Noise-Induced Randomization in Regression Discontinuity Designs

Dean Eckles, Nikolaos Ignatiadis, Stefan Wager, Han Wu

Presented by: Sai Zhang

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Outline

1 Confidence Intervals

Confidence Intervals

Asymptotic Normality

$$\hat{\tau} = \frac{\sum_i \gamma_+ (Z_i) Y_i}{\sum_i \gamma_+ (Z_i)} - \frac{\sum_i \gamma_- (Z_i) Y_i}{\sum_i \gamma_- (Z_i)}$$

$$\hat{\tau}_\gamma \xrightarrow{p} \theta_\gamma = \frac{\mathbb{E}\left[\alpha_{(1)} (U) h (U, \gamma_+)\right]}{\mathbb{E}\left[h (U, \gamma_+)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)} (U) h (U, \gamma_-)\right]}{\mathbb{E}\left[h (U, \gamma_-)\right]}$$

Asymptotic Normality

Theorem: Asymptotic Normality of $\hat{\tau}$

Suppose the sequence of weighting kernels $\gamma_+^{(n)}$ and $\gamma_-^{(n)}$ is deterministic, and $\exists \beta \in \left(0, \frac{1}{2}\right), C, C' > 0$ s.t. $\forall n$ large enough:

$$\sup_z \left|\gamma_\diamond^{(n)}(z)\right| < C n^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right] \qquad \sup_u \left|h\left(u, \gamma_\diamond^{(n)}\right)\right| < C' \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right], \qquad \diamond = \{+, -\}$$

Then

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{V_\gamma}} \xrightarrow{d} \mathcal{N}(0, 1)$$

where

$$V_\gamma = \frac{\mathbb{E}\left[\gamma_+^2(Z_i)\left(Y_i - \mu_{\gamma,+}\right)^2\right]}{\mathbb{E}\left[\gamma_+(Z_i)\right]^2} + \frac{\mathbb{E}\left[\gamma_-^2(Z_i)\left(Y_i - \mu_{\gamma,-}\right)^2\right]}{\mathbb{E}\left[\gamma_-(Z_i)\right]^2}$$

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$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{V_\gamma}} \xrightarrow{d} \mathcal{N}(0, 1)$$

where $V_\gamma = \frac{\mathbb{E}\left[\gamma_+^2(Z_i)(Y_i - \mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2} + \frac{\mathbb{E}\left[\gamma_-^2(Z_i)(Y_i - \mu_{\gamma,-})^2\right]}{\mathbb{E}[\gamma_-(Z_i)]^2}$

Assumption:

- The repsonse Y_i is bounded: $Y_i \in [0, 1]$
- $\inf_z \mathrm{Var}\left[Y_i \mid Z_i = z\right] > 0$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sqrt{n}\left[\left(\hat{\mu}_{\gamma,+}-\hat{\mu}_{\gamma,-}\right)-\left(\mu_{\gamma,+}-\mu_{\gamma,-}\right)\right]}{\sqrt{\frac{\mathbb{E}\left[\gamma_{+}^{2}(Z_i)(Y_i-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_{+}(Z_i)]^2}+\frac{\mathbb{E}\left[\gamma_{-}^{2}(Z_i)(Y_i-\mu_{\gamma,-})^2\right]}{\mathbb{E}[\gamma_{-}(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

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$$\frac{\sqrt{n}\left(\hat{\mu}_{\gamma,+}-\mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sqrt{n}\left[\left(\hat{\mu}_{\gamma,+}-\hat{\mu}_{\gamma,-}\right)-\left(\mu_{\gamma,+}-\mu_{\gamma,-}\right)\right]}{\sqrt{\frac{\mathbb{E}\left[\gamma_+^2(Z_i)(Y_i-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}+\frac{\mathbb{E}\left[\gamma_-^2(Z_i)(Y_i-\mu_{\gamma,-})^2\right]}{\mathbb{E}[\gamma_-(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\frac{\sqrt{n}\left(\hat{\mu}_{\gamma,+}-\mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\frac{\sqrt{n}\left(\frac{\sum_i \gamma_+(Z_i)Y_i}{\sum_i \gamma_+(Z_i)}-\mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

