Topic 13: Non-convex Learning + Lasso

by Sai Zhang

Key points: Combining the best of the two, we can use **Lasso plus Concave** method, with Lasso screening and concave component selecting variables, achieving a coordinated intrinsic two-scale learning.

Disclaimer: *The note is built on Prof. Jinchi Lv's lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.*

We are facing a tradeoff:

- **Convex** methods: have appealing prediction power and oracle inequalities, but challenging to provide tight false sign rate control
- **Concave** methods: have good variable selection properties, but challenging to establish global properties and risk properties

Here, we take advantage of the linearity of Lasso (convex *and* concave) and try to combine it with concave regularization to get the best of both.

13.1 Model Setup

Again, consider a linear regression model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where

- response vector ($n \times 1$): $\mathbf{y} = (y_1, \cdots, y_n)'$
- design matrix ($n \times p$): $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_p)$, with each column rescaled to have L_2-norm $n^{1/2}$

here, we consider a scenario where

- $\boldsymbol{\beta}_0 = (\beta_{0,1}, \cdots, \beta_{0,p})'$ is **sparse** (with many 0 components)
- ultra-**high** dimensions: $\log p = O(n^a)$, for some $0 < a < 1$

and consider the penalized least squares

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left\{ (2n)^{-1} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda_0 \|\boldsymbol{\beta}\|_1 + \|p_\lambda(\boldsymbol{\beta})\|_1 \right\} \tag{13.1}$$

where

- $\lambda_0 = c \left(\frac{\log p}{n} \right)^{1/2}$ for some $c > 0$
- $p_\lambda(\boldsymbol{\beta}) = p_\lambda(|\boldsymbol{\beta}|) = (p_\lambda(|\beta_1|), \cdots, p_\lambda(|\beta_p|))'$, with $|\boldsymbol{\beta}| = (|\beta_1|, \cdots, |\beta_p|)'$; the concave penalty $p_\lambda(t)$ is defined on $t \in [0, \infty)$, indexed by $\lambda \geq 0$, increasing in **both** t and λ, $p_\lambda(0) = 0$

the 2 penalty components

- L_1-component: minimum amount of regularization for *removing noise* in prediction
- concave component $\|p_\lambda(\boldsymbol{\beta})\|_1$: adapt model sparsity for *variable selection*

Under this set up, we can derive the hard-thresholding property as

> **Proposition 13.1.1: Hard-Thresholding Property**
>
> Assume the $p_\lambda(t)$, $t \geq 0$, is **increasing and concave** with
> - $p_\lambda(t) \geq p_{H,\lambda}(t) = \frac{1}{2}\left[\lambda^2 - (\lambda - t)_+^2\right]$ on $[0, \lambda]$
> - $p'_\lambda((1 - c_1)\lambda) \leq c_1\lambda$ for some $c_1 \in [0, 1)$
> - $-p''_\lambda(t)$ decreasing on $[0, (1 - c_1)\lambda]$
>
> then any <u>local minimizer</u> of 13.1 that is also a global minimizer in each coordinate has the **hard-thresholding** feature that each component is either 0 or of magnitude **larger** than $(1 - c_1)\lambda$

Such property is shared by a wide class of concave penalties, including hard-thresholding penalty $p_{H,\lambda}(t)$ with $c_1 = 0$, L_0−penalty, and SICA (with suitable c_1).

How to <u>understand</u> this proposition? Let $\hat{\beta} = (\hat{\beta}_1, \cdots, \hat{\beta}_p)'$, then **each $\hat{\beta}_j$** is the glocal minimizer of the corresponding univariate penalized least-square problem along the j−th coordinate. These univariate problems share a common form with (generally) different scalars z

$$\hat{\beta}(z) = \arg\min_{\beta \in \mathbb{R}} \left\{\frac{1}{2}(z - \beta)^2 + \lambda_0|\beta| + p_{H,\lambda}(|\beta|)\right\}$$

after we rescale all covariates to have L_2−norm $n^{1/2}$. The solution to these univariate problems are

$$\hat{\beta}(z) = \text{sgn}(z)(|z| - \lambda_0) \cdot \mathbf{1}_{|z| > \lambda + \lambda_0}$$

these solutions have the same feature as the hard-thresholded estimator: each component is either 0 or of magnitude larger than λ. This provides a better distinction between insignificant and significant covariates then soft-thresholding by L_1 penalty.

