

## Topic 13: Non-convex Learning + Lasso

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**Key points:** Combining the best of the two, we can use **Lasso plus Concave** method, with Lasso screening and concave component selecting variables, achieving a coordinated intrinsic two-scale learning.

**Disclaimer:** The note is built on Prof. *Jinchi Lv*'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

We are facing a tradeoff:

- **Convex** methods: have appealing prediction power and oracle inequalities, but challenging to provide tight false sign rate control
- **Concave** methods: have good variable selection properties, but challenging to establish global properties and risk properties

Here, we take advantage of the linearity of Lasso (convex *and* concave) and try to combine it with concave regularization to get the best of both.

### 13.1 Model Setup

Again, consider a linear regression model  $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ , where

- response vector ( $n \times 1$ ):  $\mathbf{y} = (y_1, \dots, y_n)'$
- design matrix ( $n \times p$ ):  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_p)$ , with each column rescaled to have  $L_2$ -norm  $n^{1/2}$

here, we consider a scenario where

- $\boldsymbol{\beta}_0 = (\beta_{0,1}, \dots, \beta_{0,p})'$  is *sparse* (with many 0 components)
- ultra-high dimensions:  $\log p = O(n^a)$ , for some  $0 < a < 1$

and consider the penalized least squares

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left\{ (2n)^{-1} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda_0 \|\boldsymbol{\beta}\|_1 + \|p_\lambda(\boldsymbol{\beta})\|_1 \right\} \quad (13.1)$$

where

- $\lambda_0 = c \left( \frac{\log p}{n} \right)^{1/2}$  for some  $c > 0$
- $p_\lambda(\boldsymbol{\beta}) = p_\lambda(|\boldsymbol{\beta}|) = (p_\lambda(|\beta_1|), \dots, p_\lambda(|\beta_p|))'$ , with  $|\boldsymbol{\beta}| = (|\beta_1|, \dots, |\beta_p|)'$ ; the concave penalty  $p_\lambda(t)$  is defined on  $t \in [0, \infty)$ , indexed by  $\lambda \geq 0$ , increasing in both  $t$  and  $\lambda$ ,  $p_\lambda(0) = 0$

the 2 penalty components

- $L_1$ -component: minimum amount of regularization for removing noise in prediction
- concave component  $\|p_\lambda(\boldsymbol{\beta})\|_1$ : adapt model sparsity for variable selection

Under this set up, we can derive the hard-thresholding property as

**Proposition 13.1.1: Hard-Thresholding Property**

Assume the  $p_\lambda(t)$ ,  $t \geq 0$ , is **increasing and concave** with

- $p_\lambda(t) \geq p_{H,\lambda}(t) = \frac{1}{2} [\lambda^2 - (\lambda - t)_+^2]$  on  $[0, \lambda]$
- $p'_\lambda((1 - c_1)\lambda) \leq c_1\lambda$  for some  $c_1 \in [0, 1]$
- $-p''_\lambda(t)$  decreasing on  $[0, (1 - c_1)\lambda]$

then any local minimizer of 13.1 that is also a global minimizer in each coordinate has the **hard-thresholding** feature that each component is either 0 or of magnitude **larger** than  $(1 - c_1)\lambda$

Such property is shared by a wide class of concave penalties, including hard-thresholding penalty  $p_{H,\lambda}(t)$  with  $c_1 = 0$ ,  $L_0$ -penalty, and SICA (with suitable  $c_1$ ).

**How to understand this proposition?** Let  $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_p)'$ , then **each  $\hat{\beta}_j$**  is the global minimizer of the corresponding univariate penalized least-square problem along the  $j$ -th coordinate. These univariate problems share a common form with (generally) different scalars  $z$

$$\hat{\beta}(z) = \arg \min_{\beta \in \mathbb{R}} \left\{ \frac{1}{2}(z - \beta)^2 + \lambda_0 |\beta| + p_{H,\lambda}(|\beta|) \right\}$$

after we rescale all covariates to have  $L_2$ -norm  $n^{1/2}$ . The solution to these univariate problems are

$$\hat{\beta}(z) = \text{sgn}(z)(|z| - \lambda_0) \cdot \mathbf{1}_{|z| > \lambda + \lambda_0}$$

these solutions have the same feature as the hard-thresholded estimator: each component is either 0 or of magnitude larger than  $\lambda$ . This provides a better distinction between insignificant and significant covariates than soft-thresholding by  $L_1$  penalty.

