Noise-Induced Randomization in Regression Discontinuity Designs

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Presented by: Sai Zhang

November 18, 2022

Outline

Introduction

RD Identification

$$\underbrace{Z_i}_{\text{running variable}}$$

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$\underbrace{W_i}_{\text{treatment}}$

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test scores	admission	outcomes

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test scores	admission	outcomes
test results	medication	outcomes

RD Identification: Continuity Argument

For potential outcomes $\{Y_i(0), Y_i(1)\}$: $Y_i = Y_i(W_i)$, a weighted **causal effect** can be identified as

$$\tau_c = \mathbb{E}\left[Y_i(1) - Y_i(0) \mid Z_i = c\right]$$

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assuming

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assuming

- the conditional response functions $\mu_w(z) = \mathbb{E}\left[Y(w) \mid Z = z\right]$ are **continuous**
- $\mu_w(z)$ to have a uniformly **bounded 2nd derivative** for CIs (Armstrong and Kolesár, 2018, 2020)

RD Identification: Problems of Continuity Argument

Assumption: **continuous** $\mu_w(z) = \mathbb{E}[Y(w) \mid Z = z]$

$$\tau_c = \lim_{z \downarrow c} \mathbb{E}[Y \mid Z = z] - \lim_{z \uparrow c} \mathbb{E}[Y \mid Z = z]$$

Where does this continuity come from?

RD Identification: Problems of Continuity Argument

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Where does this continuity come from?

Lee (2008): continuous measurement error in the running variable by units

RD Identification: Measurement Error

$$\underbrace{U_i}_{\text{latent variable}} \qquad \underbrace{Z_i}_{\text{running variable}} \xrightarrow{W_i = \mathbf{1}(\{Z_i \geq c\})} \underbrace{W_i}_{\text{treatment}} \Rightarrow \underbrace{Y_i}_{\text{outcome}}$$

	test scores	admission	outcomes
	test results	medication	outcomes

RD Identification: Measurement Error

$$\underbrace{U_i}_{\text{latent variable}} \xRightarrow{Z_i = U_i + e_i} \underbrace{Z_i}_{\text{running variable}} \xRightarrow{W_i = \mathbf{1}(\{Z_i \geq c\})} \underbrace{W_i}_{\text{treatment}} \Rightarrow \underbrace{Y_i}_{\text{outcome}}$$

ability	test scores	admission	outcomes
condition	test results	medication	outcomes

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ability	test scores	admission	outcomes
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Why don't we take advantage of the **measurement error** itself for inference?

This Paper

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Weighted treatment effects can be estimated if the measurement error in Z_i

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Weighted treatment effects can be estimated if the measurement error in Z_i

- has a **known distribution**

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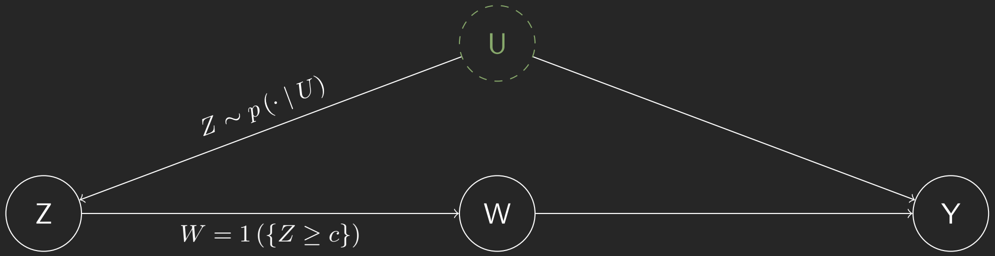
Weighted treatment effects can be estimated if the measurement error in Z_i

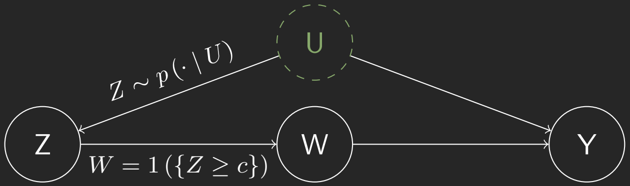
- has a **known distribution**
- is **conditionally (on U_i) independent** of potential outcomes

Key Argument

Sharp RD Design with A Noisy Running Variable



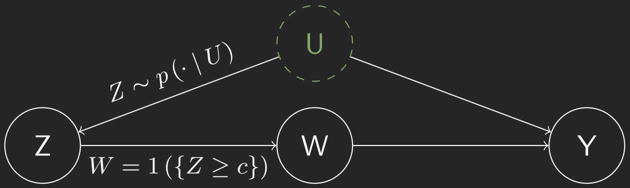
Sharp RD Design with A Noisy Running Variable



Assumption 1: Sharp RD design

- **I.I.D.** samples $\{Y_i(0), Y_i(1), Z_i\} \in \mathbb{R}^3, i = 1, \cdots, n$
- treatment assignment: $W_i = 1(\{Z_i \geq c\})$, where $c \in \mathbb{R}$ is the **cutoff**
- observation: $\{Y_i, Z_i\}$ where $Y_i = Y_i(W_i)$

Sharp RD Design with A Noisy Running Variable

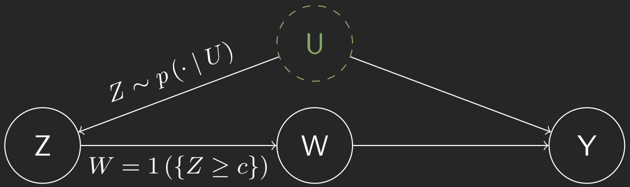


Assumption 2: Noisy running variable

$$Z_i \mid U_i \sim p(\cdot \mid U_i)$$

where $p(\cdot \mid \cdot)$ is a **known** conditional density w.r.t. to a measure λ, the latent variable U_i has an **unknown** distribution G

Sharp RD Design with A Noisy Running Variable



Assumption 2: Noisy running variable

$$Z_i \mid U_i \sim \mathcal{N}(U_i, \nu^2), \nu > 0$$

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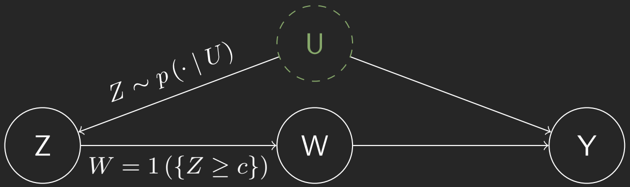
Sharp RD Design with A Noisy Running Variable



Assumption 2: Noisy running variable

$$Z_i \mid U_i \sim \text{Binomial}(K, U_i), K \in \mathbb{N}$$

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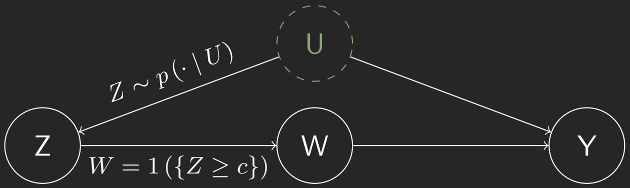
Sharp RD Design with A Noisy Running Variable



Assumption 3: Exogeneity

$$[\{Y_i(0), Y_i(1)\} \perp Z_i] \mid U_i$$

which implies $\mathbb{E}[Y_i \mid U_i, Z_i] = \alpha_{(W_i)}(u)$

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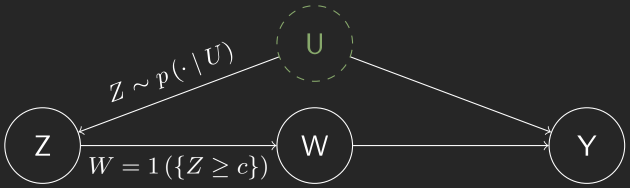

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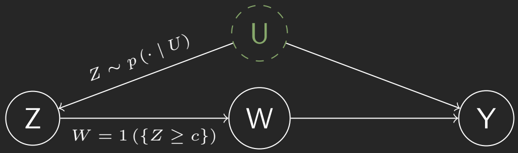
which implies $\mathbb{E}[Y_i \mid U_i, Z_i] = \alpha_{(W_i)}(u)$, where $\alpha_{(w)}(u) = \mathbb{E}[Y_i(w) \mid U_i = u]$ is the **response functions** for the potential oucomes conditional on the latent variable u

Sharp RD Design with A Noisy Running Variable



A1 **Sharp** RD

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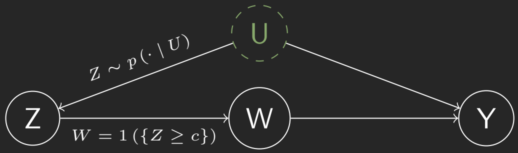

