

## Topic 5: Two-Way Cluster-Robust (TWCR) Standard Errors

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**Key points:** The validity of Two-Way Cluster-Robust (TWCR) standard errors

**Disclaimer:** This note is compiled by Sai Zhang.

### 5.1 One-Way Clustering

First, consider the case of one-way clustering. The linear model with one-way clustering

$$y_{ig} = \mathbf{x}_{ig}\boldsymbol{\beta} + u_{ig}$$

where  $i$  denotes the  $i$ th of the  $N$  individuals in the sample,  $j$  denotes the  $g$ th of the  $G$  clusters, assume that

- $\mathbb{E}[u_{ig} | \mathbf{x}_{ig}] = 0$
- error independence across clusters: for  $i \neq j$

$$\mathbb{E}[u_{ig}u_{jg'} | \mathbf{x}_{ig}, \mathbf{x}_{jg'}] = 0 \quad (5.1)$$

unless  $g = g'$ , that is, errors for individuals within the same cluster may be correlated.

Grouping observations by cluster, get

$$\mathbf{y}_g = \mathbf{X}_g\boldsymbol{\beta} + \mathbf{u}$$

where  $\mathbf{X}_g$  has dimension  $N_g \times K$  and  $\mathbf{y}_g$  has dimension  $N_g \times 1$ , with  $N_g$  observations in cluster  $g$ . Stacking over cluster, get the matrix form of the model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

with  $\mathbf{y}, \mathbf{u}$  being  $N \times 1$  vectors,  $\mathbf{X}$  being an  $N \times K$  matrix. OLS estimator gives

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y} = \left( \sum_{g=1}^G \mathbf{X}_g' \mathbf{X}_g \right)^{-1} \sum_{g=1}^G \mathbf{X}_g' \mathbf{y}_g \quad (5.2)$$

then, by CLT, we have that  $\sqrt{G}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma})$  where the variance matrix of the limit normal distribution  $\boldsymbol{\Sigma}$  is

$$\left( \lim_{G \rightarrow \infty} \frac{1}{G} \sum_{g=1}^G \mathbb{E}[\mathbf{X}_g' \mathbf{X}_g] \right)^{-1} \left( \lim_{G \rightarrow \infty} \frac{1}{G} \sum_{g=1}^G \mathbb{E}[\mathbf{X}_g' \mathbf{u}_g' \mathbf{u}_g \mathbf{X}_g] \right) \times \left( \lim_{G \rightarrow \infty} \frac{1}{G} \sum_{g=1}^G \mathbb{E}[\mathbf{X}_g' \mathbf{X}_g] \right)^{-1} \quad (5.3)$$

If the primary source of clustering is due to group-level common shocks, a useful approximation is that for the  $j$ th regressor, the default OLS variance estimate based on  $s^2(\mathbf{X}'\mathbf{X})^{-1}$  should be inflated by  $\tau_j \approx 1 + \rho_{x_j}\rho_u(\bar{N}_g - 1)$ , where

- $s$  is the estimated standard deviation of the error

- $\rho_{x_j}$  is a measure of within-cluster correlation of  $x_j$
- $\rho_u$  is the within-cluster error correlation
- $\bar{N}_g$  is the average cluster size

It's easy to see the  $\tau_j$  can be large even with small  $\rho_u$  (Kloek, 1981; Scott and Holt, 1982; Moulton, 1990). If assume the model for the cluster error variance matrices  $\mathbf{\Omega}_g = \mathbb{V}[\mathbf{u}_g | \mathbf{X}_g] = \mathbb{E}[\mathbf{u}_g \mathbf{u}_g' | \mathbf{X}_g]$ , and there is a consistent estimate  $\hat{\mathbf{\Omega}}_g$  of  $\mathbf{\Omega}_g$ , we can estimate  $\mathbb{E}[\mathbf{X}_g' \mathbf{u}_g \mathbf{u}_g' \mathbf{X}_g] = \mathbb{E}[\mathbf{X}_g' \mathbf{\Omega}_g \mathbf{X}_g]$  via GLS.