With the hard-thresholding property of Prop. 13.1.1, we can prove a basic constraint for the global optimum  $\hat{\beta}$  on an event with significant probability (Fan and Lv, 2014)

$$\|\delta_2\|_1 \leq 7\|\delta_1\|_1 \quad (13.2)$$

where  $\delta = \hat{\beta} - \beta_0 = (\hat{\beta}'_1, \hat{\beta}'_2)' - (\beta'_{0,1}, \beta'_{0,2})' = (\delta'_1, \delta'_2)'$ , with  $\delta_1 \in \mathbb{R}^s$ . Where does this constraint come from? For the penalized least square question 13.1

$$\min_{\beta \in \mathbb{R}^p} \{ (2n)^{-1} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda_0 \|\beta\|_1 + \|p_\lambda(\beta)\|_1 \}$$

the global minimizer  $\hat{\beta}$  leads to

$$\begin{aligned} (2n)^{-1} \|\mathbf{y} - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 &= (2n)^{-1} \|\mathbf{X}\beta_0 + \epsilon - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\ &= (2n)^{-1} \|\epsilon - \mathbf{X}(\hat{\beta} - \beta_0)\|_2^2 + \lambda_0 \|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\ &\leq (2n)^{-1} \|\mathbf{y} - \mathbf{X}\beta_0\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\ &= (2n)^{-1} \|\epsilon\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \end{aligned}$$

then, plug in  $\delta = \hat{\beta} - \beta_0$ , we get

$$\begin{aligned} (2n)^{-1} \|\epsilon - \mathbf{X}\delta\|_2^2 + \lambda_0 \|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq (2n)^{-1} \|\epsilon\|_2^2 + \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\ (2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq \lambda_0 \|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \end{aligned}$$

since  $\beta_{0,2} = \mathbf{0}$ ,  $\delta_2 = \beta_{0,2} + \delta_2$ , we have

$$\|\beta_0 + \delta\|_1 = \|\beta_{0,1} + \beta_{0,2} + \delta_1 + \delta_2\|_1 = \|\beta_{0,1} + \delta_1 + \delta_2\|_1 \leq \|\beta_{0,1} + \delta_1\|_1 + \|\delta_2\|_1$$

hence

$$(2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 \leq \lambda_0 \|\beta_{0,1}\|_1 - \lambda_0 \|\beta_{0,1} + \delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

and by the reverse triangle inequality  $\|\beta_{0,1}\|_1 - \|\beta_{0,1} + \delta_1\|_1 \leq \|\delta_1\|_1$ , we get

$$(2n)^{-1} \|\mathbf{X}\delta\|_2^2 - n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 \leq \lambda_0 \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

If assume the distribution of the model error  $\epsilon$  as

$$\Pr\left(\|n^{-1} \mathbf{X}' \epsilon\|_\infty > \frac{\lambda_0}{2}\right) = O(p^{-c_0})$$

conditional on the event  $\mathcal{E} = \{\|n^{-1} \mathbf{X}' \epsilon\|_\infty \leq \lambda_0/2\}$ , we have

$$-n^{-1} \epsilon' \mathbf{X}\delta + \lambda_0 \|\delta_2\|_1 - \lambda_0 \|\delta_1\|_1 \geq -\frac{\lambda_0}{2} \|\delta\|_1 + \lambda_0 \|\delta_2\|_1 - \lambda_0 \|\delta_1\|_1 = \frac{\lambda_0}{2} \|\delta_2\|_1 - \frac{3\lambda_0}{2} \|\delta_1\|_1$$

plug this result back, get

$$\frac{1}{2n} \|\mathbf{X}\delta\|_2^2 + \frac{\lambda_0}{2} \|\delta_2\|_1 \leq \frac{3\lambda_0}{2} \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \quad (13.3)$$

Now, if we further impose 2 conditions:

- **Condition 1 (eigenvalue condition)**: for some positive constant  $\kappa_0$

$$\min_{\|\delta\|_2=1, \|\delta\|_0 \leq 2s} \frac{1}{\sqrt{n}} \|\mathbf{X}\delta\|_2 \geq \kappa_0 \quad (\mathbf{A})$$

$$\kappa = \kappa(s, 7) = \min_{\delta \neq 0, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2 \vee \|\tilde{\delta}_2\|_2} \right\} > 0 \quad (\mathbf{B})$$

where  $\tilde{\delta}_2$  is the subvector of  $\delta_2$  consisting of the components with the  $s$  largest absolute values. Here

