Econometrics **May 23, 2023**

Topic 16: Graphical Network Inference

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Key points:

Disclaimer: *The note is built on Prof. Jinchi Lv's lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.*

16.1 Motivation

Consider a classic question: For n observations of dimension p, how can we capture the statistical relationships between the variables of interest? Consider the example of the multivariate Gaussian distribution:

Example 16.1.1: Multivariate Gaussian Distribution

Suppose we have n observations of dimension p, $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$. let $\mathbf{S}$ be the empirical covariance matrix. Then the probability density

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2} \det(\mathbf{\Sigma})^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})' \mathbf{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})\right\}$$

define the **inverse covariance matrix** or **precision matrix** as $\mathbf{\Omega} = \mathbf{\Sigma}^{-1}$, then we have

$$f_{\mu,\mathbf{\Omega}} = \exp\left\{\boldsymbol{\mu}'\mathbf{\Omega}\boldsymbol{x} - \left\langle \mathbf{\Omega}, \frac{1}{2}\mathbf{x}\mathbf{x}' \right\rangle - \frac{p}{2}\log(2\pi) + \frac{1}{2}\log\det(\mathbf{\Omega}) - \frac{1}{2}\boldsymbol{\mu}'\mathbf{\Omega}\boldsymbol{\mu}\right\}$$

In this example, we know that **every** multivariate Gaussian distribution can be represented by a pairwise **Gaussian Markov Random Field (GMRF)**, which an **undirected graph** $G = (V, E)$

- representing the collection of variables $\mathbf{x}$ by a vertex set $\mathcal{V} = \{1, \cdots, p\}$
- encoding correlations between variables by a set of edges $\mathcal{E} = \{(i,j) \in \mathcal{V} \mid i =\neq j, \Omega_{ij} \neq 0\}$

What's the goal? We want to estimate a **sparse** graph structure given $n \ll p$ i.i.d. observations. But what does sparsity means in this context? A sparse graph is **equivalent** to a sparse precision matrix: the precision matrix should have many 0s.

Sparse precision matrix for the Gaussian vector mentioned above $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, we have $\forall u, v$

$$x_u \perp x_v \mid \mathbf{x}_{\mathcal{V}\setminus\{u,v\}} \Leftrightarrow \Omega_{u,v} = 0$$

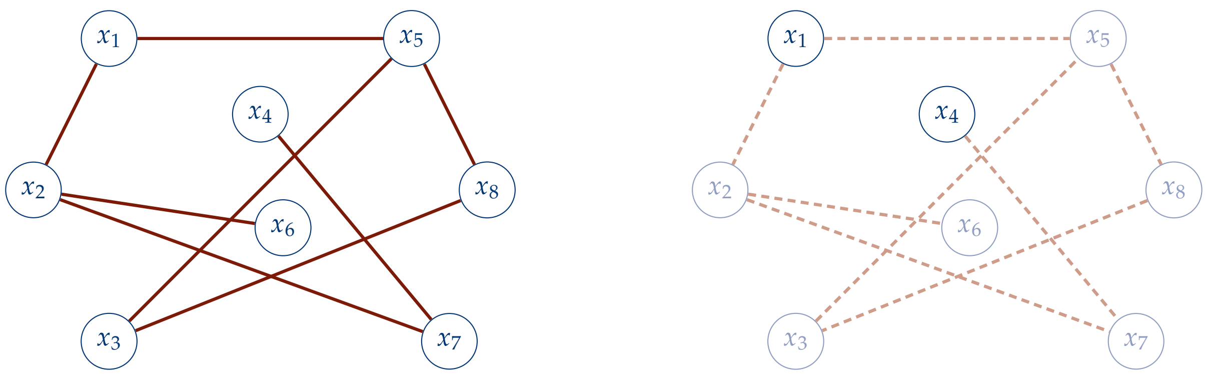
that is, sparsity of the precision matrix is equivalent to **conditional independence**[1]. Consider a graph, where x_1 and x_4 are only connected through other nodes, that is x_1 and x_4 are conditional independent,

[1]Meanwhile, for independence: $\Sigma_{u,v} = 0 \Leftrightarrow x_u \perp x_v$

then we can have the precision matrix be something like:

$$\boldsymbol{\Theta} = \begin{bmatrix} \ast & \ast & 0 & 0 & \ast & 0 & 0 & 0 \\ \ast & \ast & 0 & 0 & 0 & \ast & \ast & 0 \\ 0 & 0 & \ast & 0 & \ast & 0 & 0 & \ast \\ 0 & 0 & 0 & \ast & 0 & 0 & \ast & 0 \\ \ast & 0 & \ast & 0 & \ast & 0 & 0 & \ast \\ 0 & \ast & 0 & 0 & 0 & \ast & 0 & 0 \\ 0 & \ast & 0 & \ast & 0 & 0 & \ast & 0 \\ 0 & 0 & \ast & 0 & \ast & 0 & 0 & \ast \end{bmatrix}$$

where 0



x_1 **and** x_4 **are** ***connected*** x_1 **and** x_4 **are** ***NOT connected*****, conditionally**