

## Topic 13: Non-convex Learning + Lasso

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**Key points:** Combining the best of the two, we can use **Lasso plus Concave** method, with Lasso screening and concave component selecting variables, achieving a coordinated intrinsic two-scale learning.

**Disclaimer:** The note is built on Prof. *Jinchi Lv*'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

We are facing a tradeoff:

- **Convex** methods: have appealing prediction power and oracle inequalities, but challenging to provide tight false sign rate control
- **Concave** methods: have good variable selection properties, but challenging to establish global properties and risk properties

Here, we take advantage of the linearity of Lasso (convex *and* concave) and try to combine it with concave regularization to get the best of both.

### 13.1 Model Setup

Again, consider a linear regression model  $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ , where

- response vector ( $n \times 1$ ):  $\mathbf{y} = (y_1, \dots, y_n)'$
- design matrix ( $n \times p$ ):  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_p)$ , with each column rescaled to have  $L_2$ -norm  $n^{1/2}$

here, we consider a scenario where

- $\boldsymbol{\beta}_0 = (\beta_{0,1}, \dots, \beta_{0,p})'$  is *sparse* (with many 0 components)
- ultra-high dimensions:  $\log p = O(n^a)$ , for some  $0 < a < 1$

and consider the penalized least squares

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left\{ (2n)^{-1} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda_0 \|\boldsymbol{\beta}\|_1 + \|p_\lambda(\boldsymbol{\beta})\|_1 \right\} \quad (13.1)$$

where

- $\lambda_0 = c \left( \frac{\log p}{n} \right)^{1/2}$  for some  $c > 0$
- $p_\lambda(\boldsymbol{\beta}) = p_\lambda(|\boldsymbol{\beta}|) = (p_\lambda(|\beta_1|), \dots, p_\lambda(|\beta_p|))'$ , with  $|\boldsymbol{\beta}| = (|\beta_1|, \dots, |\beta_p|)'$ ; the concave penalty  $p_\lambda(t)$  is defined on  $t \in [0, \infty)$ , indexed by  $\lambda \geq 0$ , increasing in both  $t$  and  $\lambda$ ,  $p_\lambda(0) = 0$

the 2 penalty components

- $L_1$ -component: minimum amount of regularization for removing noise in prediction
- concave component  $\|p_\lambda(\boldsymbol{\beta})\|_1$ : adapt model sparsity for variable selection

Under this set up, we can derive the hard-thresholding property as

**Proposition 13.1.1: Hard-Thresholding Property**

Assume the  $p_\lambda(t)$ ,  $t \geq 0$ , is **increasing and concave** with

- $p_\lambda(t) \geq p_{H,\lambda}(t) = \frac{1}{2} [\lambda^2 - (\lambda - t)_+^2]$  on  $[0, \lambda]$
- $p'_\lambda((1 - c_1)\lambda) \leq c_1\lambda$  for some  $c_1 \in [0, 1]$
- $-p''_\lambda(t)$  decreasing on  $[0, (1 - c_1)\lambda]$

then any local minimizer of 13.1 that is also a global minimizer in each coordinate has the **hard-thresholding** feature that each component is either 0 or of magnitude **larger** than  $(1 - c_1)\lambda$

Such property is shared by a wide class of concave penalties, including hard-thresholding penalty  $p_{H,\lambda}(t)$  with  $c_1 = 0$ ,  $L_0$ -penalty, and SICA (with suitable  $c_1$ ).

**How to understand this proposition?** Let  $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_p)'$ , then **each  $\hat{\beta}_j$**  is the global minimizer of the corresponding univariate penalized least-square problem along the  $j$ -th coordinate. These univariate problems share a common form with (generally) different scalars  $z$

$$\hat{\beta}(z) = \arg \min_{\beta \in \mathbb{R}} \left\{ \frac{1}{2}(z - \beta)^2 + \lambda_0 |\beta| + p_{H,\lambda}(|\beta|) \right\}$$

after we rescale all covariates to have  $L_2$ -norm  $n^{1/2}$ . The solution to these univariate problems are

$$\hat{\beta}(z) = \text{sgn}(z)(|z| - \lambda_0) \cdot \mathbf{1}_{|z| > \lambda + \lambda_0}$$

these solutions have the same feature as the hard-thresholded estimator: each component is either 0 or of magnitude larger than  $\lambda$ . This provides a better distinction between insignificant and significant covariates than soft-thresholding by  $L_1$  penalty.