- Condition **(A)** is a mild sparse eigenvalue condition
- Condition **(B)** combines the restricted eigenvalue assumptions in [Bickel et al. \(2009\)](#)<sup>1</sup>. The intuition is, for OLS estimation,  $\mathbf{X}'\mathbf{X}$  should be positive definite, that is

$$\min_{\mathbf{0} \neq \delta \in \mathbb{R}^p} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta\|_2} \right\} > 0$$

however, when  $p > n$ , this condition **never** holds, hence we replace  $\|\delta\|_2$  with the  $L_2$ -norm of  $\|\delta_1\|_2$ , a subvector of  $\delta$

$$\kappa = \kappa(s, 7) = \min_{\delta \neq 0, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2} \right\} > 0$$

and for  $L_q$  loss with  $q \in (1, 2]$ , we further bound  $\|\tilde{\delta}_2\|_2$ , which leads to condition **(B)**.

<sup>1</sup>Introduced by [Candes and Tao \(2007\)](#) for studying the oracle inequalities for the Lasso estimator and Dantzig selector.

- **Condition 2 (hard-thresholding condition):** The penalty  $p_\lambda(t)$  satisfies the conditions of Prop. 13.1.1 with

$$p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$$

$$\min_{j=1, \dots, s} |\beta_{0,j}| > \max \left\{ (1 - c_1)\lambda, 2\kappa_0^{-1} p_\lambda^{1/2}(\infty) \right\}$$

Now, look back at the condition 13.3, we can upper-bound  $\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$  by  $\frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$ . Consider 2 cases:

- **Case 1:**  $\|\hat{\beta}\|_0 \geq s$ . By the hard-thresholding condition, we have  $|\beta_{0,j}| > (1 - c_1)\lambda$  and  $p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$ . Hence, for  $j = 1, \dots, s$ ,

- if  $\hat{\beta}_j \neq 0$ , we must have  $|\hat{\beta}_j| > (1 - c_1)\lambda$ . And by the mean-value theorem, we have

$$|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| = p'_\lambda(b)(|\hat{\beta}_j| - |\beta_{0,j}|) \leq p'_\lambda(b)|\delta_{0,j}|$$

where  $b$  is between  $|\beta_{0,j}|$  and  $|\hat{\beta}_j|$ , hence,  $b > |\beta_{0,j}| > (1 - c_1)\lambda$ , by the concavity of  $p_\lambda$ , we have  $p'(b) < p'((1 - c_1)\lambda) \leq \lambda_0/4$ , which leads to  $|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| \leq \frac{1}{4} \lambda_0 |\delta_j|$ .

- if  $\hat{\beta}_j = 0$ , since  $\|\hat{\beta}\|_0 \geq s$ , there must exist some  $j' > s$  s.t.  $\hat{\beta}_{j'} \neq 0$ , similarly

$$\begin{aligned} |p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_{j'}|)| &\leq |p_\lambda(|\beta_{0,j}|) - p_\lambda((1 - c_1)\lambda)| + |p_\lambda(|\hat{\beta}_{j'}|) - p_\lambda((1 - c_1)\lambda)| \\ &= p'_\lambda(b_1)(|\beta_{0,j}| - (1 - c_1)\lambda) + p'_\lambda(b_2)(|\hat{\beta}_{j'}| - (1 - c_1)\lambda) \\ &\leq p'_\lambda(b_1) \left( |\beta_{0,j}| - \underbrace{|\hat{\beta}_j|}_{=0} \right) + p'_\lambda(b_2) \left( |\hat{\beta}_{j'}| - \underbrace{|\beta_{0,j'}|}_{=0} \right) \\ &= p'_\lambda(b_1)|\delta_j| + p'_\lambda(b_2)|\delta_{j'}| \leq \frac{\lambda_0}{4} (|\delta_j| + |\delta_{j'}|) \end{aligned}$$

together, we have

$$\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq \frac{1}{4} \lambda_0 \|\delta\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$$

