

## Topic 11: Lasso And Beyond: Convex Learning

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**Key points:**

**Disclaimer:**

### 11.1 Lasso

Lasso (Least absolute Shrinkage and Selection Operator), proposed by Tibshirani (1996), aims to minimize the **SSR (sum of residual squares)** subject to the **L1-norm (sum of the absolute value)** of the coefficients being less than a constant.

#### 11.1.1 Set up

For data  $(\mathbf{x}_i, y_i)_{i=1}^n$ , where

- $y_i$  is the outcome for individual  $i$
- $\mathbf{x}_i = (x_{i1}, \dots, x_{ip})'$  is the  $p \times 1$  vector of predictors

Then the Lasso estimator  $(\hat{\alpha}, \hat{\beta})$  is defined as

$$(\hat{\alpha}, \hat{\beta}) = \arg \min_{\alpha, \beta} \left\{ \sum_{i=1}^n \left( y_i - \alpha - \sum_{j=1}^p \beta_j x_{ij} \right)^2 \right\} \quad \text{s.t.} \quad \sum_{j=1}^p |\beta_j| \leq t$$

for the  $n \times 1$  response vector  $\mathbf{y} = (y_1, \dots, y_n)'$ , the  $n \times p$  design matrix  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)'$  where  $\mathbf{x}_i = (x_{i1}, \dots, x_{ip})'$  is a  $p \times 1$  vector. Here  $\hat{\alpha} = \bar{y}$ , w.l.o.g., let  $\bar{y} = 0$  and omit  $\alpha$  for simplicity.

In matrix form, we have

- constrained form:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 \right\} \quad \text{s.t.} \quad \|\beta\|_1 \leq t$$

- unconstrained form:

$$\hat{\beta}(\lambda) = \arg \min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\beta\|_1 \right\}$$

where the regularization parameter  $\lambda \geq 0$ :

- $\lambda \rightarrow \infty$ :  $\hat{\beta}_{lasso} = \mathbf{0}$
- $\lambda = 0$ :  $\hat{\beta}_{lasso} \rightarrow \hat{\beta}_{OLS}$

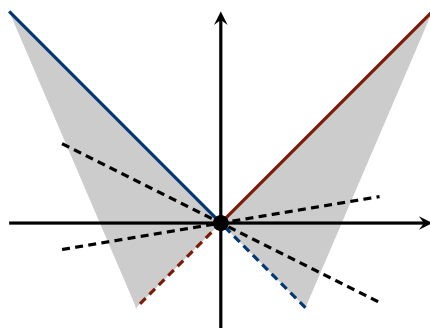
### 11.1.2 Solving Lasso

Lasso is essentially a quadratic optimization problem. Hence, the solution is given by taking the derivative (of the unconstrained question) and set it equal to 0

$$\frac{d}{d\beta} \left( \frac{1}{2n} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\beta\|_1 \right) = 0$$

$$\Rightarrow \underbrace{\frac{1}{n} \mathbf{X}'}_{p \times n} \underbrace{(\mathbf{y} - \mathbf{X}\beta)}_{= \epsilon, n \times 1} = \lambda \begin{cases} \text{sign}(\beta_j), & \beta_j \neq 0 \\ [-1, 1], & \beta_j = 0 \end{cases}$$

this result follows the fact the L-1 norm  $\|\beta\|_1$  is piecewise linear:



L1-norm (1-dimension)

For each component of the vector of the L-1 norm

$f(\beta_j) = |\beta_j|$ , we have:

- $\beta_j > 0$ :  $f'(\beta_j) = 1$
  - $\beta_j < 0$ :  $f'(\beta_j) = -1$
  - $\beta_j = 0$ :  $df \in [-1, 1]$  (shaded area)
- which gives the results stated above.

