

## Topic 3: *Moving the Goalposts* Approach

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Key points:

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**Disclaimer:** These notes are written by Sai Zhang ([email me](#) or check my [Github page](#)). The main reference for this topic is [Armstrong, Kolesár, and Kwon \(2020\)](#), I thank Prof. Armstrong for his valuable advice.

### 3.1 Finite Sample Bias-Variance Tradeoffs

#### 3.1.1 Setup

Consider the fixed design regression model

$$y_i = w_i \beta(z_i) + h(z_i) + \epsilon_i \quad (3.1)$$

where

- $w_i, z_i$  are treated as **fixed**
- $\epsilon_i$  is **independent**, with  $\mathbb{E}[\epsilon_i] = 0, \mathbb{E}[\epsilon_i^2] = \sigma_i^2$
- observation:  $\left\{ \left( y_i, w_i, z_i' \right)' \right\}_{i=1}^n$

one example is the case where  $w_i$  is **binary**, then

$$\beta(z) = f(1, z) - f(0, z)$$

which is just the ATE conditional on  $z$  under the unconfoundedness assumption. This includes the RD design, where  $z_i$  is the running variable and  $w_i$  is the treatment assignment.

Now, consider for the weighted average treatment effect

$$L_\mu [\beta(\cdot)] = \int \beta(z) d\mu(z)$$

where  $\int \mu(z) = 1$  is a **signed** measure (weight, allowing **negative** weights), construct a linear estimator

$$\hat{L}_a = \sum_{i=1}^n a_i y_i$$

where the estimation weights  $a_i$  can depend on  $\{z_i, w_i, \sigma_i^2\}_{i=1}^n$ , but **not** on  $y_i$ . Together, the bias of  $\hat{L}_a$  for  $L_\mu [\beta(\cdot)]$ , given the regression function  $\beta(\cdot), h(\cdot)$ , is

$$\mathbb{E}_{\beta(\cdot), h(\cdot)} [\hat{L}_a] - L_\mu [\beta(\cdot)] = \sum_{i=1}^n a_i [w_i \beta(z_i) + h(z_i)] - \int \beta(z) d\mu(z)$$

and its variance, given the regression function  $\beta(\cdot)$ ,  $h(\cdot)$ , is just

$$\text{Var}_{\beta(\cdot), h(\cdot)} [\hat{L}_a] = \sum_{i=1}^n a_i^2 \sigma_i^2$$

To bound the bias, assume  $h(\cdot)$  is known to belong in a class of functions  $\mathcal{H}$ , then two approaches can be adopted, for the regularity of  $\beta(\cdot)$  and the choice of  $\mu(\cdot)$ :

- 1 arbitrary  $\beta(\cdot)$ , optimizing weights  $\mu$  by *moving the goalposts*, s.t.  $L_\mu [\beta(\cdot)]$  is easy to estimate (Crump et al., 2006; Imbens and Wager, 2019) which gives the worst-case bias

$$\inf_{\mu} \sup_{\beta(\cdot), h(\cdot)} \left| \sum_{i=1}^n a_i [w_i \beta(z_i) + h(z_i)] - \int \beta(z) d\mu(z) \right| \quad \text{s.t. } h(\cdot) \in \mathcal{H}, \int d\mu(z) = 1 \quad (3.2)$$

- 2 assume constant treatment effects, i.e.,  $\beta(z) = \beta, \forall z$ , which means that  $L_\mu [\beta(\cdot)] = \beta$  regardless of  $\mu$  (Armstrong et al., 2020), and the worst-case bias is

$$\sup_{\beta, h(\cdot)} \left| \sum_{i=1}^n a_i [w_i \beta + h(z_i)] - \beta \right| \quad \text{s.t. } h(\cdot) \in \mathcal{H} \quad (3.3)$$

And, the two approaches can be linked as such:

- If  $\sum_{i=1}^n a_i w_i = 1$ , 3.2 and 3.3 are both equal to

$$\sup_{h(\cdot)} \left| \sum_{i=1}^n a_i h(z_i) \right| \quad \text{s.t. } h(\cdot) \in \mathcal{H} \quad (3.4)$$