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$$\frac{\frac{\sum_i \gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})}{\sum_i \gamma_+(Z_i)}}{\frac{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}}{n\mathbb{E}[\gamma_+(Z_i)]}} = \underbrace{\frac{\sum_i \gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})}{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}}}_{\xrightarrow{d}\mathcal{N}(0,1)} \cdot \frac{1}{\underbrace{\boxed{\frac{\frac{1}{n}\sum_i \gamma_+(Z_i)}{\mathbb{E}[\gamma_+(Z_i)]}}}_{=1+o_p(1)}} \xrightarrow{d} \mathcal{N}(0,1)$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sum_i \gamma_+(Z_i)(Y_i(1) - \mu_{\gamma,+})}{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2 (Y_i(1) - \mu_{\gamma,+})^2\right]}} \xrightarrow{d} \mathcal{N}(0,1)$$

- $\boxed{\underline{\sigma}^2 = \inf_z \mathrm{Var}[Y_i \mid Z_i = z] > 0}$: $\mathrm{Var}[\gamma_+(Z_i)(Y_i(1) - \mu_{\gamma,+})] \geq \underline{\sigma}^2 \mathbb{E}\left[\gamma_+(Z_i)^2\right]$
- $\boxed{\sup_u \left|h\left(u, \gamma_\diamond^{(n)}\right)\right| < C' \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]}$ and $\boxed{Y_i \in [0,1]}$: $|\mu_{\gamma,+}| = \left|\frac{\mathbb{E}[\alpha_{(1)}(U)h(U,\gamma_+)]}{\mathbb{E}[\gamma_+(Z_i)]}\right| \leq C'$
- $\boxed{\sup_z \left|\gamma_\diamond^{(n)}(z)\right| < Cn^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]}$ and Lyapunov CLT condition:

$$\frac{n\mathbb{E}\left[|\gamma_+(Z_i)(Y_i(1) - \mu_{\gamma,+})|^{2+q}\right]}{(n\mathrm{Var}[\gamma_+(Z_i)(Y_i(1) - \mu_{\gamma,+})])^{\frac{2+q}{2}}} \leq \left(\frac{C'+1}{\underline{\sigma}}\right)^{2+q} \left(Cn^{\beta - \frac{1}{2}}\right)^q \xrightarrow{0 \to \infty} 0$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\frac{1}{n}\sum_i \gamma_+(Z_i)}{\mathbb{E}[\gamma_+(Z_i)]} \xrightarrow{p} 1$$

- $\boxed{\sup_z \left|\gamma_\diamond^{(n)}(z)\right| < Cn^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]}$ and Chebyshev's inequality:

$$\begin{aligned}\mathbf{P}\left\{\left|\mathbb{E}_n[\gamma_+(Z_i)] - \mathbb{E}[\gamma_+(Z_i)]\right| \geq \epsilon \mathbb{E}[\gamma_+(Z_i)]\right\} &\leq \frac{\mathrm{Var}[\gamma_+(Z_i)]}{n\epsilon^2 \mathbb{E}[\gamma_+(Z_i)]^2} \\ &\leq \left(\frac{C}{\epsilon}\cdot n^{\beta-\frac{1}{2}}\right)^2 \xrightarrow{n\to\infty} 0\end{aligned}$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\frac{\sum_i \gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})}{\sum_i \gamma_+(Z_i)}}{\frac{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}}{n\mathbb{E}[\gamma_+(Z_i)]}} = \underbrace{\frac{\sum_i \gamma_+(Z_i)\left(Y_i(1)-\mu_{\gamma,+}\right)}{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2\left(Y_i(1)-\mu_{\gamma,+}\right)^2\right]}}}_{\xrightarrow{d}\mathcal{N}(0,1)} \cdot \underbrace{\frac{1}{\boxed{\frac{\frac{1}{n}\sum_i \gamma_+(Z_i)}{\mathbb{E}\left[\gamma_+(Z_i)\right]}}}}_{=1+o_p(1)} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\Rightarrow \frac{\sqrt{n}\left(\frac{\sum_i \gamma_+(Z_i)Y_i}{\sum_i \gamma_+(Z_i)} - \mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1) \Rightarrow \frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{V_\gamma}} \xrightarrow{d} \mathcal{N}(0,1)$$

Plug-in Estimator for V_γ

$$V_\gamma = \frac{\mathbb{E}\left[\gamma_+^2 \left(Z_i\right)\left(Y_i - \mu_{\gamma,+}\right)^2\right]}{\mathbb{E}\left[\gamma_+\left(Z_i\right)\right]^2} + \frac{\mathbb{E}\left[\gamma_-^2 \left(Z_i\right)\left(Y_i - \mu_{\gamma,-}\right)^2\right]}{\mathbb{E}\left[\gamma_-\left(Z_i\right)\right]^2}$$

