Econometrics **February 21, 2023**

Topic 11: Lasso And Beyond: Convex Learning

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Key points:

Disclaimer:

11.1 Lasso

Lasso (Least absolute Shrinkage and Selection Operator), proposed by Tibshirani (1996), aims to minimize the **SSR *(sum of residual squares)*** subject to the $L1-$**norm *(sum of the absolute value)*** of the coefficients being less than a constant.

11.1.1 Set up

For data $(\mathbf{x}_i, y_i)_{i=1}^n$, where

- y_i is the outcome for individual i
- $\mathbf{x}_i = (x_{i1}, \cdots, x_{ip})'$ is the $p \times 1$ vector of predictors

Then the Lasso estimator $\left(\hat{\alpha}, \hat{\boldsymbol{\beta}}\right)$ is defined as

$$\left(\hat{\alpha}, \hat{\boldsymbol{\beta}}\right) = \arg\min_{\alpha,\beta} \left\{ \sum_{i=1}^{n} \left(y_i - \alpha - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 \right\} \qquad \text{s.t. } \sum_{j=1}^{p} |\beta_j| \leq t$$

for the $n \times 1$ response vector $\mathbf{y} = (y_1, \cdots, y_n)'$, the $n \times p$ design matrix $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)'$ where $\mathbf{x}_i = (x_{i1}, \cdots, x_{ip})'$ is a $p \times 1$ vector. Here $\hat{\alpha} = \overline{y}$, w.l.o.g., let $\overline{y} = 0$ and omit α for simplicity.

In matrix form, we have

- constrained form:

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \left\{ \frac{1}{2n} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 \right\} \qquad \text{s.t. } \|\boldsymbol{\beta}\|_1 \leq t$$

- unconstrained form:

$$\hat{\boldsymbol{\beta}}(\lambda) = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \left\{ \frac{1}{2n} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_1 \right\}$$

where the regularization parameter $\lambda \geq 0$:

- $\lambda \to \infty$: $\hat{\boldsymbol{\beta}}_{lasso} = \mathbf{0}$
- $\lambda = 0$: $\hat{\boldsymbol{\beta}}_{lasso} \to \hat{\boldsymbol{\beta}}_{OLS}$

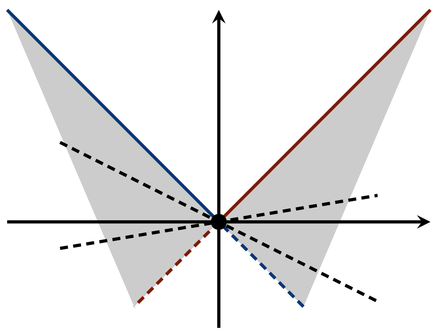
11.1.2 Solving Lasso

Lasso is essentially a quadratic optimization problem. Hence, the solution is given by taking the derivative (of the unconstrainted question) and set it equal to 0

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\beta}}\left(\frac{1}{2n}\|\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\|_2^2+\lambda\|\boldsymbol{\beta}\|_1\right)=0$$

$$\Rightarrow \qquad \frac{1}{n}\underbrace{\mathbf{X}'}_{p\times n}\underbrace{\left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right)}_{=\boldsymbol{\epsilon},n\times 1}=\lambda\begin{cases}\operatorname{sign}\left(\beta_j\right), & \beta_j\neq 0\\ [-1,1], & \beta_j=0\end{cases}$$

this result follows the fact the L-1 norm $\|\boldsymbol{\beta}\|$ is piecewise linear:



L1-norm (1-dimension)

For each component of the vector of the L-1 norm $f(\beta_j)=|\beta_j|$, we have:
- $\beta_j>0$: $f'(\beta_j)=1$
- $\beta_j<0$: $f'(\beta_j)=-1$
- $\beta_j=0$: $\mathrm{d}f\in[-1,1]$ (shaded area)

which gives the results stated above.

Take another look at this result

Proposition 11.1.1: Lasso Parameter Selection Rule

$$\frac{1}{n}\mathbf{X}'\left(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\right)=\frac{1}{n}\mathbf{X}'\boldsymbol{\epsilon}=\lambda\begin{cases}\operatorname{sign}\left(\beta_j\right), & \beta_j\neq 0\\ [-1,1], & \beta_j=0\end{cases}$$

which gives a parameter selection criterion: for $\beta_j\neq 0$, $\operatorname{sign}(\beta_j)$ **must agree** with $\operatorname{Corr}(X_j,\boldsymbol{\epsilon})$, the correlation between the j-th variable $\mathbf{X}_j$ and (full-model) residuals $\boldsymbol{\epsilon}=\mathbf{y}-\mathbf{X}\boldsymbol{\beta}$.

