

Topic 13: Non-convex Learning + Lasso

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Key points: Combining the best of the two, we can use **Lasso plus Concave** method, with Lasso screening and concave component selecting variables, achieving a coordinated intrinsic two-scale learning.

Disclaimer: The note is built on Prof. *Jinchi Lv*'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

We are facing a tradeoff:

- **Convex** methods: have appealing prediction power and oracle inequalities, but challenging to provide tight false sign rate control
- **Concave** methods: have good variable selection properties, but challenging to establish global properties and risk properties

Here, we take advantage of the linearity of Lasso (convex *and* concave) and try to combine it with concave regularization to get the best of both.

13.1 Model Setup

Again, consider a linear regression model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where

- response vector ($n \times 1$): $\mathbf{y} = (y_1, \dots, y_n)'$
- design matrix ($n \times p$): $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_p)$, with each column rescaled to have L_2 -norm $n^{1/2}$

here, we consider a scenario where

- $\boldsymbol{\beta}_0 = (\beta_{0,1}, \dots, \beta_{0,p})'$ is *sparse* (with many 0 components)
- ultra-high dimensions: $\log p = O(n^a)$, for some $0 < a < 1$

and consider the penalized least squares

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left\{ (2n)^{-1} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda_0 \|\boldsymbol{\beta}\|_1 + \|p_\lambda(\boldsymbol{\beta})\|_1 \right\} \quad (13.1)$$

where

- $\lambda_0 = c \left(\frac{\log p}{n} \right)^{1/2}$ for some $c > 0$
- $p_\lambda(\boldsymbol{\beta}) = p_\lambda(|\boldsymbol{\beta}|) = (p_\lambda(|\beta_1|), \dots, p_\lambda(|\beta_p|))'$, with $|\boldsymbol{\beta}| = (|\beta_1|, \dots, |\beta_p|)'$; the concave penalty $p_\lambda(t)$ is defined on $t \in [0, \infty)$, indexed by $\lambda \geq 0$, increasing in both t and λ , $p_\lambda(0) = 0$

the 2 penalty components

- L_1 -component: minimum amount of regularization for removing noise in prediction
- concave component $\|p_\lambda(\boldsymbol{\beta})\|_1$: adapt model sparsity for variable selection

Under this set up, we can derive the hard-thresholding property as

Proposition 13.1.1: Hard-Thresholding Property

Assume the $p_\lambda(t)$, $t \geq 0$, is **increasing and concave** with

- $p_\lambda(t) \geq p_{H,\lambda}(t) = \frac{1}{2} [\lambda^2 - (\lambda - t)_+^2]$ on $[0, \lambda]$
- $p'_\lambda((1 - c_1)\lambda) \leq c_1\lambda$ for some $c_1 \in [0, 1]$
- $-p''_\lambda(t)$ decreasing on $[0, (1 - c_1)\lambda]$

then any local minimizer of 13.1 that is also a global minimizer in each coordinate has the **hard-thresholding** feature that each component is either 0 or of magnitude **larger** than $(1 - c_1)\lambda$

Such property is shared by a wide class of concave penalties, including hard-thresholding penalty $p_{H,\lambda}(t)$ with $c_1 = 0$, L_0 -penalty, and SICA (with suitable c_1).

How to understand this proposition? Let $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_p)'$, then **each** $\hat{\beta}_j$ is the global minimizer of the corresponding univariate penalized least-square problem along the j -th coordinate. These univariate problems share a common form with (generally) different scalars z

$$\hat{\beta}(z) = \arg \min_{\beta \in \mathbb{R}} \left\{ \frac{1}{2}(z - \beta)^2 + \lambda_0 |\beta| + p_{H,\lambda}(|\beta|) \right\}$$

after we rescale all covariates to have L_2 -norm $n^{1/2}$. The solution to these univariate problems are

$$\hat{\beta}(z) = \text{sgn}(z)(|z| - \lambda_0) \cdot \mathbf{1}_{|z| > \lambda + \lambda_0}$$

these solutions have the same feature as the hard-thresholded estimator: each component is either 0 or of magnitude larger than λ . This provides a better distinction between insignificant and significant covariates than soft-thresholding by L_1 penalty.