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Proposition: Plug-in Estimator $\hat{V}_\gamma$

Under the same assumptions, we have $\frac{\hat{V}_\gamma}{V_\gamma} = 1 + o_p(1)$ where

$$\hat{V}_\gamma = \frac{\frac{1}{n}\sum_{i=1}^n \gamma_+^2(Z_i)\left(Y_i - \hat{\mu}_{\gamma,+}\right)^2}{\left[\frac{1}{n}\sum_{i=1}^n \gamma_+(Z_i)\right]^2} + \frac{\frac{1}{n}\sum_{i=1}^n \gamma_-^2(Z_i)\left(Y_i - \hat{\mu}_{\gamma,-}\right)^2}{\left[\frac{1}{n}\sum_{i=1}^n \gamma_-(Z_i)\right]^2}$$

Upper Bound for the Potential Bias $|b_\gamma| = |\theta_\gamma - \tau_w|$

$$a\text{Bias} = \theta_\gamma - \tau_w = \underbrace{\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{h(u,\gamma_-)}{\mathbb{E}_G[h(U,\gamma_-)]} \right) \alpha_{(0)}(u)\,\mathrm{d}G(u)}_{\text{Confounding bias}}$$

$$+ \underbrace{\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\,\mathrm{d}G(u)}_{\text{CATE heterogeneity bias}}$$

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$$\begin{aligned} \mathrm{aBias} = \theta_\gamma - \tau_w = & \underbrace{\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{h(u,\gamma_-)}{\mathbb{E}_G[h(U,\gamma_-)]} \right) \alpha_{(0)}(u)\,\mathrm{d}G(u)}_{\text{Confounding bias}} \\ & + \underbrace{\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\,\mathrm{d}G(u)}_{\text{CATE heterogeneity bias}} \end{aligned}$$

The distribution of $G(\cdot)$ is **unknown**

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Bound the **worst-case** bias:

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Bound the **worst-case** bias:

- Back out the class of latent variable distribution from $\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}(Z_i \leq t)$, the empirical distribution of Z_i (Massart, 1990):

$$\mathcal{G}_n = \left\{ G(\cdot) : \sup_{t \in \mathbb{R}} \left| F_G(t) - \hat{F}_n(t) \right| \leq \sqrt{\frac{\log\left(2/\alpha_n\right)}{2n}} \right\}, \qquad \alpha_n = \min\left\{0.05, n^{-1/4}\right\}$$

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- Take treatment effect heterogeneity into consideration:

$$\mathcal{T}_M = \{\tau(\cdot) \mid \tau(u) = \bar{\tau} + \Delta(u), \bar{\tau} \in \mathbb{R}, |\Delta(u)| \leq M\}, \qquad M \in [0,1]$$

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  - $\mathcal{T}_0$ $(M = 0)$: constant CATE
  - $\mathcal{T}_1$ $(M = 1)$: no assumptions
  - $\mathcal{T}_{1/2}$ $(M = 0)$: a conservative choice for a monotonicity restriction

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Proposition: Upper Bound for the Potential Bias $\hat{B}_{\gamma,M}$

Under asymptotic normality and all necessary assumptions, for $\tau(\cdot) \in \mathcal{T}_M$, the upper bound of bias is

$$\hat{B}_{\gamma,M} = \sup\left\{\left|\mathrm{Bias}\left[\gamma_\pm, \tau_w; \alpha_0\left(\cdot\right), \tau\left(\cdot\right), G\right]\right| : G \in \mathcal{G}_n, \alpha_{(0)}\left(\cdot\right) \in \left[0,1\right], \tau\left(\cdot\right) \in \mathcal{T}_M\right\}$$

then $\mathbf{P}\left(|b_\gamma| \leq \hat{B}_{\gamma,M}\right) \xrightarrow{n\to\infty} 1$

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then $\mathbf{P}\left(|b_\gamma| \le \hat{B}_{\gamma,M}\right) \xrightarrow{n\to\infty} 1$

$$\{G \in \mathcal{G}_n\} \subset \left\{|b_\gamma| \le \hat{B}_{\gamma,M}\right\} \Rightarrow \mathbf{P}\left(|b_\gamma| \le \hat{B}_{\gamma,M}\right) \ge \mathbf{P}\left(G \in \mathcal{G}_n\right)$$

