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Topic 5: Two-Way Cluster-Robust (TWCR) Standard Errors

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Key points: The validity of Two-Way Cluster-Robust (TWCR) standard errors

Disclaimer: *This note is compiled by Sai Zhang.*

5.1 One-Way Clustering

First, consider the case of one-way clustering. The linear model with one-way clustering

$$y_{ig} = \mathbf{x}_{ig}\boldsymbol{\beta} + u_{ig}$$

where i denotes the ith of the N individuals in the sample, j denotes the gth of the G clusters, assume that

- $\mathbb{E}\left[u_{ig} \mid \mathbf{x}_{ig}\right] = 0$
- error independence across clusters: for $i \neq j$

$$\mathbb{E}\left[u_{ig}u_{jg'} \mid \mathbf{x}_{ig}, \mathbf{x}_{jg'}\right] = 0 \tag{5.1}$$

  unless $g = g'$, that is, errors for individuals within the same cluster may be correlated.

Grouping observations by cluster, get

$$\mathbf{y}_g = \mathbf{X}_g\boldsymbol{\beta} + \mathbf{u}$$

where $\mathbf{X}_g$ has dimension $N_g \times K$ and $\mathbf{y}_g$ has dimension $N_g \times 1$, with N_g observations in cluster g. Stacking over cluster, get the matrix form of the model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

with $\mathbf{y}$, $\mathbf{u}$ being $N \times 1$ vectors, $\mathbf{X}$ being an $N \times K$ matrix. OLS estimator gives

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} = \left(\sum_{g=1}^{G}\mathbf{X}_g'\mathbf{X}_g\right)^{-1}\sum_{g=1}^{G}\mathbf{X}_g'\mathbf{y}_g \tag{5.2}$$

then, by CLT, we have that $\sqrt{G}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma})$ where the variance matrix of the limit normal distribution $\boldsymbol{\Sigma}$ is

$$\left(\lim_{G\to\infty}\frac{1}{G}\sum_{g=1}^{G}\mathbf{E}\left[\mathbf{X}_g'\mathbf{X}_g\right]\right)^{-1}\left(\lim_{G\to\infty}\frac{1}{G}\sum_{g=1}^{G}\mathbf{E}\left[\mathbf{X}_g'\mathbf{u}_g'\mathbf{u}_g\mathbf{X}_g\right]\right)\times\left(\lim_{G\to\infty}\frac{1}{G}\sum_{g=1}^{G}\mathbf{E}\left[\mathbf{X}_g'\mathbf{X}_g\right]\right)^{-1} \tag{5.3}$$

If the primary source of clustering is due to group-level common shocks, a useful approximation is that for the jth regressor, the default OLS variance estimate based on $s^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}$ should be inflated by $\tau_j \simeq 1 + \rho_{x_j}\rho_u\left(\overline{N}_g - 1\right)$, where

- s is the estimated standard deviation of the error

- ρ_{x_j} is a measure of within-cluster correlation of x_j
- ρ_u is the within-cluster error correlation
- $\overline{N}_g$ is the average cluster size

It's easy to see the τ_j can be large even with small ρ_u (Kloek, 1981; Scott and Holt, 1982; Moulton, 1990). If assume the model for the cluster error variance matrices $\mathbf{\Omega}_g = \mathbb{V}\left[\mathbf{u}_g \mid \mathbf{X}_g\right] = \mathbb{E}\left[\mathbf{u}_g\mathbf{u}_g' \mid \mathbf{X}_g\right]$, and there is a consistent estimate $\hat{\mathbf{\Omega}}_g$ of $\mathbf{\Omega}_g$, we can estimate $\mathbb{E}\left[\mathbf{X}_g'\mathbf{u}_g\mathbf{u}_g'\mathbf{X}_g\right] = \mathbb{E}\left[\mathbf{X}_g'\mathbf{\Omega}_g\mathbf{X}_g\right]$ via GLS.

Cluster-robust variance matrix estimate consider

$$\hat{\mathbb{V}}\left[\hat{\boldsymbol{\beta}}\right] = (\mathbf{X}'\mathbf{X})^{-1}\left(\sum_{g=1}^{G}\mathbf{X}_g'\hat{\mathbf{u}}g\hat{\mathbf{u}}_g'\mathbf{X}_g\right)(\mathbf{X}'\mathbf{X})^{-1} \tag{5.4}$$

Liang and Zeger (1986) proposed

References

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