With the hard-thresholding property of Prop. 13.1.1, we can prove a basic constraint for the global optimum $\hat{\beta}$ on an event with significant probability (Fan and Lv, 2014)

$$\|\delta_2\|_1 \leq 7\|\delta_1\|_1 \quad (13.2)$$

where $\delta = \hat{\beta} - \beta_0 = (\hat{\beta}'_1, \hat{\beta}'_2)' - (\beta'_{0,1}, \beta'_{0,2})' = (\delta'_1, \delta'_2)'$, with $\delta_1 \in \mathbb{R}^s$. Where does this constraint come from? For the penalized least square question 13.1

$$\min_{\beta \in \mathbb{R}^p} \{ (2n)^{-1} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda_0 \|\beta\|_1 + \|p_\lambda(\beta)\|_1 \}$$

the global minimizer $\hat{\beta}$ leads to

$$\begin{aligned} (2n)^{-1} \|\mathbf{y} - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 &= (2n)^{-1} \|\mathbf{X}\beta_0 + \epsilon - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\ &= (2n)^{-1} \|\epsilon - \mathbf{X}(\hat{\beta} - \beta_0)\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\ &\leq (2n)^{-1} \|\mathbf{y} - \mathbf{X}\beta_0\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\ &= (2n)^{-1} \|\epsilon\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \end{aligned}$$

then, plug in $\delta = \hat{\beta} - \beta_0$, we get

$$\begin{aligned} (2n)^{-1} \|\epsilon - \mathbf{X}\delta\|_2^2 + \lambda_0 \|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq (2n)^{-1} \|\epsilon\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\ (2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \end{aligned}$$

since $\beta_{0,2} = \mathbf{0}$, $\delta_2 = \beta_{0,2} + \delta_2$, we have

$$\|\beta_0 + \delta\|_1 = \|\beta_{0,1} + \beta_{0,2} + \delta_1 + \delta_2\|_1 = \|\beta_{0,1} + \delta_1 + \delta_2\|_1 \leq \|\beta_{0,1} + \delta_1\|_1 + \|\delta_2\|_1$$

hence

$$(2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 \leq \lambda_0 \|\beta_{0,1}\|_1 - \lambda_0 \|\beta_{0,1} + \delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

and by the reverse triangle inequality $\|\beta_{0,1}\|_1 - \|\beta_{0,1} + \delta_1\|_1 \leq \|\delta_1\|_1$, we get

$$(2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 \leq \lambda_0 \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

If assume the distribution of the model error ϵ as

$$\Pr\left(\|n^{-1} \mathbf{X}' \epsilon\|_\infty > \frac{\lambda_0}{2}\right) = O(p^{-c_0})$$

conditional on the event $\mathcal{E} = \{\|n^{-1} \mathbf{X}' \epsilon\|_\infty \leq \lambda_0/2\}$, we have

$$-n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 - \lambda_0 \|\delta_1\|_1 \geq -\frac{\lambda_0}{2} \|\delta\|_1 + \lambda_0 \|\delta_2\|_1 - \lambda_0 \|\delta_1\|_1 = \frac{\lambda_0}{2} \|\delta_2\|_1 - \frac{3\lambda_0}{2} \|\delta_1\|_1$$

plug this result back, get

$$\frac{1}{2n} \|\mathbf{X}\delta\|_2^2 + \frac{\lambda_0}{2} \|\delta_2\|_1 \leq \frac{3\lambda_0}{2} \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \quad (13.3)$$

Now, if we further impose 2 conditions:

- **Condition 1 (eigenvalue condition)**: for some positive constant κ_0

$$\min_{\|\delta\|_2=1, \|\delta\|_0 \leq 2s} \frac{1}{\sqrt{n}} \|\mathbf{X}\delta\|_2 \geq \kappa_0 \quad (\mathbf{A})$$

$$\kappa = \kappa(s, 7) = \min_{\delta \neq 0, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2 \vee \|\tilde{\delta}_2\|_2} \right\} > 0 \quad (\mathbf{B})$$

where $\tilde{\delta}_2$ is the subvector of δ_2 consisting of the components with the s largest absolute values. Here

