

Topic 14: Regularization Methods in Thresholded Parameter Space

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Key points: The connections and differences of all regularization methods and some interesting phase transition phenomena.

Disclaimer: The note is built on Prof. [Jinchi Lv's](#) lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

14.1 Model Setup

Now, consider a generalized linear model (GLM) linking a p -dimensional predictor \mathbf{x} to a scalar response Y . With canonical link, the conditional distribution of Y given \mathbf{x} has density

$$f(y; \theta, \phi) = \exp [y\theta - b(\theta) + c(y, \phi)]$$

where $\theta = \mathbf{x}'\boldsymbol{\beta}$ with $\boldsymbol{\beta}$ a p -dimensional regression coefficient vector, $b(\cdot)$ and $c(\cdot, \cdot)$ are known functions and ϕ is dispersion parameter. Again, $\boldsymbol{\beta} = (\beta_{0,1}, \dots, \beta_{0,p})'$ is sparse with many zero components, and $\log p = O(n^a)$ for some $0 < a < 1$.

The penalized negative log-likelihood is

$$Q_n(\boldsymbol{\beta}) = -n^{-1} [\mathbf{y}'\mathbf{X}\boldsymbol{\beta} - \mathbf{1}'\mathbf{b}(\mathbf{X}\boldsymbol{\beta})] + \|p_\lambda(\boldsymbol{\beta})\|_1$$

where

- $\mathbf{y} = (y_1, \dots, y_n)'$, $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)'$, each column of \mathbf{X} is rescaled to have L_2 -norm \sqrt{n}
- $\mathbf{b}(\boldsymbol{\theta}) = (b(\theta_1), \dots, b(\theta_n))'$ with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)'$
- $\|p_\lambda(\boldsymbol{\beta})\|_1 = \sum_{j=1}^p p_\lambda(|\beta_j|)$

Next, define **robust spark** κ_c

Definition 14.1.1: Robust spark κ_c

The robust spark κ_c of the $n \times p$ design matrix \mathbf{X} is defined as the smallest possible positive integer s.t. there exists an $n \times \kappa_c$ submatrix of $\frac{1}{\sqrt{n}}\mathbf{X}$ having a singular value less than a given positive constant c ([Zheng et al., 2014](#)), and

$$\kappa_c \leq n + 1$$

Bounding sparse model size can control collinearity and ensure model identifiability and stability, and as $c \rightarrow 0+$, κ_c approaches the spark. Robust spark can be some large number diverging with n :

Proposition 14.1.2: Order of κ_c

Assume $\log p = o(n)$ and that the rows of the $n \times p$ random design matrix \mathbf{X} are i.i.d. as $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ has smallest eigenvalue bounded from below by some positive constant. Then there exist

positive constants c and \tilde{c} s.t. with asymptotic probability one, $\kappa_c \geq \frac{\tilde{c}n}{\log p}$

Next, we define a thresholded parameter space

Definition 14.1.3: Thresholded parameter space

$$\mathcal{B}_{\tau,c} = \left\{ \beta \in \mathbb{R}^p : \|\beta\|_0 < \frac{\kappa_c}{2}, \text{ and for each } j, \beta_j = 0 \text{ or } |\beta_j| \geq \tau \right\}$$

where $\beta = (\beta_1, \dots, \beta_p)'$. τ is some positive threshold on parameter magnitude:

Here, τ is very important:

- τ is key to distinguishing between important covariates and noise covariates for the purpose of variable selection
- τ typically needs to satisfy $\tau \sqrt{n/\log p} \xrightarrow{n \rightarrow \infty} \infty$

It turns out that the solution to the regularization problem has the (very natural) hard-thresholding property:

Proposition 14.1.4: Hard-thresholding property

or the L_0 -penalty $p_\lambda(t) = \lambda \mathbf{1}_{t \neq 0}$, the global minimizer $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_p)'$ of the regularization problem over \mathbb{R}^p satisfies that each component $\hat{\beta}_j$ is either 0 or has magnitude larger than some positive threshold

This hard-thresholding property is shared by many other penalties such as SICA penalties. This property guarantees sparsity of the model: weak signals are generally difficult to stand out comparing to noise variables due to impact of high dimensionality

