Noise-Induced Randomization in Regression Discontinuity Designs

Dean Eckles, Nikolaos Ignatiadis, Stefan Wager, Han Wu

Presented by: Sai Zhang

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Outline

Applications

Design Estimators

The goal: Make the confidence intervals **shorter**

$$\hat{\tau}_\gamma \pm l_\alpha, \qquad l_\alpha = \min\left\{ l : \mathbf{P}\left[\left| b + n^{-\frac{1}{2}} \hat{V}_\gamma^{\frac{1}{2}} \tilde{Z} \right| \leq l \right] \geq 1 - \alpha, \forall \, |b| \leq \hat{B}_{\gamma,M} \right\}$$

by minimizing the worst-case MSE of

$$\hat{\tau} = \hat{\mu}_{\gamma,+} - \hat{\mu}_{\gamma,-} = \frac{\sum_i \gamma_+ (Z_i) Y_i}{\sum_i \gamma_+ (Z_i)} - \frac{\sum_i \gamma_- (Z_i) Y_i}{\sum_i \gamma_- (Z_i)}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_\pm(\cdot)} \frac{1}{n}\left(\int \gamma_-^2(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_+^2(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1+t_2)^2$$

s.t.

$$\begin{aligned}
|h(u,\gamma_+) - h(u,\gamma_-)| &\le t_1, && \forall u\\
M\,|h(u,\gamma_\diamond) - \bar{w}(u)| &\le t_2, && \forall u, \diamond \in \{\pm\}\\
\int \gamma_+(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_-(z)\,\mathrm{d}\bar{F}(z) &= 1\\
\gamma_-(z) &= 0, && z \ge c\\
\gamma_+(z) &= 0, && z < c\\
|\gamma_\diamond(z)| &\le Cn^\beta, && \forall z, \diamond \in \{\pm\}
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_{\pm}(\cdot)} \frac{1}{n}\left(\int \gamma_{-}^{2}(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_{+}^{2}(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1 + t_2)^2$$

s.t.

$$\begin{aligned}
|h(u,\gamma_{+}) - h(u,\gamma_{-})| &\leq t_1, & \forall u \\
M\,|h(u,\gamma_{\diamond}) - \bar{w}(u)| &\leq t_2, & \forall u, \diamond \in \{\pm\}
\end{aligned}$$

$$\int \gamma_{+}(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_{-}(z)\,\mathrm{d}\bar{F}(z) = 1$$

$$\begin{aligned}
\gamma_{-}(z) &= 0, & z \geq c \\
\gamma_{+}(z) &= 0, & z < c \\
|\gamma_{\diamond}(z)| &\leq Cn^{\beta}, & \forall z, \diamond \in \{\pm\}
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_\pm(\cdot)} \frac{1}{n}\left(\int \gamma_-^2(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_+^2(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1+t_2)^2$$

s.t.

$$\begin{aligned}
|h(u,\gamma_+) - h(u,\gamma_-)| &\le t_1, & \forall u && \text{confounding bias}\\
M\,|h(u,\gamma_\diamond) - \bar{w}(u)| &\le t_2, & \forall u, \diamond \in \{\pm\} && \text{CATE-hetrogeneity bias}\\
\int \gamma_+(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_-(z)\,\mathrm{d}\bar{F}(z) &= 1 &&\\
\gamma_-(z) &= 0, & z \ge c &&\\
\gamma_+(z) &= 0, & z < c &&\\
|\gamma_\diamond(z)| &\le Cn^\beta, & \forall z, \diamond \in \{\pm\} &&
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_{\pm}(\cdot)} \frac{1}{n}\left(\int \gamma_{-}^{2}(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_{+}^{2}(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1 + t_2)^2$$

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\int \gamma_{+}(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_{-}(z)\,\mathrm{d}\bar{F}(z) &= 1 && && \text{normalization constraint}\\
\gamma_{-}(z) &= 0, && z \ge c && \text{Sharp RD}\\
\gamma_{+}(z) &= 0, && z < c &&\\
|\gamma_{\diamond}(z)| &\le C n^{\beta}, && \forall z, \diamond \in \{\pm\} &&
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_\pm(\cdot)} \frac{1}{n}\left(\int \gamma_-^2(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_+^2(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1+t_2)^2$$

s.t.

