Econometrics **July 4, 2023**

Topic 18: Eigenvalue and Spike Models

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Key points: .

Disclaimer: *The note is built on Prof. Jinchi Lv's lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.*

18.1 Motivation

Consider n independent observations $\mathbf{X}_i \in \mathbb{R}^p$ drawn from a $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, then the covariance can be decomposed into 2 parts, white noise and low rank

$$\mathbf{\Sigma} = \mathrm{Cov}(\mathbf{X}_i) = \mathbf{I} + \sum_{k=1}^{M} \theta_k \nu_k \nu_k' = \mathbf{\Sigma}_0 + \mathbf{\Phi}$$

where M denotes the **number of spikes** in the distribution of eigenvalues. The idea is: spikes deviate from a reference model along a **small fixed number** of unknown directions. If $\mathbf{\Phi} = \mathbf{0}$, then none of the sample eigenvalues is separated from the bulk.

Why a spike model is interesting? A spike model can help determine the latent dimension of the data, some examples being

- Principal component analysis (PCA): spikes are related to the directions of the most variations of the data, i.e., the principal components
- Clustering model: M spikes is equivalent to $M + 1$ clusters
- Economic significance: M is related to the number of factor loadings

Then the question is threefold:

- How to determine M
- How to estimate ν_k
- How to test θ_k

Under rank one alternative, we would like to test the hypothesis

$$H_1 : \mathbf{\Sigma} = \mathbf{I}_p + \theta \boldsymbol{\nu}\boldsymbol{\nu}', \theta > 0$$

against the null

$$H_0 : \mathbf{\Sigma} = \mathbf{I}_p$$

with the key assumptions:

A1 Gaussian error

A2 large p: $p \leq n$ but allows $p/n \to \gamma \in (0, 1)$

Under these assumptions, for the $n \times p$ data matrix $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1' & \cdots & \mathbf{X}_n' \end{pmatrix}'$, $\mathbf{X}'\mathbf{X}$ has a $p-$dimensional **Wishart** distribution $W_p(n, \mathbf{\Sigma})$ with the degree of freedom n and covariance matrix $\mathbf{\Sigma}$, which is a *random matrix*.

If $\mathbf{Y} = \mathbf{M} + \mathbf{X}$, that is, the sum of the *random matrix* $\mathbf{X}$ and a *deterministic matrix* $\mathbf{M}$ (also $n \times p$), then $\mathbf{Y}'\mathbf{Y}$ has a $p-$dimensional Wishart distribution $W_p(n, \mathbf{\Sigma}, \mathbf{\Psi})$ with n degrees of freedom, covariance matrix $\mathbf{\Sigma}$ and non-centrality matrix $\mathbf{\Psi} = \mathbf{\Sigma}^{-1}\mathbf{M}'\mathbf{M}$.

Definition 18.1.1: Density of Wishart Distribution

The PDF of Wishart distribution is defined as

$$f(\mathbf{X}) = \frac{1}{2^{np/2}\Gamma_p\left(\frac{n}{2}\right)|\mathbf{\Sigma}|^{n/2}} |\mathbf{X}|^{(n-p-1)/2} \exp\left(-\frac{1}{2}\text{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{X}\right)\right)$$

where $\mathbf{X}$ is a symmetric positive semidefinite and $\Gamma_p\left(\frac{n}{2}\right)$ is a multivariate gamma function such that

$$\Gamma_p\left(\frac{n}{2}\right) = \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma\left(\frac{n}{2} - \frac{j-1}{2}\right)$$

Notice that the sample covariance matrix $\mathbf{S} = \frac{1}{n}\mathbf{X}'\mathbf{X}$ is just a scaled version of Wishart distribution

$$n\mathbf{S} = \mathbf{X}'\mathbf{X} \sim W_p(n, \mathbf{\Sigma})$$

For $\mathbf{\Sigma} = \mathbf{I}_p$, the empirical distribution fo eigenvalues converges to Marcenko-Pastur distribution

$$f^{\text{MP}}(x) = \frac{1}{2\pi\gamma x}\sqrt{(b_+ - x)(x - b_-)}$$

where $b_\pm = (1 \pm \sqrt{\gamma})^2$. Then:

- under $H_0 : \mathbf{\Sigma} = \mathbf{I}_p$, we have

$$n^{2/3}\left(\frac{\lambda_1 - \mu(\gamma)}{\sigma(\gamma)}\right) \xrightarrow{d} \text{TW}_1$$

  where TW_1 is the Tracy-Widom distribution
- under $H_1 : \mathbf{\Sigma} = \mathbf{I}_p + \theta\boldsymbol{\nu}\boldsymbol{\nu}', \theta > 0$, if θ is strong ($\theta \gg \sqrt{\gamma}$), then

$$n^{1/2}\left(\frac{\lambda_1 - \rho(\theta, \gamma)}{\tau(\theta, \gamma)}\right) \xrightarrow{d} \mathcal{N}(0, 1)$$

Here, the largest eigenvalue test is the best test. **But** when the signal is weak ($0 \leq \theta < \sqrt{\gamma}$), the largest eigenvalue under the alternative converges to the same distribution as null:

$$n^{2/3}\left(\frac{\lambda_1 - \rho(\theta, \gamma)}{\tau(\theta, \gamma)}\right) \xrightarrow{d} \text{TW}_1$$

which means that the largest eigenvalue test *fails*. On top of this, **resampling** also fails when p is large.

Next, we develop another test to cope with these problems.

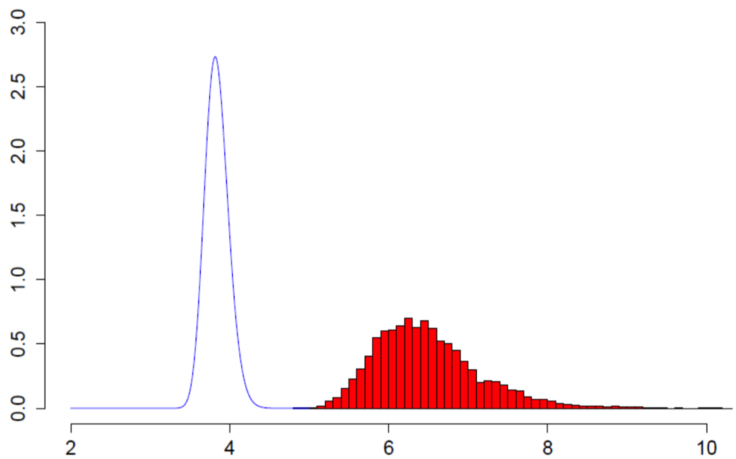

Figure 18.1: Failure of Resampling Test ($n = p = 100$)

18.2 Johnstone and Onatski (2020)

Consider the basic equation of classical multivariate statistics:

$$\det(\mathbf{H} - \mathbf{x}\mathbf{E}) = 0 \tag{18.1}$$

with $p \times p$ matrices

$$n_1\mathbf{H} = \sum_{k=1}^{n_1} \mathbf{x}_k\mathbf{x}_k' \qquad \textit{hypothesis}\ \text{SS}$$

$$n_1\mathbf{E} = \sum_{k=1}^{n_1} \mathbf{z}_k\mathbf{z}_k' \qquad \textit{error}\ \text{SS}$$

The solution $\mathbf{x}$ is generalized eigenvalues $\{\lambda_i\}_{i=1}^p$, which are the eigenvalue of F-ratio $\mathbf{E}^{-1}\mathbf{H}$. Johnstone and Onatski (2020) summarized 5 topics using $\mathbf{E}^{-1}\mathbf{H}$ relying on the five most common hypergeometric functions[1] ${}_\mathrm{p}\mathcal{F}_\mathrm{q}$

[1]Hypergeometric functions are:

- scalar inputs

$$ {}_\mathrm{p}\mathcal{F}_\mathrm{q}(a, b; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_p)_k} \frac{x^k}{k!}$$

  where $(a_j)_k$ are generalized Pochhammer symbols
- single matrix inputs, where $\mathbf{S}$ is symmetric and usually diagonal

$$ {}_\mathrm{p}\mathcal{F}_\mathrm{q}(a, b; \mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_p)_\kappa} \frac{C_\kappa(\mathbf{S})}{k!}$$

  where C_k are the zonal polynomials. Easily, ${}_0\mathcal{F}_0(\mathbf{S}) = e^{\mathrm{tr}(\mathbf{S})}$, ${}_1\mathcal{F}_0(a, \mathbf{S}) = |\mathbf{I} - \mathbf{S}|^{-a}$
- two matrix inputs, where $\mathbf{S}, \mathbf{T}$ are both symmetric

$$ {}_\mathrm{p}\mathcal{F}_\mathrm{q}(a, b; \mathbf{S}, \mathbf{T}) = \int_{O(p)} {}_\mathrm{p}\mathcal{F}_\mathrm{q}(a, b; \mathbf{SUTU}')(d)\mathbf{U}$$

