

## Topic 3: *Moving the Goalposts* Approach

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Key points:

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**Disclaimer:** These notes are written by Sai Zhang ([email me](#) or check my [Github page](#)). The main references for this topic are [Armstrong et al. \(2020\)](#); [Armstrong and Kolesár \(2018\)](#), I thank Prof. Armstrong for his valuable advice.

### 3.1 Finite Sample Bias-Variance Tradeoffs

#### 3.1.1 Setup

Consider the fixed design regression model

$$y_i = w_i \beta(z_i) + h(z_i) + \epsilon_i \quad (3.1)$$

where

- $w_i, z_i$  are treated as **fixed**
- $\epsilon_i$  is **independent**, with  $\mathbb{E}[\epsilon_i] = 0, \mathbb{E}[\epsilon_i^2] = \sigma_i^2$
- observation:  $\left\{ \left( y_i, w_i, z_i' \right)' \right\}_{i=1}^n$

one example is the case where  $w_i$  is **binary**, then

$$\beta(z) = f(1, z) - f(0, z)$$

which is just the ATE conditional on  $z$  under the unconfoundedness assumption. This includes the RD design, where  $z_i$  is the running variable and  $w_i$  is the treatment assignment.

Now, consider for the weighted average treatment effect

$$L_\mu[\beta(\cdot)] = \int \beta(z) d\mu(z)$$

where  $\int \mu(z) = 1$  is a **signed** measure (weight, allowing **negative** weights), construct a linear estimator

$$\hat{L}_a = \sum_{i=1}^n a_i y_i$$

where the estimation weights  $a_i$  can depend on  $\{z_i, w_i, \sigma_i^2\}_{i=1}^n$ , but **not** on  $y_i$ . Together, the bias of  $\hat{L}_a$  for  $L_\mu[\beta(\cdot)]$ , given the regression function  $\beta(\cdot), h(\cdot)$ , is

$$\mathbb{E}_{\beta(\cdot), h(\cdot)}[\hat{L}_a] - L_\mu[\beta(\cdot)] = \sum_{i=1}^n a_i [w_i \beta(z_i) + h(z_i)] - \int \beta(z) d\mu(z)$$

and its variance, given the regression function  $\beta(\cdot)$ ,  $h(\cdot)$ , is just

$$\text{Var}_{\beta(\cdot), h(\cdot)} [\hat{L}_a] = \sum_{i=1}^n a_i^2 \sigma_i^2$$

To bound the bias, assume  $h(\cdot)$  is known to belong in a class of functions  $\mathcal{H}$ , then two approaches can be adopted, for the regularity of  $\beta(\cdot)$  and the choice of  $\mu(\cdot)$ :

- 1 arbitrary  $\beta(\cdot)$ , optimizing weights  $\mu$  by *moving the goalposts*, s.t.  $L_\mu [\beta(\cdot)]$  is easy to estimate (Crump et al., 2006; Imbens and Wager, 2019) which gives the worst-case bias

$$\inf_{\mu} \sup_{\beta(\cdot), h(\cdot)} \left| \sum_{i=1}^n a_i [w_i \beta(z_i) + h(z_i)] - \int \beta(z) d\mu(z) \right| \quad \text{s.t. } h(\cdot) \in \mathcal{H}, \int d\mu(z) = 1 \quad (3.2)$$

- 2 assume constant treatment effects, i.e.,  $\beta(z) = \beta, \forall z$ , which means that  $L_\mu [\beta(\cdot)] = \beta$  regardless of  $\mu$  (Armstrong et al., 2020), and the worst-case bias is

$$\sup_{\beta, h(\cdot)} \left| \sum_{i=1}^n a_i [w_i \beta + h(z_i)] - \beta \right| \quad \text{s.t. } h(\cdot) \in \mathcal{H} \quad (3.3)$$

And, the two approaches can be linked as such:

- If  $\sum_{i=1}^n a_i w_i = 1$ , 3.2 and 3.3 are both equal to

$$\sup_{h(\cdot)} \left| \sum_{i=1}^n a_i h(z_i) \right| \quad \text{s.t. } h(\cdot) \in \mathcal{H} \quad (3.4)$$

