

## Topic 17: False Discovery Rate (FDR) and Knockoffs

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**Key points:** Constructing knockoff variables to control FDR when estimating regression coefficients.

**Disclaimer:** The note is built on Prof. [Jinchi Lv](#)'s lectures of the course at USC, DSO 607, High-Dimensional Statistics and Big Data Problems.

### 17.1 Motivation

Consider the classical linear regression setting

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where  $\boldsymbol{\beta} \in \mathbb{R}^p$  is the unknown vector of coefficients and  $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ . In a high-dimensional problem, we would like to just select a subset of all variables  $\hat{S} \subset \{1, \dots, p\}$  s.t. conditional on  $\{\mathbf{X}_j\}_{j \in \hat{S}}$ ,  $\mathbf{y}$  is **independent** of all other variables, we can define the **False Discovery Rate (FDR)** in can be defined as

#### Definition 17.1.1: False Discovery Rate (FDR)

$$\text{FDR} = \mathbb{E}(\text{FDP}) = \mathbb{E} \left[ \frac{|\hat{S} \cap \mathcal{H}_0|}{|\hat{S}|} = \frac{\#\{j : j \in \hat{S} \setminus S\}}{\#\{j : j \in \hat{S}\}} \right]$$

where  $\mathcal{H}_0 \subset \{1, \dots, p\}$  is the set of **null** variables:  $\mathbf{X}_j$  is **null** iff  $\mathbf{Y}$  is independent of  $\mathbf{X}_j$  conditional on the other variables  $\mathbf{X}_{-j} = \{\mathbf{X}_1, \dots, \mathbf{X}_p\} \setminus \{\mathbf{X}_j\}$ .

In this note, we consider a series of knockoff-based methods to control FDR. They all follow a common procedure:

- **Step 1:** Construct Knockoffs
- **Step 2:** Calculate test statistics for both original and knockoff variables
- **Step 3:** Calculate a threshold for the test statistics, controlling for a desired FDR level
- **Step 4:** Select variables that pass the threshold

### 17.2 Barber and Candès (2015)

**Constructing the knockoffs** [Barber and Candès \(2015\)](#) construct the knockoffs by the following procedure

- Calculate the Gram matrix  $\boldsymbol{\Sigma} = \mathbf{X}'\mathbf{X}$  for the normalized original variables, where  $\Sigma_{jj} = \|\mathbf{X}_j\|_2^2 = 1$

- Construct the knockoffs  $\tilde{\mathbf{X}}$  s.t.

$$\tilde{\mathbf{X}}'\tilde{\mathbf{X}} = \Sigma \qquad \mathbf{X}'\tilde{\mathbf{X}} = \Sigma - \text{diag}\{\mathbf{s}\}$$

where  $\mathbf{s} \in \mathbb{R}_+^p$  is a  $p$ -dimensional non-negative vector (larger  $s_j$  indicates higher power) and

- $\tilde{\mathbf{X}}$  exhibits the **same** covariance structure as the original design  $\mathbf{X}$
- The correlation between distinct original variables and knockoffs are the same as between the originals:

$$\mathbf{X}'_j\tilde{\mathbf{X}}_k = \mathbf{X}'_j\mathbf{X}_k, \quad \forall j \neq k$$

- The correlation between the original variables and their own knockoffs is **less than 1**

$$\mathbf{X}'_j\tilde{\mathbf{X}}_j = \Sigma_{jj} - s_j = 1 - s_j$$

To construct such knockoffs,

- Given a proper  $\mathbf{s}$ , if  $n \geq 2p$ , then

$$\tilde{\mathbf{X}} = \mathbf{X}(\mathbf{I} - \Sigma^{-1}\text{diag}\{\mathbf{s}\}) + \tilde{\mathbf{U}}\mathbf{C}$$

where  $\tilde{\mathbf{U}} \in \mathbb{R}^{n \times p}$  is an **orthonormal** matrix s.t.  $\tilde{\mathbf{U}}'\mathbf{X} = \mathbf{0}$  and  $\mathbf{C}'\mathbf{C} = 2\text{diag}\{\mathbf{s}\} - \text{diag}\{\mathbf{s}\}\Sigma^{-1}\text{diag}\{\mathbf{s}\} \geq \mathbf{0}$