14.2 Asymptotic Equivalence of Regularization Methods

For a universal $\lambda = c_0 \sqrt{\log p/n}$ with $c_0 > 0$ and p implicitly as $n \vee p$, consider 2 key events:

$$\mathcal{E} = \left\{ \|n^{-1} \mathbf{X}' \epsilon\|_\infty \leq \lambda/2 \right\} \quad \mathcal{E}_0 = \left\{ \|n^{-1} \mathbf{X}'_{\alpha_0} \epsilon\|_\infty \leq c_0 \sqrt{\log n/n} \right\}$$

where $\epsilon = \mathbf{y} - \mathbb{E}\mathbf{y}$, \mathbf{X}_α is a submatrix of \mathbf{X} consisting of columns in α . Here, let $\alpha_0 = \text{supp}(\beta_0)$ (non-zero variables in the true model).

For this setting, consider the following technical conditions:

- C1 **Error tail distribution**: $\Pr(\mathcal{E}^c) = O(p^{-c_1})$ and $\Pr(\mathcal{E}_0^c) = O(n^{-c_1})$ for some positive constant c_1 that can be sufficiently large for large enough c_0
- C2 **Bounded variance**: $b(\theta)$ satisfies that $c_2 \leq b''(\theta) \leq c_2^{-1}$ in its domain, where c_2 is some positive constant
- C3 **Concave penalty function**: $p_\lambda(t)$ is increasing and concave in $t \in [0, \infty)$ with $p_\lambda(0) = 0$, and is differentiable with $p'_\lambda(0+) = c_3 \lambda$ for some positive constant c_3 ¹
- C4 **Ultra-high dimensionality**: $\log p = O(n^a)$ for some constant $a \in (0, 1)$

¹A wide class of penalties, including L_1 -penalty in Lasso, SCAD, MCP and SICA, satisfy this condition.

C5 **True parameter vector**: $s = o(n^{1-a})$ and $\exists c > 0$ s.t. the **robust spark** $\kappa_c > 2s$. Moreover, $\min_{1 \leq j \leq s} |\beta_{0,j}| \gg \sqrt{\log p/n}$

Given these 5 conditions, we have that the global minimizer $\hat{\beta} = \arg \min_{\beta \in \mathcal{B}_\tau} Q_n(\beta)$ exists and satisfies oracle inequalities:

Theorem 14.2.1: Oracle Inequalities

Assume that Condition 1-5 hold and τ is chosen s.t. $\tau < \min_{1 \leq j \leq s} |\beta_{0,j}|$ and $\lambda = c_0 \sqrt{\log p/n} = o(\tau)$, then the global minimizer exists, and any such global minimizer satisfies that with probability at least $1 - O(p^{-c_1})$, it holds simultaneously that

- **False sign**:

$$FS(\hat{\beta}) \leq \frac{Cs\lambda^2\tau^{-2}}{1 - C\lambda^2\tau^{-2}}$$

- **Estimation losses**:

$$\begin{aligned} \|\hat{\beta} - \beta_0\|_q &\leq C\lambda s^{1/q}(1 - C\lambda^2\tau^{-2})^{-1/q} \\ \|\hat{\beta} - \beta_0\|_\infty &\leq C\lambda s^{1/2}(1 - C\lambda^2\tau^{-2})^{-1/2} \end{aligned} \quad \forall q \in [1, 2]$$

- **Prediction loss**:

$$\frac{1}{\sqrt{n}} \|\mathbf{X}(\hat{\beta} - \beta_0)\|_2 \leq C\lambda s^{1/2}(1 - C\lambda^2\tau^{-2})^{-1/2}$$

where C is some positive constant.

How to understand Thm.14.2.1

- These results hold uniformly over the set of all possible global minimizers
- c_1 in probability bound can be chosen arbitrarily large, affecting **only** C
- $FS(\hat{\beta}) = o(s)$ since $\lambda = o(\tau)$, while $\|\hat{\beta}\|_0 = O(\phi_{\max}s)$ where ϕ_{\max} is the largest eigenvalue of $\frac{1}{n}\mathbf{X}'\mathbf{X}$
- $\forall q \in [1, 2]$, the convergence rates of estimation losses

$$\begin{aligned} \|\hat{\beta} - \beta_0\|_q &= O\left\{s^{1/q}\sqrt{\frac{\log p}{n}}\right\} \\ \frac{1}{\sqrt{n}}\|\mathbf{X}(\hat{\beta} - \beta_0)\|_2 &= O\left(\sqrt{\frac{s \log p}{n}}\right) \end{aligned}$$

are consistent with Lasso.

