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Topic 19: Community Detection

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Key points: .

Disclaimer: *The note is built on Prof. Jinchi Lv's lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.*

19.1 Stochastic Block Model (Abbe et al., 2015)

Consider an undirected graph G, with nodes V and edges E. Let

- n be a positive integer: the number of **vertices**
- k be a positive integer: the number of **communities**
- $p = (p_1, \cdots, p_k)$ be a probability vector on $\{1, \cdots, k\} := [k]$: the **prior** on the k communities
- $\mathbf{W}$ be a $k \times k$ symmetric matrix with entries $W_{ij} \in [0, 1]$: the matrix of **connectivity probabilities**

then we have

Definition 19.1.1: Stochastic Block Model

The pair $(\mathbf{X}, G)$ is drawn under $SBM(n, p, \mathbf{W})$ if $\mathbf{X}$ is an n dimensional random vector with i.i.d. components distributed under p, and G is an $n-$vertex simple graph where vertices i and j are connected with probability W_{X_i,X_j}, **independently** of other pairs of vertices. And the **community** sets can be defined by

$$\Omega_i = \Omega_i(\mathbf{X}) := \{v \in [n] : X_v = i\}, i \in [k]$$

Immediately, we can define the symmetry of SBM as:

Definition 19.1.2: Symmetric SBM

An SBM is called symmetric if

- p is **uniform**
- $\mathbf{W}$ takes the same value **on the diagonal** and the same value **off the diagonal**

$(\mathbf{X}, G)$ is drawn under $SSBM(n, k, A, B)$ if $p = \{1/k\}^k$ and $\mathbf{W}$ takes avlue A on the diagonal and B off the diagonal.

19.1.1 Recovery

The goal of community detection is to recover the labels $\mathbf{X}$ by observing G, up to some level of accuracy. First, define **agreement** as

Definition 19.1.3: Agreement of Communities

The agreement between two community vectors $\mathbf{x}, \mathbf{y} \in [k]^n$ is obtained by maximizing the common components between $\mathbf{x}$ and any relabelling of $\mathbf{y}$, that is

$$A(\mathbf{x}, \mathbf{y}) = \max_{\pi \in S_k} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\left[x_i = \pi(y_i)\right]$$

where S_k is the group of permutations on $[k]$.

The **relabelling** permutation is used to handle symmetric communities such as in SSBM, as it is impossible to recover the actual labels in this case. But it's possible to recover the **partition**. There are 2 types of partition recovery we consider

Exact Recovery First, consider the case of **exact recovery**:

Definition 19.1.4: Exact Recovery

Let $(\mathbf{X}, G) \sim SBM(n, p, W)$, the exact recovery is solved if there exists an algorithm that takes G as an input and outpus $\hat{\mathbf{X}} = \hat{\mathbf{X}}(G)$ such that $\mathbb{P}\left\{A(\mathbf{X}, \hat{\mathbf{X}}) = 1\right\} = 1 - o_p(1)$

In the SSBM case, algorithms that guarantee

$$A(\mathbf{X}, \hat{\mathbf{X}}) \to \frac{1}{k}$$

would be trivial.

Weak Recovery On the other hand, we the case of **weak recovery** defined as

Definition 19.1.5: Weak Recovery

Weak recovery or detection is solved $SSBM(n, k, A, B)$ if for $(\mathbf{X}, G) \sim SSBM(n, k, A, B)$, then $\exists \epsilon > 0$ and an algorithm that takes G as an input and outputs $\hat{\mathbf{X}}$ such that

$$\mathbb{P}\left\{A(\mathbf{X}, \hat{\mathbf{X}}) \geq \frac{1}{k} + \epsilon\right\} = 1 - o(1)$$

19.1.2 Example: SSBM(n,2)

Let's look at the example of $SSBM(n, 2, \alpha\frac{\log n}{n}, \beta\frac{\log n}{n})$, where

- n: number of vertices (assumed to be even for simplicity)
- for each $v \in [n]$, a binary label X_v is attached s.t.

$$|\{v \in [n] : X_v = 1\}| = n/2$$

- for each pair of distinct nodes $u, v \in [n]$, an edge is placed with probability