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\int \gamma_+(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_-(z)\,\mathrm{d}\bar{F}(z) &= 1 && && \text{normalization constraint}\\
\gamma_-(z) &= 0, && z \ge c && \text{Sharp RD}\\
\gamma_+(z) &= 0, && z < c &&\\
|\gamma_\diamond(z)| &\le C n^\beta, && \forall z, \diamond \in \{\pm\} && \text{no observation is given excessive influence}
\end{aligned}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_{\pm}(\cdot)} \frac{1}{n}\left(\int \gamma_{-}^{2}(z)\,\mathrm{d}\bar{F}(z)+\int \gamma_{+}^{2}(z)\,\mathrm{d}\bar{F}(z)\right)+(t_1+t_2)^2$$

s.t.

$$M\left|h(u,\gamma_{\diamond})-\bar{w}(u)\right|\leq t_2, \qquad \forall u, \diamond\in\{\pm\} \qquad \text{CATE-hetrogeneity bias}$$

$$\int \gamma_{+}(z)\,\mathrm{d}\bar{F}(z)=\int \gamma_{-}(z)\,\mathrm{d}\bar{F}(z)=1 \qquad \text{normalization constraint}$$

Design Estimators: Quadratic Programming

Solve

$$\min_{\gamma_{\pm}(\cdot)} \frac{1}{n}\left(\int \gamma_{-}^{2}(z)\,\mathrm{d}\bar{F}(z) + \int \gamma_{+}^{2}(z)\,\mathrm{d}\bar{F}(z)\right) + (t_1 + t_2)^2$$

s.t.

$$M\left|h(u, \gamma_{\diamond}) - \bar{w}(u)\right| \leq t_2, \qquad \forall u, \diamond \in \{\pm\} \qquad \text{CATE-hetrogeneity bias}$$

$$\int \gamma_{+}(z)\,\mathrm{d}\bar{F}(z) = \int \gamma_{-}(z)\,\mathrm{d}\bar{F}(z) = 1 \qquad \text{normalization constraint}$$

$$\bar{F}(\cdot): \qquad F_G(t) = \int \mathbf{1}(\{z \leq t\}) \int p(z \mid u)\,\mathrm{d}G(u)\,\mathrm{d}\lambda(z)$$

$$\bar{w}(\cdot): \qquad \tau_w = \int \frac{w(u)}{\mathbb{E}_G[w(U)]}\tau(u)\,\mathrm{d}G(u)$$

Design Estimators: Quadratic Programming

$$\bar{F}(\cdot): \qquad F_G(t) = \int \mathbf{1}(\{z \le t\}) \int p(z \mid u)\, \mathrm{d}G(u)\, \mathrm{d}\lambda(z)$$

$$\bar{w}(\cdot): \qquad \tau_w = \int \frac{w(u)}{\mathbb{E}_G[w(U)]} \tau(u)\, \mathrm{d}G(u)$$

- $\bar{F}(\cdot)$ assigns non-trivial mass to $[c, \infty)$ and $\bar{w}(\cdot)$ is bounded: $\exists k > 1$ s.t.

$$\mathbb{P}\left[\frac{1}{k} < \bar{F}([c, \infty)) < 1 - \frac{1}{k}, \sup_u |\bar{w}(u)| < k\right] \xrightarrow{n \to \infty} 1$$

- $\int \gamma_\diamond^{(n)}(z)\, \mathrm{d}F(z)$ is asymptotically lower bounded by a strictly positive number:

$$\exists \delta > 0 \text{ s.t. } \mathbb{P}\left[\int \gamma_\diamond^{(n)}(z)\, \mathrm{d}F(z) > \delta\right] \xrightarrow{n \to \infty} 1$$

References I

Eckles, D., Ignatiadis, N., Wager, S., & Wu, H. (2020). Noise-induced randomization in regression discontinuity designs. *arXiv preprint arXiv:2004.09458.*

Thank you!