**Cluster-robust variance matrix estimate** consider

$$\hat{\mathbb{V}}[\hat{\beta}] = (\mathbf{X}'\mathbf{X})^{-1} \left( \sum_{g=1}^G \mathbf{X}_g' \hat{\mathbf{u}}_g \hat{\mathbf{u}}_g' \mathbf{X}_g \right) (\mathbf{X}'\mathbf{X})^{-1} \quad (5.4)$$

where  $\hat{\mathbf{u}}_g = \mathbf{y}_g - \mathbf{X}_g \hat{\beta}$ . This estimate is consistent if

$$G^{-1} \sum_{g=1}^G \mathbf{X}_g' \hat{\mathbf{u}}_g \hat{\mathbf{u}}_g' \mathbf{X}_g - G^{-1} \sum_{g=1}^G \mathbb{E}[\mathbf{X}_g' \mathbf{u}_g \mathbf{u}_g' \mathbf{X}_g] \xrightarrow{P} \mathbf{0}$$

as  $G \rightarrow \infty$ . An informal presentation of Eq.(5.4) is to rewrite the central matrix as

$$\hat{\mathbf{B}} = \sum_{g=1}^G \mathbf{X}_g' \hat{\mathbf{u}}_g \hat{\mathbf{u}}_g' \mathbf{X}_g = \mathbf{X}' \begin{bmatrix} \hat{\mathbf{u}}_1 \hat{\mathbf{u}}_1' & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{u}}_2 \hat{\mathbf{u}}_2' & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & & \hat{\mathbf{u}}_G \hat{\mathbf{u}}_G' \end{bmatrix} \mathbf{X} = \mathbf{X}' (\hat{\mathbf{u}} \hat{\mathbf{u}}' \otimes \mathbf{S}^G) \mathbf{X} \quad (5.5)$$

where  $\otimes$  denotes element-wise multiplication. The  $(p, q)$ th element of this matrix is

$$\sum_{i=1}^N \sum_{j=1}^N x_{ia} x_{jb} \hat{u}_i \hat{u}_j \cdot \mathbf{1}(i, j \text{ in the same cluster})$$

with  $\hat{u}_i = y_i - \mathbf{x}_i' \hat{\beta}$ .

$\mathbf{S}^G$  is an  $N \times N$  indicator matrix with  $\mathbf{S}_{ij}^G = 1$  only if the  $i$ th and  $j$ th observation belong to the same cluster: it zeros out a large amount of  $\hat{\mathbf{u}} \hat{\mathbf{u}}'$  (asymptotically equivalently,  $\mathbf{u} \mathbf{u}'$ ), specifically, only  $\sum_{g=1}^G N_g^2$  out of  $N^2 = \left( \sum_{g=1}^G N_g \right)^2$  terms are not zero (sub-matrices on the diagonal). Asymptotically

- for fixed  $N_g$ ,  $\frac{1}{N^2} \sum_{g=1}^G N_g^2 \xrightarrow{G \rightarrow \infty} 0$
- for balanced clusters  $N_g = N/G$ ,  $\frac{1}{N^2} \sum_{g=1}^G N_g^2 = \frac{1}{G} \xrightarrow{G \rightarrow \infty} 0$

A strand of literature popularizes this method:

- Liang and Zeger (1986): in a generalized estimatin equations setting
- Arellano (1987): fixed effects estimator in linear panel models
- Hansen (2007): asymptotic theory for panel data where  $T \rightarrow \infty$  in addition to  $N \rightarrow \infty$  (or  $N_g \rightarrow \infty$  in addition to  $G \rightarrow \infty$  in the notation above).

## 5.2 Two-Way Clustering

Now, consider the case of two-way clustering,

$$y_{i,gh} = \mathbf{x}'_{i,gh} \boldsymbol{\beta} + u$$

where each observation may belong to **two** dimension of groups: group  $g \in \{1, \dots, G\}$  and  $h \in \{1, \dots, H\}$ , and for  $i \neq j$

$$\mathbb{E} [u_{i,gh} u_{j,g'h'} \mid \mathbf{x}_{i,gh}, \mathbf{j}, \mathbf{g}'\mathbf{h}'] = 0 \quad (5.6)$$

unless  $g = g'$  or  $h = h'$ , that is, errors for individuals within the same group (along either  $g$  or  $h$ ) may be correlated.

**Cluster-robust variance matrix estimate** extending the one-way clustering case, keep elements of  $\hat{\mathbf{u}}\hat{\mathbf{u}}'$  where the  $i$ th and  $j$ th observations share a cluster in **any** dimension, then similar to Eq.(5.5)

$$\hat{\mathbf{B}} = \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^{GH} \right) \mathbf{X} \quad (5.7)$$

here  $\mathbf{S}^{GH}$  is an  $N \times N$  indicator matrix with  $S_{ij}^{GH} = 1$  only if the  $i$ th and  $j$ th observation share any cluster, the  $(p, q)$ th element of this matrix is

$$\sum_{i=1}^N \sum_{j=1}^N x_{ia} x_{jb} \hat{u}_i \hat{u}_j \cdot \mathbf{1}(i, j \text{ share any cluster})$$