  - $\alpha \frac{\log n}{n}$ if $X_u = X_v$
  - $\beta \frac{\log n}{n}$ if $X_u \neq X_v$

  where edges are placed independently conditionally on the vertex labels
- WLOG, $\alpha > \beta$

then we have the following theorem

Theorem 19.1.6: Exact Recovery in $SSBM(n, 2, \alpha\log(n)/n, \beta\log(n)/n)$

- Exact recovery in $SSBM(n, 2, \alpha\log(n)/n, \beta\log(n)/n)$ is solvable and efficiently so if $\left|\sqrt{\alpha} - \sqrt{\beta}\right| > \sqrt{2}$ nad unsolvable if $\left|\sqrt{\alpha} - \sqrt{\beta}\right| < \sqrt{2}$
- Exact recovery of the ground truth assignment of the partition (A, B) is also achieveable, that is: if

$$\frac{\alpha+\beta}{2} - \sqrt{\alpha\beta} > 1$$

i.e.

$$\alpha+\beta > 2, \ (\alpha-\beta)^2 > 4(\alpha+\beta) - 4$$

the maximum likelihood estimator exactly recovers the communities (up to a global flip), with high probability.

See Abbe (2017) for the proof of this theorem.

In summary, for a graph structure $G = (V, E)$ represented by adjacency matrix $\mathbf{X}_{n\times n}$, Stochastic Block Model (SBM)

- assumes that there is a symmetric matrix $\mathbf{P} = \left\{p_{ij}\right\} \in \mathbb{R}^{k\times k}$, for $k \ll n$ and a map $C : \{1, \cdots, n\} \to \{1, \cdots, k\}$, s.t. $\Pr\left(\mathbf{X}_{ij} = 1\right) = \mathbf{P}_{C(i),C(i)}$
- Define $\mathbf{\Pi} = (\pi_1, \cdots, \pi_n)' \in \mathbb{R}^{n\times k}$ where $\mathbf{\Pi}_{ij} = 1$ if $C(i) = j$, and $\mathbf{\Pi}_{ij} = 0$ otherwise
- Let $\mathbf{H} = \mathbb{E}(\mathbf{X})$ be the probability matrix, then $\mathbf{H} = \mathbf{\Pi P \Pi'}$
- A variant of SBM is degree corrected SBM which incorporates the degree heterogeneity.
  - each node is assigned a parameter $\theta_i > 0$ such that $\Pr\left(\mathbf{X}_{ij} = 1\right) = \theta_i\theta_j\mathbf{P}_{C(i),C(j)}$
  - $\mathbf{H} = \mathbf{\Theta \Pi P \Pi' \Theta}$, where $\mathbf{\Theta} = \operatorname{diag}(\theta_1, \cdots, \theta_n)$

19.2 SIMPLE Model (Fan et al., 2022)

In SBM, each $\pi_i \in \{e_1, \cdots, e_K\}$ with e_k a one entry vector whose k-th component is one. But what if each node i can belong to K different communities? We generalize π_i to be a compositional vector, and interpret it as community membership profile for node i, then

$$\Pr\left(\mathbf{X}_{ij} = 1\right) = \theta_i\theta_j \sum_{k=1}^{K}\sum_{l=1}^{K} \pi_i(k)\pi_j(l)p_{kl}$$

and $\mathbf{H} = \mathbf{\Theta \Pi P \Pi' \Theta}$. Now, consider a new statistical tests for testing whether any given pair of nodes share the same membership profiles, and providing the associated p-values.

19.2.1 Problem Setting

For an undirected graph $G = (V, E)$ with n nodes, let $\mathbf{X} = \{x_{ij}\} \in \mathbb{R}^{n\times n}$ be the **symmetric** adjacency matrix. Under a probabilistic model, assume x_{ij} is an independent realization from a Bernoulli random variable for all upper triangular entries of random matrix $\mathbf{X}$. Consider the adjacency matrix with the deterministic-random latent structure

$$\mathbf{X} = \mathbf{H} + \mathbf{W}$$

where

- $\mathbf{H} = \{h_{ij}\} \in \mathbb{R}^{n\times n}$ is the deterministic mean matrix of low rank $K \geq 1$
- $\mathbf{W} = \{w_{ij}\} \in \mathbb{R}^{n\times n}$ is a symmetric random matrix with zero mean and independent entries on and above the diagonal

Assume V is decomposed into K disjoint latent communities

$$C_1, \cdots, C_K$$

where each node i is associated with the community membership probability vector

$$\boldsymbol{\pi}_i = (\pi_i(1), \cdots, \pi_i(K))' \in \mathbb{R}^K$$

s.t.

$$\Pr(i \in C_k) = \pi_i(k),\ k = 1, \cdots, K$$

here, K is unknown but bounded away from ∞.

