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# Topic 19: Community Detection

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**Key points**: .

**Disclaimer**: *The note is built on Prof. Jinchi Lv's lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.*

## 19.1 Stochastic Block Model

Consider an undirected graph $G$, with nodes $V$ and edges $E$. Let

- $n$ be a positive integer: the number of **vertices**
- $k$ be a positive integer: the number of **communities**
- $p = (p_1, \cdots, p_k)$ be a probability vector on $\{1, \cdots, k\} := [k]$: the **prior** on the $k$ communities
- $\mathbf{W}$ be a $k \times k$ symmetric matrix with entries $W_{ij} \in [0, 1]$: the matrix of **connectivity probabilities**

then we have

**Definition 19.1.1: Stochastic Block Model**

The pair $(\mathbf{X}, G)$ is drawn under $SBM(n, p, \mathbf{W})$ if $\mathbf{X}$ is an $n$ dimensional random vector with i.i.d. components distributed under $p$, and $G$ is an $n-$vertex simple graph where vertices $i$ and $j$ are connected with probability $W_{X_i,X_j}$, **independently** of other pairs of vertices. And the **community** sets can be defined by

$$\Omega_i = \Omega_i(\mathbf{X}) := \{v \in [n] : X_v = i\}, i \in [k]$$

Immediately, we can define the symmetry of SBM as:

**Definition 19.1.2: Symmetric SBM**

An SBM is called symmetric if

- $p$ is **uniform**
- $\mathbf{W}$ takes the same value **on the diagonal** and the same value **off the diagonal**

$(\mathbf{X}, G)$ is drawn under $SSBM(n, k, A, B)$ if $p = \{1/k\}^k$ and $\mathbf{W}$ takes avlue $A$ on the diagonal and $B$ off the diagonal.

### 19.1.1 Recovery

The goal of community detection is to recover the labels $\mathbf{X}$ by observing $G$, up to some level of accuracy. First, define **agreement** as

**Definition 19.1.3: Agreement of Communities**

The agreement between two community vectors $\mathbf{x}, \mathbf{y} \in [k]^n$ is obtained by maximizing the common components between $\mathbf{x}$ and any relabelling of $\mathbf{y}$, that is

$$A(\mathbf{x}, \mathbf{y}) = \max_{\pi \in S_k} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\left[x_i = \pi(y_i)\right]$$

where $S_k$ is the group of permutations on $[k]$.

The **relabelling** permutation is used to handle symmetric communities such as in SSBM, as it is impossible to recover the actual labels in this case. But it's possible to recover the **partition**. There are 2 types of partition recovery we consider

**Exact Recovery** First, consider the case of **exact recovery**:

**Definition 19.1.4: Exact Recovery**

Let $(\mathbf{X}, G) \sim SBM(n, p, W)$, the exact recovery is solved if there exists an algorithm that takes $G$ as an input and outpus $\hat{\mathbf{X}} = \hat{\mathbf{X}}(G)$ such that $\mathbb{P}\left\{A(\mathbf{X}, \hat{\mathbf{X}}) = 1\right\} = 1 - o_p(1)$

In the SSBM case, algorithms that guarantee

$$A(\mathbf{X}, \hat{\mathbf{X}}) \to \frac{1}{k}$$

would be trivial.

**Weak Recovery** On the other hand, we the case of **weak recovery** defined as

**Definition 19.1.5: Weak Recovery**

Weak recovery or detection is solved $SSBM(n, k, A, B)$ if for $(\mathbf{X}, G) \sim SSBM(n, k, A, B)$, then $\exists \epsilon > 0$ and an algorithm that takes $G$ as an input and outputs $\hat{\mathbf{X}}$ such that

$$\mathbb{P}\left\{A(\mathbf{X}, \hat{\mathbf{X}}) \geq \frac{1}{k} + \epsilon\right\} = 1 - o(1)$$

### 19.1.2 Example: SSBM(n,2)

Let's look at the example of $SSBM(n, 2, \alpha\frac{\log n}{n}, \beta\frac{\log n}{n})$, where

- $n$: number of vertices (assumed to be even for simplicity)
- for each $v \in [n]$, a binary label $X_v$ is attached s.t.

$$|\{v \in [n] : X_v = 1\}| = n/2$$

- for each pair of distinct nodes $u, v \in [n]$, an edge is placed with probability

- $\alpha \frac{\log n}{n}$ if $X_u = X_v$
- $\beta \frac{\log n}{n}$ if $X_u \neq X_v$

where edges are place