- **Case 2:**  $\|\hat{\beta}\|_0 = s - k$  for some  $k \geq 1$ . Then we must have  $\|\delta\|_0 \leq \|\hat{\beta}\|_0 + \|\beta_0\|_0 \leq s - k + s < 2s$ , and  $\|\delta\|_2 \geq \sqrt{k} \min_{j=1, \dots, s} |\beta_{0,j}|$ . Also, there are at least  $k$  null estimates ( $\hat{\beta}_j = 0$ ), thus

$$\underbrace{\frac{1}{4n} \|\mathbf{X}\delta\|_2^2 \geq \frac{\kappa_0^2}{4} \|\delta\|_2^2}_{\text{Condition 1(A)}} \geq \underbrace{\frac{\kappa_0^2}{4} \left( \sqrt{k} \min_{j=1, \dots, s} |\beta_{0,j}| \right)^2}_{\text{Condition 2}} \geq kp_\lambda(\infty) \geq kp_\lambda(|\beta_{0,j}|)$$

similar to Case 1, we have the desired upper bound

$$\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq kp_\lambda(\infty) + \frac{1}{4} \lambda_0 \|\delta\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$$

Combining Case 1 and 2, we have  $\|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{4} \lambda_0 \|\delta\|_1$ , plug this back in 13.3, get

$$\begin{aligned} \frac{1}{n} \|\mathbf{X}\delta\|_2^2 + \lambda_0 \|\delta\|_1 &\leq 3\lambda_0 \|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \\ &\leq 3\lambda_0 \|\delta_1\|_1 + \frac{1}{2n} \|\mathbf{X}\delta\|_2^2 + \frac{1}{2} \lambda_0 \underbrace{\|\delta\|_1}_{=\|\delta_1\|_1 + \|\delta_2\|_1} \\ &\leq 7\lambda_0 \|\delta_1\|_1 \end{aligned}$$

which leads to the constraint in 13.2 and  $\frac{1}{n} \|\mathbf{X}\delta\|_2^2 \leq 7\lambda_0 \|\delta_1\|_1$ .

Now, look back at Condition 1(B)

$$\kappa = \kappa(s, 7) = \min_{\delta \neq \mathbf{0}, \|\delta_2\|_1 \leq 7\|\delta_1\|_1} \left\{ \frac{1}{\sqrt{n}} \frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2 \vee \|\tilde{\delta}_2\|_2} \right\} > 0$$

we have

$$\frac{1}{4} \kappa^2(s, 7) \|\delta_1\|_2^2 \leq \frac{1}{4} \kappa^2(s, 7) (\|\delta_1\|_2^2 \vee \|\tilde{\delta}_2\|_2^2) \leq \frac{1}{4n} \|\mathbf{X}\delta\|_2^2 \leq \underbrace{\frac{7}{4} \lambda_0 \|\delta_1\|_1}_{\text{Cauchy-Schwartz inequality}} \leq \frac{7}{4} \lambda_0 \sqrt{s} \|\delta_1\|_2$$

hence

$$\|\delta_1\|_2 \leq \frac{7\lambda_0\sqrt{s}}{\kappa^2(s, 7)} \quad \|\delta_1\|_1 \leq \sqrt{s} \|\delta_1\|_2 \leq \frac{7\lambda_0 s}{\kappa^2(s, 7)} \quad \|\delta'_2\|_2 \leq \frac{\sqrt{7\lambda_0\sqrt{s}} \|\delta_1\|_2}{\kappa(s, 7)} \quad (13.4)$$

Notice that the  $k$ -th largest absolute component of  $\delta_2$  is bounded from above by  $\|\delta_2\|_1/k$ , then for  $\delta_{2_s}$ , the subvector of  $\delta_2$  consisting of components **excluding** those with the  $s$  largest magnitudes, we have

$$\|\delta_{2_s}\|_2^2 \leq \sum_{k=s+1}^{p-s} \frac{1}{k^2} \|\delta_2\|_1^2 \leq s^{-1} \|\delta_2\|_1^2 \Rightarrow \|\delta_{2_s}\|_2 \leq \frac{1}{\sqrt{s}} \|\delta_2\|_1 \stackrel{13.2}{\leq} \frac{7}{\sqrt{s}} \|\delta_1\|_1 \stackrel{C-S}{\leq} 7 \|\delta_1\|_2$$

since  $\delta_{2_s}$  and  $\delta'_2$  are a partition of  $\delta$ , we have

$$\|\delta_2\|_2 \leq \|\delta_{2_s}\|_2 + \|\delta'_2\|_2 \leq 7 \|\delta_1\|_2 + \frac{\sqrt{7\lambda_0\sqrt{s}} \|\delta_1\|_2}{\kappa(s, 7)} \leq \frac{56\lambda_0\sqrt{s}}{\kappa^2(s, 7)} \quad (13.5)$$

