

Noise-Induced Randomization in Regression Discontinuity Designs

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November 18, 2022

Outline

1 Estimation

Estimation

Estimation of Weighted Treatment Effects

Proposition 1

Let $\gamma_+(\cdot), \gamma_-(\cdot)$ be measurable functions of Z , then under A1-A3:

$$\mathbb{E}[\gamma_+(Z)Y] = \mathbb{E}[\alpha_{(1)}(U)h(U, \gamma_+)], \quad \mathbb{E}[\gamma_-(Z)Y] = \mathbb{E}[\alpha_{(0)}(U)h(U, \gamma_-)]$$

where $h(u, \gamma) := \int \gamma(z) p(z | u) d\lambda(z)$, $\alpha_{(w)}(u) = \mathbb{E}[Y_i(w) | U_i = u]$

ratio-form estimators:

$$\hat{\mu}_{\gamma,+} = \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \underbrace{\gamma_+(Z_i)}}$$

$$\hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \underbrace{\gamma_-(Z_i)}}$$

Estimation of Weighted Treatment Effects

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Let $\gamma_+(\cdot), \gamma_-(\cdot)$ be measurable functions of Z , then under A1-A3:

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ratio-form estimators:

$$\hat{\mu}_{\gamma,+} = \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \underbrace{\gamma_+(Z_i)}_{\gamma_+(z)=0, z < c}}$$

$$\hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \underbrace{\gamma_-(Z_i)}_{\gamma_-(z)=0, z \geq c}}$$

Estimation of Weighted Treatment Effects

Ratio-form estimators:

$$\frac{\mathbb{E}[\gamma_+(Z)Y]}{\mathbb{E}[\gamma_+(Z)]} \Rightarrow \hat{\mu}_{\gamma,+} = \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)}$$
$$\frac{\mathbb{E}[\gamma_-(Z)Y]}{\mathbb{E}[\gamma_-(Z)]} \Rightarrow \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)}$$

Estimation of Weighted Treatment Effects

Ratio-form estimators:

$$\begin{aligned} \frac{\mathbb{E}[\gamma_+(Z)Y]}{\mathbb{E}[\gamma_+(Z)]} &\Rightarrow \hat{\mu}_{\gamma,+} = \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)} \\ \frac{\mathbb{E}[\gamma_-(Z)Y]}{\mathbb{E}[\gamma_-(Z)]} &\Rightarrow \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)} \end{aligned} \quad \Rightarrow \hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-}$$

Estimation of Weighted Treatment Effects

Ratio-form estimators:

$$\hat{\tau}_{\gamma} = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)}$$

What's the weighted treatment effects to be estimated?

Estimation of Weighted Treatment Effects

Ratio-form estimators:

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What's the weighted treatment effects to be estimated?

$$\tau_w = \int \frac{w(u)}{\mathbb{E}_G[w(U)]} \tau(u) dG(u), w(\cdot) \geq 0$$

Estimation of Weighted Treatment Effects

Ratio-form estimators:

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where $\tau(u)$ (**Conditional Average Treatment Effects**) is

$$\tau(u) = \mathbb{E}[Y_i(1) - Y_i(0) \mid U_i = u] = \alpha_{(1)}(u) - \alpha_{(0)}(u)$$

Weighted Treatment Effects: Examples

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■ RD paramater:

$$\tau_c = \mathbb{E}[Y_i(1) - Y_i(0) \mid Z_i = c] = \int \frac{p(c \mid u)}{\int p(c \mid u) dG(u)} \tau(u) dG(u)$$

Weighted Treatment Effects: Examples

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■ Changing the cutoff:

$$\tau_\pi = \mathbb{E}[Y_i(1) - Y_i(0) \mid c' \leq Z_i < c] = \int \tau(u) \frac{\int_{[c', c)} p(z \mid u) d\lambda(z)}{\int_{[c', c)} dF(z)} dG(u)$$

Asymptotic Bias

Theorem 2: Asymptotic Limit of $\hat{\tau}_\gamma$

$$\hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)}$$

Asymptotic Bias

Theorem 2: Asymptotic Limit of $\hat{\tau}_\gamma$

$$\begin{aligned} \hat{\tau}_\gamma = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} &= \frac{\sum_i \gamma_+(Z_i) Y_i}{\sum_i \gamma_+(Z_i)} - \frac{\sum_i \gamma_-(Z_i) Y_i}{\sum_i \gamma_-(Z_i)} \\ &\xrightarrow{p} \frac{\mathbb{E} [\alpha_{(1)}(U) h(U, \gamma_+)]}{\mathbb{E} [h(U, \gamma_+)]} - \frac{\mathbb{E} [\alpha_{(0)}(U) h(U, \gamma_-)]}{\mathbb{E} [h(U, \gamma_-)]} = \mu_{\gamma,+} - \mu_{\gamma,-} \equiv \theta_\gamma \end{aligned}$$

Asymptotic Bias

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How biased is this asymptotic limit? Comparing to

$$\tau_w = \int \frac{w(u)}{\mathbb{E}_G[w(U)]} \tau(u) dG(u), w(\cdot) \geq 0$$

Asymptotic Bias: Decomposition

$$a\text{Bias} \left[\gamma_{\pm}, \tau_w; \alpha_{(0)}(\cdot), \tau(\cdot), G \right] = \theta_{\gamma} - \tau_w$$