19.2.2 Hypothesis Testing

For any given pair of nodes $i \neq j \in V$, the goal is to infer whether they share the same community identity with quantified uncertainty level based on adjacency matrix $\mathbf{X}$, the hypothesis is

$$H_0 : \pi_i = \pi_j \qquad\qquad H_1 : \pi_i \neq \pi_j$$

More explicitly, consider the DCMM (Degree Corrected Mixed Membership) model as the underlying network model, s.t. the probability of a link between nodes i and j can be written as

$$\Pr(\mathbf{X}_{ij} = 1) = \theta_i\theta_j \sum_{k=1}^{K}\sum_{l=1}^{K} \pi_i(k)\pi_j(l)p_{kl}$$

and

$$\mathbf{H} = \mathbf{\Theta\Pi P\Pi'\Theta}$$

in matrix form, where $\mathbf{\Pi} = (\pi_1, \cdots, \pi_n)' \in \mathbb{R}^{n\times k}$ and $\mathbf{\Theta} = \operatorname{diag}(\theta_1, \cdots, \theta_n)$. Consider

- No degree homogeneity: $\mathbf{\Theta} = \sqrt{\theta}\mathbf{I}_n$, then $\mathbf{H} = \theta\mathbf{\Pi P\Pi'}$. If we eigen-decompose $\mathbf{H} = \mathbf{VDV'}$ where $\mathbf{D} = \operatorname{diag}(d_1, \cdots, d_K)$ with $|d_1| \geq |d_2| \geq \cdots |d_K| > 0$ is the matrix of all K non-zero eigenvalues and $V = (v_1, \cdots, v_K) \in \mathbb{R}^{n\times K}$ is the eigenvectors.
  - the column space spanned by $\mathbf{\Pi}$ is the same as the eigenspace spanned by the top K eigenvectors of matrix $\mathbf{H}$
  - mean matrix $\mathbf{H}$ is **not** observable: replace it with adjacency matrix $\mathbf{X}$ and conduct eigen-decomposition to get eigenvalues $\hat{d}_1, \cdots, \hat{d}_n$ and eigenvectors $\hat{v}_1, \cdots, \hat{v}_n$. We assume that

    $$|\hat{d}_1| \geq |\hat{d}_2| \geq \cdots \geq |\hat{d}_n|$$

    and let $\hat{\mathbf{V}} = (\hat{v}_1, \cdots, \hat{v}_K) \in \mathbf{R}^{n\times K}$.

Without degree heterogeneity first, consider the case where $\mathbf{\Theta} = \sqrt{\theta}\mathbf{I}_n$ and $\mathbb{E}(\mathbf{X}) = \mathbf{H} = \theta\mathbf{\Pi}\mathbf{P}\mathbf{\Pi}'$. If $\pi_i = \pi_j$, then nodes i and j are exchangeable and $\mathbf{V}(i) = \mathbf{V}(j)$. The test statistic for membership information of node i and j is given as

$$T_{ij} = \left[\hat{\mathbf{V}}(i) - \hat{\mathbf{V}}(j)\right]' \mathbf{\Sigma}_1^{-1} \left[\hat{\mathbf{V}}(i) - \hat{\mathbf{V}}(j)\right]$$

where $\mathbf{\Sigma}_1^{-1} = \mathrm{Cov}\left[(e_i - e_j)'\mathbf{WVD}^{-1}\right]$ is the asymptotic variance of $\left[\hat{\mathbf{V}}(i) - \hat{\mathbf{V}}(j)\right]$. The regularity conditions are

C1 $\exists c_0 > 0$ s.t.

$$\min\left\{\frac{|d_i|}{|d_j|} : 1 \le i \le j \le K, d_i \ne -d_j\right\} \ge 1 + c_0$$

C2 $\exists c_0 \in (0,1), c_2 \in [0, 1/2), c_1 \in (0, 1-2c_2)$ s.t. $\lambda_k(\mathbf{\Pi}'\mathbf{\Pi}) \ge c_0 n$, $\lambda_K(\mathbf{P}) \ge n^{-c_2}$ and $\theta \ge n^{-c_1}$

C3 as $n \to \infty$, all the eigenvalues of $\theta^{-1}\mathbf{D}\mathbf{\Sigma}_1\mathbf{D}$ are bounded away from 0 and ∞

and the test statistics follow the theorem

Theorem 19.2.1: Test Statistics Distribution

Under Condition **C1** and **C2**, and $\mathbf{\Theta} = \sqrt{\theta}\mathbf{I}_n$,