With the hard-thresholding property of Prop. 13.1.1, we can prove a basic constraint for the global optimum  $\hat{\beta}$  on an event with significant probability (Fan and Lv, 2014)

$$\|\delta_2\|_1 \leq 7\|\delta_1\|_1 \quad (13.2)$$

where  $\delta = \hat{\beta} - \beta_0 = (\hat{\beta}'_1, \hat{\beta}'_2)' - (\beta'_{0,1}, \beta'_{0,2})' = (\delta'_1, \delta'_2)'$ , with  $\delta_1 \in \mathbb{R}^s$ . Where does this constraint come from? For the penalized least square question 13.1

$$\min_{\beta \in \mathbb{R}^p} \{ (2n)^{-1} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda_0 \|\beta\|_1 + \|p_\lambda(\beta)\|_1 \}$$

the global minimizer  $\hat{\beta}$  leads to

$$\begin{aligned} (2n)^{-1} \|\mathbf{y} - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 &= (2n)^{-1} \|\mathbf{X}\beta_0 + \epsilon - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\ &= (2n)^{-1} \|\epsilon - \mathbf{X}(\hat{\beta} - \beta_0)\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\ &\leq (2n)^{-1} \|\mathbf{y} - \mathbf{X}\beta_0\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\ &= (2n)^{-1} \|\epsilon\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \end{aligned}$$

then, plug in  $\delta = \hat{\beta} - \beta_0$ , we get

$$\begin{aligned} (2n)^{-1} \|\epsilon - \mathbf{X}\delta\|_2^2 + \lambda_0 \|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq (2n)^{-1} \|\epsilon\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\ (2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \end{aligned}$$

since  $\beta_{0,2} = \mathbf{0}$ ,  $\delta_2 = \beta_{0,2} + \delta_2$ , we have

$$\|\beta_0 + \delta\|_1 = \|\beta_{0,1} + \beta_{0,2} + \delta_1 + \delta_2\|_1 = \|\beta_{0,1} + \delta_1 + \delta_2\|_1 \leq \|\beta_{0,1} + \delta_1\|_1 + \|\delta_2\|_1$$

hence

$$(2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 \leq \lambda_0 \|\beta_{0,1}\|_1 - \lambda_0 \|\beta_{0,1} + \delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

and by the reverse triangle inequality  $\|\beta_{0,1}\|_1 - \|\beta_{0,1} + \delta_1\|_1 \leq \|\delta_1\|_1$ , we get

$$(2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 \leq \lambda_0 \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

If assume the distribution of the model error  $\epsilon$  as

$$\Pr\left(\|n^{-1} \mathbf{X}' \epsilon\|_\infty > \frac{\lambda_0}{2}\right) = O(p^{-c_0})$$

conditional on the event  $\mathcal{E} = \{\|n^{-1} \mathbf{X}' \epsilon\|_\infty \leq \lambda_0/2\}$ , we have

$$-n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 - \lambda_0 \|\delta_1\|_1 \geq -\frac{\lambda_0}{2} \|\delta\|_1 + \lambda_0 \|\delta_2\|_1 - \lambda_0 \|\delta_1\|_1 = \frac{\lambda_0}{2} \|\delta_2\|_1 - \frac{3\lambda_0}{2} \|\delta_1\|_1$$

plug this result back, get

$$\frac{1}{2n} \|\mathbf{X}\delta\|_2^2 + \frac{\lambda_0}{2} \|\delta_2\|_1 \leq \frac{3\lambda_0}{2} \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \quad (13.3)$$

Now, if we further impose 2 conditions:

- **eigenvalue condition**: for some positive constant  $\kappa_0$

$$\min_{\|\delta\|_2=1, \|\delta\|_0 < 2s} \frac{1}{\sqrt{n}} \|\mathbf{X}\delta\|_2 \geq \kappa_0 \quad (\mathbf{A})$$

$$\kappa = \kappa(s, 7) = \min_{\delta \neq 0, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2 \vee \|\tilde{\delta}_2\|_2} \right\} > 0 \quad (\mathbf{B})$$

where  $\tilde{\delta}_2$  is the subvector of  $\delta_2$  consisting of the components with the  $s$  largest absolute values. Here

- Condition **(A)** is a mild sparse eigenvalue condition
- Condition **(B)** combines the restricted eigenvalue assumptions in [Bickel et al. \(2009\)](#)<sup>1</sup>. The intuition is, for OLS estimation,  $\mathbf{X}'\mathbf{X}$  should be positive definite, that is

$$\min_{\mathbf{0} \neq \delta \in \mathbb{R}^p} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta\|_2} \right\} > 0$$

however, when  $p > n$ , this condition **never** holds, hence we replace  $\|\delta\|_2$  with the  $L_2$ -norm of  $\|\delta_1\|_2$ , a subvector of  $\delta$

$$\kappa = \kappa(s, 7) = \min_{\delta \neq 0, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2} \right\} > 0$$

and for  $L_q$  loss with  $q \in (1, 2]$ , we further bound  $\|\tilde{\delta}_2\|_2$ , which leads to condition **(B)**.