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Proposition 1

Let $\gamma_+(\cdot), \gamma_-(\cdot)$ be measurable functions of Z, then under A1-A3:

$$\mathbb{E}[\gamma_+(Z)Y] = \mathbb{E}\left[\alpha_{(1)}(U)\, h(U, \gamma_+)\right], \qquad \mathbb{E}[\gamma_-(Z)Y] = \mathbb{E}\left[\alpha_{(0)}(U)\, h(U, \gamma_-)\right]$$

where $h(u, \gamma) := \int \gamma(z)\, p(z \mid u)\, \mathrm{d}\lambda(z)$, $\alpha_{(w)}(u) = \mathbb{E}[Y_i(w) \mid U_i = u]$

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- $\mathbb{E}\left[Y^2\right], \mathbb{E}\left[\gamma_-(Z)^2\right], \mathbb{E}\left[\gamma_+(Z)^2\right] < \infty$

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  - $\gamma_-(z) = 0$ for $z \geq c$: assign non-zero weights only to **control** units

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$$\mathbb{E}\left[\gamma_+(Z)\, Y\right] = \mathbb{E}\left[\alpha_{(1)}(U)\, h\left(U, \gamma_+\right)\right], \qquad \mathbb{E}\left[\gamma_-(Z)\, Y\right] = \mathbb{E}\left[\alpha_{(0)}(U)\, h\left(U, \gamma_-\right)\right]$$

where $h(u, \gamma) := \int \gamma(z)\, p(z \mid u)\, \mathrm{d}\lambda(z)$, $\alpha_{(w)}(u) = \mathbb{E}\left[Y_i(w) \mid U_i = u\right]$

Proof:

$$\mathbb{E}\left[\gamma_+(Z)\, Y \mid U\right]$$

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Proof:

$$\mathbb{E}\left[\gamma_+(Z)\, Y \mid U\right] = \mathbb{E}\left[\gamma_+(Z) Y \cdot \mathbf{1}\left(\{Z \geq c\}\right) \mid U\right]$$

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Proof:

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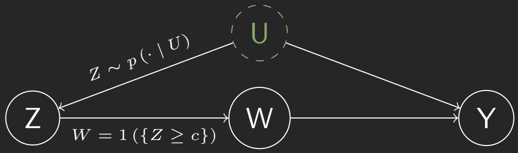
&= \mathbb{E}\left[\gamma_+(Z)\,Y(1) \cdot \mathbf{1}(\{Z \geq c\}) \mid U\right] \\
&= \underbrace{\mathbb{E}\left[Y(1) \mid U\right]}_{=\alpha_{(1)}(U)} \cdot \underbrace{\mathbb{E}\left[\gamma_+(Z)\,\mathbf{1}(\{Z \geq c\}) \mid U\right]}_{=\mathbb{E}[\gamma_+(Z)|U]=\int \gamma_+(z)p(z|U)\mathrm{d}\lambda(z)=h(U,\gamma_+)}
\end{aligned}$$

Sharp RD Design with A Noisy Running Variable



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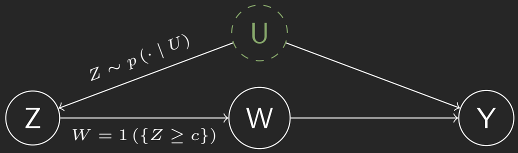

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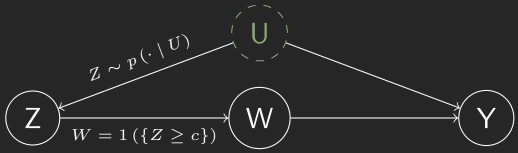
- **Need** to know $p(z \mid u)$ (conditional distribution of the noise)

Sharp RD Design with A Noisy Running Variable



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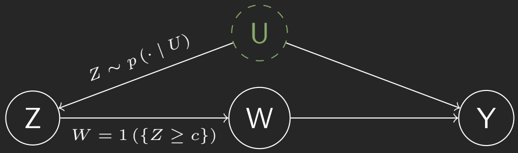
- **No need** to know G (distribution of U)
- **Need** to know $p(z \mid u)$ (conditional distribution of the noise)
  - test-retest data, prior modelling of responses to tests, physical model of the measurement device, biomedical knowledge, etc.

Sharp RD Design with A Noisy Running Variable



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- **No need** to know G (distribution of U)
- **Need** to know $p(z \mid u)$ (conditional distribution of the noise)
  - test-retest data, prior modelling of responses to tests, physical model of the measurement device, biomedical knowledge, etc.
  - still valid when **underestimating** the true noise level

Estimation

Estimation of Weighted Treatment Effects

Proposition: The Key Argument

Let $\gamma_+(\cdot), \gamma_-(\cdot)$ be measurable functions of Z, then under A1-A3:

$$\mathbb{E}\left[\gamma_+(Z)\,Y\right] = \mathbb{E}\left[\alpha_{(1)}(U)\,h(U, \gamma_+)\right], \qquad \mathbb{E}\left[\gamma_-(Z)\,Y\right] = \mathbb{E}\left[\alpha_{(0)}(U)\,h(U, \gamma_-)\right]$$

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ratio-form estimators:

$$\hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-}$$

$$= \frac{\sum_i \gamma_+(Z_i)\,Y_i}{\sum_i \underbrace{\gamma_+(Z_i)}} - \frac{\sum_i \gamma_-(Z_i)\,Y_i}{\sum_i \underbrace{\gamma_-(Z_i)}}$$

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$$= \frac{\sum_i \gamma_+(Z_i)\,Y_i}{\sum_i \underbrace{\gamma_+(Z_i)}_{\gamma_+(z)=0, z<c}} - \frac{\sum_i \gamma_-(Z_i)\,Y_i}{\sum_i \underbrace{\gamma_-(Z_i)}_{\gamma_-(z)=0, z<\geq c}}$$

Estimation of Weighted Treatment Effects

Ratio-form estimators:

$$\hat{\tau}_{\gamma} = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+ (Z_i) Y_i}{\sum_i \gamma_+ (Z_i)} - \frac{\sum_i \gamma_- (Z_i) Y_i}{\sum_i \gamma_- (Z_i)}$$

What's the weighted treatment effects to conduct inference for?

Estimation of Weighted Treatment Effects

Ratio-form estimators:

$$\hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+ (Z_i) Y_i}{\sum_i \gamma_+ (Z_i)} - \frac{\sum_i \gamma_- (Z_i) Y_i}{\sum_i \gamma_- (Z_i)}$$

What's the weighted treatment effects to conduct inference for?

$$\tau_w = \int \frac{w(u)}{\mathbb{E}_G [w(U)]} \tau(u) \mathrm{d}G(u), w(\cdot) \geq 0$$

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where $\tau(u)$ **(Conditional Average Treatment Effects)** is

$$\tau(u) = \mathbb{E}[Y_i(1) - Y_i(0) \mid U_i = u] = \alpha_{(1)}(u) - \alpha_{(0)}(u)$$

Weighted Treatment Effects: Example

$$\tau_w = \int \frac{w(u)}{\mathbb{E}_G[w(U)]} \tau(u) \mathrm{d}G(u), w(\cdot) \geq 0$$

where $\tau(u)$ **(CATE)** is $\tau(u) = \mathbb{E}[Y_i(1) - Y_i(0) \mid U_i = u] = \alpha_{(1)}(u) - \alpha_{(0)}(u)$

- RD paramater:

$$\begin{aligned}\tau_c = \mathbb{E}[Y_i(1) - Y_i(0) \mid Z_i = c] &= \mathbb{E}[\tau(U_i) \mid Z_i = c] \\ &= \int \frac{p(c \mid u)}{\int p(c \mid u) \, \mathrm{d}G(u)} \tau(u) \, \mathrm{d}G(u)\end{aligned}$$

Weighted Treatment Effects: Example

$$\tau_w = \int \frac{w(u)}{\mathbb{E}_G[w(U)]} \tau(u) \mathrm{d}G(u), w(\cdot) \geq 0$$

where $\tau(u)$ **(CATE)** is $\tau(u) = \mathbb{E}[Y_i(1) - Y_i(0) \mid U_i = u] = \alpha_{(1)}(u) - \alpha_{(0)}(u)$

Asymptotic Bias

Theorem: Asymptotic Limit of $\hat{\tau}_\gamma$

$$\hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+(Z_i)\, Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i)\, Y_i}{\sum_i \gamma_-(Z_i)}$$

Asymptotic Bias

Theorem: Asymptotic Limit of $\hat{\tau}_\gamma$

$$
\begin{aligned}
\hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = & \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)} \\
\xrightarrow{p} & \frac{\mathbb{E}[\gamma_+(Z)Y]}{\mathbb{E}[\gamma_+(Z)]} - \frac{\mathbb{E}[\gamma_-(Z)Y]}{\mathbb{E}[\gamma_-(Z)]}
\end{aligned}
$$