- A sufficient and necessary condition for  $\tilde{\mathbf{X}}$  to exist:  $\text{diag}\{\mathbf{s}\} \leq 2\Sigma$

2 types of knockoffs can be constructed, following these procedures

T1 **Equi-correlated** knockoffs: set  $s_j = 2\lambda_{\min}(\Sigma) \wedge 1$  for all  $j$ , then  $\langle \mathbf{X}_j, \tilde{\mathbf{X}}_j \rangle = 1 - 2\lambda_{\min}(\Sigma) \wedge 1$  for all  $j$ . This is essentially minimizing  $|\langle \mathbf{X}_j, \tilde{\mathbf{X}}_j \rangle|$

T2 **SDP** knockoffs: solve the convex problem

$$\arg \min_{\mathbf{s}} \sum_j (1 - s_j) \qquad \text{s.t. } 0 \leq s_j \leq 1, \text{diag}\{\mathbf{s}\} \leq 2\Sigma$$

which is essentially minimizing the average of  $\langle \mathbf{X}_j, \tilde{\mathbf{X}}_j \rangle$

**Calculate test statistics** Define and calculate test statistics  $W_j$  for each  $\beta_j \in \{1, \dots, p\}$  using  $[\mathbf{X} \quad \tilde{\mathbf{X}}]$ :

- the test statistic  $W_j$  should be constructed s.t. large positive values are evidence against the null hypothesis  $\beta_j = 0$ , for example, consider a Lasso on  $[\mathbf{X} \quad \tilde{\mathbf{X}}]$

$$\hat{\beta}(\lambda) = \arg \min_{\mathbf{b}} \left\{ \frac{1}{2} \|\mathbf{y} - [\mathbf{X} \quad \tilde{\mathbf{X}}] \mathbf{b}\|_2^2 + \lambda \|\mathbf{b}\|_1 \right\}$$

where  $\lambda$  is the point on the Lasso path at which the feature enters the model as

$$Z_j = \sup \{ \lambda : \hat{\beta}_j(\lambda) \neq 0 \}$$

$$\text{and set } W_j = (Z_j \vee \tilde{Z}_j) \cdot \begin{cases} +1, & Z_j > \tilde{Z}_j \\ -1, & Z_j < \tilde{Z}_j \end{cases}$$

- In general, the statistics  $W$  should satisfy the **sufficient** property and **anti-symmetry** property:

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<sup>1</sup>Other choices of  $W_j$  are  $W_j = |\mathbf{X}'_j\mathbf{y}| - |\tilde{\mathbf{X}}'_j\mathbf{y}|$ , or  $|\hat{\beta}_j^{\text{LS}}| - |\hat{\beta}_{j+p}^{\text{LS}}|$

**Definition 17.2.1: Property of Test Statistics  $W_j$** 

The test statistic  $W_j$  is said to obey

- the **sufficient** property if  $\mathbf{W}$  depends only on the Gram matrix and on feature-response inner products, that is

$$\mathbf{W} = f\left([\mathbf{X} \ \tilde{\mathbf{X}}]' [\mathbf{X} \ \tilde{\mathbf{X}}], [\mathbf{X} \ \tilde{\mathbf{X}}]' \mathbf{y}\right)$$

- the **antisymmetry** property if swapping the original  $\mathbf{X}_j$  and its knockoff  $\tilde{\mathbf{X}}_j$  has the effect of **switching the sign** of  $W_j$ , that is

$$W_j(Z_j, \tilde{Z}_j) = -W_j(\tilde{Z}_j, Z_j)$$

**Calculate a threshold for the test statistics** After defining the test statistic, we then

- Let  $q$  be the target FDR, define the data-dependent threshold  $T$  as

$$T = \min \left\{ t \in \mathcal{W} : \frac{\#\{j : W_j \leq -t\}}{\#\{j : W_j \geq t\} \vee 1} \leq q \right\}$$

where  $\mathcal{W} = \{|W_j| : j = 1, \dots, p\} \setminus \{0\}$  is the set of unique non-zero values attained by  $|W_j|$ 's.

