

Topic 18: Eigenvalue and Spike Models

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Key points: .

Disclaimer: The note is built on Prof. *Jinchi Lv*'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

18.1 Motivation

Consider n independent observations $\mathbf{X}_i \in \mathbb{R}^p$ drawn from a $\mathcal{N}(\mathbf{0}, \Sigma)$, then the covariance can be decomposed into 2 parts, white noise and low rank

$$\Sigma = \text{Cov}(\mathbf{X}_i) = \mathbf{I} + \sum_{k=1}^M \theta_k \mathbf{v}_k \mathbf{v}_k' = \Sigma_0 + \Phi$$

where M denotes the **number of spikes** in the distribution of eigenvalues. The idea is: spikes deviate from a reference model along a small fixed number of unknown directions. If $\Phi = \mathbf{0}$, then none of the sample eigenvalues is separated from the bulk.

Why a spike model is interesting? A spike model can help determine the latent dimension of the data, some examples being

- Principal component analysis (PCA): spikes are related to the directions of the most variations of the data, i.e., the principal components
- Clustering model: M spikes is equivalent to $M + 1$ clusters
- Economic significance: M is related to the number of factor loadings

Then the question is threefold:

- How to determine M
- How to estimate \mathbf{v}_k
- How to test θ_k

Under rank one alternative, we would like to test the hypothesis

$$H_1 : \Sigma = \mathbf{I}_p + \theta \mathbf{v} \mathbf{v}', \theta > 0$$

against the null

$$H_0 : \Sigma = \mathbf{I}_p$$

with the key assumptions:

A1 Gaussian error

A2 large p : $p \leq n$ but allows $p/n \rightarrow \gamma \in (0, 1)$

References