Asymptotic Bias

Theorem: Asymptotic Limit of $\hat{\tau}_\gamma$

$$\begin{aligned}
\hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} &= \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)} \\
&\xrightarrow{p} \frac{\mathbb{E}[\gamma_+(Z)Y]}{\mathbb{E}[\gamma_+(Z)]} - \frac{\mathbb{E}[\gamma_-(Z)Y]}{\mathbb{E}[\gamma_-(Z)]} \\
\textbf{(Prop.1)} &= \frac{\mathbb{E}\left[\alpha_{(1)}(U)\, h(U,\gamma_+)\right]}{\mathbb{E}[h(U,\gamma_+)]} - \frac{\mathbb{E}\left[\alpha_{(0)}(U)\, h(U,\gamma_-)\right]}{\mathbb{E}[h(U,\gamma_-)]} = \mu_{\gamma,+} - \mu_{\gamma,-} \equiv \theta_\gamma
\end{aligned}$$

where

$$h(u,\gamma) := \int \gamma(z)\, p(z \mid u)\, \mathrm{d}\lambda(z), \;\; \alpha_{(w)}(u) = \mathbb{E}[Y_i(w) \mid U_i = u]$$

Asymptotic Bias

Theorem: Asymptotic Limit of $\hat{\tau}_\gamma$

$$\hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)}$$
$$\xrightarrow{p} \frac{\mathbb{E}\left[\alpha_{(1)}(U) h(U, \gamma_+)\right]}{\mathbb{E}\left[h(U, \gamma_+)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)}(U) h(U, \gamma_-)\right]}{\mathbb{E}\left[h(U, \gamma_-)\right]} = \mu_{\gamma,+} - \mu_{\gamma,-} \equiv \theta_\gamma$$

How biased is this asymptotic limit? Comparing to

$$\tau_w = \int \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \tau(u) \mathrm{d}G(u), w(\cdot) \geq 0$$

Asymptotic Bias: Decomposition

$$a\text{Bias}\left[\gamma_{\pm}, \tau_w; \alpha_{(0)}\left(\cdot\right), \tau\left(\cdot\right), G\right] = \theta_{\gamma} - \tau_w$$

Asymptotic Bias: Decomposition

$$
\begin{aligned}
& a\text{Bias}\left[\gamma_{\pm}, \tau_w; \alpha_{(0)}(\cdot), \tau(\cdot), G\right] = \theta_\gamma - \tau_w \\
= & \frac{\mathbb{E}\left[\alpha_{(1)}(U)\, h(U, \gamma_+)\right]}{\mathbb{E}\left[h(U, \gamma_+)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)}(U)\, h(U, \gamma_-)\right]}{\mathbb{E}\left[h(U, \gamma_-)\right]} - \int \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \tau(u)\, \mathrm{d}G(u)
\end{aligned}
$$

Asymptotic Bias: Decomposition

$$
\begin{aligned}
& a\mathrm{Bias}\left[\gamma_{\pm}, \tau_w; \alpha_{(0)}\left(\cdot\right), \tau\left(\cdot\right), G\right] = \theta_\gamma - \tau_w \\
=& \frac{\mathbb{E}\left[\alpha_{(1)}\left(U\right) h\left(U, \gamma_+\right)\right]}{\mathbb{E}\left[h\left(U, \gamma_+\right)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)}\left(U\right) h\left(U, \gamma_-\right)\right]}{\mathbb{E}\left[h\left(U, \gamma_-\right)\right]} - \int \frac{w\left(u\right)}{\mathbb{E}_G\left[w\left(U\right)\right]} \tau\left(u\right) \mathrm{d}G\left(u\right) \\
=& \int \left(\frac{h\left(u, \gamma_+\right)}{\mathbb{E}_G\left[h\left(U, \gamma_+\right)\right]}\right) \alpha_{(1)}\left(u\right) \mathrm{d}G\left(u\right) - \int \left(\frac{h\left(u, \gamma_-\right)}{\mathbb{E}_G\left[h\left(U, \gamma_-\right)\right]}\right) \alpha_{(0)}\left(u\right) \mathrm{d}G\left(u\right) \\
& - \int \frac{w\left(u\right)}{\mathbb{E}_G\left[w\left(U\right)\right]} \tau\left(u\right) \mathrm{d}G\left(u\right)
\end{aligned}
$$

Asymptotic Bias: Decomposition

$$
\begin{aligned}
& a\text{Bias}\left[\gamma_{\pm}, \tau_w; \alpha_{(0)}(\cdot), \tau(\cdot), G\right] = \theta_\gamma - \tau_w \\
=& \frac{\mathbb{E}\left[\alpha_{(1)}(U)\, h(U, \gamma_+)\right]}{\mathbb{E}\left[h(U, \gamma_+)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)}(U)\, h(U, \gamma_-)\right]}{\mathbb{E}\left[h(U, \gamma_-)\right]} - \int \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \tau(u)\, \mathrm{d}G(u) \\
=& \int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G\left[h(U, \gamma_+)\right]}\right) \alpha_{(1)}(u)\, \mathrm{d}G(u) - \int \left(\frac{h(u, \gamma_-)}{\mathbb{E}_G\left[h(U, \gamma_-)\right]}\right) \alpha_{(0)}(u)\, \mathrm{d}G(u) \\
& - \int \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \tau(u)\, \mathrm{d}G(u)
\end{aligned}
$$

Remember? $\tau(u)$ **(Conditional Average Treatment Effects)** is

$$
\tau(u) = \mathbb{E}\left[Y_i(1) - Y_i(0) \mid U_i = u\right] = \alpha_{(1)}(u) - \alpha_{(0)}(u) \Rightarrow \boxed{\alpha_{(1)}(u) = \tau(u) + \alpha_{(0)}(u)}
$$

Asymptotic Bias: Decomposition

$$
\begin{aligned}
& a\text{Bias}\left[\gamma_{\pm}, \tau_w; \alpha_{(0)}\left(\cdot\right), \tau\left(\cdot\right), G\right] = \theta_\gamma - \tau_w \\
= & \int \left(\frac{h\left(u, \gamma_+\right)}{\mathbb{E}_G\left[h\left(U, \gamma_+\right)\right]} \right) \underbrace{\alpha_{(1)}\left(u\right)}_{=\tau(u)+\alpha_{(0)}(u)} \mathrm{d}G\left(u\right) \\
& - \int \left(\frac{h\left(u, \gamma_-\right)}{\mathbb{E}_G\left[h\left(U, \gamma_-\right)\right]} \right) \alpha_{(0)}\left(u\right) \mathrm{d}G\left(u\right) - \int \frac{w\left(u\right)}{\mathbb{E}_G\left[w\left(U\right)\right]} \tau\left(u\right) \mathrm{d}G\left(u\right)
\end{aligned}
$$

Asymptotic Bias: Decomposition

$$a\mathrm{Bias}\left[\gamma_\pm, \tau_w; \alpha_{(0)}(\cdot), \tau(\cdot), G\right] = \theta_\gamma - \tau_w$$

$$= \int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G\left[h(U,\gamma_+)\right]}\right) \underbrace{\alpha_{(1)}(u)}_{=\tau(u)+\alpha_{(0)}(u)} \, \mathrm{d}G(u)$$

$$- \int \left(\frac{h(u,\gamma_-)}{\mathbb{E}_G\left[h(U,\gamma_-)\right]}\right) \alpha_{(0)}(u) \, \mathrm{d}G(u) - \int \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \tau(u) \, \mathrm{d}G(u)$$

$$= \int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G\left[h(U,\gamma_+)\right]} - \frac{h(u,\gamma_-)}{\mathbb{E}_G\left[h(U,\gamma_-)\right]}\right) \alpha_{(0)}(u) \, \mathrm{d}G(u)$$

$$+ \int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G\left[h(U,\gamma_+)\right]} - \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]}\right) \tau(u) \, \mathrm{d}G(u)$$

Asymptotic Bias: Decomposition

$$
\begin{aligned}
&a\text{Bias}\left[\gamma_{\pm}, \tau_w; \alpha_{(0)}(\cdot), \tau(\cdot), G\right] = \theta_\gamma - \tau_w \\
=& \int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} \right) \underbrace{\alpha_{(1)}(u)}_{=\tau(u)+\alpha_{(0)}(u)} \mathrm{d}G(u) \\
&- \int \left(\frac{h(u,\gamma_-)}{\mathbb{E}_G[h(U,\gamma_-)]} \right) \alpha_{(0)}(u)\,\mathrm{d}G(u) - \int \frac{w(u)}{\mathbb{E}_G[w(U)]} \tau(u)\,\mathrm{d}G(u)
\end{aligned}
$$

$$
\begin{aligned}
=& \int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{h(u,\gamma_-)}{\mathbb{E}_G[h(U,\gamma_-)]} \right) \alpha_{(0)}(u)\,\mathrm{d}G(u) \\
&+ \int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\,\mathrm{d}G(u)
\end{aligned}
$$

Confounding bias

Asymptotic Bias: Decomposition

$$
\begin{aligned}
& a\text{Bias}\left[\gamma_{\pm}, \tau_w; \alpha_{(0)}(\cdot), \tau(\cdot), G\right] = \theta_\gamma - \tau_w \\
&= \int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G\left[h(U, \gamma_+)\right]} \right) \underbrace{\alpha_{(1)}(u)}_{=\tau(u)+\alpha_{(0)}(u)} \mathrm{d}G(u) \\
&\quad - \int \left(\frac{h(u, \gamma_-)}{\mathbb{E}_G\left[h(U, \gamma_-)\right]} \right) \alpha_{(0)}(u)\, \mathrm{d}G(u) - \int \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \tau(u)\, \mathrm{d}G(u)
\end{aligned}
$$

$$
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$$

Confounding bias

$$
+ \int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G\left[h(U, \gamma_+)\right]} - \frac{w(u)}{\mathbb{E}_G\left[w(U)\right]} \right) \tau(u)\, \mathrm{d}G(u)
$$

CATE heterogeneity bias

Asymptotic Bias: Decomposition

$$\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{h(u,\gamma_-)}{\mathbb{E}_G[h(U,\gamma_-)]} \right) \alpha_{(0)}(u)\, \mathrm{d}G(u)$$

Confounding bias

$$\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\, \mathrm{d}G(u)$$

CATE heterogeneity bias

How to minimize them?