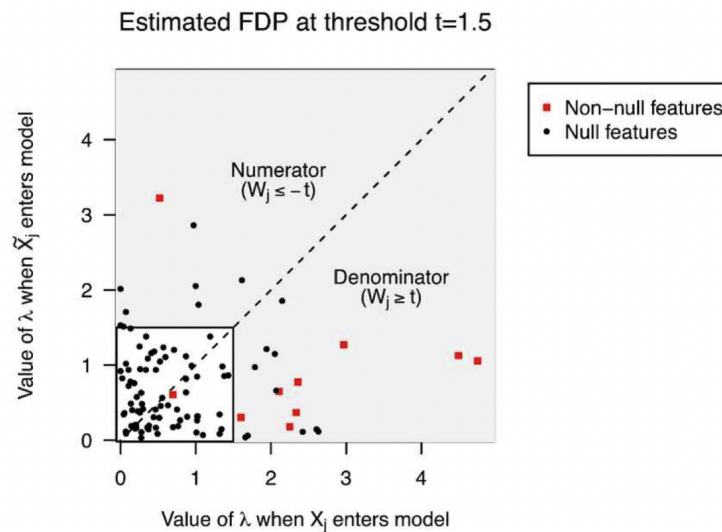


Figure 17.1: Visualizing Test Statistic Thresholding

**Variable selection** after building the threshold,

- for each  $j = 1, \dots, p$ , reject  $H_{0,j} : \beta_j = 0$  if  $W_j \geq T$ , the knockoff filter selects the model

$$\hat{S} = \{j : W_j \geq T\}$$

### 17.2.1 Intuition and Theory

#### Why knockoffs work?

- $\mathbf{W}$  is constructed (**antisymmetry** and **sufficiency**) such that the signs of the  $W_j$ 's are i.i.d. random for the null
- for any threshold  $t$ , we have

$$\#\{j : \beta_j = 0, W_j \geq t\} \stackrel{d}{=} \#\{j : \beta_j = 0, W_j \leq -t\}$$

, and the false discovery proportion (FDP) can be estimated as

$$\begin{aligned} \frac{\#\{j : \beta_j = 0, W_j \geq t\}}{\max(\#\{j : W_j \geq t\}, 1)} &\simeq \frac{\#\{j : \beta_j = 0, W_j \leq -t\}}{\max(\#\{j : W_j \geq t\}, 1)} \\ &\leq \frac{\#\{j : W_j \leq -t\}}{\max(\#\{j : W_j \geq t\}, 1)} := \widehat{\text{FDP}}(t) \end{aligned}$$

then the knockoff procedure can be interpreted as finding a threshold via  $T = \min \left\{ t \in \mathcal{W} : \widehat{\text{FDR}}(t) \leq q \right\}$

The knockoff procedure essentially controls a quantity **nearly equal** to the FDR. To control the FDR **exactly**, we have, **textbfknockoff+**, a more conservative modification of the knockoff procedure, where the threshold is

$$T = \min \left\{ t \in \mathcal{W} : \frac{1 + \#\{j : W_j \leq -t\}}{\max(\#\{j : W_j \geq t\}, 1)} \leq q \right\}$$

the **+1** part makes it harder to reject the null:

$$\begin{aligned} \text{FDP} &= \frac{\#\{j : \beta_j = 0, W_j \geq -T\}}{\#\{j : W_j \geq T\} \vee 1} \cdot \frac{1 + \#\{j : \beta_j = 0, W_j \leq -T\}}{1 + \#\{j : \beta_j = 0, W_j \leq -T\}} \\ &\leq \frac{1 + \#\{j : W_j \leq -T\}}{\#\{j : W_j \geq T\} \vee 1} \cdot \frac{\#\{j : \beta_j = 0, W_j \geq T\}}{1 + \#\{j : \beta_j = 0, W_j \leq -T\}} \\ &\leq q \cdot 1 \end{aligned}$$