- 3.2 automatically equals 3.4
- 3.3 is optimized (w.r.t.  $\mu$ ) by setting  $\mu$  to place weight  $a_i w_i$  on observation  $i$ , i.e.,  $\mu(\mathcal{Z}) = \sum_{i: z_i \in \mathcal{Z}} a_i w_i$ , which implies  $\sum_{i=1}^n a_i w_i \beta(z_i) - \int \beta(z) d\mu(z) = 0$ , hence the equality.
- Otherwise, 3.2 and 3.3 are both infinite:
  - 3.3 can be made arbitrarily large by choosing large enough  $\beta$
  - 3.2 can be made arbitrarily large by making  $\beta(\cdot)$  constant (as in 3.3) and large enough

### 3.1.2 Moving-the-goalpost Approach

### 3.1.3 Constant-treatment-effect Approach

Armstrong et al. (2020) adopt this approach, focusing on the case where  $h(\cdot)$  is a high dimensional linear function, and the penalty function is an  $l_p$  norm of the coefficients.

#### Basic setting: Homoskedastic Gaussian errors

First, consider

$$Y = w\beta + Z\gamma + \epsilon \quad (3.5)$$

where

- $\beta \in \mathbb{R}$  is the constant treatment effect to be estimated
- $\gamma \in \Gamma$  is the control coefficients, subject to the restriction (i.e., the function class  $\mathcal{H}$ )

$$\Gamma = \Gamma(C) = \{\gamma \in \mathcal{G} : \text{Pen}(\gamma) \leq C\} \quad (3.6)$$

where  $\text{Pen}(\cdot)$  is a seminorm<sup>1</sup> on some linear subspace  $\mathcal{G}$  of  $\mathbb{R}^k$ .

- $w = (w_1, \dots, w_n)' \in \mathbb{R}^n$  and  $Z = (z'_1, \dots, z'_n)' \in \mathbb{R}^{n \times k}$  are defined as before
- $\epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$  is assumed **normal and homoskedastic**, with  $\sigma^2$  known

For estimation, the goal is to construct estimators and CIs for  $\beta$ :

- estimator  $\hat{\beta}$ : consider the worst-case performance over the parameter space  $\mathbb{R} \times \Gamma$  under the **MSE** criterion

$$R_{MSE}(\hat{\beta}; \Gamma) = \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} \mathbb{E}_{\beta, \gamma} \left[ (\hat{\beta} - \beta)^2 \right]$$

- for CIs, we have 2 requirements:

A **coverage**: A  $100 \cdot (1 - \alpha)\%$  CI with half-length  $\hat{\chi} = \hat{\chi}(Y, X)$  is an interval  $\{\hat{\beta} \pm \hat{\chi}\}$  s.t.

$$\inf_{\beta \in \mathbb{R}, \gamma \in \Gamma} P_{\beta, \gamma}(\beta \in \{\hat{\beta} \pm \hat{\chi}\}) \geq 1 - \alpha$$

B **length**: the expected length of a CI  $\mathbb{E}_{\beta, \gamma}[2\hat{\chi}]$  should be as short as possible

notice that length-optimized CIs are **not** necessarily centered at an MSE-centered  $\hat{\beta}$ .

### Linear estimators and CIs

Again, consider estimators that are **linear** in the outcomes  $Y$ ,  $\hat{\beta} = a'Y$ , where  $a$  is the  $n$ -vector weights. In the vector form, the worst-case bias (as 3.3) is

$$\overline{\text{bias}}_{\Gamma}(\hat{\beta}) = \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} a'(w\beta + Z\gamma) - \beta \quad (3.7)$$

and the variance, under the assumption of homoskedasticity, is

$$\text{Var}(\hat{\beta}) = \sigma^2 a'a$$

Then the MSE is

$$R_{MSE}(\hat{\beta}; \Gamma) = \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} \mathbb{E}_{\beta, \gamma} \left[ (\hat{\beta} - \beta)^2 \right] = \overline{\text{bias}}_{\Gamma}(\hat{\beta})^2 + \text{Var}(\hat{\beta})$$