Asymptotic Bias: Decomposition

$$\int \left(\frac{h\left(u,\gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{+}\right)\right]} - \frac{h\left(u,\gamma_{-}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{-}\right)\right]} \right) \alpha_{(0)}\left(u\right) \mathrm{d}G\left(u\right)$$

Confounding bias

$$\int \left(\frac{h\left(u,\gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{+}\right)\right]} - \frac{w\left(u\right)}{\mathbb{E}_G\left[w\left(U\right)\right]} \right) \tau\left(u\right) \mathrm{d}G\left(u\right)$$

CATE heterogeneity bias

How to minimize them?

- **Confounding bias:** $h(\cdot,\gamma_{+}) \approx h(\cdot,\gamma_{-})$

Asymptotic Bias: Decomposition

$$\int \left(\frac{h\left(u,\gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{+}\right)\right]} - \frac{h\left(u,\gamma_{-}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{-}\right)\right]} \right) \alpha_{(0)}\left(u\right) \mathrm{d}G\left(u\right)$$

Confounding bias

$$\int \left(\frac{h\left(u,\gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{+}\right)\right]} - \frac{w\left(u\right)}{\mathbb{E}_G\left[w\left(U\right)\right]} \right) \tau\left(u\right) \mathrm{d}G\left(u\right)$$

CATE heterogeneity bias

How to minimize them?

- **Confounding bias:** $h(\cdot,\gamma_{+}) \approx h(\cdot,\gamma_{-})$ where $h\left(u,\gamma\right) := \int \gamma\left(z\right) p\left(z \mid u\right) \mathrm{d}\lambda\left(z\right)$
  How well the units are **balanced** via the latent variable u

Asymptotic Bias: Decomposition

$$\int \left(\frac{h\left(u,\gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{+}\right)\right]} - \frac{h\left(u,\gamma_{-}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{-}\right)\right]} \right) \alpha_{(0)}\left(u\right) \mathrm{d}G\left(u\right)$$

Confounding bias

$$\int \left(\frac{h\left(u,\gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U,\gamma_{+}\right)\right]} - \frac{w\left(u\right)}{\mathbb{E}_G\left[w\left(U\right)\right]} \right) \tau\left(u\right) \mathrm{d}G\left(u\right)$$

CATE heterogeneity bias

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  How well the units are **balanced** via the latent variable u
- **CATE heterogeneity bias:**

Asymptotic Bias: Decomposition

$$\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{h(u,\gamma_-)}{\mathbb{E}_G[h(U,\gamma_-)]} \right) \alpha_{(0)}(u)\, \mathrm{d}G(u)$$

Confounding bias

$$\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\, \mathrm{d}G(u)$$

CATE heterogeneity bias

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Asymptotic Bias: Decomposition

$$\int \left(\frac{h\left(u, \gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U, \gamma_{+}\right)\right]} - \frac{h\left(u, \gamma_{-}\right)}{\mathbb{E}_G\left[h\left(U, \gamma_{-}\right)\right]} \right) \alpha_{(0)}\left(u\right) \mathrm{d}G\left(u\right)$$

Confounding bias

$$\int \left(\frac{h\left(u, \gamma_{+}\right)}{\mathbb{E}_G\left[h\left(U, \gamma_{+}\right)\right]} - \frac{w\left(u\right)}{\mathbb{E}_G\left[w\left(U\right)\right]} \right) \tau\left(u\right) \mathrm{d}G\left(u\right)$$

CATE heterogeneity bias

How to minimize them?

- **Confounding bias:** $h(\cdot, \gamma_{+}) \approx h(\cdot, \gamma_{-})$
  How well the units are **balanced** via the latent variable u
- **CATE heterogeneity bias:**
  - $\tau(u)$ being constant w.r.t. u, a constant conditional treatment effect

Asymptotic Bias: Decomposition

$$\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{h(u,\gamma_-)}{\mathbb{E}_G[h(U,\gamma_-)]} \right) \alpha_{(0)}(u)\,\mathrm{d}G(u)$$

Confounding bias

$$\int \left(\frac{h(u,\gamma_+)}{\mathbb{E}_G[h(U,\gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\,\mathrm{d}G(u)$$

CATE heterogeneity bias

How to minimize them?

- **Confounding bias:** $h(\cdot,\gamma_+) \approx h(\cdot,\gamma_-)$
  How well the units are **balanced** via the latent variable u
- **CATE heterogeneity bias:**
  - $\tau(u)$ being constant w.r.t. u, a constant conditional treatment effect
  - $h(u,\gamma_+) = w(u), \forall u$, an absolutely “correct” weighting function

Confidence Intervals

Asymptotic Normality

$$\hat{\tau} = \frac{\sum_i \gamma_+ (Z_i) Y_i}{\sum_i \gamma_+ (Z_i)} - \frac{\sum_i \gamma_- (Z_i) Y_i}{\sum_i \gamma_- (Z_i)}$$

$$\hat{\tau}_\gamma \xrightarrow{p} \theta_\gamma = \frac{\mathbb{E}\left[\alpha_{(1)} (U) h (U, \gamma_+)\right]}{\mathbb{E}\left[h (U, \gamma_+)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)} (U) h (U, \gamma_-)\right]}{\mathbb{E}\left[h (U, \gamma_-)\right]}$$

Asymptotic Normality

Theorem: Asymptotic Normality of $\hat{\tau}$

Suppose the sequence of weighting kernels $\gamma_{+}^{(n)}$ and $\gamma_{-}^{(n)}$ is deterministic, and $\exists \beta \in \left(0, \frac{1}{2}\right), C, C' > 0$ s.t. $\forall n$ large enough:

$$\sup_z \left|\gamma_{\diamond}^{(n)}(z)\right| < C n^{\beta} \mathbb{E}\left[\gamma_{\diamond}^{(n)}(Z_i)\right] \qquad \sup_u \left|h\left(u, \gamma_{\diamond}^{(n)}\right)\right| < C' \mathbb{E}\left[\gamma_{\diamond}^{(n)}(Z_i)\right], \qquad \diamond = \{+, -\}$$

Then

$$\frac{\sqrt{n}\left(\hat{\tau}_{\gamma} - \theta_{\gamma}\right)}{\sqrt{V_{\gamma}}} \xrightarrow{d} \mathcal{N}(0, 1)$$

where

$$V_{\gamma} = \frac{\mathbb{E}\left[\gamma_{+}^{2}(Z_i)\left(Y_i - \mu_{\gamma,+}\right)^{2}\right]}{\mathbb{E}\left[\gamma_{+}(Z_i)\right]^{2}} + \frac{\mathbb{E}\left[\gamma_{-}^{2}(Z_i)\left(Y_i - \mu_{\gamma,-}\right)^{2}\right]}{\mathbb{E}\left[\gamma_{-}(Z_i)\right]^{2}}$$

Asymptotic Normality

Theorem: Asymptotic Normality of $\hat{\tau}$

Suppose the sequence of weighting kernels $\gamma_+^{(n)}$ and $\gamma_-^{(n)}$ is deterministic, and $\exists \beta \in \left(0, \frac{1}{2}\right), C, C' > 0$ s.t. $\forall n$ large enough: $\sup_z \left|\gamma_\diamond^{(n)}(z)\right| < Cn^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]$, $\sup_u \left|h\left(u, \gamma_\diamond^{(n)}\right)\right| < C'\mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]$ where $\diamond = \{+, -\}$ Then

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{V_\gamma}} \xrightarrow{d} \mathcal{N}(0, 1)$$

where $V_\gamma = \frac{\mathbb{E}\left[\gamma_+^2(Z_i)(Y_i - \mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2} + \frac{\mathbb{E}\left[\gamma_-^2(Z_i)(Y_i - \mu_{\gamma,-})^2\right]}{\mathbb{E}[\gamma_-(Z_i)]^2}$