Take another look at this result

#### Proposition 11.1.1: Lasso Parameter Selection Rule

$$\frac{1}{n} \mathbf{X}' (\mathbf{y} - \mathbf{X}\beta) = \frac{1}{n} \mathbf{X}' \epsilon = \lambda \begin{cases} \text{sign}(\beta_j), & \beta_j \neq 0 \\ [-1, 1], & \beta_j = 0 \end{cases}$$

which gives a parameter selection criterion: for  $\beta_j \neq 0$ ,  $\text{sign}(\beta_j)$  must agree with  $\text{Corr}(X_j, \epsilon)$ , the correlation between the  $j$ -th variable  $X_j$  and (full-model) residuals  $\epsilon = \mathbf{y} - \mathbf{X}\beta$ .

### 11.1.3 Algorithm: from LARS to Lasso

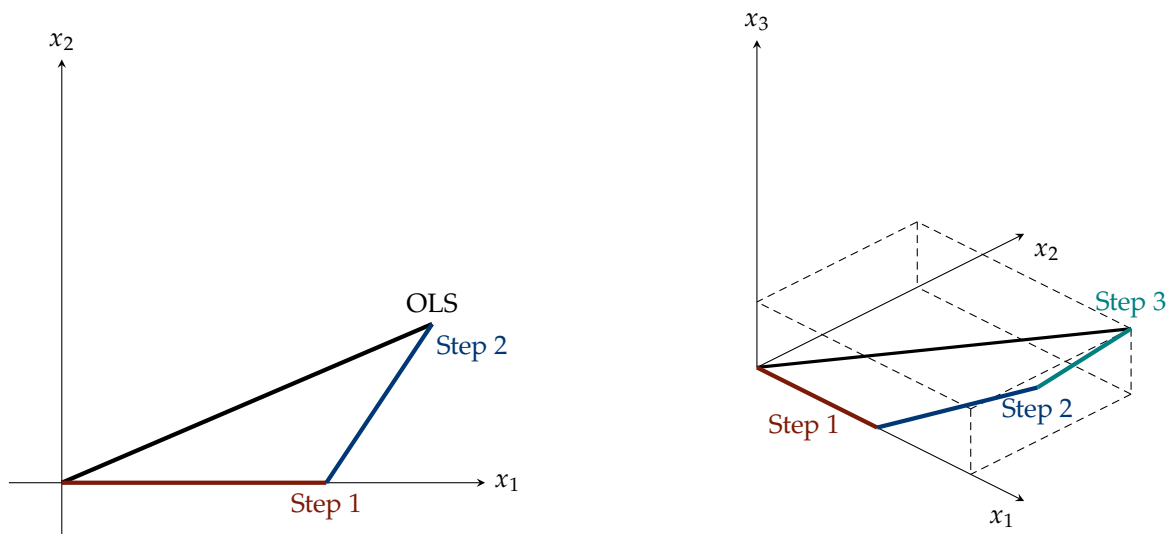
Mathematically, Lasso is quite intuitive, but computationally, it can be quite consuming. Efron et al. (2004) propose an algorithm that takes steps from a all-0 model to the biggest model (OLS), that is, **Least Angle Regression (LARS)**.

#### Intuition

The basic intuition of LARS is quite straight-forward: covariates are considered from the **highest** correlation with  $\mathbf{y}$  (*smallest* angle from  $\mathbf{y}$ ) to the **least** correlated one (*largest* angle from  $\mathbf{y}$ ) (illustrated below).

And the steps of the LARS algorithm are

- 1 start with the null model  $\hat{\beta} = \mathbf{0}$ :  $\hat{\mu} = \mathbf{X}'\mathbf{0} = 0$



2 calculate residual vector  $\mathbf{r} = \mathbf{y} - \hat{\mu}$

3 determine the correlation vector between  $\mathbf{r}$  and each parameter  $\mathbf{x}_j, \forall j = 1, \dots, p$

echos a classic model-selection method, **Forward Selection** (forward stepwise regression):

## 11.2 Penalized Least Square Estimation

Lasso is one special class of Penalized Least Square (PLS) Estimation. For the linear regression model  $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ , if  $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ , we have PLS as