We also have a sign consistency result:

Theorem 14.2.2: Sign Consistency and Oracle Inequalities

Assume the same conditions of Thm.14.2.1, further assume $\min_{1 \leq j \leq s} |\beta_{0,j}| \geq 2\tau$ and $\lambda = c_0 \sqrt{\log p/n} = o(s^{-1/2}\tau)$, and $\gamma_n = o\left(\tau \sqrt{\frac{n}{s \log n}}\right)$, then any global minimizer $\hat{\beta}$ defined satisfies that with probability at least $1 - O(n^{-c_1})$, it holds simultaneously that

- **Sign consistency**: $\text{sgn}(\hat{\beta}) = \text{sgn}(\beta_0)$
- **Estimation and prediction losses**: If the penalty function further satisfies $p'_\lambda(\tau) = O\left(\frac{\log n}{n}\right)$, then $\forall q \in [1, 2]$,

$$\|\hat{\beta} - \beta_0\|_q \leq C s^{1/q} \sqrt{\frac{\log n}{n}} \quad \|\hat{\beta} - \beta_0\|_\infty \leq C \gamma_n^* \sqrt{\frac{\log n}{n}} \quad n^{-1} D(\hat{\beta}) \leq C \frac{s \log n}{n}$$

where γ_n^* is a constant showing the behavior of $\left\| \left[\frac{1}{n} \mathbf{X}'_{\alpha_0} \mathbf{H}(\beta_1, \dots, \beta_n) \mathbf{X}_{\alpha_0} \right]^{-1} \right\|_\infty$ in a small neighborhood of β_0 , $D(\hat{\beta})$ is the Kullback-Leibler divergence, and C is some positive constant

How to understand Thm.14.2.2 Consider a linear model, where

$$\gamma_n^* = \left\| \left(\frac{1}{n} \mathbf{X}'_{\alpha_0} \mathbf{X}_{\alpha_0} \right)^{-1} \right\|_\infty \leq \sqrt{s} \left\| \left(\frac{1}{n} \mathbf{X}'_{\alpha_0} \mathbf{X}_{\alpha_0} \right)^{-1} \right\|_2 \leq \frac{\sqrt{s}}{c} \quad \gamma_n = \sup_{\alpha \subset \{s+1, \dots, p\}, |\alpha| \leq s} \left\| \frac{1}{n} \mathbf{X}'_{\alpha_0} \mathbf{X}_\alpha \right\|_\infty$$

when all true covariates are orthogonal to each other, $\gamma_n^* = 1$ and

$$\|\hat{\beta} - \beta_0\|_\infty \leq C \sqrt{\frac{\log n}{n}}$$

within a logarithmic factor $\log n$ or oracle rate. Meanwhile, the penalty function condition $p'_\lambda(\tau) = O\left(\frac{\log n}{n}\right)$ can be easily satisfied by concave penalties such as SCAD and SICA, having convergence rates improved with $\log n$ in place of $\log p$.

14.2.1 Phase Transition Phenomenon

Combining Thm.14.2.1 and 14.2.2, it's shown that

- for $p = O(n^a)$, Lasso and concave regularization methods are **asymptotically equivalent**, having the same convergence rates in the oracle inequalities, with a logarithmic factor of $\log n$
- for $\log p = O(n^a)$, concave regularization methods are **asymptotically equivalent** and still enjoy the same convergence rates in the oracle inequalities, with a logarithmic factor of $\log n^2$.

A phase diagram on how the performance of regularization methods, in the thresholded parameter space, evolves with dimensionality and penalty function.

²For Lasso, the condition $p'_\lambda(\tau) = O\left(\frac{\log n}{n}\right)$ and the choice of $\lambda = c_0 \sqrt{\frac{\log p}{n}}$ are **incompatible** with each other in this ultra-high dimensional case, and the convergence rates for Lasso (of $\log p$) are slower than those for concave regularization methods.

References

Zemin Zheng, Yingying Fan, and Jinchi Lv. High dimensional thresholded regression and shrinkage effect. *Journal of the Royal Statistical Society: Series B: Statistical Methodology*, pages 627–649, 2014.