Together, for the estimation loss  $\delta = \hat{\beta} - \beta_0$ , we have

- **$L_2$ -covar-loss-correlation:**  $\frac{1}{n} \|\mathbf{X}\delta\|_2^2 \leq 7\lambda_0 \|\delta_1\|_1 \leq \frac{(7\lambda_0)^2 s}{\kappa^2(s, 7)} \Rightarrow \frac{1}{\sqrt{n}} \|\mathbf{X}\delta\|_2 \leq \frac{7\lambda_0\sqrt{s}}{\kappa(s, 7)}$
- **$L_2$ -loss:**  $\|\delta\|_2 \leq \|\delta_1\|_2 + \|\delta_2\|_2 \leq \frac{63\lambda_0\sqrt{s}}{\kappa^2(s, 7)}$
- **$L_q$ -loss:**  $\|\delta\|_q \leq (s^{(2-q)/2} \|\delta_1\|_2^q)^{1/q} = s^{(2-q)/2q} \|\delta_1\|_2 \leq s^{(2-q)/2q} \frac{7\lambda_0\sqrt{s}}{\kappa^2(s, 7)} = \frac{7\lambda_0 s^{1/q}}{\kappa^2(s, 7)}$

←by Holder's inequality

Define the **number of falsely discovered signs** as<sup>2</sup>

$$\text{FS}(\hat{\beta}) = |\{j = 1, \dots, p : \text{sgn}(\hat{\beta}_j) \neq \text{sgn}(\beta_{0,j})\}|$$

we know from Prop.13.1.1 that  $|\hat{\beta}_j| > (1 - c_1)\lambda$  and from Condition 2 that  $|\beta_{0,j}| > (1 - c_1)\lambda$ , then if  $\text{sgn}(\hat{\beta}_j) \neq \text{sgn}(\beta_{0,j})$ , we must have  $|\delta_j| = |\hat{\beta}_j - \beta_{0,j}| \geq (1 - c_1)\lambda$ . Therefore, it follows that

$$\|\delta\|_2 \geq \left( \text{FS}(\hat{\beta}) \right)^{1/2} (1 - c_1)\lambda$$

hence

$$\text{FS}(\hat{\beta}) \leq \frac{\|\delta\|_2^2}{(1 - c_1)^2 \lambda^2} \leq \left( \frac{63}{1 - c_1} \right)^2 \left( \frac{\lambda_0}{\lambda} \right)^2 \frac{s}{\kappa^4(s, 7)}$$

The results above are all conditional on the event  $\mathcal{E} = \{\|n^{-1}\mathbf{X}'\epsilon\|_\infty \leq \lambda_0/2\}$ , hence hold simultaneously with probability  $1 - O(p^{-c_0})$ .

Altogether, we have the following theorem:

<sup>2</sup>Stronger than the total number of false positives and false negatives.

**Theorem 13.1.2: Properties of the Global Minimizer  $\hat{\beta}$** 

Assume that Condition 1 and 2 and the model error bound  $\Pr\left(\|n^{-1}\mathbf{X}'\epsilon\|_\infty > \frac{\lambda_0}{2}\right) = O(p^{-c_0})$ , and  $p_\lambda(t)$  is continuously differentiable. Then the global minimizer  $\hat{\beta}$  of 13.1 has the hard-thresholding property stated in Prop. 13.1.1, and, with probability  $1 - O(p^{-c_0})$ , satisfies simultaneously that

$$\frac{1}{\sqrt{n}} \|\mathbf{X}(\hat{\beta} - \beta_0)\|_2 = O(\kappa^{-1} \lambda_0 s^{1/2}) \quad (13.6)$$

$$\|\hat{\beta} - \beta_0\|_q = O(\kappa^{-2} \lambda_0 s^{1/q}), \quad q \in [1, 2] \quad (13.7)$$

$$\text{FS}(\hat{\beta}) = O\left(\kappa^{-4} \left(\frac{\lambda_0}{\lambda}\right)^2 s\right) \quad (13.8)$$

If in addition  $\lambda \geq \frac{56\lambda_0\sqrt{s}}{(1-c_1)\kappa^2}$ , then with probability  $1 - O(p^{-c_0})$ , we also have that

$$\text{sgn}(\hat{\beta}) = \text{sgn}(\beta_0) \quad \|\hat{\beta} - \beta_0\|_\infty = O\left(\lambda_0 \left\| \left(\frac{1}{n} \mathbf{X}_1' \mathbf{X}_1\right)^{-1} \right\|_\infty\right)$$

where  $\mathbf{X}_1$  is the  $n \times s$  submatrix of  $\mathbf{X}$  corresponding to  $s$  nonzero regression coefficients  $\beta_{0,j}$ .