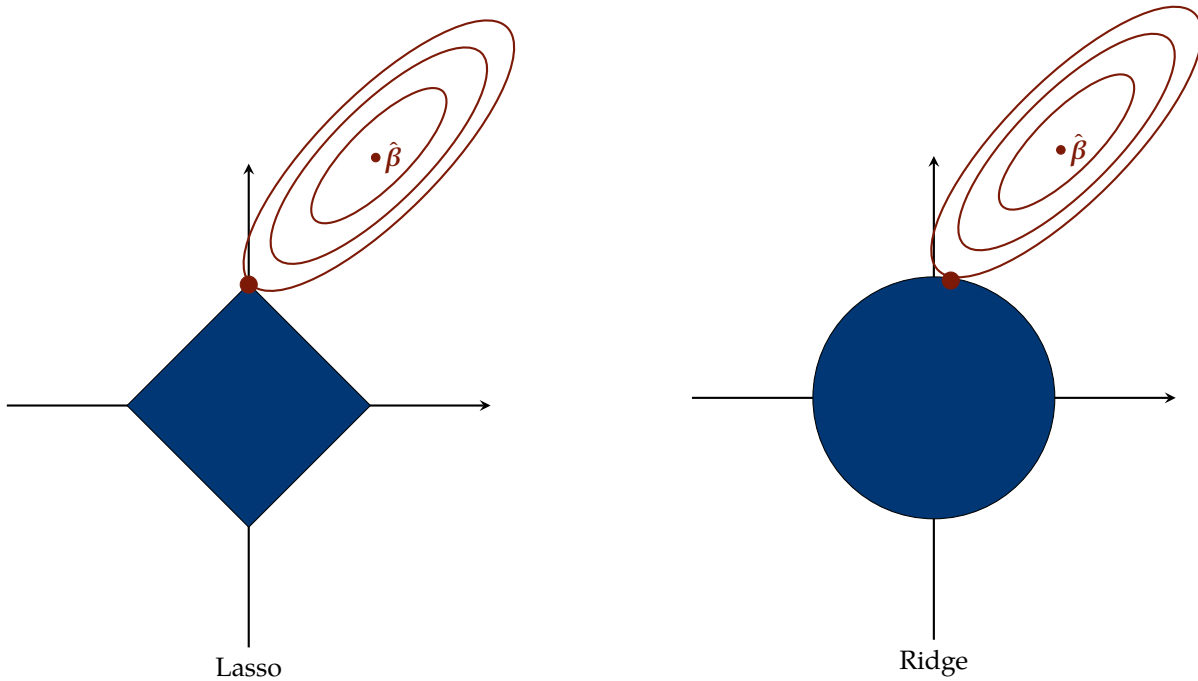
$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left\{ \frac{1}{2n} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \sum_{j=1}^p p_\lambda(|\beta_j|) \right\}$$

where  $p_\lambda(\cdot)$  is a penalty function indexed by the regularization parameter  $\lambda \geq 0$ . **Antoniadis and Fan (2001)** showed that the PLS estimator  $\hat{\boldsymbol{\beta}}$  has the following properties:

- **sparsity**: if  $\min_{t \geq 0} \{t + p'_\lambda(t)\} > 0$
- **approximate unbiasedness**: if  $p'_\lambda(t) = 0$  for  $t$  large enough
- **continuity**: iff  $\arg \min_{t \geq 0} \{t + p'_\lambda(t)\} = 0$

In general

- the **signularity** of penalty function at the origin,  $p'_\lambda(0_+) > 0$  is needed for generating **sparsity** in variable selection
- the **concavity** is needed to reduce the bias



## References

- Anestis Antoniadis and Jianqing Fan. Regularization of wavelet approximations. *Journal of the American Statistical Association*, 96(455):939–967, 2001.
- Bradley Efron, Trevor Hastie, Iain Johnstone, and Robert Tibshirani. Least angle regression. *The Annals of Statistics*, 32(2):407 – 499, 2004. doi: 10.1214/009053604000000067. URL <https://doi.org/10.1214/009053604000000067>.
- Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B (Methodological)*, 58(1):267–288, 1996.