- Condition **(A)** is a mild sparse eigenvalue condition
- Condition **(B)** combines the restricted eigenvalue assumptions in [Bickel et al. \(2009\)](#)¹. The intuition is, for OLS estimation, $\mathbf{X}'\mathbf{X}$ should be positive definite, that is

$$\min_{\mathbf{0} \neq \delta \in \mathbb{R}^p} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta\|_2} \right\} > 0$$

however, when $p > n$, this condition **never** holds, hence we replace $\|\delta\|_2$ with the L_2 -norm of $\|\delta_1\|_2$, a subvector of δ

$$\kappa = \kappa(s, 7) = \min_{\delta \neq 0, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2} \right\} > 0$$

and for L_q loss with $q \in (1, 2]$, we further bound $\|\tilde{\delta}_2\|_2$, which leads to condition **(B)**.

¹Introduced by [Candes and Tao \(2007\)](#) for studying the oracle inequalities for the Lasso estimator and Dantzig selector.

- **Condition 2 (hard-thresholding condition):** The penalty $p_\lambda(t)$ satisfies the conditions of Prop. 13.1.1 with

$$p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$$

$$\min_{j=1, \dots, s} |\beta_{0,j}| > \max \left\{ (1 - c_1)\lambda, 2\kappa_0^{-1} p_\lambda^{1/2}(\infty) \right\}$$

Now, look back at the condition 13.3, we can upper-bound $\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$ by $\frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$. Consider 2 cases:

- **Case 1:** $\|\hat{\beta}\|_0 \geq s$. By the hard-thresholding condition, we have $|\beta_{0,j}| > (1 - c_1)\lambda$ and $p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$. Hence, for $j = 1, \dots, s$,

- if $\hat{\beta}_j \neq 0$, we must have $|\hat{\beta}_j| > (1 - c_1)\lambda$. And by the mean-value theorem, we have

$$|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| = p'_\lambda(b)(|\hat{\beta}_j| - |\beta_{0,j}|) \leq p'_\lambda(b)|\delta_{0,j}|$$

where b is between $|\beta_{0,j}|$ and $|\hat{\beta}_j|$, hence, $b > |\beta_{0,j}| > (1 - c_1)\lambda$, by the concavity of p_λ , we have $p'_\lambda(b) < p'_\lambda((1 - c_1)\lambda) \leq \lambda_0/4$, which leads to $|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| \leq \frac{1}{4} \lambda_0 |\delta_j|$.

- if $\hat{\beta}_j = 0$, since $\|\hat{\beta}\|_0 \geq s$, there must exist some $j' > s$ s.t. $\hat{\beta}_{j'} \neq 0$, similarly

$$\begin{aligned} |p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_{j'}|)| &\leq |p_\lambda(|\beta_{0,j}|) - p_\lambda((1 - c_1)\lambda)| + |p_\lambda(|\hat{\beta}_{j'}|) - p_\lambda((1 - c_1)\lambda)| \\ &= p'_\lambda(b_1)(|\beta_{0,j}| - (1 - c_1)\lambda) + p'_\lambda(b_2)(|\hat{\beta}_{j'}| - (1 - c_1)\lambda) \\ &\leq p'_\lambda(b_1) \left(|\beta_{0,j}| - \underbrace{|\hat{\beta}_j|}_{=0} \right) + p'_\lambda(b_2) \left(|\hat{\beta}_{j'}| - \underbrace{|\beta_{0,j'}|}_{=0} \right) \\ &= p'_\lambda(b_1)|\delta_j| + p'_\lambda(b_2)|\delta_{j'}| \leq \frac{\lambda_0}{4} (|\delta_j| + |\delta_{j'}|) \end{aligned}$$

together, we have

$$\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq \frac{1}{4} \lambda_0 \|\delta\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$$