With the hard-thresholding property of Prop. 13.1.1, we can prove a basic constraint for the global optimum $\hat{\beta}$ on an event with significant probability (Fan and Lv, 2014)

$$\|\delta_2\|_1 \leq 7\|\delta_1\|_1 \tag{13.2}$$

where $\delta = \hat{\beta} - \beta_0 = \left(\hat{\beta}'_1, \hat{\beta}'_2\right)' - \left(\beta'_{0,1}, \beta'_{0,2}\right)' = (\delta'_1, \delta'_2)'$, with $\delta_1 \in \mathbb{R}^s$. Where does this constraint come from? For the penalized least square quesion 13.1

$$\min_{\beta \in \mathbb{R}^p} \left\{(2n)^{-1}\|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda_0\|\beta\|_1 + \|p_\lambda(\beta)\|_1\right\}$$

the global minimizer $\hat{\beta}$ leads to

$$
\begin{aligned}
(2n)^{-1}\|\mathbf{y} - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0\|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 &= (2n)^{-1}\|\mathbf{X}\beta_0 + \epsilon - \mathbf{X}\hat{\beta}\|_2^2 + \lambda_0\|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\
&= (2n)^{-1}\|\epsilon - \mathbf{X}(\hat{\beta} - \beta_0)\|_2^2 + \lambda_0\|\hat{\beta}\|_1 + \|p_\lambda(\hat{\beta})\|_1 \\
&\leq (2n)^{-1}\|\mathbf{y} - \mathbf{X}\beta_0\|_2^2 + \lambda_0\|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\
&= (2n)^{-1}\|\epsilon\|_2^2 + \lambda_0\|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1
\end{aligned}
$$

then, plug in $\delta = \hat{\beta} - \beta_0$, we get

$$
\begin{aligned}
(2n)^{-1}\|\epsilon - \mathbf{X}\delta\|_2^2 + \lambda_0\|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq (2n)^{-1}\|\epsilon\|_2^2 + \lambda_0\|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1 \\
(2n)^{-1}\|\mathbf{X}\delta\|_2^2 - n^{-1}\epsilon'\mathbf{X}\delta + \lambda_0\|\beta_0 + \delta\|_1 + \|p_\lambda(\beta_0 + \delta)\|_1 &\leq \lambda_0\|\beta_0\|_1 + \|p_\lambda(\beta_0)\|_1
\end{aligned}
$$

since $\beta_{0,2} = \mathbf{0}$, $\delta_2 = \beta_{0,2} + \delta_2$, we have

$$\|\beta_0 + \delta\|_1 = \|\beta_{0,1} + \beta_{0,2} + \delta_1 + \delta_2\|_1 = \|\beta_{0,1} + \delta_1 + \delta_2\|_1 \le \|\beta_{0,1} + \delta_1\|_1 + \|\delta_2\|_1$$

hence

$$(2n)^{-1}\|\mathbf{X}\delta\|_2^2 - n^{-1}\epsilon'\mathbf{X}\delta + \lambda_0\|\delta_2\|_1 \le \lambda_0\|\beta_{0,1}\|_1 - \lambda_0\|\beta_{0,1} + \delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

and by the reverse triangle inequality $\|\beta_{0,1}\|_1 - \|\beta_{0,1} + \delta_1\|_1 \le \|\delta_1\|_1$, we get

$$(2n)^{-1}\|\mathbf{X}\delta\|_2^2 - n^{-1}\epsilon'\mathbf{X}\delta + \lambda_0\|\delta_2\|_1 \le \lambda_0\|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1$$