Table 18.1: 5 Statistical Methods

		Statistical method	$n_1\mathbf{H}$	$n_2\mathbf{E}$	Univariate Analog
${}_0\mathcal{F}_0$	PCA	Principal components analysis	$W_p(n_1, \mathbf{\Sigma}+\mathbf{\Phi})$	$n_2\mathbf{\Sigma}$	χ^2
${}_1\mathcal{F}_0$	SigD	Signal detection	$W_p(n_1, \mathbf{\Sigma}+\mathbf{\Phi})$	$W_p(n_2, \mathbf{\Sigma})$	non-central χ^2
${}_0\mathcal{F}_1$	REG_0	Multivariate regression, with known error	$W_p(n_1, \mathbf{\Sigma}, n_1\mathbf{\Phi})$	$n_2\mathbf{\Sigma}$	F
${}_1\mathcal{F}_1$	REG	Multivariate regression, with unknown error	$W_p(n_1, \mathbf{\Sigma}, n_1\mathbf{\Phi})$	$W_p(n_2, \mathbf{\Sigma})$	non-central F
${}_2\mathcal{F}_1$	CCA	Canonical correlation analysis	$W_p(n_1, \mathbf{\Sigma}, \mathbf{\Phi}(\mathbf{Y}))$	$W_p(n_2, \mathbf{\Sigma})$	$\frac{r^2}{1-r^2}$

For ${}_0\mathcal{F}_0$ and ${}_0\mathcal{F}_1$, $\mathbf{E}$ is deterministic, $\mathbf{\Sigma}$ is known, n_2 disppears, otherwise $\mathbf{E}$ is independent of $\mathbf{H}$.

18.2.1 Definitions and global assumptions

Let $\mathbf{Z}$ be an $n \times p$ data matrix with rows (observations) drawn **i.i.d.** from $\mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma})$, and a deterministic matrix $\mathbf{M}$ of $n \times p$, then for $\mathbf{Y} = \mathbf{M} + \mathbf{Z}$,

- $\mathbf{H} = \mathbf{Y}'\mathbf{Y}$ has a p dimensional Wishart distribution $W_p(n, \mathbf{\Sigma}, \mathbf{\Psi})$ with n degrees of freedom, covariance matrix $\mathbf{\Sigma}$ and non-centrality matrix $\mathbf{\Psi} = \mathbf{\Sigma}^{-1}\mathbf{M}'\mathbf{M}$
- the corresponding central Wishart distribution with $\mathbf{M} = \mathbf{0}$ is $W_p(n, \mathbf{\Sigma})$

Johnstone and Onatski (2020) assume a relative low dimensionality $p \le \min\{n_1, n_2\}$ where n_1, n_2 are the degrees of freedom as in Table 18.1, where

- $p \le n_2$ ensures almost sure invertibility of matrix $\mathbf{E}$ in Equation 18.1
- $p \le n_1$ is not essential, but reduces the number of various situations of consideration.

With these assumptions, they established a unified statistical problem **symmetric matrix denoising (SMD)** that can be linked to the 5 classes of problems:

PCA n_1 i.i.d. observations drawn from $\mathcal{N}_p(\mathbf{0}, \mathbf{\Omega})$ to test the hull hypothesis that the population covariance $\mathbf{\Omega} = \mathbf{\Sigma}$, with the alternative of interest being

$$\mathbf{\Omega} = \mathbf{\Sigma} + \mathbf{\Phi}, \quad \text{with } \mathbf{\Phi} = \theta\boldsymbol{\phi}\boldsymbol{\phi}'$$

where $\theta > 0$, $\boldsymbol{\phi}$ are unknown, and $\boldsymbol{\phi}$ is normalized s.t. $\left\|\mathbf{\Sigma}^{-1/2}\boldsymbol{\phi}\right\| = 1$. W.L.O.G., assume $\mathbf{\Sigma} = \mathbf{I}_p$, then under the alternative, the first principal component explains a larger portion of the variation than the other principal components. Re-formulate the hypotheses in terms of the spectral *spike* parameter θ, we have

$$H_0 : \theta_0 = 0 \qquad\qquad H_1 : \theta_0 = \theta > 0$$

where θ_0 is the true value of the *spike*. A **maximal invariant statistic** consists of the solutions $\lambda_1 \ge \cdots \ge \lambda_p$ of Equation 18.1 with