- **Case 2:** $\|\hat{\beta}\|_0 = s - k$ for some $k \geq 1$. Then we must have $\|\delta\|_0 \leq \|\hat{\beta}\|_0 + \|\beta_0\|_0 \leq s - k + s < 2s$, and $\|\delta\|_2 \geq \sqrt{k} \min_{j=1, \dots, s} |\beta_{0,j}|$. Also, there are at least k null estimates ($\hat{\beta}_j = 0$), thus

$$\underbrace{\frac{1}{4n} \|\mathbf{X}\delta\|_2^2 \geq \frac{\kappa_0^2}{4} \|\delta\|_2^2}_{\text{Condition 1(A)}} \geq \underbrace{\frac{\kappa_0^2}{4} \left(\sqrt{k} \min_{j=1, \dots, s} |\beta_{0,j}| \right)^2}_{\text{Condition 2}} \geq kp_\lambda(\infty) \geq kp_\lambda(|\beta_{0,j}|)$$

similar to Case 1, we have the desired upper bound

$$\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq kp_\lambda(\infty) + \frac{1}{4} \lambda_0 \|\delta\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$$

Combining Case 1 and 2, we have $\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$, plug this back in 13.3, get

$$\begin{aligned} \frac{1}{n} \|\mathbf{X}\delta\|_2^2 + \lambda_0 \|\delta_2\|_1 &\leq 3\lambda_0 \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \\ &\leq 3\lambda_0 \|\delta_1\|_1 + \frac{1}{2n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{2} \lambda_0 \underbrace{\|\delta\|_1}_{=\|\delta_1\|_1 + \|\delta_2\|_1} \\ &\leq 7\lambda_0 \|\delta_1\|_1 \end{aligned}$$

which leads to the constraint in 13.2 and $\frac{1}{n} \|\mathbf{X}\delta\|_2^2 \leq 7\lambda_0 \|\delta_1\|_1$.

Now, look back at Condition 1(B)

$$\kappa = \kappa(s, 7) = \min_{\delta \neq \mathbf{0}, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2 \vee \|\tilde{\delta}_2\|_2} \right\} > 0$$

we have

$$\frac{1}{4} \kappa^2(s, 7) \|\delta_1\|_2^2 \leq \frac{1}{4} \kappa^2(s, 7) (\|\delta_1\|_2^2 \vee \|\tilde{\delta}_2\|_2^2) \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 \leq \underbrace{\frac{7}{4} \lambda_0 \|\delta_1\|_1}_{\text{Cauchy-Schwartz inequality}} \leq \frac{7}{4} \lambda_0 \sqrt{s} \|\delta_1\|_2$$

hence

$$\|\delta_1\|_2 \leq \frac{7\lambda_0\sqrt{s}}{\kappa^2(s, 7)} \quad \|\delta_1\|_1 \leq \sqrt{s} \|\delta_1\|_2 \leq \frac{7\lambda_0 s}{\kappa^2(s, 7)} \quad \|\delta'_2\|_2 \leq \frac{\sqrt{7\lambda_0\sqrt{s}} \|\delta_1\|_2}{\kappa(s, 7)} \quad (13.4)$$

Notice that the k -th largest absolute component of δ_2 is bounded from above by $\|\delta_2\|_1/k$, then for δ_{2_s} , the subvector of δ_2 consisting of components **excluding** those with the s largest magnitudes, we have

$$\|\delta_{2_s}\|_2^2 \leq \sum_{k=s+1}^{p-s} \frac{1}{k^2} \|\delta_2\|_1^2 \leq s^{-1} \|\delta_2\|_1^2 \Rightarrow \|\delta_{2_s}\|_2 \leq \frac{1}{\sqrt{s}} \|\delta_2\|_1 \stackrel{13.2}{\leq} \frac{7}{\sqrt{s}} \|\delta_1\|_1 \stackrel{C-S}{\leq} 7 \|\delta_1\|_2$$

since δ_{2_s} and δ'_2 are a partition of δ , we have

$$\|\delta_2\|_2 \leq \|\delta_{2_s}\|_2 + \|\delta'_2\|_2 \leq 7 \|\delta_1\|_2 + \frac{\sqrt{7\lambda_0\sqrt{s}} \|\delta_1\|_2}{\kappa(s, 7)} \leq \frac{56\lambda_0\sqrt{s}}{\kappa^2(s, 7)} \quad (13.5)$$

