

## Topic 17: False Discovery Rate (FDR) and Knockoffs

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**Key points:** Constructing knockoff variables to control FDR when estimating regression coefficients.

**Disclaimer:** The note is built on Prof. [Jinchi Lv](#)'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

### 17.1 Motivation

Consider the classical linear regression setting

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where  $\boldsymbol{\beta} \in \mathbb{R}^p$  is the unknown vector of coefficients and  $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ . In a high-dimensional problem, we would like to just select a subset of all variables  $\hat{S} \subset \{1, \dots, p\}$  s.t. conditional on  $\{\mathbf{X}_j\}_{j \in \hat{S}}$ ,  $\mathbf{y}$  is **independent** of all other variables, we can define the **False Discovery Rate (FDR)** in can be defined as

#### Definition 17.1.1: False Discovery Rate (FDR)

$$\text{FDR} = \mathbb{E}(\text{FDP}) = \mathbb{E} \left[ \frac{|\hat{S} \cap \mathcal{H}_0|}{|\hat{S}|} = \frac{\#\{j : j \in \hat{S} \setminus S\}}{\#\{j : j \in \hat{S}\}} \right]$$

where  $\mathcal{H}_0 \subset \{1, \dots, p\}$  is the set of **null** variables:  $\mathbf{X}_j$  is **null** iff  $\mathbf{Y}$  is independent of  $\mathbf{X}_j$  conditional on the other variables  $\mathbf{X}_{-j} = \{\mathbf{X}_1, \dots, \mathbf{X}_p\} \setminus \{\mathbf{X}_j\}$ .

In this note, we consider a series of knockoff-based methods to control FDR. They all follow a common procedure:

- **Step 1:** Construct Knockoffs
- **Step 2:** Calculate test statistics for both original and knockoff variables
- **Step 3:** Calculate a threshold for the test statistics, controlling for a desired FDR level
- **Step 4:** Select variables that pass the threshold

### 17.2 Barber and Candès (2015)

[Barber and Candès \(2015\)](#) construct the knockoffs by the following procedure

- Calculate the Gram matrix  $\boldsymbol{\Sigma} = \mathbf{X}'\mathbf{X}$  for the normalized original variables, where  $\Sigma_{jj} = \|\mathbf{X}_j\|_2^2 = 1$

- Construct the knockoffs  $\tilde{\mathbf{X}}$  s.t.

$$\tilde{\mathbf{X}}'\tilde{\mathbf{X}} = \mathbf{X}'\mathbf{X} - \text{diag}\{s\}$$

and

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## References

Rina Foygel Barber and Emmanuel J. Candès. Controlling the false discovery rate via knockoffs. *Annals of Statistics*, 43(5):2055–2085, 2015.