- $n_1\mathbf{H}$ equal to the sample covariance matrix
- $\mathbf{E} = \mathbf{\Sigma}$

SigD Now consider testing the **equality** of covariance matrices $\mathbf{\Omega}$ and $\mathbf{\Sigma}$, corresponding to 2 independent $p-$dimensional mean-zero Gaussian samples of size n_1 and n_2, with the alternative still

$$\mathbf{\Omega} = \mathbf{\Sigma} + \mathbf{\Phi}, \quad \text{with } \mathbf{\Phi} = \theta\boldsymbol{\phi}\boldsymbol{\phi}'$$

and again, assume $\mathbf{\Sigma} = \mathbf{I}_p$ (but NOT necessarily known), here, instead of Equation 18.1, consider

$$\det\left(\mathbf{H} - \lambda\left(\mathbf{E} + \frac{n_1}{n_2}\mathbf{H}\right)\right) = 0 \tag{18.2}$$

naturally, SigD reduces to PCA as $n_2 \to \infty$ while n_1 and p held constant.

REG$_0$ Next, consider a linear regression with multivariate response

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

with known covariance matrix $\mathbf{\Sigma}$ of the i.i.d. Gaussian rows of the error matrix $\boldsymbol{\epsilon}$. Here, to test linear restrictions on the matrix of coefficients $\boldsymbol{\beta}$, we can split the matrix of transformed response variables $\mathbf{Y}$ into 3 parts $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$, where

- $\mathbf{Y}_1$ is $n_1 \times p$ where p is the number of response variables, n_1 is the number of linear restrictions (per each of the p columns of matrix $\boldsymbol{\beta}$), under the null $H_0 : \mathbb{E}\mathbf{Y}_1 = 0$, versus the alternative

$$\mathbb{E}\mathbf{Y}_1 = \sqrt{n_1\theta}\boldsymbol{\psi}\boldsymbol{\phi}' \tag{18.3}$$

  where $\theta > 0$, $\left\|\mathbf{\Sigma}^{-1/2}\boldsymbol{\phi}\right\| = 1$ and $\|\boldsymbol{\psi}\| = 1$
- $\mathbf{Y}_2$ is $(q - n_1) \times p$, where q is the number of regressors
- $\mathbf{Y}_3$ is $(T - q) \times p$, where T is the number of observations

In this case, tests can be based on the solutions $\lambda_1, \cdots, \lambda_p$ to

$$\det(\mathbf{H} - \lambda\mathbf{E}) = \mathbf{0}$$

where $\mathbf{H} = \mathbf{Y}_1'\mathbf{Y}_1/n_1$ and $\mathbf{E} = \mathbf{\Sigma}$. The solutions represent a multivaraite analog of the difference between the sum of squared residuals in the restircted and unrestricted regressions. Under the null, $n_1\mathbf{H}$ is distributed as $W_p(n_1, \mathbf{\Sigma})$. Here,

$$\begin{aligned} n_1\mathbf{H} &\sim W_p(n_1, \mathbf{\Sigma}) && \text{under } H_0 \\ n_1\mathbf{H} &\sim W_p(n_1, \mathbf{\Sigma}, n_1\mathbf{\Phi}), \text{ where } \mathbf{\Phi} = \theta\mathbf{\Sigma}^{-1}\boldsymbol{\phi}\boldsymbol{\phi}' && \text{under } H_1 \end{aligned}$$

Again, W.L.O.G, assume $\mathbf{\Sigma} = \mathbf{I}_p$. This **canonical form** of REG$_0$ is essentially equivalent

REG Again, consider the linear regression

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

but **NOT** knowing the covariance matrix $\mathbf{\Sigma}$ of rows of $\epsilon\epsilon$.

CCA

References

Iain M Johnstone and Alexei Onatski. Testing in high-dimensional spiked models. *The Annals of Statistics*, 48(3), 2020.