If assume the distribution of the model error ϵ as

$$\Pr\left(\left\|n^{-1}\mathbf{X}'\epsilon\right\|_\infty > \frac{\lambda_0}{2}\right) = O\left(p^{-c_0}\right)$$

conditional on the event $\mathcal{E} = \left\{\left\|n^{-1}\mathbf{X}'\epsilon\right\|_\infty \le \lambda_0/2\right\}$, we have

$$-n^{-1}\epsilon'\mathbf{X}\delta + \lambda_0\|\delta_2\|_1 - \lambda_0\|\delta_1\|_1 \ge -\frac{\lambda_0}{2}\|\delta\|_1 + \lambda_0\|\delta_2\|_1 - \lambda_0\|\delta_1\|_1 = \frac{\lambda_0}{2}\|\delta_2\|_1 - \frac{3\lambda_0}{2}\|\delta_1\|_1$$

plug this result back, get

$$\frac{1}{2n}\|\mathbf{X}\delta\|_2^2 + \frac{\lambda_0}{2}\|\delta_2\|_1 \le \frac{3\lambda_0}{2}\|\delta_1\|_1 + \|p_\lambda(\beta_0)\|_1 - \|p_\lambda(\beta_0 + \delta)\|_1 \tag{13.3}$$

Now, if we further impose 2 conditions:

- **eigenvalue condition**: for some positive constant κ_0

$$\min_{\|\delta\|_2=1,\|\delta\|_0<2s} \frac{1}{\sqrt{n}}\|\mathbf{X}\delta\|_2 \ge \kappa_0 \tag{A}$$

$$\kappa = \kappa(s,7) = \min_{\delta\ne\mathbf{0},\|\delta_2\|_1\le7\|\delta_1\|_1}\left\{\frac{1}{\sqrt{n}}\frac{\|\mathbf{X}\delta\|_2}{\|\delta_1\|_2 \vee \|\tilde\delta_2\|_2}\right\} > 0 \tag{B}$$

where $\tilde\delta_2$ is the subvector of δ_2 consisting of the components with the s **largest** absolute values. Here

- Condition **(A)** is a mild sparse eigenvalue condition
- Condition **(B)** combines the restricted eigenvalue assumptions in Bickel et al. (2009)[1]. The intuition is, for OLS estimation, $\mathbf{X}'\mathbf{X}$ should be **positive definite**, that is

$$\min_{\mathbf{0}\ne\delta\in\mathbb{R}^p}\left\{\frac{1}{\sqrt{n}}\frac{\|\mathbf{X}\delta\|_2}{\|\delta\|_2}\right\} > 0$$

however, when $p > n$, this condition **never** holds, hence we replace $\|\delta\|_2$ with the L_2−norm of $\|\delta_1\|_2$, a subvector of δ

$$\kappa = \kappa(s,7)\min_{\delta\ne\mathbf{0},\|\delta_2\|_1\le7\|\delta_1\|_1}\left\{\frac{1}{\sqrt{n}}\frac{\|\mathbf{X}\delta\|_2}{\|\delta\|_2}\right\} > 0$$

and for L_q loss with $q \in (1,2]$, we further bound $\|\tilde\delta_2\|_2$, which leads to condition **(B)**.

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[1]Introduced by Candes and Tao (2007) for studying the oracle inequalities for the Lasso estimator and Dantzig selector.

- **hard-thresholding condition**: The penalty $p_\lambda(t)$ satisfies the conditions of Prop. 13.1.1 with

$$p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$$

$$\min_{j=1,\cdots,s} |\beta_{0,j}| > \max\left\{(1 - c_1)\lambda, 2\kappa_0^{-1}p_\lambda^{1/2}(\infty)\right\}$$

Now, look back at the condition 13.3, we can upper-bound $\|p_\lambda(\boldsymbol{\beta}_0)\|_1 - \|p_\lambda(\boldsymbol{\beta}_0 + \boldsymbol{\delta})\|_1$ by $\frac{1}{4n}\|\mathbf{X}\boldsymbol{\delta}\|_2^2 + \frac{1}{4}\lambda_0\|\boldsymbol{\delta}\|_1$. Consider 2 cases:

- **Case 1**: $\|\hat{\boldsymbol{\beta}}\|_0 \geq s$. By the hard-thresholding condition, we have $|\beta_{0,j}| > (1-c_1)\lambda$ and $p'_\lambda \{(1 - c_1)\lambda\} \leq \lambda_0/4$. Hence, for $j = 1, \cdots, s$,

  - if $\hat{\beta}_j \neq 0$, we must have $|\hat{\beta}_j| > (1 - c_1)\lambda$. And by the <u>mean-value theorem</u>, we have

  $$|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| = p'_\lambda(b)(|\hat{\beta}_j| - |\beta_{0,j}|) \leq p'_\lambda(b)|\delta_{0,j}|$$

  where b is between $|\beta_{0,j}|$ and $|\hat{\beta}_j|$, hence, $b > |\beta_{0,j}| > (1 - c_1)\lambda$, by the concavity of p_λ, we have $p'(b) < p'((1 - c_1)\lambda) \leq \lambda_0/4$, which leads to $|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_j|)| \leq \frac{1}{4}\lambda_0|\delta_j|$.

  - if $\hat{\beta}_j = 0$, since $\|\hat{\boldsymbol{\beta}}_0\| \geq s$, there must exist some $j' > s$ s.t. $\hat{\beta}_{j'} \neq 0$, similarly

  $$|p_\lambda(|\beta_{0,j}|) - p_\lambda(|\hat{\beta}_{j'}|)| \leq |p_\lambda\left(|\beta_{0,j}|\right) - p_\lambda\left((1 - c_1)\lambda\right)| + |p_\lambda\left(|\hat{\beta}_{j'}|\right) - p_\lambda\left((1 - c_1)\lambda\right)|$$

  $$= p'_\lambda(b_1)\left(|\beta_{0,j}| - (1 - c_1)\lambda\right) + p'_\lambda(b_2)\left(|\hat{\beta}_{j'}| - (1 - c_1)\lambda\right)$$

  $$\leq p'_\lambda(b_1)\left(|\beta_{0,j}| - \underbrace{|\hat{\beta}_j|}_{=0}\right) + p'_\lambda(b_2)\left(|\hat{\beta}_{j'}| - \underbrace{|\beta_{0,j'}|}_{=0}\right)$$

  $$= p'_\lambda(b_1)|\delta_j| + p'_\lambda(b_2)|\delta_{j'}| \leq \frac{\lambda_0}{4}\left(|\delta_j| + |\delta_{j'}|\right)$$

  together, we have

  $$\|p_\lambda(\boldsymbol{\beta}_0)\|_1 - \|p_\lambda(\boldsymbol{\beta}_0 + \boldsymbol{\delta})\|_1 \leq \frac{1}{4}\lambda_0\|\boldsymbol{\delta}\|_1 \leq \frac{1}{4n}\|\mathbf{X}\boldsymbol{\delta}\|_2^2 + \frac{1}{4}\lambda_0\|\boldsymbol{\delta}\|_1$$

- **Case 2**: $\|\hat{\boldsymbol{\beta}}\|_0 = s - k$ for some $k \geq 1$. Then we must have $\|\boldsymbol{\delta}\|_0 \leq \|\hat{\boldsymbol{\beta}}\|_0 + \|\boldsymbol{\beta}_0\|_0 \leq s - k + s < 2s$, and $\|\boldsymbol{\delta}\|_2 \geq \sqrt{k}\min_{j=1,\cdots,s}|\beta_{0,j}|$. Also, there are at least k null estimates ($\hat{\beta}_j = 0$), thus

$$\underbrace{\frac{1}{4n}\|\mathbf{X}\boldsymbol{\delta}\|_2^2 \geq \frac{\kappa_0^2}{4}\|\boldsymbol{\delta}\|_2^2}_{\text{Condition 1(A)}} \geq \frac{\kappa_0^2}{4}$$

References

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