Together, for the estimation loss $\delta = \hat{\beta} - \beta_0$, we have

- **L_2 -covar-loss-correlation:** $\frac{1}{n} \|\mathbf{X}\delta\|_2^2 \leq 7\lambda_0 \|\delta_1\|_1 \leq \frac{(7\lambda_0)^2 s}{\kappa^2(s, 7)} \Rightarrow \frac{1}{\sqrt{n}} \|\mathbf{X}\delta\|_2 \leq \frac{7\lambda_0\sqrt{s}}{\kappa(s, 7)}$
- **L_2 -loss:** $\|\delta\|_2 \leq \|\delta_1\|_2 + \|\delta_2\|_2 \leq \frac{63\lambda_0\sqrt{s}}{\kappa^2(s, 7)}$
- **L_q -loss:** $\|\delta\|_q \leq (s^{(2-q)/2} \|\delta_1\|_2^q)^{1/q} = s^{(2-q)/2q} \|\delta_1\|_2 \leq s^{(2-q)/2q} \frac{7\lambda_0\sqrt{s}}{\kappa^2(s, 7)} = \frac{7\lambda_0 s^{1/q}}{\kappa^2(s, 7)}$

←by Holder's inequality

Define the **number of falsely discovered signs** as²

$$\text{FS}(\hat{\beta}) = |\{j = 1, \dots, p : \text{sgn}(\hat{\beta}_j) \neq \text{sgn}(\beta_{0,j})\}|$$

we know from Prop.13.1.1 that $|\hat{\beta}_j| > (1 - c_1)\lambda$ and from Condition 2 that $|\beta_{0,j}| > (1 - c_1)\lambda$, then if $\text{sgn}(\hat{\beta}_j) \neq \text{sgn}(\beta_{0,j})$, we must have $|\delta_j| = |\hat{\beta}_j - \beta_{0,j}| \geq (1 - c_1)\lambda$. Therefore, it follows that

$$\|\delta\|_2 \geq \left(\text{FS}(\hat{\beta}) \right)^{1/2} (1 - c_1)\lambda$$

hence

$$\text{FS}(\hat{\beta}) \leq \frac{\|\delta\|_2^2}{(1 - c_1)^2 \lambda^2} \leq \left(\frac{63}{1 - c_1} \right)^2 \left(\frac{\lambda_0}{\lambda} \right)^2 \frac{s}{\kappa^4(s, 7)}$$

The results above are all conditional on the event $\mathcal{E} = \{\|n^{-1}\mathbf{X}'\epsilon\|_\infty \leq \lambda_0/2\}$, hence hold simultaneously with probability $1 - O(p^{-c_0})$.

Altogether, we have the following theorem:

²Stronger than the total number of false positives and false negatives.

Theorem 13.1.2: Properties of the Global Minimizer $\hat{\beta}$

Assume that Condition 1 and 2 and the model error bound $\Pr\left(\|n^{-1}\mathbf{X}'\epsilon\|_\infty > \frac{\lambda_0}{2}\right) = O(p^{-c_0})$, and $p_\lambda(t)$ is continuously differentiable. Then the global minimizer $\hat{\beta}$ of 13.1 has the hard-thresholding property stated in Prop. 13.1.1, and, with probability $1 - O(p^{-c_0})$, satisfies simultaneously that

$$\frac{1}{\sqrt{n}} \|\mathbf{X}(\hat{\beta} - \beta_0)\|_2 = O(\kappa^{-1} \lambda_0 s^{1/2}) \quad (13.6)$$

$$\|\hat{\beta} - \beta_0\|_q = O(\kappa^{-2} \lambda_0 s^{1/q}), \quad q \in [1, 2] \quad (13.7)$$

$$\text{FS}(\hat{\beta}) = O\left(\kappa^{-4} \left(\frac{\lambda_0}{\lambda}\right)^2 s\right) \quad (13.8)$$

If in addition $\lambda \geq \frac{56\lambda_0\sqrt{s}}{(1-c_1)\kappa^2}$, then with probability $1 - O(p^{-c_0})$, we also have that

$$\text{sgn}(\hat{\beta}) = \text{sgn}(\beta_0) \quad \|\hat{\beta} - \beta_0\|_\infty = O\left(\lambda_0 \left\| \left(\frac{1}{n} \mathbf{X}_1' \mathbf{X}_1\right)^{-1} \right\|_\infty\right)$$

where \mathbf{X}_1 is the $n \times s$ submatrix of \mathbf{X} corresponding to s nonzero regression coefficients $\beta_{0,j}$.