Then, we have the following theorem

#### Theorem 17.2.2: Property of the Knockoff Method

For any  $q \in [0, 1]$ , the **knockoff** method satisfies

$$\mathbb{E} \left[ \frac{\#\{j : \beta_j = 0, j \in \hat{S}\}}{\#\{j : j \in \hat{S}\} + q^{-1}} \right] \leq q$$

and the **knockoff+** method satisfies

$$\mathbb{E} \left[ \frac{\#\{j : \beta_j = 0, j \in \hat{S}\}}{\#\{j : j \in \hat{S}\}} \right] \leq q$$

in both cases, the expectation is taken over the Gaussian noise in the model, while treating original variables  $\mathbf{X}$  and knockoffs  $\tilde{\mathbf{X}}$  as fixed

### 17.3 Candès et al. (2018)

Another way of constructing knockoffs, introduced by Candès et al. (2018), is by a swapping method:

**Constructing the knockoffs** for the family of random variables  $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_p)$  are a new family of random variables  $\tilde{\mathbf{X}} = (\tilde{\mathbf{X}}_1, \dots, \tilde{\mathbf{X}}_p)$  constructed with the following 2 properties

- for any subset  $S \subset \{1, \dots, p\}$ ,

$$(\mathbf{X}, \tilde{\mathbf{X}})_{\text{swap}(S)} \stackrel{d}{=} (\mathbf{X}, \tilde{\mathbf{X}})$$

- $\tilde{\mathbf{X}} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{X}$  if there is a response  $\mathbf{Y}$

Suppose  $\mathbf{X} \sim \mathcal{N}(0, \Sigma)$ , then  $(\mathbf{X}, \tilde{\mathbf{X}})_{\text{swap}(S)}$  satisfies  $(\mathbf{X}, \tilde{\mathbf{X}})_{\text{swap}(S)} \stackrel{d}{=} (\mathbf{X}, \tilde{\mathbf{X}})$  if

$$(\mathbf{X}, \tilde{\mathbf{X}})_{\text{swap}(S)} \stackrel{d}{=} (\mathbf{X}, \tilde{\mathbf{X}}) \sim \mathcal{N}(0, \mathbf{G}), \quad \text{where } \mathbf{G} = \begin{pmatrix} \Sigma & \Sigma - \text{diag}(s) \\ \Sigma - \text{diag}(s) & \Sigma \end{pmatrix}$$

where  $\text{diag}(s)$  is any **diagonal matrix** s.t.  $\mathbf{G}$  is **positive semidefinite**. The knockoffs constructed this way are named **MX knockoffs**. For  $\mathbf{P}$ , the permutation matrix encoding the swap,

$$\mathbf{PGP} = \mathbf{G}$$

then we can sample the knockoff vector  $\tilde{\mathbf{X}}$  from the conditional distribution

$$\tilde{\mathbf{X}} \mid \mathbf{X} \stackrel{d}{=} \mathcal{N}(\mu, \mathbf{V})$$

where

$$\begin{aligned} \mu &= \mathbf{X} - \mathbf{X}\Sigma^{-1}\text{diag}(s) \\ \mathbf{V} &= 2\text{diag}(s) - \text{diag}(s)\Sigma^{-1}\text{diag}(s) \end{aligned}$$