$\hat{\mathbf{B}}$  can also be presented in one-way cluster-robust fashion:

$$\hat{\mathbf{B}} = \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^{GH} \right) \mathbf{X} = \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^G \right) \mathbf{X} + \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^H \right) \mathbf{X} - \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^{G \cap H} \right) \mathbf{X} \quad (5.8)$$

where  $\mathbf{G}^{GH} = \mathbf{G}^G + \mathbf{G}^H - \mathbf{G}^{G \cap H}$ , with

- $\mathbf{G}^G$ :  $G_{ij}^G = 1$  only if the  $i$ th and  $j$ th observation belong to the same cluster  $g \in \{1, 2, \dots, G\}$
- $\mathbf{G}^H$ :  $G_{ij}^H = 1$  only if the  $i$ th and  $j$ th observation belong to the same cluster  $h \in \{1, 2, \dots, H\}$
- $\mathbf{G}^{G \cap H}$ :  $G_{ij}^{G \cap H} = 1$  only if the  $i$ th and  $j$ th observation belong to **both** the same cluster  $g \in \{1, 2, \dots, G\}$  and the same cluster  $h \in \{1, 2, \dots, H\}$

then, similar to one-way clustering case,

$$\begin{aligned} \hat{\mathbf{V}}[\hat{\boldsymbol{\beta}}] &= (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^G \right) \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \\ &\quad + (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^H \right) \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \\ &\quad - (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \left( \hat{\mathbf{u}}\hat{\mathbf{u}}' \otimes \mathbf{S}^{G \cap H} \right) \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \end{aligned} \quad (5.9)$$

that is,

$$\hat{\mathbf{V}}[\hat{\boldsymbol{\beta}}] = \hat{\mathbf{V}}^G[\hat{\boldsymbol{\beta}}] + \hat{\mathbf{V}}^H[\hat{\boldsymbol{\beta}}] - \hat{\mathbf{V}}^{G \cap H}[\hat{\boldsymbol{\beta}}] \quad (5.10)$$

each of Eq.(5.10) can be separately computed by OLS of  $\mathbf{y}$  on  $\mathbf{X}$ , with variance matrix estimates  $\hat{\mathbf{V}}$  based on

- clustering on  $g \in \{1, 2, \dots, G\}$
- clustering on  $h \in \{1, 2, \dots, H\}$
- clustering on  $(g, h) \in \{(1, 1), \dots, (G, H)\}$

**Practical considerations** It is required to know what *ways* will be potentially important for clustering, which can be tested via checking the dimension of correlations in the errors. There are several ways to test

- estimate sample covariances of  $\mathbf{X}'\hat{\mathbf{u}}$  within dimensions, test the null that the **average** of such covariances is 0: rejecting this null is sufficient (not necessary) to reject the null of no clustering (White, 1980)
- for **small samples**, Eq. (5.4) is biased downwards. This is corrected (in Stata) by replacing  $\hat{\mathbf{u}}_g$  with  $\sqrt{c}\hat{\mathbf{u}}_g$ , where  $c = \frac{G}{G-1} \frac{N-1}{N-K} \simeq \frac{G}{G-1}$ . For two-way clustering (Eq. 5.8), there are 2 ways of correction:
  - choose correction terms for each of the 3 components:

$$c_1 = \frac{G}{G-1} \frac{N-1}{N-K}, c_2 = \frac{H}{H-1} \frac{N-1}{N-K}, c_3 = \frac{I}{I-1} \frac{N-1}{N-K}$$

with  $I$  being the number of unique clusters determined by  $G \cap H$

- choose a constant terms for all components:

$$c = \frac{J}{J-1} \frac{N-1}{N-K}$$

with  $J = \min(G, H)$

- **Var-cov matrix not positive-semidefinite:**  $\hat{\mathbf{V}}[\hat{\boldsymbol{\beta}}]$  might have negative elements on the diagonal (Eq. 5.10), informly, this is more likely to arise when clustering is done over the same groups as the fixed effects. One way to address this issue is using *eigendecomposition* technique:

$$\hat{\mathbf{V}}[\hat{\boldsymbol{\beta}}] = \mathbf{U}\mathbf{\Lambda}\mathbf{U}'$$

where

- $\mathbf{U}$  containing the eigenvectors of  $\hat{\mathbf{V}}$
  - $\mathbf{\Lambda} = \text{diag}[\lambda_1, \dots, \lambda_d]$  contains the eigenvalues of  $\hat{\mathbf{V}}$
- then create  $\mathbf{\Lambda}^+ = \text{diag}[\lambda_1^+, \dots, \lambda_d^+]$

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