- If **C3** holds too, then under the null

$$H_0 : T_{ij} \xrightarrow{\mathcal{D}} \chi^2_K$$

  as $n \to \infty$, where χ^2_K is the chi-square distribution with K degrees of freedom
- under the alternative,
  - if $n^{1/2-c_2}\sqrt{\theta}\left\|\pi_i - \pi_j\right\| \to \infty$, then for arbitrarily large constant $C > 0$, we have

$$\Pr\left(T_{ij} > C\right) \xrightarrow{n\to\infty} 1$$

  - in addition, if Condition **C3** holds, $c_2 = 0$, $\left\|\pi_i - \pi_j\right\| \sim \frac{1}{\sqrt{n\theta}}$, and

$$\left[\mathbf{V}(i) - \mathbf{V}(j)\right]' \mathbf{\Sigma}_1^{-1} \left[\mathbf{V}(i) - \mathbf{V}(j)\right] \to \mu$$

    , then

$$T_{ij} \xrightarrow{\mathcal{D}} \chi^2_K(\mu)$$

    as $n \to \infty$, where $\chi^2_K(\mu)$ is a noncentral chi-square distribution with mean μ and K degrees of freedom.

Under the joint null $H_{0,ij} : \pi_i = \pi j, \forall 1 \le i \ne j \le n$, a uniform version of Thm.19.2.1 is

$$\lim_{n\to\infty} \sup_{1\le i\ne j\le n} \left|\Pr\left(T_{ij} \le x\right) - \Pr\left(X \le x\right)\right| = 0, \forall x \in \mathbf{R}$$

where $X \sim \chi^2_K$. But the test statistic T_{ij} is not directly applicable since the population parameters K and $\mathbf{\Sigma}_1$. For consistent estimators satisfying the following condition

$$\Pr(\hat{K} = K) = 1 - o(1)$$
$$\theta^{-1}\left\|\mathbf{D}\left(\hat{\mathbf{S}}_1 - \mathbf{\Sigma}_1\right)\mathbf{D}\right\|_2 = o(1)$$

then the asymptotic results in Thm. 19.2.1 holds.

With degree heterogeneity Define componentwise ratio

$$Y(i,k) = \frac{\hat{v}_k(i)}{\hat{v}_1(i)}, \qquad 1 \le i < n, 2 \le k \le K$$

where $\hat{v}_k(i)$ is the i-th component of k-th eigenvector of $\mathbf{X}$. Due to the **exchangeability** of nodes i and j, under the null it holds that

$$\frac{v_k(j)}{v_1(j)} = \frac{v_k(j)}{v_1(j)}, \qquad 2 \le k \le K$$

Denote $\mathbf{Y}_i = (Y(i,2), \cdots, Y(i,K))'$, the new test statistics is proposed as

$$G_{ij} = \left(\mathbf{Y}_i - \mathbf{Y}_j\right)' \mathbf{\Sigma}_2^{-1} \left(\mathbf{Y}_i - \mathbf{Y}_j\right)$$

where $\mathbf{\Sigma}_2$ is the asymptotic variance of $\mathbf{Y}_i - \mathbf{Y}_j$, which is much harder to derive and estimate. So we need to impose four other conditions in addition to Condition C1-C3:

C4 $\exists c_2 \in [0, 1/2), c_3 \in (0, 1-2c_2), c_4 > 0, c_5 \in (0,1)$ s.t.

$$\lambda_K(\mathbf{P}) \ge n^{-c_2} \qquad \min_{1\le k\le K} |\mathcal{N}_k| \ge c_5 n \qquad \theta_{\max} \le c_4 \theta_{\min} \qquad \theta_{\min}^2 \ge n^{-c_3}$$

C5 $\mathbf{P} = (p_{kl})$ is positive definite, irreducible and has unit diagonal entries, moreover

$$n \min_{1\le k\le K, t=i,j} \mathrm{Var}\left(\mathbf{e}_t' \mathbf{W} \mathbf{v_k}\right) \sim n\theta_{\max}^2 \to \infty$$

C6 all the eigenvalues of

$$\left(n\theta_{\max}^2\right)^{-1} \mathbf{D}\mathrm{Cov}(f)\mathbf{D}$$

are bounded away from 0 and ∞

C7 Let η_1 be the first right singular vector of $\mathbf{P\Pi'\Theta^2\Pi}$, it holds that

$$\min_{1\le k\le K} \eta_1(k) > 0 \qquad \frac{\max_{1\le k\le K} \eta_1(k)}{\min_{1\le k\le K} \eta_1(k)} \le C$$

for some positive C, where $\eta_1(k)$ is the k-th entry of η_1.

