

## Topic 18: Eigenvalue and Spike Models

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**Key points:** .

**Disclaimer:** The note is built on Prof. *Jinchi Lv*'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

### 18.1 Motivation

Consider  $n$  independent observations  $\mathbf{X}_i \in \mathbb{R}^p$  drawn from a  $\mathcal{N}(\mathbf{0}, \Sigma)$ , then the covariance can be decomposed into 2 parts, white noise and low rank

$$\Sigma = \text{Cov}(\mathbf{X}_i) = \mathbf{I} + \sum_{k=1}^M \theta_k \mathbf{v}_k \mathbf{v}_k' = \Sigma_0 + \Phi$$

where  $M$  denotes the **number of spikes** in the distribution of eigenvalues. The idea is: spikes deviate from a reference model along a small fixed number of unknown directions. If  $\Phi = \mathbf{0}$ , then none of the sample eigenvalues is separated from the bulk.

**Why a spike model is interesting?** A spike model can help determine the latent dimension of the data, some examples being

- Principal component analysis (PCA): spikes are related to the directions of the most variations of the data, i.e., the principal components
- Clustering model:  $M$  spikes is equivalent to  $M + 1$  clusters
- Economic significance:  $M$  is related to the number of factor loadings

Then the question is threefold:

- How to determine  $M$
- How to estimate  $\mathbf{v}_k$
- How to test  $\theta_k$

Under rank one alternative, we would like to test the hypothesis

$$H_1 : \Sigma = \mathbf{I}_p + \theta \mathbf{v} \mathbf{v}', \theta > 0$$

against the null

$$H_0 : \Sigma = \mathbf{I}_p$$

with the key assumptions:

A1 Gaussian error

A2 large  $p$ :  $p \leq n$  but allows  $p/n \rightarrow \gamma \in (0, 1)$

Under these assumptions, for the  $n \times p$  data matrix  $\mathbf{X} = (\mathbf{X}'_1 \cdots \mathbf{X}'_n)'$ ,  $\mathbf{X}'\mathbf{X}$  has a  $p$ -dimensional **Wishart** distribution  $W_p(n, \mathbf{\Sigma})$  with the degree of freedom  $n$  and covariance matrix  $\mathbf{\Sigma}$ , which is a *random matrix*.

If  $\mathbf{Y} = \mathbf{M} + \mathbf{X}$ , that is, the sum of the *random matrix*  $\mathbf{X}$  and a *deterministic matrix*  $\mathbf{M}$  (also  $n \times p$ ), then  $\mathbf{Y}'\mathbf{Y}$  has a  $p$ -dimensional Wishart distribution  $W_p(n, \mathbf{\Sigma}, \mathbf{\Psi})$  with  $n$  degrees of freedom, covariance matrix  $\mathbf{\Sigma}$  and non-centrality matrix  $\mathbf{\Psi} = \mathbf{\Sigma}^{-1}\mathbf{M}'\mathbf{M}$ .

#### Definition 18.1.1: Density of Wishart Distribution

The PDF of Wishart distribution is defined as

$$f(\mathbf{X}) = \frac{1}{2^{np/2} \Gamma_p\left(\frac{n}{2}\right) |\mathbf{\Sigma}|^{n/2}} |\mathbf{X}|^{(n-p-1)/2} \exp\left(-\frac{1}{2} \text{tr}(\mathbf{\Sigma}^{-1}\mathbf{X})\right)$$

## References