Assumption:

- The repsonse Y_i is bounded: $Y_i \in [0, 1]$
- $\inf_z \operatorname{Var}\left[Y_i \mid Z_i = z\right] > 0$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sqrt{n}\left[(\hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-}) - (\mu_{\gamma,+} - \mu_{\gamma,-})\right]}{\sqrt{\frac{\mathbb{E}\left[\gamma_+^2(Z_i)(Y_i - \mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2} + \frac{\mathbb{E}\left[\gamma_-^2(Z_i)(Y_i - \mu_{\gamma,-})^2\right]}{\mathbb{E}[\gamma_-(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sqrt{n}\left[(\hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-}) - (\mu_{\gamma,+} - \mu_{\gamma,-})\right]}{\sqrt{\frac{\mathbb{E}\left[\gamma_+^2(Z_i)(Y_i - \mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2} + \frac{\mathbb{E}\left[\gamma_-^2(Z_i)(Y_i - \mu_{\gamma,-})^2\right]}{\mathbb{E}[\gamma_-(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\frac{\sqrt{n}\left(\hat{\mu}_{\gamma,+} - \mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1) - \mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sqrt{n}\left[(\hat{\mu}_{\gamma,+}-\hat{\mu}_{\gamma,-})-(\mu_{\gamma,+}-\mu_{\gamma,-})\right]}{\sqrt{\frac{\mathbb{E}\left[\gamma_+^2(Z_i)(Y_i-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}+\frac{\mathbb{E}\left[\gamma_-^2(Z_i)(Y_i-\mu_{\gamma,-})^2\right]}{\mathbb{E}[\gamma_-(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\frac{\sqrt{n}\left(\hat{\mu}_{\gamma,+}-\mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\frac{\sqrt{n}\left(\frac{\sum_i \gamma_+(Z_i)Y_i}{\sum_i \gamma_+(Z_i)}-\mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sqrt{n}\left(\frac{\sum_i \gamma_+(Z_i)Y_i}{\sum_i \gamma_+(Z_i)} - \mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sqrt{n}\left(\frac{\sum_i \gamma_+(Z_i)Y_i}{\sum_i \gamma_+(Z_i)} - \mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\frac{\frac{\sum_i \gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})}{\sum_i \gamma_+(Z_i)}}{\frac{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}}{n\mathbb{E}[\gamma_+(Z_i)]}} = \underbrace{\frac{\sum_i \gamma_+(Z_i)\left(Y_i(1)-\mu_{\gamma,+}\right)}{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2\left(Y_i(1)-\mu_{\gamma,+}\right)^2\right]}}}_{\xrightarrow{d}\mathcal{N}(0,1)} \cdot \underbrace{\frac{1}{\boxed{\frac{\frac{1}{n}\sum_i \gamma_+(Z_i)}{\mathbb{E}\left[\gamma_+(Z_i)\right]}}}}_{=1+o_p(1)} \xrightarrow{d} \mathcal{N}(0,1)$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\sum_i \gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})}{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}} \xrightarrow{d} \mathcal{N}(0,1)$$

- $\boxed{\underline{\sigma}^2 = \inf_z \mathrm{Var}\left[Y_i \mid Z_i = z\right] > 0}$: $\mathrm{Var}\left[\gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})\right] \geq \underline{\sigma}^2 \mathbb{E}\left[\gamma_+(Z_i)^2\right]$
- $\boxed{\sup_u \left|h\left(u, \gamma_\diamond^{(n)}\right)\right| < C'\mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]}$ and $\boxed{Y_i \in [0,1]}$: $|\mu_{\gamma,+}| = \left|\frac{\mathbb{E}\left[\alpha_{(1)}(U)h(U,\gamma_+)\right]}{\mathbb{E}\left[\gamma_+(Z_i)\right]}\right| \leq C'$
- $\boxed{\sup_z \left|\gamma_\diamond^{(n)}(z)\right| < Cn^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]}$ and Lyapunov CLT condition:

$$\frac{n\mathbb{E}\left[\left|\gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})\right|^{2+q}\right]}{\left(n\mathrm{Var}\left[\gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})\right]\right)^{\frac{2+q}{2}}} \leq \left(\frac{C'+1}{\underline{\sigma}}\right)^{2+q}\left(Cn^{\beta-\frac{1}{2}}\right)^q \xrightarrow{0\to\infty} 0$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\frac{1}{n}\sum_i \gamma_+(Z_i)}{\mathbb{E}\left[\gamma_+(Z_i)\right]} \xrightarrow{p} 1$$

- $\boxed{\sup_z \left|\gamma_\diamond^{(n)}(z)\right| < Cn^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}(Z_i)\right]}$ and Chebyshev's inequality:

$$\begin{aligned}\mathbf{P}\left\{\left|\mathbb{E}_n\left[\gamma_+(Z_i)\right] - \mathbb{E}\left[\gamma_+(Z_i)\right]\right| \geq \epsilon \mathbb{E}\left[\gamma_+(Z_i)\right]\right\} &\leq \frac{\operatorname{Var}\left[\gamma_+(Z_i)\right]}{n\epsilon^2 \mathbb{E}\left[\gamma_+(Z_i)\right]^2} \\ &\leq \left(\frac{C}{\epsilon} \cdot n^{\beta - \frac{1}{2}}\right)^2 \xrightarrow{n\to\infty} 0\end{aligned}$$

Asymptotic Normality: A Sketch of the Proof

$$\frac{\frac{\sum_i \gamma_+(Z_i)(Y_i(1)-\mu_{\gamma,+})}{\sum_i \gamma_+(Z_i)}}{\frac{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}}{n\mathbb{E}[\gamma_+(Z_i)]}} = \underbrace{\frac{\sum_i \gamma_+(Z_i)\left(Y_i(1)-\mu_{\gamma,+}\right)}{\sqrt{n\mathbb{E}\left[\gamma_+(Z_i)^2\left(Y_i(1)-\mu_{\gamma,+}\right)^2\right]}}}_{\xrightarrow{d}\mathcal{N}(0,1)} \cdot \underbrace{\frac{1}{\boxed{\frac{\frac{1}{n}\sum_i \gamma_+(Z_i)}{\mathbb{E}\left[\gamma_+(Z_i)\right]}}}}_{=1+o_p(1)} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\Rightarrow \frac{\sqrt{n}\left(\frac{\sum_i \gamma_+(Z_i)Y_i}{\sum_i \gamma_+(Z_i)} - \mu_{\gamma,+}\right)}{\sqrt{\frac{\mathbb{E}\left[\gamma_+(Z_i)^2(Y_i(1)-\mu_{\gamma,+})^2\right]}{\mathbb{E}[\gamma_+(Z_i)]^2}}} \xrightarrow{d} \mathcal{N}(0,1) \Rightarrow \frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{V_\gamma}} \xrightarrow{d} \mathcal{N}(0,1)$$

Plug-in Estimator for V_γ

$$V_\gamma = \frac{\mathbb{E}\left[\gamma_+^2\left(Z_i\right)\left(Y_i - \mu_{\gamma,+}\right)^2\right]}{\mathbb{E}\left[\gamma_+\left(Z_i\right)\right]^2} + \frac{\mathbb{E}\left[\gamma_-^2\left(Z_i\right)\left(Y_i - \mu_{\gamma,-}\right)^2\right]}{\mathbb{E}\left[\gamma_-\left(Z_i\right)\right]^2}$$

Plug-in Estimator for V_γ

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Proposition: Plug-in Estimator $\hat{V}_\gamma$

Under the same assumptions, we have $\frac{\hat{V}_\gamma}{V_\gamma} = 1 + o_p(1)$ where

$$\hat{V}_\gamma = \frac{\frac{1}{n}\sum_{i=1}^n \gamma_+^2\left(Z_i\right)\left(Y_i - \hat{\mu}_{\gamma,+}\right)^2}{\left[\frac{1}{n}\sum_{i=1}^n \gamma_+\left(Z_i\right)\right]^2} + \frac{\frac{1}{n}\sum_{i=1}^n \gamma_-^2\left(Z_i\right)\left(Y_i - \hat{\mu}_{\gamma,-}\right)^2}{\left[\frac{1}{n}\sum_{i=1}^n \gamma_-\left(Z_i\right)\right]^2}$$

Upper Bound for the Potential Bias $|b_\gamma| = |\theta_\gamma - \tau_w|$

$$
\begin{aligned}
a\mathrm{Bias} = \theta_\gamma - \tau_w = & \underbrace{\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{h(u, \gamma_-)}{\mathbb{E}_G[h(U, \gamma_-)]} \right) \alpha_{(0)}(u)\, \mathrm{d}G(u)}_{\text{Confounding bias}} \\
& + \underbrace{\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\, \mathrm{d}G(u)}_{\text{CATE heterogeneity bias}}
\end{aligned}
$$

