

Topic 16: Graphical Network Inference

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Key points:

Disclaimer: The note is built on Prof. [Jinchi Lv](#)'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

16.1 Motivation

Consider a classic question: For n observations of dimension p , how can we capture the statistical relationships between the variables of interest? Consider the example of the multivariate Gaussian distribution:

Example 16.1.1: Multivariate Gaussian Distribution

Suppose we have n observations of dimension p , $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \Sigma)$. let \mathbf{S} be the empirical covariance matrix. Then the probability density

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \mu)' \Sigma^{-1} (\mathbf{x} - \mu) \right\}$$

define the **inverse covariance matrix** or **precision matrix** as $\Omega = \Sigma^{-1}$, then we have

$$f_{\mu, \Omega} = \exp \left\{ \mu' \Omega \mathbf{x} - \left\langle \Omega, \frac{1}{2} \mathbf{x} \mathbf{x}' \right\rangle - \frac{p}{2} \log(2\pi) + \frac{1}{2} \log \det(\Omega) - \frac{1}{2} \mu' \Omega \mu \right\}$$

In this example, we know that **every** multivariate Gaussian distribution can be represented by a pairwise **Gaussian Markov Random Field (GMRF)**, which an undirected graph $G = (V, E)$

- representing the collection of variables \mathbf{x} by a vertex set $\mathcal{V} = \{1, \dots, p\}$
- encoding correlations between variables by a set of edges $\mathcal{E} = \{(i, j) \in \mathcal{V} \mid i \neq j, \Omega_{ij} \neq 0\}$

What's the goal? We want to estimate a **sparse** graph structure given $n \ll p$ i.i.d. observations. But what does sparsity means in this context? A sparse graph is equivalent to a sparse precision matrix: the precision matrix should have many 0s.

Sparse precision matrix for the Gaussian vector mentioned above $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \Sigma)$, we have $\forall u, v$

$$x_u \perp x_v \mid \mathbf{x}_{\mathcal{V} \setminus \{u, v\}} \Leftrightarrow \Omega_{u, v} = 0$$

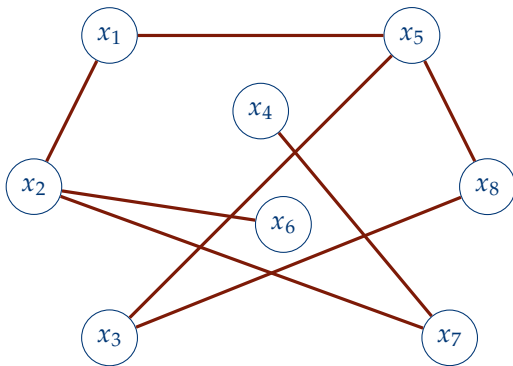
that is, sparsity of the precision matrix is equivalent to **conditional independence**¹. Consider a **graph**, where x_1 and x_4 are only connected through other nodes, that is x_1 and x_4 are conditional independent,

¹Meanwhile, for independence: $\Sigma_{u, v} = 0 \Leftrightarrow x_u \perp x_v$

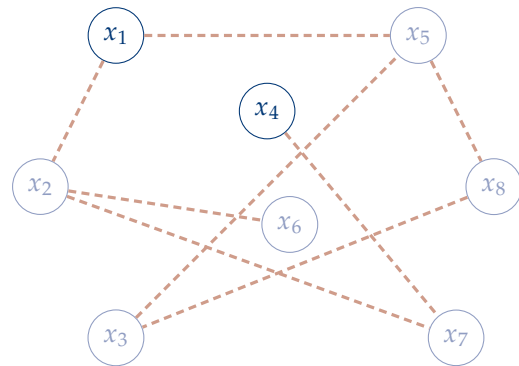
then we can have the precision matrix be something like:

$$\Theta = \begin{bmatrix} * & * & 0 & 0 & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & * & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 & 0 & * & 0 \\ * & 0 & * & 0 & * & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * & 0 & 0 & * \end{bmatrix}$$

where 0



x_1 and x_4 are connected



x_1 and x_4 are NOT connected, conditionally