The proof of the second part follows as such: by assuming  $\lambda \geq \frac{56\lambda_0\sqrt{s}}{(1-c_1)\kappa^2}$ , from Condition 2, we have  $\min_{j=1,\dots,s} |\beta_{0,j}| > \frac{56\lambda_0\sqrt{s}}{\kappa^2(s,7)}$ , combined with 13.4, we know that

$$\text{sgn}(\hat{\beta}_j) = \text{sgn}(\beta_{0,j}), \quad \forall j = 1, \dots, s$$

by a simple contradiction argument. In view of 13.5 and the hard-thresholding feature of  $\hat{\beta} = (\hat{\beta}'_{0,1}, \hat{\beta}'_{0,2})'$ , with  $\hat{\beta}_{0,1} = (\hat{\beta}_1, \dots, \hat{\beta}_s)'$ , a similar contradiction argument leads to  $\hat{\beta}_{0,2} = \mathbf{0}$ . Together, we have the sign consistency:  $\text{sgn}(\hat{\beta}) = \text{sgn}(\beta_0)$ . Under this result, applying Theorem 1 of Lv and Fan (2009), the estimation  $\hat{\beta}_{0,1}$  solves the following equation for  $\gamma \in \mathbb{R}^s$

$$\gamma = \tilde{\beta}_{0,1} - (n^{-1} \mathbf{X}_1' \mathbf{X}_1)^{-1} \mathbf{b}$$

where

- $\mathbf{X}_1$  is the  $n \times s$  submatrix of  $\mathbf{X}$  corresponding to the  $s$  non-zero regression coefficients  $\beta_{0,j}$
- $\mathbf{b} = \{\lambda_0 \mathbf{1}_s + p'_\lambda(|\gamma|)\} \circ \text{sgn}(\tilde{\beta}_{0,1}) - n^{-1} \mathbf{X}_1' \epsilon$ , with componentwise derivative and product.

From the concavity and monotonicity of  $p_\lambda(t)$  and Condition 2, we have

$$0 \leq p'_\lambda(t) \leq p'_\lambda\{(1-c_1)\lambda\} \leq \lambda_0/4$$

this gives that each component of  $\tilde{\beta}_{0,1}$  has magnitude larger than  $(1-c_1)\lambda$ . Since  $\|n^{-1} \mathbf{X}_1' \epsilon\|_\infty \leq \|n^{-1} \mathbf{X}' \epsilon\|_\infty \leq \frac{\lambda_0}{2}$  on the event  $\mathcal{E}$ , hence we have

$$\text{sgn}(\mathbf{b}) = \text{sgn}(\tilde{\beta}_{0,1}), \quad \frac{\lambda_0}{2} \leq \|\mathbf{b}\|_\infty \leq \frac{7\lambda_0}{4}$$

which completes the proof for Theorem 13.1.2.