The proof of the second part follows as such: by assuming $\lambda \geq \frac{56\lambda_0\sqrt{s}}{(1-c_1)\kappa^2}$, from Condition 2, we have $\min_{j=1,\dots,s} |\beta_{0,j}| > \frac{56\lambda_0\sqrt{s}}{\kappa^2(s,7)}$, combined with 13.4, we know that

$$\text{sgn}(\hat{\beta}_j) = \text{sgn}(\beta_{0,j}), \quad \forall j = 1, \dots, s$$

by a simple contradiction argument. In view of 13.5 and the hard-thresholding feature of $\hat{\beta} = (\hat{\beta}'_{0,1}, \hat{\beta}'_{0,2})'$, with $\hat{\beta}_{0,1} = (\hat{\beta}_1, \dots, \hat{\beta}_s)'$, a similar contradiction argument leads to $\hat{\beta}_{0,2} = \mathbf{0}$. Together, we have the sign consistency: $\text{sgn}(\hat{\beta}) = \text{sgn}(\beta_0)$. Under this result, applying Theorem 1 of Lv and Fan (2009), the estimation $\hat{\beta}_{0,1}$ solves the following equation for $\gamma \in \mathbb{R}^s$

$$\gamma = \tilde{\beta}_{0,1} - (n^{-1} \mathbf{X}_1' \mathbf{X}_1)^{-1} \mathbf{b}$$

where

- \mathbf{X}_1 is the $n \times s$ submatrix of \mathbf{X} corresponding to the s non-zero regression coefficients $\beta_{0,j}$
- $\mathbf{b} = \{\lambda_0 \mathbf{1}_s + p'_\lambda(|\gamma|)\} \circ \text{sgn}(\tilde{\beta}_{0,1}) - n^{-1} \mathbf{X}_1' \epsilon$, with componentwise derivative and product.

From the concavity and monotonicity of $p_\lambda(t)$ and Condition 2, we have

$$0 \leq p'_\lambda(t) \leq p'_\lambda\{(1-c_1)\lambda\} \leq \lambda_0/4$$

this gives that each component of $\tilde{\beta}_{0,1}$ has magnitude larger than $(1-c_1)\lambda$. Since $\|n^{-1} \mathbf{X}_1' \epsilon\|_\infty \leq \|n^{-1} \mathbf{X}' \epsilon\|_\infty \leq \frac{\lambda_0}{2}$ on the event \mathcal{E} , hence we have

$$\text{sgn}(\mathbf{b}) = \text{sgn}(\tilde{\beta}_{0,1}), \quad \frac{\lambda_0}{2} \leq \|\mathbf{b}\|_\infty \leq \frac{7\lambda_0}{4}$$

which completes the proof for Theorem 13.1.2.

How to understand Theorem 13.1.2?