<sup>1</sup>Introduced by [Candes and Tao \(2007\)](#) for studying the oracle inequalities for the Lasso estimator and Dantzig selector.

- **hard-thresholding condition**: The penalty  $p_\lambda(t)$  satisfies the conditions of Prop. 13.1.1 with

$$p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$$

$$\min_{j=1, \dots, s} |\beta_{0,j}| > \max \left\{ (1 - c_1)\lambda, 2\kappa_0^{-1} p_\lambda^{1/2}(\infty) \right\}$$

Now, look back at the condition 13.3, we can upper-bound  $\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$  by  $\frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$ . Consider 2 cases:

- **Case 1**:  $\|\hat{\beta}\|_0 \geq s$ . By the hard-thresholding condition, we have  $|\beta_{0,j}| > (1 - c_1)\lambda$  and  $p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$ . Hence, for  $j = 1, \dots, s$ ,

- if  $\hat{\beta}_j \neq 0$ , we must have  $|\hat{\beta}_j| > (1 - c_1)\lambda$ . And by the mean-value theorem, we have

$$|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| = p'_\lambda(b)(|\hat{\beta}_j| - |\beta_{0,j}|) \leq p'_\lambda(b)|\delta_{0,j}|$$

where  $b$  is between  $|\beta_{0,j}|$  and  $|\hat{\beta}_j|$ , hence,  $b > |\beta_{0,j}| > (1 - c_1)\lambda$ , by the concavity of  $p_\lambda$ , we have  $p'(b) < p'((1 - c_1)\lambda) \leq \lambda_0/4$ , which leads to  $|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| \leq \frac{1}{4} \lambda_0 |\delta_{0,j}|$ .

- if  $\hat{\beta}_j = 0$ , since  $\|\hat{\beta}_0\| \geq s$ , there must exist some  $j' > s$  s.t.  $\hat{\beta}_{j'} \neq 0$ , similarly

$$\begin{aligned} |p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_{j'}|)| &\leq |p_\lambda(|\beta_{0,j}|) - p_\lambda((1 - c_1)\lambda)| + |p_\lambda(|\hat{\beta}_{j'}|) - p_\lambda((1 - c_1)\lambda)| \\ &= p'_\lambda(b_1)(|\beta_{0,j}| - (1 - c_1)\lambda) + p'_\lambda(b_2)(|\hat{\beta}_{j'}| - (1 - c_1)\lambda) \\ &\leq p'_\lambda(b_1) \left( |\beta_{0,j}| - \underbrace{|\hat{\beta}_j|}_{=0} \right) + p'_\lambda(b_2) \left( |\hat{\beta}_{j'}| - \underbrace{|\beta_{0,j'}|}_{=0} \right) \\ &= p'_\lambda(b_1)|\delta_j| + p'_\lambda(b_2)|\delta_{j'}| \leq \frac{\lambda_0}{4} (|\delta_j| + |\delta_{j'}|) \end{aligned}$$

together, we have

$$\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq \frac{1}{4} \lambda_0 \|\delta\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$$

- **Case 2**:  $\|\hat{\beta}\|_0 = s - k$  for some  $k \geq 1$ . Then we must have  $\|\delta\|_0 \leq \|\hat{\beta}\|_0 + \|\beta_0\|_0 \leq s - k + s < 2s$ , and  $\|\delta\|_2 \geq \sqrt{k} \min_{j=1, \dots, s} |\beta_{0,j}|$ . Also, there are at least  $k$  null estimates ( $\hat{\beta}_j = 0$ ), thus

$$\underbrace{\frac{1}{4n} \|\mathbf{X}\delta\|_2^2 \geq \frac{\kappa_0^2}{4} \|\delta\|_2^2}_{\text{Condition 1(A)}} \geq \underbrace{\frac{\kappa_0^2}{4} \left( \sqrt{k} \min_{j=1, \dots, s} |\beta_{0,j}| \right)^2}_{\text{Condition 2}} \geq kp_\lambda(\infty) \geq kp_\lambda(|\beta_{0,j}|)$$

similar to Case 1, we have the desired upper bound

$$\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq kp_\lambda(\infty) + \frac{1}{4} \lambda_0 \|\delta\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$$