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and for $G \in \mathcal{G}_n$, Dvoretzky-Kiefer-Wolfowitz (DKW) inequality gives:

$$\mathbf{P}\left(G \in \mathcal{G}_n\right) \geq \mathbf{P}\left[\sup_{t\in\mathbb{R}}\left|F_G(t) - \hat{F}_n(t)\right| \leq \sqrt{\frac{\log\left(2/\alpha_n\right)}{2n}}\right] \geq 1 - \alpha_n \xrightarrow[n\to\infty]{\alpha_n = \min\left\{0.05, n^{-1/4}\right\}} 1$$

Recap: What We Have Derived

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- Asymptotic limit:

$$\hat{\tau}_\gamma \xrightarrow{p} \theta_\gamma = \frac{\mathbb{E}\left[\alpha_{(1)}(U)\, h(U, \gamma_+)\right]}{\mathbb{E}\left[h(U, \gamma_+)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)}(U)\, h(U, \gamma_-)\right]}{\mathbb{E}\left[h(U, \gamma_-)\right]}$$

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- Upper bound of the asymptotic bias:

$$\hat{B}_{\gamma,M} = \sup\left\{\left|\mathrm{Bias}\left[\gamma_\pm, \tau_w; \alpha_0(\cdot), \tau(\cdot), G\right]\right| : G \in \mathcal{G}_n, \alpha_{(0)}(\cdot) \in [0,1], \tau(\cdot) \in \mathcal{T}_M\right\}$$

Bias-aware Confidence Intervals

Corollary: Valid Confidence Intervals

Under asymptotic normality and all necessary assumptions, for $\tau(\cdot) \in \mathcal{T}_M$, consider the CIs

$$\hat{\tau}_\gamma \pm l_\alpha, \qquad l_\alpha = \min\left\{l : \mathbf{P}\left[\left|b + n^{-\frac{1}{2}}\hat{V}_\gamma^{\frac{1}{2}}\tilde{Z}\right| \le l\right] \ge 1-\alpha, \forall\,|b| \le \hat{B}_{\gamma,M}\right\}$$

where

- $\tilde{Z}$ is a standard Guassian random variable
- $\alpha \in (0,1)$ is the significant level
- $\hat{V}_\gamma$ is an estimate of the sampling variance V_γ

then

$$\lim\inf_{n\to\infty} \mathbf{P}\left[\tau_w \in \hat{\tau}_\gamma \pm l_\alpha\right] \ge 1-\alpha$$

Bias-aware Confidence Intervals

CLT (with bias considered) is

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma-\theta_\gamma\right)}{\hat{V}_\gamma^{1/2}}=\frac{\sqrt{n}\left(\hat{\tau}_\gamma-\tau_w-b_\gamma\right)}{\hat{V}_\gamma^{1/2}}\xrightarrow{d}\mathcal{N}\left(0,1\right)$$

where $b_\gamma=\theta_\gamma-\tau_w$, then let $\tilde{Z}\sim\mathcal{N}\left(0,1\right)$ we have

$$\begin{aligned}\mathbf{P}\left[\tau_w\in\hat{\tau}_\gamma\pm l_\alpha\right]=&\mathbf{P}\left[-l_\alpha-b_\gamma\leq\hat{\tau}_\gamma-\tau_w-b_\gamma\leq l_\alpha-b_\gamma\right]\\=&\mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha+b_\gamma\right)\leq\sqrt{n}\hat{V}_\gamma^{-1/2}\left(\hat{\tau}_\gamma-\tau_w-b_\gamma\right)\leq\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha-b_\gamma\right)\right]\\=&\mathbb{E}\left(\mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha+b_\gamma\right)\leq\tilde{Z}\leq\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha-b_\gamma\right)\right]\mid\hat{V}_\gamma,\hat{B}_{\gamma,M},\hat{\tau}_\gamma\right)+o\left(1\right)\\=&\mathbb{E}\left[\mathbf{P}\left(-l_\alpha\leq n^{-1/2}\hat{V}_\gamma^{1/2}\tilde{Z}+b_\gamma\leq l_\alpha\right)\mid\hat{V}_\gamma,\hat{B}_{\gamma,M},\hat{\tau}_\gamma\right]+o\left(1\right)\\=&1-\alpha+o\left(1\right)\end{aligned}$$