11.1.3 Algorithm: from LARS to Lasso

Mathematically, Lasso is quite intuitive, but computionally, it can be quite consuming. **?** propose an alogrithm that takes steps from a all-0 model to the biggest model, OLS

11.2 Penalized Least Square Estimation

Lasso is one special class of Penalized Least Square (PLS) Estimation. For the linear regression model $\mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\epsilon}$, if $\boldsymbol{\epsilon}\sim\mathcal{N}(\mathbf{0},\sigma^2\mathbf{I}_n)$, we have PLS as

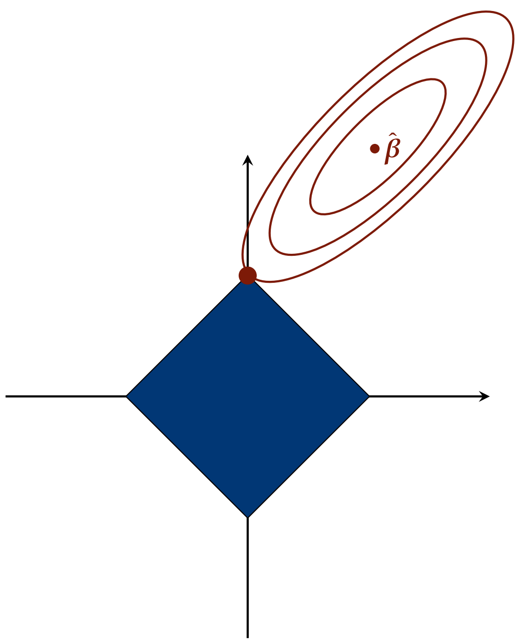
$$\min_{\boldsymbol{\beta}\in\mathbb{R}^p}\left\{\frac{1}{2n}\|\mathbf{y}-\mathbf{X}\boldsymbol{\beta}\|_2^2+\sum_{j=1}^{p}p_\lambda\left(|\beta_j|\right)\right\}$$

where $p_\lambda(\cdot)$ is a penalty function indexed by the regularization parameter $\lambda \geq 0$. Antoniadis and Fan (2001) showed that the PLS estimator $\hat{\boldsymbol{\beta}}$ has the following properties:

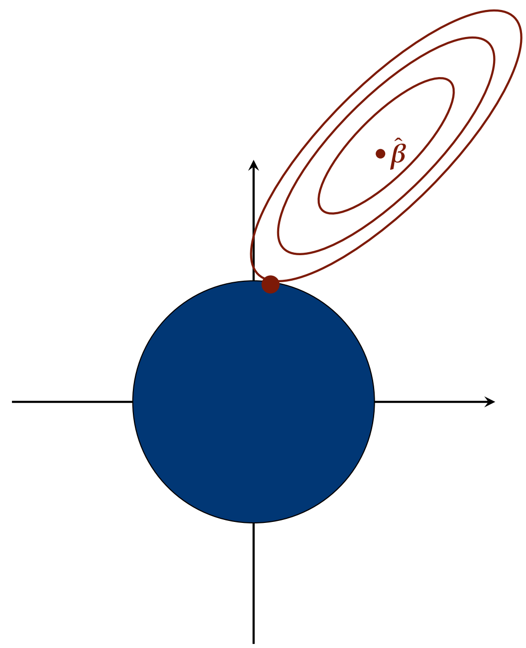
- **sparsity**: if $\min_{t\geq 0}\left\{t + p'_\lambda(t)\right\} > 0$
- **approximate unbiasedness**: if $p'_\lambda(t) = 0$ for t large enough
- **continuity**: iff $\arg\min_{t\geq 0}\left\{t + p'_\lambda(t)\right\} = 0$

In general

- the **sigularity** of penalty function at the origin, $p'_\lambda(0_+) > 0$ is needed for generating **sparsity** in variable selection
- the **concavity** is needed to reduce the bias



Lasso



Ridge

References

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