The  $t$ -statistic is

$$\frac{\hat{\beta} - \beta}{\sqrt{\text{Var}(\hat{\beta})}} \sim \mathcal{N}(b, 1), \quad |b| \leq \frac{\overline{\text{bias}}_{\Gamma}(\hat{\beta})}{\sqrt{\text{Var}(\hat{\beta})}}$$

and a two-sided CI can then be formed as

$$\hat{\beta} \pm \chi, \quad \text{where } \chi = \sqrt{\text{Var}(\hat{\beta})} \cdot \text{cv}_{\alpha} \left( \frac{\overline{\text{bias}}_{\Gamma}(\hat{\beta})}{\sqrt{\text{Var}(\hat{\beta})}} \right) \quad (3.8)$$

and the  $\text{cv}_{\alpha}(B)$  denotes the  $1 - \alpha$  quantile of a  $|\mathcal{N}(B, 1)|$ . This is a **fixed-length confidence interval (FLCI)**, with a fixed length of  $2\chi$ . It depends on  $X$  and  $\sigma^2$ , but not on  $Y$  or  $(\beta, \gamma)'$ .

<sup>1</sup>Seminorm satisfies **triangle inequality**  $\text{Pen}(\gamma + \tilde{\gamma}) \leq \text{Pen}(\gamma)$  and **homogeneity**  $\text{Pen}(c\gamma) = |c|\text{Pen}(\gamma), \forall c$ , but **NOT** necessarily positive definite ( $\text{Pen}(\gamma) = 0$  does not imply  $\gamma = 0$ ). Essentially, any convex set  $\Gamma$  that is symmetric satisfies this definition.

### Optimal weights

We have two optimization goals

- minimizing MSE:  $R_{MSE}(\hat{\beta}; \Gamma) = \overline{\text{bias}}_{\Gamma}(\hat{\beta})^2 + \text{Var}(\hat{\beta})$
- minimizing CI length:  $\chi = \sqrt{\text{Var}(\hat{\beta})} \cdot \text{cv}_{\alpha} \left( \overline{\text{bias}}_{\Gamma}(\hat{\beta}) / \sqrt{\text{Var}(\hat{\beta})} \right)$

They both increasing in  $\text{Var}(\hat{\beta})$  and  $\overline{\text{bias}}_{\Gamma}(\hat{\beta})$ , hence to find the optimal weights, it suffices to minimize variance subject to a bound  $B$  on worst-case bias, which can be written as:

$$\min_{a \in \mathbb{R}} a' a \text{ s.t. } \sup_{\beta \in \mathbb{R}, \gamma \in \Gamma} a' (w\beta + Z\gamma) - \beta \leq B \quad (3.9)$$

The optimal weight is then given by:

#### Theorem 3.1.1: Optimal Weight

Let  $\pi_{\lambda}^*$  be a solution to<sup>a</sup>

$$\min_{\pi} \|w - Z\pi\|_2^2 \text{ s.t. } \text{Pen}(\pi) \leq t_{\lambda} \quad (3.10)$$

and suppose that  $\|w - Z\pi\|_2 > 0$ ,  $\text{Pen}(\cdot)$  is continuous, then the optimal weight solving 3.9 is

$$a_{\lambda}^* = \frac{w - Z\pi_{\lambda}^*}{\left(w - Z\pi_{\lambda}^*\right)' w}$$

with the bound

$$B = \frac{C}{t_{\lambda}} \cdot \frac{\left(w - Z\pi_{\lambda}^*\right)' Z\pi_{\lambda}^*}{\left(w - Z\pi_{\lambda}^*\right)' w}$$

Consequently, we have

- estimator

$$\hat{\beta}_{\lambda} = a_{\lambda}^* Y = \frac{\left(w - Z\pi_{\lambda}^*\right)' Y}{\left(w - Z\pi_{\lambda}^*\right)' w}$$

- worst-case bias

$$\overline{\text{bias}}_{\Gamma}(\hat{\beta}_{\lambda}) = C\bar{B}_{\lambda} = \frac{C}{\text{Pen}(\pi_{\lambda}^*)} \frac{\left(w - Z\pi_{\lambda}^*\right)' Z\pi_{\lambda}^*}{\left(w - Z\pi_{\lambda}^*\right)' w}$$

- variance of estimator

$$V_{\lambda} = \frac{\sigma^2 \|w - Z\pi_{\lambda}^*\|_2^2}{\left[\left(w - Z\pi_{\lambda}^*\right)' w\right]^2}$$

<sup>a</sup>This regression can be referred to as a regularized propensity score regression (but  $w_i$  need not be binary) with penalty  $\text{Pen}(\pi)$

This result follows by applying Donoho (1994), Low (1995) and Armstrong and Kolesár (2018), rewriting 3.9

as a convex optimization problem.