Combining Case 1 and 2, we have  $\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$ , plug this back in 13.3, get

$$\begin{aligned} \frac{1}{n} \|\mathbf{X}\delta\|_2^2 + \lambda_0 \|\delta_2\|_1 &\leq 3\lambda_0 \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \\ &\leq 3\lambda_0 \|\delta_1\|_1 + \frac{1}{2n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{2} \lambda_0 \underbrace{\|\delta\|_1}_{=\|\delta_1\|_1 + \|\delta_2\|_1} \\ &\leq 7\lambda_0 \|\delta_1\|_1 \end{aligned}$$

which leads to the constraint in 13.2 and  $\frac{1}{n} \|\mathbf{X}\boldsymbol{\delta}\|_2^2 \leq 7\lambda_0 \|\boldsymbol{\delta}_1\|_1$ .

Now, look back at Condition 1(B)

$$\kappa = \kappa(s, 7) = \min_{\boldsymbol{\delta} \neq \mathbf{0}, \|\boldsymbol{\delta}_2\|_1 \leq 7\|\boldsymbol{\delta}_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\boldsymbol{\delta}\|_2}{\|\boldsymbol{\delta}_1\|_2 \vee \|\tilde{\boldsymbol{\delta}}_2\|_2} \right\} > 0$$

we have

$$\frac{1}{4} \kappa^2(s, 7) \|\boldsymbol{\delta}_1\|_2^2 \leq \frac{1}{4} \kappa^2(s, 7) (\|\boldsymbol{\delta}_1\|_2^2 \vee \|\tilde{\boldsymbol{\delta}}_2\|_2^2) \leq \frac{1}{4n} \|\mathbf{X}\boldsymbol{\delta}\|_2^2 \leq \underbrace{\frac{7}{4} \lambda_0 \|\boldsymbol{\delta}_1\|_1}_{\text{Cauchy-Schwartz inequality}} \leq \frac{7}{4} \lambda_0 \sqrt{s} \|\boldsymbol{\delta}_1\|_2$$

hence

$$\|\boldsymbol{\delta}_1\|_2 \leq \frac{7\lambda_0\sqrt{s}}{\kappa^2(s, 7)} \quad \|\boldsymbol{\delta}_1\|_1 \leq \sqrt{s} \|\boldsymbol{\delta}_1\|_2 \leq \frac{7\lambda_0 s}{\kappa^2(s, 7)} \quad \|\boldsymbol{\delta}'_2\|_2 \leq \frac{\sqrt{7\lambda_0\sqrt{s}} \|\boldsymbol{\delta}_1\|_2}{\kappa(s, 7)}$$

Notice that the  $k$ -th largest absolute component of  $\boldsymbol{\delta}_2$  is bounded from above by  $\|\boldsymbol{\delta}_2\|_1/k$ , then for  $\boldsymbol{\delta}_{2_s}$ , the subvector of  $\boldsymbol{\delta}_2$  consisting of components **excluding** those with the  $s$  largest magnitudes, we have

$$\|\boldsymbol{\delta}_{2_s}\|_2^2 \leq \sum_{k=s+1}^{p-s} \frac{1}{k^2} \|\boldsymbol{\delta}_2\|_1^2 \leq s^{-1} \|\boldsymbol{\delta}_2\|_1^2 \Rightarrow \|\boldsymbol{\delta}_{2_s}\|_2 \leq \frac{1}{\sqrt{s}} \|\boldsymbol{\delta}_2\|_1 \stackrel{13.2}{\leq} \frac{7}{\sqrt{s}} \|\boldsymbol{\delta}_1\|_1 \stackrel{\text{C-S}}{\leq} 7 \|\boldsymbol{\delta}_1\|_2$$

since  $\boldsymbol{\delta}_{2_s}$  and  $\boldsymbol{\delta}'_2$  are a partition of  $\boldsymbol{\delta}$ , we have

$$\|\boldsymbol{\delta}_2\|_2 \leq \|\boldsymbol{\delta}_{2_s}\|_2 + \|\boldsymbol{\delta}'_2\|_2 \leq 7 \|\boldsymbol{\delta}_1\|_2 + \frac{\sqrt{7\lambda_0\sqrt{s}} \|\boldsymbol{\delta}_1\|_2}{\kappa(s, 7)} \leq 7 \|\boldsymbol{\delta}_1\|_2 \leq \frac{56\lambda_0\sqrt{s}}{\kappa^2(s, 7)}$$

## References

- Peter J Bickel, Ya'acov Ritov, and Alexandre B Tsybakov. Simultaneous analysis of lasso and dantzig selector. *The Annals of Statistics*, pages 1705–1732, 2009.
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