Upper Bound for the Potential Bias $|b_\gamma| = |\theta_\gamma - \tau_w|$

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\text{aBias} = \theta_\gamma - \tau_w = \underbrace{\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{h(u, \gamma_-)}{\mathbb{E}_G[h(U, \gamma_-)]} \right) \alpha_{(0)}(u)\, \mathrm{d}G(u)}_{\text{Confounding bias}}
$$

$$
+ \underbrace{\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u)\, \mathrm{d}G(u)}_{\text{CATE heterogeneity bias}}
$$

The distribution of $G(\cdot)$ is **unknown**

Upper Bound for the Potential Bias $|b_\gamma| = |\theta_\gamma - \tau_w|$

Bound the **worst-case** bias:

Upper Bound for the Potential Bias $|b_\gamma| = |\theta_\gamma - \tau_w|$

Bound the **worst-case** bias:

- Back out the class of latent variable distribution from $\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(Z_i \leq t)$, the empirical distribution of Z_i (Massart, 1990):

$$\mathcal{G}_n = \left\{ G(\cdot) : \sup_{t\in\mathbb{R}} \left| F_G(t) - \hat{F}_n(t) \right| \leq \sqrt{\frac{\log(2/\alpha_n)}{2n}} \right\}, \qquad \alpha_n = \min\left\{0.05, n^{-1/4}\right\}$$

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- Take treatment effect heterogeneity into consideration:

$$\mathcal{T}_M = \{\tau(\cdot) \mid \tau(u) = \bar{\tau} + \Delta(u), \bar{\tau} \in \mathbb{R}, |\Delta(u)| \leq M\}, \qquad M \in [0,1]$$

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  - $\mathcal{T}_0$ $(M = 0)$: constant CATE
  - $\mathcal{T}_1$ $(M = 1)$: no assumptions
  - $\mathcal{T}_{1/2}$ $(M = 0)$: a conservative choice for a monotonicity restriction

Upper Bound for the Potential Bias $|b_\gamma| = |\theta_\gamma - \tau_w|$

Proposition: Upper Bound for the Potential Bias $\hat{B}_{\gamma,M}$

Under asymptotic normality and all necessary assumptions, for $\tau(\cdot) \in \mathcal{T}_M$, the upper bound of bias is

$$\hat{B}_{\gamma,M} = \sup\left\{|\mathrm{Bias}\left[\gamma_\pm, \tau_w; \alpha_0\left(\cdot\right), \tau\left(\cdot\right), G\right]| : G \in \mathcal{G}_n, \alpha_{(0)}\left(\cdot\right) \in \left[0,1\right], \tau\left(\cdot\right) \in \mathcal{T}_M\right\}$$

then $\mathbf{P}\left(|b_\gamma| \le \hat{B}_{\gamma,M}\right) \xrightarrow{n\to\infty} 1$

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then $\mathbf{P}\left(|b_\gamma| \leq \hat{B}_{\gamma,M}\right) \xrightarrow{n\to\infty} 1$

$$\{G \in \mathcal{G}_n\} \subset \left\{|b_\gamma| \leq \hat{B}_{\gamma,M}\right\} \Rightarrow \mathbf{P}\left(|b_\gamma| \leq \hat{B}_{\gamma,M}\right) \geq \mathbf{P}(G \in \mathcal{G}_n)$$

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$$\{G \in \mathcal{G}_n\} \subset \left\{|b_\gamma| \leq \hat{B}_{\gamma,M}\right\} \Rightarrow \mathbf{P}\left(|b_\gamma| \leq \hat{B}_{\gamma,M}\right) \geq \mathbf{P}\left(G \in \mathcal{G}_n\right)$$

and for $G \in \mathcal{G}_n$, Dvoretzky-Kiefer-Wolfowitz (DKW) inequality gives:

$$\mathbf{P}\left(G \in \mathcal{G}_n\right) \geq \mathbf{P}\left[\sup_{t\in\mathbb{R}}\left|F_G(t) - \hat{F}_n(t)\right| \leq \sqrt{\frac{\log(2/\alpha_n)}{2n}}\right] \geq 1 - \alpha_n \xrightarrow[n\to\infty]{\alpha_n = \min\{0.05, n^{-1/4}\}} 1$$

Recap: What We Have Derived

Recap: What We Have Derived

- Asymptotic limit:

$$\hat{\tau}_\gamma \xrightarrow{p} \theta_\gamma = \frac{\mathbb{E}\left[\alpha_{(1)}\left(U\right)h\left(U,\gamma_+\right)\right]}{\mathbb{E}\left[h\left(U,\gamma_+\right)\right]} - \frac{\mathbb{E}\left[\alpha_{(0)}\left(U\right)h\left(U,\gamma_-\right)\right]}{\mathbb{E}\left[h\left(U,\gamma_-\right)\right]}$$

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- Asymptotic normality:

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{\hat{V}_\gamma}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Recap: What We Have Derived

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- Asymptotic normality:

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{\hat{V}_\gamma}} \xrightarrow{d} \mathcal{N}\left(0, 1\right)$$

- Upper bound of the asymptotic bias:

$$\hat{B}_{\gamma,M} = \sup\left\{\left|\mathrm{Bias}\left[\gamma_\pm, \tau_w; \alpha_0\left(\cdot\right), \tau\left(\cdot\right), G\right]\right| : G \in \mathcal{G}_n, \alpha_{(0)}\left(\cdot\right) \in \left[0, 1\right], \tau\left(\cdot\right) \in \mathcal{T}_M\right\}$$

Bias-aware Confidence Intervals

Corollary: Valid Confidence Intervals

Under asymptotic normality and all necessary assumptions, for $\tau(\cdot) \in \mathcal{T}_M$, consider the CIs

$$\hat{\tau}_\gamma \pm l_\alpha, \qquad l_\alpha = \min\left\{ l : \mathbf{P}\left[\left| b + n^{-\frac{1}{2}} \hat{V}_\gamma^{\frac{1}{2}} \tilde{Z} \right| \leq l \right] \geq 1 - \alpha, \forall\, |b| \leq \hat{B}_{\gamma,M} \right\}$$

where

- $\tilde{Z}$ is a standard Guassian random variable
- $\alpha \in (0, 1)$ is the significant level
- $\hat{V}_\gamma$ is an estimate of the sampling variance V_γ

then

$$\lim_{n\to\infty} \inf \mathbf{P}\left[\tau_w \in \hat{\tau}_\gamma \pm l_\alpha\right] \geq 1 - \alpha$$

Bias-aware Confidence Intervals

CLT (with bias considered) is

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\hat{V}_\gamma^{1/2}} = \frac{\sqrt{n}\left(\hat{\tau}_\gamma - \tau_w - b_\gamma\right)}{\hat{V}_\gamma^{1/2}} \xrightarrow{d} \mathcal{N}\left(0,1\right)$$

where $b_\gamma = \theta_\gamma - \tau_w$, then let $\tilde{Z} \sim \mathcal{N}\left(0,1\right)$ we have

$$\begin{aligned}
\mathbf{P}\left[\tau_w \in \hat{\tau}_\gamma \pm l_\alpha\right] =& \mathbf{P}\left[-l_\alpha - b_\gamma \le \hat{\tau}_\gamma - \tau_w - b_\gamma \le l_\alpha - b_\gamma\right] \\
=& \mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha + b_\gamma\right) \le \sqrt{n}\hat{V}_\gamma^{-1/2}\left(\hat{\tau}_\gamma - \tau_w - b_\gamma\right) \le \sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha - b_\gamma\right)\right] \\
=& \mathbb{E}\left(\mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha + b_\gamma\right) \le \tilde{Z} \le \sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha - b_\gamma\right)\right] \mid \hat{V}_\gamma, \hat{B}_{\gamma,M}, \hat{\tau}_\gamma\right) + o\left(1\right) \\
=& \mathbb{E}\left[\mathbf{P}\left(-l_\alpha \le n^{-1/2}\hat{V}_\gamma^{1/2}\tilde{Z} + b_\gamma \le l_\alpha\right) \mid \hat{V}_\gamma, \hat{B}_{\gamma,M}, \hat{\tau}_\gamma\right] + o\left(1\right) \\
=& 1 - \alpha + o\left(1\right)
\end{aligned}$$

Bias-aware Confidence Intervals

CLT (with bias considered) is

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\hat{V}_\gamma^{1/2}} = \frac{\sqrt{n}\left(\hat{\tau}_\gamma - \tau_w - b_\gamma\right)}{\hat{V}_\gamma^{1/2}} \xrightarrow{d} \mathcal{N}\left(0,1\right)$$

where $b_\gamma = \theta_\gamma - \tau_w$, then let $\tilde{Z} \sim \mathcal{N}\left(0,1\right)$ we have