Then we have

Theorem 19.2.2: Test Statistic Distribution with Degree Heterogeneity

Under Condition **C1, C4-C7**, with degree heterogeneity,

- under the null,

$$G_{ij} \xrightarrow{\mathcal{D}} \chi^2_{K-1}$$

- under the alternative with $\lambda_2\left(\boldsymbol{\pi}_i\boldsymbol{\pi}_i' + \boldsymbol{\pi}_i\boldsymbol{\pi}_j'\right) \gg \frac{1}{n^{1-2c}\theta_{\min}^2}$, for any arbitrarily large constant $C > 0$,

$$\Pr(G_{ij} > C) \xrightarrow{n\to\infty} 1$$

notice that K and $\mathbf{\Sigma}_2$ are both unknown, we must have

- for estimator $\hat{\mathbf{S}}_2$ of $\mathbf{\Sigma}_2$, we need

$$\left(n\theta_{\max}^2\right)^{-1} \left\|\mathbf{D}\left(\hat{\mathbf{S}}_2 - \mathbf{\Sigma}^2\right)\mathbf{D}\right\|_2 = o_p(1)$$

  replace $\mathbf{\Sigma}_2$ with $\hat{\mathbf{S}}_2$

- for K, under Condition **C1**, and $|d_K| \gg \sqrt{\log(n)}\alpha_n$ and $\alpha_n \geq n^{c_5}$ for some positive constant c_5, a consistent thresholding estimator is defined

$$\hat{K} = \left|\left\{\hat{d}_i : \hat{d}_i^2 > 2.01(\log n)\check{d}_n, i \in [n]\right\}\right|$$

  where the constant 2.01 can be replace with any other constant slightly larger than 2, and

$$\check{d}_n = \max_{1 \leq l \leq n} \sum_{j=1}^{n} \mathbf{X}_{lj}$$

  is the maximum degree of the network. For $\hat{K}$ to be consistent, we need
  - Condition **C1** holds
  - $|d_K| \gg \sqrt{\log(n)}\alpha_n$, where $\alpha_n \geq n^{c_5}$ for some constant $c_5 > 0$

19.3 Rank Inference via Residual Subsampling

Again, consider $n \times n$ symmetric random matrix $\tilde{\mathbf{X}}$ and its decomposition

$$\tilde{\mathbf{X}} = \mathbf{H} + \mathbf{W}$$

where

- $\mathbf{H} = \mathbb{E}\left(\tilde{\mathbf{X}}\right)$ with some fixed but unknown rank $K \ll n$, it can be eigen-decomposed as

$$\mathbf{H} = \mathbf{V}\mathbf{D}\mathbf{V}'$$

  where $\mathbf{D} = \operatorname{diag}(d_1, \cdots, d_K)$ are the non-zero eigenvalues of $\mathbf{H}$ in decreasing magnitude and $\mathbf{V} = (\mathbf{v}_1, \cdots, \mathbf{v}_K)$ are the corresponding eigenvectors
- $\mathbf{W}$ has bounded and independent entries on and above the diagonals

for a simple case (networks with self loops), when the observed data matrix $\mathbf{X} = \tilde{\mathbf{X}} = \mathbf{H} + \mathbf{W}$, then we have

$$\begin{array}{l} \dfrac{\sum_{i=1}^{n} w_{ii}}{\sqrt{\sum_{i=1}^{n} \mathbb{E}w_{ii}^2}} \xrightarrow{d} \mathcal{N}(0,1) \\ \dfrac{\sum_{i=1}^{n} \mathbb{E}w_{ii}^2}{\sum_{i=1}^{n} w_{ii}^2} \xrightarrow{p} 1 \end{array} \quad \Rightarrow \quad \frac{\sum_{i=1}^{n} w_{ii}}{\sqrt{\sum_{i=1}^{n} w_{ii}^2}} \xrightarrow{d} \mathcal{N}(0,1)$$

Let $\hat{\mathbf{V}}\hat{\mathbf{D}}\hat{\mathbf{V}}' = \sum_{k=1}^{K_0}$

References

Emmanuel Abbe. Community detection and stochastic block models: recent developments. *The Journal of Machine Learning Research*, 18(1):6446–6531, 2017.