Econometrics	**April 3, 2023**

Topic 13: Non-convex Learning + Lasso

by Sai Zhang

Key points: Combining the best of the two, we can use **Lasso plus Concave** method, with Lasso screening and concave component selecting variables, achieving a coordinated intrinsic two-scale learning.

Disclaimer: *The note is built on Prof. Jinchi Lv's lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.*

We are facing a tradeoff:

- **Convex** methods: have appealing prediction power and oracle inequalities, but challenging to provide tight false sign rate control
- **Concave** methods: have good variable selection properties, but challenging to establish global properties and risk properties

Here, we take advantage of the linearity of Lasso (convex *and* concave) and try to combine it with concave regularization to get the best of both.

13.1 Model Setup

Again, consider a linear regression model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where

- response vector ($n \times 1$): $\mathbf{y} = (y_1, \cdots, y_n)'$
- design matrix ($n \times p$): $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_p)$

here, we consider a scenario where

- $\boldsymbol{\beta}_0 = (\beta_{0,1}, \cdots, \beta_{0,p})'$ is ***sparse*** (with many 0 components)
- ultra-**high** dimensions: $\log p = O(n^a)$, for some $0 < a < 1$

and consider the penalized least squares

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \left\{(2n)^{-1}\|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda_0\|\boldsymbol{\beta}\|_1 + \|p_\lambda(\boldsymbol{\beta})\|_1\right\} \tag{13.1}$$

where

- $\lambda_0 = c\left(\frac{\log p}{n}\right)^{1/2}$ for some $c > 0$
- $p_\lambda(\boldsymbol{\beta}) = p_\lambda(|\boldsymbol{\beta}|) = (p_\lambda(|\beta_1|), \cdots, p_\lambda(|\beta_p|))'$, with $|\boldsymbol{\beta}| = (|\beta_1|, \cdots, |\beta_p|)'$