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=& \mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha + b_\gamma\right) \leq \sqrt{n}\hat{V}_\gamma^{-1/2}\left(\hat{\tau}_\gamma - \tau_w - b_\gamma\right) \leq \sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha - b_\gamma\right)\right] \\
=& \mathbb{E}\left(\mathbf{P}\left[-\sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha + b_\gamma\right) \leq \tilde{Z} \leq \sqrt{n}\hat{V}_\gamma^{-1/2}\left(l_\alpha - b_\gamma\right)\right] \mid \hat{V}_\gamma, \hat{B}_{\gamma,M}, \hat{\tau}_\gamma\right) + o\left(1\right) \\
=& \mathbb{E}\left[\mathbf{P}\left(-l_\alpha \leq n^{-1/2}\hat{V}_\gamma^{1/2}\tilde{Z} + b_\gamma \leq l_\alpha\right) \mid \hat{V}_\gamma, \hat{B}_{\gamma,M}, \hat{\tau}_\gamma\right] + o\left(1\right) \\
=& 1 - \alpha + o\left(1\right)
\end{aligned}$$

Robustness to CATE Heterogeneity Misspecification

$$\hat{\tau}_\gamma \pm l_\alpha, l_\alpha = \min\left\{l : \mathbf{P}\left[\left|b + n^{-\frac{1}{2}}\hat{V}_\gamma^{\frac{1}{2}}\tilde{Z}\right| \leq l\right] \geq 1-\alpha, \forall\, |b| \leq \hat{B}_{\gamma,M}\right\}$$

where

$$\mathcal{T}_M = \{\tau(\cdot) \mid \tau(u) = \bar{\tau} + \Delta(u), \bar{\tau} \in \mathbb{R}, |\Delta(u)| \leq M\},\ M \in [0,1]$$

Consider an extreme misspecification of CATE heterogeneity: $M = 0$, are the CIs robust?

Robustness to CATE Heterogeneity Misspecification

$$\hat{\tau}_\gamma \pm l_\alpha, l_\alpha = \min \left\{ l : \mathbf{P} \left[\left| b + n^{-\frac{1}{2}} \hat{V}_\gamma^{\frac{1}{2}} \tilde{Z} \right| \le l \right] \ge 1 - \alpha, \forall \, |b| \le \hat{B}_{\gamma,M} \right\}$$

where

$$\mathcal{T}_M = \{ \tau(\cdot) \mid \tau(u) = \bar{\tau} + \Delta(u), \bar{\tau} \in \mathbb{R}, |\Delta(u)| \le M \}, \quad M \in [0,1]$$

Corollary: Robustness to CATE Heterogeneity Misspecification

The CIs under the misspecification of $M = 0$ is still valid, but only for the convenience-weighted treatment effect:

$$\tau_{h,+} := \int \frac{h(u, \gamma_+)}{\mathbb{E}_G [h(U, \gamma_+)]} \tau(u) \, \mathrm{d}G(u)$$

Robustness to CATE Heterogeneity Misspecification

Corollary: Robustness to CATE Heterogeneity Misspecification

The CIs under the misspecification of $M = 0$ is still valid, but only for:

$$\tau_{h,+} := \int \frac{h(u, \gamma_+)}{\mathbb{E}_G [h(U, \gamma_+)]} \tau(u) \, \mathrm{d}G(u)$$

$$\begin{aligned} a\text{Bias} = \theta_\gamma - \tau_w = & \int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G [h(U, \gamma_+)]} - \frac{h(u, \gamma_-)}{\mathbb{E}_G [h(U, \gamma_-)]} \right) \alpha_{(0)}(u) \, \mathrm{d}G(u) \\ & + \underbrace{\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G [h(U, \gamma_+)]} - \frac{w(u)}{\mathbb{E}_G [w(U)]} \right) \tau(u) \, \mathrm{d}G(u)}_{\text{CATE heterogeneity bias}} \end{aligned}$$

Applications

Design Estimators

The goal: Make the confidence intervals **shorter**

$$\hat{\tau}_\gamma \pm l_\alpha, \qquad l_\alpha = \min\left\{ l : \mathbf{P}\left[\left| b + n^{-\frac{1}{2}} \hat{V}_\gamma^{\frac{1}{2}} \tilde{Z} \right| \leq l\right] \geq 1 - \alpha, \forall\, |b| \leq \hat{B}_{\gamma,M} \right\}$$

by minimizing the worst-case MSE of

$$\hat{\tau} = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+(Z_i)\, Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i)\, Y_i}{\sum_i \gamma_-(Z_i)}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_{\pm}(\cdot)} \frac{1}{n}\left(\int \gamma_{-}^{2}(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_{+}^{2}(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1 + t_2)^2$$

s.t.

$$\begin{aligned}
|h(u,\gamma_{+}) - h(u,\gamma_{-})| &\le t_1, & \forall u \\
M\,|h(u,\gamma_{\diamond}) - \bar{w}(u)| &\le t_2, & \forall u, \diamond \in \{\pm\}
\end{aligned}$$

$$\int \gamma_{+}(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_{-}(z)\,\mathrm{d}\bar{F}(z) = 1$$

$$\begin{aligned}
\gamma_{-}(z) &= 0, & z \ge c \\
\gamma_{+}(z) &= 0, & z < c \\
|\gamma_{\diamond}(z)| &\le C n^{\beta}, & \forall z, \diamond \in \{\pm\}
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_{\pm}(\cdot)} \frac{1}{n}\left(\int \gamma_{-}^{2}(z)\,\mathrm{d}\bar{F}(z)+\int \gamma_{+}^{2}(z)\,\mathrm{d}\bar{F}(z)\right)+(t_1+t_2)^2$$

s.t.

$$\begin{aligned}
|h(u,\gamma_+)-h(u,\gamma_-)| &\le t_1, && \forall u\\
M\,|h(u,\gamma_\diamond)-\bar{w}(u)| &\le t_2, && \forall u, \diamond\in\{\pm\}
\end{aligned}$$

$$\int \gamma_+(z)\,\mathrm{d}\bar{F}(z)=\int \gamma_-(z)\,\mathrm{d}\bar{F}(z)=1$$

$$\begin{aligned}
\gamma_-(z) &= 0, && z\ge c\\
\gamma_+(z) &= 0, && z< c\\
|\gamma_\diamond(z)| &\le Cn^{\beta}, && \forall z, \diamond\in\{\pm\}
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_\pm(\cdot)} \frac{1}{n}\left(\int \gamma_-^2(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_+^2(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1+t_2)^2$$

s.t.

$$\begin{aligned}
|h(u,\gamma_+) - h(u,\gamma_-)| &\le t_1, && \forall u && \text{confounding bias}\\
M\,|h(u,\gamma_\diamond) - \bar{w}(u)| &\le t_2, && \forall u, \diamond \in \{\pm\} && \text{CATE-hetrogeneity bias}\\
\int \gamma_+(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_-(z)\,\mathrm{d}\bar{F}(z) &= 1 && &&\\
\gamma_-(z) &= 0, && z \ge c &&\\
\gamma_+(z) &= 0, && z < c &&\\
|\gamma_\diamond(z)| &\le Cn^\beta, && \forall z, \diamond \in \{\pm\} &&
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_{\pm}(\cdot)} \frac{1}{n}\left(\int \gamma_{-}^{2}(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_{+}^{2}(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1 + t_2)^2$$

s.t.

$$\begin{aligned}
|h(u,\gamma_+) - h(u,\gamma_-)| &\le t_1, & \forall u \quad & \text{confounding bias}\\
M\,|h(u,\gamma_\diamond) - \bar{w}(u)| &\le t_2, & \forall u, \diamond \in \{\pm\} \quad & \text{CATE-hetrogeneity bias}\\
\int \gamma_+(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_-(z)\,\mathrm{d}\bar{F}(z) &= 1 & & \text{normalization constraint}\\
\gamma_-(z) &= 0, & z \ge c \quad & \text{Sharp RD}\\
\gamma_+(z) &= 0, & z < c \quad & \\
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|\gamma_\diamond(z)| &\le Cn^\beta, && \forall z, \diamond \in \{\pm\} && \text{no observation is given excessive influence}
\end{aligned}$$

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$\bar{F}(\cdot)$: $$F_G(t) = \int \mathbf{1}(\{z \le t\}) \int p(z \mid u)\,\mathrm{d}G(u)\,\mathrm{d}\lambda(z)$$

$\bar{w}(\cdot)$: $$\tau_w = \int \frac{w(u)}{\mathbb{E}_G[w(U)]}\tau(u)\,\mathrm{d}G(u)$$

Design Estimators: Quadratic Programming

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- $\bar{F}(\cdot)$ assigns non-trivial mass to $[c, \infty)$ and $\bar{w}(\cdot)$ is bounded: $\exists k > 1$ s.t.

$$\mathbb{P}\left[\frac{1}{k} < \bar{F}([c, \infty)) < 1 - \frac{1}{k}, \sup_u |\bar{w}(u)| < k\right] \xrightarrow{n \to \infty} 1$$