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$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\hat{V}_\gamma^{1/2}} = \frac{\sqrt{n}\left(\hat{\tau}_\gamma - \tau_w - b_\gamma\right)}{\hat{V}_\gamma^{1/2}} \xrightarrow{d} \mathcal{N}\left(0,1\right)$$

where $b_\gamma = \theta_\gamma - \tau_w$, then let $\tilde{Z} \sim \mathcal{N}\left(0,1\right)$ we have

$$\begin{aligned}
\mathbf{P}\left[\tau_w \in \hat{\tau}_\gamma \pm l_\alpha\right] =& \mathbf{P}\left[-l_\alpha - b_\gamma \leq \hat{\tau}_\gamma - \tau_w - b_\gamma \leq l_\alpha - b_\gamma\right] \\
=& \mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha + b_\gamma\right) \leq \sqrt{n}\hat{V}_\gamma^{-1/2}\left(\hat{\tau}_\gamma - \tau_w - b_\gamma\right) \leq \sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha - b_\gamma\right)\right] \\
=& \mathbb{E}\left(\mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha + b_\gamma\right) \leq \tilde{Z} \leq \sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha - b_\gamma\right)\right] \mid \hat{V}_\gamma, \hat{B}_{\gamma,M}, \hat{\tau}_\gamma\right) + o\left(1\right) \\
=& \mathbb{E}\left[\mathbf{P}\left(-l_\alpha \leq n^{-1/2}\hat{V}_\gamma^{1/2}\tilde{Z} + b_\gamma \leq l_\alpha\right) \mid \hat{V}_\gamma, \hat{B}_{\gamma,M}, \hat{\tau}_\gamma\right] + o\left(1\right) \\
=& 1 - \alpha + o\left(1\right)
\end{aligned}$$

Robustness to CATE Heterogeneity Misspecification

$$\hat{\tau}_\gamma \pm l_\alpha, l_\alpha = \min \left\{ l : \mathbf{P} \left[\left| b + n^{-\frac{1}{2}} \hat{V}_\gamma^{\frac{1}{2}} \tilde{Z} \right| \le l \right] \ge 1 - \alpha, \forall \left| b \right| \le \hat{B}_{\gamma, M} \right\}$$

where

$$\mathcal{T}_M = \{ \tau(\cdot) \mid \tau(u) = \bar{\tau} + \Delta(u), \bar{\tau} \in \mathbb{R}, |\Delta(u)| \le M \}, \ M \in [0, 1]$$

Consider an extreme misspecification of CATE heterogeneity: $M = 0$, are the CIs robust?

Robustness to CATE Heterogeneity Misspecification

$$\hat{\tau}_\gamma \pm l_\alpha, l_\alpha = \min\left\{l : \mathbf{P}\left[\left|b + n^{-\frac{1}{2}}\hat{V}_\gamma^{\frac{1}{2}}\tilde{Z}\right| \leq l\right] \geq 1-\alpha, \forall\, |b| \leq \hat{B}_{\gamma,M}\right\}$$

where

$$\mathcal{T}_M = \{\tau(\cdot) \mid \tau(u) = \bar{\tau} + \Delta(u), \bar{\tau} \in \mathbb{R}, |\Delta(u)| \leq M\}, \quad M \in [0,1]$$

Corollary: Robustness to CATE Heterogeneity Misspecification

The CIs under the misspecification of $M = 0$ is still valid, but only for the convenience-weighted treatment effect:

$$\tau_{h,+} := \int \frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]}\tau(u)\,\mathrm{d}G(u)$$

Robustness to CATE Heterogeneity Misspecification

Corollary: Robustness to CATE Heterogeneity Misspecification

The CIs under the misspecification of $M = 0$ is still valid, but only for:

$$\tau_{h,+} := \int \frac{h\left(u, \gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U, \gamma_{+}\right)\right]} \tau(u) \, \mathrm{d}G(u)$$

$$\begin{aligned} a\text{Bias} = \theta_\gamma - \tau_w = & \int \left(\frac{h\left(u, \gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U, \gamma_{+}\right)\right]} - \frac{h\left(u, \gamma_{-}\right)}{\mathbb{E}_G\left[h\left(U, \gamma_{-}\right)\right]} \right) \alpha_{(0)}(u) \, \mathrm{d}G(u) \\ & + \underbrace{\int \left(\frac{h\left(u, \gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U, \gamma_{+}\right)\right]} - \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \right) \tau(u) \, \mathrm{d}G(u)}_{\text{CATE heterogeneity bias}} \end{aligned}$$

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Thank you!