Asymptotic Bias: Decomposition

$$\begin{aligned}
 a\text{Bias} [\gamma_{\pm}, \tau_w; \alpha_{(0)}(\cdot), \tau(\cdot), G] &= \theta_{\gamma} - \tau_w \\
 &= \frac{\mathbb{E} [\alpha_{(1)}(U) h(U, \gamma_+)]}{\mathbb{E} [h(U, \gamma_+)]} - \frac{\mathbb{E} [\alpha_{(0)}(U) h(U, \gamma_-)]}{\mathbb{E} [h(U, \gamma_-)]} - \int \frac{w(u)}{\mathbb{E}_G[w(U)]} \tau(u) dG(u)
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 &= \int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} \right) \alpha_{(1)}(u) dG(u) - \int \left(\frac{h(u, \gamma_-)}{\mathbb{E}_G[h(U, \gamma_-)]} \right) \alpha_{(0)}(u) dG(u) \\
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Remember? $\tau(u)$ **(Conditional Average Treatment Effects)** is

$$\tau(u) = \mathbb{E}[Y_i(1) - Y_i(0) \mid U_i = u] = \alpha_{(1)}(u) - \alpha_{(0)}(u) \Rightarrow \boxed{\alpha_{(1)}(u) = \tau(u) + \alpha_{(0)}(u)}$$

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 a\text{Bias} [\gamma_{\pm}, \tau_w; \alpha_{(0)} (\cdot), \tau (\cdot), G] &= \theta_{\gamma} - \tau_w \\
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 &\quad - \int \left(\frac{h(u, \gamma_-)}{\mathbb{E}_G [h(U, \gamma_-)]} \right) \alpha_{(0)}(u) dG(u) - \int \frac{w(u)}{\mathbb{E}_G [w(U)]} \tau(u) dG(u)
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 &= \int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G [h(U, \gamma_+)]} - \frac{h(u, \gamma_-)}{\mathbb{E}_G [h(U, \gamma_-)]} \right) \alpha_{(0)}(u) dG(u) \\
 &\quad + \int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G [h(U, \gamma_+)]} - \frac{w(u)}{\mathbb{E}_G [w(U)]} \right) \tau(u) dG(u)
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 &\quad - \int \left(\frac{h(u, \gamma_-)}{\mathbb{E}_G [h(U, \gamma_-)]} \right) \alpha_{(0)}(u) dG(u) - \int \frac{w(u)}{\mathbb{E}_G [w(U)]} \tau(u) dG(u) \\
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 \end{aligned}$$

Confounding bias

CATE heterogeneity bias

Asymptotic Bias: Decomposition

$$\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{h(u, \gamma_-)}{\mathbb{E}_G[h(U, \gamma_-)]} \right) \alpha_{(0)}(u) dG(u)$$

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CATE heterogeneity bias

How to minimize them?

Asymptotic Bias: Decomposition

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CATE heterogeneity bias

How to minimize them?

- **Confounding bias:** $h(\cdot, \gamma_+) \approx h(\cdot, \gamma_-)$

Asymptotic Bias: Decomposition

$$\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{h(u, \gamma_-)}{\mathbb{E}_G[h(U, \gamma_-)]} \right) \alpha_{(0)}(u) dG(u) \quad \text{Confounding bias}$$

$$\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{w(u)}{\mathbb{E}_G[w(U)]} \right) \tau(u) dG(u) \quad \text{CATE heterogeneity bias}$$

How to minimize them?

- **Confounding bias:** $h(\cdot, \gamma_+) \approx h(\cdot, \gamma_-)$ where $h(u, \gamma) := \int \gamma(z) p(z | u) d\lambda(z)$
How well the units are **balanced** via the latent variable u

Asymptotic Bias: Decomposition

$$\int \left(\frac{h(u, \gamma_+)}{\mathbb{E}_G[h(U, \gamma_+)]} - \frac{h(u, \gamma_-)}{\mathbb{E}_G[h(U, \gamma_-)]} \right) \alpha_{(0)}(u) dG(u)$$

Confounding bias

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Asymptotic Bias: Decomposition

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Asymptotic Bias: Decomposition

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- **CATE heterogeneity bias:**

- $\tau(u)$ being constant w.r.t. u , a constant conditional treatment effect

Asymptotic Bias: Decomposition

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CATE heterogeneity bias

How to minimize them?

- **Confounding bias:** $h(\cdot, \gamma_+) \approx h(\cdot, \gamma_-)$

How well the units are **balanced** via the latent variable u

- **CATE heterogeneity bias:**

- $\tau(u)$ being constant w.r.t. u , a constant conditional treatment effect
- $h(u, \gamma_+) = w(u), \forall u$, an absolutely correct weighting function

References I

Eckles, D., Ignatiadis, N., Wager, S., & Wu, H. (2020). Noise-induced randomization in regression discontinuity designs. *arXiv preprint arXiv:2004.09458*.

Thank you!