- False sign rate $\text{FS}(\hat{\beta}) = O\left(\kappa^{-4} \left(\frac{\lambda_0}{\lambda}\right)^2 s\right)$ is asymptotically vanishing when $\lambda_0/\lambda \rightarrow 0$, outperforming Lasso, whose false sign rate is generally bounded by $O(\lambda_{\max})$ with λ_{\max} being the largest eigenvalue of Gram matrix $n^{-1}\mathbf{X}'\mathbf{X}$; also outperforming concave method, whose false sign rate is generally of order $O(1)$. When **signal strength is stronger** and **λ is chosen suitably**, sign consistency is stronger as well.
- Convergence rates of $\frac{1}{\sqrt{n}} \|\mathbf{X}(\hat{\beta} - \beta_0)\|_2$ and $\|\hat{\beta} - \beta_0\|_q$ are the same as those in [Bickel et al. \(2009\)](#) for the L_1 -component, and are consistent with the concave component of [Zhang and Zhang \(2012\)](#). The bounds $O(\kappa^{-1} \lambda_0 s^{1/2})$, $O(\kappa^{-2} \lambda_0 s^{1/q})$ depend only on the universal regularization parameter $\lambda_0 = c \sqrt{\frac{\log p}{n}}$ for L_1 -component, and are independent of λ for concave component.
- The L_∞ -bound $\|\hat{\beta} - \beta_0\|_\infty = O\left(\lambda_0 \left\| \left(\frac{1}{n} \mathbf{X}'_1 \mathbf{X}_1\right)^{-1} \right\|_\infty\right)$ involves $\left\| \left(\frac{1}{n} \mathbf{X}'_1 \mathbf{X}_1\right)^{-1} \right\|_\infty$, which is bounded from above by $\sqrt{s} \left\| \left(\frac{1}{n} \mathbf{X}'_1 \mathbf{X}_1\right)^{-1} \right\|_2 \leq \sqrt{s} \kappa_0^{-2}$ and can be **dimension-free** in certain scenarios.
- **Oracle property**: Under all conditions of Theorem 13.1.2 hold, and let $\tilde{\beta}$ be the refitted least-squares estimator given by covariates in $\text{supp}(\hat{\beta})$, with $\hat{\beta}$ being the estimator in Theorem 13.1.2. The with probability $1 - O(p^{-c_0})$, $\tilde{\beta}$ equals the oracle estimator, and has the oracle property if the oracle estimator is asymptotic normal.

Theorem 13.1.3: Further Properties of the Global Minimizer $\hat{\beta}$

Under the same regularity conditions, the global minimizer $\hat{\beta}$ in Theorem 13.1.2 satisfies that $\forall \tau > 0$

$$\mathbb{E} \left\{ \frac{1}{n} \left\| \mathbf{X}(\hat{\beta} - \beta_0) \right\|_2^2 \right\} = O \left(\kappa^{-2} \lambda_0^2 s + m_{2,\tau} + \gamma \lambda_0 p^{-c_0} \right) \quad (13.9)$$

$$\mathbb{E} \left\{ \left\| \hat{\beta} - \beta_0 \right\|_q^q \right\} = O \left[\kappa^{-2q} \lambda_0^q s + (2-q) \lambda_0^{-1} m_{2,\tau} + (q-1) \lambda_0^{-2} m_{4,\tau} + ((2-q)\gamma + (q-1)\gamma^2) p^{-c_0} \right] \quad (13.10)$$

$$\mathbb{E} \left\{ \text{FS}(\hat{\beta}) \right\} = O \left[\kappa^{-4} \left(\frac{\lambda_0}{\lambda} \right)^s s + \lambda^{-2} m_{2,\tau} + \left(\frac{\gamma \lambda_0}{\lambda^2} + s \right) p^{-c_0} \right] \quad (13.11)$$

where $m_{q,\tau} = \mathbb{E}(|\epsilon_0|^q \mathbf{1}_{\{|\epsilon_0| > \tau\}})$ denotes the tail moment and $\gamma = \|\beta_0\|_1 + s \lambda_0^{-1} p_\lambda(\infty) + \tau^2 \lambda_0^{-1}$. If in addition $\lambda \geq 56(1 - c_1)^{-1} \kappa^{-2} \lambda_0 \sqrt{s}$, then we have

$$\begin{aligned} \mathbb{E} \left\{ \text{FS}(\hat{\beta}) \right\} &= O \left\{ \lambda^{-2} m_{2,\tau} + \left(\gamma \frac{\lambda_0}{\lambda^2} + s \right) p^{-c_0} \right\} \\ \mathbb{E} \left\{ \left\| \hat{\beta} - \beta_0 \right\|_\infty \right\} &= O \left\{ \lambda_0 \left\| (n^{-1} \mathbf{X}'_1 \mathbf{X}_1)^{-1} \right\|_\infty + \lambda_0^{-1} m_{2,\tau} + \gamma p^{-c_0} \right\} \end{aligned}$$

Again, λ_0 enters all bounds for the oracle risk inequalities, λ only enters the risk bound for the variable selection loss.

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