An important lemma is

#### Lemma 17.3.1: MX Knockoff Construction

For **MX knockoffs**, swapping **null** covariates with their knockoffs would **not** change the joint distribution of the original covariate  $\mathbf{X}$  and their knockoffs  $\tilde{\mathbf{X}}$ , conditional on the response  $\mathbf{Y}$ : Take any subset  $S \subset \mathcal{H}_0$  of nulls, then

$$(\mathbf{X}, \tilde{\mathbf{X}}) \mid \mathbf{y} \stackrel{d}{=} (\mathbf{X}, \tilde{\mathbf{X}})_{\text{swap}(S)} \mid \mathbf{y}$$

and this leads to

#### Proposition 17.3.2: Conditional Exchangeability of MX Knockoffs

The random variables  $(\tilde{\mathbf{X}}_1, \dots, \tilde{\mathbf{X}}_p)$  are **MX knockoffs** for  $(\mathbf{X}_1, \dots, \mathbf{X}_p)$  if and only if for any  $j \in \{1, \dots, p\}$ , the pair  $(\mathbf{X}_j, \tilde{\mathbf{X}}_j)$  is **exchangeable** conditional on all the other variables and their knockoffs.

under Prop.17.3.2, we can use the following algorithm to construct the MX Knockoffs

**Algorithm 17.3.3: Sequential Conditional Independent Pairs**

```

j = 1
while j ≤ p do
  sample  $\tilde{X}_j$  from  $\mathcal{L}(X_j \mid X_{-j}, \tilde{X}_{1:j-1})$ 
  j = j + 1
enda

```

<sup>a</sup>Example with  $p = 3$

- $j = 1$ : sample  $\tilde{X}_1$  from  $\mathcal{L}(X_1 \mid X_{2:3})$
- $j = 2$ : sample  $\tilde{X}_2$  from  $\mathcal{L}(X_2 \mid X_1, X_3, \tilde{X}_1)$
- $j = 3$ : sample  $\tilde{X}_3$  from  $\mathcal{L}(X_3 \mid X_{1:2}, \tilde{X}_{1:2})$

And an **approximate** construction can be achieved via matching the first 2 moments of  $(X, \tilde{X})_{\text{swap}(S)}$  and  $(X, \tilde{X})$ ,

$$\text{cov}(X, \tilde{X}) = G \quad G = \begin{pmatrix} \Sigma & \Sigma - \text{diag}(s) \\ \Sigma - \text{diag}(s) & \Sigma \end{pmatrix}$$

which can be achieved through 2 ways:

- **equicorrelated** construction

$$s_j^{\text{EQ}} = 2\lambda_{\min}(\Sigma) \wedge 1, \forall j$$

minimizing the **correlation between variable knockoff pairs** subject to the constraint that all such pairs must have the same correlation

- **semidefinite programme** construction

$$\begin{aligned} & \text{minimize} && \sum_j |1 - s_j^{\text{SDP}}| \\ & \text{subject to} && s_j^{\text{SDP}} \geq 0, \text{diag}(s^{\text{SDP}}) \leq 2\Sigma \end{aligned}$$

minimizing the sum of the absolute values of variable knockoff correlations between all suitable s

in high-dimensional situation

**Calculate test statistics** After constructing the knockoffs, we can construct the feature importance statistics by imposing a **flip sign** property: swapping the  $j$ th variable with its knockoff has the effect of changing the sign of  $W_j$

$$w_j \{ (X, \tilde{X})_{\text{swap}(S)}, y \} = \begin{cases} w_j \{ (X, \tilde{X}), y \}, & j \notin S \\ -w_j \{ (X, \tilde{X}), y \}, & j \in S \end{cases}$$

consider a statistic  $T$  for each original and knockoff variable

$$T \triangleq (Z, \tilde{Z}) = (Z_1, \dots, Z_p, \tilde{Z}_1, \dots, \tilde{Z}_p) = t \{ (X, \tilde{X}), y \}$$

if the components of  $T$  are switched in the same way:

$$(Z, \tilde{Z})_{\text{swap}(S)} = t \{ (X, \tilde{X})_{\text{swap}(S)}, y \}$$

then the **flip sign** property can be achieved by setting

$$W_j = f_j(Z_j, \tilde{Z}_j)$$

where  $f_j$  is any **antisymmetric** function  $f(v, u) = -f(u, v)$ .

