$ | $b$ | $\mathbf{\Phi}_{11}$ |
|---|---|---|---|---|---|
| SMD | ${}_0\mathcal{F}_0$ | $\exp(-p\theta^2/4)$ | - | - | $\theta p/2$ |
| PCA | ${}_0\mathcal{F}_0$ | $(1+\theta)^{-n_1/2}$ | - | - | $\theta n_1/(2(1+\theta))$ |
| SigD | ${}_1\mathcal{F}_0$ | $(1+\theta)^{-n_1/2}$ | - | - | $\theta n_1/(n_2(1+\theta))$ |
| REG$_0$ | ${}_0\mathcal{F}_1$ | $\exp(-n_1\theta/2)$ | - | $n_1/2$ | $\theta n_1^2/4$ |
| REG | ${}_1\mathcal{F}_1$ | $\exp(-n_1\theta/2)$ | $n/2$ | $n_1/2$ | $\theta n_1^2/(2n_2)$ |
| CCA | ${}_2\mathcal{F}_1$ | $(1+n_1\theta/n)^{-n/2}$ | $(n/2, n/2)$ | $n_1/2$ | $\theta n_1^2/(n_2^2 + n_2 n_1(1+\theta))$ |

Some links in Fig.18.2 can also be established via asymptotic relations between hypergeometric functions.

**Asymptotic behavior of the likelihood ratios** consider that as $n_1, n_2, p$ go to infinity so that

$$c_1 \equiv \frac{p}{n_1} \to \gamma_1 \in (0,1) \qquad c_2 \equiv \frac{p}{n_2} \to \gamma_2 \in (0,1] \tag{18.7}$$

which can be denoted as $\mathbf{n}, p \to_{\gamma} \infty$ where $\mathbf{n} = \{n_1, n_2\}$ and $\boldsymbol{\gamma} = \left\{\gamma_1, \gamma_2\right\}$. Under the null (the true value of the spike $\theta_0 = 0$), $\lambda_1, \cdots, \lambda_p$ are the eigenvalues of

- $GOE/\sqrt{p}$, for **SMD**
- $W_p(n_1, \mathbf{I}_p)/n_1$, for **PCA** and **REG**$_0$
- a $p-$dimensional multivariate beta matrix with parameters $n_1/2$ and $n_2/2$ (here scaled by a factor of $n_2/n_1$), for **SigD**, **REG**, **CCA**

and the empirical distribution of $\lambda_1, \cdots, \lambda_p$ follows

$$\hat{F} = \frac{1}{p}\sum_{j=1}^{p} \mathrm{I}\left\{\lambda_j \le \lambda\right\} \xrightarrow{a.s.} F_{\gamma} = \begin{cases} F^{SC} & \text{semi-circle distribution, for SMD} \\ F^{MP} & \text{Marchenko-Pastur distribution, for PCA, REG}_0 \\ F^{W} & \text{Wachter distribution, for SigD, REG, CCA} \end{cases}$$

A summary of the 3 classes of distributions is shown in Table 18.3. And the cumulative distribution functions $F_{\gamma}^{\lim}(\lambda)$ are linked in the sense that

$$\begin{aligned} F_{\boldsymbol{\gamma}}^{W}(\lambda) &\to F_{\gamma_1}^{MP}(\lambda) & \gamma_2 \to 0 \\ F_{\gamma_1}^{MP}(\sqrt{\gamma_1}\lambda + 1) &\to F^{SC}(\lambda) & \gamma_1 \to 0 \end{aligned}$$

If $\varphi$ is a *well-behaved* function, the centered **linear spectral statistic**

$$\sum_{j=1}^{p} \varphi(\lambda_j) - p\int \varphi(\lambda)(d)F_{\mathbf{c}}^{\lim}(\lambda) \tag{18.8}$$

converges in distribtuion to a Gaussian random variable in each of the semicircle, Marchenko-Pastur and Wachter cases.[3]

Table 18.3: Semi-circle, Marchenko-Pastur, scaled Wachter distributions

| Case | $F_{\gamma}^{\lim}$ | Density, $\lambda \in [\beta_-, \beta_+]$ | $\beta_\pm$ | Threshold $\hat{\theta}$ |
|---|---|---|---|---|
| SMD | SC | $\frac{R(\lambda)}{2\pi}$ | $\pm 2$ | 1 |
| PCA, REG$_0$ | MP | $\frac{R(\lambda)}{2\pi\gamma_1\lambda}$ | $(1 \pm \sqrt{\gamma_1})^2$ | $\sqrt{\gamma_1}$ |
| SigD, REG, CCA | W | $\frac{(\gamma_1+\gamma_2)R(\lambda)}{2\pi\gamma_1\lambda(\gamma_1-\gamma_2\lambda)}$ | $\gamma_1\left(\frac{\rho\pm 1}{\rho\pm\gamma_2}\right)^2$ | $\frac{\rho+\gamma_2}{1-\gamma_2}$ |

where $R(\lambda) = \sqrt{(\beta_+ - \lambda)(\lambda - \beta_-)}$, $\rho = \sqrt{\gamma_1 + \gamma_2 - \gamma_1\gamma_2}$

[3]The centering constant is defined in terms of $F_{\mathbf{c}} = \{c_1, c_2\}$, that is, the *correct centering* can be computed using the densities in Tab.18.3, where $\gamma_1, \gamma_2$ are replaced by $c_1 \equiv p/n_1, c_2 \equiv p/n_2$ respectively.

## References

Iain M Johnstone and Alexei Onatski. Testing in high-dimensional spiked models. *The Annals of Statistics*, 48(3), 2020.