- 3.2 automatically equals 3.4
- 3.3 is optimized (w.r.t.  $\mu$ ) by setting  $\mu$  to place weight  $a_i w_i$  on observation  $i$ , i.e.,  $\mu(\mathcal{Z}) = \sum_{i: z_i \in \mathcal{Z}} a_i w_i$ , which implies  $\sum_{i=1}^n a_i w_i \beta(z_i) - \int \beta(z) d\mu(z) = 0$ , hence the equality.
- Otherwise, 3.2 and 3.3 are both infinite:
  - 3.3 can be made arbitrarily large by choosing large enough  $\beta$
  - 3.2 can be made arbitrarily large by making  $\beta(\cdot)$  constant (as in 3.3) and large enough

### 3.1.2 Moving-the-goalpost Approach

### 3.1.3 Constant-treatment-effect Approach

Armstrong et al. (2020) adopt this approach, focusing on the case where  $h(\cdot)$  is a high dimensional linear function, and the penalty function is an  $l_p$  norm of the coefficients.

**Basic setting: Homoskedastic Gaussian errors**

First, consider

$$Y = w\beta + Z\gamma + \epsilon \quad (3.5)$$

where

- $\beta \in \mathbb{R}$  is the constant treatment effect to be estimated
- $\gamma \in \Gamma$  is the control coefficients, subject to the restriction (i.e., the function class  $\mathcal{H}$ )

$$\Gamma = \Gamma(C) = \{\gamma \in \mathcal{G} : \text{Pen}(\gamma) \leq C\} \quad (3.6)$$

where  $\text{Pen}(\cdot)$  is a seminorm<sup>1</sup> on some linear subspace  $\mathcal{G}$  of  $\mathbb{R}^k$ .

- $w = (w_1, \dots, w_n)' \in \mathbb{R}^n$  and  $Z = (z'_1, \dots, z'_n)' \in \mathbb{R}^{n \times k}$  are defined as before
- $\epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$  is assumed **normal and homoskedastic**, with  $\sigma^2$  known

For estimation, the goal is to construct estimators and CIs for  $\beta$ :

- estimator  $\hat{\beta}$ : consider the worst-case performance over the parameter space  $\mathbb{R} \times \Gamma$  under the **MSE** criterion

$$R_{MSE}(\hat{\beta}; \Gamma) = \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} \mathbb{E}_{\beta, \gamma} [(\hat{\beta} - \beta)^2]$$

- for CIs, we have 2 requirements:

A **coverage**: A  $100 \cdot (1 - \alpha)\%$  CI with half-length  $\hat{\chi} = \hat{\chi}(Y, X)$  is an interval  $\{\hat{\beta} \pm \hat{\chi}\}$  s.t.

$$\inf_{\beta \in \mathbb{R}, \gamma \in \Gamma} P_{\beta, \gamma}(\beta \in \{\hat{\beta} \pm \hat{\chi}\}) \geq 1 - \alpha$$

B **length**: the expected length of a CI  $\mathbb{E}_{\beta, \gamma} [2\hat{\chi}]$  should be as short as possible, which means that length-optimized CIs are not necessarily centered at an MSE-centered  $\hat{\beta}$

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<sup>1</sup>Seminorm satisfies **triangle inequality**  $\text{Pen}(\gamma + \tilde{\gamma}) \leq \text{Pen}(\gamma) + \text{Pen}(\tilde{\gamma})$  and **homogeneity**  $\text{Pen}(c\gamma) = |c| \text{Pen}(\gamma), \forall c$ , but **NOT** necessarily positive definite ( $\text{Pen}(\gamma) = 0$  does not imply  $\gamma = 0$ ). Essentially, any convex set  $\Gamma$  that is symmetric satisfies this definition.

## References

- Timothy B Armstrong, Michal Kolesár, and Soonwoo Kwon. Bias-aware inference in regularized regression models. *arXiv preprint arXiv:2012.14823*, 2020.
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- Guido Imbens and Stefan Wager. Optimized regression discontinuity designs. *Review of Economics and Statistics*, 101(2):264–278, 2019.