**Lemma 17.3.4: Feature Statistics: Lasso Coefficient Difference (LCD)**

Consider the Lasso **augmented with knockoffs**

$$\min_{b \in \mathbb{R}^{2p}} \frac{1}{2} \|y - (\mathbf{X}, \tilde{\mathbf{X}})b\|_2^2 + \lambda \|b\|_1$$

which has solution  $\hat{b}(\lambda) = (\hat{b}_1(\lambda), \dots, \hat{b}_p(\lambda), \hat{b}_{p+1}(\lambda), \dots, \hat{b}_{2p}(\lambda))$ , then the statistic can be constructed as

$$W_j = Z_j - \tilde{Z}_j = |\hat{b}_j(\lambda)| - |\hat{b}_{j+p}(\lambda)|$$

and conditional on  $(|W_1|, \dots, |W_p|)$ , the sign of the null  $W_j$ s ( $j \in \mathcal{H}_0$ ) are i.i.d. coin flips<sup>a</sup>.

<sup>a</sup>Proof: for a sequence independent random variables  $\epsilon = (\epsilon_1, \dots, \epsilon_p)$  s.t.  $\epsilon_j = \pm 1$  with probability  $\frac{1}{2}$  if  $j \in \mathcal{H}_0$ , and  $\epsilon_j = 1$  otherwise, put  $S = \{j : \epsilon_j = -1\} \subset \mathcal{H}_0$

- flip sign property:  $W_{\text{swap}(S)} \stackrel{\Delta}{=} w \left\{ (\mathbf{X}, \tilde{\mathbf{X}})_{\text{swap}(S)}, \mathbf{y} \right\} \stackrel{\Delta}{=} \epsilon \odot W = (\epsilon_1 W_1, \dots, \epsilon_p W_p)$

- Lemma 17.3.1:  $W_{\text{swap}(S)} \stackrel{d}{=} W$

which establishes  $W \stackrel{d}{=} \epsilon \odot W$

- a large positive value of  $W_j$  provides some evidence that the distribution of  $\mathbf{Y}$  depends on  $\mathbf{X}_j$
- value of  $\lambda$  can be chosen in any data-dependent fashion for a pair of  $\mathbf{y}$  and  $(\mathbf{X}, \tilde{\mathbf{X}})$

Why **i.i.d. coin flips**? the null  $W_j$ s are **symmetric**

$$\# \{j : W_j \leq -t, j \in \mathcal{H}_0\} \stackrel{d}{=} \# \{j : W_j \geq t, j \in \mathcal{H}_0\}$$

and for any fixed threshold  $t > 0$

$$\# \{j : W_j \leq -t\} \geq \# \{j : W_j \leq -t, j \in \mathcal{H}_0\}$$

so for the false discovery proportion (FDP)

$$\text{FDP}(t) = \frac{\# \{j : W_j \geq t, j \in \mathcal{H}_0\}}{\# \{j : W_j \geq t\}}$$

an **upward-biased** estimate is

$$\widehat{\text{FDP}}(t) = \frac{\# \{j : W_j \leq -t\}}{\# \{j : W_j \geq t\}}$$

then Theorem 17.2.2 applies.

## References

- Rina Foygel Barber and Emmanuel J. Candès. Controlling the false discovery rate via knockoffs. *Annals of Statistics*, 43(5):2055–2085, 2015.
- Emmanuel J Candès, Jianqing Fan, Lucas Janson, and Jinchi Lv. Panning for gold: ‘model- $x$ ’ knockoffs for high dimensional controlled variable selection. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80(3):551–577, 2018.