## A Proofs

### A.1 Proof of Theorem 3.1.1

Following [Armstrong and Kolesár \(2018, Equation \(25\)\)](#), the modulus of continuity is given by

$$\omega(\delta) = \sup_{\beta, \gamma} 2\beta \quad \text{s.t. } \|w\beta + Z\gamma\|_2 \leq \frac{\delta}{2}, \quad \text{Pen}(\gamma) \leq C$$

Introducing a substitution (rescaling  $\gamma$  by  $\beta$ )  $\pi = -\frac{\gamma}{\beta}$ , get

$$\omega(\delta) = \sup_{\beta, \pi} 2\beta \quad \text{s.t. } \beta\|w - Z\pi\|_2 \leq \frac{\delta}{2}, \quad \beta\text{Pen}(\pi) \leq C \quad (3.11)$$

recall the optimization problem in Theorem 3.1.1:

$$\min_{\pi} \|w - Z\pi\|_2^2 \quad \text{s.t. } \text{Pen}(\pi) \leq t_\lambda$$

We can relate the two problems via the following logic: we want to make  $\|w - Z\pi\|_2$  and  $\text{Pen}(\pi)$  small so that large values of  $\beta$  satisfy the constraint of 3.11. Formally:

#### Lemma A.1

- If  $\exists \pi \in \mathcal{G}$  s.t.  $w = Z\pi$  and  $\text{Pen}(\pi) = 0$ , then  $w(\delta) = \infty, \forall \delta \geq 0$  (automatic)
- **Otherwise:**
  - (i)  $\forall \delta > 0$ , the problem 3.11 has a solution  $\beta_\delta^{mod}, \pi_\delta^{mod}$  with  $\beta_\delta^{mod} > 0$ . For  $t_\lambda = \frac{C}{\beta_\delta^{mod}} = \frac{2C}{w(\delta)}$ ,  $\pi_\delta^{mod}$  is also a solution to the penalized regression (3.10)

$$\min_{\pi} \|w - Z\pi\|_2^2 \quad \text{s.t. } \text{Pen}(\pi) \leq t_\lambda$$

with optimized objective

$$\|w - Z\pi\|_2 = \frac{\delta}{2\beta_\delta^{mod}} = \frac{\delta}{w(\delta)} > 0$$

- (ii)  $\forall t_\lambda > 0$ , the penalized regression above has a solution  $\pi_\lambda^*$ . Setting

$$\beta_\lambda^* = \frac{C}{t_\lambda}$$

$$\delta_\lambda = 2\beta_\lambda^* \|w - Z\pi\|_2 = \frac{2C}{t_\lambda} \|w - Z\pi\|_2$$

the pair  $(\beta_\lambda^*, \pi_\lambda^*)$  solves the modulus problem 3.11 at  $\delta = \delta_\lambda$ , with optimized objective  $w(\delta_\lambda) = \frac{2C}{t_\lambda}$ , as long as  $\|w - Z\pi\|_2 > 0$

**Proof of Lemma A.1:** we prove the lemma with the following steps:

A the penalized problem 3.10 has a solution:

Let  $\mathcal{G}^{(0)}$  denote the linear subspace of vectors  $\pi \in \mathcal{G}$  s.t.  $Z\pi = 0$ ,  $\text{Pen}(\pi) = 0$ ; let  $\mathcal{G}^{(1)}$  be a subspace s.t.  $\mathcal{G} = \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)}$ . Then, we can write  $\pi \in \mathcal{G}$  uniquely as  $\pi = \pi^{(0)} + \pi^{(1)}$  where  $\pi^{(0)} \in \mathcal{G}^{(0)}$ ,  $\pi^{(1)} \in \mathcal{G}^{(1)}$ .