- $\int \gamma_{\diamond}^{(n)}(z)\, \mathrm{d}F(z)$ is asymptotically lower bounded by a strictly positive number:

$$\exists \delta > 0 \text{ s.t. } \mathbb{P}\left[\int \gamma_{\diamond}^{(n)}(z)\, \mathrm{d}F(z) > \delta\right] \xrightarrow{n \to \infty} 1$$

Design Estimators: Quadratic Programming

$$\begin{array}{r} \dfrac{1}{k} < \bar{F}\left([c,\infty)\right) < 1 - \dfrac{1}{k}, \sup_u \left|\bar{w}\left(u\right)\right| < k \\ \displaystyle\int \gamma_\diamond^{(n)}\left(z\right) \mathrm{d}F\left(z\right) > \delta \end{array} \Rightarrow \begin{array}{l} \sup_z \left|\gamma_\diamond^{(n)}\left(z\right)\right| < Cn^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}\left(Z_i\right)\right] \\ \sup_u \left|h\left(u, \gamma_\diamond^{(n)}\right)\right| < C'\mathbb{E}\left[\gamma_\diamond^{(n)}\left(Z_i\right)\right] \end{array}$$

Design Estimators: Quadratic Programming

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Theorem: Asymptotic Normality of $\hat{\tau}$

Suppose the sequence of weighting kernels $\gamma_+^{(n)}$ and $\gamma_-^{(n)}$ is deterministic, and $\exists \beta \in \left(0, \frac{1}{2}\right), C, C' > 0$ s.t. $\forall n$ large enough: $\sup_z \left|\gamma_\diamond^{(n)}\left(z\right)\right| < Cn^\beta \mathbb{E}\left[\gamma_\diamond^{(n)}\left(Z_i\right)\right]$, $\sup_u \left|h\left(u, \gamma_\diamond^{(n)}\right)\right| < C'\mathbb{E}\left[\gamma_\diamond^{(n)}\left(Z_i\right)\right]$ where $\diamond = \{+,-\}$ Then

$$\frac{\sqrt{n}\left(\hat{\tau}_\gamma - \theta_\gamma\right)}{\sqrt{V_\gamma}} \xrightarrow{d} \mathcal{N}\left(0,1\right)$$

Design Estimators: Procedure

- **Input:**
  - samples $\{Z_i, Y_i, W_i\}$ and cutoff c
  - sensitivity model $\mathcal{T}_M$, estimand of interest τ_w
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- **Input:**
  - samples $\{Z_i, Y_i, W_i\}$ and cutoff c
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  - nominal significance level α
- **Procedure:**
  - S1 guess/estimate $\bar{F}(\cdot)$ and $\bar{w}(\cdot)$ via nonparametric maximum likelihood
  - S2 solve the minimax program, get γ_+, γ_-
  - S3 form the point estimate $\hat{\tau}_\gamma$ and its variance $\hat{V}_\gamma$
  - S4 estimate the worst-case bias $\hat{B}_\gamma = \sup\left\{\left|\text{Bias}\left[\gamma_\pm, \tau_w; \alpha_0(\cdot), \tau(\cdot), G\right]\right| : G \in \mathcal{G}_n, \alpha_{(0)}(\cdot) \in [0,1], \tau(\cdot) \in \mathcal{T}_M\right\}$
  - S5 form the bias-aware CIs at level α as $\hat{\tau}_\gamma \pm l_\alpha, l_\alpha = \min\left\{l : \mathbf{P}\left[\left|b + n^{-\frac{1}{2}}\hat{V}_\gamma^{\frac{1}{2}}\tilde{Z}\right| \leq l\right] \geq 1-\alpha, \forall |b| \leq \hat{B}_{\gamma,M}\right\}$

Discussion

Literature: Continuity-Based RD

Most popular: local linear regression (Hahn et al., 2001; G. W. Imbens and Lemieux, 2008)

$$\hat{\tau}_c = \arg\min_{\tau} \left\{ \sum_{i=1}^{n} \underbrace{K}_{\text{weighting}} \left(\frac{|Z_i - c|}{\underbrace{h_n}_{\text{bandwidth}}} \right) \left(Y_i - a - \tau W_i - \beta_- (Z_i - c)_- - \beta_+ (Z_i - c)_+ \right)^2 \right\}$$

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- $\mu_{(w)}(z) = \mathbb{E}\left[Y(w) \mid Z = z\right]$ is **smooth**
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Robust CIs (Armstrong and Kolesár, 2020; Calonico et al., 2014; Kolesár and Rothe, 2018); Data-adaptive bandwidths (G. Imbens and Kalyanaraman, 2012)

Literature: Continuity-Based RD extended

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If further assume **convexity** of $\mu_{(w)}(z)$, e.g.:

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Optimization-based RD: the treatment effect τ_c can be estimated (minimax linear estimation) via **numerical convex optimization** (Armstrong and Kolesár, 2018; G. Imbens and Wager, 2019)

Link Noise-Induced RD and Continuity-Based RD

$$\mu_{(w)}(z) = \mathbb{E}\left[Y_i(w) \mid Z_i = z\right] = \frac{\int \mathbb{E}\left[Y_i(w) \mid U_i = u, Z_i = z\right] p(z \mid u)\,\mathrm{d}G(u)}{f_G(z)} = \frac{\int \alpha_{(w)}(u)\, p(z \mid u)\mathrm{d}G(u)}{\int p(z \mid u)\mathrm{d}G(u)}$$

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Then the worst-case possible curvature is:

$$\mathrm{Curv}(z, \rho, p) = \sup\left\{\left|\frac{\mathrm{d}^2\mu_{(w)}(z)}{\mathrm{d}z^2}\right| : f_G(z) = \int p(z \mid u)\, \mathrm{d}G(u) \geq \rho > 0, \alpha_{(w)}(\cdot) \in [0, 1]\right\}$$

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Armstrong and Kolesár (2020): fit 4th-degree polynomials to $\mu_{(0)}(z)$ and $\mu_{(1)}(z)$, and take the largest estimated curvature obtained anywhere

Literature: Randomization Inference RD

Posit a non-trivial interval $\mathcal{I}$ with $c \in \mathcal{I}$ s.t.

$$\{Y_i(0), Y_i(1)\} \perp Z_i \mid \{Z_i \in \mathcal{I}\}$$

then focus on this interval, perform classical **randomized** study inference

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- Design-based approach (Rubin, 2008)
- Strong assumption
  No **data-driven way** of choosing $\mathcal{I}$
  If the interval $\mathcal{I}$ is known a-priori, the problem collapses to a **RCT**

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- **NOT** assuming prior knowledge of the noise distribution $p(\cdot \mid u)$
- A stronger assumption: observing at least 3 noisy measurements of the latent variable U_i, $\{Z_i, Z_i', Z_i''\}$
  - (U_i, Z_i, Z_i', Z_i'') is **joint normal**
  - $\alpha_{(w)}(u) = \mathbb{E}\left[Y_i(w) \mid U_i = u\right]$ is **linear** w.r.t. u

RD with Ordinal Running Variables

Similarly, ordinal Z_i (bond rating, custody security score, etc.) can be seen as a noisy measurement of a latent variable U_i.
Li et al. (2021) assume

$$U_i = \mathbf{X}_i \beta$$

then use **inverse-propensity weighting** with estimated propensities $e(u) = \mathbb{P}\left[Z_i \geq c \mid U_i = u\right]$ for inference.

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Assuming: U_i can be observed, and well predicted by $\mathbf{X}_i$

Measurement Errors

- The **running variable** is unobserved, only a noisy measurement is observed
  Bartalotti et al. (2021), Davezies and Le Barbanchon (2017), Dong and Kolesár (2021), and Pei and Shen (2017)
- Measurement error in causal inference beyond RD
  Jiang and Ding (2020), Kuroki and Pearl (2014), and Pearl (2012)

A Comparison

RD designs	Assumptions
Noise-induced RD	a **known distribution** of the measurement error $p(\cdot \mid u)$
Noise-induced RD (Rokkanen, 2015)	**multiple joint-normal** noisy measurements (U_i, Z_i, Z_i', Z_i'')
	linear $\alpha_{(w)}(u) = \mathbb{E}\left[Y_i(w) \mid U_i = u\right]$
Continuity-based RD	$\mu_{(w)} = \mathbb{E}\left[Y(w) \mid Z = z\right]$ is **smooth**
OPtimization-based RD	**convexity** of $\mu_{(w)}(z)$: $\left\lvert \mu_{(w)}''(z) \right\rvert \leq B, \forall z \in \mathbb{R}$
Randomization inference RD	an **"RCT"** interval $\mathcal{I}$: $\{Y_i(0), Y_i(1)\} \perp Z_i \mid \{Z_i \in \mathcal{I}\}$
RD with ordinal Z_i	U_i can be observed, and well predicted by $\mathbf{X}_i$

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Thank you!