**How to understand Theorem 13.1.2?**

- False sign rate  $\text{FS}(\hat{\beta}) = O\left(\kappa^{-4} \left(\frac{\lambda_0}{\lambda}\right)^2 s\right)$  is asymptotically vanishing when  $\lambda_0/\lambda \rightarrow 0$ , outperforming Lasso, whose false sign rate is generally bounded by  $O(\lambda_{\max})$  with  $\lambda_{\max}$  being the largest eigenvalue of Gram matrix  $n^{-1}\mathbf{X}'\mathbf{X}$ ; also outperforming concave method, whose false sign rate is generally of order  $O(1)$ . When **signal strength is stronger** and  **$\lambda$  is chosen suitably**, sign consistency is stronger as well.
- Convergence rates of  $\frac{1}{\sqrt{n}} \|\mathbf{X}(\hat{\beta} - \beta_0)\|_2$  and  $\|\hat{\beta} - \beta_0\|_q$  are the same as those in [Bickel et al. \(2009\)](#) for the  $L_1$ -component, and are consistent with the concave component of [Zhang and Zhang \(2012\)](#). The bounds  $O(\kappa^{-1} \lambda_0 s^{1/2})$ ,  $O(\kappa^{-2} \lambda_0 s^{1/q})$  depend only on the universal regularization parameter  $\lambda_0 = c \sqrt{\frac{\log p}{n}}$  for  $L_1$ -component, and are independent of  $\lambda$  for concave component.
- The  $L_\infty$ -bound  $\|\hat{\beta} - \beta_0\|_\infty = O\left(\lambda_0 \left\| \left(\frac{1}{n} \mathbf{X}'_1 \mathbf{X}_1\right)^{-1} \right\|_\infty\right)$  involves  $\left\| \left(\frac{1}{n} \mathbf{X}'_1 \mathbf{X}_1\right)^{-1} \right\|_\infty$ , which is bounded from above by  $\sqrt{s} \left\| \left(\frac{1}{n} \mathbf{X}'_1 \mathbf{X}_1\right)^{-1} \right\|_2 \leq \sqrt{s} \kappa_0^{-2}$  and can be **dimension-free** in certain scenarios.
- **Oracle property**: Under all conditions of Theorem 13.1.2 hold, and let  $\tilde{\beta}$  be the refitted least-squares estimator given by covariates in  $\text{supp}(\hat{\beta})$ , with  $\hat{\beta}$  being the estimator in Theorem 13.1.2. The with probability  $1 - O(p^{-c_0})$ ,  $\tilde{\beta}$  equals the oracle estimator, and has the oracle property if the oracle estimator is asymptotic normal.

**Theorem 13.1.3: Further Properties of the Global Minimizer  $\hat{\beta}$** 

Under the same regularity conditions, the global minimizer  $\hat{\beta}$  in Theorem 13.1.2 satisfies that  $\forall \tau > 0$

$$\mathbb{E} \left\{ \frac{1}{n} \left\| \mathbf{X}(\hat{\beta} - \beta_0) \right\|_2^2 \right\} = O \left( \kappa^{-2} \lambda_0^2 s + m_{2,\tau} + \gamma \lambda_0 p^{-c_0} \right) \quad (13.9)$$

$$\mathbb{E} \left\{ \left\| \hat{\beta} - \beta_0 \right\|_q^q \right\} = O \left[ \kappa^{-2q} \lambda_0^q s + (2-q) \lambda_0^{-1} m_{2,\tau} + (q-1) \lambda_0^{-2} m_{4,\tau} + ((2-q)\gamma + (q-1)\gamma^2) p^{-c_0} \right] \quad (13.10)$$

$$\mathbb{E} \left\{ \text{FS}(\hat{\beta}) \right\} = O \left[ \kappa^{-4} \left( \frac{\lambda_0}{\lambda} \right)^s s + \lambda^{-2} m_{2,\tau} + \left( \frac{\gamma \lambda_0}{\lambda^2} + s \right) p^{-c_0} \right] \quad (13.11)$$

where  $m_{q,\tau} = \mathbb{E}(|\epsilon_0|^q \mathbf{1}_{\{|\epsilon_0| > \tau\}})$  denotes the tail moment and  $\gamma = \|\beta_0\|_1 + s \lambda_0^{-1} p_\lambda(\infty) + \tau^2 \lambda_0^{-1}$ . If in addition  $\lambda \geq 56(1 - c_1)^{-1} \kappa^{-2} \lambda_0 \sqrt{s}$ , then we have

$$\begin{aligned} \mathbb{E} \left\{ \text{FS}(\hat{\beta}) \right\} &= O \left\{ \lambda^{-2} m_{2,\tau} + \left( \gamma \frac{\lambda_0}{\lambda^2} + s \right) p^{-c_0} \right\} \\ \mathbb{E} \left\{ \left\| \hat{\beta} - \beta_0 \right\|_\infty \right\} &= O \left\{ \lambda_0 \left\| (n^{-1} \mathbf{X}'_1 \mathbf{X}_1)^{-1} \right\|_\infty + \lambda_0^{-1} m_{2,\tau} + \gamma p^{-c_0} \right\} \end{aligned}$$

Again,  $\lambda_0$  enters all bounds for the oracle risk inequalities

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