Therefore, we have  $Z\pi = Z\pi^{(1)}$ , and

$$\begin{aligned} \text{Pen}(\pi^{(1)}) &= \text{Pen}(\pi^{(1)}) - \text{Pen}(-\pi^{(0)}) \leq \text{Pen}(\pi) \\ \text{Pen}(\pi^{(1)}) &= \text{Pen}(\pi^{(1)}) + \text{Pen}(\pi^{(0)}) \geq \text{Pen}(\pi) \end{aligned} \Rightarrow \text{Pen}(\pi^{(1)}) = \text{Pen}(\pi)$$

Then, 3.10 can be written in terms of  $\pi^{(1)} \in \mathcal{G}^{(1)}$  only. The level sets of this optimization problem are bounded and closed (by continuity of the seminorm  $\text{Pen}(\cdot)$ ), so it has a solution, which is also the solution to the original problem.

B the modulus problem 3.11 has a solution: for the problem 3.11, feasible values of  $\beta$  are bounded as:

$$\beta \leq \frac{\delta}{2} \cdot \frac{1}{\|w - Z\pi\|_2} \qquad \beta \leq C \cdot \frac{1}{\text{Pen}(\pi)}$$

i.e.,  $\beta$  is bounded by the inverse of the minimum of  $\max\{\|w - Z\pi\|_2, \text{Pen}(\pi)\}$  over  $\pi$ , and it is strictly positive. Hence,  $\beta, \tilde{\pi}^{(1)}$  can be restricted to a compact set without changing the optimization problem.

C proof of statement (i): Proof by contradiction, if it's not true, then  $\exists \tilde{\pi}$  s.t.

$$\text{Pen}(\tilde{\pi}) \leq \frac{C}{\beta_\delta^{mod}} \equiv t_\lambda, \qquad \|w - Z\tilde{\pi}\|_2 \leq \|w - Z\pi_\delta^{mod}\|_2 - v$$

then for some  $\eta$ , let  $\tilde{\pi}_\eta = (1 - \eta)\tilde{\pi}$ , we have

$$\begin{aligned} \|w - Z\tilde{\pi}_\eta\|_2 &= \|w - Z(1 - \eta)\tilde{\pi}\|_2 \\ &\leq \|w - X\tilde{\pi}\|_2 + \eta\|Z\tilde{\pi}\|_2 \\ &\leq \|w - X\pi_\delta^{mod}\|_2 - v + \eta\|Z\tilde{\pi}\|_2 \\ &\leq \frac{\delta}{2\beta_\delta^{mod}} - v + \eta\|Z\tilde{\pi}\|_2 \end{aligned}$$

Hence,  $\exists \eta$  small enough, s.t.

$$\|w - Z\tilde{\pi}_\eta\|_2 < \frac{\delta}{2\beta_\delta^{mod}}, \qquad \text{Pen}(\tilde{\pi}_\eta) \leq (1 - \eta)\frac{C}{\beta_\delta^{mod}} < \frac{C}{\beta_\delta^{mod}}$$

therefore, by setting  $\pi = \tilde{\pi}_\eta$ , we can allow a strictly bigger  $\beta$ , which is a contradiction.

D proof of statement (ii): This result follows immediately.

Next, use Lemma A.1 to prove Theorem 3.1.1. Following Armstrong and Kolesár (2018), the class of bias-variance optimizing estimators is

$$\frac{(w\beta_{delta}^{mod} + Z\gamma_\delta^{mod})' Y}{(w\beta_{delta}^{mod} + Z\gamma_\delta^{mod})' w}$$

## References

- Timothy B Armstrong and Michal Kolesár. Optimal inference in a class of regression models. *Econometrica*, 86(2):655–683, 2018.
- Timothy B Armstrong, Michal Kolesár, and Soonwoo Kwon. Bias-aware inference in regularized regression models. *arXiv preprint arXiv:2012.14823*, 2020.
- Richard K Crump, V Joseph Hotz, Guido Imbens, and Oscar Mitnik. Moving the goalposts: Addressing limited overlap in the estimation of average treatment effects by changing the estimand, 2006.
- David L Donoho. Statistical estimation and optimal recovery. *The Annals of Statistics*, 22(1):238–270, 1994.
- Guido Imbens and Stefan Wager. Optimized regression discontinuity designs. *Review of Economics and Statistics*, 101(2):264–278, 2019.
- Mark G Low. Bias-variance tradeoffs in functional estimation problems. *The Annals of Statistics*, 23(3): 824–835, 1995.